Colloquium: Quantum and classical discrete time crystals

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The spontaneous breaking of time-translation symmetry has led to the discovery of a new phase of matter: the discrete time crystal. Discrete time crystals exhibit rigid subharmonic oscillations that result from a combination of many-body interactions, collective synchronization, and ergodicity breaking. This Colloquium reviews recent theoretical and experimental advances in the study of quantum and classical discrete time crystals. The breaking of ergodicity is focused upon as the key to discrete time crystals and the delaying of ergodicity as the source of numerous phenomena that share many of the properties of discrete time crystals, including the ac Josephson effect, coupled map lattices, and Faraday waves. Theoretically, there is a diverse array of strategies to stabilize time-crystalline order in both closed and open systems, ranging from localization and prethermalization to dissipation and error correction. Experimentally, many-body quantum simulators provide a natural platform for investigating signatures of time-crystalline order; recent work utilizing trapped ions, solid-state spin systems, and superconducting qubits are reviewed. Finally, this Colloquium concludes by describing outstanding challenges in the field and a vision for new directions on both the experimental and theoretical fronts.

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I. INTRODUCTION: SPONTANEOUS BREAKING OF TIME-TRANSLATION SYMMETRY

Spontaneous symmetry breaking is a noteworthy collective phenomenon: an assembly of constituents, each interacting with only its nearby neighbors, manages to align its behavior across large spatial and temporal separations. The concept has wide-ranging applications, from crystalline and magnetic ordering to superfluidity, superconductivity, and the generation of particle masses. Recently there has been a burst of activity surrounding the spontaneous breaking of symmetries that involve time translation (Shapere and Wilczek, 2012b; Wilczek, 2012; Sacha, 2015; Else, Bauer, and Nayak, 2016; Khemani *et al.*, 2016; Yao *et al.*, 2017). These explorations have exposed new phenomena and new opportunities, but also important subtleties. In this Colloquium, we discuss both the resulting sharpening of theoretical concepts and the discovery of previously unsuspected new phases of matter. We also discuss open questions and potential applications.

Landau-Ginzburg theory is the starting point for many theoretical treatments of spontaneous symmetry breaking. In a simple but representative example of this framework, one considers the theory of a complex scalar field $\phi(x, t)$ whose equations are invariant under a phase transformation $\phi \rightarrow e^{i\lambda}\phi$. Physically, ϕ might represent the field associated with the creation and annihilation of bosonic particles (such as ^{4}He atoms), s-wave spin singlet Cooper pairs, or the distribution of a planar spin density. Within the Landau-Ginzburg paradigm, a key role is played by the energy (or more generally the free energy) of a field configuration. The simplest energy functional consistent with symmetries is given by

$$V(\phi) = b_4|\nabla\phi|^4 + b_2|\nabla\phi|^2 + a_2|\phi|^2 + a_4|\phi|^4 = H = -L, \tag{1}$$

where $\{a_2, a_4, b_2, b_4\}$ are real-valued parameters, while H and L denote the Hamiltonian and Lagrangian densities, respectively. Taking $a_4, b_4 > 0$ insures stability against rapid spatial variations and large fields.

If $a_2 < 0$, states with $\langle\phi\rangle \neq 0$ are energetically favorable, spontaneously breaking the phase rotation symmetry. If $b_2 < 0$, states with $\langle\nabla\phi\rangle \neq 0$ are favorable, spontaneously breaking the spatial translation symmetry and indicating the emergence of spatial patterns. By reducing the symmetry, the system gains a nonzero energy per unit volume. In the limit of infinite volume, the symmetry of all physically realizable states is less than the symmetry of the original equations. This is the mechanism of spontaneous symmetry breaking in equilibrium.

Physicists are accustomed, in many contexts, to treating time and space on an equal footing. Thus, one could imagine generalizing Eq. (1) to include time derivatives. In a Lagrangian framework, it is natural to consider adding the leading-order kinetic terms,

$$L_{\rm kin} = c_4|\partial_t\phi|^4 + c_2|\partial_t\phi|^2, \tag{2}$$

which corresponds to the energy density $H_{\rm kin} = 3c_4|\partial_t\phi|^4 + c_2|\partial_t\phi|^2$. For $c_2 < 0$ and $c_4 > 0$, it is energetically favorable to have $\langle\partial_t\phi\rangle \neq 0$, thereby implying the spontaneous breaking of time-translation symmetry.

This line of reasoning suggests that spontaneous time-translation symmetry breaking (SτB) is straightforward to achieve. But subtleties abound (Shapere and Wilczek, 2012b; Wilczek, 2012). Quantum mechanics requires Hamiltonians, and for nonsingular Hamiltonians, Hamilton's equations $\partial_t\phi = \partial_p H$, $\partial_t p = -\partial_\phi H$ imply that a system is stationary at any energy minima. Thus, a ground state cannot exhibit periodic oscillations. The behavior of Eq. (2) is able to evade this conclusion because $\partial_t\phi$ is not a single-valued function of the canonical momentum [i.e., $p(\partial_t\phi) = \delta L_{\rm kin}/\delta\partial_t\phi$ does not have a unique inverse $\partial_t\phi(p)$] and, consequently, neither is the Hamiltonian (Shapere and Wilczek, 2012b). This is not necessarily an insurmountable problem. Indeed, for such singular Hamiltonians it is often possible to construct consistent (in particular, unitary) quantum theories that realize the corresponding classical dynamics in the limit of large quantum numbers (Shapere and Wilczek, 2012a, 2012b; Zhao, Yu, and Xu, 2013; Chi and He, 2014; Choudhury and Guha, 2019). Moreover, singular Hamiltonians that support time-dependent minima can arise as limiting cases of nonsingular Hamiltonians as appropriate parameters are taken large.

A limiting theory can be valuable because it is often more tractable than the full theory while being accurate within a large range of parameter space (Dai *et al.*, 2019; Shapere and Wilczek, 2019; Alekseev, Dai, and Niemi, 2020; Dai, Niemi, and Peng, 2020). However, away from this limit several studies (Bruno, 2013; Nozières, 2013; Watanabe and Oshikawa, 2015) have argued that persistent oscillations cannot arise in a quantum system that is in equilibrium with respect to a local Hamiltonian.

These subtleties suggest that the application of the Landau paradigm to SτB may not be as easy as Eq. (2) would lead us to believe. An additional necessary ingredient for SτB is a robust mechanism for *ergodicity breaking* that enables the dynamics of the system to "remember" the initial condition (such as the complex phase of ϕ) out to infinite times. Spontaneous symmetry breaking is a special case of ergodicity breaking. Focusing on this aspect of spontaneous symmetry breaking leads to the following formulations of SτB: it is a form of ergodicity breaking in which a generic ensemble of initial conditions exhibits persistent oscillations with a temporal phase shift that remembers the initial condition; cf. Eq. (3).

In recent years, a nonequilibrium route to SτB has been discovered in the context of Hamiltonians periodically driven at frequency ω_D. Such "Floquet" systems (Floquet, 1883) feature a reduced, discrete time-translation symmetry $H(t + 2\pi/\omega_D) = H(t)$, where $T_0 = 2\pi/\omega_D$ is the driving period. When a simple harmonic oscillator is driven at frequency ω_D, it responds at frequency ω_D regardless of its natural frequency. This is true even of many nonlinear systems that are far more complex: when driven at frequency ω_D, their observable properties respond at integer harmonics of ω_D regardless of their detailed structure. In this case, the observable behavior evolves with the same discrete time-translation symmetry as the underlying equations of motion. In contrast, a system is said to exhibit a *subharmonic* response if there are properties that oscillate at frequency ω_D/m for some integer $m > 1$. Such a subharmonic response spontaneously breaks the discrete time-translation symmetry down to the smaller subgroup $t \to t + 2\pi n/\omega_D$ with $n \in m\mathbb{Z}$. If this SτB is stable to perturbations, the system is a "discrete time crystal" (Sacha and Zakrzewski, 2018; Khemani, Moessner, and Sondhi, 2019; Else *et al.*, 2020; Guo and Liang, 2020; Sacha, 2020).

Owing to the periodic drive, it is not obvious how to stabilize such behavior. Within the Landau paradigm, spontaneous symmetry-breaking phases retain their order in the face of fluctuations (both thermal and quantum) by virtue of the energy penalty associated with misaligned regions. But this is not possible for a discrete time crystal, since energy is not conserved in a driven system. The oscillations in distant parts of the system must remain in lockstep even though the energy penalty for failing to do so can easily be overridden by the energy supplied by the drive.

The aim of this Colloquium is to give a unifying perspective on various nonequilibrium mechanisms for discrete SτB in closed (Fig. 1) and open systems (Fig. 2). By centering our discussion in the language of dynamical systems, we highlight the connections (and contrasts) between the rich literature on subharmonic responses in dynamical systems and more recent developments in the context of closed quantum systems. Until

recently many-body systems exhibiting SτB, such as period doubling in coupled map lattices, were found only in open systems that rely fundamentally on dissipation. They could be viewed as a type of engine: the energy supplied by the frequency-ω_D drive is converted to frequency-(ω_D/m) motion while releasing heat to a cold bath. With the discovery of many-body localized (MBL) discrete time crystals, a qualitatively different form of SτB was discovered: one that does not generate any entropy at all.

In a MBL time crystal, the timescale of SτB oscillations diverges exponentially as the system size alone is increased. However, for practical purposes it is also of interest to know whether there are other physical parameters whose limiting behavior can drive such an exponential increase, for example, the drive frequency $\omega_D \to \infty$, the temperature $T \to 0$, or the particle density $n \to \infty$. This will lead us to the notion of prethermal, activated, and driven Bose-Einstein condensate (BEC) time crystals, respectively. A key tool that is common to these "exponentially good" time crystals and to their MBL counterpart is the emergence of an effective time-independent *Floquet Hamiltonian* H_{eff}, which governs the dynamics out to exponentially long timescales. H_{eff} can exhibit emergent symmetries that are protected by the underlying time-translation symmetry of the drive. In a pleasing return to form, SτB can then be understood in terms of the breaking of these emergent symmetries and as an application of the Landau paradigm not to $H(t)$ but rather to H_{eff}.

In the setting of Floquet dynamics, the environment couples to the system only via a coherent drive. But in any physical experiment, the environment itself is a many-body system, and thus coupling to it inevitably comes along with dissipation and noise. Indeed, all experiments on discrete time crystals to date have shown their fingerprints. The possibility of perfect SτB in the presence of a drive, dissipation, and noise remains an open question.

The outlines of a possible answer to this question are provided by deep results from theoretical computer science and nonequilibrium statistical physics. As emphasized, SτB is fundamentally a form of ergodicity breaking, and there is a history of rigorous results from these communities that show that ergodicity breaking can be generically stable to noise (Toom, 1980; Gács, 2001). In that context, the motivation was to understand whether "reliable systems can emerge from unreliable components" (Von Neumann, 1952). The answer to this question is inextricably linked to the physical possibility of the most radical form of ergodicity breaking of all: classical and quantum error correction. From this perspective, if purely dissipative time crystals are a form of engine and MBL time crystals are a type of idealized perpetual motion, then SτB in open systems is an embryonic example of an error-corrected computer program: repeatedly applying the same NOT operation to all registers, the computer settles into the period-2 output …0101010101….

II. WHAT DEFINES A TIME-CRYSTALLINE PHASE OF MATTER?

Discrete SτB leads to regular and long-lived oscillations with a period that is a multiple of the drive's. Oscillations in general, however, are ubiquitous throughout nature, so in this section we formalize when such behavior signals the emergence of a genuine phase of matter. Our starting point is any
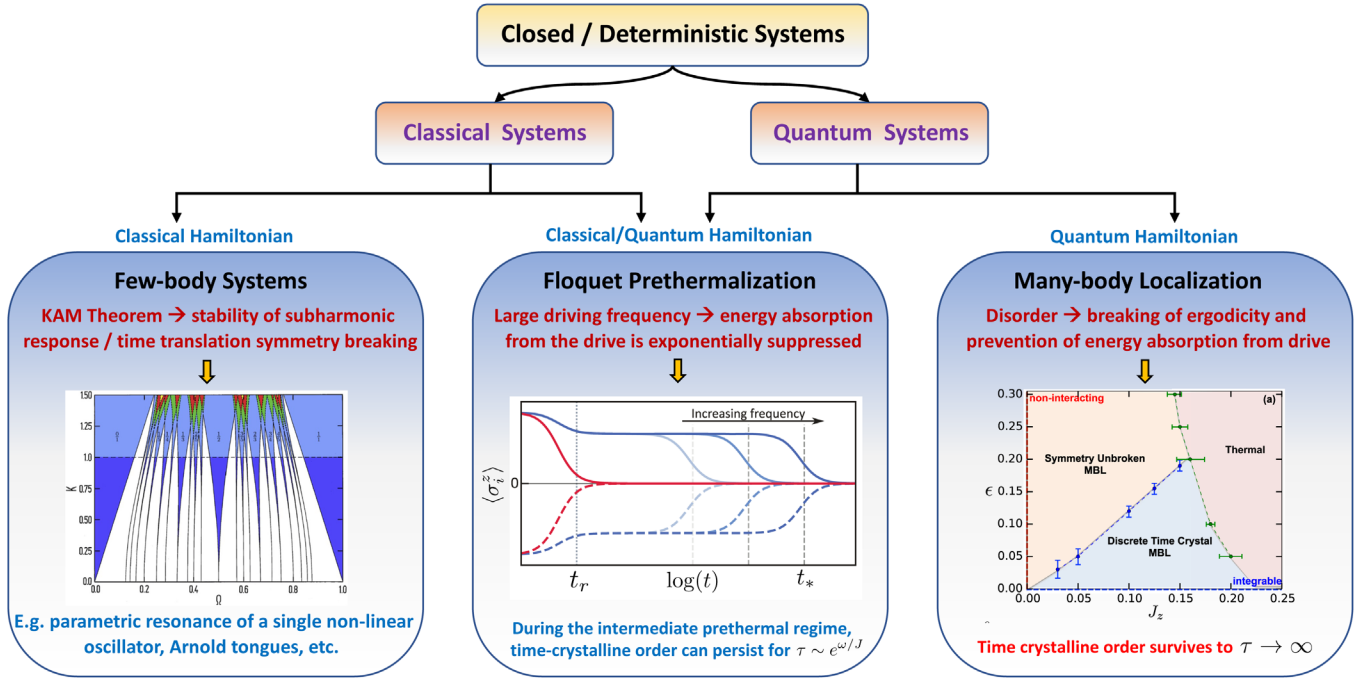
FIG. 1. Schematic depicting strategies for stabilizing time-crystalline order in periodically driven, closed systems evolving via deterministic dynamics. In few-body classical systems such as a parametrically driven nonlinear oscillator, stable subharmonic responses are ubiquitous. This stability can be understood as a time-dependent extension of the Kolmogorov-Arnold-Moser (KAM) theorem (Kolmogorov, 1954; Möser, 1962; Givental *et al.*, 2009), which proves that quasiperiodic orbits of dynamical systems remain robust to small perturbations (Sec. III.B.4). There is no quantum analog to this classical few-body strategy. For many-body systems, both classical and quantum dynamics can exhibit Floquet prethermalization in the limit of large driving frequencies (Sec. IV.D). During an intermediate prethermal window of time (i.e., before Floquet heating occurs), the system can exhibit discrete time-crystalline order. The lifetime of this order scales exponentially with the frequency of the drive ($\tau \sim e^{\omega_D/J}$). In strongly disordered, quantum many-body systems, a phenomenon known as many-body localization can occur. This prevents Floquet heating (see Sec. III.C) and also provides a mechanism for the many-body system to avoid becoming ergodic. By breaking ergodicity, time-crystalline order can persist to infinite times in the thermodynamic limit. Since many-body localization relies upon the discreteness of quantum mechanical levels, there is no classical analog to this strategy. Left panel inset: adapted from Heltberg *et al.*, 2016. Middle panel inset: adapted from Kyprianidis *et al.*, 2021. Right panel inset: adapted from Yao *et al.*, 2017.

dynamical system (Birkhoff, 1927; Katok and Hasselblatt, 1997; Strogatz, 2018) whose state x evolves under a discrete-time update rule $\Phi: x \to \Phi(x)$. The state of the system at time step t is thus $x(t) = \Phi^{(t)}(x(0))$, where the superscript denotes iteration. The time independence of Φ implies that there is a discrete time-translation symmetry. While the notation highlights a discrete time step, Φ may arise from viewing a continuous-time system "stroboscopically." In the context of classical and quantum dynamics defined by the time-periodic Hamiltonian $H(t + T) = H(t)$, x is the classical or quantum state and Φ corresponds to integration of Hamilton's equations or the Schrödinger equation over one period of the drive.

The dynamics exhibit m-fold time-translation symmetry breaking if there is a local observable O that exhibits periodic oscillations out to infinite times for a measurable volume of initial conditions x:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \sum_{n=1}^{\tau} O(\Phi^{(mn)}(x)) \neq \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{n=1}^{\tau} O(\Phi^{(mn+p)}(x)), \quad (3)$$

where $0 < p < m$ corresponds to the m phases of the orbit, with equality restored for $p = m$. A time crystal thus remembers which of the m initial conditions it is in. This implies that SτB is a particular form of ergodicity breaking

(Sinai, 1959; Walters, 2000; Cornfeld, Fomin, and Sinai, 2012): the time-averaged behavior of the m-fold iterated map $\Phi^{(m)}$ depends on the initial condition. This behavior has been called asymptotic periodicity in the literature on classical many-body dynamical systems (Lasota, Li, and Yorke, 1984; Losson and Mackey, 1996; Lasota and Mackey, 2013). While Eq. (3) implies infinitely long-lived oscillations, we may also consider a relaxed condition in which the timescale τ remains finite but diverges exponentially with a control parameter such as the drive frequency or temperature. This will be the case for prethermal (Sec. IV.D) and activated (Sec. V.A) time crystals, respectively.

We emphasize that SτB does not require each state x to itself undergo perfectly periodic motion, only that there is an observable O that oscillates on average. For example, in the context of a single variable x, the observable $O(x) = \text{sgn}(x)$ might exhibit regular oscillations even while the motion of x itself is quite chaotic. This accounts for the fact that we rarely have direct access to the microstate of the system, only to coarse-grained measurements. This limitation also motivates the requirement that oscillations be observable for a finite volume of initial conditions, over which we rarely have exact control. Taken together, an equivalent formulation of Eq. (3) takes a statistical point of view by considering the evolution of
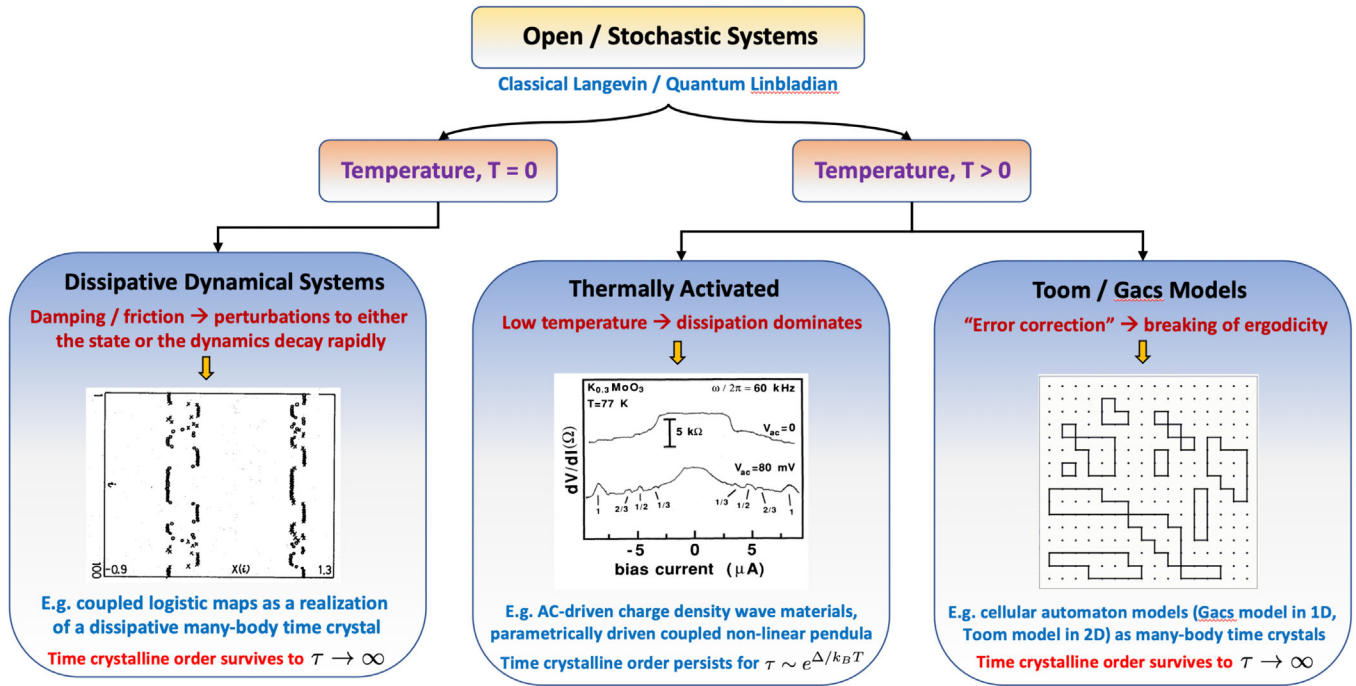
FIG. 2. Schematic depicting strategies for stabilizing time-crystalline order in periodically driven, open systems evolving via stochastic dynamics. Dissipative nonlinear dynamical systems have long been known to exhibit stable, many-body time-translation symmetry breaking. Indeed, coupled map lattices can satisfy all of the stated requirements for being a discrete time crystal. The key ingredient for the stability of SτB in this setting is a generalized version of dissipation; effectively, one can think of the microscopic dynamics in such systems as being coupled to a zero-temperature bath (alternatively, one can say that the microscopic dynamics are not information or measure preserving). This ensures that a finite volume of initial conditions contracts toward a period-doubled fixed point (Sec. II.A). The stability of time-crystalline order in many-body systems coupled to a finite-temperature bath is significantly more subtle. For example, in classical Langevin dynamics or quantum Lindbladian dynamics, at any $T > 0$ dissipation always comes with noise. In such systems (for instance, finite-temperature, parametrically driven, coupled nonlinear pendula), at low temperatures time-crystalline order can survive for an "activated" timescale $\sim e^{\Delta/k_B T}$ (Sec. V.A). Somewhat surprisingly, by considering more generic stochastic dynamics (i.e., probabilistic cellular automata), it is possible to realize finite-temperature, time-crystalline order with an infinite lifetime (Sec. V.E). Left panel inset: adapted from Kaneko, 1984. Middle panel inset: adapted from Hundley and Zettl, 1989. Right panel inset: adapted from Vasmer, 2020.

distributions over microstates $\rho(x)$; we return to this formulation in our discussion of stochastic systems in Sec. V.

Even a single degree of freedom can trivially exhibit SτB: take the map $\Phi(x) = -x$. The problem gains its richness when we demand stability: SτB is a property of a dynamical phase of matter if it is robust to any small locality-preserving perturbation of either the initial condition x or the dynamics Φ. Under the perturbation $\Phi(x) \rightarrow -x(1-\epsilon)$, oscillations are damped at times beyond $\tau \sim \epsilon^{-1}$ and SτB is destroyed.

When discussing stability, we need to be clear which class of perturbations we demand stability against. Correspondingly this defines different possible classes of SτB phases (Hamiltonian, unitary, Langevin, quantum Lindbladian, etc.) that would exhibit SτB robust to arbitrary small perturbations within that dynamical class. Depending on how broadly or narrowly we define stability, there are more or fewer examples, but weaker or stronger implications.

A. Illustrative examples

The requirements surrounding Eq. (3) contain several subtleties, so it will prove helpful to walk through several examples of "is X a time crystal" with these in mind. We first focus on examples that do exhibit oscillations (and are interesting in their own right) but which do not satisfy the strict requirements of Eq. (3).

Consider first the dynamics of an undriven oscillator $H = p^2/2m + (\omega_0^2/2m)q^2 + (\epsilon/4)q^4$. Any initial condition $x = (q, p)$ will oscillate forever: do such oscillations constitute a discrete time crystal? Since H is time independent, for the purposes of testing Eq. (3) we may choose to stroboscopically observe the dynamics at whichever period T we suspect harbors the oscillations (say, $T = 2\pi/\omega_0$). However, stability requires us to account for the nonlinearity ϵ, which causes the oscillation frequency to depend on the amplitude of the initial condition. As a result, there is no fixed period T for which Eq. (3) is satisfied for a finite volume of initial conditions: a generic ensemble of states will instead dephase, with no SτB after some characteristic time τ.

The timescale τ for such dephasing may be large, in which case the dynamics may appear to exhibit SτB in practice even while they do not satisfy Eq. (3) in principle. Consider, for example, the ac Josephson effect,

$$H = -E_J \cos\phi + 2eVn + \frac{n^2}{2C}. \tag{4}$$

In Eq. (4) ϕ is the superconducting phase difference across a Josephson junction, which is conjugate to the Cooper pair number n; E_J is the Josephson energy for the tunneling of a pair across the junction, C is the junction capacitance, and V is the voltage difference across the junction. For a large junction, the capacitance approaches $C = \infty$, and the equations of motion then give $\phi(t) = \phi_0 + 2eVt$, implying that the super-current $I_s = \dot{n} = E_J \sin(\phi)$ oscillates indefinitely in response to a dc voltage. However, for the generic case in which C is finite, a shift of variables from $n \to n - 2eVC$ brings H to the form of a pendulum whose behavior is similar to the previously discussed nonlinear oscillator. The timescale for the resulting dephasing $\tau = \sqrt{C/E_J}$ may be large, though we point out that this dependence is not an exponential. In experiments, Josephson junctions are generally resistively shunted, and related interesting phenomena arise in this open context, as discussed in Sec. V.

Another interesting limit of Eq. (4) is $E_J = 0$, in which case H has an internal U(1) phase symmetry that guarantees that $\dot{\phi} = 2eV + n/C$ will remain constant. In this case, the oscillation period is $T = 2\pi/\dot{\phi}$, and this is an example of SτB that "piggybacks" on the spontaneous symmetry break-ing of an unrelated internal symmetry. Equivalent examples are superfluids at nonzero chemical potential and an *XY* magnet in a perpendicular field. Such systems have a U(1) order parameter that precesses in time for generic initial conditions. The precession is unobservable (for example, $I_s = 0$) unless the U(1) symmetry, which makes all angles equivalent, is broken. But in the absence of U(1) symmetry, there is no barrier to energy dissipating from the macroscopic precession of the order parameter into the internal motion of its many-body constituents. At long times the system then relaxes to a state (e.g., vanishing particle number in the superfluid or spin aligned with the field in the magnet) wherein macroscopic motion ceases. To navigate this dichotomy, one must shield the system from explicit sym-metry breaking apart from brief, intermittent measurement events, as in the experiment of Urbina, Jacquinot, and Goldman (1982). In this case, SτB is stable only to perturba-tions that preserve the internal symmetry. In contrast, as we explore in this Colloquium, a stronger form of SτB is possible that depends only on the time-translation symmetry itself.

A many-body example that does satisfy all the requirements surrounding Eq. (3) is period doubling in deterministic dissi-pative systems. An iterated map $x \to f(x)$ such as the logistic map (Strogatz, 2018) provides an idealized model of the unit-time evolution of such a system. Taking, for example, $f(x) = -x(1 + a - x^2)$, there is a basin of attraction of initial states that settle into a limiting oscillation between $x = \pm\sqrt{a}$ when $a > 0$. This simple one-body example can be promoted to a many-body model (a "coupled map lattice") by considering an array of variables x_i that evolve under $\Phi: x_i \to f(x_i) + v(x_{i-1}, x_i, x_{i+1})$ with some generic local interaction v. The behavior of coupled map lattices is very rich, since they include cellular automata (Gutowitz, 1991) as a special case (Kaneko, 1984, 1992; Kapral, 1985; Kaneko and Konishi, 1987; Bunimovich and Sinai, 1988). Coupled map lattices can support collective subharmonic responses (Losson, Milton, and Mackey, 1995; Losson and Mackey, 1996; Gielis

and MacKay, 2000) that are stable to smooth perturbations of f or v.

There are many other examples of dissipative systems with spontaneous oscillations. For example, the van der Pol oscillator, $\ddot{x} = \mu(1 - x^2)\dot{x} - x$ is a continuous-time system that exhibits a closed "limit cycle" orbit with $x(t + T) = x(t)$ (van der Pol, 1920; Odani, 1995). Because of dissipation, small perturbations to the limit cycle decay, so that at long times a finite volume of initial states (i.e., neighboring trajectories) flow toward the limit cycle (Strogatz, 2018). Much like the logistic map, this single-body behavior can be extended to many degrees of freedom by coupling such systems together, leading to arrays of coupled limit-cycle oscillators (Aizawa, 1976; Kuramoto and Nishikawa, 1987; Matthews and Strogatz, 1990; Palacios *et al.*, 2005; Lauter *et al.*, 2015). Despite the continuous-time dynamics, from the perspective of time-translation symmetry breaking, the phys-ics of limit cycles is conceptually similar to the previously discussed discrete-time examples, such as the logistic map. This connection can be made precise by viewing the differ-ential equation at stroboscopic times (via a so-called first-recurrence or Poincaré mapping), wherein the continuous-time dynamics defines a coupled map lattice. One difference from examples of discrete SτB is that the period T of the cyclic orbit varies continuously with the model parameters. This is precisely because there is no external drive to set the frequency of the response. However, by adding a small periodic drive, the period T can be "entrained" (locked) to a subharmonic of the drive (Flaherty and Hoppensteadt, 1978; Storti and Rand, 1988), resulting in rigid SτB.

III. FLOQUET HAMILTONIAN SYSTEMS

As noted, if we consider systems with dissipation but no noise, such as a coupled map lattice, it is relatively straight-forward to obtain stable, many-body SτB. This is the classical zero-temperature limit of the general open system problem that we discuss in Sec. V. These systems illustrate that the existence and stability of time crystals depends strongly on the class of dynamical system considered (open or closed, classical or quantum). The presence of dissipation leads to a fixed-point orbit about which the linearized dynamics have eigenvalues $|\lambda| < 1$, such that a finite volume of initial conditions contracts toward the fixed point. In contrast, for the canonical transformation that arises from integrating Hamilton's equations over one Floquet period, the eigenvalues always come in conjugate pairs $\lambda_1 \lambda_2 = 1$ in order to ensure that the phase-space volume $dq \wedge dp$ is preserved in accord with Liouville's theorem. In this section, we discuss why stable SτB, i.e., an infinitely long-lived time crystal, is not expected to occur in such a situation.

A. Faraday waves

To illustrate why Hamiltonian structure makes the possibil-ity of SτB considerably more challenging, we consider the concrete example of Faraday-wave instabilities in shaken surface waves. Faraday observed that when a container of water with a liquid-air interface is shaken vertically at frequency ω_D, surface waves develop that oscillate at frequen-cies $\omega = (n/m)\omega_D$, which are rational subharmonics of the

drive (Faraday, 1831; Rayleigh, 1883a). The subharmonics can be understood as standing-wave modes that are forced via parametric resonance (Rayleigh, 1883b). A precise mathematical understanding of the instability is obtained by linearizing the incompressible Euler equations for surface waves (Benjamin and Ursell, 1954) in the presence of periodic acceleration. This reduces the problem to a set of "Mathieu equations" for each Fourier component of the surface height q,

$$\ddot{q}_k = -[\omega_k^2 + \delta_k \cos(\omega_D t)]q_k, \tag{5}$$

where q_k is the amplitude of the surface wave at wave vector k, ω_k is the natural frequency of the surface wave, and δ_k is proportional to the driving amplitude. The Mathieu equation features an exponentially growing solution (which looks precisely like a subharmonic response) of the approximate form $\sim e^{t\Gamma}\cos(t\omega_D/2 + \theta)$ when a mode satisfies $\omega_D \sim 2\omega_k$ (McLachlan, 1947). As we discuss, the nonlinearity then regulates the blowup and stabilizes SτB. Since the Euler equations are Hamiltonian (Olver, 1982), it might seem that Faraday-wave instabilities then provide a Hamiltonian example of SτB.

However, while this analysis implies a linear subharmonic instability, it does not resolve whether the motion at long times is an example of SτB in the strong sense. While the nonlinearities that are generically present have the favorable property of regulating the exponential blowup, they also couple different k modes. A generic initial condition will contain some energy in these high-k modes, which will then act as an effectively noisy force on the motion of the k mode in which one is hoping to observe stable SτB. The key question can then be summarized as follows: Does SτB survive when treating the Faraday-wave problem as a genuine nonlinear many-body system and, if not, what governs the autocorrelation time of the long-lived subharmonic response manifestly seen in experiments?

B. Floquet Hamiltonians and emergent symmetries

To answer this question, it proves helpful to walk through a specific example and develop some intuition along the way. To this end, we work through the example of a shaken nonlinear pendulum (i.e., a parametrically driven nonlinear oscillator); for a single pendulum (and even in the few-body case), we see in this section that one naturally gets stable SτB as a consequence of the Kolmogorov-Arnold-Moser (KAM) theorem (Kolmogorov, 1954; Möser, 1962; Givental *et al.*, 2009). In a truly many-body system, however, SτB becomes unstable. We argue that there is an intuitive general mechanism underlying this obstruction: ergodicity. Along the way, we introduce and develop a particularly important construct, namely, the Floquet Hamiltonian, which will provide a unifying framework for understanding many versions of SτB in closed Hamiltonian systems (both quantum and classical).

Consider the following Hamiltonian for a shaken nonlinear pendulum:

$$H(q, p, t) = \frac{p^2}{2} + \frac{\omega_0^2}{2}[1 + \delta \cos(\omega_D t)]q^2 + \frac{\epsilon}{4}q^4. \tag{6}$$

To connect to the previous Faraday-wave discussion, the reader may think of this pendulum as a specific k mode, where we are now explicitly accounting for the nonlinearity but neglecting the intermode coupling. The linearized equations of motion give rise to the Mathieu equation [cf. Eq. (5)] and the goal, with respect to time-crystalline order, would be to prove that the nonlinearity stabilizes SτB.

1. The rotating frame: Constructing the Floquet Hamiltonian

Near a period-doubled solution of Eq. (6), one expects on physical grounds that trajectories should approximately take the form

$$q(t) - ip(t) = \sqrt{2P(t)}e^{i[\omega_D t/2 + Q(t)]}, \tag{7}$$

where $Q(t)$ and $P(t)$ are slowly varying in comparison to ω_D. The transformation that takes $(q, p) \to (Q, P)$ is a time-dependent canonical transformation, which we denote by $\mathcal{K}_0(t)$; this transformation can be thought of as taking the system to a particular "rotating frame" [Fig. 3(a)]. Note that the period of the rotation is subharmonic, $\mathcal{K}_0(mT) = \mathcal{K}_0(0)$ (in the period-doubled case, $m = 2$). In the rotating frame, the Hamiltonian $H(Q, P, t)$ is still time dependent, but the natural frequencies of the system [i.e., those that govern the slowly varying motion of $Q(t)$, $P(t)$ are now off resonant with ω_D. Thus, one can attempt to follow up with a second transformation in order to generate a completely time-independent Floquet Hamiltonian H_F (Holthaus, 1995; Buchleitner, Delande, and Zakrzewski, 2002; Else, Bauer, and Nayak, 2017; Else, Ho, and Dumitrescu, 2020). The second transformation takes the form of a so-called Magnus expansion and the composite transformation is given by $\mathcal{K}(t) = \mathcal{K}_{\text{Magnus}}(t) \circ \mathcal{K}_0(t)$, again with $\mathcal{K}(mT) = \mathcal{K}(0)$. In summary, the strategy is
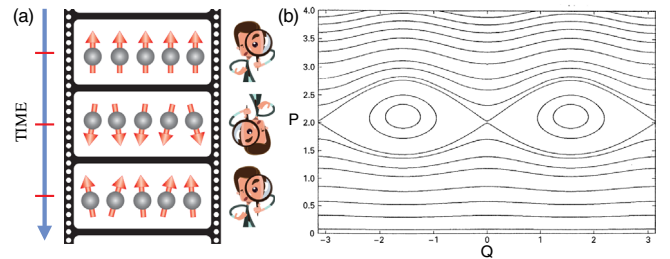
FIG. 3. (a) Schematic depiction of a time-dependent rotating frame, which transforms the original time-dependent $H(t)$ into the time-independent Floquet Hamiltonian H_F. Time-translation symmetry breaking (here with a period $m = 2$) can be understood as spontaneous symmetry breaking of an internal $\mathbb{Z}_m$ symmetry of the Floquet Hamiltonian. Adapted from Yao and Nayak, 2018. (b) Contours of equal "quasienergy" $H_F(Q, P)$ for a nonlinear parametric oscillator near a 2∶1 resonance. The original coordinates of the oscillator q and p are related to the coordinates Q and P through a time-dependent canonical transformation given approximately by Eq. (7). The two minima of H_F, at which (Q, P) are time independent, map back to the two exactly period-doubled orbits, $q_*(t) = -q_*(t + T_0)$. As predicted by Eq. (11), time-translation symmetry manifests as a $\mathbb{Z}_2$ symmetry $H_F(Q + \pi, P) = H_F(Q, P)$. Adapted from Zounes and Rand, 2002.

to find a canonical transformation $\mathcal{K}(t):(q,p) \to (Q,P)$ such that the transformed Hamiltonian $H_F(Q,P)$ is time independent. This idea that $H(q,p,t)$ might actually be equivalent to an equilibrium system (i.e., governed by H_F) in an appropriate time-dependent rotating frame was referred to as "crypto-equilibrium" by Yao and Nayak (2018) and Else *et al.* (2020).

We emphasize that *a priori* the Magnus expansion may not converge, in which case the pair of objects $\{\mathcal{K}(t), H_F\}$ need not exist: this will be the precise obstruction to stable SτB in the many-body Faraday-wave case. Note that in the quantum setting (which we soon return to), one can in principle always define a Floquet Hamiltonian by taking the logarithm of the Floquet unitary; thus, in this case, one should replace the just-mentioned "need not exist" with the fact that H_F will be highly nonlocal.

As a preview, we note that the distinction between a true discrete time crystal (Sec. IV.B) and the prethermal discrete time crystal (Sec. IV.D) can precisely be understood as whether the Magnus expansion converges. In the former case, it does and one has a well-defined Floquet Hamiltonian H_F, while in the latter case it almost does, leading to

$$H(Q,P,t) = H_{\text{eff}}(Q,P) + V(t). \tag{8}$$

In Eq. (8) H_{eff} is an effective Hamiltonian that captures the system's stroboscopic dynamics for exponentially long time-scales and $V(t)$ is the exponentially small residual time dependence that cannot be "rotated" away by the Magnus expansion.

The concept of a rotating frame transformation in the quantum case is completely analogous. In particular, within the quantum setting the dynamics are captured by a unitary time-evolution operator

$$U(t_1, t_0) = \mathcal{T} \exp\left(-i \int_{t_0}^{t_1} H(t)dt\right), \tag{9}$$

where $\mathcal{T}$ represents time ordering. In this case, $\mathcal{K}(t)$ is a unitary transformation chosen such that the time-evolution operator $U(t_1, t_0)$ decomposes as

$$U(t_1, t_0) = \mathcal{K}^{-1}(t_1)e^{-i(t_1-t_0)H_F}\mathcal{K}(t_0), \tag{10}$$

with $\mathcal{K}(mT) = \mathcal{K}(0)$ and H_F suitably local. $\mathcal{K}(t)$ is referred to as the "micromotion" since it contains the components of the dynamics at harmonics of ω_D/m, while the "slow" part of the motion resides in H_F (Shirley, 1965; Zel'Dovich, 1973; Buchleitner, Delande, and Zakrzewski, 2002; Rahav, Gilary, and Fishman, 2003; Bukov, D'Alessio, and Polkovnikov, 2015; Eckardt and Anisimovas, 2015). When one restricts to the stroboscopic dynamics, the "Floquet unitary" that implements a single discrete time step is given by $U_F = U(T,0) = \mathcal{K}^{-1}(T)e^{-iTH_F}\mathcal{K}(0)$. To ensure that the connection to the notation in Eq. (3) is clear, we note that Floquet unitary U_F then furnishes the dynamical map Φ, which acts on the quantum state $x = |\psi\rangle$.

2. Time-translation symmetry breaking as internal symmetry breaking of the Floquet Hamiltonian

If the micromotion is periodic $[\mathcal{K}(T) = \mathcal{K}(0)]$, there can be no SτB since the stroboscopic dynamics become canonically equivalent to the undriven dynamics of H_F, and SτB is forbidden in equilibrium (Bruno, 2013; Watanabe and Oshikawa, 2015). But in a time crystal, the frame rotates subharmonically [Fig. 3(a)], i.e., $\mathcal{K}(mT) = \mathcal{K}(0)$, and the transformation $X = \mathcal{K}(0)\mathcal{K}^{-1}(T)$ is nontrivial. The transformation X then plays a crucial role since it permutes through the m cycles of the subharmonic response. In the specific case of the shaken nonlinear pendulum [cf. Eq. (7)], for example, $X:(Q,P) \to (Q+\pi, P)$. We expand upon this discussion in Sec. III.B.4.

Under rather general conditions, the Magnus expansion can be constructed to ensure that $X = \mathcal{K}(nT)\mathcal{K}^{-1}(nT+T)$ is independent of n (Else, Bauer, and Nayak, 2017; Else, Ho, and Dumitrescu, 2020; Machado *et al.*, 2020), which then ensures that $X^m = \mathbb{I}$. Substituting Eq. (10) into the time-translation symmetry $U(T,0) = U(2T,T)$, one finds that

$$X^{-1}e^{-iTH_F}X = e^{-iTH_F}. \tag{11}$$

In other words, X is an internal $\mathbb{Z}_m$ symmetry of the Floquet Hamiltonian (Holthaus, 1995; Buchleitner, Delande, and Zakrzewski, 2002; von Keyserlingk, Khemani, and Sondhi, 2016; Else, Bauer, and Nayak, 2017). While the emergence of an internal symmetry may seem mysterious, it is really just a reexpression of the original discrete time-translation symmetry in a frame that is rotated by X at each step [Fig. 3(a)]. Else, Ho, and Dumitrescu (2020) referred to this intertwining as twisted time translation. However, the emergence of such a symmetry was pointed out well before the recent interest in SτB; see the extensive literature on nonspreading wave packets in periodically driven systems (Holthaus and Flatté, 1994; Holthaus, 1995; Buchleitner, Delande, and Zakrzewski, 2002).

The existence of a Floquet Hamiltonian H_F and an emergent symmetry X immediately leads to the possibility of SτB. In particular, suppose that the dynamics of H_F spontaneously break the X symmetry. Intuitively, this means that the state space breaks up into m "superselection" sectors that are permuted by the action of X but that are not connected by evolution under H_F. In the rotating frame, a trajectory thus gets stuck within a sector and, mapping back to the lab frame using $\mathcal{K}^{-1}(t)$, one finds that the average behavior will oscillate with period m. More rigorously symmetry breaking implies that there are m different initial conditions $y_j = (Q_j, P_j)$ that are related by the symmetry $(y_{j+1} = Xy_j)$. The time-averaged behaviors of these different initial conditions are distinct (in the rotating frame); for instance, there are local observables O with $\bar{O}(y_j) \equiv (1/\tau)\sum_{n=0}^{\tau} O(y_j(nT)) \neq \bar{O}(y_{j+1})$. Transforming back to the lab frame $x_j(t) = \mathcal{K}^{-1}(t)y_j(t)$ and substituting in the definitions, one immediately finds that the definition of time-crystalline order [cf. Eq. (3)] is satisfied.

One thus arrives at a correspondence between m-fold SτB and the more familiar notion of $\mathbb{Z}_m$-spontaneous symmetry breaking with respect to H_F (in the rotating frame). This illustrates a general principle of all closed Hamiltonian many-body time crystals discovered thus far (Fig. 1): In a Hamiltonian time crystal, the discrete time-translation symmetry of $H(t)$ manifests as a spontaneously broken internal $\mathbb{Z}_m$ symmetry of the Floquet Hamiltonian H_F (von Keyserlingk, Khemani, and Sondhi, 2016; Else, Bauer, and

Nayak, 2017; Else, Ho, and Dumitrescu, 2020). In a sense this bring us back full circle to our starting inspiration, the Landau paradigm (Sec. I), with the crucial replacement $H \to H_F$.

3. The spectral perspective

While the previously mentioned formalism applies equally well to the classical and quantum settings, in the quantum case an equivalent definition can be given in terms of the spectral properties of the Floquet operator U_F alone (Holthaus, 1995; Buchleitner, Delande, and Zakrzewski, 2002; Sacha, 2015): a discrete time crystal is a phase of matter in which the eigenstates of U_F are necessarily "cat states," i.e., superpositions of macroscopically distinct states (Else, Bauer, and Nayak, 2016). This is a direct generalization of the statement that when a conventional symmetry is spontaneously broken in an equilibrium system, the low-energy eigenstates that transform simply under the symmetry [such as eigenstates of the center or Cartan subalgebra (Fulton and Harris, 2013) of the symmetry group] are necessarily cat states.

To unpack this definition we investigate the implications of Sec. III.B.2 for the "Floquet eigenstates" $U_F|\varepsilon_F\rangle = e^{-iT\varepsilon_F}|\varepsilon_F\rangle$. Here the quasienergies ε_F are defined modulo the driving frequency $\varepsilon_F \equiv \varepsilon_F + \omega_D$. As discussed, U_F exhibits SτB when there is a decomposition $U_F = Xe^{-iTH_F}$ for which H_F spontaneously breaks the symmetry $[X, H_F] = 0$. In the thermodynamic limit, spontaneous symmetry breaking implies that the eigenstates of H_F come in degenerate pairs permuted by the symmetry $X|\uparrow\rangle = |\downarrow\rangle$ (we focus on $X^2 = 1$ for simplicity).[1] However, in a finite system the eigenstates of H_F must simultaneously diagonalize X, so they come in pairs $|\pm\rangle = (1/\sqrt{2})(|\uparrow\rangle \pm |\downarrow\rangle)$ with eigenvalues $\epsilon^\pm = \epsilon \pm \Delta$, which are split by an amount $\Delta \sim e^{-L/\xi}$ that is exponentially small in the system size L if the system is to exhibit true SτB. The eigenstates of H_F are thus cat states since the constituent $|\uparrow/\downarrow\rangle$ are macroscopically distinct. The quasienergies of U_F are then obtained by including their eigenvalue under X: $\epsilon_F^+ = \epsilon + \Delta$ and $\epsilon_F^- = \epsilon - \Delta + \omega_D/2$. The spectrum of U_F thus consists of pairs of cat states with eigenvalues that differ by $\omega_D/2$ up to exponential accuracy in L.

To relate this spectral property back to Eq. (3), note that an initial state $|x\rangle$ will generically have amplitudes on both the $|+\rangle$ and $|-\rangle$ states of each pair. Owing to the eigenvalue pairing, the system will thus coherently oscillate at frequency $\omega_D/2$, up to a dephasing time $\tau \sim \Delta^{-1}$ that diverges exponentially in L.

4. Application to many-body parametric resonance

We now see how the formalism from Secs. III.B.1 and III.B.2 plays out for a single parametrically driven nonlinear pendulum; cf. Eq. (6). Chirikov (1979) and

Zounes and Rand (2002) showed that $H_F(Q, P)$ indeed exists within a finite volume of phase space (calling it the resonance Kamiltonian) and that, courtesy of the KAM theorem, its existence is stable to small perturbations. The contours of equal quasienergy $H_F(Q, P)$ are shown in Fig. 3(b). As mentioned, we see that H_F exhibits a $\mathbb{Z}_2$ symmetry $X: Q \to Q + \pi$, which exchanges two local minima enclosed by a separatrix. The detailed construction of $\mathcal{K}(t)$ is involved, but the approximate intuition is as follows: One starts with the canonical transformation in Eq. (7) and removes the residual time dependence order by order through a sequence of transformations whose convergence is guaranteed by the KAM theorem. The two minima of H_F $[Q, P = (\pm\pi/2, P_*)]$ are fixed points of H_F; mapping back to the lab frame, one obtains two precisely period-doubled orbits related by time translation, $\mathcal{K}^{-1}(t): (\pm\pi/2, P_*) \to q_*(t \pm T/2), p_*(t)$. Small deviations from the minima circulate around quasienergy contours, corresponding to slow oscillations about the period-doubled motion. Similar behavior is seen in other periodically driven problems, such as the kicked rotor model, whose stroboscopic motion Φ reduces to Chirikov's standard map (Chirikov, 1979) or a particle bouncing off an oscillating mirror (Holthaus and Flatté, 1994; Holthaus, 1995). In this 0D case, the SτB arises because KAM stability allows the quasienergy to "split up" the $\{Q, P\}$ space into disconnected basins [Fig. 3(b)], rather than because of a collective phenomena. Furthermore, in 0D there is long-range order in time but not in space, and as such one may prefer to not consider this case a time crystal. This analysis also illustrates why the quantum version of a parametrically driven nonlinear oscillator will not feature infinitely long-lived SτB: while a quantum Floquet Hamiltonian with a $\mathbb{Z}_2$ symmetry may exist, for any finite barrier height there will be quantum tunneling between the two minima at a rate $\sim 1/\tau$. This leads to a unique $\mathbb{Z}_2$-symmetric steady state, and the subharmonic response will have a correlation time of τ. Holthaus and Flatté (1994), Holthaus (1995), Buchleitner, Delande, and Zakrzewski (2002), and Sacha (2015) analyzed several examples of such quantum subharmonic responses and provided estimates of the tunneling rate, including a particle bouncing off an oscillating mirror, a topic that we return to in Sec. IV.E.

The fly in the ointment comes when we attempt to construct H_F in the many-body setting. In particular, we now consider an array of coupled pendula by adding a nearest-neighbor interaction to Eq. (6),

$$H(q, p, t) = \sum_i \left[\frac{p_i^2}{2} + \frac{\omega_0^2}{2}[1 + \delta\cos(\omega_D t)]q_i^2 + \frac{\epsilon}{4}q_i^4 \right] - g\sum_{\langle i,j \rangle}(q_i - q_j)^2. \tag{12}$$

Physically, one can think of this as treating the pendulum as a *macroscopic* object composed of atoms [i.e., the Frenkel-Kontorova model (Kontorova and Frenkel, 1938)]. It differs from the Faraday problem in the details of the dispersion and nonlinearity but is otherwise conceptually similar.

If the initial condition is uniform $(q_i, p_i = q, p)$, the problem exactly reduces to Eq. (6), which we now know exhibits stable period doubling. But stability for a finite

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[1]In conventional symmetry breaking, only eigenstates with energy densities below the symmetry-breaking temperature T_c have this structure. However, if many-body localization is able to exist in Floquet Hamiltonians (Ponte *et al.*, 2015; Šuntajs *et al.*, 2020; Sels and Polkovnikov, 2021), it is believed that symmetry breaking must manifest at all energy densities (De Roeck *et al.*, 2016; Moudgalya, Huse, and Khemani, 2020; Sahay *et al.*, 2021).

volume of initial conditions requires one to consider initial conditions of the form $q_i = q_* + \delta q_i$, in which the δq_i are small but independent. When the language we introduced near Eq. (5) is used, there are now weights in the higher-k modes q_k that are all-to-all coupled through the $\sum_i q_i^4$ nonlinearity, so the center-of-mass mode (i.e., $k = 0$) cannot be examined in isolation. While one can attempt to employ the same time-dependent canonical transformation $\mathcal{K}(t)$ site by site, for generic initial conditions the coupling $g(q_i - q_j)^2$ will not be exactly time independent after this transformation; thus, since the transformed Hamiltonian $H(Q_i, P_i, t)$ remains time dependent we cannot appeal to the conservation of H_F to ensure stability. One can attempt to remedy this by adjusting the canonical transformation via a perturbative expansion in g, but there is a general reason to expect this expansion will fail to converge in the thermodynamic limit: the ubiquity of ergodicity.

C. Ergodicity, destroyer of time crystals

As discussed, Hamiltonian systems are distinguished from more general dynamical systems like coupled map lattices by the existence of a phase-space volume element $d\mu = dq \wedge dp$, which is invariant under the dynamics, i.e., the dynamics are "measure preserving." An invertible measure-preserving system is said to be ergodic if the only finite-volume subset A that is invariant under the discrete-time update rule Φ [i.e., $\Phi(A) = A$] is the entire phase space. An equivalent statement is that the orbit of any volume $\cup_m \Phi^{(m)}(A)$ eventually fills all of phase space (Cornfeld, Fomin, and Sinai, 2012). Full ergodicity is incompatible with the existence of a time-independent $H_F(Q, P)$ because the conservation of the quasienergy H_F partitions phase space into Φ-invariant quasienergy contours (though the dynamics may be ergodic within each contour). A time-dependent many-body system with a bounded state space is generically expected to be ergodic in the thermodynamic limit.[2]

Or, in the unbounded case, the generic fate of a driven, many-body Hamiltonian system is to heat up. In an ergodic system, Birkhoff's ergodic theorem (Birkhoff, 1931; Cornfeld, Fomin, and Sinai, 2012) equates a temporal average of the sort defined in Eq. (3) with averages over the entire phase space. Thus, SτB necessarily requires $\Phi^{(m)}$ to be nonergodic: a time crystal remembers its initial condition for an infinitely long time because it is encoded in the phase of the period-m oscillations. SτB is thereby deeply connected with another profound phenomena: ergodicity breaking.

When the interactions are weak or the driving is strong, the timescale required to explore all of phase space can be large, so in practice SτB may persist out to very long times τ.

This behavior is sometimes referred to as slow *Floquet heating*. While energy is not conserved, in systems described by an effective Floquet Hamiltonian [Eq. (8)], the quasienergy $\langle H_{\text{eff}} \rangle$ will change slowly due to the residual drive $V(t)$, eventually allowing the system to explore its full phase space. We return to such anomalously long timescales when we discuss prethermal time crystals in Sec. IV.D.

The existence of such a long timescale is certainly one ingredient in Faraday's observation of period doubling in a system that is Hamiltonian. An additional role is played by viscosity, which converts the high-k oscillations of the surface into heat that is eventually dissipated into the environment. When viscosity alone is added to the equations of motion, the system is no longer measure preserving and becomes analogous to the coupled map lattices where true SτB is possible. However, this might seem contradictory because viscosity is, after all, a phenomenological treatment of microscopic degrees of freedom that are themselves Hamiltonian. The key is that at any finite temperature (energy density), a microscopic bath invariably generates noise in addition to viscosity, as required by the fluctuation-dissipation theorem. Understanding the fate of SτB in the presence of dissipation and noise is a particularly rich direction; we return to the possibility of such "open" time crystals in Sec. V.

To summarize, in dissipative dynamical systems like the coupled map lattices or the limit cycle of a van der Pol oscillator, stable, many-body SτB is possible, and time crystals in this context have a long history. But their realization in measure-preserving systems that describe the Universe at the microscopic level is far more subtle. While stable SτB is possible in few-body classical systems such as a parametric resonator (i.e., the shaken pendulum example in Sec. III.B.4), their existence in many-body systems requires, at minimum, a generic mechanism for breaking ergodicity. In closed classical systems, this is not thought to be possible except when the system is fine-tuned. This is where the magic of quantum mechanics comes in, through the physics of many-body localization. Many-body localization is widely believed to allow for stable ergodicity breaking in the thermodynamic limit, and where there is ergodicity breaking there will be time crystals.

IV. CLOSED, PERIODICALLY DRIVEN QUANTUM SYSTEMS

A. Introduction to quantum Floquet phases

As noted, ergodicity breaking is necessary for the stability of a time crystal. In fact, with the insights of Sec. III in hand, one can utilize almost any generic, robust form of ergodicity breaking to stabilize a time crystal. However, this was not immediately apparent when many-body localization (Nandkishore and Huse, 2015; Abanin *et al.*, 2019) was originally discovered. We note at the outset that there remains uncertainty regarding the stability of many-body localization, with certain studies arguing for a thermalization timescale that diverges exponentially in the strength of the disorder W, $\tau \sim e^{W/\Omega}$, where Ω is a constant (Šuntajs *et al.*, 2020; Abanin *et al.*, 2021). If this is indeed the case, it would naturally place many-body localized SτB in the category of prethermal discrete time crystals (Sec. IV.D).

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[2]The ideas emanating from Boltzmann's "ergodic hypothesis" postulate that time-independent many-body systems are generically ergodic within each conserved energy shell (Von Plato, 1991). This hypothesis relates to the Floquet setting by converting the time-dependent Hamiltonian $H(t)$ into a time-independent $\mathcal{H} = H(t) + \pi_t$, where (t, π_t) is an additional canonical pair living on the cylinder $t \sim t + T$. If $\mathcal{H}$ is ergodic on an energy shell, then the stroboscopic dynamics of $H(t)$ are fully ergodic.

Many-body localization is a phenomenon that is believed to occur in 1D systems with strong quenched disorder and short-range interactions (Choi *et al.*, 2016; Smith *et al.*, 2016). It represents a failure of the eigenstate thermalization hypothesis (Deutsch, 1991; Srednicki, 1994) that occurs when disorder prevents equilibration by localizing the degrees of freedom of a system. Locally injected energy cannot spread throughout the system (Nandkishore and Huse, 2015; Abanin and Papić, 2017; Abanin *et al.*, 2019). Many-body localization was originally formulated as a property of highly disordered time-independent Hamiltonians $\hat{H}$ in which local observables fail to relax to their thermal average with respect to $e^{-\beta \hat{H}}$ and, instead, the long-time behavior depends on the details of the initial state, an example of ergodicity breaking. It was subsequently realized that many-body localization may also occur in 1D periodically driven systems with strong disorder for appropriate driving parameters (Lazarides, Das, and Moessner, 2015; Ponte *et al.*, 2015; Abanin, Roeck, and Huveneers, 2016; Sierant *et al.*, 2023). In such Floquet-MBL systems, energy absorbed during one part of the drive cycle must be returned before its completion (D'Alessio and Polkovnikov, 2013; Ponte *et al.*, 2015; Bordia *et al.*, 2017). Consequently, the energy spreading and thermalization necessary for drive-induced Floquet heating cannot occur (Sec. III.C).

In the meantime, interest in Floquet systems had been reenergized by seminal work demonstrating that periodic driving can induce topological properties in an otherwise nominally trivial system (Inoue and Tanaka, 2010; Jiang *et al.*, 2011; Lindner, Refael, and Galitski, 2011; Thakurathi *et al.*, 2013; Wang *et al.*, 2013; Potter, Morimoto, and Vishwanath, 2016; Roy and Harper, 2016, 2017; von Keyserlingk and Sondhi, 2016a, 2016b; Potirniche *et al.*, 2017). In particular, Jiang *et al.* (2011) and Thakurathi *et al.* (2013) considered periodically driven versions of a Majorana chain in which the time-dependent Hamiltonian toggles between topological superconducting and trivial insulating states. These free fermion models avoid Floquet heating by virtue of their integrability, which does not survive generic perturbations involving interactions. For such drives, the Floquet operator takes on a particularly simple form $U_{\mathrm{F}} = \exp[ic\sum_j i\gamma_j^A \gamma_j^B] \times \exp[iJ\sum_j i\gamma_j^B \gamma_{j+1}^B]$ (Prosen, 1998; Thakurathi *et al.*, 2013). For c near $\pi/2$, there are "Floquet Majorana fermions" at quasienergy π. In other words, there are locally indistinguishable states of opposite fermion parity associated with edge modes. In an undriven system, Majorana edge modes would be degenerate (i.e., they would be zero modes), but the periodic driving splits their Floquet eigenvalues by π.

This property of the spectrum does not imply a subharmonic response of the system for generic initial conditions (and moreover it is not stable to interactions). However, a Jordan-Wigner transformation of the driven Majorana chain yields a driven Ising model that exhibits subharmonic oscillations of the Ising order parameter; moreover, by adding strong disorder leading to many-body localization, the system could be stabilized against interactions (Khemani *et al.*, 2016). Initially it was not believed that the subharmonic response of the resulting "π spin glass" could survive in the absence of the Ising symmetry, implying that, in analogy with the discussions of the *XY* magnet in Sec. II.A, a time crystal stable against generic perturbations did not exist (Khemani *et al.*, 2016).

This sentiment was overturned by Else, Bauer, and Nayak (2016), who showed that discrete time crystals are a stable
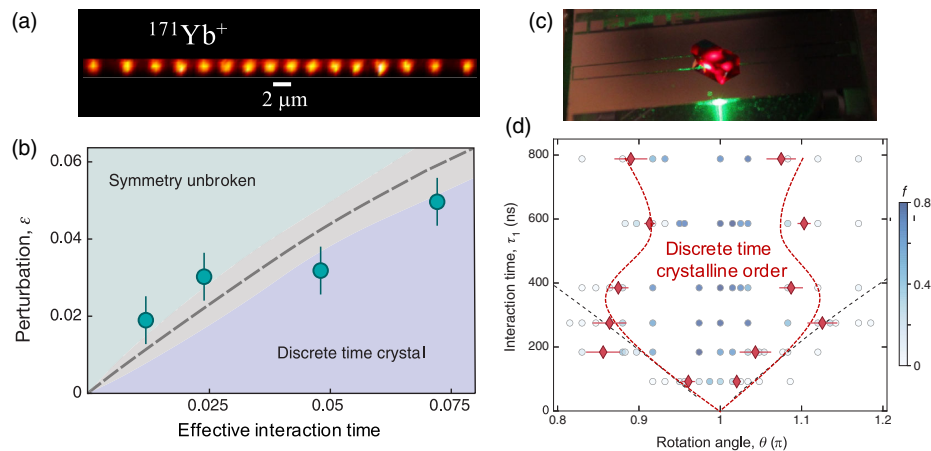

FIG. 4.   (a) Signatures of time-crystalline order were observed in a one-dimensional chain of ytterbium ions. In this system, the effective spin degree of freedom is encoded in two hyperfine "clock" states of a ^{171}Yb$^+$ ion. (b) By varying the interaction strength (*x* axis), which is effectively parametrized by an interaction timescale, and the strength of the perturbation on the π pulse (*y* axis), Zhang *et al.* (2017) were able to observe the system's behavior change from a discrete time crystal phase to a time-translation symmetry-unbroken phase. Adapted from Zhang *et al.*, 2017. (c) Choi *et al.* (2017) utilized a diamond sample containing a high density of nitrogen-vacancy color centers (~15 ppm) (Choi *et al.*, 2020). In this system, the spin consists of two m_s sublevels, and N-*V* centers interact with one another via magnetic dipole-dipole interactions. The interplay between the long-range dipolar interaction and the three-dimensional nature of the sample is conjectured to lead to critically slow thermalization (Ho *et al.*, 2017; Kucsko *et al.*, 2018), during which discrete time-crystalline order can be observed. (d) Experimentally measured phase boundary (red dashed line through diamond data points added as a guide) of the critical discrete time crystal as a function of the effective interaction strength and the π-pulse imperfection. Adapted from Choi *et al.*, 2017.

phase of matter in Floquet-MBL systems with only time-translation symmetry. In particular, a discrete time crystal in a closed system must satisfy Eq. (3) in the absence of a bath and in a manner that is stable against arbitrary weak perturbations respecting time-translation symmetry. Other symmetries may or may not be present, but Eq. (3) should hold even when those symmetries are violated. These developments led rapidly to the first phase diagram for a time crystal and to proposals describing how to create and measure discrete time crystals in experiments (Yao *et al.*, 2017), culminating in the first observations of time-crystalline behavior (Fig. 4) (Choi *et al.*, 2017; Zhang *et al.*, 2017).

Thus far we have focused on many-body localization as a possible strategy for stabilizing discrete time crystals. One of the caveats of using many-body localization is that the approach depends on the robustness of many-body localization itself. While many-body localization has purportedly been proven to exist in certain undriven, one-dimensional spin chain models with random local interactions (Imbrie, 2016), its stability in $d > 1$ and in long-range interacting systems remains an open question (Yao *et al.*, 2014; De Roeck and Huveneers, 2017; Luitz, Huveneers, and Roeck, 2017; Thiery *et al.*, 2018; Khemani, Moessner, and Sondhi, 2021; Yao *et al.*, 2021); to date, there has been no mathematical proof for the stability of many-body localized SτB in a Floquet system in any dimension. An alternate strategy for stabilizing discrete time crystalline (DTC) order called Floquet prethermalization can occur in any dimension and in the presence of long-range interactions, so long as the driving frequency ω_D is sufficiently high. To that end, in the remainder of this section on closed systems (Fig. 1), we focus in Secs. IV.B and IV.C on MBL discrete time crystals in 1D and in Sec. IV.D on prethermal discrete time crystals in arbitrary dimension and with power-law interactions.

B. Many-body localized discrete time crystal

Our focus in this section is on highlighting the key features of time-crystalline order in Floquet-MBL systems, by describing recent observations from three distinct experimental platforms: trapped atomic ions (Yao *et al.*, 2017; Zhang *et al.*, 2017), spins in solid-state materials (Abobeih *et al.*, 2019; Randall *et al.*, 2021), and superconducting qubits (Arute *et al.*, 2019; Mi *et al.*, 2021; Frey and Rachel, 2022).

Before describing the experiments, we begin by introducing a standard model for a period-doubled discrete time crystal in a spin-1/2 chain and discuss several limits of this model. While the experiments do not implement this specific Floquet unitary, the spirit of their Floquet dynamics are captured by the example. In particular, consider Floquet time evolution (with a period $T_0 = t_1 + t_2$) governed by alternating between two time-independent Hamiltonians:

$$H(t) = \begin{cases} H_1, & \text{for time } t_1, \\ H_2, & \text{for time } t_2. \end{cases} \quad (13)$$

We take H_1 and H_2 to be given by

$$H_1 = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_i (h_i^z\sigma_i^z + h_i^y\sigma_i^y + h_i^x\sigma_i^x),$$

$$H_2 = g\sum_i \sigma_i^x, \quad (14)$$

with $\vec{\sigma}$ being Pauli spin operators and $J_{ij}, h_i^x, h_i^y, h_i^z$ sufficiently disordered to ensure that the spin chain is many-body localized. The simplest way to see the trivial emergence of a period-doubled (i.e., $\nu = 1/2$ subharmonic) response is to consider the decoupled limit ($J_{ij} = 0$) with only a longitudinal field h_z along $\hat{z}$. For any individual spin initially aligned along the $\hat{z}$ axis, the spin will Larmor precess around the $\hat{x}$ field during the second portion H_2 of the Floquet evolution. When timed appropriately ($t_2 = \pi/2g$), this evolution implements a so-called π pulse, which flips the spin and causes a period-doubled response. However, much like our discussion of the map $\Phi(x) = -x$ in Sec. II, this period doubling is not rigid to perturbations and not indicative of a many-body phase. In particular, any imperfections in the timing of the π pulse will immediately lead to the breakdown of the period-doubled response (Yao *et al.*, 2017). However, by turning on the Ising interactions between the spins, in conjunction with the disorder which leads to many-body localization, the system becomes robust to small imperfections, and the subharmonic response is rigid to arbitrary, weak perturbations of both the initial state and the Hamiltonian (so long as these perturbations respect the period of the Floquet evolution). We note that this last parenthetical condition distinguishes time-crystalline order in closed systems from time-crystalline order in open systems, where perturbations that explicitly break the discrete time-translation symmetry of the dynamics can still be allowed; see Sec. V.E.

C. Experimental signatures of disordered time crystals

1. Trapped ion spin chains

Zhang *et al.* (2017) observed and characterized time-crystalline behavior in a spin chain [Fig. 4(a)] composed of trapped atomic ions. In the experiment, each ion acts as an effective spin, and interactions are controlled through external, optical spin-dependent forces (Monroe *et al.*, 2021). The system is periodically driven by successively applying three time-independent Hamiltonians corresponding to a global drive, interactions, and disorder (Smith *et al.*, 2016). The disorder is programmed by individually addressing each spin with a tunable laser beam. The ion chain was composed of a relatively small number of effective spins (where $L = 10–14$) and the time-crystalline response exhibited some sensitivity to the initial conditions (Zhang *et al.*, 2017; Khemani, Moessner, and Sondhi, 2019), suggesting that long-range interactions could be leading to a time-crystalline response stabilized by prethermalization (Sec. IV.D) rather than localization. The experiment was able to observe a crossover between the DTC regime and the symmetry-unbroken phase [Fig. 4(b)]. To do so, Zhang *et al.* (2017) measured the Fourier spectrum of each individual spin and studied the variance of the amplitude of the $\nu = 1/2$ subharmonic response as a function of the π-pulse imperfection ϵ. By increasing the strength of the interactions between the ion spins, they demonstrated that the location of the variance peak shifted toward larger values of ϵ, which is consistent with the expectation that many-body interactions are essential for stabilizing time-crystalline order.

2. Spins in condensed matter

Signatures of time-crystalline order in solid-state systems have been observed in both disordered nitrogen-vacancy
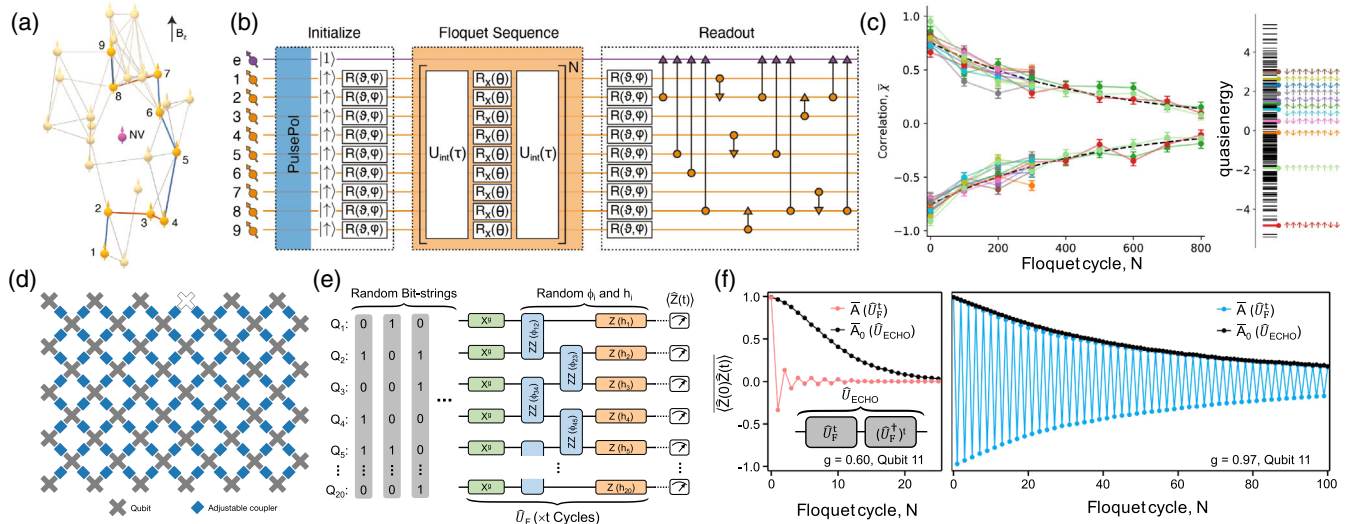
FIG. 5. (a) Schematic image depicting the nine-spin subset of the 27 precharacterized nuclear spins surrounding the single N-*V* center. (b) Each nuclear spin can be individually initialized and read out via coupling to the N-*V* center. The Floquet sequence consists of U_{int}, which includes a disordered long-range Ising interaction and a disordered longitudinal field followed by an approximate π pulse $R_x(\theta)$. The disorder is naturally inherited from the random positioning of the nuclear spins within the diamond lattice. (c) In the MBL discrete time-crystal phase, the system exhibits robust subharmonic oscillations lasting $\sim 10^3$ Floquet cycles. The various colors are associated with different initial states, which are characterized by their quasienergy in the right panel. The time-crystalline behavior is independent of the initial state, as expected for a MBL time crystal. (d) Schematic depicting the 53 transmon qubit Sycamore chip. Mi *et al.* (2021) isolated one-dimensional chains of $L = 8$, 12, 16, and 20 qubits from this two-dimensional grid. (e) Digital Floquet sequence consisting of random longitudinal fields, disordered nearest-neighbor Ising interactions, and approximate π pulses are applied to random "bit-string" initial states. (f) In the thermal phase (left panel), the disorder-averaged autocorrelation function quickly decays to zero. In the MBL time-crystal phase, the same correlation function exhibits subharmonic oscillations lasting $\sim 10^2$ Floquet cycles. Mi *et al.* (2021) also implemented a benchmarking "echo" sequence U_{echo}, which demonstrated that the time crystal's lifetime is consistent with being limited by experimental imperfections and decoherence. (a)–(c) Adapted from Randall *et al.*, 2021. (d)–(f) Adapted from Arute *et al.*, 2019, and Mi *et al.*, 2021.

(N-*V*) ensembles (Choi *et al.*, 2017; Ho *et al.*, 2017) and NMR systems (Pal *et al.*, 2018; Rovny, Blum, and Barrett, 2018a, 2018b; Luitz *et al.*, 2020). In the context of N-*V* centers [Figs. 4(c) and 4(d)], despite the presence of strong disorder it is known that dipolar interactions in three dimensions cannot lead to localization; however, the resulting time-crystalline order can exhibit an anomalously long lifetime owing to critically slow thermalization (Ho *et al.*, 2017). One of the puzzles arising from NMR experiments on phosphorus nuclear spins in ammonium dihydrogen phosphate was the observation of period doubling despite the lack of disorder and a high temperature initial state. These observations were ultimately understood as consequences of an approximate long-lived U(1) conservation law (Luitz *et al.*, 2020; Stasiuk and Cappellaro, 2023), which leads to effective prethermal time-crystalline order; see Sec. IV.D.

Recent work in the solid state has focused on pushing toward a realization of time-crystalline order in a regime compatible with many-body localization (Randall *et al.*, 2021). As depicted in Fig. 5(a), Randall *et al.* (2021) utilized a platform consisting of a precisely characterized array of 27 nuclear spins surrounding a single N-*V* center in diamond (Abobeih *et al.*, 2019). Owing to differences in the hyperfine interaction strengths, each nuclear spin is individually addressable (for both initialization and readout [Fig. 5(b)]). To work with an effective one-dimensional geometry (thus avoiding issues regarding the stability of many-body

localization in $d > 1$), Randall *et al.* (2021) selected a specific nine-spin subset of the 27 nuclear spins. By preparing a variety of product initial states, Randall *et al.* (2021) demonstrated that dipolar interactions between the nuclear spins lead to robust period doubling for $\sim 10^3$ Floquet cycles (note that each Floquet cycle lasts approximately 10 ms), regardless of the initial state [Fig. 5(c)]. Moreover, they showed that local thermalization occurs within ~ 10 Floquet cycles, demonstrating that the observed time-crystalline order is not a result of slow thermalization.

3. Superconducting transmon qubits

Signatures of time-crystalline order have also recently been observed in three experiments utilizing superconducting transmon qubits (Mi *et al.*, 2021; Xu *et al.*, 2021; Frey and Rachel, 2022). Working with Google's "Sycamore" processor, Mi *et al.* (2021) employed a quantum-circuit-based approach to demonstrating MBL time-crystalline order (Ippoliti *et al.*, 2020). In particular, to work with a one-dimensional system, Mi *et al.* (2021) isolated a nearest-neighbor coupled chain of $L = 20$ qubits from the two-dimensional array [Fig. 5(d)]; an analogous strategy was taken by Frey and Rachel (2022), who isolated an $L = 57$ qubit chain on IBM's quantum processors: ibmq_manhattan and ibmq_brooklyn. Both Mi *et al.* (2021) and Frey and Rachel (2022) utilized one- and two-qubit gates in order to implement a digital Floquet sequence. For the

experiment performed on the Sycamore processor, the specific Floquet sequence was given by [Fig. 5(e)]

$$U_F = e^{-(i/2)\sum_i h_i \sigma_i^z} e^{-(i/4)\sum_i J_i \sigma_i^z \sigma_{i+1}^z} e^{-(i/2)\pi g \sum_i \sigma_i^x}, \quad (15)$$

which consists of random longitudinal fields, disordered nearest-neighbor Ising interactions, and approximate π pulses. Deviations from the ideal π-pulse limit [i.e., $g = 1$ in Eq. (15)] are used to control the transition between the discrete time crystal phase [i.e., $g = 0.97$ in Fig. 5(f)] and the trivial thermal phase [i.e., $g = 0.6$ in Fig. 5(f)]. To demonstrate that the observed subharmonic response is not affected by the choice of initial states, both Mi *et al.* (2021) and Frey and Rachel (2022) probed the Floquet dynamics starting with random initial bit strings. In addition, Mi *et al.* (2021) also experimentally implemented a finite-size-scaling analysis by varying the length of their one-dimensional chain between contiguous subsets of 8, 12, and 16 transmon qubits; this reveals a transition between the MBL discrete time crystal and the thermal phase at a critical value of the π-pulse imperfections, $0.83 \gtrsim g_c \lesssim 0.88$. Finally, to characterize the intrinsic gate errors and decoherence of their system [with $T_1, T_2 \sim 100$ μs (Arute *et al.*, 2019; Kjaergaard *et al.*, 2020)], Mi *et al.* (2021) implemented a benchmarking Floquet sequence that reversed the digital time evolution after a set number of Floquet cycles. Dividing by this benchmarking sequence leads to signatures of time-crystalline order that exhibit minimal effective decay.

D. Prethermal discrete time crystal

1. Floquet prethermalization

While many-body localization provides a quantum approach to realizing a time crystal with an infinite lifetime in the thermodynamic limit ($\tau \sim e^L \to \infty$), in this section we focus on an alternate, disorder-free strategy called Floquet prethermalization (Fig. 1). Here the lifetime of the time-crystalline order does not scale with system size but can be exponentially long in a particular control parameter, namely, the ratio of the driving frequency to local energy scales J within the system: $\tau \sim e^{\omega_D/J}$.

In static systems, the concept of prethermalization is a powerful framework for understanding the thermalization of systems with disparate energy scales: the dynamics (and thus thermalization) of "fast moving" degrees of freedom are governed by an effective Hamiltonian where the slow degrees of freedom remain basically frozen (Kagan and Klinger, 1974; Berges, Borsányi, and Wetterich, 2004; Gring *et al.*, 2012). In driven Floquet systems, the presence of two distinct energy scales is particularly natural since the frequency of the drive and the interaction energy scales within the Hamiltonian are independent. Indeed, it was recently established that Floquet prethermalization is a generic feature of driven systems in the high-frequency regime (Abanin, Roeck, and Huveneers, 2015; Kuwahara, Mori, and Saito, 2016; Mori, Kuwahara, and Saito, 2016; Abanin *et al.*, 2017a, 2017b; Else, Bauer, and Nayak, 2017; Weidinger and Knap, 2017). From the perspective of heating, as discussed in Sec. IV.A, one of the central consequences of Floquet prethermalization is

that the frequency of the drive exponentially controls the heating timescale t_* (Abanin, Roeck, and Huveneers, 2015; Mori, Kuwahara, and Saito, 2016; Machado *et al.*, 2019). The physical intuition for this exponential scaling is as follows: at large frequencies $\omega_D \gg J$ (where J is the local energy scale of the many-body system), the system must undergo $\sim \omega_D/J$ local rearrangements in order to absorb a single unit of energy from the drive. This intuition holds for both quantum and classical systems, although in the remainder of Sec. IV.D.1 we focus on the quantum setting; we return to the classical case in the outlook (Sec. VI.B.1).

Kuwahara, Mori, and Saito (2016) and Abanin *et al.* (2017a) proved that up until the timescale $t_* \sim e^{\omega_D/J}$, the system does not absorb energy from the drive and the stroboscopic Floquet dynamics are (up to exponentially small corrections) captured by an effective static Hamiltonian H_{eff}. Unlike in many-body localization, these features of Floquet prethermalization are largely independent of disorder, dimension, and the range of interactions (Machado *et al.*, 2019, 2020; Fan *et al.*, 2020; Rubio-Abadal *et al.*, 2020; Peng *et al.*, 2021; Pizzi, Knolle, and Nunnenkamp, 2021).

While the ability to exponentially delay the onset of Floquet heating is crucial for realizing and stabilizing a discrete time crystal, one additional key insight is still needed, namely, the observation that H_{eff} can exhibit an emergent symmetry (which need not be present in the original Floquet evolution) protected by the discrete time-translation symmetry of the drive (Else, Bauer, and Nayak, 2017; Machado *et al.*, 2020). The presence of such an emergent symmetry in H_{eff} allows one to sharply define phases of matter (i.e., via symmetry breaking) in the prethermal regime (Fig. 6). A mathematical description of prethermal discrete time crystals in the quantum setting closely follows the discussion of Secs. III.B.1 and III.B.2.

This lays the foundation for the connection to time-crystalline order. If a many-body system prethermalizes to
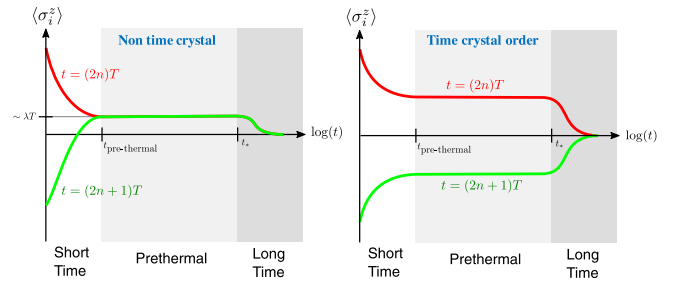

FIG. 6. Schematic depicting prethermal time-crystalline order identified via the stroboscopic behavior of the magnetization as a function of time. Left panel: in the non-time-crystal prethermal phase, the magnetization during even and odd Floquet cycles becomes identical after a short time (corresponding to local thermalization). During the intermediate prethermal regime, this magnetization may remain small, but nonzero, until it decays to zero at the Floquet heating timescale. Right panel: in the prethermal discrete time-crystal phase, the magnetization oscillates throughout the prethermal regime, leading to stroboscopic curves that exhibit plateau behavior. The prethermal time crystal ultimately melts at exponentially late times, controlled by the frequency of the driving field. Adapted from Else, Bauer, and Nayak, 2017.

a state that spontaneously breaks the emergent symmetry of H_{eff}, it will also exhibit time-crystalline order, corresponding to a subharmonic oscillation between the different symmetry sectors (Else, Bauer, and Nayak, 2017; Machado *et al.*, 2020).

To recap, within the framework of Floquet prethermalization, realizing a time crystal requires a few ingredients: First, the Floquet drive must induce an emergent symmetry in the prethermal effective Hamiltonian H_{eff}. Second, H_{eff} must be able to host a symmetry-broken phase with respect to the emergent symmetry. Note that owing to Landau-Peierls-type arguments, this naturally places constraints on the interaction range and dimensionality for realizing a prethermal discrete time crystal (PDTC) (Machado *et al.*, 2020; Kyprianidis *et al.*, 2021; Pizzi, Knolle, and Nunnenkamp, 2021). Finally, the initial state of the many-body system must have a sufficiently low energy density (measured with respect to H_{eff}) that it equilibrates to the spontaneously symmetry-broken phase during the prethermal regime (Fig. 6).

2. Prethermal discrete time crystal in a 1D trapped ion chain

To highlight the dynamical signatures of a prethermal discrete time crystal, as well as the distinctions from a MBL time crystal (Sec. IV.B), we turn to a recent experiment performed on a trapped ion quantum simulator (Kyprianidis *et al.*, 2021). As discussed in Sec. IV.C.1, some of the first experimental observations of time-crystalline behavior [Fig. 4(a) and 4(b)] were originally observed in small-scale trapped ion experiments (Zhang *et al.*, 2017), and one of the central advances in recent works is the ability to experimentally distinguish between local thermalization ("short time" in Fig. 6) and late-time dynamics in the prethermal regime.

The experiment consists of a one-dimensional chain of $N = 25$ ytterbium ions, and the Floquet evolution alternates between two types of dynamics. First, a global π pulse is applied and, second, the system evolves under a disorderless, long-range, mixed-field Ising model. At leading order in the Floquet-Magnus expansion, the stroboscopic dynamics of the ions are captured by the effective Hamiltonian

$$H_{\text{eff}} = \sum_{i,j} J_{ij}\sigma_i^x\sigma_j^x + B_y\sum_i \sigma_i^y, \qquad (16)$$

which exhibits an emergent Ising symmetry. To begin, Kyprianidis *et al.* (2021) demonstrated that, regardless of the initial state of the ion chain, the system exhibits slow, frequency-dependent heating to infinite temperature. As mentioned, for the system to exhibit time-crystalline order, the initial state of the ion chain must reach a pseudoequilibrium state (within the prethermal regime) in which it breaks the emergent Ising symmetry (Fig. 7). For a one-dimensional system at finite temperature, this is possible only for sufficiently long-range interactions (Thouless, 1969; Fisher, Ma, and Nickel, 1972; Kosterlitz, 1976). The trapped ion system generates such long-range interactions using a pair of Raman laser beams to couple the internal spin states to motional modes of the ion chain (Sørensen and Mølmer, 1999).

Initializing the system with a Néel state [Fig. 7(a), top panel], Kyprianidis *et al.* (2021) observed that the magnetization $M(t) = (1/N)\sum_{i=1}^N \langle\sigma_i^x(t)\rangle\langle\sigma_i^x(0)\rangle$ quickly decays to zero,
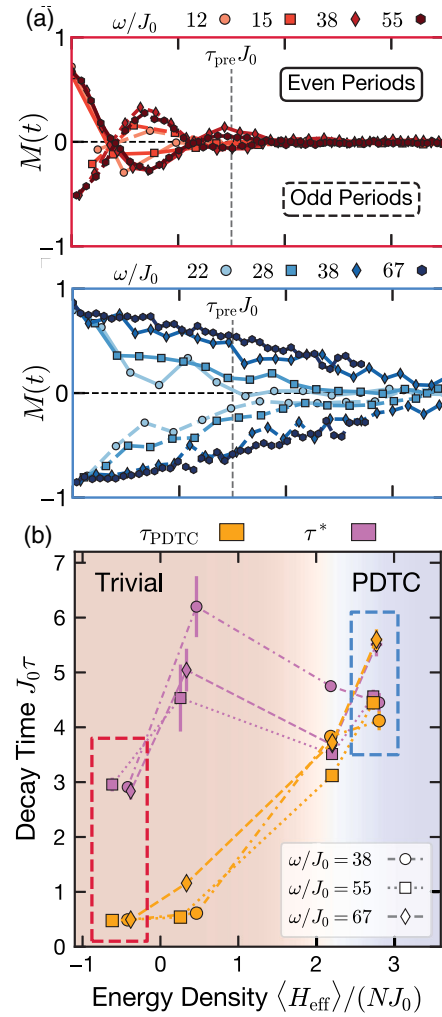

FIG. 7. Prethermal discrete time crystal in a disorder-free chain of $L = 25$ ^{171}Yb$^+$ ions. (a) For initial states whose energy density (measured with respect to H_{eff}) lies in the trivial phase of the effective Hamiltonian (top panel), the dynamics of the magnetization are largely independent of the drive frequency and quickly equilibrate to zero. Despite the rapid decay of the magnetization, the energy density exhibits slow Floquet heating. For initial states whose energy density lies in the symmetry-broken phase of H_{eff} (bottom panel), the stroboscopic dynamics of the magnetization exhibit subharmonic oscillations (solid curves correspond to even periods, while dashed curves correspond to odd periods), whose lifetime extends with increasing frequency. The lifetime of this prethermal time-crystalline order is consistent with being cut off by the slow Floquet heating timescale. (b) Experimentally measured phase diagram of a prethermal discrete time crystal as a function of the energy density of the initial state. States near the edge of the spectrum (i.e., which order with respect to H_{eff}) exhibit PDTC order, while states in the trivial phase of H_{eff} do not exhibit prethermal time-crystalline order. (a),(b) Adapted from Kyprianidis *et al.*, 2021.

in agreement with the expectation that the system equilibrates to a symmetry-unbroken paramagnetic state. On the other hand, starting with a polarized initial state, $M(t)$ exhibits period doubling [Fig. 7(a), bottom panel], whose lifetime is directly controlled by the frequency of the drive. Moreover, in the latter case, the lifetime of the time-crystalline order matches with t_*,
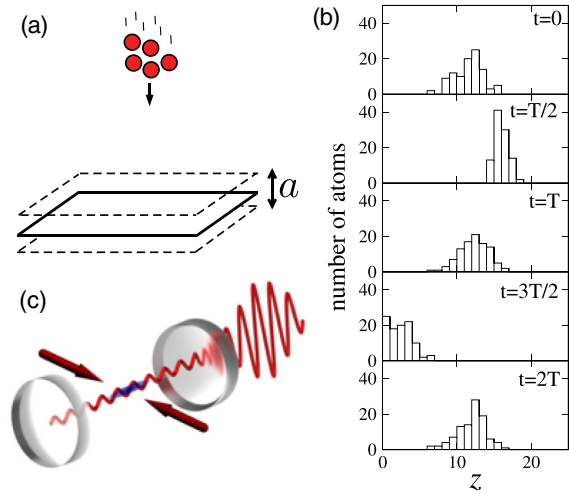
FIG. 8. (a) Schematic depicting N atoms bouncing off an oscillating mirror (Sacha, 2015). (b) Numerical simulations of the scenario in (a) shows that the time evolution of the atomic density exhibits SτB. At each time shown, the positions of 10^2 atoms (out of 10^4) are measured and a histogram of those positions is depicted. (c) A Bose-Einstein condensate in a transversely pumped high-finesse optical cavity can realize signatures of SτB in an open system. In particular, the atom-cavity system can exhibit a period-doubled oscillation between distinct density wave patterns (Keßler *et al.*, 2021). (a)–(c) Adapted from Sacha, 2015, Sacha and Zakrzewski, 2018, and Keßler *et al.*, 2021.

which is consistent with the intuition that Floquet heating ultimately melts the PDTC at late times.

An intriguing open question is whether this melting can be delayed or fully arrested by coupling the system to a cold bath (Else, Bauer, and Nayak, 2017). More broadly, the stability and rigidity of many-body time-crystalline order in dissipative, open quantum systems [Fig. 8(c)] remains an active area of exploration (Gong, Hamazaki, and Ueda, 2018; Iemini *et al.*, 2018; Tucker *et al.*, 2018; Buča and Jaksch, 2019; Buča, Tindall, and Jaksch, 2019; Dogra *et al.*, 2019; Gambetta, Carollo, Marcuzzi *et al.*, 2019; Keßler *et al.*, 2019; Lledó, Mavrogordatos, and Szymańska, 2019; Zhu *et al.*, 2019; Booker, Buča, and Jaksch, 2020; Chinzei and Ikeda, 2020; Keßler *et al.*, 2020; Lazarides *et al.*, 2020; Lledó and Szymańska, 2020; Seibold, Rota, and Savona, 2020; Keßler *et al.*, 2021; Kongkhambut *et al.*, 2022). We return to this topic in Sec. V.

E. Periodically driven Bose-Einstein condensates

Long-lived subharmonic responses have been explored extensively in driven Bose-Einstein condensates without disorder (Holthaus and Flatté, 1994; Holthaus, 1995; Sacha, 2015, 2020; Autti, Eltsov, and Volovik, 2018; Liao *et al.*, 2019; Giergiel *et al.*, 2020; Smits, Stoof, and van der Straten, 2020, 2021; Autti *et al.*, 2021; Hodson and Jarzynski, 2021; Hannaford and Krzysztof, 2022; Kongkhambut *et al.*, 2022; Autti *et al.*, 2023). A paradigmatic example is provided by a BEC bouncing off an oscillating mirror (Flatté and Holthaus, 1996; Sacha, 2015). To understand this scenario, consider a particle under the influence of gravity that bounces

off an oscillating mirror [Fig. 8(a)]. In the reference frame of the mirror (in which the mirror is held constant at $z = 0$), the particle's gravitational potential energy is given by $U(z) = m[g + a\cos(\omega_D t)]z$, with the position of the particle at $z > 0$. Classically, a single particle in this potential exhibits a $2:1$ subharmonic response that is conceptually equivalent to that of the driven nonlinear pendulum discussed in Sec. III.B (Flatté and Holthaus, 1996). In the quantum case, this leads to two "nonspreading wave packets" localized to the double minima of the effective Floquet Hamiltonian, corresponding to the two period-doubled orbits (Bialynicki-Birula, Kaliński, and Eberly, 1994; Buchleitner and Delande, 1995; Flatté and Holthaus, 1996; Buchleitner, Delande, and Zakrzewski, 2002). However, in contrast to the classical case, quantum tunneling between the two minima results in a finite lifetime τ for the subharmonic oscillations. From the spectral perspective (Sec. III.B.3), this manifests as two Floquet eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, whose relative Floquet eigenvalues are perturbed away from $\pi \to \pi + 1/\tau$. Intuitively, these two Floquet eigenstates correspond to the "bonding" and "antibonding" configurations of the effective double-well potential. When $\tau \to \infty$, the configurations $|\psi_\pm\rangle = (1/\sqrt{2})(|\psi_1\rangle \pm |\psi_2\rangle)$ are exchanged under each period, and a system prepared with a majority amplitude in one or the other will thus exhibit SτB. But with tunneling, these configurations slowly precess into each other over the tunneling time τ, restoring the symmetry (Holthaus and Flatté, 1994; Holthaus, 1995; Flatté and Holthaus, 1996; Buchleitner, Delande, and Zakrzewski, 2002; Sacha, 2015).

However, τ can be radically extended by replacing a single particle with an N-particle Bose-Einstein condensate with an attractive contact interaction $H_{\text{int}} = -(g_0/4)|\Psi(z)|^4$ (Sacha, 2015). In appropriate units, the Hamiltonian for the field operator $\hat{\Psi}$ is given by

$$\hat{H} = \int_0^\infty dz \left\{ \hat{\Psi}^\dagger(z) \left[-\frac{1}{2}\partial_z^2 + \left(1 + \frac{a}{g}\cos(\omega_D t)\right)|z| \right] \hat{\Psi}(z) \right.$$
$$\left. - \frac{g_0}{4}|\hat{\Psi}(z)|^4 \right\}. \tag{17}$$

Two approximations can be used to analyze the resulting dynamics (Sacha, 2015): (i) Taking $N \to \infty$ while holding $g_0 N$ fixed, the dynamics reduce to a periodically driven Gross-Pitaevskii equation (GPE). The GPE is a classical Hamiltonian field theory over a complex field $\psi(t, z)$ and can therefore be efficiently simulated. (ii) One can project the interacting dynamics onto the two nearly degenerate ground states (i.e., the aforementioned double minima) of the single particle Floquet Hamiltonian, effectively reducing the problem to two bosonic modes.

These approximations can be combined into three cases. When both approximations are made (i.e., taking $N \to \infty$ and also projecting to the two-mode picture), the dynamics exhibit a bifurcation transition that produces period doubling (Sacha, 2015). Intuitively, owing to the attractive interaction, the Floquet energy is minimized when all particles inhabit the same well; tunneling between the wells is thereby suppressed because all $N \to \infty$ bosons must tunnel sequentially.

If one considers only the second approximation (i.e., finite N within the two-mode picture), quantum fluctuations in the mode occupation remain, and it was shown that the tunneling time increases exponentially with N, leading to a scaling $\tau \propto e^{\alpha N}$ (Sacha, 2015; Wang *et al.*, 2021). Thus, SτB is achieved not in the limit of large system size, but rather in the limit of large particle number. However, the two-mode approximation automatically prevents Floquet heating (which is generally presumed to occur in a driven many-body system) from occurring (D'Alessio and Rigol, 2014; Khemani, Moessner, and Sondhi, 2019; Else *et al.*, 2020).

Finally, if one considers only the first approximation (i.e., $N \to \infty$ but without taking the two-mode limit), SτB effectively requires that the periodically driven Gross-Pitaevskii equation is nonergodic; see Sec. III.C. This is in contrast to the expectation that classical nonlinear field theories are generically ergodic (the integrability of the isotropic, undriven GPE being a fine-tuned exception). However, this expectation may not apply to the present situation. Owing to the gravitational confining potential, the dynamics are effectively confined to a zero-dimensional region near the minima of the double well. This is in contrast to the Gross-Pitaevskii equation in an isotropic potential or an array of coupled pendula, which each have an extensive density of states.

Thus, while the number of degrees of freedom is formally infinite, it may be that the KAM theorem still applies because there are effectively only a few degrees of freedom that are resonant with the periodic drive. With this in mind, KAM might protect SτB in this scenario, as it does for a single particle in a double-well Floquet potential (Sec. III.B.4). In the absence of periodic driving, this is indeed the case; specifically, it has been proven that the undriven GPE in a double-well potential breaks ergodicity. Below a critical energy, a condensate becomes "self-trapped" in one of the minima (Jackson and Weinstein, 2004; Albiez *et al.*, 2005). It is perhaps then plausible that the driven GPE can exhibit the same phenomena, but in the rotating frame of the $2:1$ resonance [Fig. 8(b)]. Numerical simulations of the driven GPE support this interpretation (Sacha, 2015; Kuroś *et al.*, 2020).

Finally, there is the question of what occurs when one makes neither of the two aforementioned approximations. Exact answers are difficult to obtain in this regime, as it is a full quantum many-body problem, but numerical results have been obtained within time-dependent Bogoliubov theory (Kuroś *et al.*, 2020) and the truncated Wigner approximation (Wang, Hannaford, and Dalton, 2021). Both conclude that out to thousands of driving periods, the boson population remains almost entirely within the two modes, with no evidence for heating.

These findings support the conclusion of the two-mode approximation (with finite N), where $\tau \propto e^{\alpha N}$. Such numerical simulations cannot definitively rule out exponentially slow depopulation and heating (Sec. IV.D), as might occur in a prethermal time crystal (Else, Bauer, and Nayak, 2017; Machado *et al.*, 2019; Sacha, 2020). However, if one accepts the conclusion that as $N \to \infty$, the GPE can generically break ergodicity in this setting (as it does in a static double-well potential), the possibility that Floquet heating is absent out to

infinite times even for the driven GPE, seems like a possibility (albeit still quite surprising).

Finally, we comment on the contention that time crystals in driven BECs are "effectively few body" (Khemani, Moessner, and Sondhi, 2019). This is partly a matter of nomenclature. On the one hand, these systems are clearly not few body as the interaction g_0 and the limit $N \to \infty$ are necessary for SτB. Furthermore, while not rigorously proven (much like Floquet many-body localization has also not been rigorously proven), we find it plausible that SτB could be stable up to times $\tau \propto e^{\alpha N}$, even when the physics is treated as a full many-body problem. Again, much like the situation with many-body localization, this "plausibility" is bolstered by the fact that the undriven GPE in a double-well potential does in fact break ergodicity.

On the other hand, it seems that SτB is stable only because of the reduced dimensionality induced by the gravitational confinement, which effectively ensures the accuracy of the two-mode approximation; no such picture would be obtainable for the MBL setting (Sec. IV.B). We venture that a more useful distinction is the nature of the thermodynamic limit being taken. In the case of MBL time crystals (or the driven, open time crystals of Sec. V), true SτB is achieved in the thermodynamic limit $L \to \infty$, with $\tau \sim e^{L/\xi}$, keeping intensive quantities fixed. In the case of the driven BEC, due to the confining gravitational potential, the system size L is not relevant (in this sense the problem is zero dimensional) and the problem reduces to being effectively few mode, even though the physics is realized in a full, many-body system. In this setting, SτB is recovered in the limit $N \to \infty$, keeping $g_0 N$ fixed, with $\tau \sim e^{\alpha N}$.

An interesting probe of this distinction would be to consider a 2D generalization of the driven BEC: bosons are gravitationally confined in the z direction but propagate freely along x. In this case, the density of states is extensive in x, and domain walls between the two period-doubled orbits can form. It is natural to suppose that SτB is prethermal in this case, as could be evaluated in the $N \to \infty$ limit using the 2D GPE.

V. OPEN, PERIODICALLY DRIVEN SYSTEMS AND STOCHASTIC DYNAMICS

Our discussion thus far has focused on closed systems in which the dynamics are deterministic, $x \to \Phi(x)$. However, when a system is coupled to an environment, it is often fruitful to model the effect of the environment's chaotic motion as noise (most conveniently taken to be Markovian) so that the dynamics become effectively stochastic. Rather than focusing on a particular microstate, one instead considers a probability distribution over microstates $\rho(x, t)$ that evolves under a "master equation," for example, the Fokker-Planck equation. Integrating the master equation over one Floquet cycle then produces a discrete update of the distribution $\rho(x, t + T) = \Phi[\rho(x, t)]$. In classical mechanics this results in a probability distribution over canonical coordinates $\rho(x = \{p, q\}, t)$ that evolves under a Markov process $\rho(x', t + T) = \int \Phi(x'|x)\rho(x, t)dx$, where $\Phi(x'|x)$ are the Markov transition probabilities. In the quantum case, we

have a density matrix $\hat{\rho}(t)$ that evolves under a "quantum channel" $\hat{\rho} \to \Phi[\hat{\rho}]$. One can also in principle consider non-Markovian baths, a point to which we later return.

The definition of SτB given in Eq. (3) generalizes to the open case by measuring the local observable O in expectation, and stability can be defined by requiring SτB to be robust to perturbations of the stochastic dynamics subject to locality and any other dynamical constraints that one is interested in [Fig. 8(c)]. The environment is both good and bad for SτB. On the one hand, coupling to an environment introduces friction: the energy and entropy produced by the periodic drive can now be absorbed by the bath, which can prevent the long-time Floquet heating that would otherwise destroy SτB in the absence of many-body localization. This tends to help stabilize spontaneous time-translation symmetry breaking. On the other hand, if the environment is at finite temperature, the inevitable noise that results may occasionally conspire to cause phase slips in the period-doubled motion. Roughly speaking, if noise nucleates phase slips at a rate of $1/\tau$, the SτB has a finite autocorrelation time τ and there is no true long-range order. The interplay of a periodic drive, interactions, dissipation, and noise results in a truly nonequilibrium situation that is exceptionally rich, just like the world around us (Gong, Hamazaki, and Ueda, 2018; Iemini *et al.*, 2018; Tucker *et al.*, 2018; Buča and Jaksch, 2019; Buča, Tindall, and Jaksch, 2019; Dogra *et al.*, 2019; Gambetta, Carollo, Marcuzzi *et al.*, 2019; Lledó, Mavrogordatos, and Szymańska, 2019; Zhu *et al.*, 2019; Booker, Buča, and Jaksch, 2020; Chinzei and Ikeda, 2020; Keßler *et al.*, 2019, 2020, 2021; Lazarides *et al.*, 2020; Lledó and Szymańska, 2020; Seibold, Rota, and Savona, 2020; Kongkhambut *et al.*, 2022; Reichhardt *et al.*, 2022; Nie and Zheng, 2023).

A. Activated time crystals

A classical realization of this interplay is given by *Langevin dynamics*. Owing to the coupling with the environment, each degree of freedom (q, p) experiences an additional Langevin force $F_L(t) = -\eta\dot{q} + f(t)$. Here η is the friction coefficient and $f(t)$ is a white-noise stochastic force with autocorrelation function $\langle f(t)f(t')\rangle = 2\eta T\delta(t - t')$, where T is the temperature of the bath. The total force is obtained by adding F_L to the time-dependent Hamilton's equations: $\dot{p} = -\partial_q H(t) + F_L(t)$. This leads to a master equation (the Fokker-Planck equation) of the general form $\partial_t \rho(\{p, q\}, t) = \mathcal{L}[\rho]$, where $\mathcal{L}$ is the Fokker-Planck operator. Integrating the Fokker-Planck equation over one Floquet cycle of the drive then gives the discrete-time update $\rho(t + T) = \Phi[\rho(t)]$.

Note that we are assuming the environment remains in equilibrium at temperature T, and thus instantaneously satisfies the fluctuation-dissipation theorem that relates the magnitude of the friction η and the noise $2\eta T$. If $H(t)$ were time independent, this would ensure (via detailed balance) that the system relaxes to the canonical ensemble at long times where SτB is forbidden (Bruno, 2013; Nozières, 2013; Watanabe and Oshikawa, 2015). But when $H(t)$ is periodically driven, this need not be the case: a nonequilibrium steady state can develop in which energy and entropy flow from the drive to the bath via the system.
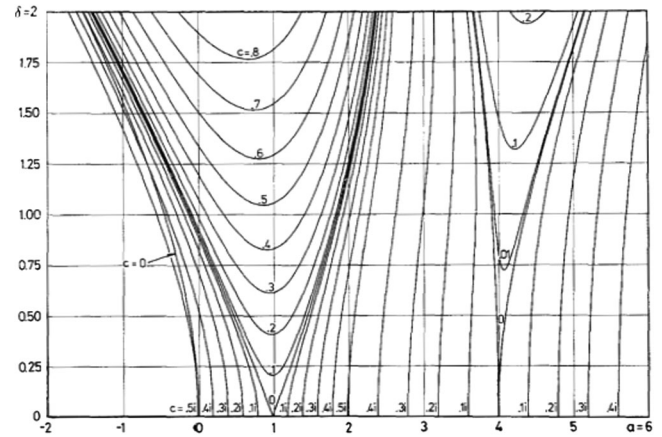

FIG. 9. Arnold tongues of the damped Matthieu equation. The x axis is the drive frequency expressed via the dimensionless ratio $a = (2\omega_0)^2/\omega_D^2$, while the y axis is the drive amplitude δ. Contours indicate the critical drive amplitude required for resonance when the coefficient of friction takes a given value c. When $c > 0$, the period-doubled solution onsets only at finite δ. Adapted from Pedersen, 1980.

We can now return to the question of period doubling in open classical many-body systems such as the Faraday-wave instability and coupled pendula; see the earlier discussions surrounding Eq. (12) and Sec. III.A. For small oscillations and weak damping, the friction results in a damped Mathieu equation of the general form $\ddot{q}_k = -[\omega_k^2 + \delta_k \cos(\omega_D t)]q_k - \gamma_k \dot{q}_k$. The subharmonic responses of this model have been extensively studied (Pedersen, 1935, 1980; Hayashi, 1953; Taylor and Narendra, 1969). As shown in Fig. 9, for $\gamma_k > 0$ the subharmonic response acquires a finite threshold value for the drive amplitude δ_k, and modes outside the resonance tongues are damped. Inside the resonance tongue, friction causes small perturbations to the exact subharmonic orbit to decay, providing an example of a stable limit cycle (Strogatz, 2018). As discussed, such stability is forbidden in nondissipative systems due to the conservation of phase-space volume. Absent noise, SτB is then realized as a sharp many-body bifurcation transition as some combination of driving frequency, amplitude, and damping strength is tuned into the 2∶1 parametric resonance of a k mode. For strongly viscous Faraday waves, the equations of motion are somewhat more complex, but the conclusions are similar (Edwards and Fauve, 1994; Kumar and Tuckerman, 1994).

However, noise (i.e., finite temperatures $T > 0$) induces fluctuations between the different cycles of the SτB, leading to a finite autocorrelation time. Recall from our discussion of prethermal systems (Secs. III.B.1 and IV.D) that we may transform to a rotating frame $\mathcal{K}(t): H(q, p, t) \to H(P, Q, t) = H_{\text{eff}}(Q, P) + V(t)$, in which the driving $V(t)$ is weak. For a single pendulum [Eq. (6)], H_{eff} takes the form of a double-well potential. The noise will then lead to activated hopping between the minima at rate $1/\tau = e^{-\Delta/k_B T}$, where Δ is the quasienergy barrier between the two minima. Thus, at finite temperature a single pendulum is no longer a true time crystal, but rather an activated time crystal in which the autocorrelation time of the SτB diverges exponentially with temperature.
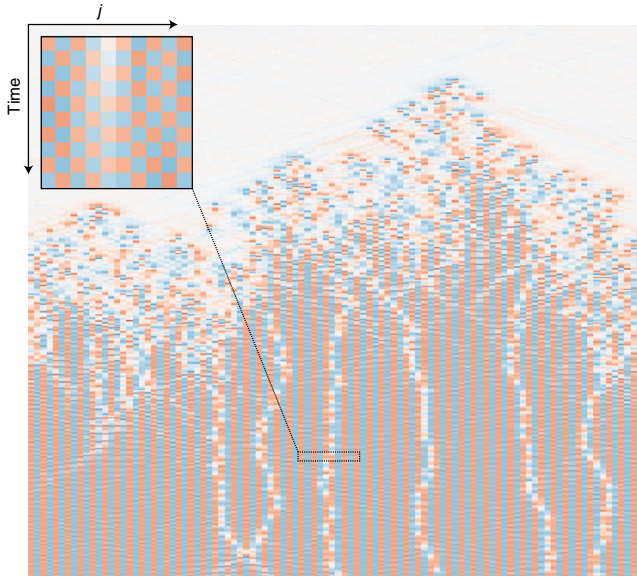
FIG. 10. Domain walls between different period-doubled solutions of the parametrically driven 1D Frenkel-Kontorova model [Eq. (12)] in contact with a finite-temperature Langevin bath. Colors indicate the amplitude of the *j*th oscillator observed at even stroboscopic times $q_j(t = 2nT)$, with time running vertically. Inset: oscillations at stroboscopic times $q_j(nT)$, which reveal the period doubling. Adapted from Yao *et al.*, 2020.

The problem is richer when the pendula are coupled into an array, where H_{eff} approximately takes the form of an Ising model (recall Sec. III.B.4 for the definition of P_*),

$$H_{\text{eff}} \approx \sum_i \left[\frac{(P_i - P_*)^2}{2} + a \cos(2Q_i) \right] + \frac{\tilde{g}}{2} \sum_{\langle i,j \rangle} (Q_i - Q_j)^2,$$

(18)

where the parameter a depends on $\{\delta, \epsilon, \omega_D/\omega_0\}$ [from Eq. (12)] and $\tilde{g}$ is the transformed coupling strength.

The low quasienergy configurations are then localized domain walls between the different period-*m* orbits. These domain walls are easily observed when numerically simulating a 1D coupled array of pendula [Eq. (18)] in the presence of the Langevin noise, as shown in Fig. 10. Indeed, one observes a gas of domain walls undergoing Brownian motion due to the noise, occasionally nucleating or annihilating in pairs. Yao *et al.* (2020) studied the motion of these domain walls in detail and found that it leads to a decay in the SτB autocorrelation function: $\langle q_i(t = nT) q_i(t = 0) \rangle \sim (-1)^n e^{-t/\tau}$. The decay rate τ is proportional to the density of domain walls, which is in turn related to the temperature of the bath through an Arrhenius law $\tau \sim e^{\Delta/k_B T}$, where Δ is now the quasienergy activation barrier required to nucleate a domain wall.

Yao *et al.* (2020) found numerically that in one dimension collective effects can cause the barrier Δ to drop discontinuously as the parameters (for instance, δ or ω_D) are varied, indicating a nonequilibrium phase transition at which SτB is completely destroyed. Without careful examination of the temperature dependence at low T, this transition might easily be mistaken as evidence for the existence of a "true" SτB

phase ensconced within a nonequilibrium phase transition, while in reality the low temperature behavior is always activated.

Naively, the existence of a finite critical temperature T_c for equilibrium Ising symmetry breaking in two or more dimensions suggests that the SτB domain walls cannot proliferate for $T < T_c$, which would thereby stabilize time-crystalline order to infinite times. However, in the damped system this is not correct. To see why, Yao *et al.* (2020) evaluated the effect of the Langevin force in the rotating frame and found two contributions. First, $\{Q, P\}$ experience a Langevin force at the same temperature T, which locally equilibrates H_{eff}, explaining the previously discussed activated processes. Second, the effective Hamiltonian acquires a new contribution from the friction η, of the form $\sum_i \eta P_* Q_i$. Its physical origin is that, in the rotating frame, P encodes the amplitude of the oscillation [cf. Eq. (7)] and, because the friction prefers to damp the amplitude, this equates to a net force on Q. The net potential $V \approx a \cos(2Q) + \eta P_* Q$ then biases Q to "roll downhill," which following Eq. (7) would unlock the response from $\omega_D/2$. The bias leads to a net force on the domain walls that causes nucleated islands to grow (Yao *et al.*, 2020), even though both sides of the domain wall are related by the $\mathbb{Z}_2$ time-translation symmetry. This net force destroys the possibility of phase coexistence. Thus, even in two or more dimensions, the time-crystalline response in this and similar models still exhibits an activated autocorrelation time. A closely related effect was studied in the context of probabilistic cellular automata (Bennett *et al.*, 1990), which are introduced in Sec. V.D.

As $\eta \to 0$, the magnitude of the aforementioned force goes to zero and the density of domain walls can then undergo a transition analogous to the equilibrium Ising transition. However, once $\eta \to 0$, one needs to again worry about heating from the residual time-dependent part of the drive (this is precisely related to the prethermal situation discussed in Sec. IV.D).

In summary, the existence of H_{eff} provides a unifying framework for three regimes of time-crystalline behavior: in the closed cased ($\eta = T = 0$), one has the prethermal scenario where heating destroys SτB after an exponentially long time $\tau \sim e^{\omega_D/J}$; in the open case ($\eta > 0$, $T > 0$), the environment nucleates domain walls that destroy the SτB on an exponentially activated timescale $\tau \sim e^{\Delta/k_B T}$, where Δ is a quasienergy barrier; and finally for the purely dissipative case $\eta > 0$, $T = 0$, there is true SτB conceptually analogous to that of the coupled map lattice (Sec. II.A).

B. Experimental realizations of activated time crystals: Pendula, ac-driven charge density waves, and fractional Shapiro steps

Several experiments on open many-body systems have observed subharmonic responses that may be interpreted within the framework of activated SτB. One example is a shaken pendulum treated as a macroscopic object of $\sim 10^{23}$ particles. The coupled oscillator array of Eq. (12) can be understood as a macroscopic pendulum composed of atoms q_i when one accounts only for its 1D width. The effect of noise on such systems is of practical interest; for example, parametric resonance of a mesoscopic mechanical oscillator such

as an AFM tip (Rugar and Grütter, 1991) is a standard tool for mass and force sensing. In perfect isolation, driving will generate stresses on the oscillator, causing the phonon modes ($q_{k\neq0}$) to absorb a portion of the driving power, heating up the system and at long times melting the oscillator. This is the prethermal scenario, though some work remains to rigorously prove that there is a regime in which the timescale increases exponentially (Mori, 2018; Ye, Machado, and Yao, 2021). Contact with a bath (for instance, air at $T = 293$ K) prevents the oscillator from melting, and it is helpful to estimate the resulting activation time. In particular, the nucleation of a "domain wall" corresponds to ripping the oscillator in half, an energy scale so many orders of magnitude above $T = 293$ K, that the timescale is inconceivable. However, collisions with an unlikely conspiracy of air molecules can also cause the entire oscillator to collectively hop between the two quasienergy minima; using Eq. (12), one can estimate the quasienergy barrier for such a fluctuation as $\Delta \sim 2\delta(2 - \omega_D/\omega_0)$.

At scales of kilograms, of meters, and at room temperature, $\Delta/k_B T$ again results in an inconceivably long timescale. However, for mesoscopic resonators it can be highly relevant. An application of this effect is found in nanoscale mass sensing based on the parametric oscillation of an AFM tip. To weigh an object attached to the tip, one detects the sharp jump in the amplitude of oscillations as the detuning $2 - \omega_D/\omega_0$ is driven near parametric resonance; the precise location of the jump is highly sensitive to the total mass. However, finite-T noise rounds out the transition into a nonlinear crossover, an effect that is experimentally observable and limits the practical resolution of parametric-resonance-based sensing (Zhang and Turner, 2004; Prakash *et al.*, 2012).

Two more intrinsically mesoscopic examples, which can be recast in the language of activated time-crystalline order, are ac-driven charge density waves (CDWs) and fractional Shapiro steps in Josephson junction arrays. Both have received extensive experimental attention; since the two systems are conceptually equivalent (MacDonald and Plischke, 1983; Bohr, Bak, and Jensen, 1984), we focus largely on the former here. In a charge density wave material such as NbSe$_3$ (Grüner, 1988), the electron density spontaneously develops a charge density modulation at twice the Fermi wave vector: $n(x) \sim n_0 \cos[2k_F x + \theta(x)]$, where θ is the slowly varying phase of the CDW. Treating θ as a single macroscopic degree of freedom, one obtains a phenomenological equation of motion given by (Grüner, Zawadowski, and Chaikin, 1981)

$$\ddot{\theta} + (\omega_0\tau)^{-1}\dot{\theta} + \sin\theta = E(t)/E_T, \qquad (19)$$

where $\sin\theta$ accounts for the potential that pins the CDW, $\omega_0\tau$ is a relaxation time (note that time is rescaled by the natural oscillation frequency ω_0) due to dissipation from the material, $E(t)$ is the applied bias, and E_T is the threshold bias for transport. This is the equation of motion for a damped and driven pendulum. When $E(t)$ is larger than the threshold E_T, the system enters into a sliding state with $\langle\partial_t\theta\rangle \neq 0$, generating a finite current.
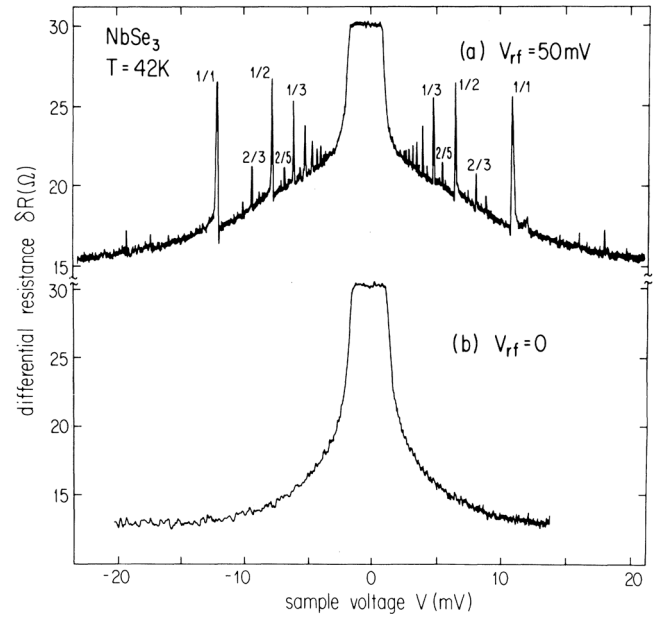

FIG. 11. Differential resistance measurements dV/dI in the charge density wave system NbSe$_3$. (a) A bias voltage $V(t) = V_{\mathrm{dc}} + V_{\mathrm{rf}}\cos(\omega_D t)$ is applied, and the dc component of the resistance $dV_{\mathrm{dc}}/dI_{\mathrm{dc}}$ is measured, holding V_{rf} and ω_D fixed. Sharp peaks in the resistance correspond to plateaus in the *I-V* curve. The location of these peaks can be attributed to a motion of the CDW in which it shifts by p/q wave vectors per driving period, resulting in a subharmonic response between the ac components of the voltage and current. (b) In the absence of the ac drive, the subharmonic response is absent. Adapted from Brown, Mozurkewich, and Grüner, 1984.

In the experiments of interest, a bias is applied with both dc and ac components, $E(t) = E_{\mathrm{dc}} + E_{\mathrm{ac}}\cos(\omega_D t)$. In the sliding state, the motion of the CDW over the periodic pinning potential defines a frequency scale $\omega_{\mathrm{dc}} \equiv \langle\dot{\theta}\rangle \approx \omega_0\tau(E_{\mathrm{dc}}/E_T)$. When $\omega_{\mathrm{dc}} = (p/q)\omega_D$, commensuration effects allow the ac oscillations to assist the motion of θ over the potential. Since the current I_{dc} is proportional to ω_{dc}, commensuration can lead to a plateau in I_{dc} whenever $\omega_{\mathrm{dc}} \approx (p/q)\omega_D$. This leads to the emergence of a "Devil's staircase" of plateaus (Zettl and Grüner, 1983; Brown, Mozurkewich, and Grüner, 1984; Thorne *et al.*, 1987b; Hundley and Zettl, 1989) in the *I-V* curve as E_{dc} is swept at fixed E_{ac}; see Fig. 11, which depicts dV/dI versus the voltage V.

Most notably, for $q \neq 1$ (see the prominent peak in Fig. 11 for $p/q = 1/2$) the response is subharmonic: the CDW shifts by $1/q$ of a wavelength with each period of the ac drive.

While Eq. (19) treats θ as a single macroscopic variable, in the experiment there are spatial fluctuations that can be accounted for by considering a 2D or 3D array of $\theta(\mathbf{r})$ coupled through an elastic stiffness. Middleton *et al.* (1992) showed theoretically that the subharmonic response remains robust in this many-body setting. In fact, in the overdamped regime with a purely sinusoidal pinning potential, subharmonic responses are found only when the problem is treated as a many-body one.

The phenomenological equation of motion [Eq. (19)], which predicts perfect SτB, neglects the effect of thermal

fluctuations that are present in experiments such as Fig. 11. As discussed by Thorne *et al.* (1987a, 1987b), a careful examination of the experimental *I-V* curves reveals that some (but not all) of the subharmonic plateaus are rounded out in a sample dependent fashion. Complementary to this, the spectral distribution $I(\omega)$ [or $V(\omega)$ in current biased mode] shows peaks at $\omega = (p/q)\omega_D$ with finite width, rather than perfect Bragg peaks. This broadening is attributed to a distribution of velocities $\partial_t\theta$ throughout the sample, which, in the language of time-crystalline order, implies the absence of perfect SτB. Indeed, numerical simulations of driven 1D CDWs confirm that finite temperature broadens the plateaus (Mali *et al.*, 2012). However, owing to thermal gradients generated due to Ohmic heating, ac-driven CDWs do not appear to be particularly well suited for quantitatively investigating the temperature dependence of the broadening at low T; see the discussion in Sec. V.A.

Mathematically equivalent physics is found in the ac Josephson effect (recall the related discussions in Sec. I). In the McCumber model (McCumber, 1968) for a resistively and capacitively shunted Josephson junction, the superconducting phase difference ϕ across a Josephson junction obeys the equation of motion

$$\omega_p^{-2}\ddot{\phi} + \omega_c^{-1}\dot{\phi} + \sin\phi = I(t)/I_c, \qquad (20)$$

where $\omega_p \equiv \sqrt{2I_c/C}$, $\omega_c \equiv 2I_cR_N$, and $V/R_N = \dot{\phi}/2R_N$ is the normal current across the junction. Here C is the capacitance, I is the current across the Josephson junction, I_c is the current, and R_N is the normal resistance. The situation considered in Sec. I occurs in experiments in which the junction is underdamped. The dc current is carried entirely by normal quasiparticles $I_{dc} = I_n$ satisfying $I_{dc}/I_c = \omega_c^{-1}\dot{\phi}$ or, simply, $I_{dc} = V/R_N$. An ac supercurrent flows in response to the voltage drop across the junction $I_{ac}(t)/I_c = \sin(2I_{dc}R_Nt)$. However, in this underdamped regime the *I-V* curve is hysteretic, and there is a branch in which the voltage drop across the junction is zero as long as the dc current is less than the critical current: $I_{dc} = I_c \sin\phi$. This vertical step in the *I-V* curve is followed by a plateau at $I = I_c$ from $V = 0$ to $V = I_cR_N$, where it rejoins the other branch.

Returning to the general case of Eq. (20), we note that this equation of motion is formally equivalent to the CDW [Eq. (19)], and by analogy one also expects steps in the *I-V* curve when the ac component of the current $I(t) = I_{dc} + I_{ac}\cos(\omega_Dt)$ is commensurate with the rational harmonics of the dc voltage $\omega_J = \langle\dot{\phi}\rangle = 2eV_{dc}$. The vertical step in the superconducting branch of the hysteretic *I-V* curve in the underdamped case is a trivial version of such a step. These "Shapiro steps" were first observed at integer harmonics in single Josephson junctions (Shapiro, 1963). Later experiments (Benz *et al.*, 1990; Lee *et al.*, 1991) on arrays of Josephson junctions revealed subharmonic Shapiro steps.

C. Ergodicity in open systems

At timescales $t > e^{\Delta/k_BT}$, an activated time crystal loses memory of the initial condition that distinguishes between the m cycles of the SτB oscillation: its dynamics are ultimately ergodic. Earlier in this Colloquium, ergodicity was introduced as a property of a measure-preserving deterministic system. The precise definition works differently in the stochastic case, but intuitively still captures whether a system inevitably "forgets" its initial condition. See Gielis and MacKay (2000) and Gray (2001) for discussions of the technical aspects required to make the definitions precise in the thermodynamic limit.

In the stochastic setting, a probability distribution ρ_s is an "invariant measure" of the dynamics if it is a fixed point of the stochastic update $\Phi[\rho_s] = \rho_s$; i.e., it is a steady-state distribution. A stochastic system is said to be ergodic if the following two properties hold: (1) the dynamics have a unique steady state ρ_s and (2) the long-time behavior of any initial state relaxes to this steady state [$\lim_{t\to\infty}\rho(t) = \rho_s$]. As in the closed, deterministic case, SτB in the sense of Eq. (3) requires Φ to be nonergodic. Otherwise, at long times, the m-possible orbits become indistinguishable and any oscillations will decay.

Instead, a SτB phase will exhibit so-called asymptotic periodicity (Lasota, Li, and Yorke, 1984; Losson and Mackey, 1996). At long times the distribution relaxes to a convex combination of m locally distinguishable distributions ρ_p, $\rho(t) \to \sum_{p=1}^m \alpha_p\rho_{p+t}$ that are cyclically permuted under the evolution $\Phi[\rho_p] = \rho_{p+1}$ (with $\rho_{p+m} = \rho_p$). While $\rho_s = (1/m)\sum_p\rho_p$ is a unique steady state, in the generic case where the α_p values are unequal, $\rho(t)$ will continue to oscillate so that the limit $\lim_{t\to\infty}\rho(t) = \rho_s$ fails to exist.[3]

The existence of true SτB in open dynamical systems thus hinges on a far more fundamental question: Can a locally interacting stochastic system generically break ergodicity? This line of questioning has a deep history in the fields of nonlinear dynamics, mathematical physics, and computer science. The answer is intimately related to the stability of phase coexistence and phase transitions in stochastic systems.

We can illustrate the ideas at play by stripping away the complexity of Hamiltonian dynamics and instead considering a classical spin system. For specificity we start in equilibrium with a classical Ising model $H_I[\{\sigma\}]$. While H_I itself does not define any dynamics, we may define a "kinetic Ising model," such as Glauber or Hasting-Metropolis rules, which update the spin configuration $\sigma \to \sigma'$ according to a local conditional probability $\Phi(\sigma'|\sigma)$ that obeys detailed balance with respect to H_I. Detailed balance ensures that the distribution $\rho_s = e^{-\beta H}/Z$ is a steady state. On a finite system at $T > 0$, standard results guarantee that ρ_s is the unique steady state with a finite relaxation time, so the process is ergodic (Feller, 1957).

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[3]In a deterministic system, "ergodicity" governs the invariance of Cesáro sums: $\lim_{\tau\to\infty}(1/\tau)\sum_{t=1}^\tau O(\Phi^{(t)}(x))$. In this case, S$\tau$B implies that the m-fold iterated map $\Phi^{(m)}$ is not ergodic, but the map Φ itself generally is ergodic (though it is not mixing). In the stochastic case, however, ergodicity is usually defined to imply the stronger form of convergence $\lim_{\tau\to\infty} = \rho_s$ rather than $\lim_{\tau\to\infty}(1/\tau)\sum_{t=1}^\tau\Phi^{(t)}[\rho_0] = \rho_s$. By this definition, Φ itself is nonergodic. While the steady state $\rho_s = (1/m)\sum_{p=1}^m\rho_p$ is unique and one has $\lim_{\tau\to\infty}(1/\tau)\times\sum_{t=1}^\tau\Phi^{(t)}[\rho_0] = \rho_s$, the limit $\lim_{\tau\to\infty}\rho(t)$ does not exist.

However, the thermodynamic limit leads to richer possibilities. When $h = 0$ and T is below the critical Ising temperature (for example, when tuned to a first-order phase transition), the phase coexistence of the two magnetized phases implies the existence of two steady states $\rho(\sigma)_+ = \rho(-\sigma)_-$, and the dynamics are not ergodic. However, for any finite h phase coexistence is lost and the steady state is unique. This illustrates a general principle in equilibrium: ergodicity breaking requires one or more parameters to be tuned to a first-order phase transition, and in this sense is fine-tuned (or requires a symmetry beyond time-translation invariance).

We note that one route around this equilibrium no go is to consider a spin Hamiltonian with multibody interactions (Kozin and Kyriienko, 2019) or a power-law interaction of sufficiently slow falloff so that the energy penalty for domain walls can be made to grow faster than the extensive energy of the symmetry-breaking external field (Liggett, 2012). Or, taking this to the extreme, one could instead consider models with all-to-all couplings in the spirit of a mean-field Hamiltonian, such that locality is lost entirely (Morita and Kaneko, 2006; Russomanno *et al.*, 2017; Lyu *et al.*, 2020; Ojeda Collado *et al.*, 2021, 2023; Pizzi, Nunnenkamp, and Knolle, 2021a; Yang and Cai, 2021). However, such models are not local in the usual sense: their thermodynamic properties are not extensive in system size, among other anomalies. Therefore, while such models can be used as the basis for stabilizing even continuous SτB, the starting point is rather different than the usual notion of locality implied in the classification of phases.

D. Probabilistic cellular automata

Moving beyond equilibrium, one can consider more general local Markov updates $\Phi(\sigma'|\sigma)$ of a discrete spin system without reference to any particular H. Such models are a type of probabilistic cellular automata (PCA) (Dobrušin *et al.*, 1990); the continuous-time generalization of PCA are called interacting particle systems (Liggett, 2012). One particularly convenient way to obtain a PCA is to start with a deterministic cellular automata (CA) (Gutowitz, 1991) defined by a local update rule $\sigma \to \Phi_{CA}(\sigma)$, and then follow each CA update with random spin flips at an "error rate" ε. For Markovian errors we then obtain a Markov process that is a perturbation of the deterministic CA: $\Phi = \Phi_{\varepsilon} \circ \Phi_{CA}$ (we note that mathematical results on PCA are even more general, allowing for non-Markovian errors). These models have been of interest to the theoretical computer science community because CA are Turing complete, so the study of the possible generic behaviors of a PCA have implications for the generic behaviors obtainable by a classical computer perturbed by errors: for instance, can reliable systems emerge from unreliable components (Von Neumann, 1952)? For a review of these connections between theoretical computer science and more familiar notions in nonequilibrium statistical physics, see Gray (2001).

From the perspective of time crystals, the relevant question becomes the following: Are there CA that remain nonergodic for generic small perturbations Φ_{ε}? And if so, might such a CA be the basis for spontaneous time-translation symmetry breaking? These questions were answered in the affirmative

by Toom (1980) and Bennett *et al.* (1990). Before examining the first question, we can sketch Bennett's answer to the second. Suppose there is a local, nonergodic CA, Φ_{CA}, with m-distinct steady states. For simplicity, we further assume that these states are exchanged by a $\mathbb{Z}_m$ symmetry, but (unlike a kinetic Ising or Potts model) we assume that the nonergodicity is stable to generic (and possibly $\mathbb{Z}_m$ breaking) errors. We can then define a rotating version of the CA by composing Φ_{CA} with the action, $G_{2\pi/m}$, of $\mathbb{Z}_m$, yielding $\Phi_{(2\pi/m)CA} \equiv G_{2\pi/m} \circ \Phi_{CA}$. By construction, if we compose m steps of the rotating CA, it will be equivalent to composing m steps of the original CA: $\Phi_{CA}^m = \Phi_{(2\pi/m)CA}^m$, where Φ_{CA}^m corresponds to applying Φ_{CA} m times. Thus, if Φ_{CA} is proven to be stably nonergodic, $\Phi_{(2\pi/m)CA}$ inherits this property and by construction exhibits the "asymptotic periodicity" of a time crystal, cycling through the m fixed steady states of Φ_{CA}.

A particularly important point is that (while convenient) it is not necessary to assume that Φ_{CA} exhibits a $\mathbb{Z}_m$ symmetry for stability. Indeed, if the nonergodicity of Φ_{CA} is stable to $\mathbb{Z}_m$-breaking perturbations, one could just as well have started out with a perturbed version of Φ_{CA} that breaks the $\mathbb{Z}_m$ symmetry. The construction thus satisfies the important conceptual requirement that SτB does not depend on any further internal symmetries.

Returning to the first question, the possibility of a local and generically nonergodic PCA was a long-standing question for several decades [see the reviews by Dobrušin *et al.* (1990), Lebowitz, Maes, and Speer (1990), and Toom (1995)] but was finally answered in the affirmative in the groundbreaking work of Toom (1980).

E. An absolutely stable open time crystal: The π-Toom model

The Toom CA is a 2D binary-spin model with a simple "northeast corner" (NEC) majority rule that is defined as follows (Toom, 1974, 1980). At each step, each spin follows a majority vote among itself and its two north and east neighbors: $\sigma_{x,y}(t + 1) = \text{maj}(\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1})$. This is a seemingly innocuous modification of the $T = 0$ nearest-neighbor kinetic Ising rule, in which each spin would follow a vote among its four {north, east, south, west} neighbors. Note, however, that, due to the spatial asymmetry, the Toom rule cannot be understood as minimizing the energy of any Hamiltonian and does not obey detailed balance (Grinstein, Jayaprakash, and He, 1985). This asymmetry ensures that if an island of the minority spin nucleates, the NEC rule will cause the island to shrink linearly in time from the NE direction inward. This is far faster than the Ising model, where the thermodynamic force on a domain wall depends on its local curvature, and hence decays with the size of the island (Bennett and Grinstein, 1985).

The Toom PCA is then obtained by perturbing with biased errors in which, for example, spins flip up with error rate ε_p and down with error rate ε_q. The bias $b = (\varepsilon_p - \varepsilon_q)/(\varepsilon_p + \varepsilon_q)$ breaks the Ising symmetry. There is nevertheless a finite volume in $\{\varepsilon_p, \varepsilon_q\}$ space in which two magnetized steady states persist despite a bias that prefers one over the other (Fig. 12). Rigorous results prove that the ergodicity breaking is robust to essentially any sufficiently small perturbation,
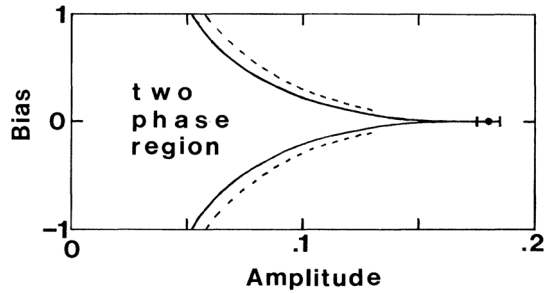
FIG. 12.   Phase diagram of the Toom model. After each CA update, spins randomly flip up or down with rate ε_p or ε_q, respectively, with "bias" $(\varepsilon_p - \varepsilon_q)/(\varepsilon_p + \varepsilon_q)$ and "amplitude" $\varepsilon_p + \varepsilon_q$. Within the two-phase region, there are two distinct steady-state distributions and the dynamics are not ergodic. Unlike the Ising model, the system can thus remember 1 bit of information even in the presence of biased noise. Adapted from Bennett and Grinstein, 1985.

even a spatiotemporally correlated one or a noise distribution that is not itself time-translation invariant (Toom, 1980; Berman and Simon, 1988; Gacs, 2021). We note that this last point distinguishes the "absolute stability" of the open system π-Toom time crystal (which is defined next) from its closed system counterparts (i.e., Sec. IV), where noise that breaks the underlying discrete time-translation symmetry would immediately destroy the time crystal.

With Toom having done the difficult part, the π-Toom model is then defined as the antimajority NEC rule $\sigma_{x,y}(t+1) = \mathrm{antimaj}(\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1})$. The π-Toom model is a time crystal, which is stable to arbitrary perturbations (Yao *et al.*, 2020; Zhuang *et al.*, 2021). The SτB of the π-Toom model was pointed out over two decades ago by Bennett *et al.* (1990), and later by Gielis and MacKay (2000), who referred to it as an example of a "type-G phase transition."

While the original Toom model exhibits $m = 2$ steady states, and hence is the basis for period $m = 2$ SτB, it generalizes to any $m > 2$ (Bennett *et al.*, 1990). Furthermore, Toom-like models exist in all dimensions $D > 1$ simply by stacking the 2D version. In fact, an extension in $D = 3$ provides the basis for error-corrected classical computing (Gács and Reif, 1988). The existence of stable ergodicity breaking in 1D PCA was shown only more recently by Gács (2001). Taken together, PCA support absolutely stable SτB of any period $m \geq 2$ in all dimensions $D \geq 1$.

Is the full error-correcting capability of the Toom and Gács models really necessary for stable SτB? For example, in the construction $\Phi_{(2\pi/m)\mathrm{CA}} \equiv G_{2\pi/m}\Phi_{\mathrm{CA}}$ we may instead take Φ_{CA} to be the kinetic Ising model or its generalization to $\mathbb{Z}_m$. The ferromagnetic interactions of the Ising model are a mild form of error correction in that they cause minority islands to shrink due to surface tension. However, Bennett *et al.* (1990) argued that this equilibrium form of error correction is insufficient for stabilizing $m > 2$ SτB.

The reason for this is that for $m > 2$ the chirality of the periodic evolution between steady states $n \to n + 1 \bmod m$ implies that there is no symmetry that forbids a stochastic force per unit length from acting on a domain wall between regions of n- and $(n+1)$-type steady states (such as those

favoring the growth of n at the expense of $n + 1 \bmod m$). Such a force is thus generic and will drive minority droplets to nucleate and grow. This force is counteracted by the surface tension of the ferromagnetism, which, however, decays with the local curvature. Thus, the former force will always win out for sufficiently large droplets, causing minority droplets to proliferate and destroy SτB at long times. The Toom model escapes this reasoning because its error correction effectively exerts a force on domain walls that is independent of their curvature, and can thus shrink minority droplets of any size (Bennett and Grinstein, 1985).

However, the $m = 2$ case is an exception because there is no handedness to the periodic evolution. Even if there is a force that favors, say, type $n = 0$ over type $n = 1$, because the domains effectively switch type at each step, the net force on a domain wall averages to zero over one period, and islands can thus be shrunk via ferromagnetic surface tension (Bennett *et al.*, 1990). This implies that the binary-spin π-Ising model can support SτB below the Ising critical temperature, as was recently studied in detail (Gambetta, Carollo, Lazarides *et al.*, 2019).

It is natural to contrast this discussion with the classical activated time crystal of Sec. V.A, which was not stable against the proliferation of minority islands even though $m = 2$. To see why, recall that the emergent Ising degree of freedom was effectively embedded into a larger continuous state space such as $\sigma = \mathrm{sgn}[\sin(Q)]$. A domain wall between $\sigma = 1, -1$ could thus be either of two types, depending on whether Q is wound clockwise or counterclockwise. Because this handedness is preserved under the oscillation $Q \to Q + \pi$, there is no symmetry that forbids a domain wall from experiencing a net force that does not average to zero over $m = 2$ periods, even when the identities of the domains themselves are exchanged.

Does this imply that if the binary-spin π-Ising model is embedded in a model with additional degrees of freedom, the SτB may not be generically stable? Or instead, may there be a parameter regime of models like the Langevin-driven parametric oscillator arrays (12) that exhibit true, rather than activated, SτB? Only time will tell.

F. Open Hamiltonian and quantum dynamics

While we have discussed the existence of stable time crystals in the stripped-down setting of a PCA (Secs. V.D and V.E), our original microscopic starting point (Sec. V) was the periodically driven Langevin equation or its quantum analog, the Lindblad equation. An interesting open question is the following: Do the rigorous results for SτB in probabilistic cellular automata translate to these settings? Since PCA are motivated as a coarse-grained description of the Hamiltonian world around us, it is tempting to speculate that they do. Two recent studies have shown how the π-Toom model can be embedded in classical Langevin (Zhuang *et al.*, 2021) or quantum Lindblad dynamics (McGinley, Roy, and Parameswaran, 2021), and in both cases there is strong numerical evidence for true SτB. However, there is not yet a fully rigorous proof as in Toom's and Gács's work. While Toom-like stability for a continuous-time version of a PCA was proved by Gács (2001), the state space in that model is

discrete. As we discuss in Sec. VI, the answer to this question is a special case of a deeper one: Is error-corrected computing physically realizable in this Universe?

VI. OUTLOOK AND FUTURE DIRECTIONS

A. New venues for time crystals

In Sec. III.C, we introduced the following mantra: Where there is ergodicity breaking, there will be time crystals. This suggests that new developments in the ergodicity breaking of many-body systems, ranging from Hilbert space fragmentation (Sala *et al.*, 2020; Yang *et al.*, 2020) and shattering (Khemani, Hermele, and Nandkishore, 2020) to quantum scars (Turner *et al.*, 2018b; Serbyn, Abanin, and Papić, 2021), confinement (Collura *et al.*, 2022), and Stark localization (Schulz *et al.*, 2019; van Nieuwenburg, Baum, and Refael, 2019; Doggen, Gornyi, and Polyakov, 2021), represent fertile ground for exploring novel formulations of time-crystalline order (Zhao, Knolle, and Moessner, 2022).

1. Quantum many-body scars

Interacting quantum systems can exhibit a weak breakdown of thermalization, where certain initial conditions exhibit persistent many-body revivals in time (Bernien *et al.*, 2017; Turner *et al.*, 2018a). This phenomenon, called quantum many-body scars, is associated with the presence of anomalous, nonthermal eigenstates; see Serbyn, Abanin, and Papić (2021) for a recent review on the topic. From the perspective of time-crystalline order, the presence of scar states can cause the system to undergo periodic entanglement and disentanglement cycles following a quench (Ho *et al.*, 2019; Michailidis *et al.*, 2020; Serbyn, Abanin, and Papić, 2021). In practice, however, quantum many-body scar states are fragile since they rely upon the existence of a dynamically disconnected subspace of nonthermalizing eigenstates; indeed, the presence of generic interactions is expected to eventually lead to thermalization (Lin, Chandran, and Motrunich, 2020).

The connection between quantum many-body scars and time-crystalline behavior has recently been explored in a Rydberg-based quantum simulation platform (Bluvstein *et al.*, 2021; Maskara *et al.*, 2021). In particular, the researchers observed that the coherent revivals associated with quantum many-body scars could be stabilized by additional periodic driving. This periodic driving leads to both an increase in the lifetime of the scarred oscillations and the emergence of a period-doubled, subharmonic response. While this is reminiscent of prethermal time-crystalline order (Sec. IV.D), we note two important differences. First, the experiments are performed in a regime where the driving frequency is of similar order as the local energy scales of the many-body system. Second, the subharmonic response exists only for Néel-like initial states (associated with the quantum scars) (Maskara *et al.*, 2021). Looking forward, it will be interesting to explore the possibility that periodic driving can enhance the stability of nonergodic dynamics. In addition to novel settings for observing time-crystalline order, the ability to parametrically control the lifetime of this order could also lead to potential applications in areas such as quantum metrology and quantum information science.

2. Stark time crystals

In the presence of an electric field, the wave function of electrons in a material is localized to a region whose size decreases as the field increases. In the absence of interactions, this phenomenon, which arises without disorder, is known as Wannier-Stark localization (Wannier, 1959). Recent theoretical and experimental investigations have explored whether such localization can persist in the presence of many-body interactions, leading to Stark many-body localization (Schulz *et al.*, 2019; Guo *et al.*, 2020; Morong *et al.*, 2021; Scherg *et al.*, 2021).

Kshetrimayum, Eisert, and Kennes (2020) discussed the possibility of discrete time crystals protected from Floquet heating by Stark many-body localization. They probed the existence of a Stark time crystal using numerical simulations of a one-dimensional spin chain. Focusing on two specific initial states (i.e., the ferromagnetic and antiferromagnetic states), Kshetrimayum, Eisert, and Kennes (2020) observed that for a sufficiently strong linear potential the states exhibit time-crystalline order, which seems to be robust to π-pulse imperfections (recall the discussions in Sec. IV.B). However, when differences in the linear potential coincide with integer multiples of the drive frequency, they observed the coherent self-destruction of time-crystalline behavior. An advantage of the clean, disorder-free setting is that one can imagine the possibility of order that spontaneously breaks both spatial and time-translation symmetries. This theme has been studied in a number of contexts (Smits *et al.*, 2018; Träger *et al.*, 2021; Žlabys *et al.*, 2021) and remains an active area of exploration.

B. Prethermalization beyond Floquet quantum systems

1. Classical prethermal discrete time crystals

In Sec. IV.D, we focused our attention on prethermal time crystals in closed (i.e., unitarily evolving) quantum systems. However, as previously discussed, one of the central features of Floquet prethermalization (i.e., exponentially slow heating) is also expected to occur in classical many-body systems (Mori, 2018; Howell *et al.*, 2019; Hodson and Jarzynski, 2021). This immediately begs the following question: Can such systems also host prethermal time crystals (Mori, 2018; Howell *et al.*, 2019; Hodson and Jarzynski, 2021; Pizzi, Nunnenkamp, and Knolle, 2021b; Ye, Machado, and Yao, 2021)?

Recent work by Pizzi, Nunnenkamp, and Knolle (2021b) and Ye, Machado, and Yao (2021) answered this question in the affirmative. A practical advantage of the generalization to classical systems is that it immediately enables the study of PDTCs in dimensions $d > 1$. For example, Ye, Machado, and Yao (2021) studied a nearest-neighbor-interacting classical Floquet spin model on the square lattice. Although the original Floquet evolution does not exhibit any symmetries, H_{eff} exhibits an emergent $\mathbb{Z}_2$ Ising symmetry. Unlike in the one-dimensional case discussed in Sec. IV.D.2, in two dimensions this Ising symmetry can be broken at finite temperatures.

2. Higher-*M* discrete time crystals

In both the quantum and classical settings, the emergent symmetry in H_{eff} is not restricted to just a $\mathbb{Z}_2$ symmetry and different subharmonic PDTCs (i.e., beyond period doubling)

can be realized (Giergiel *et al.*, 2018, 2020; Surace *et al.*, 2019; Kelly *et al.*, 2021; Pizzi, Knolle, and Nunnenkamp, 2021). The simplest approach to doing this is to utilize a Floquet evolution where the "π pulse" [see the discussion following Eq. (14)] is adjusted to be close to a π/M pulse. In certain scenarios, this choice naturally leads to an effective Hamiltonian H_{eff} that exhibits a $\mathbb{Z}_M$ symmetry (Pizzi, Nunnenkamp, and Knolle, 2021b). When this $\mathbb{Z}_M$ symmetry is spontaneously broken, the system will oscillate at a subharmonic frequency $\sim \omega_D/M$ throughout the prethermal regime. Using this strategy, Pizzi, Nunnenkamp, and Knolle (2021b) constructed a "phase diagram" for an $\omega_D/4$ PDTC in a three-dimensional classical spin system.

When one utilizes a fractional M in this strategy, the dynamics can be even richer than in the integer case. Indeed, Pizzi, Knolle, and Nunnenkamp (2021) explored this scenario in the presence of long-range interactions (Russomanno *et al.*, 2017; Huang, Li, and Yin, 2018; Kozin and Kyriienko, 2019) and observed signatures of PDTC order at a variety of fractional frequencies.

3. Prethermal time quasicrystals

Building upon generalizations to higher-M PDTCs and fractional PDTCs (Matus and Sacha, 2019; Pizzi, Knolle, and Nunnenkamp, 2021; Pizzi, Nunnenkamp, and Knolle, 2021b; Ye, Machado, and Yao, 2021), a natural question to ask is whether there can be nontrivial phases of matter in isolated many-body systems that are driven in a way that is not periodic in time. Here our focus is on the closed system, prethermal setting (Fig. 1), where a periodic drive is replaced by a quasiperiodic drive (Ringot *et al.*, 2000; Gommers, Denisov, and Renzoni, 2006; Crowley, Martin, and Chandran, 2019; Zhao, Mintert, and Knolle, 2019; Mori *et al.*, 2021; Zhao *et al.*, 2021). Recent work has shown that such systems can host prethermal quasiperiodically driven phases of matter and, in particular, can give rise to discrete time quasicrystals (Dumitrescu, Vasseur, and Potter, 2018; Else, Ho, and Dumitrescu, 2020).

Before proceeding, however, we note that related phenomena have been explored in nonlinear dissipative dynamical systems (see the coupled map lattice discussions in Sec. II.A) driven at two incommensurate frequencies (Sethna and Siggia, 1984; Held and Jeffries, 1986; Romeiras and Ott, 1987; Ding, Grebogi, and Ott, 1989; Flicker, 2018).

The spectral content of a periodic drive contains peaks only at a single drive frequency ω_D and its harmonics $n\omega_D$ for integer n. By contrast, the spectral content of a quasiperiodic drive has peaks at integer linear combinations: $n_1\omega_1 + n_2\omega_2 + \cdots + n_m\omega_m$ for some $m > 1$ number of incommensurate frequencies $\{\omega_1, \ldots, \omega_m\}$. As in Sec. IV.D, in order to discuss prethermal phases, it is necessary for the system to exhibit slow heating (i.e., energy absorption) from the drive. For quasiperiodic driving, this is strictly more challenging than the original Floquet setting since the drive is technically able to supply energy in arbitrary units. Nevertheless, an analogous slow-heating result was proven by Else, Ho, and Dumitrescu (2020). Before heating occurs (Fig. 13), the dynamics of the quasiperiodically driven system are well approximated (in a rotating frame) by an effective static Hamiltonian [in direct analogy to Sec. IV.D and Eq. (16)].
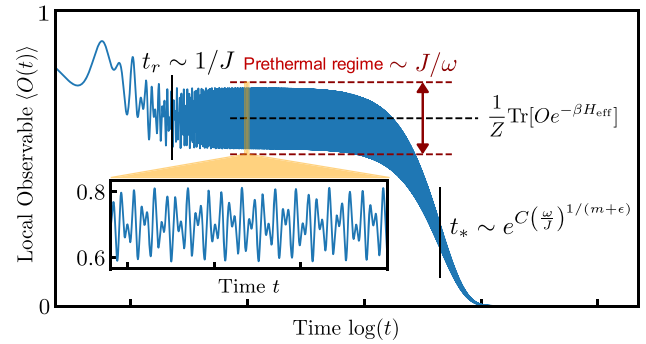

FIG. 13. Depiction of a prethermal time quasicrystal emerging from a quasiperiodically driven quantum system. During the prethermal regime, local observables exhibit discrete time quasicrystalline order, which ultimately melts at late times as the system Floquet heats to a featureless infinite-temperature state. Adapted from Else, Ho, and Dumitrescu, 2020.

To discuss the possibility of prethermal time quasicrystals (which are stable during the prethermal timescale $t < t_*$), Else, Ho, and Dumitrescu (2020) began by defining what an "order parameter" for such a phase would look like. The subtlety is that a quasiperiodically driven system does not, strictly speaking, have any remaining time-translation symmetry to break. However, such a system can still exhibit a well-defined notion of a fractional or subharmonic frequency response (Fig. 13). In particular, an observable can respond in a quasiperiodic manner with base frequencies: $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_n\}$. When $\tilde{\omega}_j$'s are not harmonics of the original driving frequencies, then the system is said to exhibit a fractional frequency response and discrete time quasicrystalline order (Else, Ho, and Dumitrescu, 2020). Connecting with our previous discussions about emergent symmetries in H_{eff}, one can think of the discrete time quasicrystal as emerging from the spontaneous breaking of a different finite Abelian group symmetry (i.e., one that replaces the $\mathbb{Z}_M$ symmetry discussed in Secs. III.B.2 and VI.B.1).

We end this section by pointing the reader in a number of related directions that fall outside the prethermal context. Many of these connect to the idea of quasiperiodic pattern formation in parametrically driven systems. For example, in the context of two-frequency forcing, even when the frequencies are commensurate (implying that they do not exhibit a quasiperiodic response in time), in certain scenarios one can realize stable Faraday-wave patterns that are analogous to a two-dimensional quasicrystal (Edwards and Fauve, 1993; Besson, Stuart Edwards, and Tuckerman, 1996; Silber, Topaz, and Skeldon, 2000). Various incarnations of a quasiperiodic response in time have also been explored in both theoretical proposals and experiments on cold atomic systems (Autti, Eltsov, and Volovik, 2018; Giergiel *et al.*, 2018; Giergiel, Kuroś, and Sacha, 2019; Pizzi, Knolle, and Nunnenkamp, 2019; Chinzei and Ikeda, 2020; Zhao *et al.*, 2022). For example, Giergiel, Kuroś, and Sacha (2019) explored the dynamics of an ultracold atomic ensemble bouncing between two orthogonal harmonically oscillating mirrors. By tuning the bare frequencies of the unperturbed particle motion, the system can reproduce fragments of the Fibonacci sequence encoded via the bounces of the atomic ensemble off of the two mirrors.

C. Applications: From metrology to quantum information benchmarking

The periodic control underlying the discrete time crystal is, more generally, an indispensable tool in the context of nuclear magnetic resonance spectroscopy (Bloembergen, Purcell, and Pound, 1947, 1948; Waugh, Huber, and Haeberlen, 1968), quantum information science (Khodjasteh and Lidar, 2005; Biercuk *et al.*, 2009), and quantum simulation (Bloch, Dalibard, and Nascimbene, 2012; Goldman *et al.*, 2014). Building upon this connection, Choi, Yao, and Lukin (2017) explored the possibility of engineering a Floquet system where quasienergy gaps can protect entangled states from static perturbations while still ensuring their sensitivity to an oscillating signal. In a sense, this idea can be understood as a generalization of spin echo spectroscopy that utilizes many-body states. It is also related to the conventional concept of using phases of matter with spontaneously broken symmetries for sensing (Frérot and Roscilde, 2018; Bennett *et al.*, 2020).

Choi, Yao, and Lukin (2017) investigated the use of a time-crystalline Floquet sequence to stabilize Schrödinger cat states that are typically extremely fragile against local perturbations. They analyzed a technique that allows for the enhancement of metrological bandwidth while maintaining the sensitivity and discussed an example in the context of the precision measurement of ac magnetic fields. More recently these ideas have inspired new many-body driving protocols aimed at circumventing the so-called interaction limit for quantum sensing (Zhou *et al.*, 2020).

Beyond metrology, the fact that signatures of time-crystalline order have been observed in a diverse array of physical platforms (see Secs. IV.C.1, IV.C.2, and IV.C.3) suggests the possibility of cross platform benchmarking for the performance of near term quantum devices (Preskill, 2018). This idea builds on the more general thread that exploring nonequilibrium dynamical phenomena may represent a particularly natural strategy for verifying and validating noisy intermediate-scale quantum technology. Finally, suggestions for utilizing time-crystalline order as a frequency standard or for beyond-standard-quantum-limit quantum sensing have also been discussed (Lyu *et al.*, 2020), although explorations in this direction remain relatively nascent.

D. Conclusions

As we have seen in this Colloquium, recent evidence points to at least two venues where infinitely long-lived time crystals may exist: (i) SτB that is stable to time-translation-invariant perturbations might occur in periodically driven closed 1D quantum systems in the presence of strong disorder (Sec. IV.B) and (ii) SτB that is stable to arbitrary perturbations can occur in periodically driven $D \geq 1$ open systems, such as the π-Toom model, whose dynamics effectively implements a form of error correction (Sec. V.E). These two flavors of time crystals evade ergodicity through vastly different means, either by strongly localizing the degrees of freedom or by actively shrinking potentially ergodicity-generating fluctuations. Between these two extremes lie two regimes of time-crystalline behavior that are exponentially long-lived: prethermal time crystals in closed systems (both classical and

quantum) and activated time crystals in open systems. Although ergodicity is deferred in both cases, it is inevitable in the long run (Palmer, 1982; Petersen, 1989; Walters, 2000).

The possibility of SτB in driven open systems can be understood as one consequence of a more radical form of ergodicity breaking: fault-tolerant computation. There is a long history of understanding computation as a fundamentally physical process, and the subsequent constraints that arise from thermodynamics: Computers may be thought of as engines for transforming free energy into waste heat and mathematical work (Landauer, 1961; Bennett, 1982; Wolpert, 2019). In this point of view, a time crystal can be understood as a physical realization of a simple computer program: while true, apply a global NOT gate. If the program can execute perfectly despite faulty (noisy) gates, and with a physical implementation that relies only on local interactions, the execution of such a program can be understood as a nonequilibrium "phase of matter," and the error threshold for fault tolerance can be understood as a nonequilibrium phase transition into a time-crystalline phase.

Gács and Reif (1988) and Gács (2001) showed that fault-tolerant classical computation can indeed be realized as a locally interacting autonomous process in the thermodynamic limit, and this work was later used to prove that fault-tolerant quantum computing can be as well (Harrington, 2004; Dauphinais and Poulin, 2017). One could thus realize a time-crystalline phase by repeatedly running the program NOT on such an error-corrected computer, either quantum or classical. In this sense, the existence of time crystals in open systems is an elementary application of deeper results regarding the physical realizability of error correction in autonomous, locally interacting systems.

One open question involves understanding how the transition from exponentially to infinitely long-lived time crystals precisely occurs. Indeed, as disorder is increased in closed 1D quantum systems, one expects a sharp phase transition to occur from the prethermal discrete time crystal (Sec. IV.D) to the Floquet-MBL discrete time crystal (Sec. IV.B). One expects a similar transition from the activated DTC to the Toom-type time crystal, as one tunes a uniform antimajority-vote rule toward the antimajority NEC rule.

One possibility is that the exponentially long-lived time crystals' lifetimes [for instance, scaling as $\sim A e^{\omega_D/J}$ in the prethermal case (Sec. IV.D) or as $\sim A e^{\Delta/T}$ in the activated case (recall that Δ is the quasienergy barrier defined in Sec. V.A)] exhibit a diverging prefactor. For example, as the transition is approached, A could diverge as $A \sim |g - g_c|^{-y}$, where g is the tuning parameter. An alternate possibility is that the effective energy scale $1/J$ or Δ might diverge at the transition.

A second set of open questions is whether a direct transition is possible from a Floquet-MBL discrete time crystal to a π-Toom or Gács-type time crystal (or something continuously connected to it). This possibility seems a bit unlikely since such a transition would have to occur precisely when the coupling to a bath is zero, which is where the system goes from closed to open. On the other hand, a Floquet-MBL discrete time crystal is what we would create with a perfect quantum computer executing precisely the same set of gates at

each time step. Meanwhile, a π-Toom time crystal is what we might create with a noisy but fault-tolerant quantum computer in which error correction was governed by a probabilistic cellular automaton. But since the former could be simulated by the latter, one might also logically expect them to be separated by no more than a second-order phase transition at which the rate of entropy production rises from zero with a discontinuous derivative.

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