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Two Applications of Axion Electrodynamics

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The equations of axion electrodynamics are studied. Variations in the axion field can give rise to peculiar distributions of charge and current. These effects provide a simple understanding of the fractional electric charge on dyons and of some recently discovered oddities in the electrodynamics of antiphase boundaries in PbTe. Some speculations regarding the possible occurrence of related phenomena in other solids are presented.

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Whether or not axions[1] have any physical reality, their study can be a useful intellectual exercise. For by having a field which modulates the effects of anomalies and instantons and calculating the consequences of its variation in space and time, we can get some intuitive feeling for these important, but often subtle and obscure, things. Also, it is (I shall argue) not beyond the realm of possibility that fields whose properties partially mimic those of axion fields can be realized in condensed-matter systems. In this spirit, I will consider in this paper two situations where the equations of axion electrodynamics seem to illuminate otherwise surprising phenomena, and then speculate briefly on potential generalizations.

To begin, let us recall the equations of axion electrodynamics. They are generated by adding to the ordinary Maxwell Lagrangean an additional term

$$\Delta\mathcal{L} = \kappa a \mathbf{E}\cdot\mathbf{B}, \tag{1}$$

where κ is a coupling constant. The resulting equations are

$$\nabla\cdot\mathbf{E} = \tilde{\rho} - \kappa\nabla a\cdot\mathbf{B}, \tag{2}$$

$$\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t, \tag{3}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{4}$$

$$\nabla\times\mathbf{B} = \partial\mathbf{E}/\partial t + \tilde{\mathbf{j}} + \kappa(\dot{a}\mathbf{B} + \nabla a\times\mathbf{E}), \tag{5}$$

where $\tilde{\rho},\tilde{j}$ are the ordinary (nonaxion) charge and current. We see that there is an extra charge density proportional to $-\nabla a\cdot\mathbf{B}$, and current density proportional $\nabla a\times\mathbf{E} + \dot{a}\mathbf{B}$. The form of these terms reflects the discrete symmetries of a: a is P and T odd. Also, these terms depend only on space-time *gradients* of the axion field. This is because with a = const, $\Delta\mathcal{L}$ in Eq. (1) becomes a perfect derivative, and does not affect the equations of motion.

Dyon charge.— Consider a magnetic monopole surrounded by a spherical ball in which $a=0$, modulating within a thin shell into $a=\theta$ at large distances (Fig. 1). Now because of the axion term in (2) one finds that the domain wall carries electric charge density $-\kappa\nabla a\cdot\mathbf{B}$, or charge/unit length $-\kappa\nabla a\Phi$ when integrated over direction, where Φ is the magnetic flux. The total charge seen by observers far from the monopole is

$$q = -\kappa\theta\Phi. \tag{6}$$

The Witten effect,[2] that in a θ vacuum magnetic monopoles become dyons with fractional charge to their magnetic charge and to θ, is essentially contained in (6). By our introducing axions, and allowing θ to become a

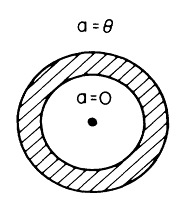

FIG. 1. Monopole surrounded by a shell of axion domain wall.

dynamical variable, the physical origin of this remarkable effect is illuminated. It arises because in an external B field axions acquire an electric dipole moment.

Let us discuss the numerical value of κ. The simplest model in which to discuss monopoles is a gauge model in which SU(2) is broken to U(1) by a triplet of Higgs fields. Denote the electric charge of the W bosons by $\pm 2e$. Then a doublet will have charges $\pm e$. The basic monopole in the theory carries magnetic charge $g=2\pi/e$, saturating the Dirac condition $ge=2\pi\times$integer. The θ term is conventionally introduced[3] in the form

$$\Delta\mathcal{L}=(\theta e^2/4\pi^2)\,\mathrm{tr}F_{\mu\nu}\tilde{F}_{\mu\nu}$$
$$=(\theta e^2/2\pi^2)(\mathbf{E}\cdot\mathbf{B}+\dots), \qquad (7)$$

where in the second equality the ordinary U(1) "electromagnetic" fields are isolated, as is appropriate for the long-distance effective Lagrangean. With this normalization, the theory is invariant under $\theta\to\theta+2\pi$. With the choice $\kappa=e^2/2\pi^2$ the axion field is numerically equal to the properly normalized θ parameter. The induced charge on the dyon is therefore

$$-\kappa\theta\Phi=-(e^2/2\pi^2)\theta g=-e\theta/\pi. \qquad (8)$$

Now imagine changing $\theta\to\theta+2\pi$, adiabatically. In this process the charge of the dyon changes, formally, by $-2e$. However, this does not imply that the theory has changed, since there are monopole-W-boson composites with the shifted quantum numbers. The states have merely been relabeled.

Mass-phase walls.—Suppose that N Dirac fermions interact with two real fields m,a according to the effective Lagrangean

$$\Delta\mathcal{L}_{\mathrm{int}}=m(x)e^{ia(x)}\bar{\psi}[(1-\gamma_5)/2]\psi+\mathrm{H.c.} \qquad (9)$$

According to the anomaly equation,[4] this is equivalent to having the interaction

$$\Delta\mathcal{L}_{\mathrm{int}}=m(x)\bar{\psi}\psi+[Ne^2a(x)/4\pi^2]\mathbf{E}\cdot\mathbf{B}, \qquad (10)$$

with the electrodynamic consequences implied by Eqs. (2) and (5). (Strictly speaking, the equivalence is true only in the limit $m\to\infty$. There are additional terms suppressed by powers of $\partial a/m$.)

In particular, consider the properties of a domain wall such that a changes by Δ from one side to the other. We find that when a magnetic field $\mathbf{B}$ is applied perpendicular to the wall, charge per unit area

$$Q/A=Ne^2B\Delta/4\pi^2 \qquad (11)$$

accumulates on the wall. Similarly, when an electric field $\mathbf{E}$ is applied to the wall, we find that a current per unit length

$$I/L=Ne^2E\Delta/4\pi^2 \qquad (12)$$

flows in the plane of the wall, perpendicular to $\mathbf{E}$.

Fradkin, Dagotto, and Boyanovsky[5] have written a paper very relevant to this subject. They argue, first, that the Lagrangean for electrons at the band edge in PbTe can be rewritten in terms of four independent fermion fields satisfying the relativistic (massive) Dirac equation. These fermion fields are special linear combinations of the electron field on different sites. They argue further that at an antiphase boundary, where the lattice ordering of Pb and Te is reversed, the sign of the mass changes.

If we do not look at the interior of the wall, but only at its effect on boundary conditions, then it can be replaced by an axion domain wall with $\Delta=\pi$. So if this replacement is legitimate (and I shall argue that it is) we should expect that charge accumulates on the wall in a normal magnetic field, and current flows in a tangential electric field, according to Eqs. (11) and (12) with $N=4$.

Fradkin, Dagotto, and Boyanovsky found these effects by a different argument. They let m change sign through real values (so that, in particular, there is a zero of m within the wall). Then, as is well known, in the absence of electromagnetic fields there is a zero-energy solution of the Dirac equation localized at the wall.[6] The zero modes, or in other words midgap states, have overlap $\frac{1}{2}$ with states that were negative energy without the wall, and overlap $\frac{1}{2}$ with states that were positive without the wall. Since there is only one zero mode no finite charge accumulates. In the presence of a normal magnetic field, however, there is a whole Landau level of zero modes with degeneracy $Be/2\pi$ per unit area per species of fermion. This represents charge $\pm(e/2)\times(Be/2\pi)\times 4$ per unit area relative to the Fermi sea for no domain wall, the sign depending on whether the Fermi level is above or below midgap. This result agrees, modulo its sign ambiguity, with the previous one.

Actually the procedure used above, of replacing a soliton field configuration by another more convenient one which is equal to it at spatial infinity, is commonly used in the computation of soliton quantum numbers.[7,8] The underlying idea is that the difference between the true field configuration and the convenient one can be interpolated by fictitious *localized* intermediate fields. Such fictitious fields do not induce current flow at infinity and so the accumulated charge on the soliton is the same as that of the convenient configuration.

In the case at hand, it is most convenient to consider the class of models with both $m(x)$ and $a(x)$ variable, i.e., a complex effective mass. It is not difficult to show, following Ref. 7, that the charge accumulated by a domain wall where the complex mass varies in an arbitrary way, provided only that it does not pass through zero, is simply proportional to the change in its phase.

The case of a purely *real* mass, passing through zero, is in a sense a degenerate limit. Tiny changes in the imaginary part of the mass can make the accumulated phase either $\pm\pi$ (Fig. 2). This instability reflects the existence of the zero mode, and is related to the abrupt change in the charge as the Fermi level sweeps through it. Note that the magnitude of the charges and currents

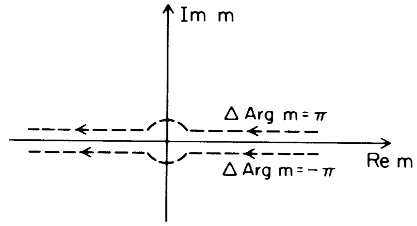

FIG. 2. Phase changes through $\pm\pi$ depending on the sign of the imaginary part.

will depend on small perturbations—including both terms, such as Zeeman splitting, not retained in the model Lagrangean (9), and effects of impurities and doping—when the Fermi level is near midgap.

For truly complex-valued masses the mechanism whereby charges and currents are generated need not be connected with the existence of 0 modes, or midgap states. For instance, if $|m(x)|$ is constant, and its phase varies slowly in the sense that $|\partial m/\partial x|/|m| \lesssim |m|$, then the local magnitude of the gap will remain $\sim 2|m|$ everywhere. Nevertheless, the charges and currents discussed above will be produced; they are associated with the behavior of the Fermi sea as a whole. Their magnitude in this case is robust.

The current associated with an electric field in the plane of the wall is so remarkable that it deserves further discussion. First, let us note its properties:

(i) Since $\mathbf{j} \propto \mathbf{E}$, the current is *voltage controlled.*

(ii) Since $\mathbf{j} \perp \mathbf{E}$, the current is *nondissipative.* Of course I have ignored impurities, etc., so that this statement is only approximate. At the level of analysis in this paper many-body effects have been ignored, and the current is not a supercurrent.

(iii) The direction of the current is determined by the unit vector $\mathbf{B}/|\mathbf{B}|$. This form follows from time-reversal symmetry. It indicates that the direction of current can reverse, ideally, in response to changes in magnetic fields of tiny magnitude. This peculiar dependence is associated, in the axion picture [Eq. (12)], with the fact that a change in the sign of the mass is ambiguous; it can mean $\Delta = +\pi$ or $-\pi$ across the wall.

And now let me discuss in more detail how these properties arise. I will first discuss the case where m is an odd, real function of z and $m(\infty)$ is positive. I use the conventions of Bjorken and Drell.[9] The zero-mode solutions of the Dirac equation are then of the form

$$\psi = \exp\left[\int_0^z m(z)dz\right][\chi_1 f(x,y) + \chi_2 g(x,y)], \qquad (13a)$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}. \qquad (13b)$$

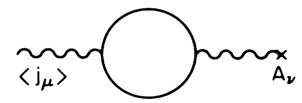

FIG. 3. Expectation of the current in a background field is derived from the vacuum polarization.

Other modes have energy $\sim |m(\infty)|$ and can be neglected. The effective two-dimensional dynamics of the low-energy modes are described by the effective Hamiltonian

$$H = i\sigma_3 \boldsymbol{\sigma}\cdot\boldsymbol{\partial}$$

acting on the two-component spinor $\eta = \binom{f}{g}$. A magnetic field normal to the wall leads to a series of Landau levels at energies $n(eB/2\pi)^{1/2}$ with degeneracy $eB/2\pi$ per unit area. If the Fermi level is slightly above 0, then one has charge accumulated on the wall, as discussed above, because the $n=0$ mode is composed half of positive-energy and half of negative-energy states relative to the free (i.e., no domain wall) Hamiltonian. The independence of the magnitude of j on the magnitude of $\mathbf{B}$ is now easily understood heuristically as follows. The basic phenomena is a drift of the zero-mode plasma in crossed electric and magnetic fields. In this situation, the drift velocity of a charged particle is proportional to E and inversely proportional to B. But since the number of states in the zero-mode plasma, as discussed above, is itself proportional to B this dependence cancels out.

An infinitesimal complex (3+1)-dimensional mass, i.e., an effective axion field, generates a (2+1)-dimensional mass μ, leading to similar results. Indeed, the effect of the axion field in the z direction is simply to induce an effective mass term

$$\Delta H = \mu\sigma_3, \quad \mu \propto \Delta, \qquad (14)$$

as we see by sandwiching the effective coupling

$$\propto \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \partial_3 a$$

in (15).

So in this case the interaction of the low-energy modes with a planar electric field reduces to a problem in the electrodynamics of (2+1)-dimensional massive fermions. Now there is a peculiarity in the vacuum polarization of (2+1)-dimensional electrodynamics that is relevant here.[10] That is, the current induced by an external field, calculated from the Feynman diagram in Fig. 3, is of the form

$$j_\rho \propto (\mu/|\mu|)\epsilon_{\rho\sigma\tau}F_{\sigma\tau}. \qquad (15)$$

This gives us the current discussed above, with the properties (i)–(iii).

Note that the (2+1)-dimensional mass μ is P and T odd; it is this which explains how it can be proportional to a magnetic field, and which allows the peculiar-looking equation (15) to be consistent with these discrete symmetries. Similarly, there is no current in the absence of a magnetic field (this point seems confused in Ref. 5).

All this analysis is based on a one-particle picture, and requires modification if condensations among the zero modes occur, as in the quantized Hall effect.[11] Such condensation is not unlikely, since the dynamics within the zero-mode plasma maps directly onto the usual two-dimensional electron-gas problem (except for the factor $N=4$ and $e \rightarrow e/2$ in the current).

Generalizations.— Is there a class of solids which realizes the Dirac equation with complex adjustable masses? Unfortunately, the mapping leading from a standard hopping Hamiltonian to effective Lagrangeans like (9) is not very transparent, and I have not succeeded in finding examples although I believe they exist. A relevant observation is that the complex mass terms violate P and T symmetry, and so they will arise only where one has nontrivial spin order in the solid.

More generally, as condensed-matter experimentalists achieve ever more exquisite control over material fabrication at the molecular level, it may be fruitful to consider what other "artificial solitons," with interesting or useful electronic properties, can be manufactured.

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[1]A very complete review of axions has recently appeared: J. E. Kim, in Seoul National University Report No. SNUHE 86/09, 1986 (to be published). See also my Erice Lectures of 1983 (to be published).

[2]E. Witten, Phys. Lett. **86B**, 283 (1979). Discussions closely related to the one here have been given by S. Coleman, in *Unity of the Fundamental Interactions,* edited by A. Zichichi (Plenum, New York, 1983), and by K. Lee, Fermilab Report No. 86/148-T, 1986 (to be published).

[3]G 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); C. Callan, R. Dashen, and D. Gross, Phys. Lett. **63B**, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).

[4]S. Adler, Phys. Rev. **177**, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969).

[5]E. Fradkin, E. Dagotto, and D. Boyanovsky, Phys. Rev. Lett. **57**, 2967 (1986).

[6]R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976); W. P. Su, J. R. Schrieffer, and A. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).

[7]J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 960 (1981).

[8]C. Callan and J. Harvey, Nucl. Phys. **B250**, 427 (1985).

[9]J. Bjorken and B. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[10]A. Redlich, Phys. Rev. D **29**, 2366 (1984).

[11]K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).