Universal Steering Criteria

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We propose a general framework for constructing universal steering criteria that are applicable to arbitrary bipartite states and measurement settings of the steering party. The same framework is also useful for studying the joint measurement problem. Based on the data-processing inequality for an extended Rényi relative entropy, we then introduce a family of steering inequalities, which detect steering much more efficiently than those inequalities known before. As illustrations, we show unbounded violation of a steering inequality for assemblages constructed from mutually unbiased bases and establish an interesting connection between maximally steerable assemblages and complete sets of mutually unbiased bases. We also provide a single steering inequality that can detect all bipartite pure states of full Schmidt rank. In the course of study, we generalize a number of results intimately connected to data-processing inequalities, which are of independent interest.

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Steering is a nonclassical phenomenon that formalizes what Einstein called "spooky action at a distance" [1,2]. For a long time, it was studied under the name of Einstein-Podolsky-Rosen paradox [3–5]. Recently, it was recognized as a form of nonlocality that sits between entanglement and Bell nonlocality [6–9] and that is intrinsically asymmetric [10,11]. Interestingly, steering can be characterized by a simple quantum information processing task, namely, entanglement verification with an untrusted party [6,7]. In addition, steering has been found useful in a number of applications, such as subchannel discrimination [12] and one-sided device-independent quantum key distribution [13,14].

Recently, detection and characterization of steering have attracted increasing attention [4–7,12,15–22]. Various steering criteria and inequalities have been derived, such as linear steering inequalities [15,18], inequalities based on multiplicative variances [4,5,15], entropic uncertainty relations [16,17], fine-grained uncertainty relations [19], and hierarchy of steering criteria based on moments [20]. However, most of these results are tailored to deal with specific scenarios; majority criteria are only applicable to given numbers of measurement settings and outcomes. In addition, many criteria (including most linear criteria) rely heavily on numerical optimization and lack a clear physical meaning and simple interpretation.

In this Letter, we propose a general framework for constructing universal steering criteria that are applicable to arbitrary measurement settings of the steering party. In particular, we introduce nonlinear steering inequalities based on the data-processing inequality for an extended Rényi relative entropy [23,24], which detect steering more systematically and efficiently than criteria in the literature. The same framework is also useful for studying the joint measurement problem [25–30]. In addition, our criteria have a clear information theoretic meaning and simple interpretation. They are closely related to steering robustness under white noise. As illustrations of the general framework, we show unbounded violation of a steering inequality by virtue of mutually unbiased bases (MUB) [31,32] and establish an interesting connection between maximally steerable settings and complete sets of MUB. We also provide a single steering inequality that can detect all bipartite pure states of full Schmidt rank.

Suppose Alice and Bob share a bipartite state ρ with reduced states ρ_A and ρ_B. Alice can perform local measurements described by a collection of positive-operator-valued measures (POVMs) $\{A_{a|x}\}$, which is known as a measurement assemblage. If Alice obtains outcome a for measurement x, then the unnormalized reduced state of Bob is $\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes 1)\rho]$. In the following, we discuss steering of Bob's system by Alice's measurements in terms of Bob's states $\rho_{a|x}$. The set of states $\rho_{a|x}$ for a given x is called an ensemble for ρ_B and denoted by $\{\rho_{a|x}\}_a$, while the whole collection of ensembles is called a state assemblage [18] and denoted by $\{\rho_{a|x}\}$; see Fig. 1. The assemblage $\{\rho_{a|x}\}$ is steerable if it does not admit a local hidden state model [6,7] as $\rho_{a|x} = \sum_\lambda p(a|x,\lambda)\sigma_\lambda$ for all a, x, where $\{\sigma_\lambda\}$ is an ensemble for ρ_B and $p(a|x,\lambda)$ are a collection of stochastic maps with $p(a|x,\lambda) \geq 0$ and $\sum_a p(a|x,\lambda) = 1$. The state ρ is steerable from Alice to Bob if there exists a measurement assemblage for Alice such that the resulting state assemblage for Bob is steerable. In this Letter, we

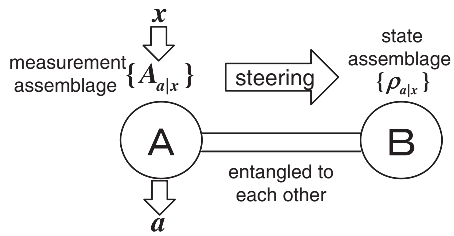

FIG. 1. Steering scenario. Alice can affect Bob's state via her measurements following the relation $\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes 1)\rho]$. Entanglement is necessary but not sufficient for steering.

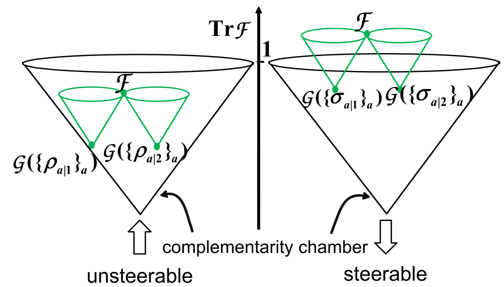

FIG. 2. Simple idea behind universal steering criteria. Here, the green cone at $\mathcal{G}(\{\rho_{a|x}\}_a)$ represents the set of superoperators $\mathcal{F}$ satisfying $\mathcal{F} \geq \mathcal{G}(\{\rho_{a|x}\}_a)$. When the assemblage $\{\rho_{a|x}\}$ is unsteerable, $\mathcal{G}(\{\rho_{a|x}\}_a)$ have a common upper bound in the complementarity chamber (left plot). Violation of this condition implies steerability (right plot).

shall focus on steerability of assemblages, no matter how they are constructed.

The steering problem is closely related to the joint measurement problem [27–30,33]. Up to a scaling, a POVM may be seen as an ensemble for the completely mixed state. A measurement assemblage is compatible or jointly measurable if the corresponding state assemblage (for the completely mixed state) is unsteerable. So many results on steering can be turned into corresponding results on POVMs, and vice versa. We shall make use of this connection without further comments.

To set the stage, we need to introduce suitable order relations on ensembles and assemblages. Given two ensembles $\{\rho_a\}$ and $\{\sigma_b\}$ for ρ_B, which may represent two preparation procedures, the ensemble $\{\rho_a\}$ is a coarse graining of $\{\sigma_b\}$, denoted by $\{\rho_a\} \preceq \{\sigma_b\}$ or $\{\sigma_b\} \succeq \{\rho_a\}$, if the former can be derived from the latter by data processing, that is, $\rho_a = \sum_b p(a|b)\sigma_b$, where the stochastic map $p(a|b)$ characterizes the data-processing procedure. In that case, $\{\sigma_b\}$ is a refinement of $\{\rho_a\}$. Intuitively, coarse graining usually leads to a less informative ensemble. Two ensembles are equivalent if they are coarse graining (refinement) of each other. The relation of coarse graining (refinement) forms a partial order on equivalent classes of ensembles for a given state. For example, the trivial ensemble $\{\rho_B\}$ is a coarse graining of all ensembles for ρ_B. However, there is no ensemble that is a refinement of all ensembles except when ρ_B is pure, in which case all ensembles are equivalent.

The order relation on ensembles can be generalized to assemblages in a natural way. Given two assemblages $\{\rho_{a|x}\}$ and $\{\sigma_{b|y}\}$ for ρ_B, the assemblage $\{\rho_{a|x}\}$ is a coarse graining of $\{\sigma_{b|y}\}$, denoted by $\{\rho_{a|x}\} \preceq \{\sigma_{b|y}\}$ or $\{\sigma_{b|y}\} \succeq \{\rho_{a|x}\}$, if each ensemble in $\{\rho_{a|x}\}$ is a coarse graining of an ensemble in $\{\sigma_{b|y}\}$. In that case, $\{\sigma_{b|y}\}$ is called a refinement of $\{\rho_{a|x}\}$. An assemblage is unsteerable if and only if it has a refinement that contains only one ensemble, that is, all its ensembles possess a common refinement. By definition, any coarse graining of an unsteerable assemblage is unsteerable. Conversely, any refinement of a steerable assemblage is steerable.

A function f on ensembles is order monotonic (or order preserving) if $f(\{\rho_a\}) \preceq f(\{\sigma_b\})$ whenever $\{\rho_a\} \preceq \{\sigma_b\}$. Order-monotonic functions on assemblages can be defined in a similar manner. Here, the image of f could be any space with a partial order, although we use the same notation for the partial order as that on ensembles. The image of all ensembles for a given state under an order-monotonic function f is called the complementarity chamber and denoted by $\mathcal{C}_f$. In those cases of interest to us, the chambers are usually finite-dimensional compact convex sets, and their shapes reflect the information tradeoff among different ensembles, hence, the name. For any unsteerable assemblage $\{\rho_{a|x}\}$ with a common refinement, say, $\{\sigma_\lambda\}$, we have $f(\{\rho_{a|x}\}_a) \preceq f(\{\sigma_\lambda\}) \in \mathcal{C}_f$ for all x. So $f(\{\rho_{a|x}\}_a)$ have a common upper bound in $\mathcal{C}_f$. Violation of this condition is a signature of steerability; see Fig. 2 for an illustration.

To unleash the potential of the above idea, it is essential to construct order-monotonic functions that are easy to characterize as follows. Let $\{\rho_a\}$ be an ensemble for ρ_B and Q a positive operator of full rank; define

$$\mathcal{G}_Q(\{\rho_a\}) \coloneqq \sum_a \frac{|\rho_a\rangle\rangle\langle\langle\rho_a|}{\mathrm{tr}(Q\rho_a)}, \qquad \bar{\mathcal{G}}_Q(\{\rho_a\}) \coloneqq \sum_a \frac{|\bar{\rho}_a\rangle\rangle\langle\langle\bar{\rho}_a|}{\mathrm{tr}(Q\rho_a)}, \tag{1}$$

where $\bar{\rho}_a = \rho_a - \mathrm{tr}(\rho_a)/d$, and d is the dimension of the Hilbert space. Here, we consider the Hilbert space of operators on the physical space, i.e., the Hilbert-Schmidt space. The kets in this space are denoted by the double-ket notation to distinguish them from ordinary kets. Superoperators, such as the outer product $|A\rangle\rangle\langle\langle B|$, act on the operator space just as ordinary operators act on the usual Hilbert space; for example $(|A\rangle\rangle\langle\langle B|)|C\rangle\rangle = |A\rangle\rangle\mathrm{tr}(B^\dagger C)$ (cf. Refs. [30,34]).

For a positive real vector $\boldsymbol{p}$ and a real vector $\boldsymbol{v}$ of the same length, we introduce extended Rényi relative entropy of order 2 as $D_2(\boldsymbol{v}\|\boldsymbol{p}) \coloneqq \log\sum_k(v_k^2/p_k)$, which reduces to Rényi relative entropy of order 2 when $\boldsymbol{p}$ and $\boldsymbol{v}$ represent probability distributions [23,24]. As shown in the Supplemental Material [35], which includes Refs. [36–48], this quantity obeys a generalized data-processing inequality. Let C be a Hermitian operator; then $\langle\langle C|\mathcal{G}_Q(\{\rho_a\})|C\rangle\rangle = \sum_a(v_a^2/p_a)$, where $v_a = \mathrm{tr}(\rho_a C)$ and

$p_a = \mathrm{tr}(\rho_a Q)$. It follows that $\langle\langle C|\mathcal{G}_Q(\{\rho_a\})|C\rangle\rangle$ is order monotonic, which, in turn, implies the following theorem.

Theorem 1.—The functions $\mathcal{G}_Q(\cdot)$ and $\bar{\mathcal{G}}_Q(\cdot)$ are order monotonic for any positive operator Q of full rank.

When Q is the identity, Eq. (1) reduces to

$$\mathcal{G}(\{\rho_a\}) := \sum_a \frac{|\rho_a\rangle\rangle\langle\langle\rho_a|}{\mathrm{tr}(\rho_a)}, \qquad \bar{\mathcal{G}}(\{\rho_a\}) := \sum_a \frac{|\bar{\rho}_a\rangle\rangle\langle\langle\bar{\rho}_a|}{\mathrm{tr}(\rho_a)}. \tag{2}$$

We have

$$\mathrm{Tr}[\mathcal{G}(\{\rho_a\})] = \sum_a \frac{\mathrm{tr}(\rho_a^2)}{\mathrm{tr}(\rho_a)} \leq \sum_a \mathrm{tr}(\rho_a) = \mathrm{tr}(\rho_B),$$
$$\mathrm{Tr}[\bar{\mathcal{G}}(\{\rho_a\})] = \mathrm{Tr}[\mathcal{G}(\{\rho_a\})] - \frac{\mathrm{tr}(\rho_B)}{d} \leq \left(1-\frac{1}{d}\right)\mathrm{tr}(\rho_B), \tag{3}$$

where "Tr" denotes the trace of superoperators, while "tr" denotes the trace of ordinary operators. Here, the upper bounds are saturated if and only if the ensemble is rank 1, that is, all the ρ_a have rank 1. Recall that $\{\rho_B\}$ is a coarse graining of all ensembles for ρ_B, we also have

$$\mathcal{G}(\{\rho_a\}) \geq \frac{|\rho_B\rangle\rangle\langle\langle\rho_B|}{\mathrm{tr}(\rho_B)}, \qquad \bar{\mathcal{G}}(\{\rho_a\}) \geq \frac{|\bar{\rho}_B\rangle\rangle\langle\langle\bar{\rho}_B|}{\mathrm{tr}(\rho_B)}, \tag{4}$$

which imply that

$$\mathrm{Tr}[\mathcal{G}(\{\rho_a\})] \geq \frac{\mathrm{tr}(\rho_B^2)}{\mathrm{tr}(\rho_B)}, \qquad \mathrm{Tr}[\bar{\mathcal{G}}(\{\rho_a\})] \geq \frac{\mathrm{tr}(\bar{\rho}_B^2)}{\mathrm{tr}(\rho_B)}. \tag{5}$$

So the purity of ρ_B sets lower bounds for $\mathrm{Tr}[\mathcal{G}(\{\rho_a\})]$ and $\mathrm{Tr}[\bar{\mathcal{G}}(\{\rho_a\})]$.

Define

$$\tau(\{\rho_{a|x}\}) = \min\{\mathrm{Tr}(\mathcal{F})|\mathcal{F} \geq \mathcal{G}(\{\rho_{a|x}\}_a) \quad \forall\ x\},$$
$$\bar{\tau}(\{\rho_{a|x}\}) = \min\{\mathrm{Tr}(\mathcal{F})|\mathcal{F} \geq \bar{\mathcal{G}}(\{\rho_{a|x}\}_a) \quad \forall\ x\}. \tag{6}$$

Note that $\mathcal{G}(\{\rho_{a|x}\}_a) - \bar{\mathcal{G}}(\{\rho_{a|x}\}_a)$ is independent of x and its trace is equal to $\mathrm{tr}(\rho_B)/d$, we deduce that

$$\tau(\{\rho_{a|x}\}) = \bar{\tau}(\{\rho_{a|x}\}) + \frac{\mathrm{tr}(\rho_B)}{d}. \tag{7}$$

Theorem 2.—The functions $\tau(\cdot)$ and $\bar{\tau}(\cdot)$ are order monotonic on assemblages. Any unsteerable assemblage $\{\rho_{a|x}\}$ for ρ_B satisfies the inequalities $\tau(\{\rho_{a|x}\}) \leq \mathrm{tr}(\rho_B)$ and $\bar{\tau}(\{\rho_{a|x}\}) \leq (1-1/d)\mathrm{tr}(\rho_B)$.

Proof of Theorem 2.—Suppose $\{\rho_{a|x}\} \preceq \{\sigma_{b|y}\}$. Then, any ensemble x in $\{\rho_{a|x}\}$ is a coarse graining of an ensemble y in $\{\sigma_{b|y}\}$, so $\mathcal{G}(\{\rho_{a|x}\}_a) \leq \mathcal{G}(\{\sigma_{b|y}\}_b)$ according to Theorem 1. Therefore, $\mathcal{F} \geq \mathcal{G}(\{\rho_{a|x}\}_a)$ for all x whenever $\mathcal{F} \geq \mathcal{G}(\{\sigma_{b|y}\}_b)$ for all y. It follows that $\tau(\{\rho_{a|x}\}) \leq \tau(\{\sigma_{b|y}\})$ and $\tau(\cdot)$ is order monotonic. By the same reasoning, so is $\bar{\tau}(\cdot)$.

The ensembles in $\{\rho_{a|x}\}$ possess a common refinement, say $\{\sigma_\lambda\}$, so that $\mathcal{G}(\{\sigma_\lambda\}) \geq \mathcal{G}(\{\rho_{a|x}\}_a)$ for all x. On the other hand, $\mathrm{Tr}[\mathcal{G}(\{\sigma_\lambda\})] \leq \mathrm{tr}(\rho_B)$ according to Eq. (3). It follows that $\tau(\{\rho_{a|x}\}) \leq \mathrm{tr}(\rho_B)$. The other inequality in Theorem 2 follows from similar reasoning or Eq. (7).

The two steering inequalities in Theorem 2 are equivalent in view of Eq. (7). In practice, one may be easier to analyze than the other. These inequalities are also applicable to unnormalized assemblages. In particular, they can serve as compatibility inequalities on POVMs if ρ_B is replaced by the identity (cf. Ref. [30]); the same remark applies to several other results in this Letter. Figure 2 elucidates the simple idea behind these steering inequalities: unsteerable assemblages cannot be too informative. More general steering inequalities can be constructed by virtue of the order-monotonic functions $\mathcal{G}_Q$ and $\bar{\mathcal{G}}_Q$ as shown in the Supplemental Material [35].

Most known steering inequalities [15,18], such as linear steering inequalities, rely on numerical calculation for each assemblage even to write down the inequalities explicitly, and the behavior of the bounds are unpredictable. Remarkably, the inequalities in Theorem 2 are applicable to arbitrary assemblages. In addition, for normalized assemblages, the upper bounds in $\tau(\{\rho_{a|x}\}) \leq 1$ and $\bar{\tau}(\{\rho_{a|x}\}) \leq (1-1/d)$ depend at most on the dimension, but not on any other detail. In this sense, our steering criteria $\tau(\{\rho_{a|x}\})$ and $\bar{\tau}(\{\rho_{a|x}\})$ are universal.

Besides wide applicability, the steering criteria $\tau(\{\rho_{a|x}\})$ and $\bar{\tau}(\{\rho_{a|x}\})$ are also superior with regard to computational complexity because they can be computed efficiently with semidefinite programming (SDP), whose size increases only linearly with the number of ensembles. Although the steerability of an assemblage can be determined by SDP [18,21], the size of such SDP increases exponentially with the number of ensembles. So our approach is attractive from both conceptual and practical perspectives.

The criterion $\bar{\tau}(\{\rho_{a|x}\})$ also quantifies the steering robustness of the assemblage $\{\rho_{a|x}\}$ with regard to white noise. Let $\rho_{a|x}(\eta) = \eta\rho_{a|x} + (1-\eta)\mathrm{tr}(\rho_{a|x})/d$ with $0 \leq \eta \leq 1$; then the assemblage $\{\rho_{a|x}(\eta)\}$ may be seen as the assemblage $\{\rho_{a|x}\}$ corrupted by white noise. It is also a coarse graining of $\{\rho_{a|x}\}$ when the latter is an assemblage for the completely mixed state. Calculation shows that $\bar{\tau}[\{\rho_{a|x}(\eta)\}] = \eta^2\bar{\tau}(\{\rho_{a|x}\})$, so the assemblage $\{\rho_{a|x}(\eta)\}$ is steerable as long as $\eta^2\bar{\tau}(\{\rho_{a|x}\}) > 1 - 1/d$.

As an illustration, let us take the qubit as an example. Let $\{\rho_a\}$ be an ensemble for $\rho_B = \frac{1}{2}(1 + \boldsymbol{s}_B \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ is the vector composed of the three Pauli matrices and $\boldsymbol{s}_B$ is the Bloch vector of ρ_B. Each member in the ensemble can be written as $\rho_a = \frac{1}{2}p_a(1 + \boldsymbol{s}_a \cdot \boldsymbol{\sigma})$, where $\boldsymbol{s}_a$ is the Bloch vector corresponding to ρ_a, and $\{p_a\}$ is a probability distribution satisfying $\sum_a p_a \boldsymbol{s}_a = \boldsymbol{s}_B$. Define

$$G(\{\rho_a\}) = \sum_a p_a \boldsymbol{s}_a \boldsymbol{s}_a^{\mathrm{T}} \tag{8}$$

as the matrix representation (up to a scaling) of the superoperator $\bar{\mathcal{G}}(\{\rho_a\}) = \frac{1}{4}\sum_a p_a |\boldsymbol{s}_a \cdot \boldsymbol{\sigma}\rangle\rangle\langle\langle \boldsymbol{s}_a \cdot \boldsymbol{\sigma}|$ with respect to the Pauli basis. Then $G(\{\rho_a\})$ is order monotonic and satisfies

$$s_B s_B^{\mathrm{T}} \le G(\{\rho_a\}) \le 1, \qquad s_B^2 \le \mathrm{tr}[G(\{\rho_a\})] \le 1. \quad (9)$$

Note that $\mathrm{tr}[G(\{\rho_a\})] \le 1$ implies $G(\{\rho_a\}) \le 1$, so the complementarity chamber lies in a truncated cone of dimension 6.

Now the steering inequalities in Theorem 2 reduce to $2\bar{\tau}(\{\rho_{a|x}\} \le 1$ with

$$2\bar{\tau}(\{\rho_{a|x}\}) = \min\{\mathrm{tr}(F) | F \ge G(\{\rho_{a|x}\}_a) \quad \forall\ x\}. \quad (10)$$

For example, consider the state ρ_B with $s_B = (0, 0, \cos\theta)$ $(0 \le \theta \le \pi/2)$ and let $\{\rho_{\pm|x} = \frac{1}{2} p_{\pm|x}(1 + s_{\pm|x} \cdot \boldsymbol{\sigma})\}$ for $x = 1,\ 2$ be an assemblage for ρ_B, where $p_{\pm|1} = \frac{1}{2}(1 \pm \cos\theta)$, $s_{\pm|1} = (0, 0, \pm 1)^{\mathrm{T}}$, $p_{\pm|2} = \frac{1}{2}$, $s_{\pm|2} = (\pm\eta \sin\theta, 0, \cos\theta)^{\mathrm{T}}$ with $0 \le \eta \le 1$. The assemblage forms a cross when represented on the Bloch ball. Calculation shows that $G(\rho_{\pm|1}) = \mathrm{diag}(0, 0, 1)$ and $G(\rho_{\pm|2}) = \mathrm{diag}(\eta^2 \sin^2\theta, 0, \cos^2\theta)$, so that $2\bar{\tau}(\{\rho_{a|x}\}) = 1 + \eta^2 \sin^2\theta$. Thus, the assemblage is steerable whenever $\eta \sin\theta \ne 0$.

To further demonstrate the power of our steering inequalities, we need to introduce several concepts. Two ensembles $\{\rho_a\}$ and $\{\sigma_b\}$ are mutually orthogonal if $\mathrm{tr}(\bar{\rho}_a \bar{\sigma}_b) = 0$ for all a, b or, equivalently, if $\bar{\mathcal{G}}(\{\rho_a\})$ and $\bar{\mathcal{G}}(\{\sigma_b\})$ have mutually orthogonal support. The same definition applies to POVMs. For rank 1 projective measurements, orthogonality is equivalent to mutually unbiasedness. Recall that two bases $\{|\psi_j\rangle\}$ and $\{|\varphi_k\rangle\}$ in dimension d are mutually unbiased if $|\langle\psi_j|\varphi_k\rangle|^2 = 1/d$ for all j, k [31,32]. The following two propositions are proved in the Supplemental Material [35].

Proposition 1.—Any assemblage $\{\rho_{a|x}\}$ for ρ_B satisfies $\tau(\{\rho_{a|x}\}) \le d\mathrm{tr}(\rho_B)$ and $\bar{\tau}(\{\rho_{a|x}\}) \le (d - 1/d)\mathrm{tr}(\rho_B)$.

Proposition 2.—Any assemblage $\{\rho_{a|x}\}$ for ρ_B with m ensembles satisfies $\bar{\tau}(\{\rho_{a|x}\}) \le m(1 - 1/d)\mathrm{tr}(\rho_B)$. The upper bound is saturated if and only if the ensembles are rank 1 and mutually orthogonal.

In view of Proposition 1, state assemblages saturating the upper bound $\bar{\tau}(\{\rho_{a|x}\}) \le d - 1/d$ [or $\tau(\{\rho_{a|x}\}) \le d$] are maximally steerable. When the assemblage $\{\rho_{a|x}\}$ is constructed from the basis states of m projective measurements (with suitable scaling), the upper bound $\bar{\tau}(\{\rho_{a|x}\}) \le m(1 - 1/d)$ is saturated if and only if the bases are mutually unbiased. The inequality $\bar{\tau}(\{\rho_{a|x}\}) \le d - 1/d$ in Proposition 1 means that each set of MUB can contain, at most, $d + 1$ bases, in agreement with the well-known bound [31,32]. When d is a prime power, a complete set of MUB can be constructed [31,32], so the steering inequality $\bar{\tau}(\{\rho_{a|x}\}) \le 1 - 1/d$ can be violated by a factor of $d + 1$, which is unbounded as d grows. Such assemblages remain steerable even under white noise with ever increasing strength. In contrast with unbounded violations of linear steering inequalities shown in Refs. [49,50], our result follows from a universal recipe, and the degree of violation can be determined precisely. An intriguing problem left open is how many bases are needed to construct a maximally steerable assemblage when complete sets of MUB cannot be found, say, in dimension 6.

More general steering inequalities can be derived by considering the effect of filtering. The following proposition is an easy generalization of a result in Ref. [29]; see, also, Refs. [9,51] for similar conclusions.

Proposition 3.—The two assemblages $\{V\rho_{a|x}V^\dagger\}$ and $\{V\rho_{a|x}^{\mathrm{T}}V^\dagger\}$ (unnormalized) are both unsteerable for any operator V if $\{\rho_{a|x}\}$ is unsteerable. When V is invertible, $\{V\rho_{a|x}V^\dagger\}$, $\{V\rho_{a|x}^{\mathrm{T}}V^\dagger\}$, and $\{\rho_{a|x}\}$ are simultaneously steerable or not.

When Bob's state ρ_B is invertible, Theorem 2 and Proposition 3 imply that any unsteerable assemblage $\{\rho_{a|x}\}$ satisfies

$$\bar{\tau}(\{\rho_B^{-1/2}\rho_{a|x}\rho_B^{-1/2}\}) \le d - 1. \quad (11)$$

Until now, we have discussed steering in terms of Bob's state assemblage $\{\rho_{a|x}\}$. At this point, it is instructive to consider steering of Bob's state by Alice's measurements as described by the assemblage $\{A_{a|x}\}$, which is the physical situation illustrated in Fig. 1. Suppose they share a pure bipartite state ρ of full Schmidt rank, which has Schmidt decomposition $\rho = \sum_{j,k}\lambda_j\lambda_k|jj\rangle\langle kk|$. Then, the reduced states of Alice and Bob have the same form with respect to the Schmidt basis $\rho_A = \rho_B = \sum_j \lambda_j^2 |j\rangle\langle j|$, and the state assemblage $\{\rho_{a|x}\}$ for Bob takes on the form $\rho_{a|x} = \rho_B^{1/2} A_{a|x}^{\mathrm{T}} \rho_B^{1/2}$ [24,29]. Therefore,

$$\bar{\tau}(\{\rho_B^{-1/2}\rho_{a|x}\rho_B^{-1/2}\}) = \bar{\tau}(\{A_{a|x}^{\mathrm{T}}\}) = \bar{\tau}(\{A_{a|x}\}), \quad (12)$$

where the second equality follows from Lemma S6 in the Supplemental Material [35]. As a consequence of Eqs. (11) and (12), $\bar{\tau}(\{A_{a|x}\}) \le d - 1$ if Alice cannot steer Bob's system. If the assemblage $\{A_{a|x}\}$ is composed of two MUB, then $\bar{\tau}(\{A_{a|x}\}) = 2(d - 1)$, which violates the bound by a factor of 2. Remarkably, the single steering inequality $\bar{\tau}(\{\rho_B^{-1/2}\rho_{a|x}\rho_B^{-1/2}\}) \le d - 1$ with two measurement settings can detect the steerability of all bipartite pure states of full Schmidt rank, whereas infinitely many inequalities linear in $\rho_{a|x}$ (note that our inequalities are not linear in $\rho_{a|x}$) are needed to achieve the same task [15,18]. Also, no general recipe is known for constructing linear steering inequalities without numerical optimization. Therefore, our approach provides a dramatic improvement over those alternatives in the literature.

As another example, consider an isotropic state in dimension $d \times d$, $\rho(\alpha) = \alpha|\Phi\rangle\langle\Phi| + (1 - \alpha)(1/d^2)$, where $|\Phi\rangle = (\sum_j |jj\rangle)/\sqrt{d}$ and $0 \le \alpha \le 1$. Suppose Alice has the measurement assemblage $\{A_{a|x}\}$; then, $\bar{\tau}(\{\rho_{a|x}\}) = (\alpha^2/d)\bar{\tau}(\{A_{a|x}\})$; cf. the Supplemental Material [35]. The isotropic state is steerable with respect to $\{A_{a|x}\}$ when $\alpha^2\bar{\tau}(\{A_{a|x}\}) > (d-1)$. If the measurement assemblage for Alice is composed of m MUB, then $\bar{\tau}(\{A_{a|x}\}) = m(d-1)$, so the isotropic state is steerable if $m\alpha^2 > 1$. In the case of two qubits, this condition turns out to be both sufficient and necessary [15,20].

In summary, we proposed a general framework for detecting and characterizing steering based on simple information theoretic ideas. By virtue of the data-processing inequality for the extended Rényi relative entropy of order 2, we then introduced a family of steering inequalities that are applicable to arbitrary assemblages and have a simple interpretation. These steering criteria are also closely related to steering robustness under white noise. As illustrations, we showed unbounded violation of a steering inequality for assemblages constructed from MUB and provided a single steering inequality that can detect all bipartite pure states of full Schmidt rank. Our Letter established intriguing connections among a number of fascinating subjects, including information theory, quantum foundations, and the geometry of quantum state space, which are of interest to researchers from diverse fields. In addition, our work has an intimate connection to quantum estimation theory. Indeed, our Theorem 1 can be applied to proving the data-processing inequality for Fisher information, and vice versa (cf. the Supplemental Material [35]). Also, our study allows us to derive and generalize many results in quantum estimation theory, which will be presented in another paper.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] E. Schrödinger, Math. Proc. Cambridge Philos. Soc. **31**, 555 (1935).
[3] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[4] M. D. Reid, Phys. Rev. A **40**, 913 (1989).
[5] M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Andersen, and G. Leuchs, Rev. Mod. Phys. **81**, 1727 (2009).
[6] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[7] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Phys. Rev. A **76**, 052116 (2007).
[8] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Nat. Phys. **6**, 845 (2010).
[9] M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Phys. Rev. A **92**, 032107 (2015).
[10] S. L. W. Midgley, A. J. Ferris, and M. K. Olsen, Phys. Rev. A **81**, 022101 (2010).
[11] J. Bowles, T. Vértesi, M. T. Quintino, and N. Brunner, Phys. Rev. Lett. **112**, 200402 (2014).
[12] M. Piani and J. Watrous, Phys. Rev. Lett. **114**, 060404 (2015).
[13] M. Tomamichel and R. Renner, Phys. Rev. Lett. **106**, 110506 (2011).
[14] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, Phys. Rev. A **85**, 010301 (2012).
[15] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A **80**, 032112 (2009).
[16] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Phys. Rev. Lett. **106**, 130402 (2011).
[17] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Phys. Rev. A **87**, 062103 (2013).
[18] M. F. Pusey, Phys. Rev. A **88**, 032313 (2013).
[19] T. Pramanik, M. Kaplan, and A. S. Majumdar, Phys. Rev. A **90**, 050305 (2014).
[20] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, and G. Adesso, Phys. Rev. Lett. **115**, 210401 (2015).
[21] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, Phys. Rev. Lett. **112**, 180404 (2014).
[22] I. Kogias, A. R. Lee, S. Ragy, and G. Adesso, Phys. Rev. Lett. **114**, 060403 (2015).
[23] A. Rényi, in *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, Berkeley, 1961) pp. 547–561.
[24] M. Hayashi, *Quantum Information: An Introduction* (Springer, Berlin, 2006).
[25] P. Busch, T. Heinonen, and P. Lahti, Phys. Rep. **452**, 155 (2007).
[26] T. Heinosaari and M. M. Wolf, J. Math. Phys. (N.Y.) **51**, 092201 (2010).
[27] M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. **113**, 160402 (2014).
[28] R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. **113**, 160403 (2014).
[29] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, Phys. Rev. Lett. **115**, 230402 (2015).
[30] H. Zhu, Sci. Rep. **5**, 14317 (2015).

[31] W. K. Wootters and B. D. Fields, Ann. Phys. (N.Y.) **191**, 363 (1989).
[32] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, Int. J. Quantum. Inform. **08**, 535 (2010).
[33] M. F. Pusey, J. Opt. Soc. Am. B **32**, A56 (2015).
[34] H. Zhu, Ph.D. thesis, National University of Singapore 2012, http://scholarbank.nus.edu.sg/bitstream/handle/10635/35247/ZhuHJthesis.pdf.
[35] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.070403 for additional proofs and discussions, which includes Refs. [36–48].
[36] I. Csiszár, Stud. Sci. Math. Hung. **2**, 299 (1967).
[37] C. H. Bennett, G. Brassard, C. Crepeau, and U. M. Maurer, IEEE Trans. Inf. Theory **41**, 1915 (1995).
[38] J. Håstad, R. Impagliazzo, L. A. Levin, and M. Luby, SIAM J. Comput. **28**, 1364 (1999).
[39] R. Zamir, IEEE Trans. Inf. Theory **44**, 1246 (1998).
[40] J. Combes, C. Ferrie, Z. Jiang, and C. M. Caves, Phys. Rev. A **89**, 052117 (2014).
[41] C. Ferrie, Phys. Rev. A **90**, 014101 (2014).
[42] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[43] D. Petz and C. Sudár, J. Math. Phys. (N.Y.) **37**, 2662 (1996).
[44] G. Zauner, Int. J. Quantum. Inform. **09**, 445 (2011).
[45] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. (N.Y.) **45**, 2171 (2004).
[46] A. J. Scott, J. Phys. A: Math. Gen. **39**, 13507 (2006).
[47] H. Zhu and B.-G. Englert, Phys. Rev. A **84**, 022327 (2011).
[48] D. M. Appleby, C. A. Fuchs, and H. Zhu, Quantum Inf. Comput. **15**, 61 (2015).
[49] M. Marciniak, A. Rutkowski, Z. Yin, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **115**, 170401 (2015).
[50] P. Skrzypczyk and D. Cavalcanti, Phys. Rev. A **92**, 022354 (2015).
[51] R. Gallego and L. Aolita, Phys. Rev. X **5**, 041008 (2015).