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Asymptotic Freedom in Hořava-Lifshitz Gravity

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We use the Wetterich equation for foliated spacetimes to study the renormalization group flow of projectable Hořava-Lifshitz gravity coupled to *n* Lifshitz scalars. Using novel results for anisotropic heat kernels, the matter-induced beta functions for the gravitational couplings are computed explicitly. The renormalization group flow exhibits an UV attractive anisotropic Gaussian fixed point where Newton's constant vanishes and the extra scalar mode decouples. This fixed point ensures that the theory is asymptotically free in the large–*n* expansion, indicating that projectable Hořava-Lifshitz gravity is perturbatively renormalizable. Notably, the fundamental fixed point action does not obey detailed balance.

Introduction.—Classical general relativity provides an excellent description for gravitational phenomena ranging from submillimeter up to astrophysical scales [1]. Despite these remarkable successes, a fundamental theory capturing the gravitational force on all length scales is still missing: quantizing the Einstein-Hilbert action along the same lines as QCD shows that the theory is perturbatively nonrenormalizable [2]. In terms of Wilson's modern formulation of renormalization [3], this feature is equivalent to the observation that the Gaussian fixed point (GFP), representing the free theory, does not act as an UV attractor for the renormalization group (RG) flow of Newton's constant G, a fact that is easily deduced from the negative mass dimension of G. This raises the question regarding whether or not there are other fixed points of the RG flow which may serve as a (nonperturbative) UV completion of gravity.

One proposal along these lines is the asymptotic safety program, initially advocated by Weinberg [4]. This scenario considers all action functionals built from the metric field and preserving (background) diffeomorphism invariance. By now, there is substantial evidence that the theory space spanned by these action functionals possesses a non-Gaussian fixed point (NGFP) with a finite number of relevant deformations [5]. Thus, it is conceivable that gravity constitutes a consistent and predictive quantum theory within the framework of nonperturbatively renormalizable quantum field theories.

Hořava-Lifshitz (HL) gravity [6,7] (see [8–11] for reviews) constitutes an alternative to this scenario. The construction performs an Arnowitt-Deser-Misner decomposition of the $D = d + 1$ dimensional metric, encoding the gravitational degrees of freedom in the lapse function N, a shift vector N_i, and a metric on the d-dimensional spatial slices σ_{ij}, thereby introducing a foliation structure on spacetime. As its key ingredient, HL gravity relaxes the symmetry requirements underlying asymptotic safety and considers action functionals invariant with respect to foliation preserving diffeomorphisms only.

The reduced symmetry group of HL gravity allows us to replace the relativistic flat-space scaling by a scaling relation that is anisotropic between space and time

$$t \to bt, \qquad x \to b^{1/z}x. \tag{1}$$

This modifies the standard dimensional analysis, and for $z > 1$, interactions including higher powers of spatial derivative terms become power-counting relevant or marginal, thereby improving the UV behavior of the theory. In particular, at criticality, $z = d$, Newton's constant is dimensionless so that the theory is power-counting renormalizable [7]. Thus, it is conceivable that HL gravity is perturbatively renormalizable (asymptotically free). In the language of the Wilsonian renormalization group, this conjecture entails the existence of an anisotropic Gaussian fixed point (AGFP) which acts as an UV attractor for Newton's constant at high energies. In [7], it was speculated that the AGFP may be located at the conformal point at which the extra scalar mode intrinsic of HL gravity decouples and, in addition, preserves detailed balance. Given that the AGFP may constitute a valid UV completion of gravity, which is supported by recent Monte Carlo simulations [12–15], and the considerable attention devoted to HL gravity, it is rather surprising that its properties remained largely mysterious and only partial results are available [16–21].

This situation clearly calls for the use of the RG to clarify the structure of HL gravity at high energies. A convenient way to perform such computations is the functional RG equation for the gravitational effective average action Γ_k [22]. The metric construction can readily be adapted to the case of projectable Hořava-Lifshitz (PHL) gravity [23], yielding the Wetterich equation governing the scale dependence of Γ_k

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr}[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \partial_t \mathcal{R}_k]. \tag{2}$$

Here, $t \equiv \ln(k/k_0)$ is the "RG time," $\Gamma_k^{(2)}$ is the Hessian of Γ_k, and $\mathcal{R}_k$ is a regulator term designed to suppress low energy modes. In the sequel, we will work with the Litim regulator [24], setting $\mathcal{R}_k(y) = (k^2 - y)\theta(k^2 - y)$.

In this Letter, we use the flow equation (2) to study the ultraviolet behavior of PHL gravity coupled to n aniso-tropic Lifshitz scalar fields in the large-n limit. In this limit, the beta functions are dominated by the contributions originating from integrating out the scalar fields. Since gravity couples universally to itself as well as to matter, the contribution of the graviton propagating in loops becomes negligible with respect to the contribution of the n matter fields and the beta functions become exact in the large-n limit. As we will show, this ansatz allows us to identify the AGFP underlying the perturbative renormalizability of HL gravity in $D = 3 + 1$ dimensions.

Setup.—Concretely, the effective average action of the system is given by projectable Hořava–Lifshitz gravity coupled to n Lifshitz scalars (LS)

$$\Gamma_k[N, N_i, \sigma, \phi] = \Gamma_k^{\text{HL}}[N, N_i, \sigma] + S^{\text{LS}}[N, N_i, \sigma, \phi]. \tag{3}$$

Here,

$$S^{\text{LS}} \equiv \frac{1}{2} \int dt d^d x N \sqrt{\sigma} \phi [\Delta_t + (\Delta_x)^z] \phi \tag{4}$$

is the action of the Lifshitz scalars with anisotropic scaling z in a curved background, $\Delta_t \equiv -(1/N\sqrt{\sigma})\partial_t N^{-1}\sqrt{\sigma}\partial_t$, and $\Delta_x \equiv -\sigma^{ij}D_i D_j$ is the positive definite covariant Laplacian on the spatial slice with D_i containing the d-dimensional Christoffel symbols constructed from $\sigma_{ij}(t, x)$. The gravitational part of Γ_k is given by the classical action for PHL gravity [7]

$$\Gamma_k^{\text{HL}} = \frac{1}{16\pi G_k} \int dt d^d x N \sqrt{\sigma}[K_{ij}K^{ij} - \lambda_k K^2 + V_k], \tag{5}$$

where all couplings $g_i(k)$ have been promoted to depend on the RG scale k. The potential $V_k[\sigma]$ contains all power-counting relevant and marginal operators constructed from the intrinsic curvature tensors on the spatial slices. Following [11], V_k for $d = 2$ has the form

$$V_k^{(d=2)} = g_0 + g_1 R + g_2 R^2, \tag{6}$$

while the case $d = 3$ includes all interactions with up to six spatial derivatives

$$V_k^{(d=3)} = g_0 + g_1 R + g_2 R^2 + g_3 R_{ij} R^{ij} - g_4 R \Delta_x R$$
$$- g_5 R_{ij} \Delta_x R^{ij} + g_6 R^3 + g_7 R R_{ij} R^{ij} + g_8 R_j^i R_k^j R_i^k. \tag{7}$$

Thus, the setup contains two wave-function renormalizations (G_k, λ_k) and three ($d = 2$) and eight ($d = 3$) running parameters in the potential, respectively. Our conventions for parametrizing V_k in terms of coupling constants follow the ones typically adopted in higher-derivative gravity and are tailored to exhibit the properties of potential fixed points. Note that we have not implemented detailed balance to simplify V_k.

Beta functions.—Now, we compute the beta functions, $\beta_{g_i} \equiv \partial_t g_i$, capturing the running of the gravitational coupling constants induced by integrating out the scalar fields. For this purpose, it is convenient to express the flow in terms of dimensionless couplings $\tilde{g}_i \equiv g_i k^{-[g_i]}$, where $[g_i]$ is the canonical mass dimension of g_i. In particular, the dimensionless Newton's constant is given by

$$g_k \equiv G_k k^{2\eta}, \qquad \eta \equiv \frac{d}{2z} - \frac{1}{2}, \tag{8}$$

so that G_k becomes dimensionless at criticality $d = z$. Actually, it is the shift of the canonical mass dimension induced by the anisotropy z which underlies the power-counting renormalizability of HL gravity.

The beta functions are obtained by substituting (3) into (2) and reading off the coefficients multiplying the extrinsic and intrinsic curvatures. In the case where the RG flow is driven by the scalar fields, the operator trace is given by $\text{Tr}W(\Box)$ with $W(y) = [y + R_k(y)]^{-1}\partial_t R_k(y)$ and $\Box \equiv \Delta_t + (\Delta_x)^z$ being the anisotropic Laplace operator.

The evaluation of the trace can be linked to the short-time expansion of the heat kernel $\text{Tr}e^{-s\Box}$ by reexpressing $W(z)$ through its Laplace transform. The expansion of the heat kernel can be found in a systematic way by the repeated use of the Baker-Hausdorff lemma combined with the off-diagonal heat-kernel techniques [25,26]. The result is given by

$$\text{Tr}e^{-s\Box} \simeq (4\pi)^{-(d+1)/2} s^{-(1+d/z)/2} \int dt d^d x N \sqrt{\sigma}$$
$$\times \left[\frac{s}{6}(e_1 K^2 + e_2 K_{ij}K^{ij}) + \sum_{n \geq 0} s^{n/z} b_n a_{2n} \right]. \tag{9}$$

Here, a_{2n} are the standard heat-kernel coefficients containing the intrinsic curvatures constructed from σ_{ij}, possibly subject to geometrical constraints arising from working on a low-dimensional manifold. Their values have, e.g., been given in [27,28]: $a_0 = 1$, $a_2 = R/6$, etc. The coefficients e_i and b_n depend on d, z and encode the corrections originating from the anisotropic differential operator. The ones in the extrinsic curvature sector are

$$e_1 = \frac{d - z + 3}{d + 2} \frac{\Gamma(\frac{d}{2z})}{z\Gamma(\frac{d}{2})}, \qquad e_2 = -\frac{d + 2z}{d + 2} \frac{\Gamma(\frac{d}{2z})}{z\Gamma(\frac{d}{2})}, \tag{10}$$

while the $b_n(d, z)$ for $0 \leq n \leq \lfloor d/2 \rfloor$ are given by

$$b_n = \frac{\Gamma(\frac{d-2n}{2z} + 1)}{\Gamma(\frac{d-2n}{2} + 1)}. \tag{11}$$

The coefficients $b_n(d, z)$ for $n > \lfloor d/2 \rfloor$ can be computed on a case by case basis. The ones relevant for the present computation are

$$b_2(2, 2) = 0, \qquad b_2(3, 3) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{6}\right),$$

$$b_3(3, 3) = -\frac{1}{2}. \tag{12}$$

For $z = 1$, these results reproduce the standard heat kernel written in terms of Arnowitt-Deser-Misner variables, while for $d = z = 2$, our formulas coincide with the special case considered in [29]. The systematic computation of the early-time expansion of the heat kernel of an anisotropic Laplace operator constitutes the main technical breakthrough of our work.

Based on the expansion (9), it is straightforward to obtain the desired beta functions by applying the Mellin transform techniques reviewed, e.g., in [30]. Defining $\phi_n \equiv 1/\Gamma(n) \int_0^1 dx\, x^{n-1}$, the scale dependence of the two (dimensionless) wave-function renormalizations is governed by

$$\beta_g = 2\eta g - \frac{2}{3}(4\pi)^{-(d-1)/2} \phi_\eta e_2 g^2,$$

$$\beta_\lambda = -\frac{2}{3}(4\pi)^{-(d-1)/2} \phi_\eta (e_1 + \lambda e_2) g, \tag{13}$$

while the beta functions for the cosmological constant $\tilde{g}_0$ and for $\tilde{g}_1$ read

$$\beta_{\tilde{g}_0} = -2\tilde{g}_0 + \frac{4g}{(4\pi)^{(d-1)/2}} \left(b_0 \phi_{n+1} - \frac{1}{6} e_2 \phi_\eta \tilde{g}_0 \right),$$

$$\beta_{\tilde{g}_1} = \left(\frac{2}{z} - 2\right) \tilde{g}_1 + \frac{2g}{3(4\pi)^{(d-1)/2}} (b_1 \phi_{n+1-1/z} - e_2 \phi_\eta \tilde{g}_1). \tag{14}$$

At criticality $d = z = 2$, the system is supplemented by

$$\beta_{\tilde{g}_2} = \frac{g}{4} \tilde{g}_2. \tag{15}$$

For $d = z = 3$, the additional beta functions read

$$\beta_{\tilde{g}_2} = -\frac{2}{3}\tilde{g}_2 + \frac{g}{5\pi}\left(\frac{1}{8\sqrt{\pi}}\frac{\Gamma(5/6)}{\Gamma(1/3)} + \tilde{g}_2\right),$$

$$\beta_{\tilde{g}_3} = -\frac{2}{3}\tilde{g}_3 + \frac{g}{5\pi}\left(\frac{1}{4\sqrt{\pi}}\frac{\Gamma(5/6)}{\Gamma(1/3)} + \tilde{g}_3\right),$$

$$\beta_{\tilde{g}_i} = \frac{g}{\pi}\left(\frac{1}{5}\tilde{g}_i - \frac{1}{2}c_i\right), \qquad i = \{4, 5, 6, 7, 8\}, \tag{16}$$

with the numerical constants

$$c_4 = \frac{1}{336}, \qquad c_5 = \frac{1}{840},$$

$$c_6 = -\frac{1}{560}, \qquad c_7 = \frac{1}{105}, \qquad c_8 = -\frac{1}{180}. \tag{17}$$

We, now, summarize the most important properties of these flow equations.

Isotropy: $z = 1$.—In the isotropic case, Eqs. (13) and (14) exhibit a fixed plane $\{\lambda^* = 1, \tilde{g}_1^* = -1\}$ which is invariant under RG transformations. This plane possesses a NGFP situated at

$$\tilde{g}_0^* = -\frac{12(d-1)}{(d+1)^2},$$

$$g^* = -\frac{3(d-1)}{2}(4\pi)^{(d-1)/2}\Gamma\left(\frac{d+1}{2}\right). \tag{18}$$

For $d = 3$, this fixed point coincides with the one found in the metric computations [30,31]. On the fixed plane, the two-derivative terms of Γ_k^{HL} combine into the $d + 1$-dimensional Einstein-Hilbert action. Thus, a flow starting on this subspace will not generate Lorentz violating interactions. This result provides first hand evidence that gravitational interactions preserving full diffeomorphism invariance span a subspace in the theory space underlying HL gravity which is closed under RG flows.

Criticality: $z = d$.—For $z > 1$, the NGFP (18) is shifted towards smaller values $\lambda^* < 1$, see Fig. 1. At criticality, $\eta = 0$, $e_2 = -de_1$ so that the fixed point is located at

$$\text{AGFP: } g^* = 0, \qquad \lambda^* = \frac{1}{d}. \tag{19}$$

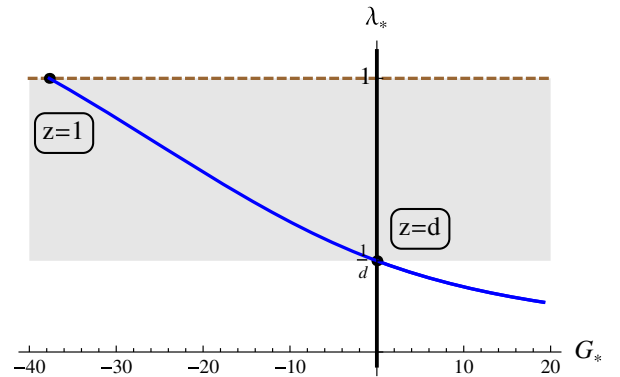In $d = 2$, the AGFP is completed by

FIG. 1 (color online). z dependence of the fixed point arising from (13). The dashed horizontal line represents the general relativistic case, while the thick vertical line represents the line of Gaussian fixed points. In the shaded area, the extra scalar mode of HL gravity is classically unstable [10].

$$\tilde{g}_1^* = 0, \qquad \tilde{g}_2^* = 0, \qquad \tilde{g}_3^* = 0, \qquad (20)$$

while in $d = 3$, the couplings determining V_* are fixed to

$$\tilde{g}_1^* = \tilde{g}_2^* = \tilde{g}_3^* = 0, \qquad \tilde{g}_i^* = \frac{5}{2} c_i. \qquad (21)$$

Newton's constant vanishes at the AGFP justifying the label "Gaussian." Moreover, the fixed point is precisely situated at the point where the extra scalar mode becomes nondynamical. It is this $z = d$ Lifshitz fixed point that underlies the conjectured perturbative renormalizability of HL gravity.

The scale dependence of the coupling constants at criticality $z = d = 3$ is shown in Fig. 2. Clearly, the
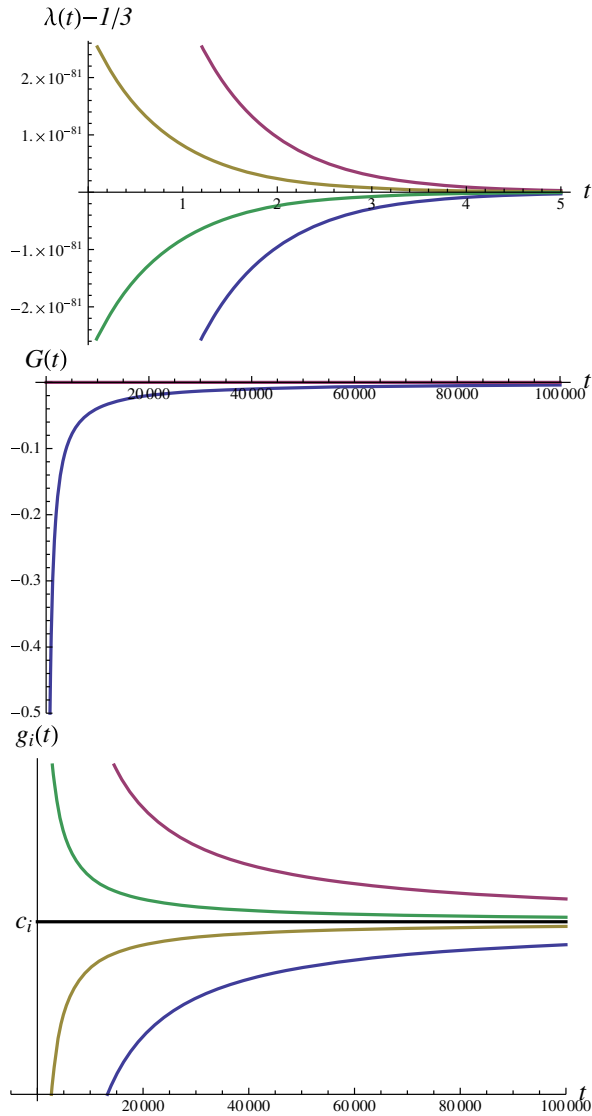

FIG. 2 (color online).   Scale-dependence of the wave-function renormalizations G_k and λ_k and the marginal couplings $\tilde{g}_i$ in $d = z = 3$. The AGFP (19), (21) acts as an UV attractor of the RG flow.

AGFP acts as an UV attractor of the RG flow, supporting the hypothesis that the theory is asymptotically free. Somewhat intriguingly, G_k flows to zero for $G_k < 0$ and the regime $G_k > 0$ is separated from the AGFP by a Landau pole. Based on the analogous computations within the asymptotic safety program [30,31], where the inclusion of gravity loops shifts the matter induced NGFP from $g^* < 0$ to $g^* > 0$, we expect that this feature will be cured by taking gravitational fluctuations into account. We hope to come back to this point in the future. For the time being, we will content ourselves with discussing two eminent consequences of our findings.

Flat space propagators at criticality.—It is illustrative to expand the $3 + 1$-dimensional fixed point action Γ_*^{HL} obtained from the AGFP around (Euclidean) flat space. Since the scalar mode is nondynamical at the AGFP, we focus on the transverse traceless (TT) fluctuations only (see also [21] for a more detailed discussion). Expanding $\sigma_{ij} = \delta_{ij} + \epsilon h_{ij}$ with $\epsilon = \sqrt{16\pi g_*}$ yields the two-point propagator

$$\mathcal{G}_*^{\mathrm{TT}} \propto (\omega^2 - \tilde{g}_5^* \vec{p}^6)^{-1}, \qquad (22)$$

where ω and $\vec{p}$ denote the energy and spatial momentum of the graviton, respectively. The higher order vertices vanish at the AGFP. Thus, the AGFP describes a noninteracting theory with a $z = 3$ anisotropic dispersion relation. At this stage, the following remarks are in order. The sign of $\tilde{g}_5^*$ indicates that the propagator is unstable in Euclidean space. Moreover, performing the same analysis in $d = 2$ shows that the propagator contains only an energy term, owed to the vanishing of $\tilde{g}_2^*$. We expect that the inclusion of gravity loops will resolve these issues. Our results, then, indicate the existence of an upper bound on the number of scalar fields that can consistently be coupled to PHL gravity without destabilizing the gravitational sector.

Detailed balance.—A conjecture already put forward in the seminal work [7] is that the potentials (6), (7) satisfy detailed balance, stating that V can be derived from a variational principle, $V \propto (\delta W[\sigma]/\delta\sigma_{ij})H_{ijkl}(\delta W[\sigma]/\delta\sigma_{kl})$ with H being the de Witt supermetric. For $z = d = 3$, this conjecture implies that the superpotential W generating the six-derivative terms is the Chern-Simons action, so that the potential is given by the square of the Cotton tensor, $V_{\mathrm{db}} \propto C_{ij}C^{ij}$, with $C^{ij} = \epsilon^{ikl}D_k(R_l^j - \frac{1}{4}R\delta_l^j)$. Rewriting V_{db} in terms of the basis (7) yields

$$V_{\mathrm{db}} \propto R_{ij}\Delta_x R^{ij} - \frac{3}{8}R\Delta_x R$$
$$+ 3R_j^i R_k^j R_i^k - \frac{5}{2}RR^{ij}R_{ij} + \frac{1}{2}R^3. \qquad (23)$$

Constructing V_* from the fixed point couplings (21) shows that the scalar induced AGFP does not obey detailed balance. This feature is actually independent of the

correction coefficients b_n and solely relies on the sixth order heat-kernel coefficients a_6.

Conclusions.—In this Letter, we used the functional renormalization group equation for PHL gravity [23] to study the matter-induced RG flow of the gravitational coupling constants in the large-n limit. As a key result, we identified the $z = d$ Lifshitz critical point (AGFP) underlying the perturbative renormalizability of the theory and showed that it acts as an UV attractor for the RG flow. The AGFP is situated at the conformal point where the extra degree of freedom intrinsic to HL gravity is non-dynamical and the underlying fixed point action does not preserve detailed balance. While the computation has been carried out in the framework of PHL gravity, we expect that the AGFP also acts as UV fixed point for the nonproject-able theory, the difference being that, owing to the enlarged theory space, the latter case will give rise to a larger number of relevant deformations. Clearly, it is interesting to complement the present computation by the inclusion of gravity loops, and we hope to come back to this point in the near future.

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