Noisy Metrology beyond the Standard Quantum Limit

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Parameter estimation is of fundamental importance in areas from atomic spectroscopy and atomic clocks to gravitational wave detection. Entangled probes provide a significant precision gain over classical strategies in the absence of noise. However, recent results seem to indicate that any small amount of realistic noise restricts the advantage of quantum strategies to an improvement by at most a multiplicative constant. Here, we identify a relevant scenario in which one can overcome this restriction and attain superclassical precision scaling even in the presence of uncorrelated noise. We show that precision can be significantly enhanced when the noise is concentrated along some spatial direction, while the Hamiltonian governing the evolution which depends on the parameter to be estimated can be engineered to point along a different direction. In the case of perpendicular orientation, we find superclassical scaling and identify a state which achieves the optimum.

Estimation of an unknown parameter is essential across disciplines from atomic spectroscopy and clocks [1–3] to gravitational wave detection [4]. It is typically achieved by letting a probe, e.g., light, interact with the system under investigation, picking up information about the desired parameter. As seen in Fig. 1, a metrology protocol can be understood in four main steps [5,6]: (i) preparation of the probe, (ii) interaction with the system, (iii) readout of the probe, and (iv) construction of an estimate of the unknown parameter from the results. Steps (i)–(iii) may be repeated many times before the final construction of the estimate.

The estimate uncertainty will depend on the available resources, here the probe size N and the total time T available for the experiment (other choices are possible [7]). By the central limit theorem, for N uncorrelated particles, the best uncertainty scales as $1/\sqrt{\nu N}$, where $\nu = T/t$ is the number of evolve-and-measure rounds. This bound is known as the shot-noise or standard quantum limit (SQL). By making use of quantum phenomena, a metrology protocol may surpass the SQL, reaching instead the limits imposed by the quantum uncertainty relations. For probes of noninteracting particles, the best possible scaling compatible with these relations is $1/(\sqrt{\nu}N)$, known as the Heisenberg limit.

Without noise, the Heisenberg limit can be attained using entangled input states, e.g., Greenberger-Horne-Zeilinger (GHZ) states for atomic spectroscopy [8]. In the presence of noise, however, the picture is much less clear, as the optimal strategy depends strongly on the model of decoherence considered. Nevertheless, the SQL has been significantly surpassed in experiments of optical magnetometry [9,10], which proved that some sources of noise can be effectively

counterbalanced [11,12]. However, unless one can keep improving the interaction strength or readout efficiency with probe size, e.g., increasing the optical depth with the atom number in atomic vapors, destructive effects of uncorrelated noise are bound to dominate at high N. In this regime of fixed noise (independent of the particle number), a number of no-go results exists, demonstrating that for most types of noisy channels acting independently on each probe particle, an infinitesimally small amount of decoherence limits any quantum improvement over the SQL to at most a constant factor [1,13–18]. In particular, any noisy evolution described by N full rank channels, i.e., channels for which no subspace of the probe state space is free of decoherence, belongs to this class [15–18]. This is arguably the most likely evolution in experiments, suggesting that the precision is always bound to scale classically for large enough N.

Here, for a frequency estimation task undergoing uncorrelated noise and for a configuration where the Hamiltonian and the noise have preferred spatial directions transversal to
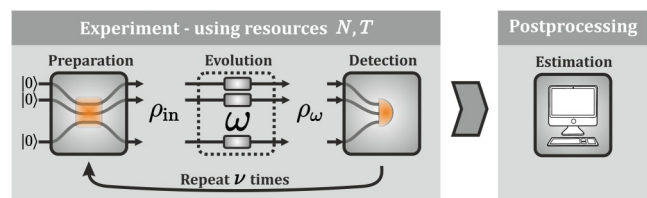

FIG. 1 (color online).   General metrology protocol where a known probe state evolves according to a physical evolution depending on an unknown parameter ω. After a sufficient amount of data are collected, an estimate for the parameter is constructed.

each other, we show that the restriction to SQL-like scaling can be surpassed. This is achieved by optimizing the duration t of the evolve-and-measure rounds. As N increases, the quantum channel describing the single-particle evolution varies due to this optimization. This allows circumventing the conditions of previous no-go results which were derived assuming a *fixed* form of the N single-particle channels [15–18]. Although t optimization has been considered previously and was not sufficient on its own [1,14], we demonstrate that in combination with directionality of the noise, it enables beating the SQL. The corresponding channel is full rank, yet we find that a GHZ-state input attains a precision scaling asymptotically as $1/N^{5/6}$. This is found numerically and confirmed by a semianalytical argument. We further demonstrate numerically that this scaling is optimal by identifying an upper bound on the precision which is saturated by the GHZ state. We also analyze deviations from perfectly directional noise. The asymptotic scaling is then again restricted to be SQL-like. However, for small deviations, we observe a much higher precision gain than for parallel noise. Note that here the noise is purely Markovian. Exploiting non-Markovianity can also lead to improved precision scaling, as previously shown [19].

Model for noisy frequency estimation.—We consider a Hamiltonian $H = (\omega/2)\sum_{k=1}^{N}\sigma_z^k$, where σ_z^k is a Pauli operator acting on the kth spin-1/2 particle (qubit), and ω is an unknown frequency to be estimated. To account for noise, we model the evolution by a master equation of Lindblad form

$$\frac{\partial\rho(t)}{\partial t} = \mathcal{H}(\rho) + \mathcal{L}(\rho). \tag{1}$$

Here, $\mathcal{H}(\rho) = -i[H, \rho]$ describes unitary evolution and the Liouvillian $\mathcal{L}(\rho)$ describes noise. We consider uncorrelated noise, such that $\mathcal{L} = \sum_k \mathcal{L}^k$, and for a single qubit we have

$$\mathcal{L}^k\rho = -\frac{\gamma}{2}[\rho - \alpha_x\sigma_x^k\rho\sigma_x^k - \alpha_y\sigma_y^k\rho\sigma_y^k - \alpha_z\sigma_z^k\rho\sigma_z^k], \tag{2}$$

where γ is the overall noise strength and $\alpha_{x,y,z} \geq 0$ with $\alpha_x + \alpha_y + \alpha_z = 1$. For $\alpha_z = 1$, this describes the situation considered by Huelga *et al.* [1], namely, dephasing along the direction of the unitary, while $\alpha_x = 1$ corresponds to dephasing transversal to the unitary. The latter model resembles the magnetometry setup of Ref. [10], in which the estimated magnetic field is directed perpendicularly to the dominant dephasing dictating the spin decoherence time and the spin relaxation is ignored. For $\alpha_x = \alpha_y = \alpha_z = 1/3$, we have an isotropic depolarizing channel.

The uncertainty $\delta\omega$ in the estimate of ω can be expressed in terms of the quantum Fisher information [20] (QFI) $\mathcal{F}(\rho_\omega)$ of the probe state ρ_ω after evolution. According to the quantum Cramér-Rao inequality [21],

$$\delta\omega\sqrt{T} \geq \frac{1}{\sqrt{\mathcal{F}(\rho_\omega)/t}}, \tag{3}$$

a bound achievable asymptotically for $\nu \gg 1$ [22,23]. One can explicitly solve the master equation (1), obtaining a map for the corresponding channel (see the Supplemental Material [24] for details). Applying the map to a given input, one obtains ρ_ω, and the QFI is computed through the diagonalization of this state. Alternatively, bounds on the QFI can be computed from the Kraus representation of the channel [14,15]. We write $\mathcal{F}_N = \mathcal{F}(\rho_\omega)$ for probes of N particles.

For a general input state, it is a difficult task to compute the QFI, as the size of ρ_ω grows exponentially with N. However, for a GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \tag{4}$$

the calculation simplifies dramatically (see Ref. [24]), and we are then able to optimize $\mathcal{F}_N/t$ (and hence $\delta\omega$) for large N. To obtain results valid for general inputs, we resort to bounds on the precision. A first indication that performance is better under transversal than parallel noise is given by the geometric classical simulation (CS) method [15] (see Fig. 2). We obtain tighter bounds by adapting the finite-N channel extension (CE) method of Kołodyński *et al.* [25] to allow for optimization over t (see Ref. [24]). As for previous asymptotic methods [14,15] without t optimization, we cannot *a priori* guarantee our bound to be saturable. However, for parallel noise ($\alpha_z = 1$), our bound is known to be tight in the asymptotic N limit both with [26] and without [25] t optimization. As we find below, the bound is also tight for transversal noise.

Parallel noise.—To understand the role of noise, we compare parallel and perpendicular noise, as well as noise which is directional but not fully concentrated on any of the axes. We start by parallel dephasing, which has been studied before [1].

For a single qubit, the QFI for an optimal input state is given by $\mathcal{F}_1^{\text{opt}} = e^{-2t\gamma}t^2$. In the noiseless case ($\gamma = 0$), as expected, the longer the probe state is allowed to evolve, the more information can be extracted about the parameter. In the noisy case, after a time $1/2\gamma$, the dephasing process wins over the unitary evolution and the extractable information is degraded. For a classical strategy using N independent qubits, this is the optimal time, and the SQL for parallel noise is

$$\delta\omega\sqrt{T} \geq \sqrt{\frac{2\gamma e}{N}}. \tag{5}$$

It has been proven [14] that when quantum strategies are allowed, the precision is instead bounded by

$$\delta\omega\sqrt{T} \geq \sqrt{\frac{1 + (e^{2\gamma t} - 1)N}{tN^2}} \geq \sqrt{\frac{c_z(\gamma, t)}{N}}. \tag{6}$$
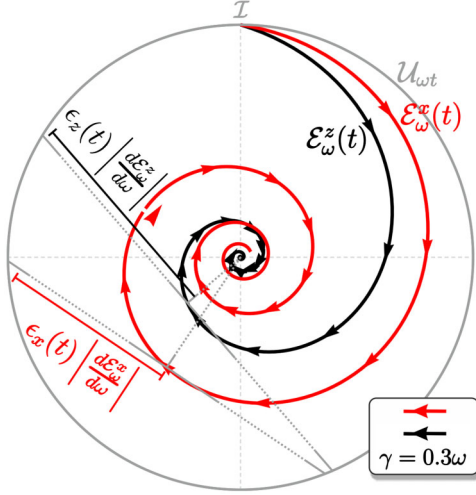
FIG. 2 (color online). Cut through the space of all valid qubit maps. The identity channel $\mathcal{I}[\varrho] = \varrho$ is at the top, and (extremal) unitary channels $\mathcal{U}_{\omega t}$ are at the circumference. One can think of phase ωt as increasing along the angular direction and decoherence γt along the radial direction. The spirals represent trajectories of channels for parallel $\mathcal{E}_\omega^z(t)$ and transversal $\mathcal{E}_\omega^x(t)$ noise, approaching the completely mixing channel at the disk center. Arrows are separated by fixed time steps. We see that $\mathcal{E}_\omega^x(t)$ loses coherence slower, being always closer to the boundary, and also has larger angular "speed" $|d\mathcal{E}_\omega^x/d\omega|$. The classical simulation method [15] provides a geometric bound on estimation precision based on the distance to the boundary: $\delta\omega\sqrt{T} \gtrsim \sqrt{[\epsilon(t)^2 t]/N}$. For asymptotic N, the bound is determined by the behavior for small t at the start of the spirals. We get $\epsilon_z(t)^2 = 2\gamma/t + O(1)$ and $\epsilon_x(t)^2 = (\gamma^2 t^2)/12 + O(t^4)$.

The tighter bound coincides with the finite-N CE method [25] for this channel and asymptotically reduces to the weaker bound. Optimizing, we find $t_z^{\mathrm{opt}} = w[N]/2\gamma$, where $w[N] = 1 + W[1 - N/eN]$ and $W[z]$ is the Lambert W function. Asymptotically, t_z^{opt} approaches zero as $1/\sqrt{N}$. Hence, the asymptotic scaling of Eq. (6) is dictated by the small-time behavior of $c_z(\gamma, t)$, which is fully determined by the CS method [15], i.e., by the geometrical location of the channel in the convex set of all quantum qubit maps (see Fig. 2):

$$c_z(\gamma, t) = \epsilon_z(t)^2 t = 2\gamma + 2\gamma^2 t + O(t^2). \quad (7)$$

Using Eqs. (5) and (6), we see that the asymptotic scaling is SQL-like and that quantum strategies provide only a constant factor improvement of $\sqrt{e}$, as found earlier [1,14]. Thus, for purely parallel dephasing, optimizing t leads only to a minor improvement of precision. The asymptotic scaling imposed by Eq. (6) is shown in Fig. 3(a).

The asymptotic SQL-like scaling is known to be saturable with spin squeezed states [25,26]. For comparison with transversal noise, we note that a GHZ input state gives no improvement over the SQL Eq. (5). A GHZ strategy is thus useless for parallel noise.

Transversal noise.—We now turn our attention to perfectly transversal noise. While we do not have an analytical expression, we can efficiently compute the finite-N CE bound and determine the optimal evolve-and-measure duration t_x^{opt} for each N numerically [see Fig. 3(a)].

Without optimization, the asymptotic scaling is still restricted to being SQL-like, as indeed it must be since transversal noise corresponds to a full rank channel [15–18]. However, unlike for parallel noise, the asymptotic quantum improvement factor is not bounded by the geometry of the set of qubit maps (see Fig. 2). In consequence, at short times, the asymptotic SQL factor $c_x(\gamma, t)$ from the CS method is

$$c_x(\gamma, t) = \epsilon_x(t)^2 t = \frac{\gamma^2}{12}t^3 + O(t^5), \quad (8)$$

and, as opposed to Eq. (7), $c_x(\gamma, t) \to 0$, as $t \to 0$.

Optimizing the finite-N CE bound over t for each N, we find that superclassical scaling is maintained (verified numerically up to $N = 10^8$ for γ/ω between 0.001 and 100) and follows an asymptotic behavior very well described by

$$\delta\omega\sqrt{T} \gtrsim \sqrt{\frac{c_x(\gamma)}{N^{5/3}}}. \quad (9)$$

For large N, $t_x^{\mathrm{opt}} \to 0$, since otherwise the restriction to SQL-like scaling applies. From our numerics, we obtain that $t_x^{\mathrm{opt}} = (3/\gamma N)^{1/3}$ and

$$c_x(\gamma) = \frac{3^{2/3}}{2}\gamma^{1/3}. \quad (10)$$

Thus, at the level of bounds, the t-optimized quantum strategies provide a scaling rather than a constant factor improvement over classical schemes for transversal noise. To confirm this scaling, as we have not yet demonstrated that the bound is tight, we examine a specific strategy based on a GHZ state.

For given N, we can analytically compute the QFI corresponding to a GHZ input (see Ref. [24]). The expression becomes cumbersome with larger N, but we can numerically determine t_x^{opt} and the minimum $\delta\omega$, for values of N up to several thousands. The result is shown in Fig. 3(a). Clearly, for the displayed values of γ/ω, the GHZ state is optimal. What is more, the GHZ strategy shows no sign of returning to SQL-like scaling for large N (verified for N up to 5000 and γ/ω between 0.001 and 10). Note that $t_x^{\mathrm{opt}} \to 0$ with increasing N. If we expand the QFI for GHZ inputs to first order in t, we find that $\mathcal{F}_N/t = N^2 t + O(t^2)$. As a semianalytical check, substituting the numerically obtained $t_x^{\mathrm{opt}} = (3/\gamma N)^{1/3}$, one recovers the scaling behavior of the CE bound. Based on this strong numerical and semianalytical evidence, we conjecture that the finite-N CE bound is indeed tight for sufficiently large N and that the asymptotic scaling is superclassical, as predicted by Eq. (9).
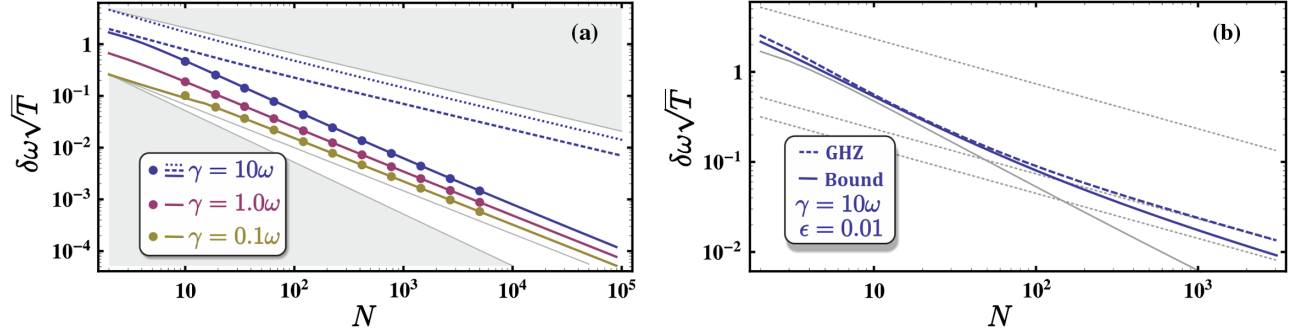
FIG. 3 (color online).   (a) Precision scaling for bounds and GHZ strategy. The bound for parallel noise with optimization of the single-round duration is shown (dotted line), as well as the bounds for transversal noise without (dashed line) and with (solid lines) optimization. Without optimization (taking t optimal for a single qubit), the asymptotic scaling is still SQL-like, but with optimization, superclassical scaling is maintained. In the latter case, the bound is saturated by the GHZ-state strategy (dots). The thin line shows $1/N^{5/6}$ scaling for reference, and the borders of the upper and lower shaded regions show SQL-like and Heisenberg scaling, respectively. (b) Bound (solid lines) and GHZ-state strategy (dashed line) for noise with a dominant transversal and a small parallel component $\alpha_x/\alpha_z = 99$. The scaling is superclassical for moderate N, while the z component determines the asymptotic scaling, which is hence SQL-like. The CE bound approaches $\sqrt{2\gamma\epsilon/N}$ (lower thin dotted line) and the GHZ-state $\sqrt{2\gamma\epsilon/N}$ (middle thin dotted line). The finite-N CE bound for parallel noise with the same strength is shown for reference (upper thin dotted line) as well as for transversal noise (thin solid line).

Intermediate noise.—In a realistic implementation of a setup with transversal noise, most likely, there will be deviations from perfect directionality. To account for such imperfections, in the following, we consider deviations along the z axis, such that $\alpha_x = 1 - \epsilon$ and $\alpha_z = \epsilon$ (we note, however, that similar conclusions hold for more general deviations, i.e., taking also $\alpha_y \neq 0$ [27]). Once there is some z noise, the asymptotic scaling must return to being SQL-like. If this were not the case, then by starting with a bit of z noise and adding x noise until α_x dominates, one could recover superclassical scaling. Hence, for large enough N, the precision would be improved by adding noise, which is clearly unphysical. Nevertheless, superlinear scaling can still persist over a large region of N.

Following this reasoning, the asymptotic bound must be of the form

$$\delta\omega\sqrt{T} \geq \sqrt{\frac{c_{xz}(\gamma, \epsilon)}{N}} \qquad (11)$$

with $c_{xz}(\gamma, \epsilon) \geq c_z(\epsilon\gamma, 0) = 2\epsilon\gamma$. In fact, we expect that equality must hold. This is because $t_{xz}^{\mathrm{opt}} \to 0$ for asymptotic N, but for very short times, noncommutativity effects seize to apply and our model is equivalent to a process in which unitary evolution and noise along each axis are applied sequentially in any order. Equality is confirmed by the numerical results in Fig. 3(b), where we see that the CE bound attains the asymptotic scaling $\sqrt{2\gamma\epsilon/N}$. The same argument applies to the GHZ-state strategy. In this case, we can see explicitly what happens at short times. Expanding up to second order in t, we have $\mathcal{F}_N/t = N^2 t - [(2N - 1)(1 - \epsilon) + 4N^2\epsilon]N\gamma t^2/2$. For large N, the expression reduces to the case of pure parallel noise with an effective noise strength of $\epsilon\gamma$, and asymptotically the

precision scales as $\sqrt{2\epsilon\gamma e/N}$. Thus, for $\epsilon > 0$, the GHZ state no longer saturates the bound. In Fig. 3(b), we explore the transition from superclassical to asymptotic SQL-like scaling for both the bound and the GHZ state. Intuitively, we expect that the transition point $N_{xz}(\gamma, \epsilon)$ to SQL-like behavior increases smoothly to infinity for $\epsilon \to 0$. Indeed, from Eqs. (9)–(11), we can estimate N_{xz} as the intersection of the superclassical and asymptotic SQL-like asymptotes $N_{xz}(\gamma, \epsilon) \sim 3/(8\gamma\epsilon^{3/2})$. Beyond $N_{xz}(\gamma, \epsilon)$, no significant gain is obtained by increasing initial entanglement; i.e., increasing the size of the entangled probe or using copies of disentangled probes of the same size leads to the same improvement in precision.

Finally, we remark that while t^{opt} tends to zero for large N, in practice, t may be bounded from below by a finite resolution t^{min}. In this case, one can optimize t for N's up to the point where $t^{\mathrm{opt}} = t^{\mathrm{min}}$ and obtain the $1/N^{5/6}$ scaling of Fig. 3(a) in this region. Beyond this point, one sets $t = t^{\mathrm{min}}$, and the precision becomes restricted to being SQL-like. Thus, the effect of $t^{\mathrm{min}} > 0$ for transversal noise, despite its different nature, has similar consequences to the effect of $\epsilon > 0$. Both lead to nonzero asymptotic constant factors $c_x(\gamma, t)$ and $c_{xz}(\gamma, \epsilon)$, respectively, which can approach zero when $t^{\mathrm{min}} \to 0$ or $\epsilon \to 0$. More generally, when both $\epsilon > 0$ and $t^{\mathrm{min}} > 0$, the ultimate precision is limited by the bound (6) with $c_z(\gamma\epsilon, t^{\mathrm{min}})$.

Conclusions.—Although recent results have shown that realistic noise prevents quantum metrology strategies from outperforming their classical counterparts by more than a constant factor, when the noise is independent of the probe size, it is nevertheless possible to observe scaling beyond the standard quantum limit in the presence of uncorrelated noise. Adapting the classical simulation and finite-N channel extension methods [15,25], we have shown that

optimizing the duration of evolve-and-measure rounds significantly enhances the precision. In particular, we have considered a unitary evolution with a well-defined direction and noise with a preferential direction transversal to the unitary. In this setting, we have analyzed a frequency estimation protocol and showed that the GHZ state achieves maximal precision in the presence of noise, providing a precision scaling of $1/N^{5/6}$. Furthermore, we have demonstrated that although the asymptotic scaling returns to being SQL-like when the noise deviates from being perfectly transversal, the constant factor improvement of the precision can be significantly enhanced for small deviations.

We believe that our work opens an avenue towards useful quantum metrology protocols in realistic, noisy settings. In particular, we expect our model to capture the essential features of anisotropic noise occurring, e.g., in quantum magnetometry [9,10]. Our results indicate a gap in scaling between the SQL and the attainable precision when the geometry of the noise is accounted for, and we hope to stimulate further research, as there is ample room for particular measurement schemes to achieve precisions inside this gap.

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[27] We find (analytically for the GHZ state and numerically for the finite-N CE bound) that when any two noise components are nonzero, asymptotic scaling returns to being SQL-like. In particular, having two transversal components present simultaneously is enough. However, as discussed in the main text, it is the parallel noise that, if present, dictates the asymptotic behavior and is less likely to be eliminated in realistic experimental scenarios.