Efficiency at maximum power of a discrete feedback ratchet

Javier Jarillo,[1,*] Tomás Tangarife,[2,†] and Francisco J. Cao[1,‡]
[1]Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, Avenida Complutense s/n, 28040 Madrid, Spain
[2]Laboratoire de Physique de l'ENS Lyon, CNRS UMR 5672 46, allée d'Italie, 69007 Lyon, France
(Received 1 August 2015; published 22 January 2016)

Efficiency at maximum power is found to be of the same order for a feedback ratchet and for its open-loop counterpart. However, feedback increases the output power up to a factor of five. This increase in output power is due to the increase in energy input and the effective entropy reduction obtained as a consequence of feedback. Optimal efficiency at maximum power is reached for time intervals between feedback actions two orders of magnitude smaller than the characteristic time of diffusion over a ratchet period length. The efficiency is computed consistently taking into account the correlation between the control actions. We consider a feedback control protocol for a discrete feedback flashing ratchet, which works against an external load. We maximize the power output optimizing the parameters of the ratchet, the controller, and the external load. The maximum power output is found to be upper bounded, so the attainable extracted power is limited. After, we compute an upper bound for the efficiency of this isothermal feedback ratchet at maximum power output. We make this computation applying recent developments of the thermodynamics of feedback-controlled systems, which give an equation to compute the entropy reduction due to information. However, this equation requires the computation of the probability of each of the possible sequences of the controller's actions. This computation becomes involved when the sequence of the controller's actions is non-Markovian, as is the case in most feedback ratchets. We here introduce an alternative procedure to set strong bounds to the entropy reduction in order to compute its value. In this procedure the bounds are evaluated in a quasi-Markovian limit, which emerge when there are big differences between the stationary probabilities of the system states. These big differences are an effect of the potential strength, which minimizes the departures from the Markovianicity of the sequence of control actions, allowing also to minimize the departures from the optimal performance of the system. This procedure can be applied to other feedback ratchets and, more in general, to other control systems.

DOI: 10.1103/PhysRevE.93.012142

I. INTRODUCTION

Brownian (or Feynman-Smoluchowski) ratchets are devices which rectify the thermal fluctuations of a stochastic system producing a net flux. In the last decades they have been widely studied, due to their theoretical importance in nonequilibrium statistical mechanics [1] and their applications in a wide range of fields, such as nanotechnology, condensed matter, or biology [1–3].

One type of Brownian ratchets are the ones known as flashing ratchets. They consist of a spatially periodic potential which can be switched on and off, acting on a collection of Brownian particles. The switching of the potential makes the system change its steady state, and a net flux of particles might be generated. Flashing ratchets can be subdivided into two different classes, depending on the nature of the protocol used for the on/off switching of the potential. On one hand, the "open-loop flashing ratchets" [1], where the potential is switched on and off with an open-loop protocol, which is a protocol that does not use information of the positions of the Brownian particles to take the switching decision, for example a periodic or a random switching of the potential. On the other hand, the "closed-loop" or "feedback flashing ratchets" [4,5] are those for which a subsystem, named the controller, gathers information about the state of the system, and it uses this information to decide when the potential is turned on or off.

Flux induced by feedback ratchets has been computed for different protocols [4–7], even taking into account the possible presence of time delays [8–10]. In addition, flux and power performance have been studied as a function of the amount of information used by the controller [11,12]. In this paper, we address the computation of the efficiency of a feedback flashing ratchet as an application of the recent results on the thermodynamics of feedback-controlled systems presented in Ref. [13]. This and other relevant recent results [14–23] have been applied to compute the efficiency of other feedback systems [13,21,24–26]. This recent burst of theoretical works on the thermodynamics of feedback systems has been accompanied by experimental realizations [27–29], and they have made it possible to start to solve several of the open questions in the field [30,31]. These developments are built over the proposal of the Maxwell demon [32], the important early contributions done by Szilard, Landauer, and Bennett, among others [33–35], and the concepts and formalism provided by the theory of information [36–38].

This paper is structured as follows. In Sec. II we describe the ratchet system and the proposed feedback protocol. Later, in Sec. III we compute the stationary flux of the system with the protocol and the maximal output power attainable. In Sec. IV we compute the entropy reduction due to the transfer of information, the average on and off times, and the efficiency. Then, in Sec. V we compare the results of maximum power and efficiency at maximum power for feedback and open-loop

*jjarillo@ucm.es
†Deceased.
‡francao@fis.ucm.es

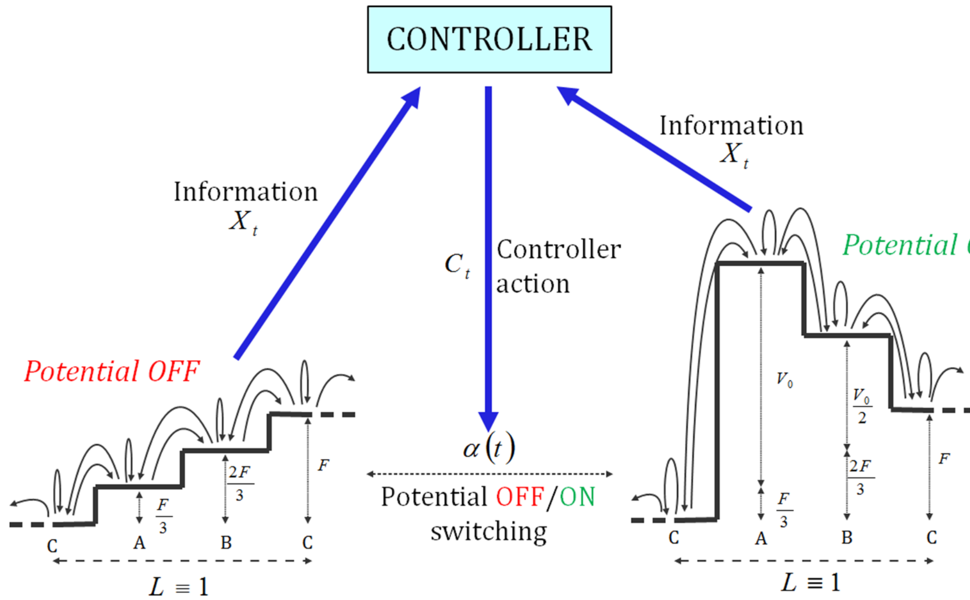

FIG. 1. Scheme of the dynamics of the particle in the ON and OFF potentials and of the information gathering and actuation of the controller. The system evolves with the ON or the OFF potential during a time interval $\Delta t = n\delta t$. After this time, the controller measures the position of the particle. If it is at a site A or B, the controller switches the potential ON, whereas if the particle is at site C, the potential is switched OFF. Then the particle evolves again during another time interval Δt until the next measurement.

protocols. Finally, in Sec. VI, we summarize and discuss the main results of the work.

II. THE MODEL

We consider the one dimensional motion of one Brownian particle subjected to an external constant force F. We also consider a discretization in both space and time: the distance between possible positions of the particles is δx, and the particle can jump to the nearest positions or stay still each time step δt. In this situation, the particle has a higher probability to move in the sense of F than in the opposite one, so we expect a flux in this sense. In order to force the particle to move against F, we introduce a potential, which is alternatively switched on and off.

In absence of the external force, the potential $\tilde{V}_{\rm ON}(x)$ is periodic in space with a period $L, \tilde{V}_{\rm ON}(x) = \tilde{V}_{\rm ON}(x+L)$, and in each period there are three discrete positions separated by a distance $l = \delta x = L/3$. We consider units of $L = 1$, and we refer to these three positions as A, B, and C. For each position the potential reads

$$\tilde{V}_{\rm ON}(x) = \begin{cases} V_0 & \text{if } x \in A, \\ \frac{V_0}{2} & \text{if } x \in B, \\ 0 & \text{if } x \in C. \end{cases} \quad (1)$$

The potential can be switched ON and OFF by a controller, introducing a temporal dependency,

$$\tilde{V}(x,t) = \alpha(t)\, \tilde{V}_{\rm ON}(x)\,, \quad (2)$$

$\alpha(t)$ being the parameter which controls the switching of the potential,

$$\alpha(t) = \begin{cases} 1 & \text{if at time } t \text{ the potential is ON,} \\ 0 & \text{if at time } t \text{ the potential is OFF.} \end{cases} \quad (3)$$

The potential $\tilde{V}(x,t)$ represents the potential energy of the particle as function of the position and time in absence of an external force. In the presence of this force F, the total potential will be given by

$$V_{\rm tot}(x,t) = \alpha(t)\, \tilde{V}_{\rm ON}(x) + F \cdot x. \quad (4)$$

See Fig. 1 for a representation of the resulting total potentials.

The probability of finding the particle at location x at time t evolves according to the master equation

$$p(x,t+\delta t) = P_{\delta t}(x|x)p(x,t) + P_{\delta t}(x|x-\delta x)p(x-\delta x,t) + P_{\delta t}(x|x+\delta x)p(x+\delta x,t), \quad (5)$$

where the transition probability between locations x and $x' = x \pm \delta x$ is [according to Kramers' method (see Ref. [39] and Appendix A) and using $l = \delta x = L/3$ and $L \equiv 1$]

$$P_{x\to x'}(\delta t) = P_{\delta t}(x'|x) = \frac{1-\exp\left(-18D\delta t\right)}{1+\exp\left[\frac{V_{\rm tot}(x',t)-V_{\rm tot}(x,t)}{k_B T}\right]}, \quad (6)$$

with D the diffusion coefficient, k_B the Boltzmann constant, and T the temperature.

If at time t the potential is fixed to either ON or OFF, the dynamics of the particle can be summarized as the dynamics of a system with just three states (A,B,C) with jumps between them. Denoting $p_t(I)$ the probability of finding the particle located at site $I \in \{A,B,C\}$ at time t, the evolution of $p_t(I)$ between consecutive time steps may be written as

$$\mathbf{p}_{t+\delta t} \equiv \begin{pmatrix} p_{t+\delta t}(A) \\ p_{t+\delta t}(B) \\ p_{t+\delta t}(C) \end{pmatrix} = \mathbf{M}_C \cdot \begin{pmatrix} p_t(A) \\ p_t(B) \\ p_t(C) \end{pmatrix} \equiv \mathbf{M}_C \cdot \mathbf{p}_t\,, \quad (7)$$

with $\mathcal{C} \in \{\text{ON,OFF}\}$. The transition matrices may be obtained by employing Eq. (6), and they are given by the expressions

$$\mathbf{M}_{\text{OFF}} = \mathbb{1} + [1 - \exp(-18\delta t)] \begin{pmatrix} -1 & \frac{b}{1+b} & \frac{1}{1+b} \\ \frac{1}{1+b} & -1 & \frac{b}{1+b} \\ \frac{b}{1+b} & \frac{1}{1+b} & -1 \end{pmatrix} \quad (8)$$

and

$$\mathbf{M}_{\text{ON}} = \mathbb{1} + [1 - \exp(-18\delta t)] \times \begin{pmatrix} -\frac{\sqrt{a}}{\sqrt{a}+b} - \frac{ab}{1+ab} & \frac{b}{\sqrt{a}+b} & \frac{1}{1+ab} \\ \frac{\sqrt{a}}{\sqrt{a}+b} & -1 & \frac{b}{\sqrt{a}+b} \\ \frac{ab}{1+ab} & \frac{\sqrt{a}}{\sqrt{a}+b} & -\frac{1}{1+ab} - \frac{b}{\sqrt{a}+b} \end{pmatrix}, \quad (9)$$

in units of $D/L^2 \equiv 1$ and $k_B T \equiv 1$, and where we have defined the quantities a and b as

$$a \equiv \exp(V_0), \quad b \equiv \exp\left(\frac{F}{3}\right). \quad (10)$$

From Eq. (7), the evolution of the system after n time steps while the potential remains ON or OFF is given (for $\mathcal{C} = \text{ON}$ or OFF) by

$$\mathbf{p}_{t+n\delta t} = (\mathbf{M}_{\mathcal{C}})^n \cdot \mathbf{p}_t, \quad (11)$$

and then the transition probability from state I to state I' in n time steps, $P_{\mathcal{C}}^{(n)}(I'|I)$, is given by

$$\begin{pmatrix} P_{\mathcal{C}}^{(n)}(A|A) & P_{\mathcal{C}}^{(n)}(A|B) & P_{\mathcal{C}}^{(n)}(A|C) \\ P_{\mathcal{C}}^{(n)}(B|A) & P_{\mathcal{C}}^{(n)}(B|B) & P_{\mathcal{C}}^{(n)}(B|C) \\ P_{\mathcal{C}}^{(n)}(C|A) & P_{\mathcal{C}}^{(n)}(C|B) & P_{\mathcal{C}}^{(n)}(C|C) \end{pmatrix} \equiv (\mathbf{M}_{\mathcal{C}})^n. \quad (12)$$

We propose the following protocol for the controller's actions. On one hand, if the controller measures the position of the particle at C at time t, it sets the potential OFF between t and the next controller's action at $t + n\delta t$, and thus the probability of the particle to go to the left is slightly higher than to go to the right. On the other hand, if the controller measures the particle at site A or B, it sets the potential ON, so that the probability of the particle going to the right is increased. This protocol is summarized in Fig. 1.

Let us explain the protocol in more detail. Assume at time t the particle is at site C, and the controller does a measure of its position. The potential is then turned to OFF, and it remains OFF during a time interval $\Delta t \equiv n\delta t$, where we have defined Δt as the time interval between two consecutive controller's actions. Then the probability to measure the particle at site I at time $t + \Delta t$ is, by definition, $P_{\text{OFF}}^{(n)}(I|C)$. Analogously, if the particle is initially located at site B, the controller switches the potential to ON, and the probability to find it at location I at time $t + \Delta t$ is given by $P_{\text{ON}}^{(n)}(I|B)$. Hence, the transition matrix with the controller's actions each Δt is given by

$$\mathbf{M}^{(n)} = \begin{pmatrix} P_{\text{ON}}^{(n)}(A|A) & P_{\text{ON}}^{(n)}(A|B) & P_{\text{OFF}}^{(n)}(A|C) \\ P_{\text{ON}}^{(n)}(B|A) & P_{\text{ON}}^{(n)}(B|B) & P_{\text{OFF}}^{(n)}(B|C) \\ P_{\text{ON}}^{(n)}(C|A) & P_{\text{ON}}^{(n)}(C|B) & P_{\text{OFF}}^{(n)}(C|C) \end{pmatrix}. \quad (13)$$

It is possible to obtain an analytical expression of this evolution matrix by computing the eigenvalues and eigenvectors of Eqs. (8) and (9), decomposing them to their diagonal form and then computing the n power of the matrices. We do not reproduce this result because its expression is very involved. However, in the limit $\Delta t \ll 1$ (i.e., in the limit of very frequent controller's actions), this evolution matrix is much simpler, and it reads

$$\mathbf{M}^{(n)} = \mathbb{1} + 18\Delta t \begin{pmatrix} -\frac{\sqrt{a}}{\sqrt{a}+b} - \frac{ab}{1+ab} & \frac{b}{\sqrt{a}+b} & \frac{1}{1+b} \\ \frac{\sqrt{a}}{\sqrt{a}+b} & -1 & \frac{b}{1+b} \\ \frac{ab}{1+ab} & \frac{\sqrt{a}}{\sqrt{a}+b} & -1 \end{pmatrix} + \mathcal{O}(\Delta t^2). \quad (14)$$

In most of the cases we consider through this work this first order approximation on Δt is sufficient. However, in some other cases it might be interesting to consider an extra term on this Taylor expansion series. The resulting expression is very involved, so we do not reproduce it. Nevertheless, in the limit that the potential height is much higher than the external force (more precisely, in the limit $a \gg b^2$), it is reduced to

$$\mathbf{M}^{(n)} \underset{a \gg b^2}{\simeq} \mathbb{1} + 18\Delta t \begin{pmatrix} -2 & 0 & \frac{1}{1+b} \\ 1 & -1 & \frac{b}{1+b} \\ 1 & 1 & -1 \end{pmatrix} + 162\Delta t^2 \begin{pmatrix} 4 & 0 & \frac{b^2-2b-2}{(1+b)^2} \\ -3 & 1 & \frac{-2b^2-2b+1}{(1+b)^2} \\ -1 & -1 & \frac{b^2+4b+1}{(1+b)^2} \end{pmatrix} + 162\delta t^2 \begin{pmatrix} -2 & 0 & \frac{-b^2+b+1}{(1+b)^2} \\ 2 & 0 & \frac{b^2+b+1}{(1+b)^2} \\ 0 & 0 & \frac{-2b}{(1+b)^2} \end{pmatrix} + \mathcal{O}(\Delta t^3) + \mathcal{O}(\delta t^3). \quad (15)$$

As we can see from this expression, the evolution depends both on the time interval between the controller's actions Δt and on the time discretization δt. In the continuous time limit, the terms proportional to δt and higher orders go to zero, and we get an evolution matrix that depends solely on the time interval between control actions Δt. (The continuous time limit is taken making $\delta t \to 0$ and $n \to \infty$ with fixed $\Delta t = n\delta t$.) In the next section we discuss why this limiting case corresponding to $a \gg b^2$ is physically relevant.

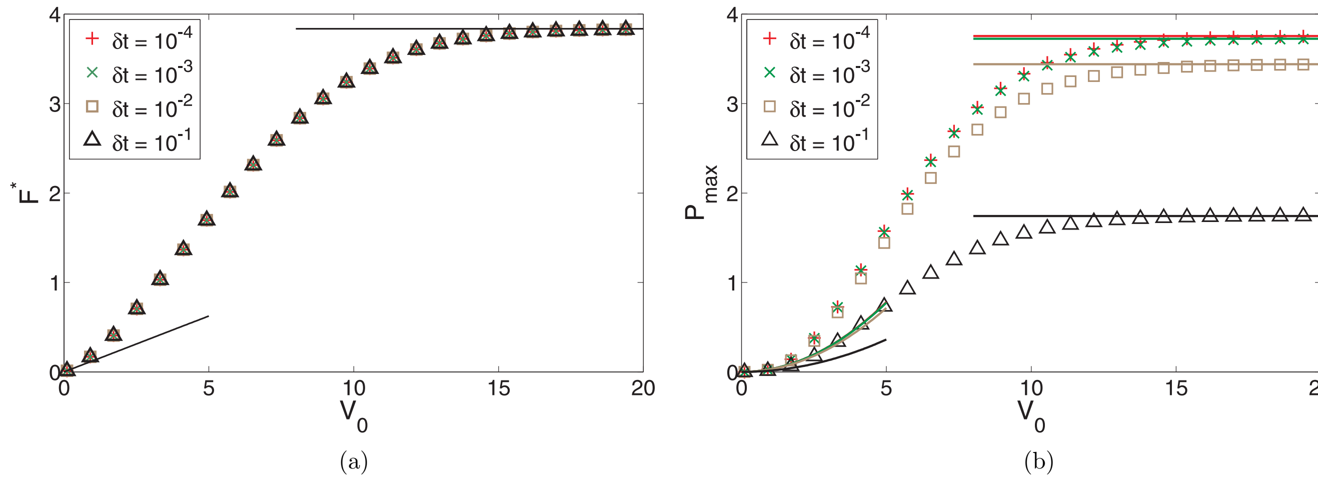

FIG. 2. (a) Force that gives the maximal output F^* and (b) maximal power output $P_{\rm max}$, both as functions of the potential height V_0 for different temporal discretizations δt for the case $n = 1$, i.e., when the time between the controller's actions Δt coincides with the discretization time $\delta t, \Delta t = n\delta t = \delta t$. Solid lines correspond to the analytical approximations for the limiting cases $V \ll 1$ and $V \gg 1$ [respectively, Eqs. (20) and (21)].

III. DYNAMICS: FLUX OF PARTICLES

The evolution equation of $p_t(I)$, with the controller's actions each Δt,

$$\mathbf{p}_{t+\Delta t} = \mathbf{M}^{(n)} \cdot \mathbf{p}_t, \tag{16}$$

defines a time-invariant Markov process. This process converges to a stationary state [40], described by the stationary distribution $\mathbf{p}_s^{(n)}$, which is given by the system of equations

$$\begin{aligned} \mathbf{p}_s^{(n)} &= \mathbf{M}^{(n)} \cdot \mathbf{p}_s^{(n)}, \\ p_s^{(n)}(A) + p_s^{(n)}(B) &+ p_s^{(n)}(C) = 1. \end{aligned} \tag{17}$$

In this stationary state, the average net flux of the particle to the right is

$$J_s^{(n)} = p_s^{(n)}(A) J_{\rm ON}^{(n)}(A) + p_s^{(n)}(B) J_{\rm ON}^{(n)}(B) + p_s^{(n)}(C) J_{\rm OFF}^{(n)}(C), \tag{18}$$

where $J_{\rm OFF}^{(n)}(C)$ is the mean flux during n time steps if the particle starts at site C and the potential is OFF, and analogously for $J_{\rm ON}^{(n)}(A)$ and $J_{\rm ON}^{(n)}(B)$ if the potential is ON and the particle starts from A or B, respectively. The computation of Eq. (18) is given in Appendix B.

A. Results for $n = 1$

For $n = 1$, the time between the controller's actions Δt coincides with the discretization time δt, $\Delta t = n\delta t = \delta t$, and the flux in the stationary state is

$$J_s^{(1)} = \frac{1 - \exp(-18\delta t)}{\delta t} \frac{(-\sqrt{a}b^3 - b^4 + ab + 1)a}{4a^2b^2 + 6a^{3/2}b^3 + 3ab^4 + 4a^2b + 5a^{3/2}b^2 + 2ab^3 + 2ab + 2\sqrt{a}b^2 + b^3 + 3a + 3\sqrt{a}b + b^2}. \tag{19}$$

From this expression, one can obtain the stopping force, which is defined as the force that cancels the flux. For big enough potential heights, $V_0 \gg k_B T$, the stopping force is $F_{\rm stop} \simeq \frac{3}{4} V_0$, whereas for small potential heights, $V_0 \ll k_B T$, it is $F_{\rm stop} \simeq \frac{1}{4} V_0$. In both limiting cases, the stopping force is proportional to V_0, as found in the continuous case [11,12]. We can also look for the force F^* that maximizes the mean power $P = J_s^{(1)} F$ at fixed V_0. The equation that gives $F^*(V_0)$ is very involved, so we do approximations for small and large enough values of V_0. Then we find that for $V_0 \ll 1$ (in units of $k_B T$),

$$\begin{aligned} F^* &\sim \frac{V_0}{8}, \quad J_{\rm max}^{(1)} \sim \frac{1 - \exp(-18\delta t)}{72\delta t} V_0, \\ P_{\rm max} &\sim \frac{1 - \exp(-18\delta t)}{576\delta t} V_0^2, \end{aligned} \tag{20}$$

whereas for $V_0 \gg 1$,

$$\begin{aligned} F^* &\sim 3.835, \quad J_{\rm max}^{(1)} \sim 0.05446 \frac{1 - \exp(-18\delta t)}{\delta t}, \\ P_{\rm max} &\sim 0.2088 \frac{1 - \exp(-18\delta t)}{\delta t}. \end{aligned} \tag{21}$$

The force F^* is represented in Fig. 2(a). We observe that the numerical value of this optimal force fits well with the approximations given in Eqs. (20) and (21), in the corresponding regimes of small enough and big enough V_0. Figure 2(b) plots the dependency on V_0 of the maximal power output. As for F^*, this maximal power output fits well with the approximations in Eqs. (20) and (21) in their corresponding regimes. In particular, it is important to remark the existence of an upper bound of the power output, which depends on the discretization time δt but for small enough δt converges

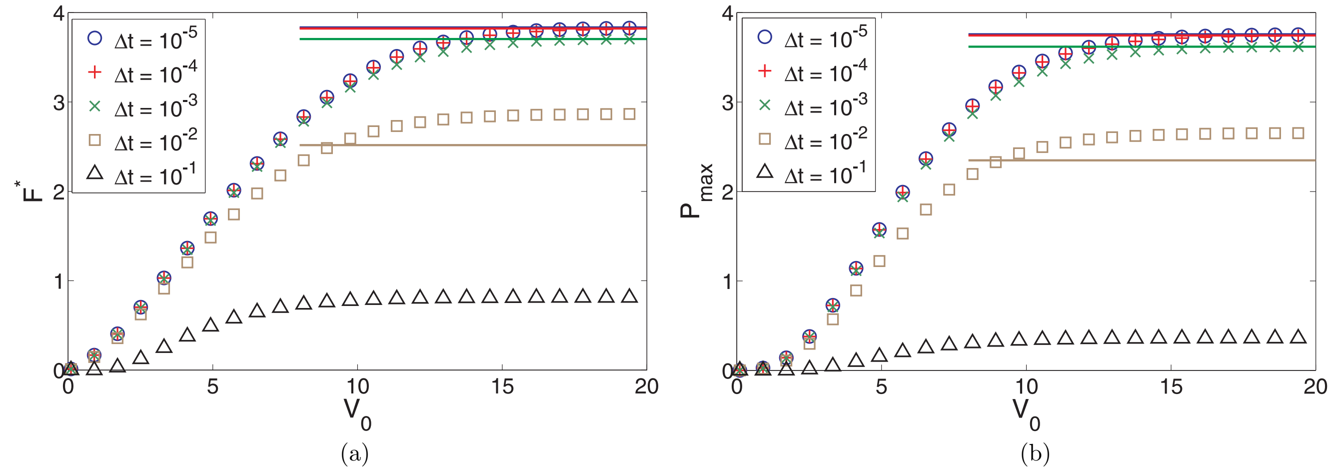

FIG. 3. (a) Force that gives the maximal output F^* and (b) maximal power output $P_{\max}$, both as functions of the potential height V_0 for different measurement times Δt. In both figures, we have taken values of $\delta t \ll \Delta t$ (exactly, $\delta t = \Delta t/100$), so the discretization effects are negligible. (The limit $\delta t \to 0$ and $n \to \infty$ with fixed $\Delta t = n\delta t$ corresponds to the continuous time limit.) In the limit $V_0 \to \infty$ the maximal power output reaches a constant value which depends on Δt (see Sec. III); i.e., the particle cannot extract an arbitrary large power with this protocol. Moreover, the power output reaches its maximum values for $V_0 \gtrsim 10$ whereas the force that maximizes this power output is $F \lesssim 3.8$. Hence, the limit $a \gg b^3$ is justified when the system works at maximum power output. Solid lines represent the analytical expressions obtained for F^* and $P_{\max}$ in the limit $a \gg b^3$ (see Sec. III). For big enough potential heights, they work well for small enough Δt, whereas for $\Delta t \gtrsim 10^{-2}$ the differences between the numerical results and the analytical expressions become greater. The analytical limit for $\Delta t = 10^{-1}$ is not represented since it differs greatly from the numerical result.

asymptotically to a maximum value. Hence, we can conclude that the particle is not able to extract an arbitrarily large power with this control protocol.

B. Results for $n > 1$

For $n > 1$, the time between the controller's actions Δt is greater than the discretization time δt ($\Delta t = n\delta t > \delta t$). In this case, we also start by computing the different contributions to the average net flux in Eq. (18). The term $J_{\mathrm{OFF}}^{(n)}(C)$ is independent on n, and can be computed explicitly (see Appendix B),

$$J_{\mathrm{OFF}}^{(n)}(C) = \frac{1-\exp(-18\delta t)}{3\delta t}\,\frac{1-b}{1+b}. \tag{22}$$

Nevertheless, the computation of $J_{\mathrm{ON}}^{(n)}(A)$ and $J_{\mathrm{ON}}^{(n)}(B)$ is more complicated, and in general it has to be done numerically, using, for example, the method described in Appendix B. Nevertheless, one can obtain an analytic expression for a limiting case. We see in Fig. 3 that for big enough values of V_0 the maximal power output reaches its maximum value and the force that maximizes the power does not depend on V_0. In this regime, the optimal force F^* is 3.8, while $V_0 \gg 3.8$. Then, in order to maximize the power output the value of the potential height should be greater than the force, and due to the properties of the exponential function, this implies that a should be some orders greater than b [defined in Eq. (10)]. Hence, the approximation $a \gg b^3$ should work very well close to the maximum power output state of our system. In this limiting case we achieve simple expressions for the flux terms of Eq. (18),

$$\begin{aligned} J_{\mathrm{ON}}^{(n)}(A) &\underset{a\gg b^3}{\simeq} -54\delta t + 54\Delta t + \mathcal{O}(\Delta t^2), \\ J_{\mathrm{ON}}^{(n)}(B) &\underset{a\gg b^3}{\simeq} 6 - 54\Delta t + \mathcal{O}(\Delta t^2). \end{aligned} \tag{23}$$

Then, replacing Eqs. (22) and (23) and the solution of Eq. (17) in Eq. (18), we achieve the expression for the stationary flux. However, if one does the approximation made in Eq. (14), i.e., keeping just the leading order terms in Δt in the evolution matrix, one does not obtain the dependency on Δt of the flux. One must take into account an extra order [Eq. (15)] in order to obtain

$$\begin{aligned} J_s^{(n)} \underset{a\gg b^3}{\simeq} &\frac{9}{2(1+b)} - \frac{81}{2}\frac{b(b^2+3b+1)}{(1+b)^3}\Delta t \\ &- \frac{81}{2}\frac{1+b-2b^2-b^3}{(1+b)^3}\delta t + \mathcal{O}(\Delta t^2). \end{aligned} \tag{24}$$

The third term of this expression, which is proportional to δt, is due to the discretization of the time coordinate and it would vanish in the continuous time limit $\delta t \to 0$ together with terms of higher powers of δt. (The continuous time limit is taken making $\delta t \to 0$ and $n \to \infty$ with fixed $\Delta t = n\delta t$.) Using this expression, we obtain that in the limit $a \gg b^3$, the force that maximizes the power output and this maximal power output read

$$\begin{aligned} F^* &\sim 3.835 - 133.1(\Delta t - \delta t) + \mathcal{O}(\Delta t^2), \\ P_{\max} &\sim 3.759 - 142.2\Delta t + 108.4\delta t + +\mathcal{O}(\Delta t^2). \end{aligned} \tag{25}$$

In Fig. 3 the maximal power output and its corresponding force are represented as functions of the potential height V_0 for different measurement times Δt in the case $\delta t \ll \Delta t$ (negligible discretization effect). From Fig. 3 it can be seen that, in the limit $V_0 \gg 1$, the force needed to maximize the power output is always much less than the potential height, so the approximation $a \gg b^3$ suits well the maximum power configuration. We can also note that the maximal power output reaches a constant maximum value (which depends on Δt) for big enough V_0; i.e., it is not possible to extract an arbitrary quantity of power from this system.

IV. THERMODYNAMICS: ENTROPY REDUCTION AND EFFICIENCY

In feedback-controlled systems the controller uses information on the state of the system to operate on the system. This information effectively implies an additional specification of the macrostate and thus a reduction of the microstates compatible with the macrostate. This implies an effective reduction of the entropy of the system [13], which has implication on the attainable power output and gives an effective free energy contribution that has to be taken into account to consistently study the thermodynamics of feedback-controlled systems, in particular in the computation of their efficiency. In fact, the main difficulty in the study of the thermodynamics of feedback-controlled systems is the computation of the entropy reduction due to the measurements and actions of the controller. We use here the method developed in Ref. [13] to compute the entropy reduction when successive controller's actions are correlated. We represent the successive controller's actions by a stochastic process $\mathcal{C}_t$, with $\mathcal{C}_t = 0$ if the potential is set OFF (i.e., the particle is measured at C) at time t and $\mathcal{C}_t = 1$ if the potential is set ON (i.e., the particle is measured at A or B).

A. Entropy reduction in feedback-controlled systems

For frequent feedback control the correlation between the controller's action diminishes the entropy reduction. In Ref. [13] it was shown that in these cases the entropy reduction depends on the whole history of the controller's actions and on the state of the system at the measurements times. Then the entropy reduction due to N consecutive actions of the controller, happening at times $t_1,t_2,\dots,t_N$, is given by

$$T\Delta S_{\text{info}} = -H(\mathcal{C}_{t_N},\dots,\mathcal{C}_{t_1}) + \sum_{k=1}^{N} H(\mathcal{C}_{t_k}|\mathcal{C}_{t_{k-1}},\dots,\mathcal{C}_{t_1},X_{t_k}), \tag{26}$$

where X_t denotes the state of the system at time t ($X_t \in \{A, B, C\}$), $H(\mathcal{C}_{t_N},\dots,\mathcal{C}_{t_1})$ is the joint entropy of the variables $\{\mathcal{C}_{t_N},\dots,\mathcal{C}_{t_1}\}$, and $H(\mathcal{C}_{t_k}|\mathcal{C}_{t_{k-1}},\dots,\mathcal{C}_{t_1},X_{t_k})$ is the conditional entropy as usually defined in information theory, $H(Y|X) = \sum_{x\in X} p(x)H(Y|X=x) = -\sum_{x\in X,y\in Y} p(x,y)\ln p(y|x)$ (see Ref. [36]), but with the logarithm in base e. A detailed discussion of Eq. (26) and its consequences is provided in Ref. [13].

In the case under consideration, the controller is said to be deterministic, as the controller's action $\mathcal{C}_{t_k}$ is perfectly determined by the state of the system X_{t_k} and the past control actions $(\mathcal{C}_{t_{k-1}},\dots,\mathcal{C}_{t_1})$. Consequently, the second term on the right-hand side of Eq. (26) is exactly zero. Thus, the entropy reduction is then just given by minus the joint entropy,

$$\begin{aligned} T\Delta S_{\text{info}} &= -H(\mathcal{C}_{t_N},\dots,\mathcal{C}_{t_1}) \\ &= \sum_{c_1,\dots,c_N} p(\mathcal{C}_{t_N}=c_N,\dots,\mathcal{C}_{t_1}=c_1) \\ &\quad\times \ln p(\mathcal{C}_{t_N}=c_N,\dots,\mathcal{C}_{t_1}=c_1). \end{aligned} \tag{27}$$

In information theory, this is the Shannon entropy of the sequence of the controller's actions (the difference is simply a factor ln 2 for the change of base of the logarithm).

The explicit computation of Eq. (27) is still a complicated issue, as it involves the joint probability $p(\mathcal{C}_{t_N}=c_N,\dots,\mathcal{C}_{t_1}=c_1)$. Starting from an arbitrary initial distribution, a large amount of simulations would be needed, so in practice it is almost unrealizable. However, when the system is in the stationary state it is possible to obtain some bounds, as we show in the following.

B. Entropy reduction rate of the system in the stationary state

We denote by $\tilde{p}_t(c)$ the probability to have at time t the controller's action $\mathcal{C}_t = c$, with $c = 0$or 1. These probabilities can be computed directly from the probability distribution of the particle location,

$$\tilde{p}_t(0) = p_t(C), \quad \tilde{p}_t(1) = p_t(A) + p_t(B). \tag{28}$$

where we have used that in our protocol the potential is turned to OFF when the particle is at site C and to ON when it is elsewhere.

The transition probability between two controller's actions c at time t and c' at time $t+\Delta t$ is denoted by $\tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=c'|\mathcal{C}_t=c)$. If the action at time t is $c=0$, the particle must be at site C and the potential must be OFF, so the transition probabilities from $c=0$ are given by

$$\begin{aligned} \tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=0|\mathcal{C}_t=0) &= P^{(n)}_{\text{OFF}}(C|C), \\ \tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=1|\mathcal{C}_t=0) &= P^{(n)}_{\text{OFF}}(A|C) + P^{(n)}_{\text{OFF}}(B|C). \end{aligned} \tag{29}$$

If the action at time t is $c=1$, the particle might be located at A or B, and the potential has to be ON. Then the probability to go to C (or to have the action $c'=0$) at time $t+\Delta t$ involves the conditional probability to be at A or B knowing that $c=1$, and the transition probabilities from A or B to C with the potential ON,

$$\begin{aligned} \tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=0|\mathcal{C}_t=1) &= \frac{p_t(A)}{p_t(A)+p_t(B)}P^{(n)}_{\text{ON}}(C|A) \\ &\quad+ \frac{p_t(B)}{p_t(A)+p_t(B)}P^{(n)}_{\text{ON}}(C|B). \end{aligned} \tag{30}$$

In the same way, the transition probability from $c=1$ to $c'=1$ is

$$\begin{aligned} &\tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=1|\mathcal{C}_t=1) \\ &= \frac{p_t(A)}{p_t(A)+p_t(B)}\big[P^{(n)}_{\text{ON}}(A|A)+P^{(n)}_{\text{ON}}(B|A)\big] \\ &\quad+ \frac{p_t(B)}{p_t(A)+p_t(B)}\big[P^{(n)}_{\text{ON}}(A|B)+P^{(n)}_{\text{ON}}(B|B)\big]. \end{aligned} \tag{31}$$

Then we can express the evolution of the probability distribution of the controller's action as

$$\tilde{\mathbf{p}}^{(n)}_{t+\Delta t} = \tilde{\mathbf{M}}^{(n)}_t \cdot \tilde{\mathbf{p}}^{(n)}_t, \tag{32}$$

where $\tilde{\mathbf{p}}^{(n)}_t$ is the vector containing the $\tilde{p}^{(n)}_t(c)$ [Eq. (28)], and $\tilde{\mathbf{M}}^{(n)}_t$ is the 2×2 matrix with elements $\tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=c'|\mathcal{C}_t=c)$ [Eqs. (29)–(31)]. The transition matrix $\tilde{\mathbf{M}}^{(n)}_t$ depends on time through the location probabilities $p_t(I)$, and as a consequence the evolution defined by Eq. (32) is not time invariant. However, in the stationary state of the system, the location probabilities $p_t(I)$ do not depend on time anymore, and therefore the transition probabilities $\tilde{P}^{(n)}(\mathcal{C}_{t+\Delta t}=c'|\mathcal{C}_t=c)$ do not depend on time either. We denote by $\tilde{P}_s^{(n)}(c'|c)$ the transition probabilities when the system is in the stationary state. The controller's actions then satisfy

$$\tilde{\mathbf{p}}_s^{(n)} = \tilde{\mathbf{M}}_s^{(n)} \cdot \tilde{\mathbf{p}}_s^{(n)}, \tag{33}$$

where $\tilde{\mathbf{p}}_s^{(n)}$ is the vector containing the $\tilde{p}_s^{(n)}(c)$ and $\tilde{\mathbf{M}}_s^{(n)}$ is the 2×2 matrix with elements $\tilde{P}_s^{(n)}(c'|c)$ which represent the time evolution of the system in the stationary state. This matrix can be expressed in frequent control $\Delta t \ll 1$ limiting case as

$$\tilde{\mathbf{M}}_s^{(n)} = \mathbb{1} + 18\Delta t \begin{pmatrix} -1 & \chi \\ 1 & -\chi \end{pmatrix} + \mathcal{O}(\Delta t^2), \tag{34}$$

where we have defined the parameter χ as

$$\chi \equiv \frac{a(1+b)(2a^{3/2}b + 3b^3\sqrt{a} + b^4 + 4ab^2 + \sqrt{a} + b)}{(\sqrt{a}+b)(2a^2b^2 + 4a^{3/2}b^3 + 2ab^4 + 2a^2b + 3a^{3/2}b^2 + ab^3 + ab + 2\sqrt{a}b^2 + b^3 + 2a + 3b\sqrt{a} + b^2)}. \tag{35}$$

From Eq. (34) it is easy to prove that the zero order approximation on Δt for the stationary state is

$$\tilde{\mathbf{p}}_s^{(n)} = \frac{1}{\chi+1}\begin{pmatrix} \chi \\ 1 \end{pmatrix} + \mathcal{O}(\Delta t). \tag{36}$$

This frequent control limit $\Delta t \ll 1$ is taken after the continuous time limit ($\delta t - > 0$ and $n \to \infty$ with fixed $\Delta t = n\delta t$); therefore, $\delta t \ll \Delta t \ll 1$.

Unlike X_t (the position of the particle at time t), $\mathcal{C}_t$ is not a time-invariant Markov process, since the probability to go to the state $\mathcal{C}_{t+\Delta t}$ does not depend solely on the previous state $\mathcal{C}_t$. For example, provided the potential at time t is set ON, we cannot know the probability of remaining ON in the next time step, since we do not know if the particle was at site A or B at time t. The evolution matrix in Eq. (34) is solely valid in the stationary regime, when the probability distribution of the particle's position is fixed. Hence, the joint entropy in Eq. (27) cannot be computed using general results about time-invariant Markov processes [36].

Assuming that the system is in the stationary state at t_0, the reduction of entropy after N actions of the controller, for N large, is given by

$$T\Delta S_{\text{info}}^{(n)} = -H\left(\mathcal{C}_{t_0+(N-1)\Delta t},\ldots,\mathcal{C}_{t_0}\right) \equiv -N\bar{H}(\mathcal{C}), \tag{37}$$

with $\bar{H}(\mathcal{C})$ the entropy reduction rate. Since the process $\mathcal{C}_t$ is not a Markov chain, this rate cannot be reduced to the conditional entropy $H(\mathcal{C}_{t_0+\Delta t}|\mathcal{C}_{t_0})$. That is the reason why, instead of computing the exact value of the entropy reduction rate, we set bounds to it. Although $\mathcal{C}_t$ is not a Markov chain, it is a deterministic function of the position X_t, and X_t is a Markov chain. Hence, the entropy reduction rate has upper and lower bounds (see Appendix C and Ref. [36] for details),

$$\begin{aligned} &H\left(\mathcal{C}_{t_0+N'\Delta t} \mid \mathcal{C}_{t_0+(N'-1)\Delta t},\ldots,\mathcal{C}_{t_0},X_{t_0}\right) \\ &\quad \leqslant \bar{H}(\mathcal{C}) \leqslant H\left(\mathcal{C}_{t_0+N'\Delta t} \mid \mathcal{C}_{t_0+(N'-1)\Delta t},\ldots,\mathcal{C}_{t_0}\right), \end{aligned} \tag{38}$$

and, in particular,

$$\bar{H}_{\text{low}} \leqslant \bar{H}(\mathcal{C}) \leqslant \bar{H}_{\text{up}}, \tag{39}$$

where we have defined $\bar{H}_{\text{low}}$ and $\bar{H}_{\text{up}}$ as

$$\bar{H}_{\text{low}} \equiv H\left(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0},X_{t_0}\right), \quad \bar{H}_{\text{up}} \equiv H\left(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0}\right). \tag{40}$$

It is important to note that the upper bound $\bar{H}_{\text{up}}$ coincides with the expression of the entropy reduction rate for a Markovian sequence. Thus, the difference $\bar{H}_{\text{up}} - \bar{H}_{\text{low}}$ will give an estimate of the departure from the estimation obtained with the Markovian expression $\bar{H}_{\text{up}}$.

Using both Eq. (39) and the expression of the conditional Shannon entropy,

$$H(I|J) = -\sum_{j\in J} P(j) \sum_{i\in I} P(i|j)\ln P(i|j), \tag{41}$$

we are going to compute these upper and lower bounds for the entropy reduction rate. On one hand, the upper bound of this entropy reduction rate (the expression for a Markovian sequence) reads

$$\begin{aligned} \bar{H}_{\text{up}} = &-18\frac{\chi}{\chi+1}[4\ln 3 + 2\ln 2 - 2 + \ln\chi \\ &+ 2\ln(\Delta t)]\Delta t + \mathcal{O}(\Delta t^2). \end{aligned} \tag{42}$$

On the other hand, in the computation of the lower bound one must take into account that $H(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0},X_{t_0}) = H(\mathcal{C}_{t_0+\Delta t} \mid X_{t_0})$, due to the fact that the evolution is fully determined by the position and not by the ON-OFF state. The resulting expression is very involved and we are not going to display it. We just comment that for small Δt (frequent control) it behaves as the upper bound, with terms proportional to Δt and $\Delta t \ln \Delta t$, when the second order terms in Δt are negligible. Instead of writing the expression of $\bar{H}_{\text{low}}$, we show the difference between both bounds, $\bar{H}_{\text{up}} - \bar{H}_{\text{low}}$, when the probability of the particle being at A is much smaller than the probability of being at B, $p_s^{(n)}(A) \ll p_s^{(n)}(B)$. Although this approximation might seem arbitrary, it suits well the maximum power output configuration. For example, for a force near 3.8 and a potential height much greater than 3.8, the set of values which maximize the power output (see Fig. 3) gives $p_s^{(n)}(A)/p_s^{(n)}(B) \simeq 0.12$. Using the approximation $p_s^{(n)}(A)/p_s^{(n)}(B) \ll 1$, we obtain

$$\bar{H}_{\text{up}} - \bar{H}_{\text{low}} \underset{p_s^{(n)}(A)\ll p_s^{(n)}(B)}{\simeq} -p_s^{(n)}(A)\sum_{c=0}^{1}\left[P^{(n)}(c|A)\ln\frac{P^{(n)}(c|B)}{P^{(n)}(c|A)} + P^{(n)}(c|A) - P^{(n)}(c|B)\right] + \mathcal{O}\left\{\left[\frac{p_s^{(n)}(A)}{p_s^{(n)}(B)}\right]^2\right\}, \tag{43}$$

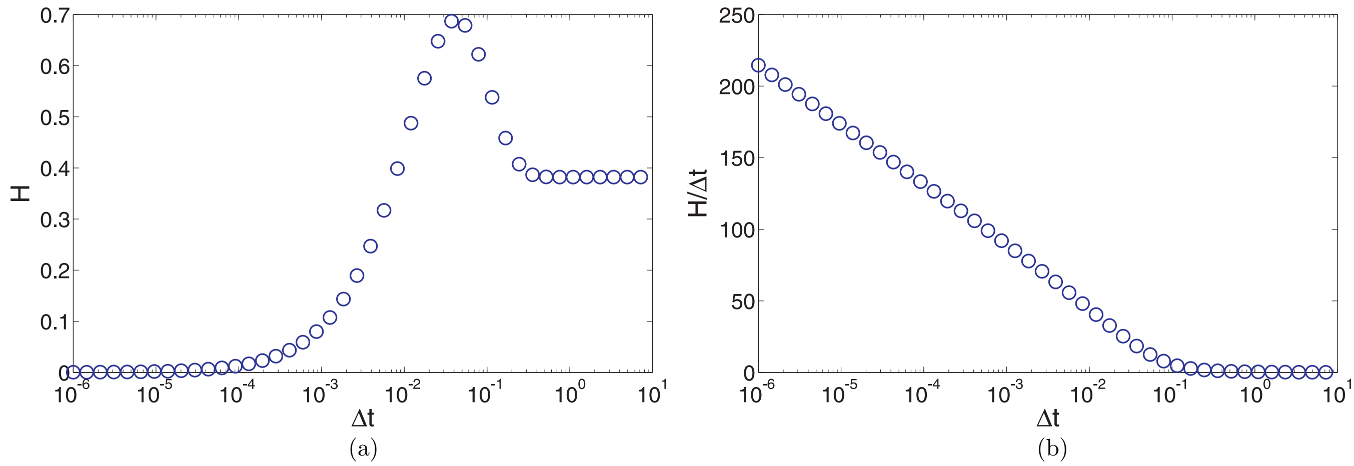

FIG. 4. Lower bounds of entropy reduction rate per control action $\bar{H}_{\rm low}$ in panel (a) and per time unit $\bar{H}_{\rm low}/\Delta t$ in panel (b) as functions of the time between the controller's actions Δt for $V_0 = 20$ and $F = F^*(V_0,\Delta t)$ (force that maximizes the power output). The time step has been chosen small enough ($\delta t = \Delta t/100$) to make the discretization effects vanish. (The limit $\delta t \to 0$ and $n \to \infty$ with fixed $\Delta t = n\delta t$ corresponds to the continuous time limit.) The upper bounds, not represented, present a maximal deviation from the lower ones of 2×10^{-8} % (because the evolution is very close to be Markovian). We can see how in the limit $\Delta t \to 0$ (continuous control) the entropy reduction rate per controller's action goes to zero, whereas the entropy reduction rate per unit time diverges logarithmically. Moreover, for $\Delta t \gtrsim 1$ the entropy reduction per control action reaches a constant value, since in this case the system reaches equilibrium before the controller acts and then the value is the same regardless of the time interval between control actions; and the entropy reduction per unit time goes to zero due to the large time interval between consecutive controller's actions.

where we have restated the difference in terms of $P^{(n)}(c|A)$ and $P^{(n)}(c|B)$, which give the probabilities to have the controller's action $c = \{0,1\}$ in the next measurement if the particle is in A or B in the present measurement. It is important to note that in the limit $p_s^{(n)}(A)/p_s^{(n)}(B) \to 0$ we have $\bar{H}_{\rm up} - \bar{H}_{\rm low} \to 0$, and $\bar{H}_{\rm low}$ becomes equal to the value for a Markovian sequence, $\bar{H}_{\rm up}$. Therefore, $p_s^{(n)}(A)/p_s^{(n)}(B) \ll 1$ is a quasi-Markovian approximation.

In the maximum power output configuration, we have $a \gg b^2$ and $\Delta t \ll 1$, implying $P^{(n)}(c|A) = P^{(n)}(c|B)$ for $c = 0$ or 1 [see Eq. (15)], so the difference between the upper and lower bounds of $\bar{H}(\mathcal{C})$, given by Eq. (43), is of second order in $p_s^{(n)}(A)/p_s^{(n)}(B)$. Hence, we expect a very small difference between $\bar{H}_{\rm low}$ and $\bar{H}_{\rm up}$ in the maximum power output configuration.

Since both bounds $\bar{H}_{\rm low}$ and $\bar{H}_{\rm up}$ go to zero in the limit $\Delta t \to 0$, the entropy reduction rate $\bar{H}(\mathcal{C})$ goes to zero in the limit $\Delta t \to 0$; i.e., the entropy reduction per controller's action, when it acts continuously, is negligible. On the other hand, and with the same reasoning, the entropy reduction rate per unit time $[\bar{H}(\mathcal{C})/\Delta t]$ diverges logarithmically in this limit. In Fig. 4 are represented the lower bounds of both $\bar{H}(\mathcal{C})$ and $\bar{H}(\mathcal{C})/\Delta t$ as functions of Δt for $V_0 = 20$ and its corresponding optimal force $F = F^*$. The upper bounds are not represented since the relative distance with the lower bound is extremely small,with a maximum around 2×10^{-8} % for the represented case. (For other cases the deviation might not be so negligible. For example, for $V = 1$ the deviation is of the order of 0.07% or less.) This fact shows that, although the evolution of the variable $\mathcal{C}_t$ is not strictly Markovian, the error made if one assumes a Markovian evolution is extremely low for configurations at maximum power output.

C. Characteristic ON and OFF times

Before obtaining the efficiency of the system with this feedback protocol, we compute other important variables: the average times that the potential will be set ON or OFF before being switched by the controller. We refer to these times as ON and OFF times, respectively, and they are denoted by $\langle t_{\rm ON}\rangle$ and $\langle t_{\rm OFF}\rangle$.

The average time in which the potential is ON is just

$$\langle t_{\rm ON}\rangle = \sum_{N=1}^{\infty} \mathcal{P}_{\rm ON}^{(N)} N \Delta t, \tag{44}$$

where we defined $\mathcal{P}_{\rm ON}^{(N)}$ as the probability of having the potential ON during $N-1$ controller's actions and then switched to OFF in the Nth action,

$$\mathcal{P}_{\rm ON}^{(N)} = \tilde{P}^{(n)}(1|1)^{N-1}\tilde{P}^{(n)}(0|1). \tag{45}$$

Then, by replacing now in these expressions Eq. (29) and computing the summation of the series, we get

$$\langle t_{\rm ON}\rangle = \frac{\Delta t}{\tilde{P}^{(n)}(0|1)} = \frac{1}{18\chi} + \mathcal{O}(\Delta t), \tag{46}$$

with χ given by Eq. (35). Similarly, the average time in which the potential is set OFF is

$$\langle t_{\rm OFF}\rangle = \frac{\Delta t}{\tilde{P}^{(n)}(1|0)} = \frac{1}{18} + \mathcal{O}(\Delta t). \tag{47}$$

Since the parameter χ is always less than 1, we have $\langle t_{\rm OFF}\rangle < \langle t_{\rm ON}\rangle$. However, for big enough potential heights χ tends to 1, and then the ON and OFF times become similar. For example, for $V_0 = 20$ and $F = 3.8$ (a maximum power output configuration), $\langle t_{\rm ON}\rangle/\langle t_{\rm OFF}\rangle \simeq 1.0001 + \mathcal{O}(\Delta t)$.

D. Efficiency

Now we have all the elements required to compute the efficiency of this system in the stationary state. We follow the definition of efficiency for isothermal feedback-controlled ratchets provided in Ref. [13]. We denote W the average output work achieved per time step, Q the eventual heat transfer, $-\Delta U_{\mathrm{cont}}$ the internal energy transfer from the controller to the system, and ΔS_{cont} the entropy increase of the controller. With this notation the first and second laws of thermodynamics read

$$\begin{aligned} \Delta U_{\mathrm{cont}} + Q + W &= 0, \\ T\Delta S_{\mathrm{cont}} + Q &\geq 0. \end{aligned} \tag{48}$$

It is then natural to define the efficiency as

$$\eta = \frac{W}{T\Delta S_{\mathrm{cont}} - \Delta U_{\mathrm{cont}}} \leq 1, \tag{49}$$

where $T\Delta S_{\mathrm{cont}} - \Delta U_{\mathrm{cont}}$ gives the maximum energy that can be converted into work, the free energy.

In this expression, the mean output work extracted from the system in Δt is just

$$W = \Delta t J_s^{(n)} F. \tag{50}$$

When it comes to the energy transfer from the controller to the system, $-\Delta U_{\mathrm{cont}}$, there are two contributions. The first one arises when the particle is at site C at time t, with the potential OFF, and at site A at time $t + \Delta t$ and the controller turns the potential to ON, since the particle then feels a potential increase of V_0 and carries an energy transfer of

$$\Delta U_{\mathrm{cont}}^{C,\mathrm{OFF}\to A,\mathrm{ON}} = -P_{\mathrm{OFF}}^{(n)}(A|C)p_s^{(n)}(C)V_0. \tag{51}$$

The second one happens when the particle, starting from a site C and with the potential OFF, reaches a site B and the controller turns the potential to ON, the particle feels an increase of its energy of $V_0/2$, and then the energy transfer reads

$$\Delta U_{\mathrm{cont}}^{C,\mathrm{OFF}\to B,\mathrm{ON}} = -P_{\mathrm{OFF}}^{(n)}(B|C)p_s^{(n)}(C)\frac{V_0}{2}. \tag{52}$$

Thus, the energy transfer during the turning on, $\Delta U_{\mathrm{cont}}^{\mathrm{OFF}\to\mathrm{ON}}$, is given by the sum of Eqs. (51) and (52). The energy transfer during the turning off of the potential is zero, since at sites C the energy levels do not change.

Finally, the entropy increase of the controller is at least the entropy it takes from the system during the controls, $T\Delta S_{\mathrm{cont}} \geq -T\Delta S_{\mathrm{info}}^{(n)} = \bar{H}(\mathcal{C})$. Thus, the maximum attainable efficiency is

$$\eta_{\mathrm{up}} = -\frac{W}{-\bar{H}_{\mathrm{low}} + \Delta U_{\mathrm{cont}}^{\mathrm{OFF}\to\mathrm{ON}}}. \tag{53}$$

The efficiency at maximal power ($F = F^*$) as a function of V_0 for different measurement times is shown in Fig. 5. There we can see that there exists a maximum attainable efficiency for a value of Δt near 10^{-2}. We can also appreciate that, in the most favorable scenario, working at maximal power the system seems very inefficient, with a maximum value of η around 2.7%.

In order to show that there is a value of Δt for which the efficiency at maximal power presents a maximum, in Fig. 6 we represent the maximal efficiency at $P_{\max}$ for different values of Δt. In this figure it can be seen that the efficiencies are not very high, with a maximum value around 2.7% for $\Delta t \sim 10^{-2}$. Moreover, for large enough Δt the maximal efficiency is zero; i.e., the controller acts so rarely on the system that it does not succeed to move the particle against the load force F.

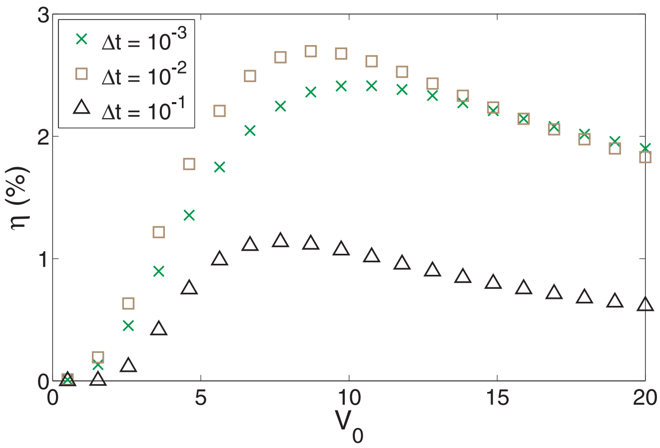

FIG. 5. Upper bound of the efficiency, η_{up}, at maximal power as function of the potential height V_0 for different values of Δt. The time step has been chosen small enough ($\delta t = \Delta t/100$) to avoid discretization effects.

V. COMPARISON WITH THE OPEN-LOOP PROTOCOL

In previous sections, we have studied the maximal power output and the efficiency at this maximal power output configuration in the feedback protocol proposed in Sec. II. Now we will compare these results with those corresponding to an open-loop protocol, for which the potential is switched alternatively to ON and OFF periodically. The relation of the results of both protocols will highlight whether the feedback protocol is worthy or not for this kind of system.

First of all, the evolution matrix in a period of n_{off} time steps with the potential OFF and then n_{on} time steps with the

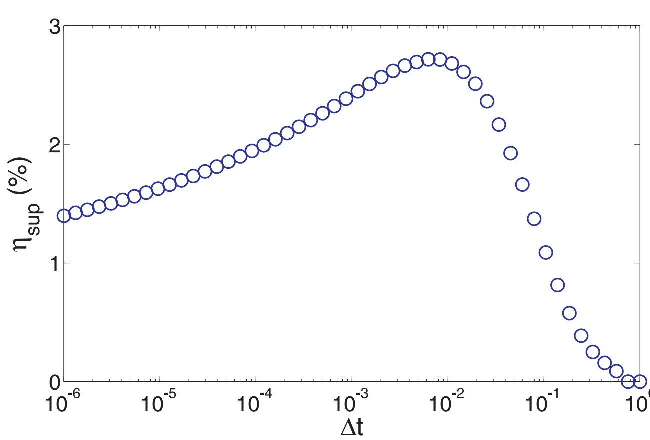

FIG. 6. Maximal efficiency at maximal power output, η_{sup}, as function of Δt. For each Δt the discretization time step δt is two orders smaller, in order to avoid discretization effects. Near $\Delta t \sim 10^{-2}$ there exists a maximum of efficiency, with a value 2.7%. On the other hand, for big enough Δt the maximal efficiency is zero, which means that the time between control actions is so large that the controller is not able to force the particle to move against the external force F.

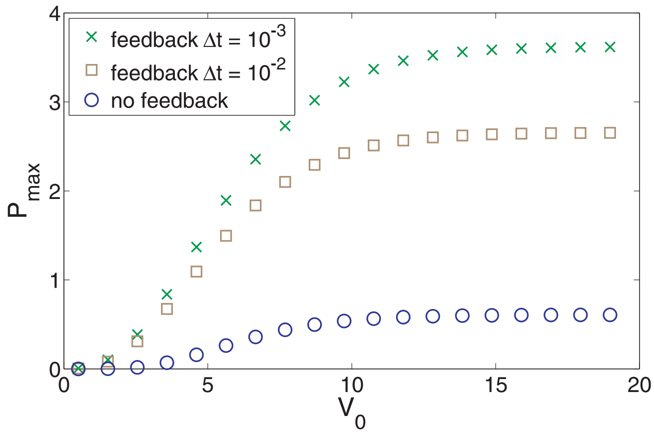

FIG. 7. Maximal power output $P_{\max}$ as a function of the potential height V_0 for the open-loop flashing ratchet with a temporal discretization of $\delta t = 5\times 10^{-8}$, and for the feedback flashing ratchet with measurement interval of $\Delta t = 10^{-2}$ and 10^{-3}. We chose these measurement time intervals because for the open-loop protocol the maximal power output occurs at $t_{\text{off}} \sim t_{\text{on}} \sim 10^{-3}$, and the maximum efficiency for the feedback ratchet was achieved at $\Delta t \sim 10^{-2}$ (see Fig. 6). In this figure it is shown that the feedback protocol increases notably the maximal output power for the system.

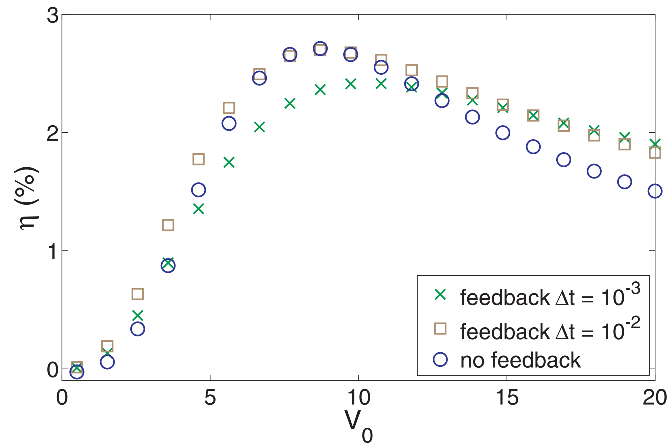

FIG. 8. Efficiencies η for the cases plotted in Fig. 7. This figure shows that the efficiencies for the open-loop protocol are similar to the efficiencies for the feedback protocol.

potential ON is

$$\mathbf{p}_{t+(n_{\text{on}}+n_{\text{off}})\delta t} = \mathbf{M}_{\text{ON}}^{n_{\text{on}}}\mathbf{M}_{\text{OFF}}^{n_{\text{off}}}\mathbf{p}_t, \tag{54}$$

with $\mathbf{M}_{\text{OFF}}$ and $\mathbf{M}_{\text{ON}}$ the matrices expressed in Eqs. (8) and (9), respectively. In the stationary state of the system,

$$\mathbf{p}_s = \mathbf{M}_{\text{ON}}^{n_{\text{on}}}\mathbf{M}_{\text{OFF}}^{n_{\text{off}}}\mathbf{p}_s, \tag{55}$$

we obtain the flux of particles

$$J_s = p_s(A)J(A) + p_s(B)J(B) + p_s(C)J(C), \tag{56}$$

where $J(A), J(B)$ and $J(C)$ are the fluxes when the particle is initially at A, B, and C, respectively. The flux $J(A)$ reads

$$J(A) = \frac{1}{n_{\text{off}}+n_{\text{on}}}\left[n_{\text{off}}J_{\text{OFF}} + n_{\text{on}}\sum_{X=\{A,B,C\}} P_{\text{OFF}}^{(n_{\text{off}})}(X|A)J_{\text{ON}}^{(n_{\text{on}})}(X)\right], \tag{57}$$

where J_{OFF} is the flux when the potential is OFF, and is given by Eq. (22), while $J_{\text{ON}}^{(n_{\text{on}})}(X)$ is the flux when the particle is initially set at X and evolves during n_{on} time steps. $J_{\text{ON}}^{(n_{\text{on}})}(B)$ is computed in Appendix B 2, and $J_{\text{ON}}^{(n_{\text{on}})}(A)$ and $J_{\text{ON}}^{(n_{\text{on}})}(C)$ can be obtained in the same way. The other quantities $J(B)$ and $J(C)$ have analogous expressions.

Once the stationary flux has been obtained, the power output is just $P = J_s F$. The efficiency η of this open-loop protocol is obtained following the procedure described in Sec. IV D, now without any entropy reduction terms due to information,

$$\eta = \frac{W}{-\Delta U_{\text{cont}}} \leqslant \frac{P_{\max}(n_{\text{off}}+n_{\text{on}})\delta t}{V_0\{q_s(A) - p_s(A) + \frac{1}{2}[q_s(B) - p_s(B)]\}}, \tag{58}$$

where $\mathbf{q}_s$ is the stationary state just before the potential is turned ON, $\mathbf{q}_s = \mathbf{M}_{\text{OFF}}^{n_{\text{off}}}\mathbf{p}_s$.

Figure 7 shows the maximal power output for an open-loop flashing ratchet with a temporal discretization of $\delta t = 5\times 10^{-8}$ and for a feedback ratchet with measurement times of $\Delta t = 10^{-2}$ and 10^{-3} as function of the potential height, V_0. We chose these measurement times since the open-loop protocol operates at maximal power for OFF and ON times around 10^{-3}, and for the feedback protocol the maximum efficiency is reached at $\Delta t \sim 10^{-2}$. We note that the power output is notably greater for the feedback protocol than for the open-loop protocol, showing that the use of information increases the attainable power output. Figure 8 shows the efficiency at maximal power output for the previous cases. The efficiency for the open-loop protocol is of the same order as for the feedback protocol. Hence, we can conclude that feedback ratchets can extract more power than open-loop flashing ratchets, with a similar efficiency. The question now is where this power increase comes from. In Fig. 9, a comparison of the input energy rate for open-loop and feedback ratchets $[-\Delta U_{\text{cont}}/(t_{\text{on}}+t_{\text{off}})$ and $-\Delta U_{\text{cont}}/\Delta t$, respectively] shows that feedback increases the energy input rate. We also see in this figure that, in the feedback ratchet, the contribution to the free energy rate from the entropy reduction $T\Delta S_{\text{cont}}/\Delta t$ due to the controller is of the same order as the input energy rate,

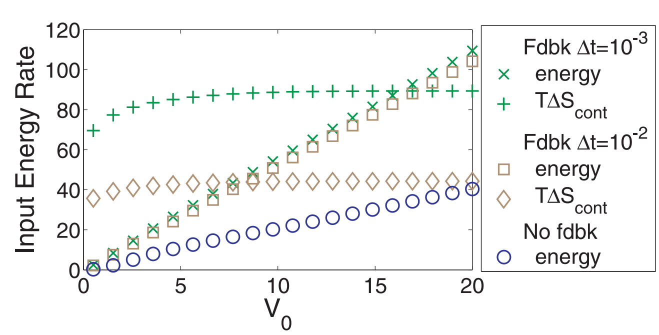

FIG. 9. Input energy rate $-\Delta U_{\text{cont}}/\Delta t$ and contribution to the free energy rate from the entropy reduction $T\Delta S_{\text{cont}}/\Delta t$ for the feedback ratchet compared with the input energy rate for the open-loop ratchet $-\Delta U_{\text{cont}}/(t_{\text{on}}+t_{\text{off}})$, as a function of the potential strength V_0 for the cases plotted in Figs. 7 and 8.

for the case that gives the maximum efficiency, $\Delta t \sim 10^{-2}$ and $V_0 \sim 8$.

VI. CONCLUSIONS

In this paper we have computed the efficiency at maximum power of a feedback ratchet. The main difficulty is the computation of the entropy reduction when, as in this case, the controller's actions are correlated. The recent paper [13] showed how to compute this entropy reduction in terms of probabilities of the possible sequence of the controller's actions, and applied it to solve simple cases. However, for most feedback ratchets the sequence of the controller's actions is non-Markovian and the computation becomes involved. Here we have introduced an alternative procedure to set strong bounds to the entropy reduction in order to compute its value. This procedure can also be applied to other feedback ratchets, and more generally to other control systems.

The maximum power output depends not only on the time interval between consecutive measurements of the position of the particles, but also on the time discretization. However, when the discretization time is some orders of magnitude less than the time between measurements, the continuous time limit is reached; i.e., the maximum power output becomes independent of the discretization time.

Once the output power was computed, we aimed to obtain the efficiency of this feedback ratchet. From the first and second laws of thermodynamics, it is possible to define an efficiency [13]. In this efficiency the increase of entropy of the controller due to the process of measurement plays a crucial role. This increase of entropy of the controller is at least (if not greater than) minus the reduction of entropy of the system. This computation of the entropy reduction is involved when, as in this case, the sequence of the controller's actions is non-Markovian. However, we have obtained an upper and a lower bound for the entropy reduction. The upper bound is the same as for a Markovian sequence, while the lower one is evaluated in a quasi-Markovian limit, which emerges when there are big differences between the stationary probabilities of the system states. These big differences are an effect of potential strength, which minimizes the departures from the Markovianicity of the sequence of control actions, also making it possible to minimize the departures from the optimal performance of the system. We found that both bounds are very close to each other when the system works at maximum power. Hence, we could estimate the efficiency at maximum power with great precision.

The reduction of entropy per control action depends on the time between measurements (i.e., depends on the time between the controller's actions). However, if the time between measurements is large enough the system reaches a steady state before the controller acts, and then the decrease of entropy is always the same and it does not depend on the time between measurements. On the other hand, when the time between measurements, Δt, is small enough, the entropy reduction per control action goes to zero. However, it decreases as $\Delta t \ln \Delta t$, and then the entropy reduction of the system per unit time diverges logarithmically in this limit, $\Delta S_{\text{info}}^{(n)}/\Delta t \xrightarrow[\Delta t \to 0]{} \ln \Delta t$. Nevertheless, we do not expect that in experimental realizations this divergence appears, since our model will represent the real system until a certain small value of Δt, and this value gives a lower cutoff for the theory.

With the previous result, we obtained an upper bound to the efficiency of this proposed system, and we got that they are lower than or similar to 2.7% when the system acts at maximal power output. We also showed that the efficiency has a maximum for measurement times $\Delta t \sim 10^{-2}$ in the temporal units of L^2/D, with L the spatial period of the potential and D the diffusion coefficient of the system. As L^2/D is the characteristic time of diffusion over a distance L, this means that maximum efficiency operating at maximum power is obtained measuring and actuating with the control at time intervals 100 times shorter than the characteristic time of diffusion over a period. This implies for spherical particles of radius r in a medium of viscosity η, a time interval of $\Delta t = 10^{-2}\frac{L^2}{D} = 10^{-2}\frac{6\pi\eta}{k_B T}L^2 r$. This range of values of Δt can be tested experimentally. For example, if the diffusion coefficient is of the order of 0.1 $\mu\text{m}^2/\text{s}$ (approximately that given by the Stokes-Einstein formula for particles of 1 μm inside water at 300 K) and L is of the order of 10 μm, the optimal measurement time is around 10 s, and it is very easy to design a controller which measures positions every 10 s.

Finally, we compare the result of the proposed feedback ratchet with its open-loop counterpart. We found that the feedback protocol increases notably the maximal attainable power output, with a similar efficiency at maximum power.

ACKNOWLEDGMENTS

We would like to thank Hugo Touchette for comments at the beginning of the redaction of this paper and for stimulating discussions. We also acknowledge Juan Pedro García-Villaluenga for a critical reading of the final version of the manuscript. This work has been supported by the Projects No. GR35/14-920911 (Banco Santander and Universidad Complutense de Madrid, Spain) and No. FIS2010-17440 (Ministerio de Economía y Competitividad, Spain). J.J. also acknowledges the financial support through Grant No. FPU-13/02934 (Ministerio de Educación, Cultura y Deportes, Spain).

APPENDIX A: JUMPING PROBABILITIES

Consider a Brownian particle in a one-dimensional double-well potential, like the one represented in Fig. 10(a).

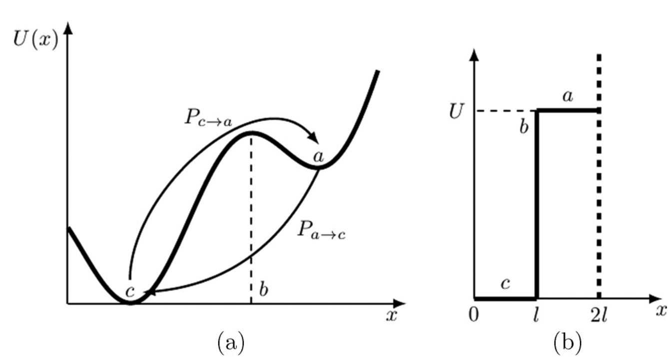

FIG. 10. Potentials and jumps considered in Appendix A. In panel (a), a scheme of the double well is shown. In panel (b), a scheme of the finite jump potential is plotted.

The probability density function satisfies the Fokker-Planck equation,

$$\partial_t \rho(x,t) = \frac{1}{\gamma}\,\partial_x[U'(x)\rho(x,t)] + D\,\partial_x^2 \rho(x,t), \tag{A1}$$

with $\rho(x,t)$ the probability density at time t, and D and γ the diffusion and friction coefficients (which are related through Einstein's formula $k_B T = D\gamma$). Following Kramer's method as described in [39], we want to reduce the population density dynamics to a two-discrete-states system dynamics, described by the master equations

$$\begin{aligned}\rho_a(t+\Delta t) &= P_{c\to a}\cdot\rho_c(t) + (1-P_{a\to c})\cdot\rho_a(t),\\ \rho_c(t+\Delta t) &= P_{a\to c}\cdot\rho_a(t) + (1-P_{c\to a})\cdot\rho_c(t).\end{aligned} \tag{A2}$$

In the above equation, $\rho_a(t)$ and $\rho_c(t)$ are defined by

$$\rho_c(t) \equiv \int_{-\infty}^{b} dx\;\rho(x,t), \quad \rho_a(t) \equiv \int_{b}^{+\infty} dx\;\rho(x,t). \tag{A3}$$

We also define the stationary probabilities

$$\rho_c^s \equiv \int_{-\infty}^{b} dx\;\rho^s(x), \quad \rho_a^s = \int_{b}^{+\infty} dx\;\rho^s(x), \tag{A4}$$

with the stationary distribution function

$$\rho^s(x) = \mathcal{N}\exp\left[-\frac{U(x)}{k_B T}\right], \tag{A5}$$

where $\mathcal{N}$ is a normalization constant. We assume that the relaxation is fast in each well, so that the distribution $\rho(x,t)$ is given by the stationary distribution, with different amplitudes in wells a and c, so

$$\rho(x,t) = \begin{cases}\rho^s(x)\cdot f_c(t) & \text{for } x < b,\\ \rho^s(x)\cdot f_a(t) & \text{for } x > b.\end{cases} \tag{A6}$$

Integrating this relation on each side of b and using Eqs. (A3) and (A4), we get

$$\rho(x,t) = \begin{cases}\rho^s(x)\cdot\frac{\rho_c(t)}{\rho_c^s} & \text{for } x < b,\\ \rho^s(x)\cdot\frac{\rho_a(t)}{\rho_a^s} & \text{for } x > b.\end{cases} \tag{A7}$$

Replacing this expression into the Fokker-Planck equation [Eq. (A1)] gives

$$\begin{aligned}\partial_t\rho_a(t) &= -r_a\,\rho_a(t) + r_c\,\rho_c(t),\\ \partial_t\rho_c(t) &= r_a\,\rho_a(t) - r_c\,\rho_c(t),\end{aligned} \tag{A8}$$

with

$$r_a \equiv D\left[\rho_a^s\int_c^a \frac{dx}{\rho^s(x)}\right]^{-1}, \quad r_c \equiv D\left[\rho_c^s\int_c^a \frac{dx}{\rho^s(x)}\right]^{-1}. \tag{A9}$$

Using these expressions in Eqs. (A2) and (A8), we get that the jump probabilities read

$$\begin{aligned}P_{c\to a} &= \frac{r_c}{r_a+r_c}\left(1-e^{-(r_a+r_c)\Delta t}\right),\\ P_{a\to c} &= \frac{r_a}{r_a+r_c}\left(1-e^{-(r_a+r_c)\Delta t}\right).\end{aligned} \tag{A10}$$

In the case of the potential of Fig. 10(b) with a finite jump U at $x = l$ and absorbing boundary conditions at $x = 0, 2l$, and assuming that the equilibrium positions c and a are located at $x = l/2$ and $x = 3l/2$, respectively, we find the jumping probabilities

$$\begin{aligned}P_{c\to a} &= \frac{1}{1+\exp(-U/\gamma D)}\left(1-e^{-2D\Delta t/l^2}\right),\\ P_{a\to c} &= \frac{\exp(-U/\gamma D)}{1+\exp(-U/\gamma D)}\left(1-e^{-2D\Delta t/l^2}\right).\end{aligned} \tag{A11}$$

Finally, using Einstein's relation $\gamma D = k_B T$ and units of $L = 1$, these equations are reduced to Eq. (6) for $l = L/3$.

APPENDIX B: COMPUTATION OF THE FLUX

The flux of particles to the right between times t and $t + n\delta t$ is defined by

$$\begin{aligned}J_t &= \frac{X_m(t+n\delta t) - X_m(t)}{n\delta t}\\ &= \frac{1}{n\delta t}\sum_x x[p(x,t+n\delta t) - p(x,t)],\end{aligned} \tag{B1}$$

where $X_m(t)$ is the mean position of the particle at time t. Using the master equation

$$p(x,t+n\delta t) = \sum_{k=-n}^{n} P^{(n)}(x|x+k\delta x)p(x+k\delta x,t), \tag{B2}$$

we have

$$J_t = \frac{1}{n\delta t}\sum_x x\left[\sum_{k=-n}^{n} P^{(n)}(x|x+k\delta x)p(x+k\delta x,t) - p(x,t)\right]. \tag{B3}$$

With the change of variables $x \to x - k\delta x$, we get

$$\begin{aligned}J_t &= \frac{1}{n\delta t}\sum_x\left[\sum_{k=-n}^{n}(x-k\delta x)P^{(n)}(x-k\delta x|x)p(x,t) - xp(x,t)\right]\\ &= \frac{1}{n\delta t}\sum_x p(x,t)\left[\sum_{k=-n}^{n}(x-k\delta x)P^{(n)}(x-k\delta x|x) - x\right]\\ &= \frac{1}{n\delta t}\sum_x p(x,t)\left[x\left(\sum_{k=-n}^{n}P^{(n)}(x-k\delta x|x) - 1\right) - \sum_{k=-n}^{n}k\delta x\,P^{(n)}(x-k\delta x|x)\right].\end{aligned} \tag{B4}$$

From the conservation of probabilities,

$$\sum_{k=-n}^{n} P^{(n)}(x - k\delta x|x) = 1, \tag{B5}$$

the first term in Eq. (B4) vanishes, and redefining $k \to -k$ we get

$$J_t = \sum_x p(x,t) \sum_{k=-n}^{n} \frac{k}{n}\frac{\delta x}{\delta t} P^{(n)}(x + k|x). \tag{B6}$$

We define $\mathcal{A}$ as the set of values of x of type A (and analogously $\mathcal{B}$ and $\mathcal{C}$ for B and C). Decomposing the above sum into A, B, and C contributions and using the translational invariance of the jumping probabilities, we have

$$J_t = p_t(A)J^{(n)}(A) + p_t(B)J^{(n)}(B) + p_t(C)J^{(n)}(C), \tag{B7}$$

where $p_t(A) = \sum_{x\in\mathcal{A}} p(x,t)$ [analogously $p_t(B)$ and $p_t(C)$] and where

$$J^{(n)}(A) = \frac{\delta x}{\delta t}\sum_{k=-n}^{n} \frac{k}{n} P^{(n)}(x + k|x) \quad \forall\, x \in \mathcal{A}, \tag{B8}$$

and $J^{(n)}(B)$ and $J^{(n)}(C)$ have analogous expressions.

$J^{(n)}(I)$ is the flux after n time steps for a particle initially at site I. If a control action happens at time t, then the jumping probabilities in $J^{(n)}(A)$ and $J^{(n)}(C)$ are given by the potential OFF [in fact $J^{(n)}(A) = J^{(n)}(C)$], and the jumping probabilities in $J^{(n)}(B)$ are given by the potential ON. We thus rewrite the expression of the flux as

$$J_t = p_t(A)J^{(n)}_{\rm OFF}(A) + p_t(B)J^{(n)}_{\rm ON}(B) + p_t(C)J^{(n)}_{\rm OFF}(C). \tag{B9}$$

1. Computation of $J^{(n)}_{\rm OFF}(C)$

If the particle is initially placed in C, the potential is set OFF, so it is invariant under translations $x \to x + \delta x$ as we can see in Fig. 1. Thus, due to this symmetry, the flux after n time steps is the same as the flux after just one,

$$J^{(n)}_{\rm OFF}(C) = J^{(1)}_{\rm OFF}(C). \tag{B10}$$

Then, using Eq. (B8) we obtain

$$J^{(n)}_{\rm OFF}(C) = \frac{\delta x}{\delta t}(P_+ - P_-), \tag{B11}$$

where we have defined P_+ as the probability $P^{(1)}(x + \delta x|x)$ and P_- as $P^{(1)}(x - \delta x|x)$. These jumping probabilities are shown in Eq. (8), and by replacing them in Eq. (B11) we finally obtain

$$J^{(n)}_{\rm OFF}(C) = \frac{\delta x}{\delta t}[1 - \exp(-18\Delta t)]\frac{1-b}{1+b}. \tag{B12}$$

2. Computation of $J^{(n)}_{\rm ON}(A)$ and $J^{(n)}_{\rm ON}(B)$

Because the potential is no longer invariant under translations $x \to x + \delta x$ when the potential is ON, the procedure used for $J^{(n)}_{\rm OFF}(C)$ does not apply in this case, and the fluxes $J^{(n)}_{\rm ON}(A)$ and $J^{(n)}_{\rm ON}(B)$ have to be computed numerically.

Let us obtain first $J^{(n)}_{\rm ON}(B)$. The flux is, by definition, $J^{(n)}_{\rm ON}(B) = \frac{\delta x}{\delta t}\sum_{k=-n}^{n}\frac{k}{n}P^{(n)}_{\rm ON}(x + k\delta x|x)$ for $x \in \mathcal{B}$. The method we find more convenient in order to compute this quantity recursively is the one based on the Chapman-Kolmogorov equation,

$$P^{(n)}_{\rm ON}(x_0 + k\delta x|x_0) = \sum_{j=-1}^{1} P^{(1)}_{\rm ON}[x_0 + k\delta x|x_0 + (k+i)\delta x] \times P^{(n-1)}_{\rm ON}[x_0 + (k+i)\delta x|x_0]. \tag{B13}$$

The flux $J^{(n-1)}(B)$ is computed from the $P^{(n-1)}(x'|x_0)$ for a fixed $x_0 \in \mathcal{B}$ and $x' \in \{x_0 - (n-1)\delta x, \ldots, x_0 + (n-1)\delta x\}$. Then, using the above equation, we compute the $P^{(n)}(x_0 + k\delta x|x_0)$ for $k \in \{-n, \ldots, n\}$ and compute $J^{(n)}(B)$. As a simple example, the case $n = 1$ takes the form

$$J^{(1)}_{\rm ON}(B) = \frac{\delta x}{\delta t}[1 - \exp(-18\Delta t)]\frac{\sqrt{a} - b}{\sqrt{a} + b}. \tag{B14}$$

Furthermore, although for a general n we could not provide a simple expression of this flux, we found that in the limiting case $a \gg b^2$ this flux can be written as

$$J^{(n)}_{\rm ON}(B) \underset{a\gg b^2}{\simeq} 6 - 54n\delta t + \mathcal{O}(\delta t^2). \tag{B15}$$

The computation of $J^{(n)}_{\rm ON}(A)$ is completely analogous, and with the same method the final result is

$$J^{(n)}_{\rm ON}(A) \underset{a\gg b^2}{\simeq} -54\delta t + 54\Delta t + \mathcal{O}(\Delta t^2). \tag{B16}$$

APPENDIX C: ENTROPY REDUCTION RATE BOUNDS

Following Ref. [36] we can define two entropy reduction rates given by two limits: the per control entropy of the N controller actions,

$$\bar{H}(\mathcal{C}) \equiv \lim_{N\to\infty} \frac{H(\mathcal{C}_{t_0+(N-1)\Delta t}, \ldots, \mathcal{C}_{t_0})}{N}, \tag{C1}$$

and the conditional entropy of the last controller action given the past controller actions,

$$\bar{H}'(\mathcal{C}) \equiv \lim_{N\to\infty} H(\mathcal{C}_{t_0+(N-1)\Delta t} \mid \mathcal{C}_{t_0+(N-2)\Delta t}, \ldots, \mathcal{C}_{t_0}). \tag{C2}$$

Both limits exist and give the same value [36]:

$$\bar{H}(\mathcal{C}) = \bar{H}'(\mathcal{C}). \tag{C3}$$

The upper bound is found from the fact that conditioning reduces the entropy

$$\begin{aligned} &H(\mathcal{C}_{t_0+(N'+1)\Delta t} \mid \mathcal{C}_{t_0+N'\Delta t}, \ldots, \mathcal{C}_{t_0}) \\ &\quad \leqslant H(\mathcal{C}_{t_0+(N'+1)\Delta t} \mid \mathcal{C}_{t_0+N'\Delta t}, \ldots, \mathcal{C}_{t_0+\Delta t}) \\ &\quad = H(\mathcal{C}_{t_0+N'\Delta t} \mid \mathcal{C}_{t_0+(N'-1)\Delta t}, \ldots, \mathcal{C}_{t_0}). \end{aligned} \tag{C4}$$

Therefore, from Eqs. (C2), (C3), and (C4) we get the upper bound

$$\bar{H}(\mathcal{C}) \leqslant H(\mathcal{C}_{t_0+N'\Delta t} \mid \mathcal{C}_{t_0+(N'-1)\Delta t}, \ldots, \mathcal{C}_{t_0}). \tag{C5}$$

In particular, for $N' = 1$ it gives

$$\bar{H}(\mathcal{C}) \leqslant H(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0}), \tag{C6}$$

and we define

$$\bar{H}_{\rm up} \equiv H(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0}). \tag{C7}$$

It is important to note that the upper bound $\bar{H}_{\rm up}$ coincides with the expression for the entropy rate for Markovian processes. However, here as $\mathcal{C}_t$ is non-Markovian, this expression gives only an upper bound.

The lower bound is obtained considering that although $\mathcal{C}_t$ is not a Markov chain, it is a deterministic function of the position X_t, and X_t is an stationary Markovian process. Using the results for the bounds of hidden Markov processes [36] we get the lower bound

$$H\left(\mathcal{C}_{t_0+N'\Delta t} \mid \mathcal{C}_{t_0+(N'-1)\Delta t},\ldots,\mathcal{C}_{t_0},X_{t_0}\right) \leqslant \bar{H}(\mathcal{C}). \quad \text{(C8)}$$

This lower bound also for $N' \to \infty$ converges to the entropy rate, while for $N' = 1$ gives the simpler bound

$$H\left(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0},X_{t_0}\right) \leqslant \bar{H}(\mathcal{C}), \quad \text{(C9)}$$

and we define

$$\bar{H}_{\rm low} \equiv H\left(\mathcal{C}_{t_0+\Delta t} \mid \mathcal{C}_{t_0},X_{t_0}\right). \quad \text{(C10)}$$

The previous discussion explains how the bounds for $\bar{H}(\mathcal{C})$ used in the main text are obtained, in particular those in Eq. (39).

---

[1] P. Reimann, Phys. Rep. **361**, 57 (2002).
[2] H. Linke, Appl. Phys. A **75**, 167 (2002).
[3] P. Hänggi and F. Marchesoni, Rev. Mod. Phys. **81**, 387 (2009).
[4] E. M. Craig, N. J. Kuwada, B. J. Lopez, and H. Linke, Ann. Phys. (Berlin) **17**, 115 (2008).
[5] F. J. Cao, L. Dinis, and J. M. R. Parrondo, Phys. Rev. Lett. **93**, 040603 (2004).
[6] M. Feito and F. J. Cao, Phys. Rev. E **74**, 041109 (2006).
[7] M. Feito, J. P. Baltanás, and F. J. Cao, Phys. Rev. E **80**, 031128 (2009).
[8] M. Feito and F. J. Cao, Phys. Rev. E **76**, 061113 (2007).
[9] E. M. Craig, B. R. Long, J. M. R. Parrondo, and H. Linke, Europhys. Lett. **81**, 10002 (2008).
[10] S. A. M. Loos, R. Gernert, and S. H. L. Klapp, Phys. Rev. E **89**, 052136 (2014).
[11] M. Feito and F. J. Cao, Eur. Phys. J. B **59**, 63 (2007).
[12] F. J. Cao, M. Feito, and H. Touchette, Phys. A (Amsterdam, Neth.) **388**, 113 (2009).
[13] F. J. Cao and M. Feito, Phys. Rev. E **79**, 041118 (2009).
[14] T. Sagawa and M. Ueda, Phys. Rev. Lett. **100**, 080403 (2008).
[15] T. Sagawa and M. Ueda, Phys. Rev. Lett. **104**, 090602 (2010).
[16] J. M. Horowitz and S. Vaikuntanathan, Phys. Rev. E **82**, 061120 (2010).
[17] T. Sagawa, Prog. Theor. Phys. **127**, 1 (2012).
[18] T. Sagawa and M. Ueda, Phys. Rev. Lett. **109**, 180602 (2012).
[19] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[20] D. Mandal and C. Jarzynski, Proc. Natl. Acad. Sci. USA **109**, 11641 (2012).
[21] S. Deffner and C. Jarzynski, Phys. Rev. X **3**, 041003 (2013).
[22] A. C. Barato and U. Seifert, Phys. Rev. Lett. **112**, 090601 (2014).
[23] T. Sagawa, J. Stat. Mech. (2014) P03025.
[24] M. Bauer, D. Abreu, and U. Seifert, J. Phys. A Math. Theor. **45**, 162001 (2012).
[25] A. C. Barato and U. Seifert, Europhys. Lett. **101**, 60001 (2013).
[26] T. Kosugi, Phys. Rev. E **88**, 032144 (2013).
[27] B. J. Lopez, N. J. Kuwada, E. M. Craig, B. R. Long, and H. Linke, Phys. Rev. Lett. **101**, 220601 (2008).
[28] V. Serreli, C.-F. Lee, E. R. Kay, and D. A. Leigh, Nature (London) **445**, 523 (2007).
[29] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, Nat. Phys. **6**, 988 (2010).
[30] J. Bechhoefer, Rev. Mod. Phys. **77**, 783 (2005).
[31] F. J. Cao and M. Feito, Entropy **14**, 834 (2012).
[32] H. Leff and A. F. Rex, *Maxwell's Demon 2 Entropy, Classical and Quantum Information, Computing* (CRC Press, Boca Raton, FL, 2010).
[33] L. Szilard, Z. Phys. **53**, 840 (1929).
[34] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961).
[35] C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).
[36] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley & Sons, New York, 2012).
[37] H. Touchette and S. Lloyd, Phys. Rev. Lett. **84**, 1156 (2000).
[38] H. Touchette and S. Lloyd, Phys. A (Amsterdam, Neth.) **331**, 140 (2004).
[39] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Springer Series in Synergetics (Springer, Berlin, 1994).
[40] All the coefficients of the matrices $M_{\rm ON}$ and $M_{\rm OFF}$—and thus of $M^{(n)}$—are strictly positive. Thus, the convergence to the stationary state is insured by the Perron-Frobenius theorem [41].
[41] F. R. Gantmacher, *Theory of Matrices* (AMS Chelsea, Providence, RI, 2000), Vol. 2.