Kerr-de Sitter spacetime, Penrose process, and the generalized area theorem

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(Received 5 January 2018; published 26 April 2018)

We investigate various aspects of energy extraction via the Penrose process in the Kerr-de Sitter spacetime. We show that the increase in the value of a positive cosmological constant, Λ, always reduces the efficiency of this process. The Kerr-de Sitter spacetime has two ergospheres associated with the black hole and the cosmological event horizons. We prove by analyzing turning points of the trajectory that the Penrose process in the cosmological ergoregion is never possible. We next show that in this process both the black hole and cosmological event horizons' areas increase, and the latter becomes possible when the particle coming from the black hole ergoregion escapes through the cosmological event horizon. We identify a new, local mass function instead of the mass parameter, to prove this generalized area theorem. This mass function takes care of the local spacetime energy due to the cosmological constant as well, including that which arises due to the frame-dragging effect due to spacetime rotation. While the current observed value of Λ is quite small, its effect in this process could be considerable in the early Universe scenario where its value is much larger, where the two horizons could have comparable sizes. In particular, the various results we obtain here are also evaluated in a triply degenerate limit of the Kerr-de Sitter spacetime we find, in which radial values of the inner, the black hole and the cosmological event horizons are nearly coincident.

DOI: 10.1103/PhysRevD.97.084049

I. INTRODUCTION

The notion of a conserved positive energy of a particle or any system is associated with a future-directed timelike Killing vector field, e.g., the time-translational Killing vector fields in Minkowski spacetime or in the exterior of a static black hole. For a stationary rotating black hole, e.g., the Kerr spacetime, however, the surface at which the timelike Killing vector field becomes null has only nonzero intersections with the horizon at the axial points, $\theta = 0, \pi$, where the effect of the rotation vanishes. The axisymmetric surface where the timelike Killing vector field becomes null is known as the ergosphere and the black hole event horizon (BEH) is located within it. Thus in the region between the ergosphere and the BEH, known as the ergoregion, the timelike Killing vector field is spacelike (see e.g., Refs. [1–3]). Due to this nonexistence of a future-directed timelike Killing vector field, there can be negative-energy particles within the ergoregion, even classically.

The existence of such negative-energy particles gives rise to a classical mechanism for energy extraction from a black hole, namely the Penrose process; see Refs. [1–3] and references therein. Let a particle carrying positive energy enter the ergoregion of a Kerr black hole. We imagine that it breaks into two pieces there: one carrying negative energy and the other, positive. The negative-energy particle enters the BEH and the positive-energy particle, after reaching a turning point, comes out of the ergosphere and finally gets intercepted by an outside observer. With respect to such an observer the usual (positive) energy conservation must be valid. Thus it is clear that the ejecta will carry more energy than the initial incoming particle, effectively extracting energy from the black hole. It turns out that the energy thus extracted is largely rotational and the process can only continue until the black hole settles down into the Schwarzschild spacetime, where no ergosphere is present.

Since the Penrose process reduces the rotation of a black hole more than its mass, a Kerr black hole can never become a naked curvature singularity under this process; see Ref. [4] for a formal proof of this. In Ref. [5], it was proved that the black hole horizon area must increase under this process, thereby providing evidence in favor of the second law of black hole mechanics. We further refer the reader to Refs. [6,7] for various inequalities (respectively, the Wald and the Bardeen-Press-Teukolsky inequality) regarding the local speed of the fragments within the ergoregion. In order that the process indeed occurs, those inequalities show that the fragments must be moving with considerable relativistic speed. We further refer the reader to Ref. [8] for an interesting account of how electrostatic energy could be extracted from a charged black hole, via a Penrose-like process. It is also relevant to note that if we consider a magnetic field associated with the black hole and compute the Penrose process for a test charge particle, the

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energy extraction turns out to be much more efficient than the uncharged case [9,10].

A variant of the Penrose process using classical fields instead of particles exists, namely, the superradiance or the superradiant scattering (see e.g., Ref. [1]). If ω is the frequency of a scalar field and Ω_H is the angular speed on the horizon, for $(\omega - \lambda\Omega_H) < 0$ (λ denotes the azimuthal eigenvalue), the flux of the energy-momentum going through the horizon turns out to be negative whereas the flux at infinity is positive. It thus effectively extracts energy from the hole. We refer the reader to e.g., Refs. [11–28] for most recent trends and developments in this topic. See also Ref. [29] and references therein for a discussion of superradiance in the acoustic analogue gravity paradigm. We further refer the reader to Ref. [30] for a review and an exhaustive list of references. In particular, superradiance has been studied for the anti-de Sitter black hole spacetimes [20–23], including nonlinear backreaction effects [22]. For de Sitter black holes on the other hand, which is our focus, we refer the reader to Refs. [24–28]. The condition for energy extraction for such black holes is given by $(\omega - \lambda\Omega_C) > 0$ where Ω_C is the angular speed on the cosmological event horizon, along with the usual $(\omega - \lambda\Omega_H) < 0$. In particular, massless scalar wave superradiance was studied in Ref. [25] in the doubly degenerate Nariai limit of the Kerr-de Sitter spacetime.

In this paper, we shall study the Penrose process in the Kerr-de Sitter (KdS) spacetime [31–34], and will address a couple of questions which, to the best our knowledge, have been hitherto unaddressed in the literature. First, it is relevant to ask, how does a positive Λ affect the process' efficiency, keeping in mind the results of refs. [6,7] pertaining to the asymptotically flat spacetime? We shall see that in KdS this process becomes *always* less efficient with increasing Λ (cf. Secs. III and IV). Second, unlike the Kerr spacetime, KdS is endowed with two ergospheres (cf. discussions in Sec. II), associated with the black hole and the cosmological event horizon (CEH). It is natural to ask then, can there be any Penrose process in the cosmological ergosphere as well? Clearly, in such a process one expects to steal rotational energy from regions beyond the CEH, which is induced onto the cosmological constant due to frame-dragging effects (cf. Sec. III A). However, we shall see by a simple analysis of the turning points of the trajectory that such a process is never possible.

The third question concerns the second law of the de Sitter black hole thermodynamics; see e.g., Refs. [35–53] and references therein for recent developments. Precisely, the two horizons (BEH and CEH) in such spacetimes have individual temperatures and entropies associated with them [35]. A rigorous proof of the area theorem for the cosmological event horizon can be seen in Ref. [37]. Clearly, such a two-temperature system makes the usual notion of the black hole thermodynamics as a grand canonical ensemble ambiguous [38]. To an observer residing within the region bounded by the two horizons, a total entropy of the spacetime can be attributed, equaling one quarter of the sum of the two horizon areas [39] (see also e.g., Refs. [40,41,43] and references therein). We refer the reader to Ref. [50] for a derivation of such a total entropy using near horizons' conformal symmetries. We also refer the reader to Ref. [54] for a proposal to further modify such an entropy via the correlation between the two horizons. Quite interestingly, the variation of the total entropy gives rise to a *single* effective temperature in such spacetimes (see e.g., Refs. [40,41,43]). Interesting results, including calculations pertaining to the phase transition using such an effective temperature can be seen e.g., in Refs. [48,49,51,52]. It is thus a very important task to check whether the second law of de Sitter black hole mechanics actually holds in a physical scenario i.e., whether in such a process the aforementioned total entropy increases, which is known as the generalized area theorem [39]. There has been an attempt to check this theorem under different physical processes; however, to the best of our knowledge, unlike the asymptotically flat spacetimes, a completely general or topological proof of this theorem is hitherto absent.

Motivated by this, in Sec. V, we shall prove by introducing a suitable local mass function that in the Penrose process, not only does the area of the BEH increase, but also when the ejecta escapes through the CEH, its area increases too, thereby proving the generalized area theorem in this process. Finally we conclude in Sec. VI. We shall use a mostly positive signature for the metric and will set $c = 1 = G$ below.

II. THE METRIC AND SOME GENERAL FEATURES OF HORIZON SIZES

The KdS metric in the Boyer-Lindquist coordinates reads [31–34],

$$\begin{aligned} ds^2 = &-\frac{\Delta_r - a^2\sin^2\theta\Delta_\theta}{\rho^2}dt^2 \\ &-\frac{2a\sin^2\theta}{\rho^2\Xi}((r^2+a^2)\Delta_\theta - \Delta_r)dtd\phi \\ &+\frac{\sin^2\theta}{\rho^2\Xi^2}((r^2+a^2)^2\Delta_\theta - \Delta_r a^2\sin^2\theta)d\phi^2 \\ &+\frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \end{aligned} \tag{1}$$

where,

$$\Delta_r = (r^2+a^2)(1-H_0^2r^2) - 2Mr, \qquad \Delta_\theta = 1 + H_0^2a^2\cos^2\theta,$$
$$\Xi = 1 + H_0^2a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta \tag{2}$$

where $H_0^2 = \Lambda/3$ with Λ being the positive cosmological constant and M and a are usually respectively called the mass and angular momentum parameters. Setting $a = 0$ recovers the Schwarzschild-de Sitter spacetime whereas setting further $M = 0$ recovers the de Sitter spacetime written in the static patch. Setting $M = 0$ alone results in a line element diffeomorphic to the de Sitter spacetime [33]. The cosmological and the black hole event horizons are respectively given by the largest (say, r_C) and the next to the largest (say, r_H) roots of $\Delta_r = 0$. The smallest positive root, $r = r_-$ of $\Delta_r = 0$ corresponds to the inner or the Cauchy horizon.

The location of the ergosphere for a stationary axisymmetric spacetime is given by $g_{tt} = 0$ [1]. Unlike the $\Lambda \leq 0$ cases, the KdS spacetime is endowed with two, instead of one ergosphere as follows. Since $g_{tt} > 0$ at $\Delta_r = 0$, the Killing vector field $(\partial_t)^a$ is spacelike on the horizons. Thus $(\partial_t)^a$ must be null at some points off the horizons. Since there exists a region in between the black hole and the cosmological event horizon where $(\partial_t)^a$ is timelike, it is clear that we must have two surfaces on which $g_{tt} = 0$, giving two ergospheres associated with the two Killing horizons. We shall call them the black hole ergosphere and the cosmological ergosphere and will investigate the Penrose process for both of them.

The surface gravities of the BEH and the CEH of Eq. (1) are respectively given by

$$\kappa_H = \frac{r_H(1 - 2H_0^2 r_H^2 - H_o^2 a^2) - M}{r_H^2 + a^2},$$
$$-\kappa_C = \frac{r_C(1 - 2H_0^2 r_C^2 - H_o^2 a^2) - M}{r_C^2 + a^2} \tag{3}$$

where κ_C is positive and the minus sign in front of it indicates repulsive effects. The areas of the BEH and CEH are respectively given by

$$A_H = \frac{4\pi(r_H^2 + a^2)}{1 + H_0^2 a^2}, \qquad A_C = \frac{4\pi(r_C^2 + a^2)}{1 + H_0^2 a^2}. \tag{4}$$

We have the angular speeds $(= -g_{t\phi}/g_{\phi\phi})$ on the horizons,

$$\Omega_H = \frac{a\Xi}{r_H^2 + a^2}, \qquad \Omega_C = \frac{a\Xi}{r_C^2 + a^2} \tag{5}$$

and also the horizon Killing vector fields,

$$\chi_H^a = (\partial_t)^a + \Omega_H(\partial_\phi)^a, \qquad \chi_C^a = (\partial_t)^a + \Omega_C(\partial_\phi)^a, \tag{6}$$

which are future directed and null on the respective horizons.

We further note below some general features regarding the comparative horizons' radii of the KdS, Kerr, Schwarzschild-de Sitter (SdS) and empty de Sitter spacetimes, which are useful for our future purposes, and are derived in Appendix A. a) The CEH of KdS is smaller than the empty de Sitter radius, H_0^{-1}. b) The BEH of KdS is smaller than that of SdS but the CEH of KdS is larger than the CEH of SdS (the parameter M is held fixed). c) The BEH of KdS is larger than the BEH of the Kerr spacetime, with the parameters M and a held fixed. Since the minimum horizon length of the Kerr spacetime is M or a (the extremal case), clearly for KdS we have a lower bound on the BEH radius, $r_H > M$ or $r_H > a$. Also, as we increase H_0, the BEH radius of KdS increases whereas the CEH radius decreases. All these conclusions on horizon sizes in Appendix A are derived by playing with the sign of the function Δ_r [Eq. (2)] only, in different spacetime regions.

We also note here the limits of the various parameters for a regular KdS spacetime derived in Appendix B, starting from a triply degenerate limit in which the inner horizon, the BEH and CEH coincide,

$$MH_0 \leq 0.2435 \quad \text{and} \quad \frac{a}{M} \leq 1.01 \quad \text{and} \quad aH_0 \leq 0.246. \tag{7}$$

Clearly only any two of the above inequalities can be independent. Note that the upper bound on aH_0 matches the one found numerically in Ref. [25]. Note also that $a = M$ is not the extremal limit of KdS, which was pointed out earlier in Ref. [34] via analyzing the parameter space.

III. CALCULATION OF THE PENROSE PROCESS

We shall briefly sketch below the derivation of the variable separated geodesic equation for the KdS spacetime, first done in Ref. [31]. The stationarity and axisymmetry of the KdS spacetime permits two conserved quantities for a geodesic u^a: the energy and the orbital angular momentum, respectively given by

$$E = -g_{ab}(\partial_t)^a u^b, \qquad L = g_{ab}(\partial_\phi)^a u^b. \tag{8}$$

In order to analyze geodesics and perform a variable separation in Eq. (1), the Hamilton-Jacobi equation is useful [2,31],

$$2\frac{\partial S}{\partial \lambda} = g^{ab}\left(\frac{\partial S}{\partial x^a}\right)\left(\frac{\partial S}{\partial x^b}\right) \tag{9}$$

where λ is an affine parameter along the geodesic and we take the ansatz for Hamilton's principle function S as

$$S = -\frac{1}{2}\delta\lambda - Et + L\phi + S_r(r) + S_\theta(\theta) \tag{10}$$

where $\delta = 1(0)$ for timelike (null) geodesics. Equation (9) can be written explicitly as

$$\rho^2\delta=\frac{(r^2+a^2)^2\Delta_\theta-\Delta_r a^2\sin^2\theta}{\Delta_r\Delta_\theta}E^2 -\frac{2\Xi a((r^2+a^2)\Delta_\theta-\Delta_r)}{\Delta_r\Delta_\theta}EL-\frac{\Xi^2(\Delta_r-a^2\sin^2\theta\Delta_\theta)}{\Delta_r\Delta_\theta\sin^2\theta}L^2 -\Delta_\theta\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2-\Delta_r\left(\frac{\partial S_r(r)}{\partial r}\right)^2 \tag{11}$$

which, after a little rearrangement could be written as

$$\Delta_r\left(\frac{\partial S_r(r)}{\partial r}\right)^2-\frac{(r^2+a^2)^2}{\Delta_r}\left(E-\frac{a\Xi L}{r^2+a^2}\right)^2 +(\Xi L-aE)^2+\delta r^2 =-\Delta_\theta\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2-\frac{\sin^2\theta}{\Delta_\theta}\left(aE-\frac{\Xi L}{\sin^2\theta}\right)^2 +(\Xi L-aE)^2-\delta a^2\cos^2\theta. \tag{12}$$

Thus the left-(right-)hand side of the above equation is a function of $r(\theta)$ only. This is only possible if each side equals a constant, known as the Carter constant, say, $-\zeta$. Also, the constant $(\Xi L-aE)^2$ appearing on both sides guarantees the recovery of the correct equation in the static limit ($a=0$). A proof that $\zeta\geq 0$ is given in Appendix C.

The momenta along the radial and polar directions are given by

$$p_r=\frac{\partial S_r}{\partial r},\qquad p_\theta=\frac{\partial S_\theta}{\partial\theta}. \tag{13}$$

Thus, $p^r\equiv\dot r=g^{rr}p_r$ and $p^\theta\equiv\dot\theta=g^{\theta\theta}p_\theta$, giving us the radial and polar equations for the geodesic,

$$\rho^4\dot r^2=(r^2+a^2)^2\left(E-\frac{a\Xi L}{r^2+a^2}\right)^2 -\Delta_r(\zeta+\delta r^2+(\Xi L-aE)^2),$$
$$\rho^4\dot\theta^2=-\frac{1}{\sin^2\theta}(Ea\sin^2\theta-\Xi L)^2 +\Delta_\theta(\zeta-\delta a^2\cos^2\theta+(\Xi L-aE)^2). \tag{14}$$

Let us now derive the Penrose process following Ref. [2]. Imagine that one massive particle moving along a timelike geodesic enters the ergosphere and reaches one turning point ($\dot r=0$) and there it breaks into two massless pieces with negative and positive energies. For $\dot r=0$, we obtain from the first line of Eq. (14),

$$E=\frac{\Xi aLx\pm[\Xi^2a^2L^2x^2+(r^2+a^2(1+x))(\Xi^2L^2(1-x)+\Delta_r(\delta+\zeta/r^2))]^{1/2}}{[r^2+a^2(1+x)]} \tag{15}$$

or alternatively,

$$L=\frac{-aEx\pm[a^2E^2x^2+(1-x)(E^2(r^2+a^2(1+x))-\Delta_r(\delta+\zeta/r^2))]^{1/2}}{\Xi(1-x)} \tag{16}$$

where for the sake of brevity we have written,

$$x=H_0^2(r^2+a^2)+\frac{2M}{r}. \tag{17}$$

Let us now consider a null geodesic ($\delta=0$) on the equatorial plane $\theta=\pi/2$ (i.e., $\zeta=0$) and consider Eq. (15) first, where, in order to have positive energy in the limit $a\to 0$, we must retain only the positive sign. On the other hand for $a\neq 0$, a necessary criterion for having negative energy is $L<0$ (i.e., counter-rotating orbits with respect to the direction of the black hole angular momentum). Then it is clear that along with the necessary condition $L<0$, in order to have negative energy we must also have

$$1-2M/r-H_0^2(r^2+a^2)<0. \tag{18}$$

Since $g_{tt}(\theta=\pi/2)=-(1-H_0^2(r^2+a^2)-2M/r)$, the above inequality clearly represents the inside-ergosphere region on the equatorial plane. A little algebra simplifies Eq. (16) to

$$L=\frac{E\left[-a(\frac{2M}{r}+H_0^2(r^2+a^2))\pm\sqrt{\Delta_r}\left(1-\frac{\delta(1-\frac{2M}{r}-H_0^2(r^2+a^2))}{E^2}\right)^{1/2}\right]}{\Xi(1-\frac{2M}{r}-H_0^2(r^2+a^2))}. \tag{19}$$

Let us now consider a massive particle ($\delta = 1$) with energy $E^{(0)} > 0$ and angular momentum $L^{(0)}$ entering the ergosphere. Let the particle be broken into two massless ($\delta = 0$) pieces with energies and angular momenta $(E^{(1)}, L^{(1)})$ and $(E^{(2)}, L^{(2)})$ respectively, with one crossing the black hole horizon and the other coming out of the ergoregion. We have from Eq. (19)

$$L^{(0)} = E^{(0)} \frac{\left[-a(\frac{2M}{r} + H_0^2(r^2 + a^2)) + \sqrt{\Delta_r}\left(1 - \frac{(1-\frac{2M}{r}-H_0^2(r^2+a^2))}{(E^{(0)})^2}\right)^{1/2}\right]}{\Xi(1 - \frac{2M}{r} - H_0^2(r^2 + a^2))} = \alpha^{(0)}(r, E^{(0)})E^{(0)} \quad \text{(say)},$$
$$L^{(1)} = E^{(1)} \frac{[-a(\frac{2M}{r} + H_0^2(r^2 + a^2)) - \sqrt{\Delta_r}]}{\Xi(1 - \frac{2M}{r} - H_0^2(r^2 + a^2))} = \alpha^{(1)}(r)E^{(1)} \quad \text{(say)},$$
$$L^{(2)} = E^{(2)} \frac{[-a(\frac{2M}{r} + H_0^2(r^2 + a^2)) + \sqrt{\Delta_r}]}{\Xi(1 - \frac{2M}{r} - H_0^2(r^2 + a^2))} = \alpha^{(2)}(r)E^{(2)} \quad \text{(say)}. \tag{20}$$

Using the above equations, the stationary-axisymmetric conservation laws read

$$E^{(1)} + E^{(2)} = E^{(0)},$$
$$L^{(1)} + L^{(2)} = \alpha^{(1)}(r)E^{(1)} + \alpha^{(2)}(r)E^{(2)} = L^{(0)} = \alpha^{(0)}(r, E^{(0)})E^{(0)} \tag{21}$$

which can be solved to get the energies

$$E^{(1)} = \frac{\alpha^{(0)}(r, E^{(0)}) - \alpha^{(2)}(r)}{\alpha^{(1)}(r) - \alpha^{(2)}(r)} E^{(0)},$$
$$E^{(2)} = \frac{\alpha^{(1)}(r) - \alpha^{(0)}(r, E^{(0)})}{\alpha^{(1)}(r) - \alpha^{(2)}(r)} E^{(0)}. \tag{22}$$

We find using Eq. (20),

$$E^{(1)} = -\frac{1}{2}\left[\left(1 - \frac{(1 - \frac{2M}{r} - H_0^2(r^2 + a^2))}{(E^{(0)})^2}\right)^{1/2} - 1\right]E^{(0)},$$
$$E^{(2)} = \frac{1}{2}\left[\left(1 - \frac{(1 - \frac{2M}{r} - H_0^2(r^2 + a^2))}{(E^{(0)})^2}\right)^{1/2} + 1\right]E^{(0)}. \tag{23}$$

Thus it is clear once again that energy extraction i.e., $E^{(2)} > E^{(0)}$ would be possible when we are inside the ergosphere [Eq. (18)]. The amount of energy extracted is given by

$$\delta E = -E^{(1)}. \tag{24}$$

Thus the maximum of the energy extracted would correspond to the minimum (negative) value of the function $(1 - 2M/r - H_0^2(r^2 + a^2))$, which certainly corresponds to the horizon, $\Delta_r = 0$ and we have

$$\delta E_{\max} = \frac{1}{2}\left[\left(1 + \frac{a^2}{r_H^2 (E^{(0)})^2}\right)^{1/2} - 1\right]E^{(0)}. \tag{25}$$

The above expression is formally similar for black hole spacetimes with or without a Λ, the effect of which implicitly comes through the value of r_H. For the extremal Kerr black hole ($a = M, H_0 = 0$), one has $r_H = M$ and $1 + a^2/r_H^2 = 2$. Taking $E^{(0)} = 1$, we get $\delta E_{\max} = 0.207$ [2].

For KdS, as we have seen that r_H increases as we increase H_0 (Sec. II), we conclude that a positive Λ *always* reduces the efficiency of the Penrose process, for given values of a and $E^{(0)}$. In particular as a special case, if we consider the triply degenerate limit of KdS (Appendix B), we get using Eq. (B7) and Eq. (B8),

$$\delta E_{\max} = \frac{1}{2}\left[\left(1 + \frac{0.469}{(E^{(0)})^2}\right)^{1/2} - 1\right]E^{(0)}. \tag{26}$$

For the customary value $E^{(0)} = 1$, we find $\delta E_{\max} = 0.106$ which is half of the result for the extremal Kerr black hole. We shall come back to this issue in Sec. IV.

A. Is energy extraction using the cosmological ergosphere possible?

Let us imagine that a particle carrying positive energy and angular momentum breaks into two fragments in the *cosmological ergoregion*. One of the fragments, carrying negative energy and angular momentum crosses the CEH and escapes while the other carrying positive energy and angular momentum enters the region inside. The rotation of the black hole induces a frame-dragging effect on Λ, as is evident from the $H_0^2 a^2$ term appearing in various metric functions. Thus in the cosmological ergoregion, the Penrose process, if any, is supposed to steal rotational kinetic energy from the cosmological constant, in the region beyond the CEH. However, such a process is never possible, as can be seen below.

First for the black hole, it is clear that in order to ensure that the particle carrying positive energy and angular momentum indeed comes out of the ergosphere instead of falling into the hole, we must have a turning point, say $r = r_T$, somewhere in between the ergosphere and the horizon along its trajectory [2].[1] Let us consider the first line in Eq. (14) with $\dot{r}^2(r = r_T + \delta r_T; \delta r_T > 0) = 0^+$. Thus if we move inward, for $E, L > 0$, we must have $\dot{r}^2(r = r_T) = 0$, for the turning point r_T to exist. Since both Δ_r and $(E - a\Xi L/(r^2 + a^2))^2$ decrease with decreasing r, clearly it is possible to have such turning points. In the cosmological ergoregion on the other hand, for $E, L > 0$, let us imagine a point where $\dot{r}^2(r = r_T - \delta r_T; \delta r_T > 0) = 0^+$. Since the function $(E - a\Xi L/(r^2 + a^2))$ increases with increasing r, whereas Δ_r decreases, we must have $\dot{r}^2(r = r_T) > 0^+$, always. This shows that there can be no such turning point and both positive and negative-energy fragments, if they are created in the cosmological ergoregion, would eventually cross the CEH and disappear.

Thus only in the black hole ergoregion is energy extraction via the Penrose process possible. Also, after extracting energy from the black hole, the positive-energy ejecta can reach the CEH and escape, eventually decreasing the energy and angular momentum of the spacetime region bounded by the CEH.

IV. INEQUALITIES FOR THE LOCAL SPEEDS OF FRAGMENTS

So far we have seen that in order to extract energy, we must have $L < 0$. The Wald inequality [6] and the Bardeen-Press-Teukolsky inequality [7] (see also Ref. [2]) further establish lower bounds on ejecta particles' speeds, in order that the Penrose process indeed occurs. We shall consider these inequalities in the Kerr-de Sitter spacetime below, in order to show that a positive Λ increases those lower bounds. The derivations presented below are parallel to that of the Kerr geometry and hence we shall not go into much detail, instead referring the reader to the above references.

A. The Wald inequality

Let us imagine a test particle moving along a geodesic with four-velocity u^a $(u \cdot u = -1)$ and conserved energy $E > 0$ that breaks into two fragments and let v^a $(v \cdot v = -1)$ and ϵ respectively be the four-velocity and conserved energy of one of them. Let us erect an orthonormal basis ${e_{(\mu)}}^a$ (the greek index within parentheses represents the local Lorentz frame) and let $u^a = {e_{(0)}}^a$. Expanding v^a then in the orthonormal basis $v^a = {e_{(\mu)}}^a u^{(\mu)}$, we have

$$v^a = \frac{(u^a + v^{(i)} {e_{(i)}}^a)}{\sqrt{1 - v^{(i)} v_{(i)}}} \quad (i = 1, 2, 3). \tag{27}$$

[1] We have seen in the preceding discussions that the maximum amount of energy is extracted when the turning point is located on the horizon.

We next expand the timelike Killing vector field $(\partial_t)^a$ in the orthonormal basis,

$$(\partial_t)^a = (\partial_t)^{(0)} u^a + {e_{(i)}}^a (\partial_t)^{(i)}. \tag{28}$$

Thus the conserved energy of the initial particle, $E = -g_{ab} u^a (\partial_t)^b$, can be written in the orthonormal basis as

$$E = (\partial_t)^{(0)}. \tag{29}$$

Also

$$\begin{aligned} g_{tt} = g_{ab}(\partial_t)^a(\partial_t)^b &= -((\partial_t)^{(0)})^2 + (\partial_t)^{(i)}(\partial_t)_{(i)} \\ &= -E^2 + (\partial_t)^{(i)}(\partial_t)_{(i)}. \end{aligned} \tag{30}$$

Likewise the energy ϵ of one the fragments is given by

$$\epsilon = -(\partial_t)^a v_a = \frac{E - |\mathbf{v}||\partial_\mathbf{t}| \cos\varphi}{\sqrt{1 - |\mathbf{v}|^2}} \tag{31}$$

where we have written $|\mathbf{v}|$ and $|\partial_\mathbf{t}|$ respectively as the norms of the spatial parts of the four-velocity and the timelike Killing vector field in the orthonormal frame and φ is the angle between them. Using Eq. (30) and Eq. (1) the above equation becomes

$$\epsilon = \frac{E - |\mathbf{v}| \left(E^2 - \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2}\right)^{1/2} \cos\varphi}{\sqrt{1 - |\mathbf{v}|^2}}. \tag{32}$$

This gives the Wald inequality for the Kerr-de Sitter spacetime

$$\begin{aligned} &\frac{E - |\mathbf{v}| \left(E^2 - \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2}\right)^{1/2}}{\sqrt{1 - |\mathbf{v}|^2}} \\ &\quad \leq \epsilon \leq \frac{E + |\mathbf{v}| \left(E^2 - \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2}\right)^{1/2}}{\sqrt{1 - |\mathbf{v}|^2}}. \end{aligned} \tag{33}$$

Thus in order to have a negative value of ϵ, we must have

$$|\mathbf{v}| > \frac{E}{\left(E^2 - \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2}\right)^{1/2}}. \tag{34}$$

It is clear that the above lower bound has a minimum at $\theta = \pi/2$ and on the BEH

$$|\mathbf{v}| > \frac{E}{\left(E^2 + \frac{a^2}{r_H^2}\right)^{1/2}}. \tag{35}$$

Since r_H increases with increasing H_0, the above expression shows that $|\mathbf{v}|$ would be higher with higher values of

H_0 (with E, a held fixed). We can compare the extreme cases here as well. For an extreme Kerr black hole, we have $a/r_H = 1$. On the other hand, for the triply degenerate KdS solution, using Eq. (B5), Eq. (B6) and Eq. (B7), we find $(a/r_H)^2 = (2\sqrt{3} - 3) \approx 0.464$. Thus taking the customary value $E = 1$, we find for the extreme Kerr spacetime, $|\mathbf{v}| > 0.707$ whereas in our triply degenerate limit, we obtain $|\mathbf{v}| > 0.825$.

B. The Bardeen-Press-Teukolsky inequality

The Bardeen-Press-Teukolsky inequality establishes a result analogous to the above as follows. Let us consider two particles with energies E_+ and E_- that collide at a point. We erect an orthonormal basis ${e_{(\mu)}}^a$ as earlier (with $e_{(0)} = u^a$). Let us also suppose that in this frame the particles move with equal and opposite three-velocities, $v^{(i)}$ and $-v^{(i)}$. The magnitude of their local relative speed in this frame is found by using the velocity addition formula,

$$|\mathbf{v}_{\rm rel}| = \frac{2|\mathbf{v}|}{1 + |\mathbf{v}|^2}. \tag{36}$$

We wish to find a lower bound on $|\mathbf{v}_{\rm rel}|$ such that E_- could be negative. Using steps similar to the previous subsection, one gets

$$E_\pm = \frac{(\partial_t)^{(0)} \pm |\mathbf{v}||\partial_\mathbf{t}| \cos\varphi}{\sqrt{1 - |\mathbf{v}|^2}}. \tag{37}$$

Using the above expression, we find after some algebra,

$$|\mathbf{v}|^4 (E_+ + E_-)^2 - 2|\mathbf{v}|^2 (E_+^2 + E_-^2 + 2g_{tt}) + (E_+ - E_-)^2 \le 0 \tag{38}$$

which yields

$$|\mathbf{v}| \ge \frac{\sqrt{E_+^2 - \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2}} - \sqrt{E_-^2 - \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2}}}{(E_+ + E_-)}. \tag{39}$$

Considering the marginal case, $E_- = 0^-$, we find on the horizon for $\theta = \pi/2$,

$$|\mathbf{v}| \ge \frac{1}{E_+} \left(\sqrt{E_+^2 + \frac{a^2}{r_H^2}} - \frac{a}{r_H} \right). \tag{40}$$

Partially differentiating this with respect to r_H, we find

$$\frac{\partial |\mathbf{v}|}{\partial r_H} \ge \frac{a\left(1 - \frac{1}{(1 + r_H^2 E_+^2/a^2)^{1/2}}\right)}{r_H^2 E_+} \ge 0.$$

This shows as earlier that, since the increase in H_0 increases r_H, the lower bound of Eq. (40) increases with increasing value of the cosmological constant, while E_+ and a are held fixed.

Finally, taking once again the customary value $E_+ = 1$, one obtains for the extremal Kerr black hole $|\mathbf{v}| \ge 0.414$, which [using Eq. (36)] yields $|\mathbf{v}_{\rm rel}| \ge 0.707$. On the other hand, taking the triply degenerate extremal limit of KdS once again [Eq. (B8)], we find $|\mathbf{v}| \ge 0.529$. This yields $|\mathbf{v}_{\rm rel}| \ge 0.826$.

V. GENERALIZED AREA THEOREM FOR KERR-DE SITTER SPACETIME

So far we have established two things: a) a positive Λ weakens energy extraction via the Penrose process and b) energy extraction in the cosmological ergoregion is never possible. We shall now prove below that both horizon areas increase under this process, thereby providing physical evidence in favor of the generalized area theorem for rotating de Sitter black holes. This is not at all obvious *a priori*, owing to the results summarized at the end of Sec. II. In particular, we have seen that an increase in the parameter M increases r_H but decreases r_C whereas an increase in the parameter a does the opposite. Now, when ejecta (carrying positive energy and angular momentum) comes out of the black hole ergoregion and escapes through the cosmological event horizon, it is clear that the parameters M and a describing the spacetime metric in the region bounded by the two horizons decrease, yielding separate, opposite effects on r_H and r_C. Also, the Smarr formula for de Sitter black holes reads [35],

$$\int_\Sigma \delta T_{ab} (\partial_t)^a d\Sigma^b = -\frac{\kappa_C \delta A_C + \kappa_H \delta A_H}{8\pi} - \Omega_H^{\rm Rel} \delta J_H$$

where $\Omega_H^{\rm Rel}$ denotes the relative angular speed of the black hole horizon with respect to the cosmological horizon, J_H is the angular momentum of the black hole and Σ denotes a spacelike hypersurface located in between the two horizons; the surface gravities κ_H, κ_C are given by Eq. (3). For the Penrose process in particular, one may expect the left-hand side to be negative due to the energy extraction process. Likewise we should have $\delta J_H < 0$. Thus it is not obvious *a priori* whether under this process both A_H and A_C or at least the sum of them would indeed increase.

Hence it needs to be proven that under this process, nevertheless, the area theorem is preserved. There is another nontriviality pertaining to the definition of the mass in this case, as we shall see below. But before we go into that, let us first briefly recall how the area theorem in this process is established in the Kerr spacetime [1,2,5].

Since the horizon Killing vector field χ_H^a [Eq. (6)] is future-directed null on the BEH, we must have $-u_a \chi_H^a \ge 0$ there, giving [1],

$$E - L\Omega_H \ge 0. \tag{41}$$

The energy E and the angular momentum L of the particle change respectively, the mass M and the total angular momentum $J = Ma$ of the black hole. We have from the above equation

$$(r_H^2 + a^2)\delta M \geq a\delta J \tag{42}$$

where $\delta J = a\delta M + M\delta a$. The above inequality then can be rewritten as

$$(r_H^2 \delta M - Ma\delta a) \geq 0. \tag{43}$$

For the Kerr black hole, $r_H = M + \sqrt{M^2 - a^2}$ and the horizon area is given by $A_H = 4\pi(r_H^2 + a^2)$, which is rewritten as,

$$A_H = 8\pi(M^2 + M\sqrt{M^2 - a^2}). \tag{44}$$

Taking the first-order variation of the above equation, we find after a little algebra

$$\delta A_H = \frac{16\pi}{\sqrt{M^2 - a^2}}(r_H^2 \delta M - Ma\delta a) \geq 0 \tag{45}$$

where the inequality follows from Eq. (43). This proves that in the Penrose process the black hole horizon area always increases. On the other hand since $\partial A_H/\partial a \leq 0$ and $\partial A_H/\partial M \geq 0$ [Eq. (44)], and also A_H has its greatest value for $a = 0$ and the smallest value for $M = a$, it is clear that under the Penrose process, the black hole mass decreases more slowly than the angular momentum. A Kerr black hole thus would eventually evolve towards lower-spin states and starting from the extremal limit $a = M$ we cannot reach a naked curvature singularity for which $a > M$. From Eq. (44) and the fact that $\delta A_H \geq 0$, we can define an irreducible mass $M_{\rm irr}$ [5],

$$M_{\rm irr}^2 = \frac{1}{2}M\left(M + \sqrt{M^2 - a^2}\right) \tag{46}$$

which effectively increases in the Penrose process.

Let us come back to our focus: the KdS spacetime. We first note in this case that we cannot simply interpret the mass parameter M to be the mass or energy, as follows. Within BEH, apart from the mass M itself, there should be local positive energy due to the cosmological constant as well. Second, as we have also mentioned earlier, the rotation of the black hole induces rotational kinetic energy onto the cosmological constant due to the frame dragging, as is evident from the $H_0^2 a^2$ term appearing in various metric functions. Thus since any change in M and a due to the infall of a particle changes the horizon size too, we must take into account the changes in the aforementioned local energies associated with the cosmological constant as well.

Indeed, for the Schwarzschild-de Sitter black hole, one can have a local, Tolman-like mass function [43] (see also Ref. [44]),

$$M_{\rm loc}(r) = M + \frac{H_0^2 r^3}{2}$$

in terms of which the metric function reads

$$ds^2 = -(1 - 2M_{\rm loc}(r)/r)dt^2 + (1 - 2M_{\rm loc}(r)/r)^{-1}dr^2 + r^2 d\Omega^2.$$

For KdS, we *define* the following local, continuous, positive-definite effective mass function:

$$M_{\rm loc}(r) \coloneqq M + \frac{H_0^2 r}{2}(r^2 + a^2). \tag{47}$$

For $a = 0$, the mass function reduces to that of the static case whereas putting $\Lambda = 0$ recovers the standard mass parameter of the Kerr spacetime. Clearly, the above mass function takes care of the spacetime rotation. In terms of $M_{\rm loc}(r)$, the function Δ_r in Eq. (2) can be written as

$$\Delta_r = r^2 - 2M_{\rm loc}(r)r + a^2.$$

In order to further justify this choice, let us note that intuitively, the size of a black hole should always be expected to increase with the increase of the total mass or energy contained within it. Indeed, on the horizon $\Delta_r = 0$, we can write using $M_{\rm loc}(r)$ a transcendental equation,

$$r_H = M_{\rm loc,H} + \sqrt{M_{\rm loc,H}^2 - a^2} \tag{48}$$

where $M_{\rm loc,H}$ is the value of $M_{\rm loc}(r)$ on $r = r_H$. It is very easy to see from the above equation that $\partial_{M_{\rm loc,H}} r_H \geq 0$.

Having successfully identified the mass function, the rest becomes straightforward. Equation (42) now takes the form,

$$(r_H^2 + a^2)\delta M_{\rm loc,H} \geq a\delta J_{\rm loc,H} \tag{49}$$

with $J_{\rm loc,H} \coloneqq aM_{\rm loc,H}$ and $\delta J = a\delta M_{\rm loc,H} + M_{\rm loc,H}\delta a$. Thus we have

$$(r_H^2 \delta M_{\rm loc,H} - M_{\rm loc,H} a\delta a) \geq 0. \tag{50}$$

The variation of the black hole horizon area is found from Eq. (4),

$$\delta A_H = 4\pi \frac{\delta(r_H^2 + a^2)}{1 + H_0^2 a^2} - 8\pi \frac{(r_H^2 + a^2)H_0^2 a\delta a}{(1 + H_0^2 a^2)^2}. \tag{51}$$

Using the transcendental equation (48) we can write

$$(r_H^2 + a^2) = 2\left(M_{\mathrm{loc,H}}^2 + M_{\mathrm{loc,H}}\sqrt{M_{\mathrm{loc,H}}^2 - a^2}\right).$$

A linear variation of the above equation and the use of Eq. (50) gives that the first term on the right-hand side of Eq. (51) is always greater than or equal to zero. On the other hand, since we have seen in Sec. III that the negative-energy particle entering the black hole must have negative angular momentum, we must have $\delta a < 0$ above, and thus $\delta A_H \geq 0$.

Let us now imagine that the particle moving outward carrying positive energy and angular momentum reaches the CEH and escapes through it. How does the area of the CEH change? We must use on the CEH,

$$E - \lambda\Omega_C \geq 0.$$

We also need to work with Eq. (47) evaluated at $r = r_C$,

$$M_{\mathrm{loc,C}} \coloneqq M + \frac{H_0^2 r_C}{2}(r_C^2 + a^2),$$
$$J_{\mathrm{loc,C}} \coloneqq a M_{\mathrm{loc,C}}. \tag{52}$$

We follow the same procedure as that for the BEH. Recalling that $\delta a < 0$ in this case as well (i.e., some positive angular momentum is going out off the region bounded by the CEH), we can show that $\delta A_C \geq 0$. This proves the generalized area theorem or the second law of thermodynamics under the Penrose process in the Kerr-de Sitter spacetime. To the best of our knowledge, this is the first explicit demonstration of that theorem for rotating black holes in de Sitter spacetime.

Finally, the irreducible mass function in Eq. (46) now takes two local values

$$M_{\mathrm{irr;H,C}}^2 = \frac{1}{2} M_{\mathrm{loc,H,C}}\left(M_{\mathrm{loc,H,C}} + \sqrt{M_{\mathrm{loc,H,C}}^2 - a^2}\right). \tag{53}$$

Let us now look at Eq. (A6) with $\delta H_0 = 0$,

$$\Delta_r(a + \delta a, M + \delta M) = \Delta_r(a, M) + 2a(1 - H_0^2 r^2)\delta a - 2r\delta M. \tag{54}$$

At the earlier location of the horizons, r_H, r_C [corresponding to $\Delta_r(a, M) = 0$], we have

$$\Delta_r(a + \delta a, M + \delta M)|_{r = r_H, r_C} = 2[a(1 - H_0^2 r_{H,C}^2)\delta a - r_{H,C}\delta M]. \tag{55}$$

For the Penrose process both δa and δM are negative. Now let us consider the increase in the black hole event horizon in this process, such that the earlier horizon radius r_H is now located within the new horizon radius. This means the left-hand side of the above equation is negative. Then since $a < r_H$ and $(1 - H_0^2 r^2) < 1$ for KdS (cf. Sec. II), it is clear that the change in the rotation parameter a should be faster than the change in the mass parameter M. On the other hand since the cosmological horizon also increases when the outgoing ejecta crosses it, the left-hand side of Eq. (55) must be positive on the earlier location of the CEH. Putting these all together, we have

$$\frac{a(1 - H_0^2 r_C^2)}{r_C} \leq \left(\frac{\delta M}{\delta a}\right)_{\mathrm{KdS}} \leq \frac{a(1 - H_0^2 r_H^2)}{r_H}. \tag{56}$$

Since the rotation parameter decreases faster than the mass parameter, under the Penrose process the KdS spacetime evolves to the Schwarzschild-de Sitter. Furthermore, from the area theorem we proved above along with the fact that the empty de Sitter horizon area $(4\pi/H_0^2)$ is greater than the sum of the two horizon areas of the Schwarzschild-de Sitter spacetime (see e.g., Ref. [37]), we find the same upper bound on the total horizon area of KdS,

$$(A_H + A_C)_{\mathrm{KdS}} < \frac{4\pi}{H_0^2}. \tag{57}$$

Finally, it is also clear that if we consider the Penrose process in the triply degenerate limit discussed in Appendix B, it would spin down making all the ratios a/M, aH_0 (also MH_0) [Eqs. (B6)–(B7)] smaller and would move away from the extremal point instead of creating a naked curvature singularity.

VI. SUMMARY AND OUTLOOK

In this work we have established three chief results: a) a positive Λ reduces the efficiency of the Penrose process, b) the Penrose process is never possible in the cosmological ergoregion and c) the generalized area theorem, or the second law of de Sitter black hole mechanics is satisfied in this process. Note that when one deals with the classical energy-momentum tensor T_{ab} of a matter field, the area theorem for a Killing horizon is satisfied if T_{ab} obeys the null or the strong energy condition, which essentially guarantees that the energy current corresponding to T_{ab} along any future-directed null vector is non-spacelike and future directed on the horizon [1]. Hence the causality conditions, $-u_a\chi_{H,C}^a \geq 0$, we used on the horizons may be interpreted as the analogues of such energy conditions, for the single particle picture we were concerned with. We have pointed out in relevant places, the qualitatively new local effects due to Λ, induced from the frame dragging. We also have derived various conclusions on the relative horizon sizes of rotating and nonrotating de Sitter black holes (Sec. II) and an upper bound on the total horizon area of KdS [Eq. (57)]. The inclusion of electric charge in our analysis should be absolutely straightforward.

Several new things could be investigated as follow-up works. For example, a systematic analysis of the superradiance for various spin fields (see Refs. [24–28] for scalar fields) in KdS and an evaluation of their profiles in various extremal limits would be highly interesting. Also, it seems very important to establish a topological version of the generalized area theorem for de Sitter black holes, analogous to that of the asymptotic flat spacetimes (see e.g., Ref. [1], and references therein). Apart from this, as we discuss in Appendix B, the triply degenerate extremal KdS solution, just like the doubly degenerate Nariai limit [25], might offer interesting quantum field theories. We shall come back to these issues in our future publications.

ACKNOWLEDGMENTS

I sincerely acknowledge T. Padmanabhan for various suggestions and valuable discussions, in particular, regarding the cosmological ergosphere. I would also like to thank N. K. Dadhich, K. Lochan and S. Chakraborty for useful discussions. I acknowledge anonymous referees for useful comments.

APPENDIX A: COMPARATIVE HORIZON SIZES OF EMPTY DE SITTER, STATIC AND ROTATING DE SITTER BLACK HOLES

We first note the different signs of the function Δ_r in Eq. (2) in different regions,

$$\Delta_r(r_H < r < r_C) > 0, \qquad \Delta_r(r_H, r_C, r_-) = 0,$$
$$\Delta_r(r_- < r < r_H) < 0, \qquad \Delta_r(r < r_-) > 0$$
$$\text{and} \quad \Delta_r(r > r_C) < 0. \tag{A1}$$

We write

$$\Delta_r = -H_0^2\left[r^2\left(r^2 - \frac{1}{H_0^2}\right) + a^2\left(r^2 - \frac{1}{H_0^2}\right) + \frac{2Mr}{H_0^2}\right]. \tag{A2}$$

For the empty de Sitter spacetime, the CEH is located at $r = H_0^{-1}$. Thus

$$\Delta_r(r = H_0^{-1}) = -\frac{2M}{H_0} < 0 \tag{A3}$$

which is only possible when $r = H_0^{-1}$ is located outside the CEH of KdS [Eq. (A1)]. Thus the CEH of KdS is smaller in size than that of the empty de Sitter spacetime. Likewise the horizons of the Schwarzschild-de Sitter ($a = 0$) spacetime are given by the zeros of the function

$$-\left[r\left(r^2 - \frac{1}{H_0^2}\right) + \frac{2M}{H_0^2}\right]. \tag{A4}$$

On the CEH of the empty de Sitter spacetime ($=H_0^{-1}$), the above function is negative. It is negative also at $r = 2M$, the horizon radius of the Schwarzschild spacetime. This shows that a) the CEH of the Schwarzschild-de Sitter spacetime is smaller than that of the empty de Sitter spacetime whereas b) the BEH of the same is larger than that of a Schwarzschild black hole with the same mass parameter.

On the Schwarzschild-de Sitter horizons [Eq. (A4) = 0], we have for KdS,

$$\Delta_r = -H_0^2 a^2\left(r^2 - \frac{1}{H_0^2}\right) \tag{A5}$$

which is positive because as we have seen, for both KdS and Schwarzschild-de Sitter, the cosmological horizon scale is smaller than that of empty de Sitter (H_0^{-1}). Since $\Delta_r > 0$ for KdS at the Schwarzschild-de Sitter horizon lengths, it is clear that the BEH inKdS is smaller than that of Schwarzschild-de Sitter, whereas the CEH is larger than the same (the parameter M held fixed).

Putting these all together, perhaps we could generically conjecture that a) an increase in M of a black hole increases its horizon size but decreases the size of the CEH, b) an increase in the cosmological constant increases the black hole size but does the opposite to the CEH, and c) an increase in rotation decreases the size of the black hole but increases the size of the CEH. Let us see these explicitly by making an infinitesimal change in the parameters $a \to a + \delta a$, $M \to M + \delta M$ and $H_0 \to H_0 + \delta H_0$. We have at the linear order,

$$\begin{aligned}&\Delta_r(a + \delta a, M + \delta M, H_0 + \delta H_0)\\ &= \Delta_r(a, M, H_0) - 2H_0 r^2(r^2 + a^2)\delta H_0\\ &\quad + 2a(1 - H_0^2 r^2)\delta a - 2r\delta M\end{aligned} \tag{A6}$$

and we evaluate the sign of the left-hand side at the old horizon points, $\Delta_r(a, M, H_0) = 0$. If we take $\delta M \neq 0$ only, we have $\Delta_r(a + \delta a, M + \delta M, H_0 + \delta H_0) < 0$ on the old horizons. This implies that for $\delta M > 0$ the size of the BEH/CEH has increased/decreased [Eq. (A1)]. Likewise for $\delta a \neq 0$ only, the size of the BEH/CEH decreases/increases for $\delta a > 0$, WHICH follows from our earlier result that the CEH size in KdS is always smaller than that of empty de Sitter. Finally, for $\delta H_0 > 0$ the size of the BEH/CEH increases/decreases. Thus, since the minimum horizon radius in the Kerr spacetime is either M or a, we must have for the KdS spacetime the lower bounds $r_H > M$ or $r_H > a$.

APPENDIX B: THE TRIPLY DEGENERATE LIMIT OF THE KERR-DE SITTER SPACETIME

We wish to derive a triply degenerate extremal limit of the Kerr-de Sitter spacetime. For a doubly degenerate or the Nariai limit, we refer the reader to Ref. [25]. We have to solve $\Delta_r = 0$ or equivalently,

$$r^4 - \frac{(1 - H_0^2 a^2) r^2}{H_0^2} + \frac{2Mr}{H_0^2} - \frac{a^2}{H_0^2} = 0. \tag{B1}$$

If $r_i (i = 1, 2, 3, 4)$ are the four roots of this equation, we have

$$r_1 + r_2 + r_3 + r_4 = 0, \qquad r_1 r_2 r_3 r_4 = -\frac{a^2}{H_0^2},$$
$$r_1 r_2 + r_2 r_3 + r_3 r_4 + r_4 r_1 + r_3 r_1 + r_2 r_4 = -\frac{(1 - H_0^2 a^2)}{H_0^2},$$
$$r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2 = -\frac{2M}{H_0^2}. \tag{B2}$$

The first of the above equations shows that not all of the four roots can be simultaneously real and positive. We wish to have three positive real roots which, in order of increasing size, are respectively the inner or the Cauchy horizon, the black hole event horizon and the cosmological event horizon. Then taking $r_4 < 0$ we get by using the first and second of the above equations,

$$r_1^2 + r_2^2 + r_3^2 + r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{(1 - H_0^2 a^2)}{H_0^2}. \tag{B3}$$

Since the left-hand side is positive definite, we immediately conclude that

$$a < \frac{1}{H_0}. \tag{B4}$$

We are chiefly interested here in finding analytically the triply degenerate situation. Writing $r_1 = r_2 = r_3 = r_0$, it is easy to obtain from Eq. (B2),

$$r_0 = \left(\frac{a^2}{3H_0^2}\right)^{1/4} = \frac{(1 - H_0^2 a^2)^{1/2}}{\sqrt{6} H_0} = \left(\frac{M}{4H_0^2}\right)^{1/3}. \tag{B5}$$

This shows that the root is expressible via only one parameter: M, H_0, or a. For instance, equating the first two and using Eq. (B4), we obtain

$$aH_0 = 2 - \sqrt{3} \tag{B6}$$

which is consistent with Eq. (B4). Likewise

$$MH_0 = \frac{4}{3^{3/4}} (2 - \sqrt{3})^{3/2} \quad \text{and} \quad \frac{a}{M} = \frac{3^{3/4}}{4(2 - \sqrt{3})^{1/2}} \approx 1.01 \tag{B7}$$

which shows that $a = M$ is not the extremal limit of KdS, as pointed out earlier in Ref. [34] by analysis of the parameter space of the KdS spacetime. We can reexpress the horizon length as

$$r_0 = \frac{\sqrt{3} M}{4(2 - \sqrt{3})} = \frac{(2 - \sqrt{3})^{1/2}}{3^{1/4} H_0} = \frac{a}{3^{1/4} (2 - \sqrt{3})^{1/2}} \approx 1.62M. \tag{B8}$$

If we have a charged black hole, the rotation parameter simply gets replaced by $\sqrt{a^2 + Q^2}$. We can compare Eq. (B7) with the known cases. For the extremal Kerr black hole ($H_0 = 0$), we must have $a/M = 1$. Also, for the static Nariai class of black holes ($a = 0$) we have $3M\sqrt{\Lambda} = 1 \equiv MH_0 = 0.192$.

We shall now also prove that the limits of Eqs. (B6)–(B7) are really extreme for if we try to increase the value of any of the parameters M, a and H_0, we create a naked curvature singularity, following Appendix A. First, recall that we have already proven that increasing H_0 or M would increase the BEH size but decrease the CEH size. Thus when they are coincident, increasing either or both of these parameters would certainly destroy both of them. What happens when we increase a? Let us recall that increasing a would decrease the BEH and increase the CEH, leading to no definite conclusion. However, note that the function Δ_r [Eq. (A2)] is negative in the region between the BEH and the Cauchy horizon. Then following exactly the same procedure as in Appendix A, we can show that increasing a actually increases the Cauchy horizon size. Since the BEH and the Cauchy horizon behave oppositely for $\delta a > 0$, certainly in the coincident limit, increasing a any further would destroy the horizon structure. In other words, in a regular KdS spacetime, the limits of Eq. (B6), Eq. (B7) or Eq. (7) must be satisfied. We can also compare this triply degenerate solution once again to the doubly degenerate Nariai one derived in Ref. [25], where only the BEH and the CEH are coincident. Note that for such a case, we may definitely increase a, making the CEH bigger and the BEH smaller, until the BEH becomes coincident with the Cauchy horizon.

Let us now also obtain a solution for the Reissner-Nördstrom-de Sitter solution with triply degenerate Killing horizons. The metric reads

$$ds^2 = -\left(1 - \frac{2M}{r} - H_0^2 r^2 + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} - H_0^2 r^2 + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{B9}$$

Like KdS, here also the horizons are determined by the roots of a quartic equation,

$$r^4 - \frac{r^2}{H_0^2} + \frac{2Mr}{H_0^2} - \frac{Q^2}{H_0^2} = 0. \tag{B10}$$

The analogue of Eq. (B2) now becomes

$$r_1+r_2+r_3+r_4=0, \qquad r_1r_2r_3r_4=-\frac{Q^2}{H_0^2},$$
$$r_1r_2+r_2r_3+r_3r_4+r_4r_1+r_3r_1+r_2r_4=-\frac{1}{H_0^2},$$
$$r_1r_2r_3+r_2r_3r_4+r_3r_4r_1+r_4r_1r_2=-\frac{2M}{H_0^2}. \qquad \text{(B11)}$$

We find for the triply degenerate case,

$$r_0=\left(\frac{Q^2}{3H_0^2}\right)^{1/4}=\frac{1}{\sqrt{6}H_0}=\left(\frac{M}{4H_0^2}\right)^{1/3} \qquad \text{(B12)}$$

with the conditions

$$QH_0=\frac{1}{2\sqrt{3}}, \qquad MH_0=\frac{\sqrt{2}}{3\sqrt{3}}, \qquad \frac{Q}{M}=\frac{3}{2\sqrt{2}}, \qquad \text{(B13)}$$

Note the chief difference between the known extremal black holes and the ones we have demonstrated: *all* known extremal black holes have doubly degenerate Killing horizons, whereas here we have *triply degenerate* ones, indicating a novel geometry and quantum field theory in the vicinity of such horizons. Hopefully such black holes can be important in the early Universe or in the context of the false vacuum decay scenario.

Certainly, such triply degenerate solutions are qualitatively different from their doubly degenerate Nariai counterparts, where the black hole and the cosmological event horizons are nearly coincident. For example, for the Reissner-Nördstrom-de Sitter spacetime, we can find such a Nariai solution for $Q=M$ and $4MH_0=1$, for which the black hole, cosmological and Cauchy horizons are respectively located at

$$r_H=\frac{1}{2H_0}=r_C, \qquad r_-=\frac{\sqrt{2}-1}{2H_0}. \qquad \text{(B14)}$$

APPENDIX C: A PROOF THAT CARTER'S CONSTANT ≥ 0

If the motion is confined to the equatorial plane $\theta=\pi/2$, clearly we have $\zeta=0$. Now on the right-hand side of Eq. (12), all but the third term are negative. Let us then consider the function

$$\frac{\sin^2\theta}{\Delta_\theta}\left(aE-\frac{\Xi L}{\sin^2\theta}\right)^2-(\Xi L-aE)^2.$$

Clearly the first term dominates as we move towards the axis, $\theta=0,\pi$. On the other hand on $\theta=\pi/2$ they become equal. Let us now check whether there exists any point in the intervals $0<\theta<\pi/2$ or $\pi/2<\theta<\pi$, where the second term dominates the first. If this is indeed the case, we shall have at least one zero of the above function in the above intervals and hence at least one extremum. However, it is easy to check that the only extremum of the above function is located at $\theta=\pi/2$, proving that the above function remains always greater than or equal to zero in the entire domain $0\leq\theta\leq\pi$. Thus the right-hand side of Eq. (12) is negative, proving $\zeta\geq 0$, *always*.

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