Post-Newtonian Gravitational Radiation from Point Masses in a Hyperbolic Kepler Orbit*

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The energy and the angular momentum radiated away in the form of gravitational waves from a system of two point particles with positive total energy are calculated in the lowest nonvanishing post-Newtonian approximation. By these radiations a particle arriving from infinity can be captured, and the cross section for such captures is determined as a function of the energy at infinity. The radiation from elliptic (bound) orbits can also be inferred, and is found to be in agreement with known results.

I. INTRODUCTION

The rate of radiation of energy and angular momentum by gravitational waves by a system of two point masses describing elliptic orbits about one another in the linearized theory of gravitation has been known for some time.[1,2] In this note, we shall consider the same problem for hyperbolic orbits. For such unbounded orbits one cannot, without justification, use the averaged formulas usually written for the rate of loss of energy and angular momentum by multiply periodic systems. To avoid possible ambiguity, we shall use the post-Newtonian formalism, which explicitly includes contributions that vanish on averaging.

II. THE TOTAL ENERGY RADIATED

The rate of loss of energy from a system by gravitational radiation, in the $2\frac{1}{2}$ post-Newtonian approximation,[3] is given by

$$\frac{dE}{dt}=-\frac{G}{c^5}\left[\frac{d^3I_{ij}}{dt^3}\frac{d^3I_{ij}}{dt^3}-\frac{1}{3}\frac{d^3I_{ii}}{dt^3}\frac{d^3I_{jj}}{dt^3}+\frac{d^2}{dt^2}\left(\frac{2}{3}\frac{d^3I_{ij}}{dt^3}\frac{dI_{ij}}{dt}+\frac{1}{3}\frac{d^3I_{ii}}{dt^3}\frac{dI_{jj}}{dt}-\frac{5}{6}\frac{d^2I_{ij}}{dt^2}\frac{d^2I_{ij}}{dt^2}\right)\right], \quad (1)$$

where I_{ij} is the ijth component of the moment-of-inertia tensor of the system. In the foregoing, the terms in the second derivatives will not contribute if the system is multiply periodic and we are interested only in the secular changes. These same terms in the second derivatives will also not contribute when integrated over an unbounded orbit if the terms take the same values at $+\infty$ and $-\infty$; and this clearly happens for the hyperbolic orbits.

We will assume that the orbit lies in the xy plane, and that the coordinates of the two mass points m_1 and m_2 are $(r_1\cos\theta, r_1\sin\theta)$ and $(-r_2\cos\theta, -r_2\sin\theta)$, respectively. We choose the origin at the center of mass, so that

$$r_1=\frac{m_2}{m_1+m_2}r,\quad r_2=\frac{m_1}{m_1+m_2}r.$$

Then the nonvanishing components of the moment-of-inertia tensor are

$$I_{xx}=\mu r^2\cos^2\theta,\quad I_{yy}=\mu r^2\sin^2\theta,\quad I_{xy}=I_{yx}=\mu r^2\sin\theta\cos\theta, \quad (2)$$

where $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass.

The equation of the orbit is

$$r=a(e^2-1)/(1-e\cos\theta), \quad (3)$$

where a is the semimajor axis and e is the eccentricity of the orbit; and the angular velocity along the orbit is given by

$$\frac{d\theta}{dt}=\frac{1}{r^2}[G(m_1+m_2)a(e^2-1)]^{1/2}. \quad (4)$$

The derivatives of the components of the moment-of-inertia tensor can be evaluated with the help of Eqs. (3) and (4); substituting the resulting expressions in (1), we obtain

$$\frac{dE}{dt}=-\frac{8}{15}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(e^2-1)^5}(1-e\cos\theta)^4[12(1-e\cos\theta)^2+e^2\sin^2\theta]+\frac{4}{15}\frac{G^3}{c^5}\frac{m_1{}^2m_2{}^2}{a^2(e^2-1)^2}\frac{d^2}{dt^2}[9(1-e\cos\theta)^3+3e^2\sin^2\theta(1-e\cos\theta)+\tfrac{5}{2}e^2(e-\cos\theta)^2]. \quad (5)$$

The total energy radiated is then the integral of this expression over the orbit.

The asymptotes of the hyperbolic orbit are given by

$$\cos\theta_0 = 1/e\,; \tag{6}$$

thus we have

$$\begin{aligned}\Delta E &= \int_{-\infty}^{\infty} \frac{dE}{dt}\,dt \\ &= -\frac{8}{15}\frac{G^{7/2}}{c^5}\frac{m_1^{\,2}m_2^{\,2}(m_1+m_2)^{1/2}}{a^{7/2}(e^2-1)^{7/2}}\int_{\theta_0}^{2\pi-\theta_0}(1-e\cos\theta)^2[12(1-e\cos\theta)^2+e^2\sin^2\theta]\,d\theta \\ &= -\frac{2}{15}\frac{G^{7/2}}{c^5}\frac{m_1^{\,2}m_2^{\,2}(m_1+m_2)^{1/2}}{a^{7/2}(e^2-1)^{7/2}}[(\pi-\theta_0)(96+292e^2+37e^4)+\tfrac{1}{3}e\sin\theta_0(602+457e^2)]\,.\end{aligned} \tag{7}$$

The explicit appearance of the angle θ_0 in this expression is to be noted.

The energy radiated during one period from particles describing an elliptic orbit can be obtained from Eq. (7) by the obvious substitutions $e \to -e$, $e^2-1 \to 1-e^2$. Also, the terms in θ_0 in the equation now vanish. Thus we obtain

$$\Delta E = -\frac{2}{15}\frac{G^{7/2}}{c^5}\frac{m_1^{\,2}m_2^{\,2}(m_1+m_2)^{1/2}}{a^{7/2}(e^2-1)^{7/2}}\,\pi(96+292e^2+37e^4) \tag{8}$$

for the energy radiated during one elliptic orbit. The average rate of radiation of energy, obtained from (8) by dividing the expression on the right-hand side by the period of the orbit, agrees with the formula given by Peters and Mathews.[1]

III. THE TOTAL ANGULAR MOMENTUM RADIATED

The post-Newtonian formalism[3] gives the rate of radiation of angular momentum as

$$\frac{dL_k}{dt} = \epsilon_{ijk}\frac{G}{c^5}\left\{\frac{2}{5}\frac{d^2I_{jl}}{dt^2}\frac{d^3I_{il}}{dt^3} + \frac{d}{dt}\left[\frac{3}{5}\left(\frac{d^4I_{jl}}{dt^4}I_{il} - \frac{d^3I_{jl}}{dt^3}\frac{dI_{il}}{dt}\right) + \frac{2}{3}\frac{d^3I_{il}}{dt^3}\mu x_i v_j - 2\left(\frac{d^3I_{il}}{dt^3} - \tfrac{1}{3}\delta_{il}\frac{d^3I_{mm}}{dt^3}\right)\mu v_l x_j\right]\right\}, \tag{9}$$

where $v_i = dx_i/dt$, and ϵ_{ijk} is the alternating symbol in three indices. The earlier remarks about the terms that appear as total time derivatives also apply to Eq. (9); therefore, the total angular momentum radiated during one orbit will be given by the secular terms alone. Substituting the expressions for the components of the moment-of-inertia tensor of the unbound two-particle system, we obtain

$$\begin{aligned}\frac{dL_z}{dt} &= -\frac{8}{5}\frac{G^{7/2}}{c^5}m_1^{\,2}m_2^{\,2}\left(\frac{m_1+m_2}{a^7(e^2-1)^7}\right)^{1/2}(1-e\cos\theta)^3[2(1-e\cos\theta)(2-e\cos\theta)-e^2\sin^2\theta] \\ &\quad -\frac{4}{15}\frac{G^3}{c^5}\frac{m_1^{\,2}m_2^{\,2}}{a^2(e^2-1)^2}\frac{d}{dt}\{e\sin\theta(1-e\cos\theta)[11(1-e\cos\theta)-5\sin^2\theta]\}\,.\end{aligned} \tag{10}$$

Thus we have

$$\begin{aligned}\Delta L_z &= \int_{-\infty}^{\infty}\frac{dL_z}{dt}\,dt \\ &= -\frac{8}{15}\frac{G^3}{c^5}\frac{m_1^{\,2}m_2^{\,2}}{a^2(e^2-1)^2}\int_{\theta_0}^{2\pi-\theta_0}(1-e\cos\theta)[2(1-e\cos\theta)(2-e\cos\theta)-e^2\sin^2\theta]\,d\theta \\ &= -\frac{8}{5}\frac{G^3}{c^5}\frac{m_1^{\,2}m_2^{\,2}}{a^2(e^2-1)^2}[(\pi-\theta_0)(8+7e^2)+e\sin\theta_0(13+e^2)]\,.\end{aligned} \tag{11}$$

Again we can obtain the angular momentum radiated during one period by a system describing an elliptic orbit from Eq. (11), by the substitutions $e \to -e$, $e^2-1 \to 1-e^2$; thus we have

$$\Delta L_z = -\frac{8}{5}\frac{G^3}{c^5}\frac{m_1^{\,2}m_2^{\,2}}{a^2(e^2-1)^2}\,\pi(8+7e^2)\,. \tag{12}$$

The corresponding expression for the average rate of radiation of angular momentum agrees with that found by Peters.[2]

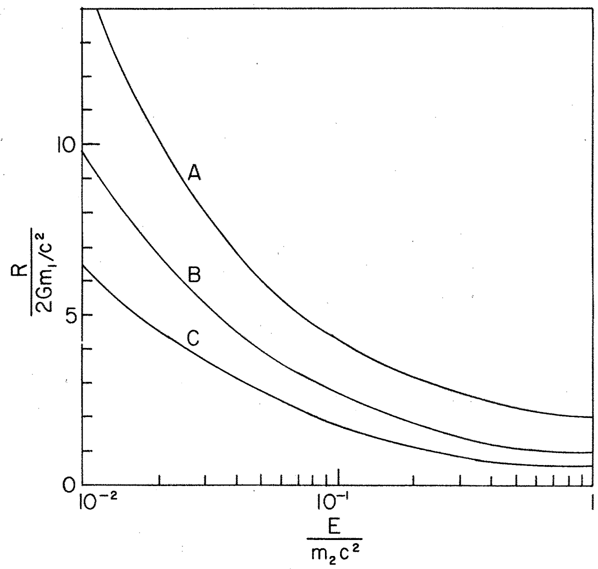

FIG. 1. Disk of capture as a function of energy. (*A*) $m_2/m_1=1$. (*B*) $m_2/m_1=0.1$. (*C*) $m_2/m_1=0.01$.

IV. THE RADIATIVE CAPTURE OF A TWO-BODY SYSTEM

The impact parameter of a Keplerian system is given by

$$R=\frac{Gm_1m_2}{2E}(e^2-1)^{1/2}, \tag{13}$$

or

$$\frac{R}{2Gm_1/c^2}=\tfrac{1}{4}(e^2-1)^{1/2}\,\frac{m_2c^2}{E}, \tag{14}$$

where

$$E=Gm_1m_2/2a. \tag{15}$$

For a particle to be just captured, the energy radiated away must equal the total energy of the system:

$$E=\frac{2}{15}\,\frac{G^{7/2}}{c^5}\,\frac{m_1^{\,2}m_2^{\,2}(m_1+m_2)^{1/2}}{a^{7/2}(e^2-1)^{7/2}}\times\{[\pi-\cos^{-1}(1/e)](96+292e^2+37e^4)+\tfrac{1}{3}e(e^2-1)^{1/2}(602+457e^2)\}\,; \tag{16}$$

or, eliminating a with the aid of Eq. (15), we have

$$\frac{E}{m_2c^2}=\left(\frac{15}{16\sqrt{2}}\right)^{2/5}\left(\frac{m_2}{m_1}\right)^{-2/5}\left(1+\frac{m_2}{m_1}\right)^{-1/5}\times\left(\frac{\pi-\cos^{-1}(1/e)}{(e^2-1)^{7/2}}(96+292e^2+37e^4)+\frac{1}{3}\,\frac{e}{(e^2-1)^3}(602+457e^2)\right)^{-2/5}. \tag{17}$$

Elimination of e between (14) and (17) will yield the required expression for the disk of capture as a function of energy; this elimination is most conveniently carried out numerically by choosing suitable values for the eccentricity of the orbit and solving for R and E from Eqs. (14) and (17). Figure 1 shows the result of this calculation. Of course, all systems with zero initial energy are captured; and the disk of capture shrinks to essentially a constant at sufficiently high energies. One cannot regard the curves toward the right-hand side of the graph as corresponding to a physically meaningful situation, as they represent relativisitic energies; furthermore, the disk of capture appears to be smaller than the Schwarzschild radius for curves B and C.

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[2]P. C. Peters, Phys. Rev. 136, B1224 (1964).

[3]S. Chandrasekhar and F. Paul Esposito, Astrophys. J. 160, 153 (1970).