Black hole perturbations in spatially covariant gravity with just two tensorial degrees of freedom

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We study linear perturbations around a static and spherically symmetric black hole solution in spatially covariant gravity with just two tensorial degrees of freedom. In this theory, gravity modification is characterized by a single time-dependent function that appears in the coefficient of K^2 in the action, where K is the trace of the extrinsic curvature. The background black hole solution is given by the Schwarzschild solution foliated by the maximal slices and has a universal horizon at which the lapse function vanishes. We show that the quadratic action for the odd-parity perturbations is identical to that in general relativity upon performing an appropriate coordinate transformation. This in particular implies that the odd-parity perturbations propagate at the speed of light, with the inner boundary being the usual event horizon. We also derive the quadratic action for even-parity perturbations. In the even-parity sector, one of the two tensorial degrees of freedom is mixed with an instantaneous scalar mode, rendering the system distinct from that in general relativity. We find that monopole and dipole perturbations, which are composed solely of the instantaneous scalar mode, have no solutions regular both at the universal horizon and infinity (except for the trivial one corresponding to the constant shift of the mass parameter). We also consider stationary perturbations with higher multipoles. By carefully treating the locations of the inner boundary, we show that also in this case there are no solutions regular both at the inner boundary and infinity. Thus, the black hole solution we consider is shown to be perturbatively unique.

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I. INTRODUCTION

Lovelock's theorem [1,2] states that a diffeomorphism-invariant theory constructed only from the metric tensor in four dimensions leads uniquely to the Einstein equations in general relativity (GR). To modify gravity, one must therefore break at least one of the postulates of the theorem, A typical way of modifying gravity is to add a new dynamical degree of freedom (d.o.f.). The simplest example is scalar-tensor gravity having one scalar and two tensorial d.o.f.s, which has been studied extensively, with a particular focus over the past decade on the Horndeski theory [3–5] and its extensions called degenerate higher-order scalar-tensor (DHOST) theories [6–11] (see Ref. [12] for a review). An apparently different approach is abandoning full diffeomorphism invariance. For example, one can consider spatially covariant theories respecting only spatial diffeomorphism invariance under the transformations of spatial coordinates, $x^i \to \tilde{x}^i(t, x^j)$ [13,14]. However, this approach is basically equivalent to adding new dynamical degrees of freedom and spatially covariant gravity is regarded as a gauge-fixed version of a fully covariant scalar-tensor theory. Indeed, starting from a spatially covariant theory, one can introduce a Stückelberg scalar field to restore full diffeomorphism invariance and write an action for a corresponding fully covariant scalar-tensor theory [15].

In scalar-tensor theories, the dynamical scalar d.o.f. (say ϕ) obeys a wavelike equation with some propagation speed c_s, the concrete form of which depends on the Lagrangian. An interesting twist is a case where ϕ is an instantaneous mode, $c_s = \infty$, which occurs when the form of the Lagrangian is chosen appropriately. In such theories, only tensorial d.o.f.s propagate, while the configuration of the instantaneous scalar d.o.f. is determined completely from boundary conditions. Within the so-called $P(\phi, X)$ theory [where $X = -(\partial\phi)^2/2$], the cuscuton theory, $P = \mu(\phi)\sqrt{X} - V(\phi)$, gives rise to such a nonpropagating scalar field [16], leading to interesting cosmology [17]. The cuscuton theory was extended in Ref. [18] by demanding that $c_s = \infty$ in the Horndeski and slightly more general theories. Performing a more rigorous Hamiltonian analysis, one can determine the subset of spatially covariant theories of gravity having two tensorial d.o.f.s (TTDOFs) only. This was done in the case where the action is linear in the lapse function in Ref. [19] and then the general conditions to render the scalar mode nondynamical were derived in Ref. [20]. Spatially covariant gravity satisfying these conditions is dubbed as a TTDOF theory. Since the dynamical d.o.f.s in TTDOF theories are the same as those

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in GR, TTDOF gravity may be thought of as minimal modification of GR. Determining the most general form of the action for TTDOF theories has turned out to be very difficult, and the authors of Ref. [20] managed to obtain a particular example of a family of TTDOF theories assuming that the action is quadratic in the extrinsic curvature and linear in the intrinsic curvature as in the Arnowitt-Deser-Misner (ADM) expression for the Einstein-Hilbert term. The resultant action contains that for the extended cuscuton theory as a special case.

A family of TTDOF theories obtained in Ref. [20] has several time-dependent parameters, which can in principle be freely chosen from a purely theoretical point of view. Some of them can however be fixed by requiring that the speed of gravitational waves is equal to that of light and the usual GR behavior is restored in the weak-gravity regime in the Solar System [21]. The TTDOF theory with the thus reduced parameter space can still mimic the background evolution of the standard ΛCDM model [22]. Interestingly, Ref. [22] studied the cosmic microwave background (CMB) constraints on the TTDOF cosmological model and reported a $\sim 4\sigma$ deviation from the ΛCDM model based on GR.

Ignoring the terms that are relevant only on cosmological scales, we have a TTDOF theory characterized by a single time-dependent parameter, the impacts of which are not seen in the propagation of gravitational waves and in the weak-gravity regime in the Solar System. The theory admits the Schwarzschild solution foliated by the maximal slices [21]. This can easily be seen by noting that the modified gravity parameter enters only in the coefficient of the K^2 term in the TTDOF action we are considering, where K is the trace of the extrinsic curvature. The situation here is quite similar to that of Einstein-Aether theory and the low-energy limit of Horava gravity [23,24]. Such a black hole solution in modified gravity is sometimes called stealth, as the geometry is described by the Schwarzschild solution in GR even though the Stückelberg scalar exhibits a nontrivial configuration. In this paper, we study linear perturbations around this stealth black hole solution in the TTDOF theory, paying particular attention to the behavior of the instantaneous scalar mode whose inner boundary conditions are imposed at the universal horizon (the location at which the lapse function vanishes) rather than the usual event horizon. One of our goals is to see whether the stealth Schwarzschild solution is unique or not. The question was partially addressed in Ref. [21], but there only static monopole perturbations were considered. In the present paper, we start with deriving quadratic actions for all (time-dependent) perturbations with higher multipoles, following closely the formulation of black hole perturbation theory in Horndeski [25,26] and degenerate higher-order scalar-tensor (DHOST) gravity [27–29] (see also Refs. [30–34]). This result itself has wider applications than merely investigating perturbative uniqueness of the background black hole solution. See also Ref. [35] for a related analysis of the behavior of the instantaneous scalar field around a black hole with a universal horizon.

This paper is organized as follows. In the next section, we briefly review the TTDOF theory and its static and spherically symmetric black hole solution. In Sec. III, we study the odd-parity perturbations. Then, we consider the even-parity perturbations, presenting the main results separately for monopole, dipole, and higher-multipole modes in Sec. IV. Finally, we draw our conclusions in Sec. V.

II. A BLACK HOLE SOLUTION IN TTDOF THEORY

In this section, we introduce the TTDOF theory [20] and its static and spherically symmetric black hole solution [21].

A. TTDOF theory

We consider modified gravity with just TTDOF developed in Ref. [20]. The symmetry of the theory is the spatial diffeomorphism invariance under $x^i \to \tilde{x}^i(t, x^j)$, and therefore we use the ADM variables to write the action. Specifically, the action that we consider in this paper is given by

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\left[K_{ij}K^{ij} - \frac{1}{3}\left(\frac{2N}{\beta + N} + 1\right)K^2 + R\right], \tag{1}$$

where N is the lapse function, R is the three-dimensional Ricci scalar calculated from the spatial metric γ_{ij}, and K_{ij} is the extrinsic curvature of constant time hypersurfaces,

$$K_{ij} \coloneqq \frac{1}{2N}(\partial_t\gamma_{ij} - D_iN_j - D_jN_i), \tag{2}$$

with N_i being the shift vector and D_i the covariant derivative operator associated with γ_{ij}. Here, $\beta = \beta(t)$ is an arbitrary function of time characterizing a modification of GR, and by setting $\beta = 0$ the action (1) reduces to the ADM expression of the Einstein-Hilbert action.

A family of TTDOF theories originally introduced in Ref. [20] has seven arbitrary functions of time, but we focus on its particular subset described by the action (1) with six of the functions being set to their "canonical" values. This is basically because we are interested in the phenomenologically interesting class of modified gravity that reproduces the standard behavior of gravity in the solar system (in the sense that the parametrized post-Newtonian parameter γ^{PPN} is given by $\gamma^{\mathrm{PPN}} = 1$) and in which gravitational waves propagate at the speed of light [21], while rendering the cosmology nontrivial [22]. We provide a more detailed discussion on this point in the Appendix.

One can recover the full four-dimensional diffeomorphism invariance by introducing the Stückelberg scalar field. The resultant covariant expression of the action (1) belongs to the so-called U-degenerate theory [21], which is a higher-order scalar-tensor theory satisfying the degeneracy conditions only when one takes the unitary gauge [36]. However, the apparent scalar degree of freedom is in fact an instantaneous mode having infinite propagation speed and obeying an elliptic equation. Therefore, the scalar field does not propagate and its behavior is determined completely by boundary conditions.

B. A black hole solution

The ADM variables for a static and spherically symmetric solution are of the form

$$N = N(r), \qquad N_i \mathrm{d}x^i = B(r)F(r)\mathrm{d}r,$$
$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = F^2\mathrm{d}r^2 + r^2\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b, \tag{3}$$

with $\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ $(a, b = \theta, \varphi)$. Note at this point that we cannot set $B = 0$ in general when working in the ADM action (i.e., in the unitary gauge), because we no longer have the freedom to perform a coordinate transformation $t \to \tilde{t}(t, r)$.

Substituting Eq. (3) to the action (1) and varying it with respect to N, B, and F, we obtain a set of the field equations for these variables. A general solution has not been derived, but one can at least find the following particular solution:

$$N = N_0\sqrt{f(r)}, \quad F(r) = \frac{1}{\sqrt{f(r)}}, \quad B(r) = \frac{N_0 b_0}{r^2}, \tag{4}$$

with

$$f(r) \coloneqq 1 - \frac{\mu_0}{r} + \frac{b_0^2}{r^4}, \tag{5}$$

where N_0, μ_0, and b_0 are integration constants. The solution represents a foliation of the Schwarzschild geometry by maximal slices ($K = 0$). Indeed, by performing the coordinate transformation

$$\tau = N_0 t - \int^r \frac{b_0/r^2}{\sqrt{f}(1 - \mu_0/r)}\mathrm{d}r, \tag{6}$$

and introducing the Stückelberg field $\phi = t(\tau, r)$ accordingly, one obtains $\mathrm{d}s^2 = -(1-\mu_0/r)\mathrm{d}\tau^2 + (1-\mu_0/r)^{-1}\mathrm{d}r^2 + r^2\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b$, which is nothing but the standard Schwarzschild metric. From this observation, we see that μ_0 is the mass parameter, b_0 characterizes the foliation, and N_0 just corresponds to the rescaling of the time coordinate. It should be emphasized that the TTDOF theory with the action (1) admits the above solution even if β is an arbitrary time-dependent function.

For $b_0 \le b_{0,c} \coloneqq 3\sqrt{3}\mu_0^2/16$, $N(r)$ vanishes at some $r(>0)$. The location at which $N(r) = 0$ is called the universal horizon, which is the causal boundary for the scalar mode with infinite propagation speed. Only for $b_0 = b_{0,c}$ the universal horizon is regular, while $f(r)$ changes its sign at the universal horizon for $b_0 < b_{0,c}$. In the rest of the paper, we basically consider the black hole solution with a regular universal horizon (though the perturbation equations can be derived without regard to the value of b_0). For $b_0 = b_{0,c}$, one has

$$f(r) = \left(1 - \frac{3\mu_0}{4r}\right)^2\left(1 + \frac{\mu_0}{2r} + \frac{3\mu_0^2}{16r^2}\right) \ge 0, \tag{7}$$

and the universal horizon is located at $r = 3\mu_0/4$. Constant t hypersurfaces and the regular universal horizon are depicted in the Penrose diagram of the Schwarzschild geometry in Fig. 1. For $b_0 > b_{0,c}$, $f(r)$ is positive everywhere, and therefore the instantaneous mode is accessible to the singularity at $r = 0$. We do not consider this case in this paper.

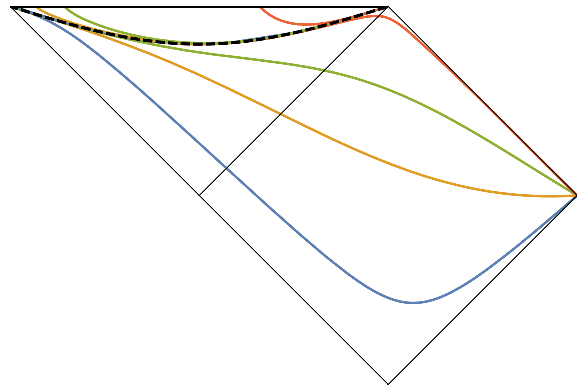

FIG. 1. Colored lines represent $t = \mathrm{const}$ hypersurfaces, and the black dashed line shows the location of the universal horizon, $r = 3\mu_0/4$.

Here it should be noted again that gravitational waves propagate at the speed of light in our TTDOF theory. While the causal boundary for the instantaneous scalar mode is given by the universal horizon at $r = 3\mu_0/4$, that for gravitational waves is still the usual event horizon at $r = \mu_0$.

In the rest of the paper, we set $\mu_0 = 1$ to simplify the expressions.

C. A test scalar field on the black hole background

We are going to discuss the metric perturbations on the black hole background introduced above, which are composed of the usual tensorial degrees of freedom and the instantaneous scalar mode. Since the system of the perturbation equations for the metric is highly involved, it is instructive to consider here a test scalar field $\pi(t, x^i)$ obeying an elliptic equation on constant t hypersurfaces as the simplest model of an instantaneous mode:

$$S_\pi \coloneqq -\frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\nabla_i\pi\nabla^i\pi. \tag{8}$$

This example helps us to see what will be done in the next two sections.

For the above black hole background, the equation of motion for π reads

$$f\pi'' + f'\pi' - \left[\frac{\ell(\ell+1)}{r^2} + \frac{f'}{r}\right]\pi = 0, \tag{9}$$

where a prime stands for differentiation with respect to r and the angular part of the Laplacian was replaced by the eigenvalue $-\ell(\ell+1)/r^2$. The natural inner boundary is seen to be the universal horizon at which $f = 0$. In the case of $\ell = 0$, it is easy to find the analytic solution

$$\pi(t,r) = r\left[c_1(t) + c_2(t)\int_r^\infty \frac{\mathrm{d}r}{r^2 f}\right], \tag{10}$$

where c_1 and c_2 are integration functions. For large r we have

$$\pi \simeq c_1 r + c_2\left(1 + \frac{1}{2r}\right), \tag{11}$$

while in the vicinity of the universal horizon, we have

$$\pi \simeq \frac{3c_2}{8(r-3/4)} - \frac{c_2}{3}\ln(r-3/4). \tag{12}$$

In order for π not to diverge at infinity, we require that $c_1 = 0$. However, if $c_2 \neq 0$ then π diverges at the universal horizon. Therefore, the only possible regular solution is given by $\pi = 0$.

We will perform a similar analysis for metric perturbations. In the case of metric perturbations, multiple components of the metric are coupled, leading to a more complicated system. Moreover, the system contains higher spatial derivatives (for even-parity perturbations with $\ell \geq 1$), allowing for more linearly independent solutions.

III. ODD-PARITY PERTURBATIONS

In this section and the next, we study linear perturbations around the black hole solution introduced in the previous section. We first consider odd-parity perturbations in this section. Only the gravitational-wave degrees of freedom participate in the odd-parity sector. Noting that the TTDOF theory (1) is built so that gravitational waves propagate in the same way as in GR, we expect that the odd-parity sector is identical to that in GR, except that the background metric is written in the nonstandard coordinate system. Indeed, a modification from GR appears only in the K^2 term in action, and it is easy to see that this term does not include any contribution from odd-parity perturbations. Below we will see this more explicitly.

Using the spherical harmonics $Y_{\ell m}(\theta,\varphi)$, we expand the odd-parity perturbations as

$$\delta N_a = \sum_{\ell,m} h_0^{(\ell m)}(t,r)\epsilon^b{}_a \partial_b Y_{\ell m}, \tag{13}$$

$$\delta\gamma_{ra} = \sum_{\ell,m} h_1^{(\ell m)}(t,r)\epsilon^b{}_a \partial_b Y_{\ell m}, \tag{14}$$

$$\delta\gamma_{ab} = \frac{1}{2}\sum_{\ell,m} h_2^{(\ell m)}(t,r)[\epsilon_a{}^c\nabla_c\nabla_b + \epsilon_b{}^c\nabla_c\nabla_a]Y_{\ell m}, \tag{15}$$

where ∇_a is the covariant derivative defined with respect to σ_{ab} and ϵ_{ab} is the Levi-Civita tensor with $\epsilon_{\theta\varphi} = \sin\theta$ and $\epsilon_\theta{}^\varphi = \epsilon_{\theta a}\sigma^{a\varphi} = 1/\sin\theta$. Note that $h_2^{(\ell m)} = 0$ for $\ell = 1$. Under a gauge transformation

$$x^a \to x^a + \xi^a, \qquad \xi_a \coloneqq \sum_{\ell,m} \xi^{(\ell m)}(t,r)\epsilon^b{}_a \partial_b Y_{\ell m}, \tag{16}$$

$h_2^{(\ell m)}$ transforms as

$$h_2^{(\ell m)} \to h_2^{(\ell m)} + 2r^2\xi^{(\ell m)}. \tag{17}$$

We thus choose to remove $h_2^{(\ell m)}$ for $\ell \geq 2$.

The quadratic action for the odd-parity perturbations can be written in the form

$$S^{(\mathrm{odd})} = \sum_{\ell,m}\int \mathrm{d}t\mathrm{d}r \mathcal{L}_{\ell m}^{(\mathrm{odd})}. \tag{18}$$

Let us consider the quadratic Lagrangian with $\ell \geq 2$ (the $\ell = 1$ sector must be treated separately). Thanks to the symmetry, it is sufficient to study the Lagrangian with $m = 0$:

$$\mathcal{L}_{\ell\geq 2, m=0}^{(\mathrm{odd})} = \frac{1}{4N_0 r^2}\left[2j^2 + \frac{c_\ell}{f}\right]h_0^2 - \frac{c_\ell N_0}{4r^2}\left(1 - \frac{1}{r}\right)h_1^2 + \frac{j^2}{4N_0}\left[(\dot{h}_1 - h_0')^2 + \frac{4}{r}h_0\dot{h}_1\right] - \frac{c_\ell}{2}\frac{b_0}{r^4\sqrt{f}}h_0 h_1, \tag{19}$$

where $j^2 \coloneqq \ell(\ell+1)$ and $c_\ell \coloneqq (\ell-1)\ell(\ell+1)(\ell+2)$. Here, a dot and a prime denote derivatives with respect to t and r, respectively. To simplify the expression, we omit the labels (ℓm) from h_0 and h_1. We find no terms that depend on β. Therefore, the Lagrangian (19) must coincide with that of the odd-parity perturbations of a Schwarzschild black hole in GR expressed in the nonstandard coordinate system. It is obvious that one can arrive at Eq. (19) even if β is a function of time.

To go back to the standard Schwarzschild coordinates, we use the time coordinate τ defined in Eq. (6) and define

$$\tilde{h}_0 \coloneqq \frac{h_0}{N_0}, \qquad \tilde{h}_1 \coloneqq h_1 + \frac{b_0/r^2}{\sqrt{f}(1-1/r)}\frac{h_0}{N_0}. \tag{20}$$

Then, the quadratic action for $\ell \geq 2$ and $m = 0$ can be written as

$$S^{(\mathrm{odd})}_{\ell\geq 2, m=0} = \frac{1}{4}\int \mathrm{d}\tau \mathrm{d}r\left\{\frac{1}{r^2}\left[2j^2 + \frac{c_\ell}{1-1/r}\right]\tilde{h}_0^2 - \frac{c_\ell}{r^2}\left(1-\frac{1}{r}\right)\tilde{h}_1^2 + j^2\left[(\partial_\tau \tilde{h}_1 - \tilde{h}_0')^2 + \frac{4}{r}\tilde{h}_0\partial_\tau\tilde{h}_1\right]\right\}. \tag{21}$$

This coincides with the quadratic action for the odd-parity perturbations of the Schwarzschild solution in GR under an appropriate identification of variables. Obviously, the same conclusion should hold for the dipole perturbation, which corresponds to a slow rotation of the black hole.

A comment is now in order. In Ref. [37], quasinormal frequencies of gravitational perturbations of a similar black hole in Einstein-Aether theory were calculated imposing the inner boundary conditions at the universal horizon. We argue, however, that the odd-parity metric perturbations in the present case obey the same equation as in GR, i.e., the Regge-Wheeler equation, and propagate at the speed of light, which indicates that the inner boundary must be set at the horizon for photons, i.e., the usual event horizon at $r = 1$.

IV. EVEN-PARITY PERTURBATIONS

Let us move to discuss the even-parity perturbations. In usual scalar-tensor theories in which the scalar field is dynamical, both the gravitational-wave and scalar degrees of freedom take part in the odd-parity sector. Therefore, even-parity perturbations with $\ell \geq 2$ are composed of a mixture of gravitational waves and the scalar, while monopole and dipole perturbations are composed solely of the scalar degree of freedom. A similar mixing occurs for $\ell \geq 2$ in a tricky way in the present case where the scalar is an instantaneous mode obeying an elliptic equation. Monopole and dipole perturbations in the present case are nondynamical and their configurations are determined entirely from boundary conditions. Below we will see these points in more detail.

We write the even-parity perturbations of the ADM variables as

$$\delta N = N(r)\sum_{\ell,m}\mathsf{H}_0^{(\ell m)}(t,r)Y_{\ell m}, \tag{22}$$

$$\delta N_r = \sum_{\ell,m}\mathsf{H}_1^{(\ell m)}(t,r)Y_{\ell m}, \tag{23}$$

$$\delta\gamma_{rr} = F^2(r)\sum_{\ell,m}\mathsf{H}_2^{(\ell m)}(t,r)Y_{\ell m}, \tag{24}$$

$$\delta N_a = \sum_{\ell,m}\mathsf{b}^{(\ell m)}(t,r)\partial_a Y_{\ell m}, \tag{25}$$

$$\delta\gamma_{ra} = \sum_{\ell,m}\mathsf{a}^{(\ell m)}(t,r)\partial_a Y_{\ell m}, \tag{26}$$

$$\delta\gamma_{ab} = r^2\sum_{\ell,m}\mathsf{K}^{(\ell m)}(t,r)\sigma_{ab}Y_{\ell m} + r^2\sum_{\ell,m}\mathsf{G}^{(\ell m)}(t,r)\nabla_a\nabla_b Y_{\ell m}. \tag{27}$$

Note that $\mathsf{a}^{(00)} = \mathsf{b}^{(00)} = \mathsf{G}^{(00)} = 0$. Under infinitesimal coordinate transformations

$$r \to r + \sum_{\ell,m}\xi_r^{(\ell m)}(t,r)Y_{\ell m},$$
$$x^a \to x^a + \sum_{\ell,m}\xi_\Omega^{(\ell m)}(t,r)\nabla^a Y_{\ell m}, \tag{28}$$

the perturbation variables transform as

$$\mathsf{H}_0 \to \mathsf{H}_0 - \frac{N'}{N}\xi_r, \qquad \mathsf{H}_1 \to \mathsf{H}_1 - (BF\xi_r)' - F^2\dot{\xi}_r,$$
$$\mathsf{H}_2 \to \mathsf{H}_2 - 2\xi_r' - \frac{2F'}{F}\xi_r,$$
$$\mathsf{b} \to \mathsf{b} - r^2\dot{\xi}_\Omega - BF\xi_r, \qquad \mathsf{a} \to \mathsf{a} - r^2\xi_\Omega' - F^2\xi_r,$$
$$\mathsf{K} \to \mathsf{K} - \frac{2}{r}\xi_r, \qquad \mathsf{G} \to \mathsf{G} - 2\xi_\Omega, \tag{29}$$

where the labels (ℓm) were omitted. In the following analysis, we will use these gauge degrees of freedom to remove some of the variables. Note that we are working in the unitary gauge and hence we do not have the freedom to change the time coordinate.

As in the analysis of the odd-parity perturbations, we calculate the quadratic action,

$$S^{(\mathrm{even})} = \sum_{\ell,m}\int \mathrm{d}t\mathrm{d}r\mathcal{L}^{(\mathrm{even})}_{\ell m}, \tag{30}$$

and study the perturbations with $\ell = 0$, $\ell = 1$, and $\ell \geq 2$ separately.

A. $\ell = 0$

For the monopole perturbations, we are left with $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$, and K, where we omit the labels (00). From the transformation rules (29) it can be seen that we can impose the gauge condition $\mathsf{K} = 0$. The quadratic Lagrangian for $\ell = 0$ is then given by

$$\mathcal{L}^{(\mathrm{even})}_{\ell=0} = \frac{\beta r^2}{12N_0(\beta + N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{2N_0 b_0}{r^2}\tilde{\mathsf{H}}_0' - \frac{2}{r^2}(r^2\tilde{\mathsf{H}}_1)'\right]^2 - \frac{2b_0}{r}\tilde{\mathsf{H}}_0'\tilde{\mathsf{H}}_1 - N_0 r f\tilde{\mathsf{H}}_0'\mathsf{H}_2 - \frac{r}{N_0\sqrt{f}}\mathsf{H}_2\dot{\tilde{\mathsf{H}}}_1, \tag{31}$$

where we defined the convenient combinations of the variables as

$$\tilde{\mathsf{H}}_0 := \mathsf{H}_0 + \frac{1}{2}\mathsf{H}_2, \tag{32}$$

$$\tilde{\mathsf{H}}_1 := \sqrt{f}\mathsf{H}_1 - \frac{N_0 b_0}{r^2}(\mathsf{H}_0 + \mathsf{H}_2). \tag{33}$$

The Euler-Lagrange equations for $\tilde{\mathsf{H}}_0, \tilde{\mathsf{H}}_1$, and H_2 read

$$\left(\frac{2b_0}{r}\tilde{\mathsf{H}}_1 + N_0 r f \mathsf{H}_2 + \frac{2N_0 b_0}{r^2}\mathsf{A}\right)' = 0, \tag{34}$$

$$-\frac{2b_0}{r}\tilde{\mathsf{H}}_0' + \frac{r}{N_0\sqrt{f}}\dot{\mathsf{H}}_2 + 2r^2\left(\frac{\mathsf{A}}{r^2}\right)' = 0, \tag{35}$$

$$N_0 r f \tilde{\mathsf{H}}_0' + \frac{r}{N_0\sqrt{f}}\dot{\tilde{\mathsf{H}}}_1 + \frac{\dot{\mathsf{A}}}{\sqrt{f}} = 0, \tag{36}$$

where A is defined as

$$\mathsf{A} := \frac{\beta r^2}{6N_0(\beta + N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{2N_0 b_0}{r^2}\tilde{\mathsf{H}}_0' - \frac{2}{r^2}(r^2\tilde{\mathsf{H}}_1)'\right]. \tag{37}$$

Equations (34)–(36) can be solved exactly [for any $\beta = \beta(t)$] as follows. First, Eq. (34) can be integrated to give

$$\frac{2b_0}{r}\tilde{\mathsf{H}}_1 + N_0 r f \mathsf{H}_2 + \frac{2N_0 b_0}{r^2}\mathsf{A} = C_1(t), \tag{38}$$

where $C_1(t)$ is a time-dependent integration function. Using Eqs. (35), (36), and (38), we obtain

$$2N_0^2 r^2 f^{3/2}\left(\frac{\mathsf{A}}{r^2}\right)' = -\dot{C}_1, \tag{39}$$

which can be integrated to give

$$\mathsf{A} = \frac{\dot{C}_1 r^2}{2N_0^2}\int_r^\infty \frac{\mathrm{d}r}{r^2 f^{3/2}} + C_2(t) r^2, \tag{40}$$

where $C_2(t)$ is another integration function. We then use Eqs. (35) and (37) to get

$$(r^2\tilde{\mathsf{H}}_1)' = -N_0 r^3\left(\frac{\mathsf{A}}{r^2}\right)' - \frac{3N_0}{\beta}(\beta + N_0\sqrt{f})\mathsf{A}. \tag{41}$$

We impose the boundary conditions $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2 \to 0$ at infinity. Then, noting that $\mathsf{A} \approx (\dot{C}_1/2N_0^2)r + C_2 r^2$ for large r, we see from Eq. (41) that $\dot{C}_1 = C_2 = 0$, i.e., $\mathsf{A} = 0$, though C_1 can still be a nonvanishing, time-independent constant. Thus, we obtain

$$\frac{\tilde{\mathsf{H}}_1}{\sqrt{4\pi}} = \frac{N_0 C_0(t)}{r^2}, \tag{42}$$

where $C_0(t)$ is an integration function. (The factor $1/\sqrt{4\pi}$ comes from Y_{00}.) It follows from Eq. (38) that

$$\frac{\mathsf{H}_2}{\sqrt{4\pi}} = \frac{1}{f}\left[\frac{\delta\mu}{r} - \frac{2b_0 C_0(t)}{r^4}\right], \tag{43}$$

where now we write $C_1 = \sqrt{4\pi}N_0\delta\mu$ with $\delta\mu$ being a *time-independent* constant. Finally, we have

$$\frac{\tilde{\mathsf{H}}_0}{\sqrt{4\pi}} = \frac{\dot{C}_0}{N_0}\int_r^\infty \frac{\mathrm{d}r}{r^2 f^{3/2}}. \tag{44}$$

All the perturbation variables are regular at the usual event horizon, $r = 1$. Notice, however, that $\tilde{\mathsf{H}}_0$ diverges as $\tilde{\mathsf{H}}_0 \sim (r - 3/4)^{-2}$ at the universal horizon. To avoid this, we are forced to set $\dot{C}_0 = 0$. Now it is easy to see that the above solution for the monopole perturbations can be reproduced by perturbing the parameters of the background solution as $\mu_0 \to \mu_0 + \delta\mu$ and $b_0 \to b_0 + C_0$, which shows that $\delta\mu$ simply corresponds to a shift of the mass parameter and nonvanishing C_0 renders the universal horizon singular by enforcing $b_0 \neq b_{0,c}$. Therefore, we conclude that no nontrivial regular solution for the monopole perturbations exists.

Let us look at the above solution from the viewpoint of the Stückelberg field. All the metric perturbations with the coefficients C_0 and $\dot{C}_0$ can be eliminated by performing the coordinate transformation

$$t \to T = t - \frac{C_0(t)}{N_0}\int_r^\infty \frac{\mathrm{d}r}{r^2 f^{3/2}}. \tag{45}$$

We thus see that the geometry remains Schwarzschild (with the mass parameter $1 + \delta\mu$), while one has the fluctuation of the Stückelberg field, $\phi = T + \delta\phi$, where $\delta\phi$ is the minus of the second term in Eq. (45). However, ϕ is singular at $r = 3/4$ unless $C_0(t) = 0$, and the same conclusion follows.

Let us give a comment on the difference between the present analysis and that in Ref. [21]. In Ref. [21], the authors investigated static monopole deformations of the same black hole solution in the same theory without paying particular attention to the regularity of the inner boundary, while in the present paper we have started from generic time-dependent monopole perturbations and showed explicitly how any time-dependent deformation is prohibited by the boundary condtions. The result obtained here is not trivial as the fluctuation of the instantaneous scalar could in principle be time dependent.

B. $\ell = 1$

In the quadratic Lagrangian for the dipole perturbations, $\mathsf{K}^{(1m)}$ and $\mathsf{G}^{(1m)}$ appears only through the combination $\mathsf{K}^{(1m)} - \mathsf{G}^{(1m)}$. We may therefore impose the gauge condition $\mathsf{K}^{(1m)} - \mathsf{G}^{(1m)} = 0$ along with $\mathsf{H}_0^{(1m)} = 0$ by choosing $\xi_r^{(1m)}$ and $\xi_\Omega^{(1m)}$ appropriately. Focusing on the $m = 0$ part without loss of generality, the quadratic Lagrangian for $\ell = 1$ is found to be

$$\mathcal{L}^{(\text{even})}_{\ell=1,m=0} = \frac{\beta r^2}{12N_0(\beta + N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{N_0 b_0}{r^2}\mathsf{H}_2' - \frac{2}{r^2}(r^2\tilde{\mathsf{H}}_1)' + \frac{4\tilde{\mathsf{b}}}{r} + \frac{6(\beta + N_0\sqrt{f})}{\beta r}(\tilde{\mathsf{H}}_1 - \tilde{\mathsf{b}})\right]^2 + \frac{r^2}{2N_0}\left[\dot{\tilde{\mathsf{a}}} - rN_0 b_0\left(\frac{\sqrt{f}}{r^3}\tilde{\mathsf{a}}\right)' - r\left(\frac{\sqrt{f}}{r}\tilde{\mathsf{b}}\right)' - \frac{\tilde{\mathsf{H}}_1}{r\sqrt{f}}\right]^2 - \frac{3\sqrt{f}}{\beta}(\tilde{\mathsf{H}}_1 - \tilde{\mathsf{b}})^2 + \frac{6b_0}{r^2}\tilde{\mathsf{H}}_1\tilde{\mathsf{a}} - \frac{3b_0}{r^2}\mathsf{H}_2\tilde{\mathsf{b}} + N_0 f\tilde{\mathsf{a}}^2 - \frac{N_0}{2}[f + (rf)']\tilde{\mathsf{a}}\mathsf{H}_2 + \frac{N_0}{4}(rf)'\mathsf{H}_2^2, \tag{46}$$

where to simplify the expression we defined

$$\tilde{\mathsf{a}} \coloneqq \frac{\mathsf{a}}{r}, \qquad \tilde{\mathsf{b}} \coloneqq \frac{\mathsf{b}}{r\sqrt{f}} - \frac{N_0 b_0}{r^3}\mathsf{a}, \qquad \tilde{\mathsf{H}}_1 \coloneqq \sqrt{f}\mathsf{H}_1 - \frac{N_0 b_0}{r^2}\mathsf{H}_2, \tag{47}$$

and omitted the labels (10).

We then introduce new variables P and Q and define the Lagrangian $\tilde{\mathcal{L}}^{(2)}_{\ell=1,m=0}$ as

$$\tilde{\mathcal{L}}^{(2)}_{\ell=1,m=0} = \mathcal{L}^{(2)}_{\ell=1,m=0} - \frac{\beta r^2}{12N_0(\beta + N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{N_0 b_0}{r^2}\mathsf{H}_2' - \frac{2}{r^2}(r^2\tilde{\mathsf{H}}_1)' + \frac{4\tilde{\mathsf{b}}}{r} + \frac{6(\beta + N_0\sqrt{f})}{\beta r}(\tilde{\mathsf{H}}_1 - \tilde{\mathsf{b}}) - \frac{\beta + N_0\sqrt{f}}{\beta}\mathsf{Q}\right]^2 + \frac{r^2}{2N_0}\left[\dot{\tilde{\mathsf{a}}} - rN_0 b_0\left(\frac{\sqrt{f}}{r^3}\tilde{\mathsf{a}}\right)' - r\left(\frac{\sqrt{f}}{r}\tilde{\mathsf{b}}\right)' - \frac{\tilde{\mathsf{H}}_1}{r\sqrt{f}} - \mathsf{P}\right]^2. \tag{48}$$

This Lagrangian is equivalent to the original one $\mathcal{L}^{(2)}_{\ell=1,m=0}$ because the additional parts vanish upon substituting the solutions to the equations of motion for P and Q. However, the new Lagrangian $\tilde{\mathcal{L}}^{(2)}_{\ell=1,m=0}$ is more useful for our anaysis. Varying $\tilde{\mathcal{L}}^{(2)}_{\ell=1,m=0}$ with respect to $\tilde{\mathsf{H}}_1, \mathsf{H}_2, \tilde{\mathsf{a}}$, and $\tilde{\mathsf{b}}$, we obtain the equations of motion for these variables, which turn out to be the constraint equations. One can easily solve them to express $\tilde{\mathsf{H}}_1$, H_2, $\tilde{\mathsf{a}}$, and $\tilde{\mathsf{b}}$ in terms of P, Q, and their first derivatives: $\tilde{\mathsf{H}}_1 = (\ldots)\mathsf{P} + (\ldots)\mathsf{Q} + (\ldots)\dot{\mathsf{P}} + (\ldots)\dot{\mathsf{Q}} + (\ldots)\mathsf{P}' + (\ldots)\mathsf{Q}'$, $\mathsf{H}_2, \tilde{\mathsf{a}}, \tilde{\mathsf{b}} = \ldots$, where the explicit expressions are messy. Substituting these back to $\tilde{\mathcal{L}}^{(2)}_{\ell=1,m=0}$, we can express it in terms of P, Q, and their derivatives. We then introduce the new variable $\chi \coloneqq \mathsf{Q} + 3\sqrt{f}\mathsf{P}$ and replace Q in the Lagrangian by χ. By doing so we can remove all the derivatives acting on P. The equation of motion for P can therefore be solved to express it in terms of χ and its derivatives. Substituting the solution back to the Lagrangian to remove P and performing integration by parts, we finally arrive at the reduced Lagrangian for a single master variable χ. Despite lengthy expressions at each intermediate step, the final form of the reduced Lagrangian is rather simple,

$$\tilde{\mathcal{L}}^{(2)}_{\ell=1,m=0} = \frac{\beta(t)}{81N_0^2}[d_1(r)(\chi'')^2 + d_2(r)(\chi')^2 + d_3(r)\chi^2], \tag{49}$$

where the coefficients d_1, d_2, and d_3 are independent of $\beta(t)$ and are given explicitly by

$$d_1(r) = \frac{r^8 f^{3/2}}{3}, \tag{50}$$

$$d_2(r) = -2r^6 f^{1/2}\left(2 - \frac{1}{r} - \frac{5b_0^2}{r^4}\right), \tag{51}$$

$$d_3(r) = \frac{r^4}{f^{1/2}}\left(\frac{2}{r} - \frac{20b_0^2}{r^4} + \frac{21b_0^2}{r^5}\right). \tag{52}$$

(We are primarily interested in the case where $b_0 = b_{0,c}$, but the above expressions are valid for any b_0.) Note that there are no time derivatives of χ in the Lagrangian, which means that there is no propagating dipole mode.

The equation of motion for χ is given by the fourth-order differential equation with respect to r,

$$(d_1\chi'')'' - (d_2\chi')' + d_3\chi = 0, \tag{53}$$

and the configuration of χ is determined once one specifies the boundary conditions. As implied by Eqs. (50)–(52), the inner boundary conditions must be imposed at the location at which f vanishes, i.e., the universal horizon. All the other variables can then be obtained straightforwardly from χ.

We have not found an analytic solution to Eq. (53). However, we can obtain solutions valid near the boundaries. Let us first look for a solution valid for large r in the form

$$\chi = \mathcal{C}_0(t) + \frac{\mathcal{C}_1(t)}{r} + \frac{\mathcal{C}_2(t)}{r^2} + \cdots + \log r\left[\mathcal{D}_0(t) + \frac{\mathcal{D}_1(t)}{r} + \frac{\mathcal{D}_2(t)}{r^2} + \cdots\right]. \quad (54)$$

Substituting this into Eq. (53), one can derive the algebraic relations among the coefficients. One may take $\mathcal{C}_0$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_5$ as independent coefficients and express all the other coefficients in terms of them. One can then obtain the metric perturbations H_1, H_2, a, and b in a series expansion similar to Eq. (54). In order for the metric perturbations to be vanishing at infinity, it turns out that $\mathcal{C}_0 = \mathcal{C}_2 = \mathcal{C}_3 = 0 = \ddot{\mathcal{C}}_5$ must be imposed. Then, $\mathcal{C}_5$ may have the form $\mathcal{C}_5 = v_1 t + v_0$, where v_1 and v_0 are constants, showing that χ diverges as $t \to \infty$ unless $\dot{\mathcal{C}}_5 = v_1 = 0$. We thus see that only a *constant* $\mathcal{C}_5$ is allowed to be nonvanishing from the boundary conditions at infinity and χ is given by

$$\chi = \frac{\mathcal{C}_5}{r^5}\left(1 + \frac{3}{4r} + \frac{9}{16r^2} + \frac{7}{16r^3} + \cdots\right). \quad (55)$$

In this way, χ is uniquely determined up to an overall constant. The dipole metric perturbations for large r are given accordingly by

$$\tilde{\mathsf{H}}_1 = \frac{\mathcal{C}_5}{r^3}\left(\frac{2}{9} + \frac{1}{12}\frac{1}{r} + \frac{1}{24}\frac{1}{r^2} + \cdots\right), \quad (56)$$

$$\mathsf{H}_2 = \frac{b_0}{N_0\mu_0}\frac{\mathcal{C}_5}{r^4}\left(\frac{8}{3} + \frac{4}{3}\frac{1}{r} + \frac{5}{6}\frac{1}{r^2} + \cdots\right), \quad (57)$$

$$\tilde{\mathsf{a}} = \frac{b_0}{N_0}\frac{\mathcal{C}_5}{r^4}\left(\frac{4}{3} + \frac{1}{3}\frac{1}{r} + \cdots\right), \quad (58)$$

$$\tilde{\mathsf{b}} = -\frac{\mathcal{C}_5}{r^3}\left(\frac{1}{9} + \frac{1}{12}\frac{1}{r} + \frac{1}{16}\frac{1}{r^2} + \cdots\right), \quad (59)$$

where β does not appear in the series expansion.

Next, let us examine the behavior of the dipole perturbations near the (regular) universal horizon at $r = 3/4$, setting now $b_0 = b_{0,c}$. Near the universal horizon, we have $d_1 \simeq (81/256\sqrt{2}) \cdot (r-3/4)^3$, $d_2 \simeq (81/256\sqrt{2}) \cdot 3(r-3/4)$, and $d_3 \simeq (81/256\sqrt{2}) \cdot 4(r-3/4)^{-1}$. It then follows that,

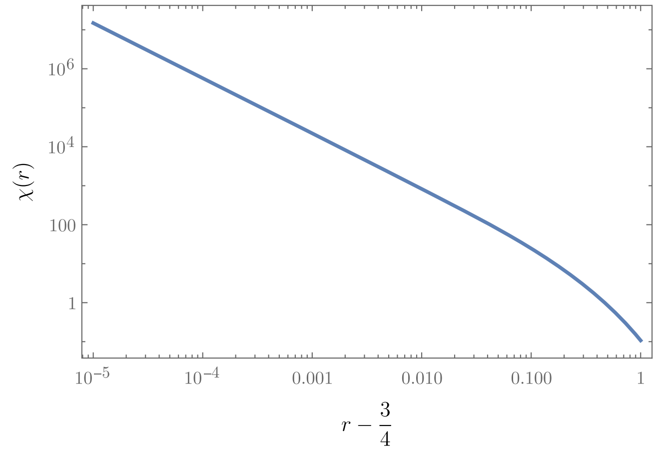

FIG. 2. The behavior of χ near the universal horizon, obtained by solving Eq. (53) numerically with the boundary condition (55) (with $\mathcal{C}_5 = 1$).

near the universal horizon, χ is given by a linear combination of the four independent solutions

$$(r-3/4)^{\sqrt{2}}, \qquad (r-3/4)^{\sqrt{2}}\ln(r-3/4),$$
$$(r-3/4)^{-\sqrt{2}}, \qquad (r-3/4)^{-\sqrt{2}}\ln(r-3/4). \quad (60)$$

By numerically solving Eq. (53) from some large r toward the universal horizon, we find that the coefficients of the latter two are nonvanishing and χ diverges, as can be seen from Fig. 2. This forces the metric perturbations reconstructed from χ to diverge as

$$(BF)^{-1}\mathsf{H}_1 \sim \mathsf{H}_2 \sim \mathsf{a} \sim \mathsf{b} \sim (r-3/4)^{-1}\chi. \quad (61)$$

One can also see that the linear perturbations of the three-dimensional Ricci scalar and the trace of the extrinsic curvature diverge as $\delta R \sim (r-3/4)^{-1}\chi$ and $K \sim (r-3/4)^{-6}\chi$. We, therefore, conclude that no regular dipole perturbations are allowed in the present setup.

C. $\ell \geq 2$

Finally, let us consider the even-parity perturbations with $\ell \geq 2$, which contains both gravitational waves and instantaneous mode. By choosing appropriately $\xi_r^{(\ell m)}$ and $\xi_\Omega^{(\ell m)}$, one can set $\mathsf{K}^{(\ell m)} = \mathsf{G}^{(\ell m)} = 0$. The remaining variables are $\mathsf{H}_0^{(\ell m)}$, $\mathsf{H}_1^{(\ell m)}$, $\mathsf{H}_2^{(\ell m)}$, $\mathsf{a}^{(\ell m)}$, and $\mathsf{b}^{(\ell m)}$. Focusing again on the perturbations with $m = 0$ and omitting the labels (ℓm), the quadratic Lagrangian is given by

$$\mathcal{L}^{(\text{even})}_{\ell\geq 2,m=0} = \frac{\beta r^2}{12N_0(\beta+N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{N_0 b_0}{r^2}\mathsf{H}_2' - \frac{2}{r^2}(r^2\tilde{\mathsf{H}}_1)' + \frac{6(\beta+N_0\sqrt{f})}{\beta r}\tilde{\mathsf{H}}_1 - \frac{6N_0 b_0}{\beta r^3}(\beta+N_0\sqrt{f})\mathsf{H}_0 - \frac{j^2(\beta+3N_0\sqrt{f})}{\beta r}\tilde{\mathsf{b}}\right]^2 + \frac{j^2 r^2}{4N_0}\left[\dot{\tilde{\mathsf{a}}} - rN_0 b_0\left(\frac{\sqrt{f}}{r^3}\tilde{\mathsf{a}}\right)' - r\left(\frac{\sqrt{f}}{r}\tilde{\mathsf{b}}\right)' - \frac{\tilde{\mathsf{H}}_1}{r\sqrt{f}}\right]^2 - \frac{3\sqrt{f}}{\beta}\left(\tilde{\mathsf{H}}_1 - \frac{N_0 b_0}{r^2}\mathsf{H}_0 - \frac{j^2}{2}\tilde{\mathsf{b}}\right)^2 + \frac{3j^2 b_0}{r^2}\tilde{\mathsf{H}}_1\tilde{\mathsf{a}} - \frac{3j^2 b_0}{2r^2}\mathsf{H}_2\tilde{\mathsf{b}} + \frac{j^2}{2}N_0 f\tilde{\mathsf{a}}^2 - \frac{j^2 N_0}{4}[f+(rf)']\tilde{\mathsf{a}}\mathsf{H}_2 + \frac{N_0}{4}(rf)'\mathsf{H}_2^2 + \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{4N_0}\tilde{\mathsf{b}}^2 + N_0 f[r\mathsf{H}_2' - j^2(r\tilde{\mathsf{a}})']\mathsf{H}_0 + N_0\left[\frac{j^2}{2} + (rf)'\right]\mathsf{H}_0\mathsf{H}_2 - \frac{j^2}{2}N_0[f+(rf)']\tilde{\mathsf{a}}\mathsf{H}_0, \tag{62}$$

where $\tilde{\mathsf{H}}_1$, $\tilde{\mathsf{a}}$, and $\tilde{\mathsf{b}}$ are defined in the same way as in the case of $\ell = 1$ [see Eq. (47)] and $j^2 \coloneqq \ell(\ell+1)$ as before.

Since we only have one of the two tensorial modes as a dynamical degree of freedom in the even-parity sector with $\ell \geq 2$ and there is no dynamical scalar degree of freedom, we expect that by integrating out nondynamical variables in the Lagrangian (62) we would end up with a reduced Lagrangian written in terms of a single master variable. Unfortunately, however, we have not been able to do so. Therefore, in what follows we focus on stationary perturbations by dropping all time derivatives and solve directly the equations of motion derived from the Lagrangian (62) without trying to rewrite the system in terms of a single variable. (In doing so we also assume that $\beta = \text{const}$) This amounts to considering stationary deformations of the black hole while discarding propagating gravitational waves. Varying the action, one can straightforwardly obtain the equations of motion for H_0, $\tilde{\mathsf{H}}_1$, H_2, $\tilde{\mathsf{a}}$, and $\tilde{\mathsf{b}}$:

$$\frac{\delta S^{(\text{even})}}{\delta \mathsf{H}_0^{(\ell 0)}} = \frac{\delta S^{(\text{even})}}{\delta \tilde{\mathsf{H}}_1^{(\ell 0)}} = \frac{\delta S^{(\text{even})}}{\delta \mathsf{H}_2^{(\ell 0)}} = \frac{\delta S^{(\text{even})}}{\delta \tilde{\mathsf{a}}^{(\ell 0)}} = \frac{\delta S^{(\text{even})}}{\delta \tilde{\mathsf{b}}^{(\ell 0)}} = 0. \tag{63}$$

First, we determine the behavior of the metric perturbations at large r, assuming the series expansion form

$$\mathsf{H}_0 = r^{-\ell}\left(\mathcal{C}_0^{(0)} + \frac{\mathcal{C}_1^{(0)}}{r} + \frac{\mathcal{C}_2^{(0)}}{r^2} + \cdots\right), \qquad \tilde{\mathsf{H}}_1 = r^{-\ell}\left(\mathcal{C}_0^{(1)} + \frac{\mathcal{C}_1^{(1)}}{r} + \frac{\mathcal{C}_2^{(1)}}{r^2} + \cdots\right),$$
$$\mathsf{H}_2 = r^{-\ell}\left(\mathcal{C}_0^{(2)} + \frac{\mathcal{C}_1^{(2)}}{r} + \frac{\mathcal{C}_2^{(2)}}{r^2} + \cdots\right), \qquad \tilde{\mathsf{a}} = r^{-\ell}\left(\mathcal{C}_0^{(a)} + \frac{\mathcal{C}_1^{(a)}}{r} + \frac{\mathcal{C}_2^{(2)}}{r^2} + \cdots\right),$$
$$\tilde{\mathsf{b}} = r^{-\ell}\left(\mathcal{C}_0^{(b)} + \frac{\mathcal{C}_1^{(b)}}{r} + \frac{\mathcal{C}_2^{(b)}}{r^2} + \cdots\right). \tag{64}$$

Substituting these to the equations of motion (63), we can derive the algebraic relations among the coefficients $\mathcal{C}_0^{(0)}, \mathcal{C}_0^{(1)}, \ldots$. By inspecting the relations, we find that only two of the coefficients are free and independent, and all the other coefficients are expressed using the two. Explicitly, we have

$$\mathsf{H}_0 = \frac{\mathcal{B}_0}{r^{\ell+4}}\frac{b_0}{N_0}\left[1 + \frac{2\ell^2+5\ell+4}{4(\ell+1)}\frac{1}{r} + \frac{2\ell^3+11\ell^2+21\ell+16}{16(\ell+1)}\frac{1}{r^2} + \cdots\right] - \frac{1}{2(\ell+1)}\frac{\mathcal{B}_2}{r^{\ell+1}}\left[1 + \frac{\ell+2}{2}\frac{1}{r} + \cdots\right], \tag{65}$$

$$\tilde{\mathsf{H}}_1 = \frac{\mathcal{B}_0}{r^{\ell+2}}\left[1 + \frac{\ell(2\ell+1)}{4(\ell+1)}\frac{1}{r} + \frac{\ell(2\ell+1)}{16}\frac{1}{r^2} + \cdots\right] - \frac{\mathcal{B}_2}{r^{\ell+3}}N_0 b_0\left[\frac{\ell+2}{4(\ell+1)} + \frac{4\ell^4+35\ell^3+85\ell^2+98\ell+48}{16(\ell+1)(\ell+2)(2\ell+3)}\frac{1}{r} + \cdots\right], \tag{66}$$

$$\mathsf{H}_2 = -\frac{2\mathcal{B}_0}{r^{\ell+3}}\frac{b_0}{N_0}\left[1 + \frac{2\ell^2+5\ell+4}{4(\ell+1)}\frac{1}{r} + \frac{2\ell^3+11\ell^2+21\ell+16}{16(\ell+1)}\frac{1}{r^2} + \cdots\right] + \frac{\mathcal{B}_2}{r^{\ell+1}}\left[1 + \frac{\ell^3+5\ell^2+7\ell+4}{2(\ell+1)(\ell+2)}\frac{1}{r} + \cdots\right], \tag{67}$$

$$\tilde{\mathsf{a}} = \frac{1}{\ell+1}\frac{\mathcal{B}_0}{r^{\ell+4}}\frac{b_0}{N_0}\left[1 + \frac{2\ell+5}{4}\frac{1}{r} + \frac{2\ell^2+13\ell+22}{16}\frac{1}{r^2} + \cdots\right] - \frac{\mathcal{B}_2}{2r^{\ell+1}}\left[\frac{1}{\ell+1} + \frac{\ell+4}{2(\ell+2)} + \cdots\right], \tag{68}$$

$$\tilde{\mathsf{b}} = \frac{1}{\ell+1}\frac{\mathcal{B}_0}{r^{\ell+2}}\left[1 + \frac{2\ell+1}{4}\frac{1}{r} + \frac{(2\ell+1)(\ell+2)}{16}\frac{1}{r^2} + \cdots\right] - \frac{\mathcal{B}_2}{r^{\ell+3}}N_0 b_0\left[\frac{1}{4(\ell+1)} + \frac{4\ell^2+13\ell+18}{16(\ell+1)(2\ell+3)}\frac{1}{r} + \cdots\right], \tag{69}$$

where we redefined the two independent coefficients and introduced the constants $\mathcal{B}_0$ and $\mathcal{B}_2$. The above expressions are valid for any b_0.

To see the physical interpretation of the two constants $\mathcal{B}_0$ and $\mathcal{B}_2$, let us perform similar calculations for the Schwarzschild solution, $\mathrm{d}s^2 = -(1-1/r)\mathrm{d}t^2 + (1-1/r)^{-1}\mathrm{d}r^2 + r^2\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b$, in GR. The quadratic Lagrangian for the even-parity perturbations of the Schwarzschild solution in GR can be reproduced by setting $N_0 = 1$, $b_0 = 0$, and $\beta = 0$ in the Lagrangian (62) and removing b with an infinitesimal coordinate transformation of t,

$$\begin{aligned}\mathcal{L}^{(\mathrm{even,GR})}_{\ell\geq 2,m=0} = {}& r\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{2}{r^2}(r^2\tilde{\mathsf{H}}_1)'\right]\tilde{\mathsf{H}}_1 + \frac{j^2r^2}{4}\left[\dot{\tilde{\mathsf{a}}} - \frac{\tilde{\mathsf{H}}_1}{r\sqrt{f}}\right]^2 \\ &+ \frac{j^2}{2}f\tilde{\mathsf{a}}^2 - \frac{j^2}{4}[f+(rf)']\tilde{\mathsf{a}}\mathsf{H}_2 + \frac{1}{4}(rf)'\mathsf{H}_2^2 \\ &+ f[r\mathsf{H}_2' - j^2(r\tilde{\mathsf{a}})']\mathsf{H}_0 + \left[\frac{j^2}{2} + (rf)'\right]\mathsf{H}_0\mathsf{H}_2 \\ &- \frac{j^2}{2}[f+(rf)']\tilde{\mathsf{a}}\mathsf{H}_0,\end{aligned} \tag{70}$$

with $f = 1 - 1/r$. Similar manipulations as described above lead to

$$\mathsf{H}_0 = -\frac{1}{2(\ell+1)}\frac{\bar{\mathcal{B}}_2}{r^{\ell+1}}\left[1 + \frac{\ell+2}{2}\frac{1}{r} + \cdots\right], \tag{71}$$

$$\tilde{\mathsf{H}}_1 = 0, \tag{72}$$

$$\mathsf{H}_2 = \frac{\bar{\mathcal{B}}_2}{r^{\ell+1}}\left[1 + \frac{\ell^3+5\ell^2+7\ell+4}{2(\ell+1)(\ell+2)}\frac{1}{r} + \cdots\right], \tag{73}$$

$$\tilde{\mathsf{a}} = -\frac{\bar{\mathcal{B}}_2}{2r^{\ell+1}}\left[\frac{1}{\ell+1} + \frac{\ell+4}{2(\ell+2)}\frac{1}{r} + \cdots\right], \tag{74}$$

where it is found that we are allowed to have only one integration constant $\bar{\mathcal{B}}_2$ in the case of GR. Equations (71)–(74) agree with Eqs. (65)–(69) with $b_0 = 0 = \mathcal{B}_0$, if one identifies $\mathcal{B}_2$ as $\bar{\mathcal{B}}_2$. This observation clearly shows that the terms proportional to $\mathcal{B}_2$ come from the tensorial degrees of freedom, while those with $\mathcal{B}_0$ originate from the instantaneous scalar mode.

Having determined the solution for large r, let us investigate the behavior of the perturbations near the inner boundaries, focusing on the case of the regular universal horizon, $b_0 = b_{0,c}$. If one sets $\mathcal{B}_2 = 0$ to kill the tensorial degree of freedom and integrates the equations of motion inwards, the metric perturbations are found to diverge at the universal horizon, $r = 3/4$, as presented in Fig. 3. This result shows that the terms with the coefficient $\mathcal{B}_0$ are indeed induced by the instantaneous mode, whose causal boundary is the universal horizon. By inspecting the equations of motion in the vicinity of $r = 3/4$, we find that the metric perturbations diverge as

$$\begin{aligned}&\tilde{\mathsf{H}}_1 \simeq \varepsilon^{1-k_\ell}(c + c'\ln\varepsilon), \qquad \mathsf{H}_2 \simeq -\frac{2\sqrt{3}}{N_0}\tilde{\mathsf{H}}_1, \\ &\mathsf{H}_0 \simeq \frac{\sqrt{3}}{N_0}\varepsilon^{1-k_\ell}\left[c - \frac{4\sqrt{2}\beta}{N_0}\frac{(2-k_\ell)(3-k_\ell)}{\ell(\ell+1)}c' + c'\ln\varepsilon\right], \\ &\tilde{\mathsf{a}} \simeq -\frac{3\sqrt{3}}{8N_0(1-k_\ell)}\varepsilon^{-k_\ell}\left(c - \frac{c'}{1-k_\ell} + c'\ln\varepsilon\right), \\ &\tilde{\mathsf{b}} \simeq \frac{4}{1-k_\ell}\varepsilon^{2-k_\ell}\left[c - c'\left(\frac{1}{1-k_\ell} + \frac{2\sqrt{2}\beta}{N_0}\right)\right. \\ &\qquad\left. + \frac{2(1-k_\ell)(3-k_\ell)}{\ell(\ell+1)}c'\ln\varepsilon\right],\end{aligned} \tag{75}$$

where $\varepsilon := r - 3/4$, $k_\ell := 2 + \sqrt{1 + \ell(\ell+1)/2}$, and c and c' are constants. One can also see that the linear perturbations of the three-dimensional Ricci scalar and the

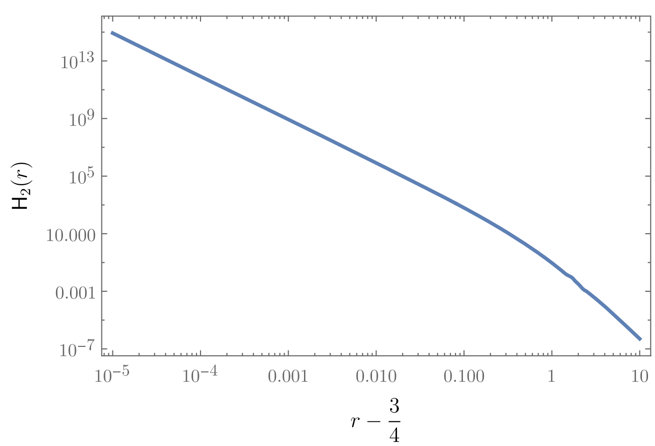

FIG. 3. The behavior of the numerical solution for H_2 near the universal horizon. The boundary condition at large r is given by $\mathcal{B}_0 = 1$ and $\mathcal{B}_2 = 0$. The modified gravity parameter is given by $\beta = 1/5$.

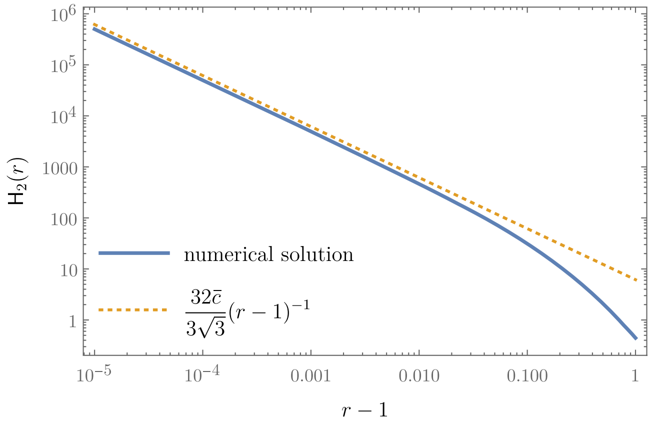

FIG. 4. The behavior of the numerical solution for H_2 near the event horizon (blue solid line). The boundary condition at large r is given by $\mathcal{B}_0 = 0$ and $\mathcal{B}_2 = 1$. The modified gravity parameter is given by $\beta = 1/5$. The analytic result is also shown as the orange dotted line, with $\bar{c} = 1$.

extrinsic curvature diverge as $\delta R \sim \varepsilon^{1-k_\ell}(\mathsf{c}_1 + \mathsf{c}_2 \ln \varepsilon)$ and $K \sim \varepsilon^{-(4+k_\ell)}(\mathsf{c}_3 + \mathsf{c}_4 \ln \varepsilon)$ as in the dipole case, where c_i's are constants.

If one sets $\mathcal{B}_2 \neq 0$ and integrates the equations of motion inwards, one finds that the metric perturbations diverge at the usual event horizon, $r = 1$, before reaching the universal horizon, as shown in Fig. 4. Recalling that the tensorial degrees of freedom propagate at the speed of light in the present theory, it is reasonable that the inner boundary in this case is given by the usual event horizon. In this case, one can study the behavior of the metric perturbations near the event horizon by expanding the equations of motion around $r = 1$ to obtain

$$\begin{aligned} &\tilde{\mathsf{H}}_1 \simeq \bar{c}(r-1)^{-1}, \qquad \mathsf{H}_2 \simeq \frac{32\bar{c}}{3\sqrt{3}N_0}(r-1)^{-1}, \\ &\mathsf{H}_0 \simeq -\frac{1184\bar{c}}{81\sqrt{3}N_0}\ln(r-1), \\ &\tilde{\mathsf{a}} \simeq \frac{32\bar{c}}{3\sqrt{3}\ell(\ell+1)N_0}(r-1)^{-1}, \quad \tilde{\mathsf{b}} \simeq \frac{2\bar{c}}{\ell(\ell+1)}(r-1)^{-1}, \end{aligned} \tag{76}$$

where $\bar{c}$ is a constant. One can see that the linear perturbations of the three-dimensional Ricci scalar and the extrinsic curvature diverge as $\delta R \sim \mathsf{d}_1(r-1)^{-1}$ and $K \sim \mathsf{d}_2(r-1)^{-1} + \mathsf{d}_3 \ln(r-1)$, where d_i's are constants. In any case the stationary perturbations diverge at the inner boundaries, leading to the conclusion that no regular perturbations with $\ell \geq 2$ are allowed.

V. CONCLUSIONS

We have considered a spatially covariant theory of gravity having just two tensorial degrees of freedom and a nonpropagating (instantaneous) scalar mode. There is no propagating scalar mode, and therefore the number of propagating degrees of freedom is the same as in GR. In a particular subset of such theories [20], the standard Newtonian behavior of gravity is reproduced and the propagation of gravitational waves in a cosmological background obeys the same equation as in GR [21]. Moreover, in the same subset of theories of [20], the Schwarzschild solution foliated by the maximal slices is a solution to the field equations, as in the case of Einstein-Aether theory [38]. The solution forms a universal horizon, which is the causal boundary for the scalar mode with infinite propagation speed [21]. In this paper, we have studied linear perturbations of this black hole solution.

First, we have studied the odd-parity perturbations. The odd-parity sector contains only one of the two tensorial modes but is devoid of contributions from the instantaneous scalar mode. We have presented the quadratic Lagrangian for the odd-parity perturbations, which turned out to be identical to that in GR after an appropriate identification of the variables. Since the tensorial modes propagate at the speed of light, the causal boundary of the odd-parity perturbations is given by the usual even horizon rather than the universal horizon inside of it.

Next, we have considered the even-parity sector of black hole perturbations, in which the instantaneous scalar mode and one of the two tensorial modes are mixed. We have derived quadratic actions for monopole, dipole, and higher multipole modes ($\ell \geq 2$), and studied them separately. The tensorial mode does not contribute to the monopole and dipole perturbations. As a result, we have found no radiative behavior in these perturbations. We have solved the set of field equations for the monopole perturbations and found no solution that is regular both at infinity and at the inner boundary, i.e., the universal horizon, except for the trivial one corresponding to a shift of the mass parameter. For the dipole perturbations, we have derived a single master equation which is a fourth-order differential equation with respect to the radial coordinate. Again, we have found no solution that is regular both at infinity and at the universal horizon.

The situation gets more complicated for the even-parity perturbations with $\ell \geq 2$. We have derived the general quadratic action. Although there is only one propagating degree of freedom in the even-parity sector with $\ell \geq 2$, we have not been able to reduce the system to a single master equation because of the complexity of the equations stemming from the mixing with the instantaneous scalar mode. We therefore focused on stationary perturbations and investigated their properties. We have found that the stationary perturbations with $\ell \geq 2$ at large r are characterized by two integration constants, $\mathcal{B}_2$ and $\mathcal{B}_0$, governing respectively the contributions from the tensorial degree of freedom and the instantaneous scalar mode. We carefully identified the locations of the appropriate inner boundaries and showed that the perturbations diverge at the inner boundaries in any case unless $\mathcal{B}_2 = \mathcal{B}_0 = 0$.

To conclude, we have developed black hole perturbation theory for the Schwarzschild solution foliated by the maximal slices in spatially covariant gravity with just two tensorial degrees of freedom and established its perturbative uniqueness. As a future direction, it would be interesting to study the wavelike behavior of the even-parity perturbations with $\ell \geq 2$ to see the impact of mixing with the instantaneous scalar mode on gravitational waves, which is technically more difficult and is left for further study.

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APPENDIX: A QUICK RECAP OF TTDOF THEORY AND ITS BLACK HOLE SOLUTIONS

1. The action of TTDOF theory

The general conditions to eliminate a propagating scalar d.o.f. from a (would-be) scalar-tensor theory have been derived in Ref. [20]. However, it seems almost unfeasible to determine the concrete form of the general action satisfying the conditions. Instead, one can start by assuming a simple ansatz for the action and derive particular solutions for the form of the action admitted by the conditions. Specifically, it is assumed in Ref. [20] that the action is quadratic in the extrinsic curvature and linear in the intrinsic curvature and does not include the time derivatives of the lapse function. A family of TTDOF theories thus obtained is described by

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\left[\frac{\beta_0 N}{\beta_2 + N}K_{ij}K^{ij} - \frac{\beta_0}{3}\left(\frac{2N}{\beta_1 + N} + \frac{N}{\beta_2 + N}\right)K^2 + \alpha_1 + \alpha_2 R + \frac{1}{N}(\alpha_3 + \alpha_4 R)\right], \tag{A1}$$

where $\alpha_{1,2,3,4}$ and $\beta_{0,1,2}$ are arbitrary time-dependent functions. This action includes previously known examples as specific cases and extends them further: the cuscuton theory [16] corresponds to the case with $\beta_0 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = \alpha_4 = 0$ (in the units of $8\pi G = 1$), while the extension cuscuton theory [18] corresponds to the case with $\beta_1 = \beta_2$. It should be noted that the action (A1) is derived by requiring that the scalar d.o.f. does not propagate rather than by symmetry principles. A symmetry-based discussion on this kind of theory can be found in Ref. [39].

From a purely theoretical point of view, the functions $\alpha_{1,2,3,4}$ and $\beta_{0,1,2}$ are free. However, it would be better to restrict ourselves to the following phenomenologically viable subset of theories. Let us consider metric perturbations around Minkowski spacetime, $N = 1 + \Phi(\vec{x})$, $N_i = \partial_i\chi(\vec{x})$, $\gamma_{ij} = [1 - 2\Psi(\vec{x})]\delta_{ij} + h_{ij}(t, \vec{x})$, where Φ, χ, and Ψ are the static part of the perturbations produced by a Newtonian source and h_{ij} is a gravitational wave. One can show from the analysis of the static part of the metric perturbations that [21]

$$8\pi G = \frac{\alpha_2 + \alpha_4}{\alpha_2^2}, \qquad \gamma^{\mathrm{PPN}} = \frac{\alpha_2}{\alpha_2 + \alpha_4}. \tag{A2}$$

The standard behavior of gravity can be reproduced by taking $\alpha_2 = 1$ and $\alpha_4 = 0$. The coefficient of $K_{ij}K^{ij}$ gives the coupling between gravitational waves and matter. We require that this coupling must be the standard one by setting $\beta_0 = \mathrm{const}$ and $\beta_2 = 0$. Then, the speed of gravitational waves is given by [21]

$$c_{\mathrm{GW}}^2 = \frac{\alpha_2 + \alpha_4}{\beta_0} = \frac{1}{\beta_0}. \tag{A3}$$

For this to be the speed of light, we take $\beta_0 = 1$. These observations tell us that, from the phenomenological point of view, theories with $\beta_0 = \alpha_2 = 1$ and $\beta_2 = \alpha_4 = 0$ are of primary interest and the action is now given by

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\left[K_{ij}K^{ij} - \frac{1}{3}\left(\frac{2N}{\beta + N} + 1\right)K^2 + R + \alpha_1 + \frac{\alpha_3}{N}\right], \tag{A4}$$

where we defined $\beta \coloneqq \beta_1$.

2. Static and spherically symmetric black hole solutions

While having said that the action (A4) is of primary interest, in this subsection we will consider static and spherically symmetric black hole solutions in a slightly different theory with

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\left[K_{ij}K^{ij} - \frac{1}{3}\left(\frac{2N}{\beta + N} + 1\right)K^2 + \alpha_2 R + \alpha_1\right], \tag{A5}$$

where α_1 and α_2 are constants, but β can be an arbitrary time-dependent function. The reason for this change is simply practical; theories with $\alpha_3 \neq 0$ do not admit analytic solutions, while we can find analytic solutions even if the condition $\alpha_2 = 1$ is relaxed.

The ADM variables for static and spherically symmetric solutions are taken to be

$$N = N(r), \qquad N_i \mathrm{d}x^i = B(r)F(r),$$
$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = F^2(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{A6}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ It is straightforward to see that the action (A5) admits the following particular solution [21]:

$$N = N_0\sqrt{f(r)}, \quad B = \frac{N_0 b_0}{r^2}, \quad F = \frac{1}{\sqrt{f(r)}}, \tag{A7}$$

where

$$f(r) = 1 + \frac{\alpha_1}{6\alpha_2}r^2 - \frac{\mu_0}{r} + \frac{b_0^2}{\alpha_2 r^4}, \tag{A8}$$

and N_0, μ_0, and b_0 are integration constants. The metric can be reduced to a diagonal form by introducing the new time coordinate defined by $\mathrm{d}\tau = N_0\{\mathrm{d}t - [BF/(N^2 - B^2)]\mathrm{d}r\}$:

$$\mathrm{d}s^2 = -\left(1 + \frac{\alpha_1}{6\alpha_2}r^2 - \frac{\mu_0}{r} + \frac{1-\alpha_2}{\alpha_2}\frac{b_0^2}{r^4}\right)\mathrm{d}\tau^2 + \left(1 + \frac{\alpha_1}{6\alpha_2}r^2 - \frac{\mu_0}{r} + \frac{1-\alpha_2}{\alpha_2}\frac{b_0^2}{r^4}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{A9}$$

In the case of $\alpha_2 = 1$ (which is phenomenologically viable), this metric describes Schwarzschild-(anti–)de Sitter spacetime. In the main text, we focus on the case with $\alpha_2 = 1$ and a vanishing cosmological constant ($\alpha_1 = 0$), yielding the Schwarzschild geometry.

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