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Two dielectric Spheres in an electric field*

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We consider two dielectric spheres in a constant electric field which may be perpendicular or parallel to the axis of the two spheres. The electrostatic potentials inside and outside the spheres are expressed as infinite series. The coefficients appearing in these series satisfy difference equations. By using perturbation expansion, we solve these difference equations. We also calculate the dipole moments of the two spheres.

I. INTRODUCTION

One-body problems (i.e., problems where a body of a given shape, either dielectric or metallic, is subjected to a constant electric field) are currently solved in books on electrostatics.[1,2] The potentials and the dipole moments for these "one-body" problems were calculated explicitly, in most cases, and have quite simple forms. Only a few two-body problems have reached the same degree of success. This comes from the fact that it is difficult to find suitable coordinates in which both the solution of the Laplace equation and the boundary conditions are expressed in a simple form. Only recently,[3] the problem of two metallic spheres was solved explicitly. The problem of finding the electrostatic potential and dipole moment of two dielectric spheres in a constant electric field was not solved till now. The problem of finding the electrostatic forces between two macroscopic dielectric spheres was treated however by several authors. In early days, Hamaker[4] calculated this force by using a r^{-6} atomic-type force between two points of the two spheres. More recently, Mitchell and Ninham[5] and Langbein[6] calculated the Van der Waals interaction between two dielectric spheres by using more sophisticated methods. In all these cases it was possible to obtain the forces without calculating explicitly the potentials.

In this paper, we obtain explicitly the electrostatic potentials and dipole moments of two dielectric spheres in a constant electric field parallel or perpendicular to their axis. We separate the Laplace equations in bispherical coordinates as was done in the case of two metallic spheres by Levine and McQuarrie.[3] We apply the usual boundary conditions at the surface of the two spheres and obtain a set of second-order difference equations (or in other words a three-term recurrence relation) for the coefficients appearing in the potentials. The difference equations have nonlinear coefficients and we have to use a perturbation expansion. The difference equations with linear coefficients resulting from the zero, first, and second order of the perturbation expansion are solved by a method due to Boole[7] and Milne-Thomson.[8] In Sec. II we separate the Laplace equation in bispherical coordinates, apply the boundary conditions, and obtain the difference equations. In Sec. III we introduce the perturbation expansion from which we obtain difference equations with linear coefficients. We obtain then the particular solution of these difference equations. In Sec. IV we calculate the dipole moments of the two spheres.

II. GENERAL SOLUTION

We consider two identical spheres of radius R and dielectric constant ϵ_i. The dielectric constant of the medium is ϵ_e. The two spheres are placed with their centers on the z axis symmetrically about the origin, respectively, at $z=+l$ and $z=-l$. The length of the tangent from the origin to one of the spheres is $a=(l^2-R^2)^{1/2}$. We define the bispherical coordinates[3,9] of a given point in space by

$$\begin{aligned} x &= a\sin\alpha\cos\phi/(\cosh\eta-\cos\alpha)\,,\\ y &= a\sin\alpha\sin\phi/(\cosh\eta-\cos\alpha)\,,\\ z &= a\sinh/(\cosh\eta-\cos\alpha)\,. \end{aligned} \tag{1}$$

We have

$$x^2+y^2+(z-a\coth\eta)^2=a^2/\sinh^2\eta, \tag{2}$$

hence the surface $\eta=$ const. represent spheres. Our two spheres of radius R and centers at $z=\pm l$ may be represented by $\eta=\eta_0$ (upper sphere) and $\eta=-\eta_0$ (lower sphere) with

$$l=a\coth\eta_0,\quad R=a/\sinh\eta_0. \tag{3}$$

For $\eta=\pm\infty$, it follows from (1) that $x=y=0$ and $z=\pm a$. It also follows from (3) that $l-R<a<l+R$, when $\eta_0>0$ and, thus, the two points $x=y=0$, $z=\pm a$ lie, respectively, inside the upper and the lower spheres.

Laplace's equation reads in bispherical coordinates[9]

$$\nabla^2 V = \frac{(\cosh\eta - \cos\alpha)^3}{a^2 \sin\alpha}\left[\sin\alpha \frac{\partial}{\partial\eta}\left(\frac{1}{\cosh\eta - \cos\alpha}\frac{\partial V}{\partial\eta}\right) + \frac{\partial}{\partial\alpha}\left(\frac{\sin\alpha}{\cosh\eta - \cos\alpha}\frac{\partial V}{\partial\alpha}\right) + \frac{1}{\sin\alpha(\cosh\eta - \cos\alpha)}\frac{\partial^2 V}{\partial^2\phi}\right] = 0. \tag{4}$$

Its most general solution is

$$V = (\cosh\eta - \cos\alpha)^{1/2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(M_n e^{\bar{n}\eta} + N_n e^{-\bar{n}\eta})\,[M'_{nm}P_n^m(\cos\alpha) + N'_{nm}Q_n^m(\cos\alpha)]\,(M''_m \sin m\phi + N''_m \cos m\phi), \tag{5}$$

where $\bar{n} = n + \frac{1}{2}$ and $P_n^m(\cos\alpha)$, $Q_n^m(\cos\alpha)$ are, respectively, Legendre functions of the first and second kinds. Our solution has to be regular in the xz plane at $\alpha = \frac{1}{2}\pi$ and this implies $N'_{nm} = 0$. We shall now consider, in detail, the two cases when the external uniform electrostatic field E is parallel and perpendicular to the z axis.

A. $\vec{E}$ parallel to z

In this case the electrostatic potential V has cylindrical symmetry about the z axis. It is therefore independent of the angle ϕ and only the term $m = 0$ has to be retained in Eq. (5); thus

$$V = (\cosh\eta - \cos\alpha)^{1/2}\sum_{n=0}^{\infty}(M_n e^{+\bar{n}\eta} + N_n e^{-\bar{n}\eta})P_n(\cos\alpha). \tag{6}$$

We represent by V_+, V_-, and V_e, respectively, the potentials inside the upper sphere, the lower sphere, and the medium. The potential due to the external field is $V_0 = -Ez$, if $\vec{E}$ points in the positive z direction. It is antisymmetric with respect to reflections through the xy plane ($z \to -z$ or $\eta \to -\eta$). The potentials V_e and V_+, V_- also possess this property, i.e.,

$$V_e(-\eta, \alpha) = -V_e(\eta, \alpha), \tag{7a}$$

$$V_+(-\eta, \alpha) = -V_-(\eta, \alpha). \tag{7b}$$

In order to obtain the potential V_e outside the spheres we use the symmetry condition (7a) and the fact that for $z \to \infty$, $V_e \to V_0$. Thus

$$V_e(\eta, \alpha) = (\cosh\eta - \cos\alpha)^{1/2} \times \sum_{n=0}^{\infty} A_n \sinh\bar{n}\eta\, P_n(\cos\alpha) - \frac{Ea\sinh\eta}{\cosh\eta - \cos\alpha}. \tag{8}$$

From (7b) and from the fact that V_+ and V_- have to be finite at the points $x = y = 0$, $z = \pm a$, where $\eta = \pm\infty$, we obtain

$$V_+(\eta, \alpha) = (\cosh\eta - \cos\alpha)^{1/2}\sum_{n=0}^{\infty} B_n e^{-\bar{n}\eta}P_n(\cos\alpha), \tag{9a}$$

$$V_-(\eta, \alpha) = -(\cosh\eta - \cos\alpha)^{1/2}\sum_{n=0}^{\infty} B_n e^{\bar{n}\eta}P_n(\cos\alpha). \tag{9b}$$

Now we introduce the two boundary conditions

$$V_+(\eta_0, \alpha) = V_e(\eta_0, \alpha) \tag{10}$$

and

$$\epsilon_i E_{n,+}(\eta_0, \alpha) = \epsilon_e E_{n,e}(\eta_0, \alpha), \tag{11}$$

representing the continuity of the potential and of the dielectric displacement at the surfaces of the spheres. The normal component of the electric field is $E_n = -[(\cosh\eta - \cos\alpha)/a](dV/d\eta)$, since the gradient, in bispherical coordinates, is given by[9]

$$\text{grad} = \frac{\cosh\eta - \cos\alpha}{a}\left(\vec{a}_\eta \frac{\partial}{\partial\eta} + \vec{a}_\alpha \frac{\partial}{\partial\alpha} + \vec{a}_\phi \frac{1}{\sinh\eta}\frac{\partial}{\partial\phi}\right), \tag{12}$$

where $\vec{a}_\eta$, $\vec{a}_\alpha$, and $\vec{a}_\phi$ are unit vectors in the η, α, and ϕ directions. Condition (11) can thus be written

$$\epsilon_i\left(\frac{\partial V_+}{\partial\eta}\right)(\eta_0, \alpha) = \epsilon_e\left(\frac{\partial V_e}{\partial\eta}\right)(\eta_0, \alpha). \tag{11$'$}$$

From Eqs. (8), (9a), and (10), we obtain

$$B_n e^{-\bar{n}\eta_0} = A_n \sinh\bar{n}\eta_0 - 2^{3/2}Ea\bar{n}\, e^{-\bar{n}\eta_0}, \tag{13}$$

where we have used the expansion[3]

$$\sinh\eta_0(\cosh\eta_0 - \cos\alpha)^{-3/2} = 2^{3/2}\sum_{n=0}^{\infty}\bar{n}\, e^{-\bar{n}\eta_0}P_n(\cos\alpha). \tag{14}$$

Replacing (8) and (9a) in (11$'$), multiplying by $(\cosh\eta_0 - \cos\alpha)^{1/2}$, and using the expansion

$$(\cosh\eta_0 - \cos\alpha)^{-1/2} = 2^{1/2}\sum_{n=0}^{\infty} e^{-\bar{n}\eta_0}P_n(\cos\alpha) \tag{15}$$

and the recurrence relation

$$\cos\alpha\, P_n = \frac{n}{2n+1}P_{n-1} + \frac{n+1}{2n+1}P_{n+1}, \tag{16}$$

we get

$$(2n+1)\cosh\eta_0(\epsilon_e A_n \cosh\bar{n}\eta_0 + \epsilon_i B_n e^{-\bar{n}\eta_0}) + \sinh\eta_0(\epsilon_e A_n \sinh\bar{n}\eta_0 - \epsilon_i B_n e^{-\bar{n}\eta_0})$$
$$- (n+1)[\epsilon_e A_{n+1}\cosh(n+\tfrac{3}{2})\eta_0 + \epsilon_i B_{n+1} e^{-(n+3/2)\eta_0}] - n[\epsilon_e A_{n-1}\cosh(n-\tfrac{1}{2})\eta_0 + \epsilon_i B_{n-1} e^{-(n-1/2)\eta_0}]$$
$$= 2^{3/2}\epsilon_e E a e^{-\bar{n}\eta_0}[\cosh\eta_0 - (2n+1)\sinh\eta_0]. \quad (17)$$

We now eliminate B_n between Eqs. (13) and (17) and obtain the following set of difference (or recurrence) relations for the coefficients A_n:

$$n[\epsilon_e \cosh(n-\tfrac{1}{2})\eta_0 + \epsilon_i \sinh(n-\tfrac{1}{2})\eta_0]A_{n-1} - [(2n+1)\cosh\eta_0(\epsilon_i \sinh\bar{n}\eta_0 + \epsilon_e\cosh\bar{n}\eta_0) + (\epsilon_e - \epsilon_i)\sinh\eta_0 \sinh\bar{n}\eta_0]A_n$$
$$+ (n+1)[\epsilon_e\cosh(n+\tfrac{3}{2})\eta_0 + \epsilon_i \sinh(n+\tfrac{3}{2})\eta_0]A_{n+1} = 2^{3/2}Ea(\epsilon_e - \epsilon_i)e^{-\bar{n}\eta_0}[ne^{\eta_0} - (n+1)e^{-\eta_0}], \quad n = 0, 1, 2, \ldots . \quad (18)$$

B. $\vec{E}$ perpendicular to z

We shall take $\vec{E}$ pointing in the positive direction of the y axis. The general form of the potential outside the spheres is then

$$V_e(\eta,\alpha,\phi) = (\cosh\eta - \cos\alpha)^{1/2} \times \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(C_{nm}\cosh\bar{n}\eta + D_{nm}\sinh\bar{n}\eta) \times (C'_m \sin m\phi + D'_m \cos m\phi) \times P_n^m(\cos\alpha) - \frac{Ea\sin\alpha\sin\phi}{\cosh\eta - \cos\alpha}, \quad (19)$$

where we have omitted the irregular function $Q_n^m(\cos\alpha)$ and added the potential $V_0 = -Ey$ due to the external field. The potential V_0 is antisymmetric with respect to reflections through the xz plane ($y \to -y$ or $\phi \to -\phi$). The same must be true for V_e; hence we must have $D'_m = 0$ in Eq. (19). Owing to the geometry of the problem, the potentials are also symmetric with respect to reflections through the xy plane.

$$V_e(-\eta,\alpha,\phi) = V_e(\eta,\alpha,\phi), \quad (20a)$$

$$V_+(-\eta,\alpha,\phi) = V_-(\eta,\alpha,\phi). \quad (20b)$$

The occurrence of the factor $\sin\phi$ in V_0 in Eq. (19) and the fact that the boundary condition

$$V_e(\eta_0,\alpha,\phi) = V_+(\eta_0,\alpha,\phi) \quad (10)$$

has to be satisfied identically in $\sin m\phi$ implies that we should retain only the term $m=1$ in Eq. (19). Making use of (20a), we then have, for the potential outside the spheres,

$$V_e(\eta,\alpha,\phi) = (\cosh\eta - \cos\alpha)^{1/2} \times \sum_{n=1}^{\infty} C_n \cosh\bar{n}\eta\, P_n^1(\cos\alpha)\sin\phi - \frac{Ea\sin\alpha\sin\phi}{\cosh\eta - \cos\alpha}. \quad (21)$$

From (20b) and the finiteness of the potential inside the spheres when $\eta = \pm\infty$ we obtain

$$V_+(\eta,\alpha,\phi) = (\cosh\eta - \cos\alpha)^{1/2} \times \sum_{n=1}^{\infty} D_n e^{-\bar{n}\eta} P_n^1(\cos\alpha)\sin\phi, \quad (22a)$$

$$V_-(\eta,\alpha,\phi) = (\cosh\eta - \cos\alpha)^{1/2} \times \sum_{n=1}^{\infty} D_n e^{\bar{n}\eta} P_n^1(\cos\alpha)\sin\phi. \quad (22b)$$

Replacing (21) and (22) in the boundary condition (10) and using expansion (14) and the relation

$$\sin\alpha\, P_n^1(\cos\alpha) = \frac{n(n+1)}{2n+1}[P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)], \quad (23)$$

we find

$$D_n e^{-\bar{n}\eta_0} = C_n \cosh\bar{n}\eta_0 - 2^{3/2}Eae^{-\bar{n}\eta_0}. \quad (24)$$

Consider now the boundary condition (11′). Multiplying it by $(\cosh\eta_0 - \cos\alpha)^{1/2}$ and using the relations

$$(2n+1)\cos\alpha\, P_n^1(\cos\alpha) = nP_{n+1}^1(\cos\alpha) + (n+1)P_{n-1}^1(\cos\alpha) \quad (25)$$

and (23), together with the expansion (14), we obtain

$$(2n+1)\cosh\eta_0(\epsilon_e C_n \sinh\bar{n}\eta_0 + \epsilon_i D_n e^{-\bar{n}\eta_0}) + \sinh\eta_0(\epsilon_e C_n\cosh\bar{n}\eta_0 + \epsilon_i D_n e^{-\bar{n}\eta_0})$$
$$- (n-1)[\epsilon_e C_{n-1}\sinh(n-\tfrac{1}{2})\eta_0 + \epsilon_i D_{n-1}e^{-(n-1/2)\eta_0}]$$
$$- (n+2)[\epsilon_e C_{n+1}\sinh(n+\tfrac{3}{2})\eta_0 + \epsilon_i D_{n+1}e^{-(n+3/2)\eta_0}] = -2^{5/2}\epsilon_e Eae^{-\bar{n}\eta_0}\sinh\eta_0. \quad (26)$$

After eliminating D_n between Eqs. (24) and (26), we obtain the following set of difference relations for the coefficients C_n:

$$(n-1)[\epsilon_e \sinh(n-\tfrac{1}{2})\eta_0 + \epsilon_i \cosh(n-\tfrac{1}{2})\eta_0]C_{n-1} - [(2n+1)\cosh\eta_0(\epsilon_e \sinh\bar{n}\eta_0 + \epsilon_i \cosh\bar{n}\eta_0) + (\epsilon_e - \epsilon_i)\sinh\eta \cosh\bar{n}\eta_0]C_n + (n+2)[\epsilon_e \sinh(n+\tfrac{3}{2})\eta_0 + \epsilon_i \cosh(n+\tfrac{3}{2})\eta_0]C_{n+1} = 2^{5/2}(\epsilon_e - \epsilon_i)Ea \sinh\eta_0 e^{-\bar{n}\eta_0}, \quad n=1,\ 2,\ 3,\ldots. \tag{27}$$

III. DIFFERENCE EQUATIONS

We shall now find the solutions of the difference equations (18) and (27). If we replace the hyperbolic sines and cosines by exponentials, introduce the parameter

$$\Delta = (\epsilon_i - \epsilon_e)/(\epsilon_i + \epsilon_e), \tag{28}$$

and set

$$A_n = 2^{5/2} Ea\Delta e^{-2n\eta_0}\bar{A}_n, \quad C_n = 2^{5/2} Ea\Delta e^{-2n\eta_0}\bar{C}_n, \tag{29}$$

these equations become

$$n(e^{\eta_0} - \Delta e^{-2(n-1)\eta_0})\bar{A}_{n-1} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0 + \Delta n e^{-2n\eta_0} + \Delta(n+1)e^{-2(n+1)\eta_0}]\bar{A}_n + (n+1)(e^{-\eta_0} - \Delta e^{-2(n+2)\eta_0})\bar{A}_{n+1} = (n+1)e^{-2\eta_0} - n \tag{18'}$$

and

$$(n-1)(e^{\eta_0} + \Delta e^{-2(n-1)\eta_0})\bar{C}_{n-1} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0 - n\Delta e^{-2n\eta_0} - (n+1)\Delta e^{-2(n+1)\eta_0}]\bar{C}_n + (n+2)(e^{-\eta_0} + \Delta e^{-2(n+2)\eta_0})\bar{C}_{n+1} = e^{-2\eta_0} - 1. \tag{27'}$$

The theory of difference equations was developed thoroughly during the nineteenth century. A good description of the properties and structure of difference equations is given in the book of Milne-Thomson.[8] It is shown there that a linear difference equation of second order has analytic solutions only if its coefficients are polynomials or at most rational functions of the discrete variable n. Equations (18) and (27′) are second-order linear difference equations whose coefficients contain exponentials of n, hence these equations do not possess analytic solutions. In order to solve these equations we have to use a perturbation expansion. We note that for nonmetallic spheres and medium ($\epsilon_i, \epsilon_e < \infty$) we have $|\Delta| < 1$ and that $e^{-\eta_0} < 1$, if the two spheres are sufficiently small or far apart. Hence all the terms in Eqs. (18′) and (27′) containing Δ and $e^{-2n\eta_0}$ may be considered first order from the point of view of perturbation theory and multiplied by the perturbation parameter ϵ. Introducing the perturbation expansion

$$\bar{A}_n = \bar{A}_n^{(0)} + \epsilon \bar{A}_n^{(1)} + \epsilon^2 \bar{A}_n^{(2)} + \cdots, \tag{30}$$

in Eq. (18′), we obtain to zero, first, and second order in the perturbation parameter ϵ the following equations, respectively:

$$ne^{\eta_0}\bar{A}_{n-1}^{(0)} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0]\bar{A}_n^{(0)} + (n+1)e^{-\eta_0}\bar{A}_{n+1}^{(0)} = (n+1)e^{-2\eta_0} - n, \tag{31a}$$

$$ne^{\eta_0}\bar{A}_{n+1}^{(1)} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0]\bar{A}_n^{(1)} + (n+1)e^{-\eta_0}\bar{A}_{n+1}^{(1)} = \Delta e^{-2n\eta_0}\{ne^{2\eta_0}\bar{A}_{n-1}^{(0)} - [n + (n+1)e^{-2\eta_0}]\bar{A}_n^{(0)} + (n+1)e^{-4\eta_0}\bar{A}_{n+1}^{(0)}\}, \tag{31b}$$

and

$$ne^{\eta_0}\bar{A}_{n-1}^{(2)} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0]\bar{A}_n^{(2)} + (n+1)e^{-\eta_0}\bar{A}_{n+1}^{(2)} = \Delta e^{-2n\eta_0}\{ne^{2\eta_0}\bar{A}_{n-1}^{(1)} - [n + (n+1)e^{-2\eta_0}]\bar{A}_n^{(1)} + (n+1)e^{-4\eta_0}\bar{A}_{n+1}^{(1)}\}. \tag{31c}$$

Introducing in Eq. (27′) the perturbation expansion

$$\bar{C}_n = \bar{C}_n^{(0)} + \epsilon \bar{C}_n^{(1)} + \epsilon^2 \bar{C}_n^{(2)} + \cdots, \tag{32}$$

we obtain to zero, first, and second order in ϵ

$$(n-1)e^{\eta_0}\bar{C}_{n-1}^{(0)} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0]\bar{C}_n^{(0)} + (n+2)e^{-\eta_0}\bar{C}_{n+1}^{(0)} = e^{-2\eta_0} - 1, \tag{33a}$$

$$(n-1)e^{\eta_0}\bar{C}_{n-1}^{(1)} + [\Delta \sinh\eta_0 - (2n+1)\cosh\eta_0]\bar{C}_n^{(1)} + (n+2)e^{-\eta_0}\bar{C}_{n+1}^{(1)} = \Delta e^{-2n\eta_0}\{-(n-1)e^{2\eta_0}\bar{C}_{n-1}^{(0)} + [n + (n+1)e^{-2\eta_0}]\bar{C}_n^{(0)} - (n+2)e^{-4\eta_0}\bar{C}_{n+1}^{(0)}\}, \tag{33b}$$

and

$$(n-1)e^{\eta_0}\overline{C}_{n-1}^{(2)}+[\Delta\sinh\eta_0-(2n+1)\cosh\eta_0]\overline{C}_n^{(2)}+(n+2)e^{-\eta_0}\overline{C}_{n+1}^{(2)}$$
$$=\Delta e^{-2n\eta_0}\{-(n-1)e^{2\eta_0}\overline{C}_{n-1}^{(1)}+[n+(n+1)e^{-2\eta_0}]\overline{C}_n^{(1)}-(n+2)e^{-4\eta_0}\overline{C}_{n+1}^{(1)}\}. \tag{33c}$$

We note that it is sufficient to find particular solutions to Eqs. (31a)–(31c) and (33a)–(33c). The solutions to the homogeneous equations will contain an undetermined factor expressing the fact that the potentials are known up to an arbitrary constant. We try to find a particular solution of the form $\alpha n+\beta$ to Eq. (31a). We obtain

$$\overline{A}_n^{(0)}=\frac{2e^{-\eta_0}}{3-\Delta}\left(n-\frac{e^{-2\eta_0}}{1-e^{-2\eta_0}}\right). \tag{34}$$

Replacing (34) in Eq. (31b) and identifying powers of e^{η_0} and $e^{-\eta_0}$, we have

$$\overline{A}_n^{(1)}=\frac{2\Delta}{3-\Delta}e^{-2n\eta_0}e^{-2\eta_0}\left(n-\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \tag{35}$$

Replacing (35) in (31c) and identifying again powers of e^{η_0} and $e^{-\eta_0}$ we find

$$\overline{A}_n^{(2)}=\frac{2\Delta^2}{3-\Delta}e^{-4n\eta_0}e^{-3\eta_0}\left(n-\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \tag{36}$$

We go now to the transverse case. Equations (33a)–(33c) may be solved in exactly the same way as it was done for the longitudinal case. We find for the particular solution of Eq. (33a):

$$\overline{C}_n^{(0)}=[2/(3-\Delta)]e^{-\eta_0}. \tag{37}$$

The solutions of Eq. (33b) then reads

$$\overline{C}_n^{(1)}=-\frac{2\Delta}{3-\Delta}e^{-2n\eta_0}e^{-2\eta_0}\left(1+\frac{1-\Delta}{2n}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \tag{38}$$

Finally, the particular solution of Eq. (33c) is

$$\overline{C}_n^{(2)}=\frac{2\Delta^2}{3-\Delta}e^{-4n\eta_0}e^{-3\eta_0}\left(1+\frac{1-\Delta}{2n}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \tag{39}$$

IV. DIPOLE MOMENTS

We shall calculate the electric-dipole moments of the two spheres in the electric field. We shall first obtain general formulas in terms of the coefficients A_n and C_n. Then, we shall calculate the zero-, first-, and second-order dipole moments $\mu_L^{(0)}$, $\mu_T^{(0)}$, $\mu_L^{(1)}$, $\mu_T^{(1)}$, $\mu_L^{(2)}$, and $\mu_T^{(2)}$.

First let us calculate the dipole moment of the two spheres when the field is parallel to their axis. In this case the x and y components of the dipole moment vanish by symmetry and the z component is given by

$$\mu_L=2\int z\sigma\,dS. \tag{40}$$

On the surface $\eta=\eta_0$ of the upper sphere we have $z=a\sinh\eta_0/(\cosh\eta_0-\cos\alpha)$, $dS=a^2\sin\alpha\,d\alpha\,d\phi/(\cosh\eta_0-\cos\alpha)^2$ and $a=R\sinh\eta_0$. The surface charge density is $(E_{+n}-E_{en})/4\pi$, where the normal fields are to be expressed in terms of the potentials V_+ and V_e of Eqs. (9a) and (8). Explicitly we can write the dipole moment

$$\mu_L=\frac{a^2}{2\pi}\sinh\eta_0\int\frac{\sin\alpha\,d\alpha\,d\phi}{(\cosh\eta_0-\cos\alpha)^2}\left(\frac{\sinh\eta_0}{2(\cosh\eta_0-\cos\alpha)^{1/2}}\sum_{n=0}^{\infty}A_n\sinh(n+\tfrac{1}{2})\eta_0P_n(\cos\alpha)\right.$$
$$+(\cosh\eta_0-\cos\alpha)^{1/2}\sum_{n=0}^{\infty}A_n(n+\tfrac{1}{2})\cosh(n+\tfrac{1}{2})\eta_0P_n(\cos\alpha)-\frac{Ea\cosh\eta_0}{\cosh\eta_0-\cos\alpha}$$
$$+\frac{Ea\sinh^2\eta_0}{(\cosh\eta_0-\cos\alpha)^2}-\frac{\sinh\eta_0}{2(\cosh\eta_0-\cos\alpha)^{1/2}}\sum_{n=0}^{\infty}B_ne^{-(n+1/2)\eta_0}P_n(\cos\alpha)$$
$$\left.+(\cosh\eta_0-\cos\alpha)^{1/2}\sum_{n=0}^{\infty}B_n(n+\tfrac{1}{2})e^{-(n+1/2)\eta_0}P_n(\cos\alpha)\right). \tag{41}$$

We may replace the coefficients B_n in (41) by their expression (13) in terms of the A_n. The integration over $d\alpha$ is done easily after using the expansions

$$\frac{\sinh\eta_0}{(\cosh\eta_0-\cos\alpha)^{1/2}}=2^{3/2}\sum_k(k+\tfrac{1}{2})e^{-(k+1/2)\eta_0}P_k(\cos\alpha), \tag{42}$$

$$\frac{\sinh^2\eta_0}{(\cosh\eta_0-\cos\alpha)^{3/2}}=\frac{2^{5/2}}{3}\sum_k (k+\tfrac{1}{2})(k+\tfrac{1}{2}+\coth\eta_0)\times e^{-(k+1/2)\eta_0}P_k(\cos\alpha), \tag{43}$$

and the orthonormality of Legendre polynomials. We find

$$\mu_L=\tfrac{8}{3}Ea^3\sum_{n=0}^{\infty}(n+\tfrac{1}{2})e^{-(2n+1)\eta_0}(\coth\eta_0+n+\tfrac{1}{2}) +2^{3/2}a^2\sum_{n=0}^{\infty}(n+\tfrac{1}{2})A_n-8Ea^3\sum_{n=0}^{\infty}(n+\tfrac{1}{2})e^{-(2n+1)\eta_0} +\tfrac{2}{3}Ea^3\frac{3\cosh^2\eta_0+1}{\sinh^3\eta_0}-2Ea\frac{\cosh^2\eta_0}{\sinh^3\eta_0}. \tag{44}$$

We can simplify (44) by introducing the geometric progression

$$\sum_{n=0}^{\infty}e^{-(2n+1)\eta_0}=\frac{1}{2\sinh\eta_0} \tag{45}$$

and the relation obtained by differentiation, namely,

$$\sum_{n=0}^{\infty}(2n+1)^2e^{-(2n+1)\eta_0}=\frac{\cosh^2\eta_0}{\sinh^3\eta_0}-\frac{1}{2\sinh\eta_0}. \tag{46}$$

We find then

$$\mu_L=2^{3/2}a^2\sum_{n=0}^{\infty}(n+\tfrac{1}{2})A_n. \tag{47}$$

The total charge $\int\sigma\,dS$ on a dielectric sphere has to be zero. Using the expression given above for σ and dS we find for the total charge $\int\sigma\,dS$ an expression similar to Eq. (41). Performing the integration in the same way as for Eq. (41) we find that the condition that the total charge on a sphere is zero is equivalent to

$$\sum_{n=0}^{\infty}A_n=0. \tag{48}$$

We can now write (47) as

$$\mu_L=2^{3/2}a^2\sum_{n=1}^{\infty}nA_n. \tag{49}$$

When the electric field is perpendicular to the axis of the two spheres, say along the y axis, the only nonzero component of the dipole moment is the y component. Calling this components μ_T we have

$$\mu_T=2\int y\sigma\,dS, \tag{50}$$

where $y=a\sin\alpha\sin\phi/(\cosh\eta_0-\cos\alpha)$ and σ and dS are given by the same expressions as in the longitudinal case. The fields E_{+n} and E_{en} are given now in terms of the potentials V_{+T} and V_{eT} of Eqs. (21) and (22a). We find

$$\mu_T=\frac{a^2}{2\pi}\int\frac{\sin^2\alpha\sin\phi}{(\cosh\eta-\cos\alpha)^2}\bigg(\frac{\sinh\eta_0}{2(\cosh\eta_0-\cos\alpha)^{1/2}}\sum C_n\cosh(n+\tfrac{1}{2})\eta_0P_n^1(\cos\alpha)\sin\phi +(\cosh\eta_0-\cos\alpha)^{1/2}\sum C_n(n+\tfrac{1}{2})\sinh(n+\tfrac{1}{2})\eta_0P_n^1(\cos\alpha)\sin\phi +\frac{Ea\sin\alpha\sin\phi\sinh\eta_0}{(\cosh\eta_0-\cos\alpha)^2}-\frac{\sinh\eta_0}{2(\cosh\eta_0-\cos\alpha)^{1/2}}\sum D_ne^{-(n+1/2)\eta_0}P_n^1(\cos\alpha)\sin\phi +(\cosh\eta_0-\cos\alpha)^{1/2}\sum D_n(n+\tfrac{1}{2})e^{-(n+1/2)\eta_0}P_n^1(\cos\alpha)\sin\phi\bigg)d\alpha\,d\phi. \tag{51}$$

The coefficients D_n may be replaced by their expression (24) in terms of the C_n. The integration in (51) is done easily by using the recurrence relations

$$\sin\alpha P_n^1(\cos\alpha)=n[P_{n-1}(\cos\alpha)-\cos\alpha P_n(\cos\alpha)], \tag{52}$$

$$(2n+1)\cos\alpha P_n(\cos\alpha)=nP_{n-1}(\cos\alpha)+(n+1)P_n(\cos\alpha), \tag{53}$$

the expansions (42) and (43) and the orthonormality of Legendre polynomials. We obtain

$$\mu_T=-\tfrac{8}{3}Ea^3\sum_{n=0}^{\infty}n(n+1)e^{-(2n+1)\eta_0}+\frac{2}{3}\frac{Ea^3}{\sinh^2\eta_0}+2^{1/2}a^2\sum_{n=0}^{\infty}n(n+1)C_n. \tag{54}$$

After the use of (45) and (46) we find that

$$\mu_T=2^{1/2}a^2\sum_{n=1}^{\infty}n(n+1)C_n. \tag{55}$$

Equations (49) and (50) give us general formulas for the dipole moments of the two spheres in terms of the coefficients A_n and C_n appearing in the potentials. We have determined in Sec. III the expression of A_n and C_n as sums of three terms $A_n\sim A_n^{(0)}+A_n^{(1)}+A_n^{(2)}$ and $C_n\sim C_n^{(0)}+C_n^{(1)}+C_n^{(2)}$. We shall now calculate explicitly the dipole moments corresponding to each of these terms. The longitudinal dipole moment corresponding to the particular solution of zero order is

$$\mu_L^{(0)}=2^{3/2}a^2\sum_{n=1}^{\infty}nA_n^{(0)}, \tag{56}$$

TABLE I. Potential due to the two spheres, for $l/R=3$, the field pointing along the z axis. The potentials do not contain the external field contribution. The values of the potential are to be multiplied by $10^{-3}C$, where $C=[2\Delta/(3-\Delta)]ER$.

r	θ	Potential due to the spheres
$2l$	0°	40(56) [a]
$5l$	0°	8(9.6)
$10l$	0°	1.6(1.6)
$2l$	30°	33.6(48)
$5l$	30°	4.8(8)
$10l$	30°	1.6(1.6)
$2l$	60°	7.7(27)
$5l$	60°	2.6(4.5)
$10l$	60°	0.6(1.1)

[a] Values in parentheses are calculated by the approximate formula $V=(2m/r^2)\cos\theta$.

where $A_n^{(0)}$ is given by Eqs. (34) and (29) as

$$A_n^{(0)}=2^{7/2}Ea\Delta(3-\Delta)^{-1}e^{-2n\eta_0}e^{-\eta_0}\times[n-e^{-2\eta_0}(1-e^{-2\eta_0})^{-1}]. \quad (57)$$

The sums in (56) are of the geometric progression type and we find

$$\mu_L^{(0)}=[4\Delta/(3-\Delta)]ER^3. \quad (58)$$

Expressing Δ in terms of ϵ_i and ϵ_e, Eq. (58) may be written

$$\mu_L^{(0)}=2[(\epsilon_i-\epsilon_e)/(\epsilon_i+2\epsilon_e)]ER^3. \quad (58')$$

Equation (58′) illustrates an interesting point: the dipole moment corresponding to the particular solution of zero order of the difference equation is equal to two times the dipole moment of a single dielectric sphere in the electric field. For the transverse case we have for the zero-order moment

$$\mu_T^{(0)}=2^{1/2}a^2\sum_{n=1}^{\infty}n(n+1)C_n^{(0)}, \quad (59)$$

where $C_n^{(0)}$ is obtained from Eqs. (29) and (37) as

$$C_n^{(0)}=2^{3/2}Ea\Delta(3-\Delta)^{-1}e^{-2n\eta_0}e^{-\eta_0}. \quad (60)$$

The sums in Eq. (55) are again of the geometric progression type and we obtain the same remarkable result as in the longitudinal case

$$\mu_T^{(0)}=\frac{4\Delta}{3-\Delta}ER^3=2\frac{\epsilon_i-\epsilon_e}{\epsilon_i+2\epsilon_e}ER^3. \quad (61)$$

We go now to the first-order perturbation terms. We obtain $A_n^{(1)}$ by using Eqs. (29) and (35), namely,

$$A_n^{(1)}=2^{7/2}Ea\Delta^2(3-\Delta)^{-1}e^{-4n\eta_0}e^{-2\eta_0}\times\left(n-\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (62)$$

Applying the general formula (49) to Eq. (62) we find, approximately,

$$\mu_L^{(1)}=8\mu_L^{(0)}\Delta\sinh^3\eta_0\,e^{-6\eta_0}\times\left(1-\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (63)$$

In the transverse case we get from Eqs. (29) and (38),

$$C_n^{(1)}=-2^{7/2}Ea\Delta^2(3-\Delta)^{-1}e^{-4\eta_0}e^{-2\eta_0}\times\left(1+\frac{1-\Delta}{2n}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (64)$$

and applying the formula (55) we find, approximately,

$$\mu_T^{(1)}=-8\mu_T^{(0)}\Delta\sinh^3\eta_0\,e^{-6\eta_0}\times\left(1+\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (65)$$

From Eqs. (28) and (36) we obtain

$$A_n^{(2)}=2^{7/2}Ea\Delta^3(3-\Delta)^{-1}e^{-6n\eta_0}e^{-3\eta_0}\times\left(n-\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (66)$$

Using (49), we find

$$\mu_L^{(2)}=8\mu_L^{(0)}\Delta^2\sinh^3\eta_0\,e^{-9\eta_0}\times\left(1-\frac{1+\Delta}{2}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (67)$$

Taking into account Eqs. (29) and (39), we have

$$C_n^{(2)}=2^{7/2}Ea\Delta^3(3-\Delta)^{-1}e^{-6n\eta_0}e^{-3\eta_0}\times\left(1+\frac{1-\Delta}{2n}e^{-2\eta_0}+O(e^{-4\eta_0})\right). \quad (68)$$

Finally, using (55) we find the second-order transverse moment

TABLE II. Potential due to the two spheres, for $l/R=3$, the field pointing along the y axis. The potentials do not contain the external field contribution. The values of the potential are to be multiplied by $10^{-3}C$, where $C=[2\Delta/(3-\Delta)]ER$; $\phi=\frac{1}{2}\pi$.

r	θ	Potential due to the spheres
$2l$	30°	61
$5l$	30°	5
$10l$	30°	1
$2l$	60°	47
$5l$	60°	7
$10l$	60°	2
$2l$	90°	40
$5l$	90°	8
$10l$	90°	2

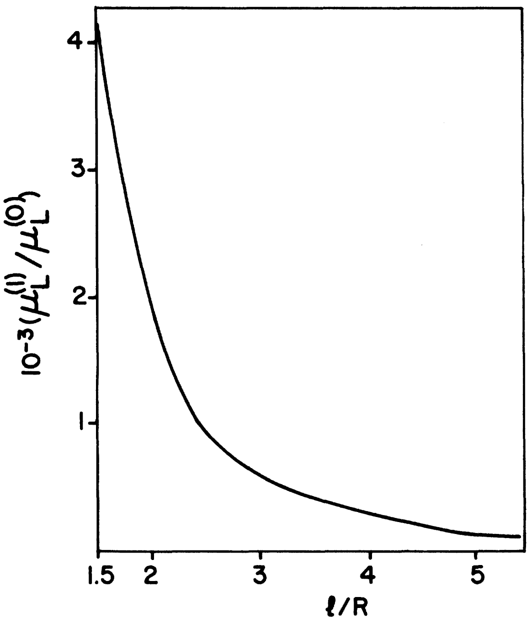

FIG. 1. Ratio $\mu_L^{(1)}/\mu_L^{(0)}$ of first-order to zero-order longitudinal dipole moments vs reduced distance l/R between two spheres.

$$\mu_T^{(2)} = 8\mu_T^{(0)}\Delta^2 \sinh^3\eta_0 e^{-9\eta_0} \times\left(1 - \frac{1-\Delta}{2} e^{-2\eta_0} + O(e^{-4\eta_0})\right). \quad (69)$$

V. DISCUSSION

We have obtained the coefficients of the potentials and the dipole moments of two dielectric spheres in an electric field up to zero, first, and second order in a perturbation expansion. From the form of the solutions at these orders we can infer that the solutions corresponding to the second order of the perturbation expansion and the corresponding dipole moments are, respectively, proportional to $\Delta^3 e^{-6n\eta_0}$ and $e^{-6\eta_0}$. Hence, if the spheres are sufficiently far apart, these contributions are very small.

In Tables I and II we give some particular values of the potentials at some points represented by spherical coordinates r, θ, ϕ. The origin of the spherical coordinates is taken at the midpoint between the two spheres. One can use, when the field points along the axis of the spheres, the approximate formula $V = (2m/r^2)\cos\theta$, where m represents the zero-order dipole moment of a sphere. This formula gives a rather good estimate of the potential, as seen in Table I, especially at large distances when the two spheres begin to behave as point dipoles. We represent, in Tables I and II, the potentials $V_e - V_0$ from Eqs. (8) and (21), due only to the two polarized spheres. The potentials V_0, owing to the external fields, are, in general, greater than the potentials $V_e - V_0$.

In Fig. 1 we have represented the variation of the ratio $\mu_L^{(1)}/\mu_L^{(0)}$ of first-order to zero-order longitudinal dipole moments with reduced distance between the spheres l/R. The ratio $\mu_T^{(1)}/\mu_T^{(0)}$ of transverse dipole moments is obtained from Fig. 1 by the relation $\mu_T^{(1)}/\mu_T^{(0)} \simeq -\mu_L^{(1)}/\mu_L^{(0)}$.

The problem of two dielectric spheres in an electric field may have several applications. One may use the results of this problem for the determination of the second dielectric virial coefficient as was done by Levine and McQuarrie[3] in the case of metallic spheres. Then the molecules are seen as dielectric spheres. The problem of the calculation of the potential of two dielectric spheres is related to the problem of the calculation of the Van der Waals attraction between two spheres. This last problem has applications in dyeing and washing processes, in pouring and sliding processes, and also in problems related to colloid stability and biological transport. If instead of dielectric spheres one considers spherical voids in a dielectric, one may foresee applications in such problems as void growth by precipitation of atomic vacancies and the void theory of the liquid state.[10]

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