Chirped chiral solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift

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We find exact solutions to the nonlinear Schrödinger equation (NLSE) in the presence of self-steepening and a self-frequency shift. These include periodic solutions and localized solutions of dark-bright type which can be chiral, the chirality being controlled by the sign of the self-steepening term. A form of self-phase-modulation that can be tuned by higher-order nonlinearities as well as by the initial conditions, distinct from the nonlinear Schrödinger equation, characterizes these solutions. In certain nontrivial parameter domains, solutions are found to satisfy the linear Schrödinger equation, indicating the possibility of linear superposition in this nonlinear system. Dark and bright solitons exist in both the anomalous and normal dispersion regimes, and a duality between the dark-bright type of solution and kinematic higher-order chirping is also seen. Localized kink solutions similar to NLSE solitons, but with very different self-phase-modulation, are identified.

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The nonlinear Schrödinger equation (NLSE)

\[ i\psi_t + a_1 \psi_{tt} + a_2 |\psi|^2 \psi = 0 \]  \hspace{1cm} (1)

governs the dynamics of picosecond pulse propagation in optical fibers [1], where \( a_1 \) is the group velocity dispersion (GVD) parameter and \( a_2 \) specifies the strength of Kerr nonlinearity. As predicted by Hasegawa and Tappert [2] and experimentally observed by Mollenauer et al. [3], this system supports stable soliton solutions owing their existence to complete integrability [4]. With the advent of high-intensity laser beams, it has become possible to generate optical pulses with width of the order of 10 fs. Higher-order effects like third-order dispersion, self-steepening of the pulse due to the dependence of the slowly varying part of the nonlinear polarization on time, and the self-frequency shift arising from the delayed Raman response become important in the study of the propagation of these pulses. In order to account for them, Kodama [5] and Kodama and Hasegawa [6] proposed a higher-order nonlinear Schrödinger equation as a generalization of the NLSE:

\[ i\psi_t + a_1 \psi_{tt} + a_2 |\psi|^2 \psi + i[a_3 \psi_{ttt} + a_4 (|\psi|^2 \psi)_t + a_5 \psi (|\psi|^2)_t] = 0, \]  \hspace{1cm} (2)

where a third-order dispersion with coefficient \( a_3 \), a self-steepening term with coefficient \( a_4 \), and a self-frequency shift effect with coefficient \( a_5 \) have been added. This model, unlike the NLSE, is not integrable in general. A few integrable cases have been identified: (i) the Sasa-Satsuma case \([a_3:a_4:(a_4+a_5)=1:6:3]\) [7], (ii) the Hirota case \([a_3:a_4:(a_4+a_5)=1:6:0]\) [8], and (iii) derivative NLSEs of types I and II [9]. Many restrictive special solutions of bright and dark type have been obtained [10–12].

The effect of third-order dispersion is significant for femtosecond pulses when the GVD is close to zero. It is negligible for optical pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero. However, in this case self-steepening as well as self-frequency shift terms are still dominant and should be retained. The effects of these higher-order terms on pulse propagation have been extensively studied numerically [1,13], and some special solutions to this system are also known [14].

In this Rapid Communication, we report the existence of a different class of localized as well as periodic solutions for the NLSE in the presence of self-steepening and self-frequency shift. The localized solutions include dark, bright, and kink-type solitons. These complex solitons are generally chirped and show a different form of self-phase-modulation. Unlike the NLSE, where chirping is of kinematic origin (controlled by initial conditions), the chirping in the present case has both kinematic and dynamic origin. The former varies as the reciprocal of the intensity, whereas the latter is directly proportional to the intensity and depends upon higher-order nonlinearities only. It is evident that kinematic chirping plays a significant role for dark solitons, whereas the dynamical chirping would be important for bright solitons. We found that for certain special values of parameters the intensity profiles of these solitons are similar to those of NLSE solitons, while retaining the nontrivial phase structure due to higher-order nonlinear terms. It is also seen that, in the same regime, kink solitons exist which otherwise are not allowed in this system. For certain parameter values, this model mimics the NLSE; however, the presence of higher-
order nonlinearities crucially affects the dynamics. And as a consequence one sees that the fundamental bright soliton of the NLSE is no longer a valid solution. Further, trivial phase dark solitons exist in the anomalous dispersion regime and are found to be chiral, with the direction of propagation set by the self-steeping term. Chirped dark and bright solitons are found to exist in the normal as well as the anomalous dispersion regime. Chirped dark solitons in the anomalous regime, whereas chirped bright solitons in the normal dispersion regime, exhibit chirality, which is controlled by the self-steeping term. Observation of chiral solitons is one of the main results of this Rapid Communication. Very interestingly, it is seen that for some nontrivial choice of parameters, this system behaves like the free-particle Schrödinger equation with appropriate constants, making this system amenable to linear superposition, which is otherwise forbidden in this nonlinear system. This system is found to be Painlevé, thereby establishing the existence of regular solutions.

Modulo a trivial kinematic phase, the complex envelope traveling wave solutions can be generally represented as

$$\psi(x,t) = \rho(x)e^{i\chi(x)}$$

where $\xi = \alpha(t-ux)$ is the traveling coordinate, and $\rho$ and $\chi$ are real functions of $\xi$. Here, $\alpha$ is a scale parameter and $u = 1/v$ with $v$ being the group velocity of the wave packet. The ansatz solution leads to the compatibility conditions

$$-au\rho' + 2a^2(1 + a_1)\rho' + \alpha^2a_1\rho + 3a_0a_1(\rho')^2 - 2a_0a_2(\rho')^3 = 0,$$

$$\alpha u\rho' + 2a^2(1 - a_1)\rho' + \alpha^2a_1\rho + a_3\rho^3 - a_4a_1\rho^3 = 0.$$ (4)

Equation (4) can be exactly integrated to yield

$$\chi' = \frac{u}{2a_1} + \frac{c}{a_0a_1} - \frac{(3a_3 + 2a_1)}{4a_1} \rho^2,$$ (6)

where $c$ is to be determined by the initial conditions. It is to be noted that the phase has a nontrivial form and has two intensity-dependent chirping terms, apart from the kinematic first term. As is evident, the second term is of kinematic origin and is common to the Schrödinger equation as well. The last term is due to higher-order nonlinearities and leads to chirping that is exactly inverse to that of the former. This is a form of self-phase-modulation that is controlled by interaction. The amplitude equation (5) reduces to

$$\theta_1\rho'' + \theta_2\rho' + \theta_3\rho^3 + \theta_4\rho^5 = \frac{c^2}{\rho^3}$$ (7)

with $\theta_1 = \alpha^2a^2_1$, $\theta_2 = (u^2 - a_1a_2 + 2a_3a_4)/4$, $\theta_3 = (2a_1a_2 - a_4)/2$, and $\theta_4 = (4a_4 - 1)(3a_4 + 2a_5)/16$. We note that the nontrivial contribution from higher-order nonlinear terms is through $\theta_4$, which is zero for $a_2 = 1/4$ and $a_5 = -2/3$ (assuming $a_4 \neq 0$ and $a_5 \neq 0$), and as we shall soon show, this results in interesting physical consequences. In the case when $a_2 = 1/4$, $a_5 = -2/3$, both the intensity as well as the phase will not have any new features due to higher-order nonlinearities and the solutions will exactly resemble NLSE solutions. However, unlike in the NLSE, both dark and bright solitons exist, in both the normal and anomalous dispersion regimes. When $c = 0$, the existence of fundamental bright solitons with $\rho = A \text{sech}(\xi)$ is forbidden, and only dark solitons with $\rho = A \text{tanh}(\xi)$ exist, with $\alpha^2 = u^2/2a_1^2$. Furthermore, these dark solitons in the anomalous dispersion regime, i.e., $a_1 > 0$, respect the inequality $a_1u \geq 2|a_1||a_4|$, which restricts them to travel only along one direction, given by the sign of $a_4$. This is an example of a chiral soliton which is absent in the NLSE. When $c = 0$, both bright and dark solitons exist, and satisfy $2a_1a_2 > ua_4$ and $2a_1a_2 < ua_4$, respectively, showing that both have mutually exclusive velocity space. As a consequence, in the anomalous dispersion regime, dark solitons exist, whereas chirped bright solitons in the normal dispersion regime respect $-2|a_1||a_4| > uA_4$ and hence are also chiral. Notice that the directionality of these solitons is due to the presence of the nonlinearity; the sign of $a_4$ decides the direction in which solitons are allowed to propagate. For $a_4 = 1/4$, the intensity profile will be the same as for the NLSE, whereas the phase will still show nontrivial chirping. In this case also both dark and bright solitons can be chiral, and can exist in the normal and anomalous dispersion regimes, which is in sharp contrast to the NLSE.

It is very intriguing to see that, when $a_4 = -2a_5/3$ and $\rho = -3a_1a_2/2a_4$, Eq. (7) combined with Eq. (6) reduces to the free-particle Schrödinger equation in $\psi$. So for this choice of parameters, in the presence of both Kerr and higher-order nonlinearities, the effective evolution equation for $\psi$ is linear, and one would expect to see phenomena like interference, which is forbidden otherwise in this system.

Equation (7) can be cast into a convenient form using $\rho = \sqrt{\sigma}$:

$$\frac{\theta_1}{2} \sigma'' + 2\theta_2 \sigma + \frac{3\theta_3}{2} \sigma^2 + \frac{4\theta_4}{3} \sigma^3 = k,$$ (8)

where $k$ is a constant fixed by initial conditions. Solutions for this equation, with $\theta_4 \neq 0$, can be found by a conformal Möbius transformation:

$$\sigma = \frac{A + Bf}{C + Df},$$ (9)

which for some suitable $A, B, C,$ and $D$ connects $\sigma$ to the elliptic function $f$. These elliptic functions, as is known, are generalizations of trigonometric and hyperbolic functions and appear in the solutions of many nonlinear equations.

Considering the importance of localized solutions, we set $f(\xi) = \text{sech}(\xi)$, and look for allowed values of $A, B, C,$ and $D$ for which (9) is a solution of Eq. (8). The consistency condition leads to $A = 8\theta_4 A \tilde{A} - 3\theta_3 B = 8\theta_2 \tilde{B} - 3\theta_3 \tilde{D}, C = 8\theta_4$, and $D = 8\theta_4 \tilde{D}$, where $\tilde{A}$, $\tilde{B}$, $\tilde{D}$, and $\alpha$ are given by

$$\left(1024 \theta_4^4 \tilde{A}^3 + 1536 \theta_4 \theta_3 \theta_2 \tilde{A} + 432 \theta_3^3 \theta_2^2 \tilde{A} - 864 \theta_2^2 \theta_4^2 \tilde{A}^2 \right)\tilde{A}^3 + (-3 \theta_3^4 + 162 \theta_3^2 \theta_2 \theta_4 - 576 \theta_2 \theta_4^2 \tilde{A}^2 - 768k \theta_2^2) = 0,$$

$$\tilde{B} = -\left(\tilde{D}(-54 \theta_3^2 + 27 \theta_2 \tilde{A} + 96 \theta_2 \theta_4 + 128 \theta_4^2 \tilde{A}^2)\right)/64 \theta_4^2.$$
A natural question arises whether the model is integrable in this regime or not. Following the Ablowitz–Ramani–Segur algorithm, we investigate the singularity structure of the ordinary differential equation (8), which is obtained from an exact reduction of the original partial differential equation (2) [22,23]. Interestingly, we found that the ordinary differential equation represented by (8) possesses poles as the only movable singularities, which implies that this system indeed has the Painlevé (P) property [22,23]. Hence, we see that this

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system passes the Painlevé test, and is Painlevé integrable, which guarantees the existence of regular solutions in general.

In conclusion, we have found a class of exact solutions to the NLSE system in the presence of self-steepening and self-frequency shift terms. These include localized solutions of dark-bright type, kink solutions, and periodic solutions. These solutions have nontrivial phase chirping which varies as a function of intensity and are different from those in Ref. [10] where the solutions had a trivial phase. A nontrivial connection of this system with the linear Schrödinger equation in appropriate limits is pointed out. A duality is seen between the dark-bright type of solution and kinematic higher-order chirping. A form of self-phase-modulation has been observed in this case that shows chirp reversal across the pulse profile. It is known that prechirping of pulses often leads to a better quality of pulse; in particular it is quite effective with a distributed GVD and nonlinearity [16,19,24,25]. In this context, the solutions having chirping due to initial conditions as well as dynamical conditions will provide a better control. It is noted that for some parameter values the intensity and phase of these solitons will exactly be the same as NLSE solitons, and are found to be chiral, with the direction of propagation controlled by self-steepening term. Both dark and bright solitons are found to exist in both the normal and anomalous dispersion regimes. It is seen in some cases that the intensity of these solitons will be like NLSE solitons and only the phase structure will be different. Kink solutions are found to exist in this system for special choices of parameters. The system is seen to possess the P property and hence is Painlevé integrable.

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