Diamagnetism of a free particle in classical electron theory with classical electromagnetic zero-point radiation

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(Received 20 March 1979)

Landau's diamagnetism for a free point charge is shown to exist within classical electron theory with classical electromagnetic zero-point radiation. The system considered is a nonrelativistic classical point charge bound harmonically in three dimensions and situated in a magnetic field when random classical electromagnetic zero-point radiation or thermal radiation is present. The average energy, angular momentum, and magnetic moment are calculated at finite temperature, and then carried to the limit at which the harmonic binding vanishes so as to obtain the behavior for a free particle. In the presence of the Rayleigh-Jeans radiation spectrum one finds that all diamagnetic effects vanish, in agreement with the results of traditional classical statistical mechanics. In the Planck radiation spectrum one finds exactly the results of quantum theory involving a Langevin function for the temperature dependence. In particular, the average angular momentum and magnetic moment for a classical point charge in classical zero-point radiation are $\langle L_z \rangle = -\hbar$ and $\langle M_z \rangle = -|e|\hbar/2mc$, where the orientation of the z axis is given by the magnetic field direction. Thus the classical results in the presence of classical zero-point radiation are entirely different from those of traditional classical electron theory, and they suggest possible classical explanations of the space quantization appearing in quantum theory. The results also suggest a derivation of Planck's spectrum from traditional classical statistical mechanics by applying Boltzmann statistics to the orientation of the average magnetic moment caused by the zero-point radiation.

INTRODUCTION

Traditional classical electron theory allows no diamagnetic behavior.[1] Van Leuwen[2] in 1919 and Van Vleck[3] in 1932 showed that if one applies the Boltzmann distribution of traditional classical statistical mechanics to a system of classical point charges, then the distribution of particle positions and velocities is the same as in the absence of a magnetic field. In the present work classical electromagnetic zero-point radiation is introduced into classical electron theory to obtain the classical theory termed "random electrodynamics."[4] Within random electrodynamics traditional classical statistic mechanics is invalid.[5] And random electrodynamics does indeed show diamagnetic effects.

The existence of diamagnetism for classical systems in the presence of classical electromagnetic zero-point radiation was first pointed out by Marshall[6] in 1963 for an isotropic harmonic-oscillator system in a weak magnetic field. However, Marshall did not treat the diamagnetism of a free classical point charge. Braffort and Taroni[7(a)] in 1967 and Surdin[7(b)] in 1970 considered a free point charge in classical electromagnetic zero-point radiation and noted that the kinetic energy takes approximately the value $\frac{1}{2}\hbar\omega_B$, where $\omega_B = eB/mc$ is the cyclotron frequency in terms of the particle charge e, mass m, and the magnetic field B. Very recently Sachidanandam[8] mentioned that the average angular momentum of a free particle in classical zero-point radiation has the magnitude $\hbar$. However, none of these authors discusses the temperature dependence of the diamagnetism of a free particle and none discusses the connection with the null result of tradiational classical statistical mechanics. In the present paper we first review some previous work without making Marshall's weak-field approximation, we correct some slips and misprints in the literature, and we then focus upon the diamagnetism of a free classical point charge in zero-point radiation and in thermal radiation.

Some of the results seem interesting. We find that free classical charges in zero-point radiation have an average angular momentum $\langle L_z \rangle = \mp\hbar$ along the direction of any external magnetic field where the negative sign is taken by all positive charges and the positive sign by negative charges. The magnitude $|\langle L_z \rangle| = \hbar$ holds independently of the magnitudes of the magnetic field, of the particle charge e, or of the particle mass m. Thus a free charge e acts as though it had an angular momentum $\hbar$ and a permanent magnetic moment of magnitude $\mu = |e|\hbar/2mc$ which always aligns itself antiparallel to the magnetic field. The system behaves in a manner reminiscent of what is termed "space quantization" in that its angular momentum always assumes a fixed average value relative to any direction of space delineated by a magnetic field. In the presence of thermal radiation given by the Planck formula with zero-point radiation the value of the angular momentum $\langle L_z \rangle$ decreases with temperature according to a Langevin function and vanishes entirely in the Rayleigh-Jeans limit. The expressions obtained for the particle kinetic energy and magnetic moment are just those of Landau's[9] work of 1930 in connection with the diamagnetism of free electrons in quantum theory.

BASIC MODEL

Random electrodynamics involves a fundamental change in the boundary conditions of classical electron theory. Random classical electromagnetic radiation with a Lorentz-invariant[10] spectrum, classical electromagnetic zero-point radiation, is introduced as the homogeneous solution of Maxwell's equations which is present irrespective of the mechanical system involved. The mechanical system we consider here is that of a classical point charge e of mass m bound in a three-dimensional isotropic harmonic potential $V = \frac{1}{2} m\omega_0^2 \vec{r}^2$ with a magnetic field $\vec{B} = \hat{K}B$ along the z direction. The system is treated nonrelativistically in the dipole approximation. We solve the system in the general case, and then take the weak-field limit $\omega_0 \gg \omega_L$, where $\omega_L = eB/2mc$ is the Larmor frequency, in order to recover Marshall's result[6] for the diamagnetic behavior of the bound charge system. In the opposite limit $\omega_L \gg \omega_0 \to 0$ of no binding, we obtain the behavior of a free charge. The energy, angular momentum, and magnetic moment for the system will be discussed for each case.

The nonrelativistic equation of motion for the point charge is given by

$$m\ddot{\vec{r}} = -m\omega_0^2 \vec{r} + (e/c)\dot{\vec{r}} \times \vec{B} + (2e^2/3c^3)\dddot{\vec{r}} + e\vec{E}^{\mathrm{in}}(0,t) , \tag{1}$$

where $-m\omega_0^2\vec{r}$ is the harmonic restoring force, $(e/c)\dot{\vec{r}} \times \vec{B}$ is the magnetic part of the Lorentz force,

$$(2e^2/3c^3)\dddot{\vec{r}} = m\Gamma\dddot{\vec{r}}$$

is the nonrelativistic radiation damping force, and $e\vec{E}^{\mathrm{in}}(0,t)$ is the force due to the random classical electromagnetic field taken in the dipole approximation. The random radiation field can be written[4] as a sum over plane waves with random phases,

$$\vec{E}^{\mathrm{in}}(\vec{r},t) = \sum_{\lambda=1}^{2} \int d^3k\, \hat{\epsilon}(\vec{k},\lambda)\mathfrak{h}(\vec{k},\lambda) \times \cos[\vec{k}\cdot\vec{r} - \omega t - \theta(\vec{k},\lambda)] , \tag{2}$$

where the random phase $\theta(\vec{k},\lambda)$ is distributed uniformly over $(0,2\pi)$ and is distributed independently for each wave vector $\vec{k}$ and polarization λ. The energy per normal mode $g(\omega)$ in the electromagnetic field is given by

$$g(\omega) = \pi^2\mathfrak{h}^2(\vec{k},\lambda) , \tag{3}$$

and the radiation spectrum $\rho(\omega)$ is obtained from $g(\omega)$ as

$$\rho(\omega) = (\omega^2/\pi^2c^3)g(\omega) . \tag{4}$$

For the Planck spectrum with zero-point radiation

$$g(\omega) = \pi^2\mathfrak{h}^2(\vec{k},\lambda) = \tfrac{1}{2}\omega\hbar \coth(\hbar\omega/2KT) . \tag{5}$$

The vector equation (1) corresponds to three first-order equations[11] for the components x, y, and z,

$$\ddot{x} + \omega_0^2 x - \omega_B \dot{y} - \Gamma\dddot{x} = (e/m)E_x^{\mathrm{in}}(0,t) , \tag{6}$$

$$\ddot{y} + \omega_0^2 y + \omega_B \dot{x} - \Gamma\dddot{y} = (e/m)E_y^{\mathrm{in}}(0,t) , \tag{7}$$

$$\ddot{z} + \omega_0^2 z - \Gamma\dddot{z} = (e/m)E_z^{\mathrm{in}}(0,t) , \tag{8}$$

where ω_B is twice Larmor's frequency

$$\omega_B = 2\omega_L = eB/mc . \tag{9}$$

TWO METHODS OF SOLUTION

We can obtain the solutions for the three coupled stochastic differential equations (6)–(8) by at least two different approaches. One method involving the explicit solutions for x and y was presented[12] in 1975. The calculation given there is suitable for the present analysis provided a few minus signs are corrected[13] and provided we do not make the weak-field approximation introduced there.[14] The other method based upon work[15] of 1978 involves the use of action-angle variables for systems without harmonics. We will follow this second method because we believe it is capable of general extension, and also because the diamagnetism section of our work in 1978 contains all too many sloppy misprints.[16]

Our general analysis of 1978 involved nonrelativistic charged systems with multiply periodic orbits located in random classical electromagnetic radiation. If the system involved only simple frequencies of oscillation with no harmonics, then the distribution

$$P(\tilde{J}_1,\tilde{J}_2,\tilde{J}_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)$$

on phase space is given by

$$P(\tilde{J}_1,\tilde{J}_2,\tilde{J}_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3) = \text{const}\exp\left[-\left(\frac{\omega_1\tilde{J}_1}{g(\omega_1)} + \frac{\omega_2\tilde{J}_2}{g(\omega_2)} + \frac{\omega_3\tilde{J}_3}{g(\omega_3)}\right)\right], \tag{10}$$

where $J_i = 2\pi\tilde{J}_i$ and $w_i = \tilde{w}_i/2\pi$ are the action and angle variables of the multiply periodic mechanical system and where $g(\omega)$ is the energy density per normal mode in the random classical electromagnetic radiation. For Planck's spectrum with zero-point radiation $g(\omega)$ is given by (5). In the special case of zero temperature where only zero-point radiation is present

$$g(\omega)=\tfrac{1}{2}\hbar\omega\,, \tag{11}$$

the distribution takes the form

$$P(\tilde{J}_1,\tilde{J}_2,\tilde{J}_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3) = \text{const}\exp[-2(\tilde{J}_1+\tilde{J}_2+\tilde{J}_3)/\hbar]\,. \tag{12}$$

Since the probability density (12) is a function of only the adiabatic invariants $\tilde{J}_1$, $\tilde{J}_2$, and $\tilde{J}_3$ of the mechanical system, the average values of the adiabatic invariants $\langle\tilde{J}_1\rangle$, and $\langle\tilde{J}_3\rangle$ remain adiabatic invariants in the presence of classical electromagnetic zero-point radiation.[15]

OBTAINING THE ACTION-ANGLE VARIABLES

The equations of motion for the mechanical system corresponding to (6)–(8) for the charged system are found by omitting the radiation damping and the random radiation field,

$$\ddot{x}+\omega_0^2 x-\omega_B\dot{y}=0\,, \tag{13}$$

$$\ddot{y}+\omega_0^2 y+\omega_B\dot{x}=0\,, \tag{14}$$

$$\ddot{z}+\omega_0^2 z=0\,. \tag{15}$$

The Hamiltonian for this system is

$$\begin{aligned} H&=[\vec{p}-(e/c)\vec{A}]^2/2m+\tfrac{1}{2}m\omega_0^2\vec{r}^2\\ &=(p_x^2+p_y^2+p_z^2)/2m+\omega_L(p_x y-p_y x)\\ &\quad+\tfrac{1}{2}m\omega_S^2(x^2+y^2)+\tfrac{1}{2}m\omega_0^2 z^2\,, \end{aligned} \tag{16}$$

where we have chosen the vector potential

$$\vec{A}=-\tfrac{1}{2}(\hat{i}\,y-\hat{j}\,x)B\,, \tag{17}$$

and have used Larmor's frequency ω_L,

$$\omega_L=\tfrac{1}{2}\omega_B=eB/2mc\,, \tag{18}$$

and the frequency ω_S,

$$\omega_S=(\omega_0^2+\omega_L^2)^{1/2}\,. \tag{19}$$

The solution to the motion is given by

$$x=\mathcal{A}\cos\tilde{w}_1+\mathcal{B}\cos\tilde{w}_2\,, \tag{20}$$

$$y=-\mathcal{A}\sin\tilde{w}_1+\mathcal{B}\sin\tilde{w}_2\,, \tag{21}$$

$$z=\mathcal{C}\cos\tilde{w}_3\,, \tag{22}$$

$$\begin{aligned} p_x&=m\dot{x}+(e/c)A_x=m(\dot{x}-\omega_L y)\\ &=-m\omega_S(\mathcal{A}\sin\tilde{w}_1+\mathcal{B}\sin\tilde{w}_2)\,, \end{aligned} \tag{23}$$

$$\begin{aligned} p_y&=m\dot{y}+(e/c)A_y=m(\dot{y}+\omega_L x)\\ &=-m\omega_S(\mathcal{A}\cos\tilde{w}_1-\mathcal{B}\cos\tilde{w}_2)\,, \end{aligned} \tag{24}$$

$$p_z=m\dot{z}=-m\omega_0\mathcal{C}\sin\tilde{w}_3\,, \tag{25}$$

where $\tilde{w}_1$, $\tilde{w}_2$, and $\tilde{w}_3$ are the angle variables

$$\tilde{w}_i=\omega_i t+\delta_i,\quad i=1,2,3 \tag{26}$$

with frequencies

$$\omega_1=\omega_L+\omega_S\,,\quad \omega_2=-\omega_L+\omega_S\,,\quad \omega_3=\omega_0\,, \tag{27}$$

and arbitrary phase constants δ_i.

The motion is immediately separable in z so that

$$\tilde{J}_3=\frac{1}{2\pi}\oint p_z dz=\frac{1}{2\pi}\int_0^{2\pi}d\tilde{w}_3 p_z\frac{\partial z}{\partial\tilde{w}_3}=\tfrac{1}{2}m\omega_0\mathcal{C}^2\,. \tag{28}$$

For multiply periodic systems which are not immediately separable we can obtain the action variables as[17]

$$\tilde{J}_l=\frac{1}{2\pi}\int_0^{2\pi}d\tilde{w}_l\sum_k p_k\frac{\partial x_k}{\partial\tilde{w}_l}\,. \tag{29}$$

In the present case the substitution of the solutions (20), (21), (23), and (24) for x, y, p_x, and p_y leads to

$$\tilde{J}_1=m\omega_S\mathcal{A}^2\,,\quad \tilde{J}_2=m\omega_S\mathcal{B}^2\,. \tag{30}$$

Thus solving for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in terms of $\tilde{J}_1$, $\tilde{J}_2$, and $\tilde{J}_3$, and then substituting back into (20)–(25), we have x, y, z, p_x, p_y, and p_z expressed as functions of the action-angle variables $\tilde{J}_1$, $\tilde{J}_2$, $\tilde{J}_3$, $\tilde{w}_1$, $\tilde{w}_2$, and $\tilde{w}_3$.

SOLUTION IN TERMS OF ACTION-ANGLE VARIABLES

Next we express the particle energy, angular momentum, and distance squared from the z axis as functions of the action-angle variables. From Eqs. (20)–(25), (28), (30), we find

$$H=\omega_1\tilde{J}_1+\omega_2\tilde{J}_2+\omega_0\tilde{J}_3\,, \tag{31}$$

$$\begin{aligned} L_x&=m(y\dot{z}-z\dot{y})\\ &=[\omega_0(\tilde{J}_1\tilde{J}_3)^{1/2}\sin\tilde{w}_1\sin\tilde{w}_3-\omega_0(\tilde{J}_2\tilde{J}_3)^{1/2}\sin\tilde{w}_2\sin\tilde{w}_3\\ &\quad+\omega_1(\tilde{J}_1\tilde{J}_3)^{1/2}\cos\tilde{w}_1\cos\tilde{w}_3\\ &\quad-\omega_2(\tilde{J}_2\tilde{J}_3)^{1/2}\cos\tilde{w}_2\cos\tilde{w}_3](2/\omega_0\omega_S)^{1/2}\,, \end{aligned} \tag{32}$$

$$\begin{aligned} L_y&=m(z\dot{x}-x\dot{z})\\ &=[\omega_0(\tilde{J}_1\tilde{J}_3)^{1/2}\cos\tilde{w}_1\sin\tilde{w}_3+\omega_0(\tilde{J}_2\tilde{J}_3)^{1/2}\cos\tilde{w}_2\sin\tilde{w}_3\\ &\quad-\omega_1(\tilde{J}_1\tilde{J}_3)^{1/2}\sin\tilde{w}_1\cos\tilde{w}_3\\ &\quad-\omega_2(\tilde{J}_2\tilde{J}_3)^{1/2}\sin\tilde{w}_2\cos\tilde{w}_3](2/\omega_0\omega_S)^{1/2}\,, \end{aligned} \tag{33}$$

$$\begin{aligned} L_z&=m(x\dot{y}-y\dot{x})\\ &=[-\omega_1\tilde{J}_1+\omega_2\tilde{J}_2-2\omega_L(\tilde{J}_1\tilde{J}_2)^{1/2}\\ &\quad\times\cos(\tilde{w}_1+\tilde{w}_2)]/\omega_S\,, \end{aligned} \tag{34}$$

and

$$x^2+y^2=[\tilde{J}_1+\tilde{J}_2+(\tilde{J}_1\tilde{J}_2)^{1/2}\times\cos(\tilde{w}_1+\tilde{w}_2)]/(m\omega_S)\,. \quad (35)$$

AVERAGE VALUES FOR THE GENERAL CASE

In order to evaluate the average values of these quantities for the charged system (6)–(8) in the presence of random classical radiation, we merely average over phase space using the distribution (10); for example, the average energy is

$$\langle H\rangle=\int_0^\infty d\tilde{J}_1\int_0^\infty d\tilde{J}_2\int_0^\infty d\tilde{J}_3\int_0^{2\pi} d\tilde{w}_1\int_0^{2\pi} d\tilde{w}_2\int_0^{2\pi} d\tilde{w}_3 \times HP(\tilde{J}_1,\tilde{J}_2,\tilde{J}_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)\,. \quad (36)$$

In the case of Planck's spectrum with zero-point radiation (5), the averages are

$$\langle H\rangle=\sum_{i=1}^{3}\omega_i\tfrac{1}{2}\hbar\coth(\hbar\omega_i/2KT)\,, \quad (37)$$

$$\langle L_x\rangle=\langle L_y\rangle=0\,, \quad (38)$$

$$\langle L_z\rangle=-\left(\frac{\omega_1}{\omega_S}\right)\tfrac{1}{2}\hbar\coth\left(\frac{\hbar\omega_1}{2KT}\right)+\left(\frac{\omega_2}{\omega_S}\right)\tfrac{1}{2}\hbar\coth\left(\frac{\hbar\omega_2}{2KT}\right)\,, \quad (39)$$

$$\langle x^2+y^2\rangle=\frac{\tfrac{1}{2}\hbar\coth(\hbar\omega_1/2KT)+\tfrac{1}{2}\hbar\coth(\hbar\omega_2/2KT)}{m\omega_S}\,. \quad (40)$$

The magnetic moment $\vec{M}$ differs from the angular $\vec{L}$ by simply the factor $e/2mc$,

$$\vec{M}=(e/2mc)\vec{L}. \quad (41)$$

Hence from Eqs. (38) and (39) we see that the average value has only a z component.

LOW-TEMPERATURE LIMIT

In the limit of low temperature $T\to 0$, we have $\coth(\hbar\omega/2KT)\to 1$, and so

$$\langle H\rangle=\tfrac{1}{2}\hbar\omega_1+\tfrac{1}{2}\hbar\omega_2+\tfrac{1}{2}\hbar\omega_0\,, \quad (42)$$

$$\langle L_z\rangle=(\omega_2-\omega_1)\hbar/2\omega_S=-\omega_L\hbar(\omega_0^2+\omega_L^2)^{-1/2}\,, \quad (43)$$

$$\langle M_z\rangle=-(e/2mc)^2B\hbar(\omega_0^2+\omega_L^2)^{-1/2}\,, \quad (44)$$

and

$$\langle x^2+y^2\rangle=\hbar/m\omega_S=\hbar(\omega_0^2+\omega_L^2)^{-1/2}\,. \quad (45)$$

We note that $\langle M_z\rangle$ in Eq. (44) involves the square of the charge e and always points in a direction opposite to the magnetic field. The sign of $\langle L_z\rangle$ can be traced back and can be seen to depend upon the sign of e. We see that when the charge e changes sign, the roles of ω_1 and ω_2 as larger and smaller frequency are interchanged; this reverses the sign of $\langle L_z\rangle$ but not of $\langle M_z\rangle$.

HIGH-TEMPERATURE (TRADITIONAL CLASSICAL) LIMIT

In the high-temperature limit $KT\gg\tfrac{1}{2}\hbar\omega$ the Planck spectrum goes over to the Rayleigh-Jeans law which is compatible with the traditional statistical mechanics of nonrelativistic particles.[18] And within traditional statistical mechanics there is no diamagnetic behavior whatsoever.[19] Thus for $T\to\infty$, $\hbar\omega/2KT\ll 1$, and we expand $\coth x$ for small x as

$$\coth x=1/x+\tfrac{1}{3}x-\tfrac{1}{45}x^3+\cdots \quad (46)$$

so that the leading term in $\langle L_z\rangle$ involves

$$\langle L_z\rangle\cong-\frac{\omega_1}{\omega_S}\frac{KT}{\omega_1}+\frac{\omega_2}{\omega_S}\frac{KT}{\omega_2}\,, \quad (47)$$

and hence vanishes leaving

$$\langle L_z\rangle\cong(\omega_2^2-\omega_1^2)\hbar^2/(12\omega_S KT)\to 0 \quad \text{for } T\to\infty\,. \quad (48)$$

Thus we recapture the traditional classical result with no diamagnetism.

Our calculation shows the mechanism for the vanishing of the diamagnetic behavior. In the Rayleigh-Jeans spectrum, and only in this spectrum, the contributions to the average angular momentum for the opposite directions of rotation give an exact cancellation no matter what the strength of the magnetic field relative to the binding potential. This corresponds to the familiar mechanical mechanism which is presented in the literature and in particular in VanVleck's monograph.[20]

LIMIT OF WEAK MAGNETIC FIELD COMPARED TO BINDING

If the magnetic field B is small compared to the binding due to the potential $|eB/2mc|=|\omega_L|\ll\omega_0$, then we find the diamagnetic situation considered by Marshall.[6] In this case

$$\omega_1=\omega_L+(\omega_0^2+\omega_L^2)^{1/2}\cong\omega_0+\omega_L\,, \quad (49)$$

$$\omega_2=-\omega_L+(\omega_0^2+\omega_L^2)^{1/2}\cong\omega_0-\omega_L\,, \quad (50)$$

$$\omega_S=(\omega_0^2+\omega_L^2)^{1/2}\cong\omega_0 \quad (51)$$

and the average angular momentum, which has only the z component in (39), becomes

$$\langle L_z\rangle=\frac{\omega_L}{\omega_0}\hbar\coth\left(\frac{\hbar\omega_0}{2KT}\right)-\omega_L\hbar\frac{\partial[\coth(\hbar\omega_0/2KT)]}{\partial\omega_0} \quad (52)$$

with $\langle M_z\rangle=(e/2mc)\langle L_z\rangle$. This result for $\langle M_z\rangle$ agrees with that of Marshall[6] and also with that obtained from quantum theory.[21]

FREE-PARTICLE LIMIT

For the opposite limit of a free particle, we allow the binding potential to become negligibly weak compared to the magnetic field trapping, $\omega_0 \ll |\omega_B| = |eB/mc|$. In this case

$$\omega_1 = \tfrac{1}{2}\omega_B + [(\tfrac{1}{2}\omega_B)^2 + \omega_0^2]^{1/2} \cong \omega_B \,, \tag{53}$$

$$\omega_2 = -\tfrac{1}{2}\omega_B + [(\tfrac{1}{2}\omega_B)^2 + \omega_0^2]^{1/2} \cong \omega_0^2/\omega_B \,, \tag{54}$$

$$\omega_S = [(\tfrac{1}{2}\omega_B)^2 + \omega_0^2]^{1/2} \cong \tfrac{1}{2}\omega_B \,, \tag{55}$$

where we have assumed $\omega_B = eB/mc$ is positive; if ω_B is negative, the roles of ω_1 and ω_2 are interchanged. Thus for a free particle $\omega_0 \to 0$, and we apply the expansion for $\coth x$ given in (46) to obtain

$$\langle H\rangle = \omega_B \tfrac{1}{2}\hbar \coth(\hbar\omega_B/2KT) + 2KT \,, \tag{56}$$

$$\langle L_z\rangle = -\hbar[\coth(\hbar\omega_B/2KT) - 2KT/\hbar\omega_B] \,. \tag{57}$$

The additive contributions $2KT$ in $\langle H\rangle$ and $2KT/\omega_B$ in $\langle L_z\rangle$ actually reflect the fact that the particle was bound at temperature T *before* the no-binding limit was taken.

In the high-temperature limit we again apply the expansion of $\coth x$ given in (46) and obtain

$$\langle H\rangle = 3KT \,, \tag{58}$$

$$\langle L_z\rangle \cong -\hbar^2\omega_B/6KT \to 0 \quad \text{as } T \to \infty \,. \tag{59}$$

The first expression corresponds to the high-temperature limit of a particle bound harmonically in three dimensions. Furthermore the vanishing of $\langle L_z\rangle$ depends crucially on the contribution $2KT/\omega_B$ in (57). Both these results show the residual effects of the binding of the particle, no matter how weak the binding is. This agrees with the emphasis in the literature[20] that the vanishing of diamagnetism within traditional classical statistical mechanics depends crucially upon the existence of finite binding no matter how weak.

In the opposite limit of zero temperature we see the effects of the zero-point radiation alone. In this case

$$\langle H\rangle = \tfrac{1}{2}\hbar\omega_B \tag{60}$$

and

$$\langle L_z\rangle = -\hbar \,, \tag{61}$$

where the sign of $\langle L_z\rangle$ is reversed if the charge is negative. In this limit there is no effect of the arbitrary weak binding, $\omega_0 \to 0$. Now we find that the charged particle assumes an average angular momentum in a direction antiparallel to the applied magnetic field, and this angular momentum has the value $\hbar$, where $\hbar$ is the constant appearing as the scale factor in the classical electromagnetic zero-point radiation. The value for $\langle L_z\rangle$ at zero temperature is independent of the particle charge and mass, and is independent of the magnetic field strength. Moreover no matter what direction is chosen for the applied magnetic field the angular momentum always has the component $\hbar$ relative to this direction.

POSSIBLE CLASSICAL DERIVATION OF PLANCK'S SPECTRUM

The magnetic moment differs from the angular momentum by the simple proportionality $e/2mc$, so that

$$\langle M_z\rangle = -|e|\hbar/2mc \,, \tag{62}$$

where the absolute value of the charge $|e|$ appears so that $\langle M_z\rangle$ always points in a direction opposite to that of the applied magnetic field $\vec{B}$. If we apply[22] Boltzmann statistics to this zero-point magnetic moment regarding it as subject to thermal fluctuations in orientation with an associated additional energy relative to the antialigned configuration

$$\mathcal{E} = +\langle\vec{M}\rangle \cdot \vec{B} = -(|e|\hbar B/2mc)\cos\theta \,, \tag{63}$$

then the average magnetic moment at temperature T is

$$\langle M_z\rangle = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta(-|e|\hbar/2mc)\cos\theta \exp(+|e|\hbar B\cos\theta/2mcKT)}{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \exp(+|e|\hbar B\cos\theta/2mcKT)}$$

$$= -(|e|\hbar/2mc)[\coth(\hbar\omega_B/2KT) - 2KT/\hbar\omega_B] \,, \tag{64}$$

with $\omega_B = |e|B/mc$, which agrees exactly with the result from (57) arising from the Planck spectrum. If one can justify this use of traditional classical statistical mechanics for this system, then this procedure will provide a derivation of Planck's radiation spectrum within classical electron theory with classical electromagnetic zero-point radiation.

It should be emphasized how entirely foreign to traditional classical electron theory are the results appearing here. The angular momentum (61) and the magnetic moment (62) of a free classical point charge are fixed values independent of the strength of the magnetic field yet always have a fixed component in the direction of the applied magnetic field. Moreover the angular momentum always has the magnitude $\hbar$, where $h = 2\pi\hbar$ is Planck's constant chosen as the scale factor in classical electromagnetic zero-point radiation. All of these results have the flavor of quantum theory. Yet they are results of classical electron theory in which we have merely changed the boundary condition to correspond to random classical electromagnetic radiation with a Lorentz-invariant spectrum, classical electromagnetic zero-point radiation.

CONNECTIONS WITH QUANTUM THEORY

The results found here for diamagnetism in classical electrodynamics with classical electromagnetic zero-point radiation correspond to Landau's diamagnetism[9] of free electrons within quantum theory. The quantum calculations look entirely different from our classical ones but the results for the average energy, angular momentum, and magnetic moment agree exactly. The general connection between the classical and quantum theories for this case follows along the lines given earlier[12] for harmonic-oscillator systems in classical and quantum theories.

The diamagnetism of a free charged particle in quantum theory was noted by Landau in 1930, and can be obtained by solving the Schrödinger equation in cylindrical polar coordinates

$$\mathcal{E}\psi = H\psi = \frac{-\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \omega_L \frac{\hbar}{i}\frac{\partial\psi}{\partial\phi} + \tfrac{1}{2}m\omega_S^2\rho^2\psi + \tfrac{1}{2}m\omega_0^2 z^2\psi \tag{65}$$

corresponding to the Hamiltonian given by our Eq. (16) for a particle in a magnetic field and in a harmonic potential. The free particle limit $\omega_0 \to 0$ must be taken after the calculation. The crucial role played at finite temperature by any arbitrarily weak binding is apparent in Landau's work though not introduced in terms of the harmonic-oscillator binding used here.

CLOSING SUMMARY

The existence of diamagnetism within classical electron theory depends crucially upon the spectrum of random radiation which is present at the charged system. In order to interact with the magnetic field, the particle must be charged. However, a charged particle which is not immersed in random radiation will radiate away all its energy and come to rest. If random radiation is present, then the particle will be put into a fluctuating motion due to the random electromagnetic forces and in general there will be residual magnetic effects. Two radiation spectra play special roles. If the Rayleigh-Jeans spectrum of radiation is present, then diamagnetism is not present because of cancelling of the angular momentum contributions from particle motion associated with arbitrarily weak binding of the charged particle. If the Lorentz-invariant spectrum of classical electromagnetic zero-point radiation is present, then for a nonrelativistic free particle the average energy, average angular momentum, and average magnetic moment agree exactly with the diamagnetism of quantum theory.

ACKNOWLEDGMENT

The present calculation was stimulated by the remark in an unpublished manuscript of S. Sachidanandam that a free classical particle in classical zero-point radiation and in a magnetic field has an average angular momentum of absolute value $\hbar$.

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[1]See, for example, J. H. van Vleck, *The Theory of Electric and Magnetic Susceptibilities* (Oxford University, London, 1966). (Republication of 1932 ed.)

[2]J. H. van Leeuwen, dissertation, Leiden, 1919 (unpublished); J. Phys. 2, 361 (1921). Apparently some of these results were anticipated by N. Bohr, dissertation, Copenhagen, 1911 (unpublished).

[3]See Ref. 1, p. 97.

[4]A review is given by T. H. Boyer, Phys. Rev. D 11, 790 (1975).

[5]T. H. Boyer, Phys. Rev. 186, 1304 (1969).

[6]T. W. Marshall, Proc. R. Soc. A 276, 475 (1963).

[7](a) P. Braffort and A. Taroni, C. R. Acad. Sci. B 264, 1437 (1967); (b) M. Surdin, *ibid.* 270, 193 (1970).

[8]S. Sachidanandam (unpublished).

[9]L. Landau, Z. Phys. 64, 629 (1930); see also Ref. 1, p. 353.

[10]See Ref. 6; and also T. H. Boyer, Phys. Rev. 182, 1374 (1969).

[11]Marshall's analysis in Ref. 6 is given in terms of a charge $-e$. There seems to be an inconsistency in sign between the two sides of Marshall's Eq. (6.8), but this does not affect his result since the right-hand side involves a random force.

[12]T. H. Boyer, Phys. Rev. D 11, 809 (1975).

[13]The following misprints appear in Ref. 12. In Eqs. (122) and (123) the sign in front of i should be reversed for each appearance except in the exponentials. In the first line of Eq. (125) the minus sign should be omitted. Equation (128) should have a minus sign before the right-hand side, and this sign error reappears in Eqs. (131), (136), and (137). Furthermore Eq. (128) requires a factor of $\frac{1}{2}$ and the insertion of a square for the factor of m. The unnumbered equation following (147) should have a square for the e. Finally Eq. (148) should involve a factor of e^2 rather than c^2 in the numerator.

[14]In Ref. 12 the weak-field approximations in (134) and

(135) are unnecessarily introduced following Eq. (136). The narrow-linewidth approximation used to evaluate the integrals can be carried out without using the weak-field expansions.

[15]T. H. Boyer, Phys. Rev. A 18, 1238 (1978).

[16]There are a series of misprints in the diamagnetism section of Ref. 15, occasioned in part by sloppy transcription and improper attention to the negative charge used by Marshall in Ref. 6. In Ref. 15 the second line for the Hamiltonian in Eq. (48) omits a factor of m as well as the term in B^2. Equations (57), (58), (64)–(66), and (70) all contain sign errors involving the charge e and the Larmor frequency $eB/2mc$. The subscript in the derivative in Eq. (59) should be changed to read ∂w_k, and the subscript on one factor of momentum in the middle of line (65) should correspond to p_y. The appearance of $2w_L$ should be omitted entirely in Eqs. (60)–(63). The plus and minus subscripts in Eq. (70) should be changed to give $\tilde{J}_1$ and $\tilde{J}_2$.

[17]See, for example, M. Born, *Mechanics of the Atom* (Ungar, New York, 1960), p. 95, Eq. (22).

[18]The Rayleigh-Jeans law is not compatible with the traditional classical statistical mechanics of relativistic particles. See T. H. Boyer, Phys. Rev. D (to be published).

[19]See Ref. 1, p. 94; or D. C. Mattis, *The Theory of Magnetism* (Harper and Row, New York, 1965), p. 21.

[20]See Ref. 1, pp. 100–102.

[21]See the last section of Ref. 12.

[22]See Ref. 1, p. 31.