PHYSICAL REVIEW A **104**, 032601 (2021)

k-uniform quantum information masking

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(Received 31 May 2021; accepted 13 August 2021; published 1 September 2021)

It is impossible to mask an arbitrary quantum state into the correlations between two subsystems such that the original information is completely unknown to each local system. This is the no-masking theorem proposed by Modi *et al.* [K. Modi, A. K. Pati, A. Sen(De), and U. Sen, Phys. Rev. Lett. **120**, 230501 (2018)]. In this work, we propose the concept of k-uniform quantum information masking in multipartite systems and indicate the relation between quantum error-correcting codes (QECCs) in heterogeneous systems and quantum information masking. As a consequence, we show that the no-masking theorem is a special case of the quantum Singleton bound for QECCs in heterogeneous systems essentially, and we give a more general no-masking theorem in multipartite systems based on the quantum Singleton bound. We also solve two open questions proposed by Li and Wang [M.-S. Li and Y.-L. Wang, Phys. Rev. A **98**, 062306 (2018)]. That is, an arbitrary state of level d cannot be masked in a tripartite system of level d such that the marginal states are not proportional to identity, and an arbitrary state of level d cannot be masked in a tripartite system of level n with $n < d$. Further, we give some methods for constructing new QECCs from old QECCs in heterogeneous systems.

DOI: 10.1103/PhysRevA.104.032601

I. INTRODUCTION

The no-go theorems like the no-cloning theorem [1–3], the no-broadcasting theorem [4], the no-deleting theorem [5], and the no-hiding theorem [6] are consequences of the linearity and the unitarity of quantum mechanics. Recently, Modi *et al.* proposed the concept of quantum information masking [7]. This is a physical process that encodes quantum information into a bipartite system, while the information is completely unknown to each local system. They highlighted another no-go theorem which is called the no-masking theorem. That is, an arbitrary quantum state cannot be masked. In Ref. [8], the authors showed that quantum information masking in multipartite systems is possible. In their masking protocol, it is also required that the original information is inaccessible to each local system. More recently, a photonic implementation of quantum information masking was given in Ref. [9].

Errors are inevitable when quantum information goes through a noisy channel [10–12]. Quantum error-correcting codes (QECCs) play a central role in quantum information processing. They can protect quantum information from various quantum noises. There are a lot of investigations for QECCs in homogeneous systems which have equal local dimensions [13–19]. It is expected that we often face a more complicated situation when the encoded states belong to heterogeneous systems which have different local dimensions, where the study of QECCs is more difficult. The most important bound, the quantum Singleton bound [14], can also be generalized for QECCs in heterogeneous systems [20]. The quantum Singleton bound can be seen as having its origins in the no-cloning theorem [17,21]. One can also expect more features for the quantum Singleton bound.

For quantum information masking in multipartite systems, collusion between some subsystems would then reveal the encoded quantum information [7]. Thus, a stronger version of quantum information masking is desirable. In this paper, we propose the concept of k-uniform quantum information masking in multipartite systems. It is required that the original information is inaccessible to each k subsystem. Specially, we refer to the $\lfloor \frac{N}{2} \rfloor$-uniform quantum information masking as the strong quantum information masking. First, we show the relation between QECCs in heterogeneous systems and k-uniform quantum information masking; that is, if all states in $\mathbb{C}^{d_0}$ can be k-uniformly masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$, then there exists an $((N, d_0, k+1))_{d_1,d_2,\ldots,d_N}$ QECC. The converse is true. As a consequence, we show that the no-masking theorem in Ref. [7] is a special case of the quantum Singleton bound for QECCs in heterogeneous systems. We also obtain a more general no-masking theorem in multipartite systems; that is, when N is even, an arbitrary state in $\mathbb{C}^{d_0}$ cannot be strongly masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$. In Ref. [22], the authors proposed two open questions.

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(i) Is it possible to (1-uniformly) mask all states of $\mathbb{C}^d$ into $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ such that the marginal states do not equal $\frac{1}{d}\mathbb{I}_d$?

(ii) Can all states of $\mathbb{C}^d$ be (1-uniformly) masked into $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ with $n < d$?

We give negative answers to both questions. Further, we give some methods for constructing new QECCs from old QECCs in heterogeneous systems.

The rest of this paper is organized as follows. In Sec. II, we introduce the concept of k-uniform quantum information masking and QECCs in heterogeneous systems. In Sec. III, we show the relation between QECCs in heterogeneous systems and k-uniform quantum information masking. In Sec. IV, we show some methods for constructing new QECCs from old QECCs in heterogeneous systems. Finally, we conclude in Sec. V.

II. PRELIMINARY

For a Hilbert space $\mathbb{C}^d$, we always assume $d \geqslant 2$ if not specified. Let $(\mathbb{C}^d)^{\otimes N}$ denote $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$, where $\mathbb{C}^d$ appears N times. Let $\mathbb{Z}_d$ denote the set of integers $\{0, 1, \ldots, d-1\}$. Given a set $B = \{A_1, A_2, \ldots, A_N\}$, we denote $A_\ell^c = B \setminus \{A_\ell\}$ for an element $A_\ell \in B$, and we denote $A^c = B \setminus A$ for a subset $A \subset B$. In Ref. [7], the authors proposed the concept of quantum information masking.

Definition 1. [7]. An operation $\mathcal{S}$ is said to mask quantum information contained in states $\{|a_j\rangle_{A_1} \in \mathcal{H}_{A_1}\}$ by mapping them to states $\{|\psi_j\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}\}$ such that all the reductions to one party of $|\psi_j\rangle$ are identical; i.e.,

$$\rho_{A_1} = \mathrm{Tr}_{A_2}|\psi_j\rangle\langle\psi_j| \text{ and } \rho_{A_2} = \mathrm{Tr}_{A_1}|\psi_j\rangle\langle\psi_j|$$

have no information about the value of j.

In Ref. [22], quantum information masking in multipartite systems is proposed.

Definition 2. [22]. An operation $\mathcal{S}$ is said to mask quantum information contained in states $\{|a_j\rangle_{A_1} \in \mathcal{H}_{A_1}\}$ by mapping them to states $\{|\psi_j\rangle \in \otimes_{\ell=1}^N \mathcal{H}_{A_\ell}\}$ such that all the reductions to one party of $|\psi_j\rangle$ are identical; i.e., for any $A_\ell \in \{A_1, A_2, \ldots, A_N\}$,

$$\rho_{A_\ell} = \mathrm{Tr}_{A_\ell^c}|\psi_j\rangle\langle\psi_j|$$

has no information about the value of j.

Definition 2 also requires that all the reductions to one party of $\{|\psi_j\rangle \in \otimes_{\ell=1}^N \mathcal{H}_{A_\ell}\}$ are identical. However, collusion between some parties would then reveal the encoded quantum information. For example, a masker $\mathcal{S}$ masks quantum information contained in $\{|0\rangle, |1\rangle, |2\rangle\} \in \mathbb{C}^3$ into $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\} \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, where

$$\begin{aligned}|\psi_0\rangle_{ABC} &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle),\\ |\psi_1\rangle_{ABC} &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle),\\ |\psi_2\rangle_{ABC} &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle).\end{aligned}$$

If Alice and Bob are collusive, then $\rho_{AB}^{(0)} = \mathrm{Tr}_C\,|\psi_0\rangle\langle\psi_0| = \frac{1}{3}(|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|)$, $\rho_{AB}^{(1)} = \mathrm{Tr}_C\,|\psi_1\rangle\langle\psi_1| = \frac{1}{3}(|02\rangle\langle 02| + |10\rangle\langle 10| + |21\rangle\langle 21|)$, and $\rho_{AB}^{(2)} = \mathrm{Tr}_C\,|\psi_2\rangle\langle\psi_2| = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$. Alice and Bob can easily distinguish $\rho_{AB}^{(0)}$, $\rho_{AB}^{(1)}$, and $\rho_{AB}^{(2)}$, and they would reveal the encoded quantum information. To avoid this collusion, we propose the k-uniform quantum information masking as follows.

Definition 3. An operation $\mathcal{S}$ is said to *k-uniformly mask* quantum information contained in states $\{|a_j\rangle_{A_0} \in \mathcal{H}_{A_0}\}$ by mapping them to states $\{|\psi_j\rangle \in \otimes_{\ell=1}^N \mathcal{H}_{A_\ell}\}$ such that all the reductions to k parties of $|\psi_j\rangle$ are identical; i.e., for all $A = \{A_{\ell_1}, A_{\ell_2}, \ldots, A_{\ell_k}\}$, which is any subset of $\{A_1, A_2, \ldots, A_N\}$ with cardinality k,

$$\rho_A = \mathrm{Tr}_{A^c}|\psi_j\rangle\langle\psi_j| \tag{1}$$

has no information about the value of j. Specially, if $k = \lfloor \frac{N}{2} \rfloor$, we refer to the k-uniform quantum information masking as the *strong* quantum information masking.

Note that when $\mathcal{H}_{A_0} = \mathcal{H}_{A_1}$, $k = 1$, and $N = 2$, Definition 3 is the same as Definition 1; when $\mathcal{H}_{A_0} = \mathcal{H}_{A_1}$ and $k = 1$, Definition 3 is the same as Definition 2. Our masking protocol also forms the basis for quantum secret sharing [23,24]. In a (k, N) threshold scheme, a secret quantum state is divided into N shares such that any k of those shares can be used to reconstruct the secret, but any set of $k-1$ or fewer shares contains absolutely no information about the secret. However, if $N \geqslant 2k$, then no (k, N) threshold scheme exists [24]. In our k-uniform quantum information masking scheme, even if $N \geqslant 2(k+1)$, it allows secret sharing of quantum information from a "boss" to his N "subordinates" such that no collaboration of k subordinates can retrieve the information. Note that we do not consider recovering the secret in our scheme.

Next, we introduce the concept of quantum error-correcting codes (QECCs) in heterogeneous systems. Let $\{e_j\}_{j\in\mathbb{Z}_{d^2}}$ be an orthogonal operator basis acting on $\mathbb{C}^d$ that includes the identity $e_0 = \mathbb{I}$, such that $\mathrm{Tr}(e_i^\dagger e_j) = \delta_{ij} d$. A local error basis $\mathcal{E} = \{E_\alpha\}$ on the N-partite heterogeneous system $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ consists of

$$E_\alpha = e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \cdots \otimes e_{\alpha_N}^{(N)},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{Z}_{d_1^2} \times \mathbb{Z}_{d_2^2} \times \cdots \times \mathbb{Z}_{d_N^2}$, $e_{\alpha_i}^{(i)}$ acts on $\mathbb{C}^{d_i}$, and $\mathrm{Tr}(E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta} d_1 d_2 \cdots d_N$. The support of a local error operator E_α is defined as $\mathrm{supp}(E_\alpha) := \mathrm{supp}(\alpha) = \#\{i \mid \alpha_i \neq 0,\ \ 1 \leqslant i \leqslant N\}$, and the weight is $\mathrm{wt}(E_\alpha) = |\mathrm{supp}(E_\alpha)|$. Any operator M acting on the space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ can be decomposed as

$$M = \frac{1}{d_1 d_2 \cdots d_N} \sum_{E_\alpha \in \mathcal{E}} \mathrm{Tr}(E_\alpha^\dagger M) E_\alpha. \tag{2}$$

QECCs in heterogeneous systems (also called QECCs over mixed alphabets [20]) must satisfy the Knill-Laflamme condition [11,12]. Thus, we give the following definition.

Definition 4. Let $\mathcal{Q}$ be a K-dimensional subspace of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$. Then $\mathcal{Q}$ is called an $((N, K, k+1))_{d_1, d_2, \ldots, d_N}$ QECC, if for any orthonormal basis $\{i_\mathcal{Q}\}_{i\in\mathbb{Z}_K}$ of $\mathcal{Q}$ and all errors $E_\alpha \in \mathcal{E}$ with $\mathrm{wt}(E_\alpha) < k+1$,

$$\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle = C(E_\alpha)\delta_{ij},$$

where the constant $C(E_\alpha)$ depends only on E_α. Here $k+1$ is called the distance of the code. If $C(E_\alpha)=\frac{\mathrm{Tr}(E_\alpha)}{d_1d_2\cdots d_N}=0$ when $0<\mathrm{wt}(E_\alpha)<k+1$, then the code is called pure. By convention, $((N,1,k+1))_{d_1,d_2,\ldots,d_N}$ refers only to pure codes. Specially, if $d_1=d_2=\cdots=d_N=d$, $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ is denoted as $((N,K,k+1))_d$.

We call the $((N,K,k+1))_d$ QECCs the QECCs in homogeneous systems. There exists an equivalent definition for QECCs in homogeneous systems [17,25]. Similarly, we can give an equivalent definition for QECCs in heterogeneous systems.

Definition 5. Let $\mathcal{Q}$ be a K-dimensional subspace of $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_N}$. Then $\mathcal{Q}$ is called an $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ QECC if for all states $|\psi\rangle\in\mathcal{Q}$, and errors $E_\alpha\in\mathcal{E}$ with $\mathrm{wt}(E_\alpha)<k+1$,

$$\langle\psi|E_\alpha|\psi\rangle=C(E_\alpha),$$

where the constant $C(E_\alpha)$ depends only on E_α. If $C(E_\alpha)=\frac{\mathrm{Tr}(E_\alpha)}{d_1d_2\cdots d_N}=0$ when $0<\mathrm{wt}(E)<k+1$, then the code is called pure.

The proof for the equivalence between Definition 4 and Definition 5 is given in Appendix A. Note that E_α can be replaced by any operator M which is the identity acting on at least $N-k$ parties [14,17]. In this case, if $C(M)=\frac{\mathrm{Tr}(M)}{d_1d_2\ldots d_N}$, then the code is called pure. For an $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ QECC, it can detect all errors acting on at most k subsystems and correct all errors acting on at most $\lfloor\frac{k}{2}\rfloor$ subsystems.

III. RELATION BETWEEN QECCS AND QUANTUM INFORMATION MASKING

In this section, we give the relation between QECCs and quantum information masking. We show that the no-masking theorem in Ref. [7] is a special case of the quantum Singleton bound for QECCs in heterogeneous systems, and we give a more general no-masking theorem based on the quantum Singleton bound. We also answer two open questions from Ref. [22]. A pure state $|\psi\rangle$ is called a *k-uniform state* in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_N}$, if all reductions to k parties of $|\psi\rangle$ are maximally mixed [26,27]. Let $\mathbb{I}_{d_{j_1},d_{j_2},\ldots,d_{j_k}}$ be the identity operator acting on the space $\mathbb{C}^{d_{j_1}}\otimes\mathbb{C}^{d_{j_2}}\otimes\cdots\otimes\mathbb{C}^{d_{j_k}}$. First, we show that an $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ QECC corresponds to a special kind of subspace of $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_N}$.

Lemma 6. Let $\mathcal{Q}$ be a subspace of $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_N}$. If $\mathcal{Q}$ is an $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ QECC, then for any k parties, the reductions of all states in $\mathcal{Q}$ to the k parties are identical. The converse is true. Further if $\mathcal{Q}$ is pure, then any state in $\mathcal{Q}$ is a k-uniform state. The converse is also true.

Proof. "$\Rightarrow$" Assume $\mathcal{Q}$ is an $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ QECC. For any state $|\psi\rangle\in\mathcal{Q}$, by Eq. (2), we have

$$\begin{aligned}|\psi\rangle\langle\psi|&=\frac{1}{d_1d_2\ldots d_N}\mathbb{I}_{d_1,d_2,\ldots,d_N}+\frac{1}{d_1d_2\ldots d_N}\\&\quad\times\sum_{1\leqslant\mathrm{wt}(E_\alpha)\leqslant k}\mathrm{Tr}(E_\alpha^\dagger|\psi\rangle\langle\psi|)E_\alpha+\frac{1}{d_1d_2\ldots d_N}\\&\quad\times\sum_{\mathrm{wt}(E_\alpha)\geqslant k+1}\mathrm{Tr}(E_\alpha^\dagger|\psi\rangle\langle\psi|)E_\alpha\\&=\frac{1}{d_1d_2\ldots d_N}\mathbb{I}_{d_1,d_2,\ldots,d_N}+\frac{1}{d_1d_2\ldots d_N}\\&\quad\times\sum_{1\leqslant\mathrm{wt}(E_\alpha)\leqslant k}\overline{\langle\psi|E_\alpha|\psi\rangle}E_\alpha+\frac{1}{d_1d_2\ldots d_N}\\&\quad\times\sum_{\mathrm{wt}(E_\alpha)\geqslant k+1}\overline{\langle\psi|E_\alpha|\psi\rangle}E_\alpha,\end{aligned}\tag{3}$$

where $\overline{\langle\psi|E_\alpha|\psi\rangle}$ is the complex conjugate of $\langle\psi|E_\alpha|\psi\rangle$. Then for any subset $S=\{j_1,j_2,\ldots,j_k\}\subset\{1,2,\ldots,N\}$ with $|S|=k$,

$$\begin{aligned}\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|&=\frac{1}{d_{j_1}d_{j_2}\ldots d_{j_k}}\mathbb{I}_{d_{j_1},d_{j_2},\ldots,d_{j_k}}+\frac{1}{d_1d_2\ldots d_N}\\&\quad\times\sum_{1\leqslant\mathrm{wt}(E_\alpha)\leqslant k}\overline{\langle\psi|E_\alpha|\psi\rangle}\mathrm{Tr}_{S^c}(E_\alpha)\\&=\frac{1}{d_{j_1}d_{j_2}\ldots d_{j_k}}\mathbb{I}_{d_{j_1},d_{j_2},\ldots,d_{j_k}}+\frac{1}{d_1d_2\ldots d_N}\\&\quad\times\sum_{1\leqslant\mathrm{wt}(E_\alpha)\leqslant k,\ \mathrm{supp}(E_\alpha)\subset S}\overline{C(E_\alpha)}\mathrm{Tr}_{S^c}(E_\alpha).\end{aligned}\tag{4}$$

Note that the third term of Eq. (3) disappears when we trace over S^c. This is because when we trace over S^c for an E_α with $\mathrm{wt}(E_\alpha)\geqslant k+1$, we must trace at least one nonidentity operator, which is zero. Since $C(E_\alpha)$ is a constant depending only on E_α by Definition 5, we obtain that $\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|$ are identical for all states $|\psi\rangle$.

Specially, if the code is pure, then $C(E_\alpha)=0$ for $0<\mathrm{wt}(E_\alpha)<k+1$. This means that $\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|=\frac{1}{d_{j_1}d_{j_2}\cdots d_{j_k}}\mathbb{I}_{d_{j_1},d_{j_2},\ldots,d_{j_k}}$; i.e., $|\psi\rangle$ is a k-uniform state in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_N}$.

"$\Leftarrow$" For any error $E_\alpha\in\mathcal{E}$ with $\mathrm{wt}(E_\alpha)<k+1$, there exists a subset $S=\{j_1,j_2,\ldots,j_k\}\subset\{1,2,\ldots,N\}$ with $|S|=k$ such that $\mathrm{supp}(E_\alpha)\subset S$. For a product operator $P=P_1\otimes P_2\otimes\cdots\otimes P_N$, where P_i acts on $\mathbb{C}^{d_i}$ for $1\leqslant i\leqslant N$, we have

$$\mathrm{Tr}(PE_\alpha)=\frac{d_{j_1}d_{j_2}\ldots d_{j_k}}{d_1d_2\ldots d_N}\mathrm{Tr}_S(\mathrm{Tr}_{S^c}P\cdot\mathrm{Tr}_{S^c}E_\alpha).\tag{5}$$

For any state $|\psi\rangle\in\mathcal{Q}$, the operator $|\psi\rangle\langle\psi|$ can be decomposed by the sum of the product operator from Eq. (2). Then by the linearity of trace operation and Eq. (5), it is implied that

$$\begin{aligned}\langle\psi|E_\alpha|\psi\rangle&=\mathrm{Tr}(|\psi\rangle\langle\psi|E_\alpha)\\&=\frac{d_{j_1}d_{j_2}\ldots d_{j_k}}{d_1d_2\ldots d_N}\mathrm{Tr}_S(\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|\cdot\mathrm{Tr}_{S^c}E_\alpha).\end{aligned}\tag{6}$$

Since $\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|$ are identical for all states $|\psi\rangle$, we have

$$\begin{aligned}\langle\psi|E_\alpha|\psi\rangle&=\frac{d_{j_1}d_{j_2}\ldots d_{j_k}}{d_1d_2\ldots d_N}\mathrm{Tr}_S(\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|\cdot\mathrm{Tr}_{S^c}E_\alpha)\\&=C(E_\alpha),\end{aligned}\tag{7}$$

where $C(E_\alpha)$ is a constant which depends only on E_α. By Definition 5, $\mathcal{Q}$ is an $((N,K,k+1))_{d_1,d_2,\ldots,d_N}$ QECC.

Specially, if $|\psi\rangle$ is a k-uniform state in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_N}$, then $\mathrm{Tr}_{S^c}|\psi\rangle\langle\psi|=\frac{1}{d_{j_1}d_{j_2}\cdots d_{j_k}}\mathbb{I}_{d_{j_1},d_{j_2},\ldots,d_{j_k}}$. This implies that

$$\langle\psi|E_\alpha|\psi\rangle=\frac{1}{d_1d_2\ldots d_N}\mathrm{Tr}_S(\mathrm{Tr}_{S^c}E_\alpha)$$

$$= \frac{1}{d_1 d_2 \dots d_N}\mathrm{Tr}(E_\alpha) = C(E_\alpha) = 0, \tag{8}$$

for $0 < \mathrm{wt}(E_\alpha) < k+1$. Thus by Definition 5, $\mathcal{Q}$ is a pure $((N, K, k+1))_{d_1,d_2,\dots,d_N}$ QECC. ■

Lemma 6 can also be viewed as the definition of QECCs [28]. Now we show some examples of QECCs.

Example 7. (i) Let $\{|i_{\mathcal{Q}}\rangle\}_{i\in\mathbb{Z}_3}$ be an orthonormal basis of the subspace $\mathcal{Q}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$, where

$$\begin{aligned}
|0_{\mathcal{Q}}\rangle &= \frac{1}{\sqrt{6}}(|00000\rangle + |12111\rangle + |01210\rangle + |22021\rangle \\
&\quad + |10220\rangle + |21101\rangle), \\
|1_{\mathcal{Q}}\rangle &= \frac{1}{\sqrt{6}}(|21020\rangle + |02201\rangle + |11100\rangle + |20211\rangle \\
&\quad + |12010\rangle + |00121\rangle), \\
|2_{\mathcal{Q}}\rangle &= \frac{1}{\sqrt{6}}(|20110\rangle + |11221\rangle + |02120\rangle + |10001\rangle \\
&\quad + |22200\rangle + |01011\rangle).
\end{aligned} \tag{9}$$

We can check that for any unit vector (v_0, v_1, v_2), $\sum_{i\in\mathbb{Z}_3} v_i|i_{\mathcal{Q}}\rangle$ is a 2-uniform state in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$. Thus $\mathcal{Q}$ is a pure $((5, 3, 3))_{3,3,3,3,2}$ QECC by Lemma 6.

(ii) Let $\{|i_{\mathcal{Q}}\rangle\}_{i\in\mathbb{Z}_2}$ be an orthonormal basis of the subspace $\mathcal{Q}$ in $(\mathbb{C}^2)^{\otimes 9}$, where

$$\begin{aligned}
|0_{\mathcal{Q}}\rangle &= \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \\
&\quad \otimes (|000\rangle + |111\rangle), \\
|1_{\mathcal{Q}}\rangle &= \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \\
&\quad \otimes (|000\rangle - |111\rangle).
\end{aligned} \tag{10}$$

Then $\mathcal{Q}$ is the Shor's nine-qubit code [10,29], which is an impure $((9, 2, 3))_2$ QECC. One can easily check that the reduction to the first two parties of $|0_{\mathcal{Q}}\rangle$ is $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, and it is not proportional to identity.

An $((N, K, k+1))_{d_1,d_2,\dots,d_N}$ QECC $\mathcal{Q}$ can encode all states in $\mathbb{C}^K$ to the subspace $\mathcal{Q}$ of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$. By Definition 3 and Lemma 6, we can show the relation between QECCs and quantum information masking.

Theorem 8. If all states in $\mathbb{C}^{d_0}$ can be k-uniformly masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$, then there exists an $((N, d_0, k+1))_{d_1,d_2,\dots,d_N}$ QECC. The converse is true. Further, if all image states are k-uniform states, then the QECC is pure. The converse is also true.

By Theorem 8, Example 7 (ii) provides us a 2-uniform masking scheme of all states $\mathbb{C}^2$ to $(\mathbb{C}^2)^{\otimes 9}$, such that reductions to any two parties are not necessarily maximally mixed. The relation between QECCs and quantum information masking is shown in Fig. 1. It seems that k-uniform quantum information masking is similar to quantum error correction. In fact, k-uniform quantum information masking is a more general concept than the quantum error correction. For k-uniform quantum information masking, the states that are masked are from a set of $\mathcal{H}_{A_0}$. For an $((N, K, k+1))_{d_1,d_2,\dots,d_N}$ QECC, the states that are encoded are from a subspace $\mathbb{C}^K$. In Ref. [7], Modi *et al.* highlighted the no-masking theorem as a new no-go theorem. That is, an arbitrary quantum state in $\mathbb{C}^{d_1}$ cannot be (1-uniformly) masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. However, one can determine that any maskable set must be on a spherical circle in certain Euclidean spaces [30,31].

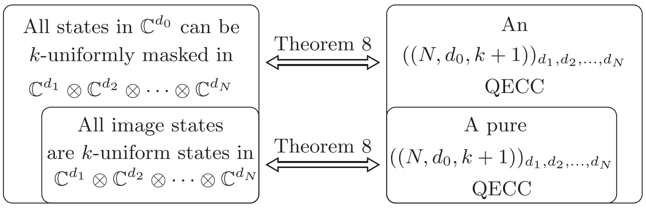

FIG. 1. The relation between quantum error-correcting codes and quantum information masking.

An $((N, K, k+1))_{d_1,d_2,\dots,d_N}$ QECC satisfies the quantum Singleton bound [20]

$$\begin{aligned}
&K \leqslant \min\left\{\prod_{j\in C} d_j \mid C \subset \{1, 2, \dots, N\}, |C| = N - 2k\right\}, \\
&\qquad \text{for } N \geqslant 2k+1, \text{ and} \\
&K \leqslant 1, \quad \text{for } N = 2k.
\end{aligned} \tag{11}$$

Specially, an $((N, K, k+1))_d$ QECC has the quantum Singleton bound [14]

$$K \leqslant d^{N-2k}. \tag{12}$$

A QECC that achieves the equality in Eq. (12) is called a *quantum maximum-distance separable (MDS) code*, i.e., having parameters $((N, d^{N-2k}, k+1))_d$. By applying Theorem 8, we have the following corollary.

Corollary 9. The no-masking theorem is a special case of the quantum Singleton bound for QECCs in heterogeneous systems.

Proof. A $((2, d_1, 2))_{d_1,d_2}$ QECC must have $d_1 \leqslant 1$ by the quantum Singleton bound in Eq. (11). Therefore, if $d_1 \geqslant 2$, a $((2, d_1, 2))_{d_1,d_2}$ QECC does not exist. By Theorem 8, this is equivalent to that an arbitrary state in $\mathbb{C}^{d_1}$ cannot be (1-uniformly) masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. ■

More generally, when N is even, an $((N, d_0, \frac{N}{2}+1))_{d_1,d_2,\dots,d_N}$ QECC does not exist, since $d_0 \leqslant 1$ by the quantum Singleton bound in Eq. (11). Then we can give a more general no-masking theorem in multipartite systems by Theorem 8.

Theorem 10. When N is even, an arbitrary state in $\mathbb{C}^{d_0}$ cannot be strongly masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$.

One can also investigate the maskable set of any masker in Theorem 10, and we leave it as an open question. The quantum Singleton bound can be seen as having its origins in the no-cloning theorem [17,21]. So we believe that the no-masking theorem can also be seen as having its origins in the no-cloning theorem. In Ref. [22], the authors left two open questions.

(i) Is it possible to (1-uniformly) mask all states of $\mathbb{C}^d$ into $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ such that the reductions to one party do not equal $\frac{1}{d}\mathbb{I}_d$?

(ii) Can all states of $\mathbb{C}^d$ be (1-uniformly) masked into $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ with $n < d$?

For the second question, in Ref. [32], the authors gave a negative answer for the case of $d = 3$ and $n = 2$. Now, we can completely negate both questions.

Corollary 11. It is impossible to (1-uniformly) mask all states of $\mathbb{C}^d$ into $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ such that the reductions to one party do not equal $\frac{1}{d}\mathbb{I}_d$.

Proof. If we can mask all states of $\mathbb{C}^d$ into $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ such that the reductions to one party do not equal $\frac{1}{d}\mathbb{I}_d$, then it is equivalent to the existence of an impure $((3, d, 2))_d$ QECC by Theorem 8. Note that a $((3, d, 2))_d$ QECC is a quantum MDS code. However, quantum MDS codes must be pure [14,25]; therefore, impure $((3, d, 2))_d$ QECCs do not exist. ■

Corollary 12. It is impossible to (1-uniformly) mask all states of $\mathbb{C}^d$ into $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ with $n < d$.

Proof. If we can mask all states of $\mathbb{C}^d$ into $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, then it is equivalent to the existence of a $((3, d, 2))_n$ QECC by Theorem 8. But by the quantum Singleton bound in Eq. (12), it must have $d \leqslant n$. ■

In Ref. [22], the authors also showed that if $d \neq 2$ or 6, then all states in $\mathbb{C}^d$ can be (1-uniformly) masked in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. In Ref. [32], the authors proved that it is impossible to mask all states of $\mathbb{C}^2$ into $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Recently, the existence of a $((3, 6, 2))_6$ quantum MDS code was given in Ref. [33], which was obtained from a $((4, 1, 3))_6$ quantum MDS code. By Theorem 8, we know that all states in $\mathbb{C}^6$ can be masked in $\mathbb{C}^6 \otimes \mathbb{C}^6 \otimes \mathbb{C}^6$, which solves the last open case in Ref. [22]. In the next section, we show some methods to obtain new QECCs from old QECCs.

IV. NEW QECCS FROM OLD QECCS

In this section, we give some constructions of new QECCs from old QECCs in heterogeneous systems. Some methods are inspired by the constructions of QECCs in homogeneous systems [13,25].

Proposition 13. If there exists a pure $((N, K, k + 1))_{d_1,d_2,\dots,d_N}$ QECC, then there exists a pure $((N - 1, d_1K, k))_{d_2,d_3,\dots,d_N}$ QECC.

Assume the subspace $\mathcal{Q}$ of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ spanned by $\{|i_{\mathcal{Q}}\rangle\}_{i\in\mathbb{Z}_K}$ is a pure $((N, K, k + 1))_{d_1,d_2,\dots,d_N}$ QECC. Let

$$|i_{\mathcal{Q}}\rangle = \frac{1}{\sqrt{d_1}} \sum_{p\in\mathbb{Z}_{d_1}} |p^{(1)}\rangle |\psi_p^{(i_{\mathcal{Q}})}\rangle, \tag{13}$$

where $\{|p^{(1)}\rangle\}_{p\in\mathbb{Z}_{d_1}}$ is an orthonormal basis of $\mathbb{C}^{d_1}$. Then the subspace of $\mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ spanned by $\{|\psi_p^{(i_{\mathcal{Q}})}\rangle\}_{p\in\mathbb{Z}_{d_1}, i\in\mathbb{Z}_K}$ is a pure $((N - 1, d_1K, k))_{d_2,d_3,\dots,d_N}$ QECC. Details of the proof of Proposition 13 are given in Appendix B. We can show an example of Proposition 13.

Example 14. Let

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|00000\rangle + |01111\rangle + |10011\rangle + |11100\rangle + |20101\rangle + |21010\rangle + |30110\rangle + |31001\rangle). \tag{14}$$

It is a 2-uniform state in $\mathbb{C}^4 \otimes (\mathbb{C}^2)^{\otimes 4}$ [34], and the subspace of $\mathbb{C}^4 \otimes (\mathbb{C}^2)^{\otimes 4}$ spanned by $\{|\psi\rangle\}$ is a (pure) $((5, 1, 3))_{4,2,2,2,2}$ QECC by Lemma 6. Let

$$|\psi_1\rangle = \tfrac{1}{2}(|0000\rangle + |1110\rangle + |2101\rangle + |3011\rangle),$$

$$|\psi_2\rangle = \tfrac{1}{2}(|0111\rangle + |1001\rangle + |2010\rangle + |3100\rangle). \tag{15}$$

By Proposition 13, the subspace of $\mathbb{C}^4 \otimes (\mathbb{C}^2)^{\otimes 3}$ spanned by $\{|\psi_i\rangle\}_{i=1}^2$ is a pure $((4, 2, 2))_{4,2,2,2}$ QECC.

Proposition 15. (i) If there exists an $((N, K, k + 1))_{d_1,d_2,\dots,d_N}$ QECC and an $((N, L, k + 1))_{s_1,s_2,\dots,s_N}$ QECC, then there exists an $((N, KL, k + 1))_{d_1s_1,d_2s_2,\dots,d_Ns_N}$ QECC, which is pure if both the original codes are pure.

(ii) If there exists an $((N_1, K, k + 1))_{d_1,d_2,\dots,d_{N_1}}$ QECC and an $((N_2, L, k + 1))_{s_1,s_2,\dots,s_{N_2}}$ QECC, then there exists an $((N_1 + N_2, KL, k + 1))_{d_1,d_2,\dots,d_{N_1},s_1,s_2,\dots,s_{N_2}}$ QECC, which is pure if both the original codes are pure.

Assume the $((N, K, k + 1))_{d_1,d_2,\dots,d_N}$ QECC $\mathcal{Q}_1$ is spanned by $\{|i\rangle_{A_1,A_2,\dots,A_N}\}_{i\in\mathbb{Z}_K}$, and the $((N, L, k + 1))_{s_1,s_2,\dots,s_N}$ QECC $\mathcal{Q}_2$ is spanned by $\{|j\rangle_{B_1,B_2,\dots,B_N}\}_{j\in\mathbb{Z}_L}$. Then the subspace $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ spanned by $\{(|i\rangle \otimes |j\rangle)_{A_1B_1,A_2B_2,\dots,A_NB_N}\}_{(i,j)\in\mathbb{Z}_K\times\mathbb{Z}_L}$ is an $((N, KL, k + 1))_{d_1s_1,d_2s_2,\dots,d_Ns_N}$ QECC, and the subspace $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ spanned by $\{(|i\rangle \otimes |j\rangle)_{A_1,A_2,\dots,A_{N_1},B_1,B_2,\dots,B_{N_2}}\}_{(i,j)\in\mathbb{Z}_K\times\mathbb{Z}_L}$ is an $((N_1 + N_2, KL, k + 1))_{d_1,d_2,\dots,d_{N_1},s_1,s_2,\dots,s_{N_2}}$ QECC. Details of the proof of Proposition 15 are given in Appendix C.

Proposition 16. If there exists an $((N, K, k + 1))_{(d_0d_1),d_2,\dots,d_N}$ QECC, then there exists an $((N + 1, K, k + 1))_{d_0,d_1,d_2,\dots,d_N}$ QECC, which is pure if the original code is pure.

We only need to replace the states in $\mathbb{C}^{d_0d_1}$ with those in $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_1}$. Details of the proof of Proposition 16 are given in Appendix D. Conversely, if we replace the states in $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_1}$ with those in $\mathbb{C}^{d_0d_1}$, we cannot guarantee that the distance is still $k + 1$. However, distance k is guaranteed as stated in Proposition 17, for which the proof can be found in Appendix D.

Proposition 17. If there exists an $((N + 1, K, k + 1))_{d_0,d_1,d_2,\dots,d_N}$ QECC, then there exists an $((N, K, k))_{(d_0d_1),d_2,\dots,d_N}$ QECC, which is pure if the original code is pure.

By Lemma 6, we know that an $((N, 1, k + 1))_{d_1,d_2,\dots,d_N}$ QECC is a k-uniform state in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$. All the constructions in this section can be used to construct k-uniform states in heterogeneous systems. For example, the authors in Ref. [34] showed that, for any $d \geqslant 3$, there exists a 2-uniform state in $(\mathbb{C}^d)^{\otimes N} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ for any $N \geqslant 7$ and $N \neq 4d + 2$ or $4d + 3$. We can show that the unknown cases $N = 4d + 2$ (or $4d + 3$) do exist by applying Proposition 15 (ii) with a 2-uniform state in $(\mathbb{C}^d)^{\otimes(2d+2)} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ (or $(\mathbb{C}^d)^{\otimes(2d+3)} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$) from Ref. [34] and a 2-uniform state in $(\mathbb{C}^d)^{\otimes 2d}$ from Ref. [8] for any $d \geqslant 3$.

V. CONCLUSION

In this paper, we have given a generalized concept of quantum information masking which is called k-uniform quantum information masking, and we have shown the relation between quantum error-correcting codes (QECCs) in heterogeneous systems and k-uniform quantum information masking. Specially, we have shown that the no-masking theorem in Ref. [7] is a special case of the quantum Singleton bound for QECCs in heterogeneous systems, and we have given a more general no-masking theorem in multipartite systems. We have also answered two open questions proposed in Ref. [22]. Further, we have shown some methods to obtain new QECCs from old QECCs in heterogeneous systems.

Although we have shown that an arbitrary state in $\mathbb{C}^{d_0}$ cannot be strongly masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ when N is even, strong quantum information masking is possible when N is odd. Strong quantum information masking is related to absolute maximally entangled (AME) states [35], which are multipartite entangled states that are maximally entangled for any possible bipartition. Note that an $\lfloor \frac{N+1}{2} \rfloor$-uniform state in $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ [i.e., an $((N+1, 1, \lfloor \frac{N+1}{2} \rfloor + 1))_{d_0, d_1, d_2, \ldots d_N}$ QECC] must be an AME state. By Proposition 13, we would obtain an $((N, d_0, \lfloor \frac{N+1}{2} \rfloor))_{d_1, d_2, \ldots d_N}$ QECC from an AME state. Then an arbitrary state in $\mathbb{C}^{d_0}$ can be strongly ($\lfloor \frac{N}{2} \rfloor$-uniformly) masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ when N is odd. Thus it is interesting to investigate AME states in heterogeneous systems [27,33,34,36–38]. There are also some other problems left. One open problem is to investigate the maskable set $\mathcal{S} \subset \mathbb{C}^{d_0}$ which can be strongly masked in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ when N is even [30,31]. Moreover, can we find a tighter bound than the quantum Singleton bound for QECCs in heterogeneous systems?

ACKNOWLEDGMENTS

We thank Felix Huber for some useful suggestions. F.S. and X.Z. were supported by the NSFC under Grants No. 11771419 and No. 12171452, and the Anhui Initiative in Quantum Information Technologies under Grant No. AHY150200. M.-S.L. was supported by the NSFC (Grant No. 12005092) and the China Postdoctoral Science Foundation (Grant No. 2020M681996). L.C. was supported by the NNSF of China (Grant No. 11871089) and the Fundamental Research Funds for the Central Universities (Grant No. ZG216S2005).

APPENDIX A: THE PROOF FOR THE EQUIVALENCE BETWEEN DEFINITION 4 AND DEFINITION 5

Proof. Definition 4 "⇒" Definition 5. We can decompose $|\psi\rangle$ in the basis $\{i_\mathcal{Q}\}_{i \in \mathbb{Z}_K}$, $|\psi\rangle = \sum_{i \in \mathbb{Z}_K} a_i |i_\mathcal{Q}\rangle$, where $\sum_{i \in \mathbb{Z}_K} |a_i|^2 = 1$. Then for $\mathrm{wt}(E_\alpha) < k+1$,

$$\begin{aligned}\langle\psi|E_\alpha|\psi\rangle &= \sum_{i \in \mathbb{Z}_K} \sum_{j \in \mathbb{Z}_K} \overline{a_i} a_j \langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle \\ &= \sum_{i \in \mathbb{Z}_K} |a_i|^2 \langle i_\mathcal{Q}|E_\alpha|i_\mathcal{Q}\rangle \\ &= \sum_{i \in \mathbb{Z}_K} |a_i|^2 C(E_\alpha) = C(E_\alpha).\end{aligned}$$

Definition 4 "⇐" Definition 5. We only need to show that $\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle = 0$ for $i \neq j$ and $\mathrm{wt}(E_\alpha) < k+1$. Let $|\psi\rangle = \lambda|i_\mathcal{Q}\rangle + u|j_\mathcal{Q}\rangle$, where $|\lambda|^2 + |u|^2 = 1$. Then

$$\begin{aligned}\langle\psi|E_\alpha|\psi\rangle &= (\overline{\lambda}\langle i_\mathcal{Q}| + \overline{u}\langle j_\mathcal{Q}|)E_\alpha(\lambda|i_\mathcal{Q}\rangle + u|j_\mathcal{Q}\rangle) \\ &= C(E_\alpha) + \overline{\lambda}u\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle + \lambda\overline{u}\langle j_\mathcal{Q}|E_\alpha|i_\mathcal{Q}\rangle \\ &= C(E_\alpha).\end{aligned}$$

It implies

$$\overline{\lambda}u\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle + \lambda\overline{u}\langle j_\mathcal{Q}|E_\alpha|i_\mathcal{Q}\rangle = 0. \tag{A1}$$

We can choose $\lambda = \frac{1}{\sqrt{2}}, u = \frac{1}{\sqrt{2}}$ and $\lambda = \frac{1}{\sqrt{2}}, u = \frac{1}{\sqrt{2}}i$, i.e.,

$$\begin{aligned}\frac{1}{2}\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle + \frac{1}{2}\langle j_\mathcal{Q}|E_\alpha|i_\mathcal{Q}\rangle &= 0, \\ \frac{1}{2}i\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle - \frac{1}{2}i\langle j_\mathcal{Q}|E_\alpha|i_\mathcal{Q}\rangle &= 0.\end{aligned} \tag{A2}$$

Thus, we have $\langle i_\mathcal{Q}|E_\alpha|j_\mathcal{Q}\rangle = 0$. ■

APPENDIX B: THE PROOF OF PROPOSITION 13

Proof. Assume the subspace $\mathcal{Q}$ of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ spanned by $\{|i_\mathcal{Q}\rangle\}_{i \in \mathbb{Z}_K}$ is a pure $((N, K, k+1))_{d_1, d_2, \ldots, d_N}$ QECC. Let

$$|i_\mathcal{Q}\rangle = \frac{1}{\sqrt{d_1}} \sum_{p \in \mathbb{Z}_{d_1}} |p^{(1)}\rangle \big|\psi_p^{(i_\mathcal{Q})}\big\rangle, \tag{B1}$$

where $\{|p^{(1)}\rangle\}_{p \in \mathbb{Z}_{d_1}}$ is an orthonormal basis of $\mathbb{C}^{d_1}$. Next, we show that the subspace of $\mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$ spanned by $\{|\psi_p^{(i_\mathcal{Q})}\rangle\}_{p \in \mathbb{Z}_{d_1}, i \in \mathbb{Z}_K}$ is a pure $((N-1, d_1K, k))_{d_2, d_3, \ldots, d_N}$ QECC.

Let

$$E_\beta = e_{\beta_2}^{(2)} \otimes e_{\beta_3}^{(3)} \otimes \cdots \otimes e_{\beta_N}^{(N)}, \tag{B2}$$

where $\beta = \{\beta_2, \beta_3, \ldots, \beta_N\} \in \mathbb{Z}_{d_2^2} \times \mathbb{Z}_{d_3^2} \times \cdots \times \mathbb{Z}_{d_N^2}$. Then $\{E_\beta\}_{\beta \in \mathbb{Z}_{d_2^2} \times \mathbb{Z}_{d_3^2} \times \cdots \times \mathbb{Z}_{d_N^2}}$ is an error basis acting on $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3} \otimes \cdots \otimes \mathbb{C}^{d_N}$. Assume the error basis $\{e_{\alpha_1}^{(1)}\}_{\alpha_1 \in \mathbb{Z}_{d_1^2}}$ acting on $\mathbb{C}^{d_1}$. Let the $d_1 \times d_1$ matrix $M_{\alpha_1} = (m_{i,j})_{i,j \in \mathbb{Z}_{d_1}}$ be the matrix representation of the error $e_{\alpha_1}^{(1)}$ under the basis $\{|p^{(1)}\rangle\}_{p \in \mathbb{Z}_{d_1}}$. We can define a row vector

$$\begin{aligned}u_{\alpha_1} = (&m_{0,0},\ m_{0,1},\ \ldots,\ m_{0,d_1-1},\ m_{1,0},\ m_{1,1},\ \ldots, \\ &\times\ m_{1,d_1-1},\ \ldots,\ m_{d_1-1,0},\ m_{d_1-1,1},\ \ldots, m_{d_1-1,d_1-1}).\end{aligned} \tag{B3}$$

Then $u_{\alpha_i}^\dagger \cdot u_{\alpha_j} = \mathrm{Tr}(M_{\alpha_i}^\dagger M_{\alpha_j}) = d_1\delta_{i,j}$. This means that

$$N = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{d_1^2-1} \end{pmatrix} \tag{B4}$$

is a $d_1^2 \times d_1^2$ full-rank matrix.

Since the QECC $\mathcal{Q}$ is pure, by Definition 4, we have

$$\begin{aligned}\langle i_\mathcal{Q}|e_{\alpha_1}^{(1)} \otimes E_\beta|j_\mathcal{Q}\rangle = \frac{1}{d_1} \sum_{p \in \mathbb{Z}_{d_1}} \sum_{q \in \mathbb{Z}_{d_1}} &\langle p^{(1)}|e_{\alpha_1}^{(1)}|q^{(1)}\rangle, \\ &\langle\psi_p^{(i_\mathcal{Q})}|E_\beta|\psi_q^{(j_\mathcal{Q})}\rangle = 0,\end{aligned} \tag{B5}$$

for $0 < \mathrm{wt}(e_{\alpha_1} \otimes E_\beta) = \mathrm{wt}(e_{\alpha_1}) + \mathrm{wt}(E_\beta) < k+1$, or $i \neq j$;

$$\langle i_\mathcal{Q}|e_{\alpha_1}^{(1)} \otimes E_\beta|i_\mathcal{Q}\rangle = \langle i_\mathcal{Q}|i_\mathcal{Q}\rangle = 1, \tag{B6}$$

for $\mathrm{wt}(e_{\alpha_1} \otimes E_\beta) = 0$. There are two cases.

(i) If $0 < \mathrm{wt}(E_\beta) < k$ and $i, j \in \mathbb{Z}_K$ or $\mathrm{wt}(E_\beta) = 0$ and $i \neq j \in \mathbb{Z}_K$, then we have

$$\sum_{p \in \mathbb{Z}_{d_1}} \sum_{q \in \mathbb{Z}_{d_1}} \langle p^{(1)}|e_{\alpha_1}^{(1)}|q^{(1)}\rangle \big\langle\psi_p^{(i_\mathcal{Q})}\big|E_\beta\big|\psi_q^{(j_\mathcal{Q})}\big\rangle = 0, \tag{B7}$$

for $\alpha_1 \in \mathbb{Z}_{d_1^2}$. Let the column vector

$$\begin{aligned} X^{(i,j)} = \big(&\langle\psi_0^{(i_\mathcal{Q})}|E_\beta|\psi_0^{(j_\mathcal{Q})}\rangle,\ \langle\psi_0^{(i_\mathcal{Q})}|E_\beta|\psi_1^{(j_\mathcal{Q})}\rangle, \dots, \\ &\langle\psi_0^{(i_\mathcal{Q})}|E_\beta|\psi_{d_1-1}^{(j_\mathcal{Q})}\rangle,), \dots, \\ &\langle\psi_{d_1-1}^{(i_\mathcal{Q})}|E_\beta|\psi_0^{(j_\mathcal{Q})}\rangle, \langle\psi_{d_1-1}^{(i_\mathcal{Q})}|E_\beta|\psi_1^{(j_\mathcal{Q})}\rangle, \dots \\ &\langle\psi_{d_1-1}^{(i_\mathcal{Q})}|E_\beta|\psi_{d_1-1}^{(j_\mathcal{Q})}\rangle\big)^T. \end{aligned} \tag{B8}$$

By Eq. (B7), we have

$$NX^{(i,j)} = 0. \tag{B9}$$

It implies that $X^{(i,j)} = 0$. This means that

$$\langle\psi_p^{(i_\mathcal{Q})}|E_\beta|\psi_q^{(j_\mathcal{Q})}\rangle = 0, \tag{B10}$$

for $0 < \mathrm{wt}(E_\beta) < k$, $i, j \in \mathbb{Z}_K$, and $p, q \in \mathbb{Z}_{d_1}$;

$$\langle\psi_p^{(i_\mathcal{Q})}|E_\beta|\psi_q^{(j_\mathcal{Q})}\rangle = \langle\psi_p^{(i_\mathcal{Q})}|\psi_q^{(j_\mathcal{Q})}\rangle = 0, \tag{B11}$$

for $\mathrm{wt}(E_\beta) = 0$ and $i \neq j \in \mathbb{Z}_K$ and $p, q \in \mathbb{Z}_{d_1}$.

(ii) If $\mathrm{wt}(E_\beta) = 0$ and $i = j \in \mathbb{Z}_K$, then

$$\sum_{p\in\mathbb{Z}_{d_1}}\sum_{q\in\mathbb{Z}_{d_1}} \langle p^{(1)}|e_{\alpha_1}^{(1)}|q^{(1)}\rangle\langle\psi_p^{(i_\mathcal{Q})}|E_\beta|\psi_q^{(i_\mathcal{Q})}\rangle = d_1\delta_{\alpha_1,0}, \tag{B12}$$

for $\alpha_1 \in \mathbb{Z}_{d_1^2}$. Let Y be a $d_1^2 \times 1$ column vector $Y = (d_1, 0, 0, \dots, 0)^T$. By Eq. (B12), we have

$$NX^{(i,i)} = Y. \tag{B13}$$

There exists a unique solution for $X^{(i,i)}$, that is,

$$\langle\psi_p^{(i_\mathcal{Q})}|E_\beta|\psi_q^{(i_\mathcal{Q})}\rangle = \langle\psi_p^{(i_\mathcal{Q})}|\psi_q^{(i_\mathcal{Q})}\rangle = \delta_{p,q}, \tag{B14}$$

for $\mathrm{wt}(E_\beta) = 0$, $i \in \mathbb{Z}_K$, and $p, q \in \mathbb{Z}_{d_1}$.

Thus, by Eqs. (B10), (B11), and (B14) and Definition 4, we obtain that the subspace of $\mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_N}$ spanned by $\{|\psi_p^{(i_\mathcal{Q})}\rangle\}_{p\in\mathbb{Z}_{d_1}, i\in\mathbb{Z}_K}$ is a pure $((N-1, d_1K, k))_{d_2,d_3,\dots,d_N}$ QECC. ■

APPENDIX C: THE PROOF OF PROPOSITION 15

Proof. (i) Assume the $((N, K, k+1))_{d_1,d_2,\dots,d_N}$ QECC $\mathcal{Q}_1$ is spanned by $\{|i\rangle_{A_1,A_2,\dots,A_N}\}_{i\in\mathbb{Z}_K}$, and the $((N, L, k+1))_{s_1,s_2,\dots,s_N}$ QECC $\mathcal{Q}_2$ is spanned by $\{|j\rangle_{B_1,B_2,\dots,B_N}\}_{j\in\mathbb{Z}_L}$. Next, we show that the subspace $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ spanned by $\{(|i\rangle \otimes |j\rangle)_{A_1B_1,A_2B_2,\dots,A_NB_N}\}_{(i,j)\in\mathbb{Z}_K\times\mathbb{Z}_L}$ is an $((N, KL, k+1))_{d_1s_1,d_2s_2,\dots,d_Ns_N}$ QECC.

Let $\{E_\alpha\}_{\alpha\in\mathbb{Z}_{d_1^2}\times\mathbb{Z}_{d_2^2}\times\dots\times\mathbb{Z}_{d_N^2}}$ and $\{E_\beta\}_{\beta\in\mathbb{Z}_{s_1^2}\times\mathbb{Z}_{s_2^2}\times\dots\times\mathbb{Z}_{s_N^2}}$ be error bases acting on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_N}$ and $\mathbb{C}^{s_1} \otimes \mathbb{C}^{s_2} \otimes \dots \otimes \mathbb{C}^{s_N}$, respectively, where

$$\begin{aligned} E_\alpha &= e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \dots \otimes e_{\alpha_N}^{(N)}, \\ E_\beta &= e_{\beta_1}^{(1)} \otimes e_{\beta_2}^{(2)} \otimes \dots \otimes e_{\beta_N}^{(N)}. \end{aligned} \tag{C1}$$

We define

$$\begin{aligned} E_{(\alpha,\beta)} = E_\alpha \otimes E_\beta = &\big(e_{\alpha_1}^{(1)} \otimes e_{\beta_1}^{(1)}\big) \\ &\otimes \big(e_{\alpha_2}^{(2)} \otimes e_{\beta_2}^{(2)}\big) \otimes \dots \otimes \big(e_{\alpha_N}^{(N)} \otimes e_{\beta_N}^{(N)}\big), \end{aligned} \tag{C2}$$

where $(\alpha, \beta) = [(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_N, \beta_N)]$. Then $\{E_{(\alpha,\beta)}\}_{(\alpha,\beta)\in(\mathbb{Z}_{d_1^2}\times\mathbb{Z}_{s_1^2})\times(\mathbb{Z}_{d_2^2}\times\mathbb{Z}_{s_2^2})\times\dots\times(\mathbb{Z}_{d_N^2}\times\mathbb{Z}_{s_N^2})}$ is an error basis acting on $\mathbb{C}^{d_1s_1} \otimes \mathbb{C}^{d_2s_2} \otimes \dots \otimes \mathbb{C}^{d_Ns_N}$. If $\mathrm{wt}(E_{(\alpha,\beta)}) < k+1$, then $\mathrm{wt}(E_\alpha) < k+1$ and $\mathrm{wt}(E_\beta) < k+1$. Then for all $\mathrm{wt}(E_{(\alpha,\beta)}) < k+1$,

$$\begin{aligned} \langle i_1|\langle j_1|E_{(\alpha,\beta)}|i_2\rangle|j_2\rangle &= \langle i_1|E_\alpha|i_2\rangle\langle j_1|E_\beta|j_2\rangle \\ &= C(E_\alpha)C(E_\beta)\delta_{i_1,i_2}\delta_{j_1,j_2} \\ &= C(E_{(\alpha,\beta)})\delta_{((i_1,j_1),(i_2,j_2))}. \end{aligned} \tag{C3}$$

Thus $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ is an $((N, KL, k+1))_{d_1s_1,d_2s_2,\dots,d_Ns_N}$ QECC by Definition 4.

If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are pure, then $C(E_\alpha) = \frac{\mathrm{Tr}(E_\alpha)}{d_1d_2\dots d_N}$ and $C(E_\beta) = \frac{\mathrm{Tr}(E_\beta)}{s_1s_2\dots s_N}$. We have

$$\begin{aligned} C(E_{(\alpha,\beta)}) = C(E_\alpha)C(E_\beta) &= \frac{\mathrm{Tr}(E_\alpha)}{d_1d_2\dots d_N}\frac{\mathrm{Tr}(E_\beta)}{s_1s_2\dots s_N} \\ &= \frac{\mathrm{Tr}(E_\alpha \otimes E_\beta)}{d_1s_1d_2s_2\dots d_Ns_N} = \frac{\mathrm{Tr}(E_{(\alpha,\beta)})}{d_1s_1d_2s_2\dots d_Ns_N}. \end{aligned}$$

Thus, $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ is also pure.

(ii) Assume the $((N_1, K, k+1))_{d_1,d_2,\dots,d_{N_1}}$ QECC $\mathcal{Q}_1$ is spanned by $\{|i\rangle_{A_1,A_2,\dots,A_{N_1}}\}_{i\in\mathbb{Z}_K}$, and the $((N_2, L, k+1))_{s_1,s_2,\dots,s_{N_2}}$ QECC $\mathcal{Q}_2$ is spanned by $\{|j\rangle_{B_1,B_2,\dots,B_{N_2}}\}_{j\in\mathbb{Z}_L}$. We show that the subspace $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ spanned by $\{(|i\rangle \otimes |j\rangle)_{A_1,A_2,\dots,A_{N_1},B_1,B_2,\dots,B_{N_2}}\}_{(i,j)\in\mathbb{Z}_K\times\mathbb{Z}_L}$ is an $((N_1+N_2, KL, k+1))_{d_1,d_2,\dots,d_{N_1},s_1,s_2,\dots,s_{N_2}}$ QECC.

Note that an $((N_1, K, k+1))_{d_1,d_2,\dots,d_{N_1}}$ QECC $\mathcal{Q}_1$ spanned by $\{|i\rangle_{A_1,A_2,\dots,A_{N_1}}\}_{i\in\mathbb{Z}_K}$ can be viewed as an $((N_1+N_2, K, k+1))_{d_1,d_2,\dots,d_{N_1},1,1,\dots1}$ QECC spanned by $\{|i\rangle_{A_1,A_2,\dots,A_{N_1},B_1,B_2,\dots,B_{N_2}}\}_{i\in\mathbb{Z}_K}$, and an $((N_2, L, k+1))_{s_1,s_2,\dots,s_{N_2}}$ QECC $\mathcal{Q}_2$ spanned by $\{|j\rangle_{B_1,B_2,\dots,B_{N_2}}\}$ can be viewed as an $((N_1+N_2, K, k+1))_{1,1,\dots,1,s_1,s_1,\dots s_{N_2}}$ QECC spanned by $\{|j\rangle_{A_1,A_2,\dots,A_{N_1},B_1,B_2,\dots,B_{N_2}}\}_{j\in\mathbb{Z}_L}$. By using (i), we obtain that the subspace $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ spanned by $\{(|i\rangle \otimes |j\rangle)_{A_1,A_2,\dots,A_{N_1},B_1,B_2,\dots,B_{N_2}}\}_{(i,j)\in\mathbb{Z}_K\times\mathbb{Z}_L}$ is an $((N_1+N_2, KL, k+1))_{d_1,d_2,\dots,d_{N_1},s_1,s_2,\dots,s_{N_2}}$ QECC. ■

APPENDIX D: THE PROOF OF PROPOSITION 16

Proof. Assume the $((N, K, k+1))_{(d_0d_1),d_2,\dots,d_N}$ QECC $\mathcal{Q}_1$ is spanned by $\{|i\rangle_{A_1,A_2,\dots,d_N}\}_{i\in\mathbb{Z}_K}$. By replacing the states in $\mathbb{C}^{d_0d_1}$ with those in $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_1}$, we show that it is an $((N+1, K, k+1))_{d_0,d_1,d_2,\dots,d_N}$ QECC $\mathcal{Q}_2$ spanned by $\{|i\rangle_{B_0,B_1,A_2,\dots,d_N}\}_{i\in\mathbb{Z}_K}$.

Let

$$E_\alpha = e_{\alpha_0}^{(0)} \otimes e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \dots \otimes e_{\alpha_N}^{(N)}, \tag{D1}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_N) \in \mathbb{Z}_{d_0^2} \times \mathbb{Z}_{d_1^2} \times \mathbb{Z}_{d_2^2} \times \dots \times \mathbb{Z}_{d_N^2}$, Then $\{E_\alpha\}_{\alpha\in\in\mathbb{Z}_{d_0^2}\times\mathbb{Z}_{d_1^2}\times\mathbb{Z}_{d_2^2}\times\dots\times\mathbb{Z}_{d_N^2}}$ is an error basis acting on $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_N}$.

Let

$$E_\beta = \big(e_{\alpha_0}^{(0)} \otimes e_{\alpha_1}^{(1)}\big) \otimes e_{\alpha_2}^{(2)} \otimes \dots \otimes e_{\alpha_N}^{(N)}, \tag{D2}$$

where $\beta = (\beta_{(\alpha_0,\alpha_1)}, \alpha_2, \dots, \alpha_N) \in (\mathbb{Z}_{d_0^2} \times \mathbb{Z}_{d_1^2}) \times \mathbb{Z}_{d_2^2} \times \cdots \times \mathbb{Z}_{d_N^2}$, $\beta_{(\alpha_0,\alpha_1)} = (\alpha_0, \alpha_1)$. Then $\{E_\beta\}_{\beta \in (\mathbb{Z}_{d_0^2} \times \mathbb{Z}_{d_1^2}) \times \mathbb{Z}_{d_2^2} \times \cdots \times \mathbb{Z}_{d_N^2}}$ is an error basis acting on $\mathbb{C}^{d_0 d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}$. If $\mathrm{wt}(E_\alpha) < k+1$, then $\mathrm{wt}(E_\beta) < k+1$. For all $\mathrm{wt}(E_\alpha) < k+1$, we have

$$
\begin{aligned}
&{}_{B_0,B_1,A_2,\dots,A_N}\langle i|E_\alpha|j\rangle_{B_0,B_1,A_2,\dots,A_N} \\
&= {}_{A_1,A_2,\dots,A_N}\langle i|E_\beta|j\rangle_{A_1,A_2,\dots,A_N} \\
&= C(E_\beta)\delta_{i,j} = C(E_\alpha)\delta_{i,j}. \qquad \text{(D3)}
\end{aligned}
$$

Thus $\mathcal{Q}_2$ is an $((N+1, K, k+1))_{d_0,d_1,d_2,\dots,d_N}$ QECC by Definition 4. If $\mathcal{Q}_1$ is pure, it is easy to see that $\mathcal{Q}_2$ is also pure. ■

From the proof of Proposition 16, Proposition 17 is true by the fact that if $\mathrm{wt}(E_\beta) < k$, then $\mathrm{wt}(E_\alpha) < k+1$.

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