A Path Integral for Spin*

LAWRENCE SCHULMAN
Physics Department, Indiana University, Bloomington, Indiana 47401
(Received 1 August 1968)

A path integral for spinning particles is developed. It is a one-particle theory, equivalent to the usual quantum mechanics. Our method employs a classical model for spin which is quantized by path integration. The model, the spherical top, is a natural one from a group-theoretic point of view and has been used before in a similar context. The curvature and multiple connectedness of the top coordinate space [$SO(3)$] provide some interesting features in the sum over paths. The Green's function which results from this procedure propagates all spins, and recovery of the usual Pauli spinors from this formalism is achieved by projection to a specific spin subspace.

1. INTRODUCTION

RECENTLY, Feynman, who invented the subject, had this to say about path integrals[1]:

> ". . . path integrals suffer most grievously from a serious defect. They do not permit a discussion of spin operators . . . in a simple and lucid way. . . . Nevertheless, spin is a simple and vital part of real quantum mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation."

This representation, for the nonrelativistic case, is our present concern. The formulation is in terms of a classical model for spin which is familiar and noncontroversial, and our efforts will be directed at path integration of this model.

To our knowledge, existing path-integral theories for spin[2] concentrate on the statistical aspects of the problem and as such are most naturally expressed as field theories. The spin properties of the fermions or bosons of these theories are somewhat secondary and not especially transparent. It would appear that nonrelativistically spin and statistics are separate questions and that a simple *spin* theory should concentrate on just that, leaving the complications of several particles to other considerations. Our goal is then a one-particle theory with optional second quantization.

The idea behind our approach is simple. In principle, there is no difficulty in using path integrals to get the spin of a polyatomic molecule composed of spinless atoms. By a change of variables it is possible to describe this path integral as being over translational, rotational, and internal coordinates. The second of these gives rise to total spin. To get the simplest spinning object we throw away the extra internal coordinates and append to translational coordinates only rotational variables. This will also give half-integral spin since, as is well known, the "ideal" top, as opposed to a bound state of several particles, possesses all spins ($j=0, \frac{1}{2}, 1, \cdots$).

The word "top" is used here because this is the archetype of a mechanical object described by rotational coordinates. Thus the position of a top is determined by a rotation (e.g., that which brings it from some fiducial position), which is to say that its position is given by an element of the group $SO(3)$.

In fact, the relation between half-integral spins and the rotation group is particularly direct in the context of path-integral theory.[3] Ray representations of $SO(3)$ arise because its fundamental group is not trivial—i.e., there are paths in the group which are not deformable into one another. The connection between homotopy theory and representation theory is made via possibly multivalued functions defined on the group manifold. In path integral theory we work directly with the paths. Distinct homotopy classes of paths enter the sum over paths with arbitrary relative phase factors. The selection of these phase factors gives rise to the various multivalued representations. Between given endpoints in $SO(3)$ there are two classes of paths. Depending on the relative sign with which these are added one obtains the propagator for a top of integral or half-integral spin. Incidentally with this viewpoint the distinction between an ideal top and an n-body bound state is evident. As long as the latter can in principle come apart its total coordinate space is R^{3n}, which is simply connected (and therefore only integral spins are allowed).

Another approach to spin theory can be obtained through the use of a Hamiltonian, and Bacry[4] presents a classical phase space and in fact uses fewer coordinates for his spinning particle than we shall. Nevertheless, our desire is to extend Feynman's theory in its most pristine form: a classical system with Lagrangian and variational principle. Furthermore, it is not clear that path integral computations in phase space are feasible for any but the most trivial coordinate systems.

Recovery of the usual Pauli spinor formalism from the top theory described above is easily accomplished by projection to a fixed angular momentum subspace. Similarly the behavior of this top in the presence of an

* Work supported in part by the National Science Foundation and the Army Research Office, Durham.

[1] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill Book Co., New York, 1965), p. 355.
[2] See J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960), and references quoted therein.
[3] M. Hamermesh, *Group Theory* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962), pp. 331, 332.
[4] H. Bacry, Argonne National Laboratory report, 1966 (unpublished); also H. Bacry, Commun. Math. Phys. **5**, 97 (1967).

electromagnetic field presents no problem peculiar to path-integral theory and this question has been satisfactorily discussed in the context of standard quantum mechanics by Bopp and Haag.[5]

It is unlikely (to say the least) that the techniques described here can compete in utility with the customary Pauli spin formalism. Prior to projection, the Green's function which we compute will propagate all spins and so carries a good deal of information which is irrelevant when dealing with particles of fixed spin. Nevertheless, there are situations where the present viewpoint is helpful. One often wishes to describe a "particle" which appears in several spin states. Our top is such a particle and, moreover, may possess additional internal coordinates [orbit labels, in case $SO(3)$ is not transitive on M—cf. Sec. 2].

To summarize then, we feel that we have brought spin within the compass of Feynman's formulation of quantum mechanics. In the second section of this paper, we motivate our selection of the spherical top as the carrier of the internal motion presumed to give rise to spin. While this may seem to be an obvious choice it should be pointed out that there are other models of spin having very little to do with tops, and this section will thus serve to emphasize our point of view and basic assumptions. As will be seen, our attitude is that spin is basically a property of rotations and we shall require that whatever gives rise to a spin also has well-defined rotation properties. Thus our spin model will bear a "natural" relation to the rotation group.

Furthermore, although we have so far been reticent about the relativistic extension of the present theory, it is clear that that theory demands precise isolation of the assumptions built in the spin model. There exist models of relativistic tops but these generally involve many degrees of freedom, degrees of freedom whose relation to spin is obscure. The approach of Sec. 2 has in fact led us to a top of the same sort as the nonrelativistic model but since the present work is directed primarily to questions of path integration this other issue will not be pursued further here.

Two aspects of the top's path integration call for special comment and this is the subject of Sec. 3. There we establish a rule for summing over paths in multiply connected spaces. We also present a formula of DeWitt[6] for the Green's function at infinitesimal times on a space with nontrivial metric. Both of these are relevant for the top, and finally in Sec. 4 the equivalence of path integration and standard quantum mechanics is demonstrated. That section closes with some remarks on the transition from the present formalism to that of Pauli spinors. Section 5 is a conclusion.

[5] F. Bopp and R. Haag, Z. Naturforsch. **5a**, 644 (1950).
[6] Bryce S. DeWitt, Rev. Mod. Phys. **29**, 377 (1957).

2. FORMULATION OF SPIN THEORY

To establish a path-integral theory for spin it is first necessary to specify what is meant by a configuration of the system supposed to give rise to the spin. Then to each pair of initial and final configurations must be assigned a number depending on the possible intermediate configurations of the system.

There is a parallel between the requirements for a path-integral theory and those for a classical model of spin. There, one also specifies some mechanical system which has well-defined configurations in time and considers its motion. In fact, this is just what we shall do, and then quantize the system by means of path integrals.

In quantum mechanics, one specifies the state of a spinning particle by s and s_z. We find this description inconvenient (for immediate transcription to classical mechanics) since the variable in question is discrete and this poses various difficulties for path integration. An example of some of these difficulties, in the relativistic case, is given by Feynman in the Appendix to one of his papers.[7] There, by putting spinors in the exponential, he achieves a theory which in a sense is equivalent to the Dirac equation but which is not fully satisfactory as a path-integral theory. At least part of this incompleteness Feynman attributes to the "geometrical mysteries involved in the superposition of hypercomplex numbers."[7]

Instead we consider this angular momentum to be the angular momentum of *something*; something is spinning. This, however, is just another way of saying that there is an internal variable. Call the space of this internal coordinate M. We intend to think of M as a more typical sort of coordinate space, a continuous one in which the system has a true path (a curve). In a sense, we are using the license provided by path-integral theory to speak more concretely of the internal coordinate space.

To determine the structure of M, we follow an argument of Finkelstein.[8] We ask for the desired transformation properties of this space. In our case we shall demand transitivity of M under the (3-dimensional, real) rotation group. Then, says Finkelstein, M can be identified with a coset space of $SO(3)$, where for simplicity the metric and topology are assumed carried over by the one-to-one correspondence which must exist. The argument relating M to quotient spaces of its transformation groups can be found in Hermann's book.[9]

Why transitivity under $SO(3)$? We know the transformation properties of the momenta to which our internal variables are to be conjugate—these are just the usual spin angular momenta. Now the variables to

[7] R. P. Feynman, Phys. Rev. **84**, 108 (1951), Appendix D.
[8] D. Finkelstein, Phys. Rev. **100**, 924 (1955).
[9] R. Hermann, *Lie Groups for Physicists* (W. A. Benjamin, Inc., New York, 1964), p. 3.

which these momenta are conjugate must have some transformation properties under rotation since if there is some underlying Lagrangian dynamics (which is what we are looking for in order to path integrate) the momenta are directly related to the coordinates and their velocities. Let the orbit of $m \in M$ be O_m [$=$the image of m under all $U \in SO(3)$]. Then the simplest assumption we can make is transitivity, i.e., $O_m = M$. If this is not the case, then we either have some discrete complication or an additional degree of freedom. Since we intend our particle to have no structure other than spin, we take $O_m = M$.

Thus the candidates for M are limited to the coset spaces of $SO(3)$. These are

(1) $SO(3)$, the group itself,
(2) $SO(3)/SO(2)$, essentially a 2-sphere, S^2,
(3) $SO(3)/SO(3)$, just a point.

The subgroup relative to which the quotient is taken is the isotropy group of the system, i.e., those rotations which leave unchanged the mechanical object to which internal coordinates are ascribed. For example, case (2) might describe a dipole. In case (3), the object is spinless. An object having $M = SO(3)$ itself is the top. Both $SO(3)$ and the position of a top can be parametrized by the Euler angles.

There is more to a classical model, however, than specifying a coordinate manifold M. A dynamics too must be given, for example, by a Lagrangian. In our case, since M is identified with a quotient space of a compact Lie group it automatically inherits a mechanics; the paths of the system are geodesics of the coset space. There is a natural metric associated with this Lie group and what we are saying is that if M is the group then the metric coincides with the Lagrangian. This, however, does not exhaust the possibilities. Geodesic motion in $SO(3)$ corresponds to a spherical top. Symmetrical and asymmetrical tops are more general and have Lagrangians differing from the $SO(3)$ metric. We have not found any reason to employ these more general objects and will concentrate on spherical tops. Therefore, if $M = SO(3)$ the spinning object is a spherical top. "Is" in the foregoing sentence means mechanically indistinguishable from. We do not imply any finite extension for our "electron." If $M = SO(3)/SO(2)$ then the metric is the (naturally) induced one, and the geodesics are in fact great circles on the sphere.

That we should come in the end to calling our spinning object a top should not be surprising in view of the parallel requirements of spin models and path integrals. In 1950, Bopp and Haag[5] studied the quantum mechanical top with just this goal in mind, namely, to establish it as a model for spin. They brought out the fact that wave functions (and differential operators) could be defined on the internal coordinate space (these wave functions are the components of the rotation matrices—the "curly D's") and that both integral and half-integral spins appeared. When this top was placed in an electromagnetic field, agreement was found with the usual Pauli spinor theory.

Thus the path-integral theory is formulated as follows: The path of the spinning object is a curve in $R^3 \times SO(3)$ with specified initial and final points in the composite space. For each of these curves use the given Lagrangian to compute the action S along this curve. Then sum the quantity $\exp(iS/\hbar)$ over paths and this is the propagator between the two endpoints. Later on, after obtaining the propagator we shall show how to recover the usual form of quantum mechanics (i.e., with spinors).

3. SIMPLE EXAMPLES

Before discussing the path integral for the top, we shall examine some simpler cases. The top differs from most systems heretofore path-integrated in two distinctive ways. The examples to be given in this section embody these features separately. First, the coordinate space of the top is multiply connected. The fixed-axis rigid rotator (FARR), whose position can be parametrized by the points on a circle, is similar in this respect and we shall examine its path integral. Second, $SO(3)$ is curved. Its metric tensor is not diagonal and is a function of the coordinates (e.g., Euler angles). Because of the stochastic character of the integral this will lead to the keeping of higher-order differentials even for free motion (i.e., $dS = Ldt$ is not sufficiently accurate). Illustrating this phenomenon will be free motion in the plane employing polar rather than Cartesian coordinates. This was studied by Edwards and Gulyaev[10] and we shall cite some of their observations. The correct short-time propagator is given by DeWitt[6] who expresses it as a function of the endpoints using the metric g of the space, its derivatives and whatever potentials are present. The meaning of "correct" and the dangers of casualness in this regard will also be discussed.

FARR is a system with one coordinate φ, $0 \leq \varphi \leq 2\pi$, and Lagrangian

$$L = \tfrac{1}{2} I \dot{\varphi}^2 . \tag{3.1}$$

This space is the group manifold of the group $SO(2)$ and with the metric corresponding to Eq. (3.1) it is flat. The interesting feature here is the identification of the points 0 and 2π. Although the "integral spin" version of this system has been treated in the literature[11] and we offer only a mild generalization, still we feel some interesting aspects of the problem were ignored and it is just these points which must be cleared up before attempting the top problem.

In setting about to path-integrate the FARR, we consider the general instructions for such a computa-

[10] S. F. Edwards and Y. V. Gulyaev, Proc. Roy. Soc. (London) **279A**, 229 (1964).

[11] W. K. Burton and A. H. DeBorde, Nuovo Cimento **2**, 197 (1955).

tion. For a given pair of points in a space (φ_1, φ_2 in our case) consider curves from t_1 to t_2 between them. For each such $\varphi(t)$ let

$$S[\varphi(t)]=\int L[\varphi(t)]dt.$$

Then the propagator K from (φ_1,t_1) to (φ_2,t_2) is the sum over the paths of $\exp(iS/\hbar)$.

This is ordinarily a sufficient prescription, aside from the decidedly nontrivial question of how to perform this sum. Given paths $\varphi(t)$ and $\psi(t)$ with the proper end points there is no question of relative phase between their contributions to the propagator. For example, we could not have

$$K\sim e^{iS[\varphi]/\hbar}-e^{iS[\psi]/\hbar}+\cdots.$$

The reason for this lack of ambiguity is evident. If we deform $\psi(t)$ continuously into $\varphi(t)$, the contribution due to ψ ($e^{iS[\psi]/\hbar}$) must continuously go over into that due to φ.

For FARR this argument breaks down. This is precisely because there are paths between given end-points which are *not* deformable into one another. (We remind the reader that paths which loop around a circle different numbers of times are in different homotopy classes—i.e., cannot be continuously deformed into one another.) Thus although paths in the same homotopy class cannot have arbitrary relative phase factors there is no *a priori* restriction on the overall sign (or phase factor) of the summands in one class relative to those in another. We thus arrive at a more general prescription: In one homotopy class, sum as usual. Then add these partial sums with some relative phase factors determined perhaps from other considerations.

In order to visualize and simplify the homotopy sum, it is easiest to path-integrate in the universal covering space of the original space which is of course simply connected. Let M have universal covering space M^* with covering projection $p:M^*\to M$ which is locally homeomorphic. (See Fig. 1.) Corresponding to paths from m to n (m, $n\in M$) in M we have paths from some fixed $m^*\in p^{-1}(m)$ to all the $n_j^*\in p^{-1}(n)$ (j running through the fundamental group of M) in M^*. In M^* we form path integrals to each of the preimages of n where p is also used to bring the Lagrangian from M to M^*. Then sum over j with phase factors which are so far arbitrary. If $M=SO(2)$, $M^*=$the real line R and p takes real numbers modulo 2π. The Lagrangian on R corresponding to (3.1) is that of a free particle in one dimension. $p^{-1}(\varphi_2)=\{\varphi_2+2n\pi|n=0, \pm1, \cdots\}$. The path integral from φ_1 to $\varphi_2+2n\pi$ in R is easy: It is just the usual free-particle propagator. Hence the propagator has the following form:

$$K(\varphi_2,t_2;\varphi_1,t_1)=\sum_{n=-\infty}^{\infty}a_nk_n(\varphi,t),\qquad \varphi=\varphi_2-\varphi_1,\quad t=t_2-t_1, \tag{3.2}$$

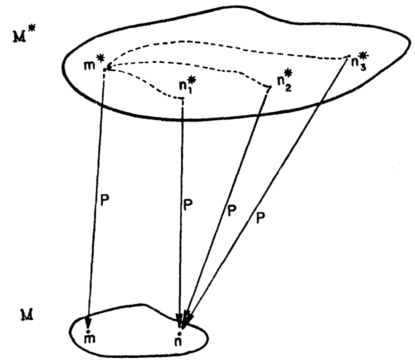

FIG. 1. Paths in M^*.

where

$$k_n(\varphi,t)=\left(\frac{I}{2\pi i\hbar t}\right)^{1/2}\exp\left(\frac{iI}{2\hbar t}(\varphi-2n\pi)^2\right),\qquad |a_n|=1. \tag{3.3}$$

Now we consider the conditions to be imposed on the phase factors a_n. In quantum mechanics one expects a transformation which is physically the identity to introduce no more than a phase factor. The choice $a_n=e^{in\delta}$ causes $K\to e^{i\delta}K$ when $\varphi_2\to\varphi_2+2\pi$. With the notation

$$K(\varphi+\varphi_1,T+t_1;\varphi_1,t_1)=K(\varphi,T)=K_\delta(\varphi,T),\qquad \gamma=I/\hbar T, \tag{3.4}$$

we have

$$K_\delta(\varphi,T)=\left(\frac{\gamma}{2\pi i}\right)^{1/2}\sum_{n=-\infty}^{\infty}e^{in\delta}e^{i\gamma(\varphi-2n\pi)^2/2}. \tag{3.5}$$

Recall the definition of the Jacobi theta function[12]:

$$\theta_3(z,t)=\sum_{n=-\infty}^{\infty}e^{i\pi tn^2}e^{2inz}, \tag{3.6}$$

and the fundamental identity which it satisfies by virtue of the Poisson summation formula:

$$\theta_3(z,t)=(-it)^{-1/2}e^{z^2/i\pi t}\theta_3(z/t,-1/t). \tag{3.7}$$

In the half-plane $\mathrm{Im}(t)>0$, θ_3 is analytic in z. Clearly K_δ is a theta function, and making use of the obvious fact that $\theta_3(z,t)=\theta_3(-z,t)$ we have:

$$K_\delta(\varphi,T)=\left(\frac{\gamma}{2\pi i}\right)^{1/2}e^{i\gamma\varphi^2/2}\theta_3(\gamma\pi\varphi-\tfrac{1}{2}\delta,2\gamma\pi). \tag{3.8}$$

Note $\mathrm{Im}\gamma=0$, depriving the series of its absolute convergence and opening the door to all sorts of pathologies, which as we shall see do not miss this opportunity. Nevertheless, for manipulative convenience we can take the moment of inertia, I (or γ), to have a small positive imaginary part, restoring analyticity. This

[12] Richard Bellman, *A Brief Introduction to Theta Functions* (Holt, Rinehart, and Winston, Inc., New York, 1961).

strategem is a familiar one in situations where one must handle a Green's function and is not at all peculiar to path integrals. However, an extension of this idea has been used by Nelson[13] to give a rigorous formulation of the idea of sum over paths (as opposed to iteration of the infinitesimal Green's function, which is what Feynman's prescription amounts to). For pure imaginary mass (or I), quantum mechanics becomes diffusion and then the Wiener integral is for Brownian motion what Feynman's sum over paths would like to be for quantum mechanics. An analytic continuation to real mass in the diffusion Green's function then provides the quantum Green's function. In any case, these questions are not the concern of the present work and we shall always be content with the prescription $I\to I+i\epsilon$.

What does conventional quantum mechanics have to say about the Green's function for this problem? In general, this can be written as a sum over stationary states

$$G(\varphi_2,t_2;\varphi_1,t_1)=\sum_m \psi_m(\varphi_2)\psi_m{}^*(\varphi_1)e^{-iE_m(t_2-t_1)/\hbar}. \quad (3.9)$$

Schrodinger's equation follows by the usual quantization methods from the Lagrangian (3.1) and yields the stationary states

$$\psi_m(\varphi)=\frac{1}{(2\pi)^{1/2}}e^{i(m+\alpha)\varphi},\quad m=0,\pm1,\pm2,\cdots \quad (3.10)$$

with

$$0\le\alpha<1,\quad E_m=(\hbar/2I)(m+\alpha)^2. \quad (3.11)$$

The change in phase of ψ under rotation by 2π is $e^{2\pi i\alpha}$. We can now use (3.9) to form G_δ, where $\delta=2\pi\alpha$. As a notational convention we call the path-integral propagator K and that computed by the usual quantum mechanics G. The object of course is to prove them equal. From (3.9)–(3.11), with obvious notation, we have

$$\begin{aligned}G_\delta(\varphi,T)&\\ &=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp\left[i\varphi\left(n+\frac{\delta}{2\pi}\right)\right]\exp\left[-\frac{i}{2\gamma}\left(n+\frac{\delta}{2\pi}\right)^2\right]\\ &=\frac{1}{2\pi}\exp\left[\frac{i\delta\varphi}{2\pi}-\frac{i\delta^2}{8\gamma\pi^2}\right]\theta_3\left(\tfrac{1}{2}\varphi-\frac{\delta}{8\gamma\pi},\frac{-1}{2\gamma\pi}\right). \quad (3.12)\end{aligned}$$

To compare (3.8) and (3.12) use the identity (3.7) to transform one into the other. Verification is direct and we find the path integral gives the "right" answer.

The cases $\delta=0$ and $\delta=\pi$ are obviously of most interest since these correspond, in some sense, to integral and half-integral spin. ±1 are the only phase factors which arise for the top since $SO(3)$ is only doubly connected, in contrast to $SO(2)$ which is infinitely connected.

The degree of pathology exhibited by this Green's function is entertaining, especially in view of the elementary nature of the example. This pathology of course has nothing to do with path integrals since the Green's function is that obtained from ordinary quantum mechanics. For $\mathrm{Im}t>0$ the zeros of the theta function are doubly periodic (the function is only quasiperiodic) forming an infinite lattice in the z plane. As $\mathrm{Im}t\to0$ the lattice collapses and the zeros move towards the real z axis. For t real and rational there will be a finite number of zeros in any interval. For t irrational the zeros are dense on the real z axis.[14] This does not imply G is identically zero since we are now very much without analyticity. In fact, we know that as an integral kernel G is norm preserving and has nonzero integrals.

In addition to its topological features, FARR serves as a preview to the top in some other respects. We first meet the theta function which reappears in the top propagator. We also see a recurrent phenomenon in path integrals. Let the "classical path" be that followed by a particle obeying its classical equations of motion. What we have seen is that the FARR path integral, like that for the free particle in Euclidean space, like that for the harmonic oscillator, and like other examples, collapses from a sum over all paths to a sum over classical paths [cf. (3.2) and (3.3)]. In fact, this phenomenon seems to occur for all time-dependent quantum mechanical Green's functions which have been evaluated in closed form (of which I am aware). This, however, is not a random subset of all Green's functions and the property appears not to hold universally[15] although a theorem,[16] proved using path-integral techniques, claims its generality. It would indeed be interesting to know under what circumstance the statement is true, particularly since similar results exist for optics.[17]

The accuracy with which one must handle infinitesimal quantities in path-integral theory is greater than that to which one is accustomed in the usual calculus. The underlying reason is that one is dealing with essentially a Brownian motion and is forming a stochastic integral. A rule of thumb in Brownian motion is $(dx)^2=dt$ (dx is the distance covered in time dt), corresponding to the fact that the square of the distance a particle diffuses is proportional to the time it has been diffusing. The erratic nature of the path makes it cover a lot of ground microscopically but not to appear to get very far macroscopically. By the rule of thumb, if one wishes first-order accuracy in the dt he must keep terms of order $(dx)^2$. Consider the action S which is the integral of L over some path. One would ordinarily

[13] Edward Nelson, J. Math. Phys. **5**, 332 (1964).

[14] This is just Kronecker's theorem: If q is irrational the set of points $((nq))$, $n=0, 1, \cdots$, is dense in the interval (0,1). $((x))=x-[x]$. See G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Clarendon Press, Oxford, 1962), p. 376.

[15] L. S. Schulman, thesis, Princeton University, 1967 (unpublished).

[16] M. Clutton-Brock, Proc. Camb. Phil. Soc. **61**, 201 (1965).

[17] J. B. Keller, J. Opt. Soc. Am. **52**, 116 (1962).

expect to drop terms like $(dx)^4/(dt)$ in approximating S. However, counting differentials we see that for a path integral or Brownian motion this is of order dt. The need to keep these higher-order differentials was noted by Edwards and Gulyaev[10] (and by Feynman[18]) who performed the path integral for a free particle using polar rather than Cartesian coordinates. The Lagrangian in two dimensions is

$$L=\tfrac{1}{2}(\dot{r}^2+r^2\dot{\varphi}^2). \qquad (3.13)$$

This is obtained from the corresponding Cartesian expression by the rules of differential calculus. If used in a path integral it is found to give the wrong answer and it is necessary to use the action integrated along the classical path [accurate to $(dx)^4/(dt)$] even for infinitesimal dt. This justifies our earlier statement, $dS\neq Ldt$. (This is essentially the reason why the theorem mentioned above[16] on the sum over classical paths is not reliable.)

In general, when maintaining this greater accuracy the infinitesimal (in time) propagator consists of more than just $\exp(iS/\hbar)$. For curved spaces and complicated mechanical systems, the normalization of the propagator, which we have so far ignored, is a function of the coordinates, which is not the case for, e.g., the harmonic oscillator. With this factor, and some others to come, the path integral looks less like a sum over paths and more like an infinite folding of the infinitesimal Green's function. Be that as it may, over the years the correct form of the short-time Green's function has evolved, tracing its earliest roots to improvements in the WKB approximation by Van Vleck,[19] continuing with the work of Morette[20] and Pauli[21] and finally given in full generality for curved spaces by DeWitt.[6] The ultimate test, marking this as the "correct" propagator, is the following: It satisfies, to first order in (dt), the equation for the Green's function of Schrödinger's equation,

$$\left(H''-i\hbar\frac{\partial}{\partial t''}\right)G(x'',t'';x',t')=-i\hbar\delta(x''-x')\delta(t''-t')$$

with H the general expression for the Hamiltonian in a curved space. We now give DeWitt's expression with a slight reduction of his generality suitable to our purposes.

A particle moves in a curved n-dimensional space. The Lagrangian is

$$L=\tfrac{1}{2}g_{ij}\dot{x}^i\dot{x}^j+a_i\dot{x}^i-v \qquad (3.14)$$

with summation convention. The coordinate is $x=(x^1,\cdots,x^n)$ with the dot indicating total time differentiation. There are potentials a_i and v which may be functions of x and t. The coefficients g_{ij} of the quadratic form for the kinetic energy will be looked upon as a metric and we shall assume the space is of constant curvature, R. Given two points x'', x' and times t'', t' there is a classical path connecting them, and the action as a function of these points, $S(x'',t'';x',t')=\int L dt$, is defined as the line integral taken over this path. S is also the solution of the Hamilton-Jacobi equation and can be expanded to order dt [meaning terms of order $(dx)^4/(dt)$ are retained] using g, its derivatives, a, its derivatives, and v. Such an expression is given by DeWitt. Using S, we define the Van Vleck determinant

$$\begin{aligned}D_{ji}&=-\partial^2S(x'',t'';x',t')/\partial x''^j\partial x'^i,\\ D&=D(x'',t'';x',t')=\det D_{ji}.\end{aligned} \qquad (3.15)$$

Define g^{mn} by $g^{mn}g_{ns}=\delta^m{}_s=$ Kronecker delta. Also define $g=\det g_{ij}$. R, the curvature, is built as usual from g_{ij}:

$$\begin{aligned}&[jk,i]=\tfrac{1}{2}(g_{ij,k}+g_{ik,j}-g_{jk,i}),\\ &R_{ikjl}=\tfrac{1}{2}(g_{ij,kl}-g_{il,kj}-g_{kj,il}+g_{kl,ij})\\ &\qquad\quad+g^{mn}([ij,m][kl,n]-[kj,m][il,n]),\\ &R_{ij}=-g^{kl}R_{ikjl},\quad R=g^{ij}R_{ij}.\end{aligned} \qquad (3.16)$$

In terms of these quantities the propagator is given by[22,23]

$$\begin{aligned}K(x'',t'';x',t')=&(2\pi i\hbar)^{-n/2}g''^{-1/4}D^{1/2}(x'',t'';x't')g'^{-1/4}\\ &\times e^{i\hbar Rt/12}\exp[iS(x'',t'';x',t')/\hbar]\end{aligned} \qquad (3.17)$$

with $g''=g(x'')$, $g'=g(x')$, $t=t''-t'$. With this formula all obstacles are removed for the final assault on the top.

4. PATH INTEGRAL FOR THE TOP

The top moves in both external position space and internal spin space. For a free top, these motions are uncoupled and the Green's function is a product of the individual Green's functions. The position-space propagator does not concern us here and we confine attention to that for spin motion.

For this motion we need the following information concerning its classical mechanics: Given initial and final configurations of the top, what is the action computed along the classical path connecting them?

---

[18] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).

[19] J. H. Van Vleck, Proc. Nat. Acad. U. S. Sci. **14**, 178 (1928).

[20] Cecile Morette, Phys. Rev. **81**, 848 (1951).

[21] W. Pauli, Feldquantisierung, lecture notes, Zurich, 1951 (unpublished), Appendix.

[22] We have put the curvature term $\exp(i\hbar Rt/12)$ in the propagator rather than modify the Lagrangian, as DeWitt (Ref. 6) suggests. For a nonconstant curvature, following his technique, the original Lagrangian L is replaced by $L+\hbar^2R/12$. Then the propagator (3.17), without the explicit $\exp(i\hbar Rt/12)$ (but with the effect of the added term in the action), is the Green's function for the Schrödinger equation obtained from L alone by the usual quantization methods. This is a clearcut difference between path integral and other forms of quantization, a difference which (1) disappears as $\hbar\to 0$, (2) disappears if $R=0$, and (3) appears to be irrelevant if $R=$ constant as in our case. [See discussion in WADC Technical Report No. 57-216, ASTIA Document No. AD 118180 (unpublished).]

[23] If the Lagrangian has a mass parameter $(L=\tfrac{1}{2}mg_{ij}\dot{x}^i\dot{x}^j+\cdots)$ keep (3.17) exactly as is, except replace R by (R/m) where R is still computed from g alone. The $D^{1/2}$ factor will then provide the $m^{n/2}$ dependence one generally finds in propagators.

(A "classical path" conforms to the classical equations of motion.)

To a top with Euler angles φ, θ, ψ is assigned the $SU(2)$ element

$$U(\varphi,\theta,\psi)=\exp(-i\varphi\sigma_z/2)\exp(-i\theta\sigma_y/2)\exp(-i\psi\sigma_z/2) \tag{4.1}$$

with the usual σ matrices.

Active rotation of the system through angle α about an axis $\hat{n}$ is effected by left multiplication of $U(\varphi,\theta,\psi)$ with the matrix

$$\exp(-i\alpha\hat{n}\cdot\boldsymbol{\sigma}/2). \tag{4.1'}$$

Passive rotation of the observer is also effected by left multiplication but with α replacing $-\alpha$. A right multiplication of U interpreted actively moves the top in its body system. Appendix A gives a rationale for this description and shows left multiplication to be a kinematic symmetry of all tops, while right multiplication leaves invariant only the Lagrangian of the spherical top.

The metric tensor for $SU(2)$ [or $SO(3)$] is $g_{\theta\theta}=g_{\varphi\varphi}=g_{\psi\psi}=1$, $g_{\varphi\psi}=g_{\psi\varphi}=\cos\theta$, other components zero. Euler angles will be denoted by E with the convention $E=(\varphi,\theta,\psi)$. Then

$$(ds)^2=g_{ij}dE_idE_j=(d\theta)^2+(d\varphi)^2+(d\psi)^2+2\cos\theta\,d\varphi\,d\psi.$$

For a spherical top the relation between metric and Lagrangian is

$$L=I(ds)^2/2(dt)^2=\tfrac{1}{2}I(\dot\theta^2+\dot\varphi^2+\dot\psi^2+2\cos\theta\,\dot\varphi\dot\psi), \tag{4.2}$$

where I is the moment of inertia. The action S from E_{initial} to E_{final} is a function of U_i and U_f (and of course $T=t_f-t_i$), $S=f(U_i,U_f)$. Rotational symmetry of the external world requires f to be invariant under left multiplication of its arguments. The spherical nature of the top demands in addition invariance under right multiplication. [Now it is clear why the spherical top Lagrangian corresponds to the metric: The (natural Lie group) metric too is invariant under both left and right multiplication.] Thus for any $A\in SU(2)$, $f(B,C)=f(BA,CA)=f(AB,AC)$. Left multiply with arbitrary A and right multiply with $U_i^{-1}A^{-1}$:

$$S=f(1,\,AU_fU_i^{-1}A^{-1}).$$

For the diagonalizable matrices with which we deal the only invariants under similarity transformation are the eigenvalues and their multiplicity (or functions of these). For $SU(2)$, the determinant and unitarity conditions leave but one independent invariant. For convenience we define

$$\cos\tfrac{1}{2}\Gamma=\tfrac{1}{2}\operatorname{Tr}U_fU_i^{-1} \tag{4.3}$$

and S must be some function of Γ. In terms of the representation (4.1)

$$\cos\tfrac{1}{2}\Gamma=\cos\tfrac{1}{2}(\theta_f-\theta_i)\cos\tfrac{1}{2}(\varphi_f-\varphi_i)\cos\tfrac{1}{2}(\psi_f-\psi_i)-\cos\tfrac{1}{2}(\theta_f+\theta_i)\sin\tfrac{1}{2}(\varphi_f-\varphi_i)\sin\tfrac{1}{2}(\psi_f-\psi_i). \tag{4.4}$$

To identify the functional dependence of S on Γ, consider the boundary conditions $\theta_i=\theta_f=0$, $\varphi_f=\varphi_i$. The motion can now obviously be taken as uniform rotation with the figure axis pointing in the $\theta=0$ direction. The resulting ambiguity in φ and ψ is resolved by taking $\varphi=$constant. Then constant$=\dot\psi=(\psi_f-\psi_i+2n\pi)/(t_f-t_i)$, where n is the number of times ψ passes through ψ_f for $t<t_f$. By (4.2), $S=(I/2T)\times(\psi_f-\psi_i+2n\pi)^2$. From (4.4) [or (4.3)] we have only

$$\Gamma\equiv\psi_f-\psi_i \pmod{2\pi}. \tag{4.4'}$$

Γ can be made a well-defined function of this orbit with the definition $\Gamma=\psi_f-\psi_i+2n\pi$. Then

$$S=(I/2T)\Gamma^2. \tag{4.5}$$

Clearly Γ is just the arc length of the geodesic in $SU(2)$ connecting U_i and U_f. The multivaluedness of the arc cosine which appears in the inverse of Eq. (4.3) corresponds to the discrete set of geodesics connecting points in $SU(2)$, each of which is but a local minimum. Physically one can imagine a top going from one configuration to another in a fixed time by spinning once at a certain rate about a certain axis, or by spinning twice about the same axis at a greater velocity or by spinning n times, etc.[24] To be more precise, in the case of the physical top the group of interest is $SO(3)$ and the relevant paths correspond to solutions of

$$\cos\tfrac{1}{2}\Gamma=\pm\tfrac{1}{2}\operatorname{Tr}U_fU_i^{-1}. \tag{4.3'}$$

For each solution of this equation (both signs) there is a motion of the top starting and ending at the prescribed Euler angles and taking the given amount of time. [(4.4′) actually corresponds to (4.3′).]

Recalling the connection between geodesics and one-parameter subgroups, it is also easy to write the position of the top on the $SU(2)$ group manifold as a function of time.

$$U(t)=\exp(-i\Gamma t\hat{n}\cdot\boldsymbol{\sigma}/2T)U_i. \tag{4.6}$$

With this solution it is clear that Γ and $\hat{n}$ are related to U_f and U_i by

$$U_fU_i^{-1}=\exp(-i\Gamma\hat{n}\cdot\boldsymbol{\sigma}/2) \tag{4.7}$$

[times ±1 for $SO(3)$]. Thus the constant angular velocity vector is $\boldsymbol{\Omega}=\Gamma\hat{n}/T$.

There is one more factor to compute for the evaluation of (3.17), the expression for the short-time propagator. This factor is $(g')^{-1/4}D^{1/2}(g'')^{-1/4}$. Like S it is a function of U_f and U_i, and like S it is rotationally invariant. In Appendix B, there is a proof of this statement and the expression is evaluated. The result is

$$\left(\frac{I}{T}\right)^{3/2}\frac{\Gamma}{2\sin\frac{1}{2}\Gamma}. \tag{4.8}$$

[24] If the top is symmetric but not spherical this picture breaks down, and while there is a discrete infinity of paths, they are not all about the same axis.

Finally, the curvature of $SU(2)$ is $R=\frac{3}{2}$ and is easily computed from (3.16).

Using (3.17) the path-integral expression for the infinitesimal propagator is

$$K=\left(\frac{I}{2\pi i\hbar T}\right)^{3/2}\frac{\Gamma}{2\sin\frac{1}{2}\Gamma}\exp\left(\frac{i\hbar T}{8I}\right)\exp\left(\frac{iI\Gamma^2}{2\hbar T}\right). \quad (4.9)$$

Γ in (4.9) is the smallest (in absolute value) solution of (4.3′). Denoting this by Γ_0, the other solutions are $\Gamma=\Gamma_0+2n\pi$, $n=\pm1, \pm2, \cdots$.

As in Sec. 3, we check this against the standard quantum mechanical result. Schrödinger's equation is

$$\frac{-\hbar^2}{2I}\Delta\psi=i\hbar\frac{\partial\psi}{\partial t} \quad (4.10)$$

with Δ the Laplacian on $SU(2)$ [or $SO(3)$]:

$$\Delta=\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial\psi^2}-2\cos\theta\frac{\partial^2}{\partial\varphi\partial\psi}\right). \quad (4.11)$$

The stationary states are labeled by j, m, k which are related respectively to the eigenvalues of J^2, J_z, J_ζ ($J_\zeta=\hat{n}_\zeta\cdot\mathbf{J}$, where $\hat{n}_\zeta$ points along the figure axis). The normalized eigenfunctions having the appropriate rotational properties together with their energy eigenvalues are

$$\left(\frac{2j+1}{8\pi^2}\right)^{1/2}D_{mk}^{j*}(\varphi,\theta,\psi),\quad E_{jmk}=\frac{\hbar^2}{2I}j(j+1). \quad (4.12)$$

(These are the $\mathfrak{D}$'s, the representation matrices of the rotation group.) As in (3.9) we have for the Green's function

$$G(E_2,T;E_1,0)=\sum_{jmk}\frac{2j+1}{8\pi^2}D_{mk}^{j*}(E_2)D_{mk}^{j}(E_1)\times\exp\left(-\frac{i\hbar T}{2I}j(j+1)\right), \quad (4.13)$$

where the sum either includes all integral (G_+) or half-integral (G_-) values of j, m, k. This is required in order that G have well-defined rotation properties.

We now show G to be related to the theta function. Consider the argument of D to be $U(E)\in SU(2)$, rather than just E. By unitarity $D_{mk}^{j*}(U)=D_{km}^{j}(U^{-1})$. For any j,

$$\sum_{mk}D_{mk}^{j*}(U_2)D_{mk}^{j}(U_1)=\sum_{mk}D_{mk}^{j*}(U_2)D_{km}^{j*}(U_1^{-1})$$
$$=\sum_{m}D_{mm}^{j*}(U_2U_1^{-1})=\mathrm{Tr}D^{j*}(U_2U_1^{-1}). \quad (4.14)$$

Since the trace is invariant under similarity transform we can take D^j diagonal. This will be the case if the z axis is taken along the direction $\hat{n}$ defined by U_2 and U_1 through Eq. (4.7). In that case $D_{mm}^{j}(U_2U_1^{-1})=\exp(im\Gamma)$, with the Γ of Eqs. (4.3). It follows that

$$\mathrm{Tr}D^{j*}(U_2U_1^{-1})=\frac{\sin(j+\frac{1}{2})\Gamma}{\sin\frac{1}{2}\Gamma} \quad (4.15)$$

and

$$G_\pm=\frac{-2}{\sin(\frac{1}{2}\Gamma)}\frac{1}{8\pi^2}\frac{\partial}{\partial\Gamma}\sum_j\cos[(j+\tfrac{1}{2})\Gamma]\times\exp\left(\frac{-i\hbar T}{2I}j(j+1)\right)$$
$$\equiv\frac{-2}{\sin(\frac{1}{2}\Gamma)}\frac{1}{8\pi^2}\frac{\partial}{\partial\Gamma}g_\pm,\quad \begin{matrix}g_+ \text{ is for } j=0,1,\cdots\\ g_- \text{ is for } j=\frac{1}{2},\frac{3}{2},\cdots\end{matrix} \quad (4.16)$$

$$g_+=\tfrac{1}{2}\theta_2(\tfrac{1}{2}\Gamma,-c/\pi)e^{ic/4}=\tfrac{1}{2}e^{i\Gamma/2}\theta_3(\tfrac{1}{2}\Gamma-\tfrac{1}{2}c,-c/\pi),$$
$$g_-=\tfrac{1}{2}e^{ic/4}[\theta_3(\tfrac{1}{2}\Gamma,-c/\pi)-1] \quad (4.17)$$

[using $\theta_2(z,t)=e^{i\pi t/4}e^{iz}\theta_3(z+\frac{1}{2}\pi t,t)$], where $c=\hbar T/2I$. Because of the $\partial/\partial\Gamma$ the term -1 in g_- can be ignored. To each of these expressions apply the identity (3.7) and rewrite the theta function as an infinite sum

$$g_\pm=\tfrac{1}{2}e^{ic/4}\left(\frac{\pi}{ic}\right)^{1/2}\sum_{-\infty}^{\infty}(\mp)^n e^{i(\Gamma+2n\pi)^2/4c}. \quad (4.18)$$

Thus,

$$G_\pm=\frac{e^{i\hbar T/8I}}{2\sin\frac{1}{2}\Gamma}\left(\frac{I}{2\pi i\hbar T}\right)^{3/2}\sum_{n=-\infty}^{\infty}(\mp)^n(\Gamma+2n\pi)\times\exp\left(\frac{iI}{2\hbar T}(\Gamma+2n\pi)^2\right). \quad (4.19)$$

For sufficiently small T, the only term contributing is $n=0$. Hence,

$$G_\pm=e^{i\hbar T/8I}\left(\frac{I}{2\pi i\hbar T}\right)^{3/2}\frac{\Gamma}{2\sin(\Gamma/2)}e^{i\Gamma^2I/2\hbar T} \quad (4.20)$$

in agreement with (4.9).

There is a familiar ring to Eq. (4.19), the exact finite-time propagator. Recalling the classical mechanics of this system, it is apparent that the sum in (4.19) is a *sum over classical paths*. Thus, given $U(E')$ and $U(E'')$, we obtain a solution Γ_0 of $\cos\frac{1}{2}\Gamma_0=\pm\mathrm{Tr}U(E'')U(E')^{-1}$ and from this an infinity of solutions $\Gamma=\Gamma_0+2n\pi$ with action $I(\Gamma_0+2n\pi)^2/2T$. The associated classical paths fall into two homotopy classes depending on whether n is even or odd. Because of the $\sin\frac{1}{2}\Gamma$ factor in $G_\pm$, these classes enter the sum with opposite sign for integral spin and with the same sign for half-integral spin.

We now perform the sum over paths using the infinitesimal propagator established in Eq. (4.9). This is superfluous to the extent that equivalence of our formalism with ordinary quantum mechanics is established with the equality of the infinitesimal propagators. Proceeding from this to the finite-time propagator is a luxury for either formulation. Nevertheless, iteration of the Green's function (which is the sum over paths) seems to be the peculiar domain of path-integral theory and we now proceed with this calculation.

By Sec. 3, the proper arena is $SU(2)$ rather than $SO(3)$. In order to write the $SU(2)$ propagator, even infinitesimally, there are a few ambiguities to be disposed of. First it is clear that given U'', $U'\in SU(2)$ it is not the solutions of (4.3′) that we look to, but rather those of (4.3). Now, which solution? The automatic answer is the smallest one. This can, however, lead to trouble because of the use of this solution in an integral. Suppose some solution Γ has been selected. The Green's function will be roughly $(\Gamma/\sin\frac{1}{2}\Gamma)\times e^{i\Gamma^2/(t_{j+1}-t_j)}$, where Γ depends on U_{j+1} and U_j, which are respectively the locations of the system at the times t_{j+1} and t_j. (The usual path-integral context is assumed with the interval $[t_i,t_f]$ being broken up into $t_i=t_0, t_1, \cdots, t_j, \cdots, t_{N+1}=t_f$. Position at t_j is U_j. The sum over paths is effected by integrating over U_j; $j=1, \cdots, N$.) For fixed U_{j-1} and U_{j+1}, as U_j varies over $SU(2)$ there is no one analytic expression for $\Gamma_{j,j+1}$ or $\Gamma_{j-1,j}$. The analog of this for FARR $[SO(2)]$ is easy to visualize (see Fig. 2). As illustrated, no one would doubt that the action is given by $I(\varphi_i-\varphi_{i+1})^2/2T$. Suppose φ_i moves counterclockwise. Eventually the action becomes $I(2\pi-\varphi_i-\varphi_{i+1})^2/2T$. The same ambiguity occurs in $SU(2)$ and while we may have started with a certain geodesic, in the course of integrating we may find ourselves with a Γ that is inaccurate by 4π. For sufficiently small $\Delta t=t_{j+1}-t_j$ this is quite unsatisfactory since it is only the smallest solution that makes a significant contribution. [Note a similar rationale in going from (4.19) to (4.20). As $\epsilon\to 0$, $\epsilon^{-1/2}e^{ix^2/\epsilon}$ is essentially a δ function in x.] To settle the dilemma we make use of this last fact by adding together *all* the solutions. When one of the terms $(\Gamma+4n\pi)^2$ is small it will contribute. If none is near zero, none will contribute to the iterated integral. Furthermore, because $SU(2)$ is simply connected there is no question of phase factors in adding together different terms. Then we can take as infinitesimal propagator on $SU(2)$

$$G=G_{SU(2)}=\left(\frac{I}{2\pi i\hbar T}\right)^{3/2}\sum_{n=-\infty}^{\infty}\frac{\Gamma+4n\pi}{2\sin\frac{1}{2}\Gamma}e^{i\hbar T/8I}\times\exp\left(\frac{iI(\Gamma+4n\pi)^2}{2\hbar T}\right) \quad (4.21)$$

with $T=t_{j+1}-t_j$ and Γ some definite (but arbitrary) solution of (4.3) (with "f"$=j+1$ and "i"$=j$).

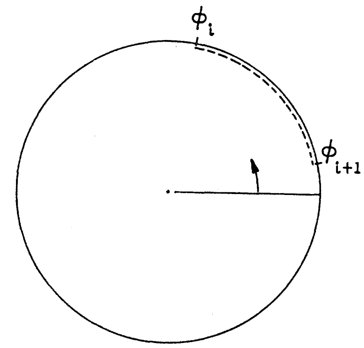

FIG. 2. Two positions of the fixed-axis rigid rotator.

Next work backwards from (4.19) to (4.13) and use the "factored" form of the propagator for a path integration. The arid stretch from Eq. (4.13) to (4.19) can be summarized by

$$G_\pm=(I/2\pi i\hbar T)^{3/2}\exp(i\hbar T/8I)\sum_{n=-\infty}^{\infty}\frac{\Gamma+2n\pi}{2\sin\frac{1}{2}\Gamma}(\mp)^n\times\exp\left(\frac{iI(\Gamma+2n\pi)^2}{2\hbar T}\right)$$

$$=\sum_{jmk}{}^{(\pm)}\frac{2j+1}{8\pi^2}D_{mk}{}^{j*}(E_2)D_{mk}{}^{j}(E_1)e^{-i\hbar j(j+1)T/2I}, \quad (4.22)$$

where $\sum^{(\pm)}$ means sum over $j-\frac{1}{2}(1\pm 1)=0, 1, \cdots$. The relation among E_2, E_1, U_2, U_1, and Γ is the usual. Comparison of (4.21) and (4.22) leads to

$$G_{SU(2)}=G_++G_-. \quad (4.23)$$

It follows that the second sum in (4.22) can be reinterpreted to apply to $G_{SU(2)}$ by taking $\sum^{[SU(2)]}$ to mean sum over all j, $j=0, \frac{1}{2}, 1, \cdots$.

The iteration of $G_{SU(2)}$ expressed as a sum of D's is trivial by virtue of the orthogonality of the D's (this iteration is the tedious part in the "sum over paths"); and the form that we already have for $G_{SU(2)}$ turns out to be correct for finite time. The $SO(3)$ propagators G_+ and G_- follow in an unsurprising way and we elaborate no further.

Finally, we indicate the connection of this formalism with that of Pauli spinors. A particle initially in a state of fixed j, m, k will retain this property when propagated by the appropriate one of $G_\pm$. Without interactions one expects nothing to happen, and indeed nothing does. Projecting, for example, to the subspace $j=k=\frac{1}{2}$ one can write G_- in spinor form. Generally, spinors are obtained by Fourier transforming (the D's) to an angular momentum representation and projecting to a fixed j, k subspace.

A magnetic field in, say, the x direction introduces an interaction term proportional to J_x. This will induce transitions from one m state to another and will modify

G appropriately. That the modifications are in quantitative agreement with spinor theory is demonstrated by Bopp and Haag.[5] Since states of different j and k are not mixed (J_x commutes with J^2 and J_ζ) the Pauli spinor formalism is adequate. This last remark reflects the fact that when we have no interactions that change the total spin it is sufficient to confine attention to that spin alone.

5. CONCLUSION

Particle models possessing spin were proposed[25] some years before intrinsic spin was recognized in nature. At first these models involved rotating objects of one sort or another (tops), but as the significance of *half*-integral spin became apparent, it came to be felt that some extra feature, some gimmick, would be necessary if one were to account for two-valued wave functions. This in turn led to what is now the generally prevailing attitude, namely, a general uncomfortableness at the mention of internal spin variables and a reliance on the more formal, but nevertheless completely adequate, spinor wave functions. These are the labeled basis vectors for a representation of $SO(3)$, but are endowed with no further properties.

If anything, these views were reinforced by the limitation of orbital angular momentum to integral values. This seemed to be one more strike against the idea of a classical motion—and thus a motion which could be reduced to the orbital motions of a composite system—giving rise to spin.

Nevertheless, as it happens, there is a classical system, which, when quantized (according to your favorite method), gives rise to half-integral spin and two-value wave functions. The system is the classical top, and its quantum mechanical analysis is presented by Bopp and Haag.[5] Moreover, as we have tried to indicate in Sec. 2 of this paper, the word "top" merely serves as a generic term for a system requiring the full rotation group to specify its position.

What is it then that causes the top to possess this anomalous behavior? Why does an operation that is physically the identity change the wave function?[26] It is here that path-integral theory is most useful. Clearly, there must be an ambiguity in the process of quantization.[27] In path integration one sums a functional of classical paths. But the usual recipe is not complete. Functionals of some paths may enter the sum with an over-all phase factor relative to those of others. The resolution of this ambiguity leads directly to the terminology of homotopy theory. If one path can be continuously deformed into another, then a relative phase can be eliminated; if not, they lie in different homotopy classes and an over-all relative phase may indeed be present. Because there are two homotopy classes[28] in $SO(3)$, there turn out to be two choices of phase, and these correspond to integral or half-integral spin. As a source of half-integral spin, this ambiguity is precisely appropriate: It is only because $SO(3)$ is multiply connected that it admits multivalued continuous functions, and hence ray representations.

It is in the nature of this spin model that one does not simply obtain spin $\frac{1}{2}$ or some other single value. There is some object requiring rotational coordinates, and it can appear in any one of many spin states. This fact enjoys a mixed reception. If one desires only a path-integral theory for, say, an electron—which does not appear to possess other spin states—then the coordinate space (φ,θ,ψ) Green's function possesses more than enough information (*a fortiori* it possesses enough). On the other hand, if one desires that a certain particle appear with several spin values, then this would seem to be an appropriate framework. Needless to say, a discussion of this sort requires a relativistic theory, since one naturally imagines spin-related mass dependences. Such an extension of the present work has been studied by the author and will appear at a later time.

There is an important technical point which arises in the path-integration of the top. The paths do not lie in a Euclidean space with trivial metric, so that some additional care is required when approximating the classical action preparatory to performing the sum over paths. This problem has been studied before, and we find that the formula given by DeWitt[6] is adequate for our purposes.

In Sec. 4, the Green's function for the top is obtained by path integration, and is written down in closed form [Eq. (4.16)] as a derivative of the Jacobi theta function.

ACKNOWLEDGMENT

It is a pleasure to thank Professor Arthur Wightman for his encouragement in this work, much of which is part of a thesis submitted to Princeton University, 1967, and written under his guidance.

APPENDIX A: SOME MECHANICS OF THE TOP

This Appendix discusses some kinematics of (not necessarily spherical) tops.

To obtain the angular velocity vector $\mathbf{\Omega}$ for the top from the form of $U(t)$ [see Eq. (4.1)], consider the effect of such a velocity on U. It is expected to rotate

[25] H. A. Lorentz, *The Theory of Electrons* (B. G. Teubner, Leipzig, Germany, 1916), p. 217. In his 1906 lectures at Columbia University, Lorentz anticipated problems arising from the rotation of his extended electron, in particular, in its interaction with a magnetic field.

[26] Actually, this situation is far less rare than might be supposed. Since massive states in nonrelativistic quantum mechanics arise from ray representations of the Galilean group [V. Bargmann, Ann. Math. **59**, 1 (1954)] a phase factor associated with the identity should hardly be considered an anomaly. Nevertheless, when a classical model is available, we maintain that it is desirable to pinpoint just where this phase factor is introduced. The parameter m (mass), which selects the ray representation for the Galilean group, is already present in the classical theory.

[27] Klauder (Ref. 2) also speaks of an "action option," a choice at one stage in the process of Feynman quantization, leading to Bose or Fermi statistics.

[28] Since we work with paths (or curves), only the fundamental homotopy group enters.

the top through an angle $|\mathbf{\Omega}|\ dt$ about an axis $\hat{\Omega}$ in time dt. The interpretation we have given left multiplication in $SU(2)$ implies

$$U(t+dt)=\exp(-i\mathbf{\Omega}\cdot\boldsymbol{\sigma}dt/2)U(t) \tag{A1}$$

or

$$\mathbf{\Omega}\cdot\boldsymbol{\sigma}=(2i/dt)[U(t+dt)U^{-1}(t)-1]. \tag{A2}$$

The effect on $\mathbf{\Omega}$ of finite left and right translations by $A\in SU(2)$ is seen to be

$$\text{Left:}\quad \mathbf{\Omega}'\cdot\boldsymbol{\sigma}=A\mathbf{\Omega}\cdot\boldsymbol{\sigma}A^{-1}, \tag{A3a}$$

$$\text{Right:}\quad \mathbf{\Omega}'=\mathbf{\Omega}. \tag{A3b}$$

For a general (free) top the Lagrangian is

$$L=\tfrac{1}{2}\mathbf{\Omega}\cdot\mathbf{I}\cdot\mathbf{\Omega}=\tfrac{1}{2}\Omega_j I_{jk}\Omega_k, \tag{A4}$$

where I_{jk} is the moment-of-inertia tensor. To see what happens to the dyadic $\mathbf{I}$ under rotation, and hence to L, we construct a simple model of the top which also provides motivation for our form of $U(\varphi,\theta,\psi)$. Mass points m_i are placed at $\pm\boldsymbol{\varrho}_i^0$ with $(\rho_i^0)_j=a\delta_{ij}$; i, $j=1$, 2, 3. These move rigidly in time and the position of the ith vector is given by

$$\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_i(t)=U(\varphi,\theta,\psi)\boldsymbol{\varrho}_i^0\cdot\boldsymbol{\sigma}U^{-1}(\varphi,\theta,\psi). \tag{A5}$$

This will connect our notation with the usual description of Euler angles. The $\boldsymbol{\varrho}_i^0$ are, in effect, the body axes. Note that $\exp(i\alpha\sigma_z/2)$, a passive rotation through the positive angle α, *decreases* φ by α. It was the desire to give passive rotations the plus sign that led to the minus signs in U and ultimately to D^* rather than D in (4.12).

Since

$$d\boldsymbol{\varrho}_i/dt=\mathbf{\Omega}\times\boldsymbol{\varrho}_i, \tag{A6}$$

the kinetic energy

$$T=2\times\tfrac{1}{2}\sum_{i=1}^{3}m_i(d\boldsymbol{\varrho}_i/dt)^2 \tag{A7}$$

can be written

$$T=\tfrac{1}{2}\mathbf{\Omega}\cdot\mathbf{I}\cdot\mathbf{\Omega} \tag{A8}$$

with

$$\mathbf{I}=2a^2\sum_{i=1}^{3}m_i(\boldsymbol{\varrho}_i^2\mathbf{1}-\boldsymbol{\varrho}_i\boldsymbol{\varrho}_i), \tag{A9}$$

the moment-of-inertia tensor. The transformation properties of $\mathbf{I}$ under rotation follow from those of $\boldsymbol{\varrho}_i$ and it will surprise no one to learn that it transforms as a tensor. We introduce the usual homomorphism of $SU(2)\to SO(3)$ and for $A\in SU(2)$ we define $R(A)\in SO(3)$ to be the rotation such that $R(A)_{kj}(\rho_i)_j=(\rho_i)'_k$, where $\boldsymbol{\varrho}'\cdot\boldsymbol{\sigma}=A\boldsymbol{\varrho}\cdot\boldsymbol{\sigma}A^{-1}$. By Eq. (A5) a left multiplication of U by A rotates $\boldsymbol{\varrho}_i$ by $R(A)$. Then

$$I'_{jk}=R_{jm}(A)R_{kn}(A)I_{mn}. \tag{A10}$$

Since under left multiplication $\mathbf{\Omega}$ suffers the same rotation, the kinetic energy T remains invariant. This conclusion also follows from the observation that left multiplication is equivalent to passive rotation of the space frame which must leave kinetic energies unchanged.

Not so for right multiplication. This transformation can be understood with the aid of the definitions (subscript i suppressed):

$$\boldsymbol{\varrho}^{0\prime\prime}\cdot\boldsymbol{\sigma}=A\boldsymbol{\varrho}^0\cdot\boldsymbol{\sigma}A^{-1},$$
$$\boldsymbol{\varrho}''\cdot\boldsymbol{\sigma}=U\boldsymbol{\varrho}^{0\prime\prime}\cdot\boldsymbol{\sigma}U^{-1}=UA\boldsymbol{\varrho}^0\cdot\boldsymbol{\sigma}A^{-1}U^{-1}. \tag{A11}$$

We interpret A as causing a rotation of the system in the body axes—or passively a relabeling of these axes—but recalling that $\mathbf{\Omega}$ does *not* change, this will not be equivalent to some left multiplication. The effect on $\mathbf{I}$ is described using B defined by $UA=BU$ (a definition depending on the specific U involved). Then $\mathbf{I}$ transforms as in equation (A10) but with $R(B)$. However, since $\mathbf{\Omega}$ does not change at all, *T is not invariant*. There is one exception to this statement, the spherical top (m_i all equal), for which $\mathbf{I}$ is a multiple of the identity. It is this exception, with its wider class of invariances, which occupies us in Sec. 4.

APPENDIX B: VAN VLECK DETERMINANT

As a notational amenity we will denote coordinates by q and momenta by p. Our interest is in the function

$$f(q'',t'';q',t)=g(q')^{-1/2}D(q'',t'';q',t')g(q'')^{-1/2} \tag{B1}$$

whose square root appears in the propagator (3.17). The system for which we evaluate this is the free spherical top and as a first step *we show this quantity to be invariant under both "left" and "right" rotations.* For a top whose position is correlated to $U\in SU(2)$ a left (right) rotation by A brings the top to $AU(UA)$.

From the definition of D and p, we have

$$D=\det D_{ij},\quad D_{ij}=-\partial p_i''/\partial q_j'. \tag{B2}$$

Consider at time t' a volume $dp'dq'$ of $2n$-dimensional phase space near the point q', p'. Suppose that at the later time t'' this has come to occupy the volume $dp''dq''$ about q'', p''. By Liouville's theorem

$$dp'dq'=dp''dq''. \tag{B3}$$

Imagine that instead of specifying q', p', dq', dp' and arriving at the respective double-primed quantities, we specify q', dq', q'', dq'' and deduce (from solutions of the equations of motion) the remaining quantities. (There is an ambiguity here since q', q'' may yield a discrete set of solutions but we fix attention on one of these.) Equation (B3) still holds. If there is an integral of some sort to be done over p' one can switch to an integral over q'' and then (B3) provides a relation between volume elements

$$dp'=(\partial p''/\partial q')dq'', \tag{B4}$$

where the Jacobian can be evaluated at q', q'' or the corresponding q', p'.

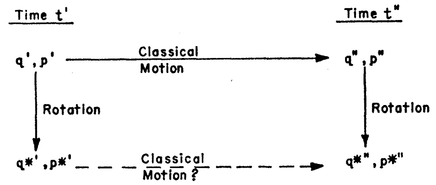

FIG. 3. Rotational invariance.

What happens under rotation? Let the rotated variables have asterisks, $q^{*\prime}$, etc. Then we have an invariant volume element:

$$[g(q)]^{1/2}dq=[g(q^*)]^{1/2}dq^*. \tag{B5}$$

[(B5) is true for more general transformations and has nothing special to do with rotations.] Either sort of rotation (left or right) is a "point transformation" and hence a canonical transformation. This implies the invariance of the phase-space volume element.

$$dq\,dp=dq^*dp^*. \tag{B6}$$

Now we are given q', p' at t' which at t'' arrives at q'', p'' with each of these surrounded by appropriate volume elements satisfying (B3) or (B4). *Let us suppose* that after rotation $q^{*\prime\prime}$, $p^{*\prime\prime}$ is the position at t'' of a system which was at $q^{*\prime}$, $p^{*\prime}$ at time t'. Taking into account ($B6$) and ($B4$) we have

$$dp^{*\prime}=(\partial p^{*\prime\prime}/\partial q^{*\prime})dq^{*\prime\prime}. \tag{B7}$$

Equation (B4) can be divided by $[g(q')]^{1/2}\equiv g'^{1/2}$ to obtain

$$\begin{aligned} g'^{-1/2}dp'&=g'^{-1/2}(\partial p''/\partial q')g''^{-1/2}g''^{1/2}dq'' \\ &=-f(q'',t'';q',t')g''^{1/2}dq''. \end{aligned} \tag{B8}$$

The same can be written for the asterisked quantities. From (B5) and (B6) it is clear that $g'^{-1/2}dp'$ is a rotational invariant. Equating the starred and unstarred versions of Eq. (B8) we have finally that

$$f(q'',t'';q',t')=f(q^{*\prime\prime},t'';q^{*\prime},t'). \tag{B9}$$

It remains to examine the supposition stated between equations (B6) and (B7) and which is represented diagrammatically in Fig. 3. If the rotation is a passive one (left multiplication) the transformation must be interpreted in its action on p (which is so far undetermined) so as to validate the supposition. This statement does not really need explicit mathematical support but since the right-multiplication case is more involved we bring in some of our $SU(2)$ notation here too. By (4.6) the path in $SU(2)$ is

$$U(t)=\exp(-i\Gamma t\hat{n}\cdot\boldsymbol{\sigma}/2T)U_i \tag{B10}$$

and $\boldsymbol{\Omega}=\Gamma\hat{n}/T$. The relation of p_θ, p_φ, and p_Ψ to $\boldsymbol{\Omega}$ is a geometrical one, i.e., from

$$\begin{aligned} \Omega_x&=\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi, \\ \Omega_y&=\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi, \\ \Omega_z&=\dot{\psi}\cos\theta+\dot{\varphi}. \end{aligned} \tag{B11}$$

[This is $\boldsymbol{\Omega}$ in the space axes. Note that our conventions differ from those of Goldstein[29] and are completely embodied in Eq. (4.1) and the rule (4.1′).] Since $L=\frac{1}{2}I\Omega^2$ it follows, e.g., that $p_\varphi=I\Omega_z$. The rotational properties of the p's follow therefore from those of $\boldsymbol{\Omega}$. Thus the validity of the diagram is equivalent to the validity of the corresponding one with Ω replacing p everywhere (e.g., $p^{*\prime}\rightarrow\Omega^{*\prime}$). The effect on $\boldsymbol{\Omega}$ of left and right multiplication is given in (A3a) and (A3b). Left multiplying Eq. (B10) with arbitrary A, we see that the entire path is left multiplied and remains a geodesic. Furthermore, the angular velocity transformation is consistent with the rules of Appendix A. This remark is justified by

$$AU(t)=[A\exp(-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t/2)A^{-1}]AU_i \tag{B12}$$

and expansion of the exponential

$$\begin{aligned} &A\exp(-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t/2)A^{-1} \\ &\qquad=\cos\tfrac{1}{2}\Omega t-iA[(\boldsymbol{\Omega}\cdot\boldsymbol{\sigma})/\Omega]A^{-1}\sin\tfrac{1}{2}\Omega t. \end{aligned} \tag{B13}$$

The same applies to right rotations insofar as Eq. (B10) is concerned. Right multiplication [of (B10)] is seen to move the top a fixed amount but leave $\boldsymbol{\Omega}$ unchanged. Again the geodesic nature of the path is unaffected. Equation (B10) takes the form

$$U'(t)\equiv U(t)A=\exp(-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t/2)U_i' \tag{B14}$$

with $U_i'\equiv U_iA$. This is consistent with the transformation law (A3b) for $\boldsymbol{\Omega}$ and verifies the supposition (the question in Fig. 3 is answered affirmatively). For nonspherical tops "right" invariance is not present [and (B10) is not valid] and one observes that it really does matter whether $\theta=0$ or $\theta=1$. In one case the top does not precess, in the other it does.

The remainder of the computation (of f) is tedious and we have failed to avoid taking lots of derivatives. After performing differentiations of $\cos\frac{1}{2}\Gamma$, the end points are specialized to

$$\theta_2=\theta_1=\tfrac{1}{2}\pi,\quad \psi_2=\psi_1,\quad \varphi_2=\Gamma,\quad \varphi_1=0.$$

This can be done only because of the rotational invariance whose proof has occupied so much space. The result of the calculation is given in Sec. 4.

[29] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), p. 161.