

Diffeomorphism invariance of the effective gravitational action

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We investigate on the diffeomorphism invariance of the effective gravitational action, focusing in particular on the path integral measure. In the literature, two different measures are mainly considered, the Fradkin-Vilkovisky and the Fujikawa one. With the help of detailed calculations, we show that, despite claims to the contrary, the Fradkin-Vilkovisky measure is diffeomorphism invariant, while the Fujikawa measure is not. In particular, we see that, contrary to naïve expectations, the presence of g^{00} factors in the Fradkin-Vilkovisky measure is necessary to ensure the invariance of the effective gravitational action. We also comment on results recently appeared in the literature, and show that formal calculations can easily miss delicate points.

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I. INTRODUCTION

If quantum gravity is described as a quantum field theory (as opposed to string theory or other formulations), two cases are possible. It might be either a UV-complete theory or an effective field theory (EFT) [1]. In each of these cases, it is defined through the path integral

$$\int d\mu e^{-S}, \quad (1.1)$$

where $d\mu$ is the measure and S the classical action. Of course, only after the theory $(d\mu, S)$ is given can the possible existence of fixed points be investigated.¹ In particular, it can be seen whether the theory is UV-complete, in which case it possesses a UV-attractive fixed point, or it is an EFT valid up to a certain energy scale, above which its UV-completion takes over. Starting from the path integral (1.1), the Wilsonian RG approach implements a nonperturbative definition of the theory (successive

elimination of infinitesimal shells of modes), no matter if the theory is UV-complete or an EFT.

In a recent paper [2], we calculate the one-loop (Euclidean) effective action Γ^{1l} in quantum gravity within the Einstein-Hilbert truncation, paying due attention to the path integral measure and to the identification of the physical UV cutoff Λ . In [3], we extend this analysis deriving the renormalization group (RG) equations for the running Newton and cosmological constant. In [2], we show that the quantum correction to the vacuum energy does not present quartically and quadratically divergent contributions.² In [3], we find that the RG flow of the Newton and cosmological constant significantly differs from the so-called asymptotic safety (AS) scenario [9–12]. In particular, we do not see any sign of a nontrivial UV-attractive fixed point, which is the distinguishing feature of this latter scenario.

In [2,3], the Fradkin-Vilkovisky (FV) path integral measure [13–15] is used. It is sometimes claimed that, since this measure contains noncovariant factors of the time-time component $g^{00}(x)$ of the inverse metric, it is not diffeomorphism invariant, and that (on the contrary) the invariant measure is the one proposed by Fujikawa [16]. This point of view has been recently taken up in [17] where, considering the case of a scalar theory in a gravitational background, the authors claim to demonstrate that the

¹Needless to say, if $d\mu'$ and S' are such that $d\mu' e^{-S'} = d\mu e^{-S}$, the couple $(d\mu', S')$ defines the same theory as $(d\mu, S)$. Obviously, the RG properties of the theory (including the presence/absence of fixed points) are not altered by such a trivial reshuffling of terms.

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²For other related approaches to the naturalness problem see [4–8].

Fujikawa measure is diffeomorphism invariant (they then extend these conclusions to the case of pure gravity).

The aim of the present paper is to bring clarity on the question related to the measure. Performing a thorough investigation, we show that, despite the presence of g^{00} factors, the FV measure is diffeomorphism invariant. With the help of detailed calculations, we will show that this result is intimately related to the *very definition* of the path integral, that involves the introduction of a time ordering parameter and of a discretization of spacetime (lattice). By the same token, we will see that the result by which the Fujikawa measure appears to be diffeomorphism invariant is due to a *formal* treatment of the path integral, resorting to which the aforementioned points are overlooked. As a consequence, nontrivial terms that appear when the path integral undergoes a general coordinate transformation are missed. Once these terms are taken into account, it turns out that the FV measure is diffeomorphism invariant, while the Fujikawa measure is not.

The rest of the paper is organized as follows. In Sec. II, we consider the contribution to the effective gravitational action $\Gamma[g]$ due to the quantum fluctuations of a scalar field and start the investigation on its diffeomorphism invariance. Section III is devoted to the detailed calculation of all the terms that appear when $\Gamma[g]$ undergoes a general coordinate transformation. With the help of the results of Sec. III, in Sec. IV we show that the FV measure is invariant, while the Fujikawa one is not. In Sec. V, we compare our results with those of previous literature, in particular with those of the recent paper [17]. Section VI is for the conclusions.

II. DIFFEOMORPHISMS AND PATH INTEGRAL

Let us consider the gravitational action $S_g[g]$ (think for instance of the Einstein-Hilbert action) in the presence of a scalar field with action $S_m[\phi, g]$. The total action is [from now on we indicate with g the metric $g_{\mu\nu}$ when it appears as the argument of a function/functional, its determinant otherwise; the signature $(-, +, +, +)$ is used]

$$\begin{aligned} S[\phi, g] &= S_g[g] + S_m[\phi, g] \\ &= S_g[g] + \int d^4x \mathcal{L}_m(\phi(x), \partial_\mu \phi(x), g(x)) \end{aligned} \quad (2.1)$$

where $\mathcal{L}_m(\phi(x), \partial_\mu \phi(x), g(x))$ is the matter Lagrangian density

$$\begin{aligned} \mathcal{L}_m(\phi(x), \partial_\mu \phi(x), g(x)) \\ = -\frac{1}{2} \sqrt{-g(x)} (g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2(x)). \end{aligned} \quad (2.2)$$

The effective gravitational action³ $\Gamma[g]$ is given by

³For general considerations on the effective gravitational action see for instance [18–20].

$$e^{i\Gamma[g]} = \int e^{iS[\phi(x), g(x)]} \prod_x [M(g(x)) d\phi(x)], \quad (2.3)$$

where $M(g(x))$ is a nontrivial term in the configuration space measure $\prod_x [M(g(x)) d\phi(x)]$. In the literature, two different expressions for $M(g(x))$ are mainly considered. For the FV measure [13],

$$M_{\text{FV}}(g(x)) = (-g^{00}(x))^{1/2} (-g(x))^{1/4}, \quad (2.4)$$

while for the Fujikawa measure [16] (μ is an arbitrary mass scale)

$$M_{\text{Fuji}}(g(x)) = \mu (-g(x))^{1/4}. \quad (2.5)$$

Within the canonical formalism, the FV measure is obtained from the phase space path integral measure $\prod_x [d\pi(x) d\phi(x)]$ (Liouville), while the Fujikawa measure [16] from $\prod_x [(g^{00}(x))^{-1/2} d\pi(x) d\phi(x)]$ [21]. After integration over the conjugate momenta, the FV configuration space measure (2.4) contains $[g^{00}(x)]^{1/2}$ factors, while the Fujikawa measure (2.5) does not.

A fundamental aspect to be stressed is the following. In (2.3), $\prod_x$ indicates that in the definition of the path integral a discretization of spacetime is introduced, which in turn implies that $\prod_x [M(g(x)) d\phi(x)]$ in (2.3) can (obviously) be written as $[\prod_x M(g(x))] [\prod_x d\phi(x)]$. This latter (trivial) observation will be useful in the following. Moreover, since in Sec. V we will compare our results with those of [17], it is useful to observe that, at the beginning of their work, the authors define the path integral measure in the same way as we do in (2.3) [see their Eqs. (2.1) and (2.2)]. However, they are not entirely coherent with this definition along their calculations, and thus end up missing important terms in the transformation of the effective action under diffeomorphisms. We will add further comments on these points in due time.

Let us go back now to the meaning of $\prod_x$, stressing a crucial point first raised in [22], and later deeply investigated in [13]. In any formulation of quantum field theory, and in particular in the formulation of gauge theories, the construction of the basic transition amplitudes $\langle \phi', t' | \phi'', t'' \rangle$ (definition of the S matrix) requires the identification of a parameter in terms of which a time ordering is introduced. Differently from the case of Yang-Mills theories, where the gauge transformation only affects the form of the fields, in gravity a general coordinate transformation acts on the coordinates, and thus on the argument of the fields. As we will see, this has a nontrivial impact on the time ordering. In the path integral formulation, the time ordering parameter can be introduced as follows. Given a coordinate system x^μ , if the points of the lattice are obtained from the intersection between hypersurfaces $x^0 = \text{const}$ and curves $x^i = \text{const}$,

the time ordering parameter is identified⁴ with x^0 [13,22]. The notation $\prod_x$ in Eq. (2.3) indicates the product over all the points Q_i of the lattice $\mathcal{E}_1$ defined in this way. Note that the construction of the lattice and the identification of the ordering parameter can be done only once the coordinate system has been specified.

Let us call Σ the reference frame with coordinates x and $\hat{\Sigma}$ another frame with coordinates $\hat{x}$. In $\hat{\Sigma}$, the action (2.1) is written as (we use the notation $\hat{\partial}_\mu \equiv \frac{\partial}{\partial \hat{x}^\mu}$)

$$\begin{aligned} \hat{S}[\hat{\phi}, \hat{g}] &= \hat{S}_g[\hat{g}] + \hat{S}_m[\hat{\phi}, \hat{g}] \\ &= \hat{S}_g[\hat{g}] + \int d^4 \hat{x} \mathcal{L}_m(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x})), \end{aligned} \quad (2.6)$$

and for the effective action $\hat{\Gamma}[\hat{g}]$ we have

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\hat{S}[\hat{\phi}, \hat{g}]} \prod_{\hat{x}} [M(\hat{g}(\hat{x})) d\hat{\phi}(\hat{x})]. \quad (2.7)$$

Similarly to what we said for (2.3), the notation $\prod_{\hat{x}}$ in (2.7) indicates the product over the points P_i of the lattice $\mathcal{E}_2$ obtained from the intersection of hypersurfaces $\hat{x}^0 = \text{const}$ with curves $\hat{x}^i = \text{const}$. In this case, the time ordering parameter is $\hat{x}^0$. We will see that, due to this change in the time ordering parameter, the path integral measure transforms nontrivially under diffeomorphisms.⁵ This important aspect is missed in [17]: the measure $\prod_x [M(g(x)) d\phi(x)]$ should not be treated formally, and attention should be paid to the discretization and to the time ordering underlying this expression.

Let us investigate now on the diffeomorphism invariance of the effective action, performing the transition from $\hat{\Sigma}$ to Σ . Our aim is to see for which choice of $M(g(x))$, either the

FV one in (2.4) or the Fujikawa one in (2.5), we have $\hat{\Gamma}[\hat{g}] = \Gamma[g]$. Since the classical action is diffeomorphism invariant, i.e. $\hat{S}[\hat{\phi}, \hat{g}] = S[\phi, g]$, from (2.3) and (2.7) we see that $\hat{\Gamma}[\hat{g}] = \Gamma[g]$ if

$$\prod_{\hat{x}} [M(\hat{g}(\hat{x})) d\hat{\phi}(\hat{x})] = \prod_x [M(g(x)) d\phi(x)]. \quad (2.8)$$

We stress again that in $\hat{\Sigma}$ and Σ two different lattices and time ordering parameters are considered; this is encoded in the two product symbols $\prod_{\hat{x}}$ of (2.7) and $\prod_x$ of (2.3). To realize the bridge between $\hat{\Gamma}[\hat{g}]$ (defined in $\hat{\Sigma}$) and $\Gamma[g]$ (defined in Σ), we now proceed in two steps.

In the first step, while remaining in the frame $\hat{\Sigma}$, we move from the lattice $\mathcal{E}_2$ of points P_i (and time ordering parameter $\hat{x}^0$) to the lattice $\mathcal{E}_1$ of points Q_i (and time ordering parameter x^0). This is realized going from $\prod_{\hat{x}} [M(\hat{g}(\hat{x})) d\hat{\phi}(\hat{x})]$ to $\prod_x [M(\hat{g}(\hat{x}(x))) d\hat{\phi}(\hat{x}(x))]$ and performing in the action $\hat{S}[\hat{\phi}, \hat{g}]$ the change of integration variables from $\hat{x}$ to x through the relations $\hat{x} = \hat{x}(x)$, $\hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}) = \hat{g}_{\mu\nu}(\hat{x}(x))$. We get [see comments below (2.10) for the factors A and B]

$$\begin{aligned} e^{i\hat{\Gamma}[\hat{g}]} &= \int e^{i\hat{S}[\hat{\phi}, \hat{g}]} \left[\prod_{\hat{x}} M(\hat{g}(\hat{x})) \right] \left[\prod_{\hat{x}} d\hat{\phi}(\hat{x}) \right] \\ &= \int e^{i\hat{S}[\hat{\phi}, \hat{g}]} \left[B \prod_x M(\hat{g}(\hat{x}(x))) \right] \left[A \prod_x d\hat{\phi}(\hat{x}(x)) \right], \end{aligned} \quad (2.9)$$

where $\hat{S}$ in the right-hand side is written as

$$\begin{aligned} \hat{S}[\hat{\phi}, \hat{g}] &= \hat{S}_g[\hat{g}] + \hat{S}_m[\hat{\phi}, \hat{g}] \\ &= \hat{S}_g[\hat{g}] - \frac{1}{2} \int d^4 x J \sqrt{-\hat{g}(\hat{x}(x))} \left(\hat{g}^{\mu\nu}(\hat{x}(x)) \frac{\partial x^\rho}{\partial \hat{x}^\mu} \frac{\partial x^\sigma}{\partial \hat{x}^\nu} \partial_\rho \hat{\phi}(\hat{x}(x)) \partial_\sigma \hat{\phi}(\hat{x}(x)) + m^2 \hat{\phi}^2(\hat{x}(x)) \right) \\ &\equiv \hat{S}_g[\hat{g}] + \int d^4 x \tilde{\mathcal{L}}_m(\hat{\phi}(\hat{x}(x)), \partial_\mu \hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x))), \end{aligned} \quad (2.10)$$

with $J \equiv |\det \frac{\partial \hat{x}}{\partial x}|$ the Jacobian. Note that $\hat{S}[\hat{\phi}, \hat{g}]$ is the action for the fields $\hat{\phi}$ and $\hat{g}_{\mu\nu}$, that is the action in the frame $\hat{\Sigma}$. We stress that up to now only a change of integration variables has been performed (not a change of reference

frame). Moreover, $\tilde{\mathcal{L}}_m$ in the third line of (2.10) is the matter Lagrangian density,⁶ still in the frame $\hat{\Sigma}$, written in terms of the variables x .

⁴The quantization can also be realized considering a more general spacelike hypersurface $\tau(x) = \text{const}$, in which case the role of ordering parameter is played by τ .

⁵With an abuse of notation, throughout the paper we will use the same symbol $\hat{x}$ to indicate both the action of the diffeomorphism on x and its corresponding image, i.e. $\hat{x} = \hat{x}(x)$.

⁶It is easy to see that $\tilde{\mathcal{L}}_m$ gives rise to the same equations of motion and the same energy-momentum tensor for the field $\hat{\phi}$ as $\mathcal{L}_m$ in (2.6). The only difference is that they are written in terms of the variables x using the relations $\hat{x} = \hat{x}(x)$, $\hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}) = \hat{g}_{\mu\nu}(\hat{x}(x))$.

Let us comment now on the factors A and B that appear in the right-hand side of (2.9). One might naively expect that the step performed above ($\prod_{\hat{x}} \rightarrow \prod_x$) consists of a trivial reshuffling of (spacetime) points, in which case one would have $A = B = 1$. This is what is implicitly implemented in [17]. However, we have seen that in going from the left to the right-hand side of (2.9), a nontrivial change of lattice and of ordering parameter occurs (see above). This is what ultimately leads to the appearance of nontrivial factors $A \neq 1$ and $B \neq 1$ in (2.9). The latter are calculated in the next section.

In the second step, we move from $\hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}(x))$ to $\phi(x)$ and $g_{\mu\nu}(x)$. This will eventually lead to the relation between $\hat{\Gamma}[\hat{g}]$ and $\Gamma[g]$. From (2.9), we get (below the invariance of the classical action, $\hat{S}[\hat{\phi}, \hat{g}] = S[\phi, g]$, is used)

$$\begin{aligned} e^{i\hat{\Gamma}[\hat{g}]} &= \int e^{i\hat{S}[\hat{\phi}, \hat{g}]} \left[B \prod_x M(\hat{g}(\hat{x}(x))) \right] \left[A \prod_x d\hat{\phi}(\hat{x}(x)) \right] \\ &= \int e^{iS[\phi, g]} \left[BC \prod_x M(g(x)) \right] \left[AE \prod_x d\phi(x) \right]. \end{aligned} \quad (2.11)$$

In the above equation, the factor C arises when we express $M(\hat{g}(\hat{x}(x)))$ in terms of $g_{\mu\nu}(x)$, while E is the Jacobian of the transformation $\hat{\phi}(\hat{x}(x)) \rightarrow \phi(x)$. As for A and B , the calculation of C and E is performed in the next section.

From (2.3) and (2.11), we see that for the effective action to be invariant under general coordinate transformations, i.e., to have $\hat{\Gamma}[\hat{g}] = \Gamma[g]$, it is necessary that $ABCE = 1$. In the coming Secs. III and IV, we will see that while this is the case for the FV measure [see M_{FV} in (2.4)], it fails to be true for the Fujikawa measure [see M_{Fuj} in (2.5)]. Let us proceed now to the calculation of A , B , C , and E .

III. TRANSFORMATION FACTORS

In the present section, we calculate the transformation factors A , B , C , and E that appear in (2.11) following [13], where similar calculations have been performed for the pure gravity case. In particular, we will find that the factors A and B , that are related to the transition $\prod_{\hat{x}} \rightarrow \prod_x$ involved in the transformation of the effective action under diffeomorphisms, turn out to be nontrivial ($\neq 1$). As anticipated, this shows that $\prod_{\hat{x}} \rightarrow \prod_x$ is not a trivial reshuffling of points, but rather a delicate step to be performed carefully while establishing how the effective action transforms under diffeomorphisms. Overlooking these terms, one is led to think that the FV measure is not diffeomorphism invariant, due to the presence of the g^{00} factors [see (2.4)]. In fact, the opposite is true. As it will be seen in Sec. IV, the invariance of the effective action emerges from a balance between these nontrivial terms and those coming from the g^{00} factors in the FV measure. Let us proceed now with the calculation.

The factor E . We begin with the factor E . Since ϕ is a scalar field $[\hat{\phi}(\hat{x}(x)) = \phi(x)]$ and E is the Jacobian of the transformation $\hat{\phi}(\hat{x}(x)) \rightarrow \phi(x)$, clearly $E = 1$.

The factor A . Let us move now to the factor A . As mentioned in the previous section, the FV measure $\prod_x [M_{\text{FV}}(g(x)) d\phi(x)]$ [see (2.4)] is obtained within the canonical formalism from the phase space path integral measure $\prod_x [d\pi(x) d\phi(x)]$, while the Fujikawa measure $\prod_x [M_{\text{Fuj}}(g(x)) d\phi(x)]$ [see (2.5)] from $\prod_x [(g^{00}(x))^{-1/2} d\pi(x) d\phi(x)]$. To calculate A , we consider both cases at once writing [below $f(g) = 1$ for the FV measure and $f(g) = (g^{00}(x))^{-1/2}$ for the Fujikawa one]

$$\begin{aligned} e^{i\Gamma_{\mathcal{E}_1}[g]} &= e^{iS_g[g]} \int e^{i \int d^4x [\partial_0 \phi(x) \pi(x) - \mathcal{H}(\pi(x), \phi(x))]} \\ &\quad \times \left[\prod_x f(g(x)) d\pi(x) d\phi(x) \right] \end{aligned} \quad (3.1)$$

where $\pi(x)$ is the momentum conjugate to the field $\phi(x)$

$$\begin{aligned} \pi(x) &= \frac{\partial \mathcal{L}_m(\phi(x), \partial_\mu \phi(x), g(x))}{\partial (\partial_0 \phi(x))} \\ &= -(-g(x))^{1/2} g^{0\mu}(x) \partial_\mu \phi(x), \end{aligned} \quad (3.2)$$

$\mathcal{L}_m(\phi(x), \partial_\mu \phi(x), g(x))$ the Lagrangian density in the frame Σ given in (2.1), and $\mathcal{H}(\pi(x), \phi(x))$ is the Hamiltonian density [here we use the shorthand notation $\mathcal{H}(\pi, \phi)$ for $\mathcal{H}(\pi, \phi, \partial_i \phi, g)$]

$$\begin{aligned} \mathcal{H}(\pi, \phi) &= \pi(x) \partial_0 \phi(x) - \mathcal{L}_m(\phi(x), \partial_\mu \phi(x), g(x)) \\ &= \frac{1}{2} (-g^{00})^{-1} (-g)^{-1/2} \pi^2 + (-g^{00})^{-1} g^{0i} \pi \partial_i \phi \\ &\quad + \frac{1}{2} (-g^{00})^{-1} (-g)^{1/2} g^{0i} g^{0j} \partial_i \phi \partial_j \phi \\ &\quad + \frac{1}{2} (-g)^{1/2} g^{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} (-g)^{1/2} m^2 \phi^2. \end{aligned} \quad (3.3)$$

As stressed in the previous section [see comments below (2.5)], $\prod_x$ in (3.1) means that the discretization is realized considering the points Q_i of the lattice $\mathcal{E}_1$ and that the time ordering parameter is x^0 (this is further stressed by the subscript $\mathcal{E}_1$ in $\Gamma_{\mathcal{E}_1}[g]$). In this respect, we stress that $\pi(x)$ is correctly defined by (3.2) since in this case x^0 is the time ordering parameter. As said above (see footnote 4), we could also quantize the theory considering a more general family of spacelike hypersurfaces $\tau(x) = \text{const}$, in which case the time ordering parameter would be τ , and the conjugate momenta would be obtained differentiating the Lagrangian with respect to $\partial_t \phi$.

For the effective action in $\hat{\Sigma}$ we have (now the lattice $\mathcal{E}_2$ of points P_i is used)

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_g[\hat{g}]} \int e^{i \int d^4\hat{x} [\hat{\partial}_0 \hat{\phi}(\hat{x}) \hat{\pi}(\hat{x}) - \mathcal{H}(\hat{\pi}(\hat{x}), \hat{\phi}(\hat{x}))]} \times \left[\prod_{\hat{x}} f(\hat{g}(\hat{x})) d\hat{\pi}(\hat{x}) d\hat{\phi}(\hat{x}) \right] \quad (3.4)$$

where $\hat{\pi}(\hat{x})$ is the momentum conjugate to the field $\hat{\phi}(\hat{x})$

$$\hat{\pi}(\hat{x}) = \frac{\partial \mathcal{L}_m(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))}{\partial(\hat{\partial}_0 \hat{\phi}(\hat{x}))} = -(-\hat{g}(\hat{x}))^{1/2} \hat{g}^{0\mu}(\hat{x}) \hat{\partial}_\mu \hat{\phi}(\hat{x}), \quad (3.5)$$

$\mathcal{L}_m(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))$ the Lagrangian density in the frame $\hat{\Sigma}$ [see (2.6)], and $\mathcal{H}(\hat{\pi}(\hat{x}), \hat{\phi}(\hat{x}))$ the corresponding Hamiltonian density [see (3.3)]. The symbol $\prod_{\hat{x}}$ indicates that the discretization is realized considering the points P_i of the lattice $\mathcal{E}_2$. The time ordering parameter is now $\hat{x}^0$, and accordingly the conjugate momentum $\hat{\pi}(\hat{x})$ is given by (3.5).

As explained in the previous section, the bridge between $\hat{\Gamma}[\hat{g}]$ and $\Gamma[g]$ is realized in two steps. In the first one (from which the factor A emerges), while remaining in the frame $\hat{\Sigma}$ we write $\hat{\Gamma}[\hat{g}]$ switching from the lattice $\mathcal{E}_2$ to the lattice $\mathcal{E}_1$ (that in turn implies switching from the time ordering

parameter $\hat{x}^0$ to x^0). Considering then $\mathcal{E}_1$, for $\hat{\Gamma}[\hat{g}]$ we have

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]} = e^{i\hat{S}_g[\hat{g}]} \int e^{i \int d^4x [\partial_0 \hat{\phi}(\hat{x}(x)) \tilde{\pi}(\hat{x}(x)) - \tilde{\mathcal{H}}(\tilde{\pi}(\hat{x}(x)), \hat{\phi}(\hat{x}(x)))]} \times \left[F \prod_x f(\hat{g}(\hat{x}(x))) \right] \left[\prod_x d\tilde{\pi}(\hat{x}(x)) d\hat{\phi}(\hat{x}(x)) \right]. \quad (3.6)$$

In the above equation, F is the factor that arises when $\prod_{\hat{x}} f(\hat{g}(\hat{x}))$ is written in terms of $\prod_x f(\hat{g}(\hat{x}(x)))$, i.e., when we switch from $\mathcal{E}_2$ to $\mathcal{E}_1$: $\prod_{\hat{x}} f(\hat{g}(\hat{x})) = F \prod_x f(\hat{g}(\hat{x}(x)))$. We will shortly see that we do not need to calculate explicitly the transformation factor F . Moreover, since in (3.6) the time ordering parameter is x^0 , the momentum $\tilde{\pi}(\hat{x}(x))$ conjugate to $\hat{\phi}(\hat{x}(x))$ is given by [see comments below (2.10) and (3.2)]

$$\tilde{\pi}(\hat{x}(x)) = \frac{\partial \tilde{\mathcal{L}}_m(\hat{\phi}(\hat{x}(x)), \partial_\mu \hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x)))}{\partial(\partial_0 \hat{\phi}(\hat{x}(x)))} = -(-\hat{g}(\hat{x}(x)))^{1/2} G^{0\mu}(\hat{x}(x)) \partial_\mu \hat{\phi}(\hat{x}(x)), \quad (3.7)$$

and the Hamiltonian density to be used is

$$\tilde{\mathcal{H}}(\tilde{\pi}(\hat{x}(x)), \hat{\phi}(\hat{x}(x))) = \tilde{\pi}(\hat{x}(x)) \partial_0 \hat{\phi}(\hat{x}(x)) - \tilde{\mathcal{L}}_m(\hat{\phi}(\hat{x}(x)), \partial_\mu \hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x))) = \frac{1}{2} (-G^{00})^{-1} (-\hat{g})^{-1/2} \tilde{\pi}^2 + (-G^{00})^{-1} G^{0i} \tilde{\pi} \partial_i \hat{\phi} + \frac{1}{2} (-G^{00})^{-1} (-\hat{g})^{1/2} G^{0i} G^{0j} \partial_i \hat{\phi} \partial_j \hat{\phi} + \frac{1}{2} (-\hat{g})^{1/2} G^{ij} \partial_i \hat{\phi} \partial_j \hat{\phi} + \frac{1}{2} J (-\hat{g})^{1/2} m^2 \hat{\phi}^2, \quad (3.8)$$

where $\tilde{\mathcal{L}}_m(\hat{\phi}(\hat{x}(x)), \partial_\mu \hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x)))$ is the Lagrangian density in (2.10) and we define ($J \equiv |\det \frac{\partial \hat{x}}{\partial x}|$ is the Jacobian related to the change of variables $\hat{x} \rightarrow x$)

$$G^{\mu\nu}(\hat{x}(x)) \equiv J \hat{g}^{\rho\sigma}(\hat{x}(x)) \frac{\partial x^\mu}{\partial \hat{x}^\rho} \frac{\partial x^\nu}{\partial \hat{x}^\sigma}. \quad (3.9)$$

Let us perform now in (3.4) the functional integration over $\hat{\pi}(\hat{x})$. We get

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_g[\hat{g}]} \left[\prod_{\hat{x}} f(\hat{g}(\hat{x})) \right] \times \int e^{i\hat{S}_m[\hat{\phi}, \hat{g}]} \left[\prod_{\hat{x}} W(\hat{g}(\hat{x})) \right] \prod_{\hat{x}} d\hat{\phi}(\hat{x}) \quad (3.10)$$

where we define

$$W(\hat{g}(\hat{x})) \equiv (-\hat{g}(\hat{x}))^{1/4} (-\hat{g}^{00}(\hat{x}))^{1/2}. \quad (3.11)$$

Similarly, the integration over $\tilde{\pi}(\hat{x}(x))$ in (3.6) gives

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]} = e^{i\hat{S}_g[\hat{g}]} F \left[\prod_x f(\hat{g}(\hat{x}(x))) \right] \times \int e^{i\hat{S}_m[\hat{\phi}, \hat{g}]} \left[\prod_x Y(\hat{g}(\hat{x}(x))) \right] \prod_x d\hat{\phi}(\hat{x}(x)), \quad (3.12)$$

where we define

$$Y(\hat{g}(\hat{x}(x))) \equiv J^{1/2} (-\hat{g}(\hat{x}(x)))^{1/4} [-\hat{g}^{\mu\nu}(\hat{x}(x)) \hat{\partial}_\mu x^0 \hat{\partial}_\nu x^0]^{1/2}. \quad (3.13)$$

Moreover, writing $\prod_{\hat{x}} d\hat{\phi}(\hat{x})$ in terms of $\prod_x d\hat{\phi}(\hat{x}(x))$ [see Eq. (2.9)], Eq. (3.10) can be written as

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{\Sigma}_{\mathcal{E}_2}[\hat{g}]} \left[\prod_{\hat{x}} f(\hat{g}(\hat{x})) \right] \int e^{i\hat{\Sigma}_m[\hat{\phi}, \hat{g}]} \times \left[\prod_{\hat{x}} W(\hat{g}(\hat{x})) \right] \left[A \prod_x d\hat{\phi}(\hat{x}(x)) \right]. \quad (3.14)$$

Since $\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]$ and $\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]$ are the same effective action in $\hat{\Sigma}$, simply written using the two lattices $\mathcal{E}_2$ and $\mathcal{E}_1$, respectively, from (3.12) and (3.14) we finally get

$$A = \frac{\prod_x Y(\hat{g}(\hat{x}(x)))}{\prod_{\hat{x}} W(\hat{g}(\hat{x}))}. \quad (3.15)$$

We now write A in a convenient form. Let us consider an infinitesimal coordinate transformation (in the following we only need to keep terms up to first order in ϵ)

$$\hat{x}^\mu = x^\mu + \epsilon^\mu(x), \quad (3.16)$$

and switch from $\prod_x$ to $\prod_{\hat{x}}$ in the numerator of (3.15). As we will see below [Eq. (3.20)], this allows to factor out a term that cancels the denominator. Observing that⁷ [as usual, $\delta^{(4)}(0)$ comes from $\sum_x \rightarrow \int d^4x$]

$$\prod_x J^{1/2} = \exp\left(\frac{\delta^{(4)}(0)}{2} \int d^4x \log(1 + \partial_\mu \epsilon^\mu)\right) = \exp\left(\frac{\delta^{(4)}(0)}{2} \int d^4x \partial_\mu \epsilon^\mu\right) = 1, \quad (3.17)$$

where we have used

$$\frac{\partial \hat{x}^\mu}{\partial x^\nu} = \delta_\nu^\mu + \partial_\nu \epsilon^\mu; \quad d^4\hat{x} = d^4x J \equiv d^4x \left| \det \frac{\partial \hat{x}}{\partial x} \right| = d^4x (1 + \partial_\mu \epsilon^\mu(x)), \quad (3.18)$$

and calculating the third factor in the right-hand side of (3.13) ($\partial_\mu x^0 = \delta_\mu^0 - \partial_\mu \epsilon^0$),

$$-\hat{g}^{\mu\nu}(\hat{x}(x)) \hat{\partial}_\mu x^0 \hat{\partial}_\nu x^0 = -\hat{g}^{00} + 2\hat{g}^{0\mu} \partial_\mu \epsilon^0 = -\hat{g}^{00} \left(1 - 2 \frac{\hat{g}^{0\mu}}{\hat{g}^{00}} \partial_\mu \epsilon^0\right), \quad (3.19)$$

from (3.11), (3.13), (3.17), (3.19) we get

$$\prod_x Y(\hat{g}(\hat{x}(x))) = \exp\left(\frac{\delta^{(4)}(0)}{2} \int d^4\hat{x} (1 - \partial_\mu \epsilon^\mu) \log\left[(-\hat{g}(\hat{x}))^{1/2} (-\hat{g}^{00}(\hat{x})) \left(1 - 2 \frac{\hat{g}^{0\mu}(\hat{x})}{\hat{g}^{00}(\hat{x})} \partial_\mu \epsilon^0\right)\right]\right) = \left[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\right] \exp\left(-\frac{\delta^{(4)}(0)}{2} \int d^4\hat{x} \left[\partial_\mu \epsilon^\mu \log((- \hat{g}(\hat{x}))^{1/2} (-\hat{g}^{00}(\hat{x}))) + 2 \frac{\hat{g}^{0\mu}(\hat{x})}{\hat{g}^{00}(\hat{x})} \partial_\mu \epsilon^0\right]\right). \quad (3.20)$$

Finally, inserting (3.20) in (3.15) we have

$$A = \exp\left(-\frac{\delta^{(4)}(0)}{2} \int d^4x \left[\partial_\mu \epsilon^\mu \log((-g(x))^{1/2} (-g^{00}(x))) + 2 \frac{g^{0\mu}(x)}{g^{00}(x)} \partial_\mu \epsilon^0\right]\right). \quad (3.21)$$

The factor B. Let us calculate now B . We begin by observing that $M(\hat{g}(\hat{x}))$ in (2.9) is nothing but the product $f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))$ in (3.10). We then have [below we make use of (3.18)]

$$\prod_{\hat{x}} M(\hat{g}(\hat{x})) = \prod_{\hat{x}} [f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))] = \exp\left(\delta^{(4)}(0) \int d^4\hat{x} \log(f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x})))\right) = \exp\left(\delta^{(4)}(0) \int d^4x \partial_\mu \epsilon^\mu \log(f(g(x))W(g(x)))\right) \prod_x [f(\hat{g}(\hat{x}(x)))W(\hat{g}(\hat{x}(x)))] = \exp\left(\delta^{(4)}(0) \int d^4x \partial_\mu \epsilon^\mu \log(M(g(x)))\right) \prod_x M(\hat{g}(\hat{x}(x))). \quad (3.22)$$

Recalling that $\prod_{\hat{x}} M(\hat{g}(\hat{x})) = B \prod_x M(\hat{g}(\hat{x}(x)))$ [see (2.9)], from (3.22) we have

⁷Note that the exponent in the third term of (3.17) has the form of a surface integral. It is then equal to zero in the case of boundary-less manifolds and/or asymptotically flat (smooth) gravitational fields, in which latter case gauge transformations such that $\epsilon \rightarrow 0$ at infinity are considered to preserve the asymptotics.

$$B = \exp\left(\delta^{(4)}(0) \int d^4x \partial_\mu \epsilon^\mu \log(M(g(x)))\right). \quad (3.23)$$

The factor C . Finally, we calculate the factor C , that is defined by $\prod_x M(\hat{g}(\hat{x}(x))) = C \prod_x M(g(x))$ [see (2.11)]. At first order in ϵ , we have

$$\begin{aligned} M(\hat{g}^{\mu\nu}(\hat{x}(x))) &= M(g^{\mu\nu}(x) + g^{\mu\sigma}(x) \partial_\sigma \epsilon^\nu(x) + g^{\nu\sigma}(x) \partial_\sigma \epsilon^\mu(x)) \\ &= M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)} (g^{\alpha\sigma}(x) \partial_\sigma \epsilon^\beta(x) + g^{\beta\sigma}(x) \partial_\sigma \epsilon^\alpha(x)), \end{aligned} \quad (3.24)$$

where we used

$$\hat{g}^{\mu\nu}(\hat{x}) = \frac{\partial \hat{x}^\mu(x)}{\partial x^\rho} \frac{\partial \hat{x}^\nu(x)}{\partial x^\sigma} g^{\rho\sigma}(x) = g^{\mu\nu}(x) + g^{\mu\sigma}(x) \partial_\sigma \epsilon^\nu(x) + g^{\nu\sigma}(x) \partial_\sigma \epsilon^\mu(x). \quad (3.25)$$

We then get

$$\begin{aligned} \prod_x M(\hat{g}(\hat{x}(x))) &= \exp\left(\delta^{(4)}(0) \int d^4x \log\left[M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)} (g^{\alpha\sigma}(x) \partial_\sigma \epsilon^\beta(x) + g^{\beta\sigma}(x) \partial_\sigma \epsilon^\alpha(x))\right]\right) \\ &= \exp\left(\delta^{(4)}(0) \int d^4x [M(g)]^{-1} \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)} (g^{\alpha\sigma}(x) \partial_\sigma \epsilon^\beta(x) + g^{\beta\sigma}(x) \partial_\sigma \epsilon^\alpha(x))\right) \prod_x M(g(x)), \end{aligned} \quad (3.26)$$

from which

$$C = \exp\left(\delta^{(4)}(0) \int d^4x [M(g)]^{-1} \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)} (g^{\alpha\sigma}(x) \partial_\sigma \epsilon^\beta(x) + g^{\beta\sigma}(x) \partial_\sigma \epsilon^\alpha(x))\right). \quad (3.27)$$

Having calculated the factors A , B , C , and E , we can now see how the FV and the Fujikawa measure transform under diffeomorphisms. This is the subject of the next section.

IV. FRADKIN-VILKOVISKY VERSUS FUJIKAWA MEASURE

As shown in Sec. II [see (2.11) and comments below], for the effective action to be diffeomorphism invariant, it must be $ABCE = 1$. Having found in the previous section the general expressions for A , B , C , and E , we can now calculate these terms for both $M(g(x)) = M_{\text{FV}}(g(x))$ and $M(g(x)) = M_{\text{Fuj}}(g(x))$. From a simple inspection of (3.21), (3.23) and (3.27) we see that, while B and C depend on the specific form of $M(g(x))$, A does not. Moreover, we have already seen at the beginning of Sec. III that $E = 1$. We have then to calculate B and C for $M_{\text{FV}}(g(x))$ and $M_{\text{Fuj}}(g(x))$, respectively.

Frادkin-Vilkovisky measure. Inserting (2.4) in (3.23) and (3.27) we get

$$B_{\text{FV}} = \exp\left(\frac{\delta^{(4)}(0)}{2} \int d^4x \partial_\mu \epsilon^\mu \log[(-g(x))^{1/2} (-g^{00}(x))]\right) \quad (4.1)$$

$$C_{\text{FV}} = \exp\left(\delta^{(4)}(0) \int d^4x \frac{g^{0\mu}(x)}{g^{00}(x)} \partial_\mu \epsilon^0\right), \quad (4.2)$$

from which [see also (3.21)]

$$B_{\text{FV}} C_{\text{FV}} = A^{-1}. \quad (4.3)$$

Equation (4.3), together with the result $E = 1$, shows that for the FV measure

$$AB_{\text{FV}} C_{\text{FV}} E = 1, \quad (4.4)$$

which means that in this case [see (2.11)]: $\hat{\Gamma}[\hat{g}] = \Gamma[g]$. As anticipated, the effective action calculated using the FV measure is diffeomorphism invariant.

Fujikawa measure. Inserting (2.5) in (3.23) and (3.27) we get

$$B_{\text{Fuj}} = \exp\left(\delta^{(4)}(0) \int d^4x \partial_\mu \epsilon^\mu \log((-g(x))^{1/4} \mu)\right) \quad (4.5)$$

$$C_{\text{Fuj}} = 1. \quad (4.6)$$

From (3.21), (4.5), and (4.6) (together with $E = 1$) we obtain

$$AB_{\text{Fuji}} C_{\text{Fuji}} E = \exp \left(\delta^{(4)}(0) \int d^4 x \left[\partial_\mu \epsilon^\mu \log \left(\frac{\mu}{(-g^{00})^{1/2}} \right) - \frac{g^{0\mu}(x)}{g^{00}(x)} \partial_\mu \epsilon^0 \right] \right), \quad (4.7)$$

which means that in this case [see (2.11)]: $\hat{\Gamma}[\hat{g}] \neq \Gamma[g]$. As anticipated, and contrary to what is claimed in [17], the effective action calculated using the Fujikawa measure is *not* diffeomorphism invariant.

V. COMPARISON WITH EXISTING LITERATURE

The main result of the analysis presented in the previous sections is that the FV measure [13] is diffeomorphism invariant, while the Fujikawa measure [16] is not. The naïve argument according to which the FV measure is claimed to be noninvariant is that it contains factors of the time-time component g^{00} of the inverse metric. Actually, we have shown that the opposite is true. As thoroughly discussed in the previous sections, the g^{00} factors in the FV measure guarantee the diffeomorphism invariance of the effective action. In fact, their presence is imposed by the necessity of having a time ordering parameter to define S matrix elements. By the same token, we have shown that the Fujikawa measure is not diffeomorphism invariant. Moreover, our calculations allow to understand why in the recent work [17] the opposite conclusion is reached, namely that the Fujikawa measure rather than the FV one is diffeomorphism invariant. The detailed analysis on the transformation of the effective action under diffeomorphisms developed in the previous sections shows that some of the terms involved in this transformation are missed in that paper.

Since the question of the diffeomorphism invariance of the effective action, sometimes source of discussions and controversies, is central to the present work, we find it worth to make a more detailed comparison between our results and those of [17]. We will see that the calculations and claims of that paper are flawed.

The authors of [17] consider the scalar theory defined by (2.1) and (2.2) above (see their Sec. II). A simple inspection of their Eqs. (2.14) and (2.19) shows that they overlook the transition between the two lattices involved in the coordinate transformation (together with the related change of time ordering parameter). This is tantamount to assume that this transition is nothing but a trivial reshuffling of points [see (2.9) and comments below]. Actually, the authors of [17] consider *only* the factors related to the transformation $(-\hat{g})^{1/4} \hat{\phi} \rightarrow (-g)^{1/4} \phi$. According to the notation that we introduced in the previous sections, this amounts to consider *only* the factors C and E in (2.11), while missing both the nontrivial factors A and B . It is precisely because they miss these latter terms that the authors of [17] are led to conclude that the Fujikawa measure is diffeomorphism invariant. The same issues arise when the authors extend

their considerations to pure quantum gravity (see their Sec. VI). Also in this case, they consider the Fujikawa measure [16], and attempt to demonstrate that the latter is Becchi-Rouet-Stora-Tyutin (BRST) invariant. Once again, they miss nontrivial transformation factors in the transition $\hat{x} \rightarrow x$, as it is immediately seen in their Eqs. (6.3)–(6.10).

Actually, all the arguments and conclusions in [17] are flawed, since they are based on the alleged diffeomorphism invariance of the Fujikawa measure. For instance, they claim that the path integral measure to be used in phase space is [21] [see Sec. III of [17], in particular the comments below Eqs. (3.9) and (3.27)]

$$\prod_x [(g^{00}(x))^{-1/2} d\pi(x) d\phi(x)]. \quad (5.1)$$

Starting with (5.1), in fact, after integration over the conjugate momenta π the Fujikawa measure in configuration space is obtained. As shown in the previous sections, however, the diffeomorphism invariant measure in configuration space is the Fradkin-Vilkovisky one. The latter is obtained after integration over π starting from the phase space path integral measure

$$\prod_x [d\pi(x) d\phi(x)], \quad (5.2)$$

that is nothing but the Liouville measure. This is what one would expect from general considerations based on the path integral construction of the basic transition amplitude $\langle \phi', t' | \phi'', t'' \rangle$. For R^2 higher-derivative theories of gravity, a similar derivation of the configuration space measure from the Liouville one was later done by Buchbinder and Lyakhovich [23].

To support their claim on the diffeomorphism invariance of the Fujikawa measure, the authors of [17] (see their Sec. IV) consider the contribution δS to the effective gravitational action from a free scalar field, obtained using this measure [μ is an arbitrary mass scale in the Fujikawa measure, see (2.5)]

$$\delta S[g] = \frac{1}{2} \text{Tr} \log \left(\frac{-\square + m^2}{\mu^2} \right), \quad (5.3)$$

where $-\square$ is the spin-0 Laplace-Beltrami operator for the metric $g_{\mu\nu}$. They evaluate the trace resorting to two different methods: (i) sum over the eigenvalues of $-\square + m^2$; (ii) proper-time formalism, and get for δS the results (4.7) and (4.15) of [17], respectively. Their calculations, however, overlook a quite delicate point (see below). This

oversight leads them in both cases to a result for δS expressed in terms of diffeomorphism invariant quantities, so they conclude that the Fujikawa measure is diffeomorphism invariant. The delicate point is that, in the case of gravitational theories, subtleties arise in the calculation of $\log(-\square + m^2)$ [that appears in their Eqs. (4.4) and (4.13)], and one should carefully take into account the distributional nature of the Green's function of $(-\square + m^2)$ [13,24,25]. When this is done, a nontrivial term contributing to the trace in (5.3) is found. This term is missed in [17]. Taking x^0 as time ordering parameter [see comments below Eqs. (2.5) and (2.7)], this latter term turns out to be $\delta^{(4)}(0) \int d^4x \log(g^{00})$ [24], which is not diffeomorphism invariant. As a consequence, the quantum correction δS calculated with the Fujikawa measure is *not* diffeomorphism invariant.

Let us consider now the result for $\delta S[g]$ obtained when the FV measure rather than the Fujikawa one is used, which is what we calculate in [2]. The result is similar to (5.3) (obtained with the Fujikawa measure), but in addition to the “Tr log” it contains an extra term [see second line of (5.4)], whose crucial importance will be soon clear. The calculation is performed taking the metric $g_{\mu\nu}^{(a)}$ of a sphere of radius a , and gives [see Eq. (48) of [2]]

$$\begin{aligned} \delta S[g^{(a)}] &= \frac{1}{2} \log [\det (-\tilde{\square} + a^2 m^2)] - \frac{1}{2} \log \left(\prod_x \tilde{g}^{00}(x) \right) \\ &= \frac{1}{2} \text{Tr} \log (-\tilde{\square} + a^2 m^2) \\ &\quad - \frac{\delta^{(4)}(0)}{2} \int d^4x \log(\tilde{g}^{00}(x)), \end{aligned} \quad (5.4)$$

where $-\tilde{\square}$ is the Laplace-Beltrami operator for the metric $\tilde{g}_{\mu\nu}$ of a unit sphere, and $\tilde{g}^{00}$ is the time-time component of the inverse of $\tilde{g}_{\mu\nu}$. To avoid misunderstandings, we stress again that, despite the presence in (5.4) of $\tilde{g}_{\mu\nu}$ and $-\tilde{\square}$, $\delta S[g^{(a)}]$ is calculated for a sphere of *generic* radius a . As clearly explained in [2,26] (to which we refer for details), these dimensionless quantities ($-\tilde{\square}$, $\tilde{g}_{\mu\nu}$, and $a^2 m^2$) appear thanks to the presence of the FV measure. As we stressed several times in [2,3,26], and contrary to what is incorrectly claimed in [27], we do not take the radius a as reference scale. This is immediately evident if for instance we note that in (5.4) the combination $a^2 m^2$ appears.

Some crucial points need to be stressed. First of all, we observe that the term $\frac{1}{2} \log(\prod_x \tilde{g}^{00}(x))$ in (5.4) comes from the exponentiation of the measure term $\prod_x (\tilde{g}^{00}(x))^{1/2}$ [see (2.3), with $M(g)$ given in (2.4)]. In this respect, we recall [see comments below (2.5)] that the discretization involved in the definition of the path integral, encoded in $\prod_x$, allows the trivial splitting $\prod_x [(g^{00}(x))^{1/2} (g(x))^{1/4} d\phi(x)] = [\prod_x (g^{00}(x))^{1/2}] [\prod_x (g(x))^{1/4} d\phi(x)]$, which is what we operate to get (5.4). Moreover, as it should be clear from

the comments below (5.3), the calculation of $\frac{1}{2} \text{Tr} \log(-\tilde{\square} + a^2 m^2)$ in (5.4) gives rise to the term $\frac{\delta^{(4)}(0)}{2} \int d^4x \log(\tilde{g}^{00}(x))$ [13,24]. This term *cancels* the last one in the second line⁸ of (5.4). All the other terms that come from the “Tr log” turn out to be diffeomorphism invariant. Therefore, the presence of the $g^{00}(x)$ factors in the measure not only does not spoil the diffeomorphism invariance of the effective action (as one would naïvely expect) but it is rather necessary⁹ to *compensate* noninvariant terms that arise from the calculation of the “Tr log.” Similar considerations apply to the pure gravity case [13].

We have just seen that δS is given by (5.4), and that the presence of $g^{00}(x)$ factors in the configuration space measure is crucial to have a diffeomorphism invariant result. In [17], the more general measure $\prod_x [(g(x))^{1/4} \Omega_g(x) d\phi(x)]$ is also considered, and the authors write (see their Sec. V)

$$\delta S[g] = \frac{1}{2} \text{Tr} \log[\Omega_g^{-2}(-\square + m^2)]. \quad (5.5)$$

Based on their idea that the diffeomorphism invariance of δS is ensured if the fluctuation operator $\Omega_g^{-2}(-\square + m^2)$ is covariant, they claim that the only possibility is for Ω_g to be a scalar. The analysis of the present work (see in particular the three previous paragraphs) shows that, contrary to this claim, the only possibility to have a diffeomorphism invariant effective action is to take $\Omega_g = (g^{00})^{1/2}$.

In fact, as explained in detail above, the diffeomorphism invariance of the effective action is not naïvely related to the fact that only covariant functions and functionals are present in the path integral. The invariance emerges from a delicate balance between *all* the elements that enter the definition of the path integral. We have seen that the delicate point concerns the product $\prod_x$ of all these elements, more specifically the time ordering and the discretization (lattice) involved in the definition of the path integral and encoded in this product.

Going back to (5.5), for a scalar Ω_g the authors of [17] manage to write $\Omega_g^{-2}(-\square + m^2)$ in the form of a minimal Laplace-type operator, i.e. an operator of the kind $(-\square + m^2)$ as the one in (5.3) [see Eqs. (A5) and (A6) of [17]]. They then calculate the “Tr log” resorting to proper-time techniques, but fail to treat the $\log(-\square)$ with the due care, thus missing relevant terms [see the thorough discussion

⁸See [22] for earlier discussions on the appearance and cancellation of $\delta(0)$ divergences.

⁹In [2], the term $\frac{1}{2} \log(\prod_x \tilde{g}^{00}(x)) = \frac{\delta^{(4)}(0)}{2} \int d^4x \log(\tilde{g}^{00}(x))$ is indicated with $\mathcal{C}$ [see Eq. (48) of [2]]. There, we do not refer to the cancellation discussed in this work since the terms involved are a -independent, and have no impact in the calculation of the quantum corrections to the Newton and cosmological constant (the objective of [2]).

below (5.3)] [24]. As a consequence, the proper-time RG equation (5.6) of [17], that the authors derive differentiating with respect to the scale Λ their (incomplete) regularized action S_Λ , cannot be trusted, and the considerations and conclusions they draw starting from this equation are flawed.

The authors then move to consider the pure gravity case (in the EH truncation; see Sec. VII of [17]), and develop considerations that mirror (as they say) those made for the contribution to the effective gravitational action $\Gamma[g]$ from the scalar field. Even in this pure gravity case, in fact, the contribution δS_{grav} to Γ from the graviton is calculated considering traces similar to (5.5). They claim that if one uses the Fujikawa measure [16] a diffeomorphism invariant RG flow is obtained that (in four dimensions) contains terms of the kind [see their Eqs. (7.1) and (7.2)]

$$\sim \Lambda^4 \int d^4x \sqrt{g} \quad (5.6)$$

and

$$\sim \Lambda^2 \int d^4x \sqrt{g} R. \quad (5.7)$$

Moreover, they observe that these terms, that depend quartically and quadratically on the running scale Λ , are at the origin of the UV-attractive fixed point of the asymptotic safety scenario. We will comment on the existence/absence of this fixed point at the end of the present section.

In [17], it is also claimed that the use of the FV measure in the path integral amounts to make in (5.6) and (5.7) (that are obtained using the Fujikawa measure) the replacement $\Lambda^2 \rightarrow N^2 g^{00}$, where N is a dimensionless (running) cutoff. They then get

$$\Lambda^4 \int d^4x \sqrt{g} \rightarrow N^4 \int d^4x (g^{00})^2 \sqrt{g} \quad (5.8)$$

and

$$\Lambda^2 \int d^4x \sqrt{g} R \rightarrow N^2 \int d^4x \sqrt{g} g^{00} R. \quad (5.9)$$

Accordingly, they claim that the FV measure leads to the presence of noninvariant operators in the running action $S_\Lambda[g]$. These claims are incorrect. In fact, from the thorough discussion above, it is clear that these claims are due to the loss of important terms in their calculation of $S_\Lambda[g]$. We have shown that when the FV measure is used, not only noninvariant terms of the kind (5.8) and (5.9) do not appear, but also that the presence of g^{00} factors in this measure is necessary to guarantee the diffeomorphism invariance of $S_\Lambda[g]$ [see (5.4) and comments therein]. More in detail, we have seen that the g^{00} factors in the FV measure turn out to

cancel terms that arise when the different $\log(-\square^{(s)})$ in δS_{grav} are correctly calculated [here $-\square^{(s)}$ are the Laplace-Beltrami operators for the different spins, $s = 0, 1, 2$], and only diffeomorphism invariant terms are left in the final result. Among them are the terms¹⁰ $N^4/3$ and $-34N^2/3$ in our Eq. (30) of [3]. Contrary to what is incorrectly claimed in [17], they are diffeomorphism invariant and do not come from (5.8) and (5.9) evaluated in a coordinate system where $g^{00} = R$.

Let us finally go back to the question of the possible existence of the UV-attractive fixed point of the AS scenario. It is true that this fixed point would exist if terms of the kind (5.6) and (5.7) appeared in the RG equation for the running action. However, thanks to a careful treatment of the path integral measure and a proper introduction of the physical running scale, in [3] we show that a term like (5.6) is absent, and that (more generally) the beta functions for the Newton and the cosmological constant are significantly different from those of the AS literature [11,12]. They do not possess the nontrivial UV-attractive fixed point of the AS scenario. Finally, concerning the beta functions of the AS literature, it is worth to add here that a gauge and parametrization dependence of these RG functions has been repeatedly reported. See for instance the very recent work [28], where it is shown that the existence of the UV attractive fixed point of the AS scenario depends on the value of the gauge fixing parameter α . In a certain range of values of α , this fixed point even disappears. A clear indication that the UV behavior of the theory depends on the gauge fixing parameter was previously given in [29], where it was shown that, depending on the value of α , the UV behavior of the Newton constant in fact changes from screening to antiscreening.

VI. CONCLUSIONS

In the present paper we have thoroughly considered the controversial issue concerning the diffeomorphism invariance of the path integral measure in quantum gravity. It is usually thought that, due to the presence of noncovariant g^{00} factors, the FV measure cannot be diffeomorphism invariant.

With the help of the detailed analysis developed in Secs. II–IV, we have shown that, contrary to this naïve expectation, the g^{00} factors that appear in the FV measure are necessary to ensure the diffeomorphism invariance of the effective action $\Gamma[g]$. Actually, we have seen that the invariance emerges from a delicate balance between *all* the elements involved in the definition of the path integral. In particular, we have shown that a crucial point is related to

¹⁰In [3], the calculations are performed with a spherical background. Moreover, the dimensionless (running) cutoff N is indicated with L . For the sake of clarity, here we uniform our notation to [17] and use N .

the necessity of introducing a time ordering parameter and a discretization (lattice) of spacetime. Differently from other gauge theories, where the gauge transformation does not affect the spacetime coordinates, when considering a general coordinate transformation $x \rightarrow \hat{x}$, two different lattices and time ordering parameters are involved in the two coordinate systems. As a consequence, nontrivial factors appear when the path integral undergoes a general coordinate transformation. The g^{00} factors of the FV measure are necessary to *exactly* compensate for these nontrivial terms, and this is what ensures the diffeomorphism invariance of the effective action.

Another important point concerns the derivation of the configuration space measure from the phase space one. The FV measure $\prod_x [(g^{00}(x))^{1/2} (g(x))^{1/4} d\phi(x)]$ is obtained after integration over the conjugate momenta π if the phase space measure is the Liouville one, namely $\prod_x [d\pi(x) d\phi(x)]$. The latter is what one would naturally expect from considerations based on the path integral construction of the basic transition amplitude $\langle \phi', t' | \phi'', t'' \rangle$. On the contrary, the Fujikawa measure is obtained in configuration space if $\prod_x [(g^{00}(x))^{-1/2} d\pi(x) d\phi(x)]$ is assumed to be the phase space measure. This rather bizarre form of the phase space measure was first proposed in [21], though the arguments presented in that paper are far from being well grounded. This proposal was recently taken up by the authors of [17]. Based on their idea that the diffeomorphism invariant measure in configuration space is the Fujikawa one, they derive this bizarre phase space measure from the request that, after integration over the

conjugate fields π , the resulting configuration space measure is the Fujikawa one. The detailed analysis presented in this work, where we have shown that it is rather the FV measure to be invariant, indicates that such an artificial distortion of the natural Liouville measure in phase space has to be rejected (as one also expects on the basis of simpler arguments).

In conclusion, we have shown that the question of the measure to be used in the calculation of the effective gravitational action $\Gamma[g]$ presents delicate aspects that can be missed if one resorts to *formal* calculations. Paying attention to the construction of the path integral involved in the very definition of $\Gamma[g]$, it turns out that the measure proposed by Fradkin and Vilkovisky ensures the diffeomorphism invariance of $\Gamma[g]$, while the measure proposed by Fujikawa does not.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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