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Caldeira-Leggett quantum time crystal: Periodic revival of quantum coherence by a sub-Ohmic environment

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We propose the Caldeira-Leggett quantum time crystal in which the interaction between a quantum system and a viscoelastic sub-Ohmic environment leads to periodic revivals of quantum coherence. Using the Caldeira-Leggett model of quantum decoherence with a sub-Ohmic environment characterized by the spectral function $J(\omega) \propto \omega^s$ ($0 \leqslant s < 1$), we discover that decreasing the sub-Ohmic parameter s enhances the coherence lifetime of Gaussian wave packets, with the purity (i.e. coherence) decaying as $t^{-s/2}$ throughout the regime $0 \leqslant s < 1$. A revival oscillation of the purity with lifetime $e^{2/s}$ occurs in the deep sub-Ohmic regime ($s < 0.1$), which we call the Caldeira-Leggett quantum time crystal. We reveal that such oscillations originate from the fact that an environment in the deep sub-Ohmic regime behaves as a gel-like fractal medium with viscosity and elasticity, suggesting the significance of fractal gel-like environments *in vivo* and in information processing.

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I. INTRODUCTION

Quantum decoherence is the process by which quantum systems lose their coherence and behave more classically due to interactions with their environment. Decoherence plays a central role in the foundation of quantum mechanics [1–3], quantum information [4], and quantum biology [5–12]. Although thermal fluctuations were long thought to suppress quantum effects in biological systems, this view has been recently reconsidered in terms of open quantum systems [5–12], which show that the persistence of coherence depends on the details of system–environment coupling. Recent studies have also shown that specific environment properties can give rise to novel quantum phenomena. In particular, in the case of quantum time crystals by decoherence, quantum oscillations that are averaged out due to strong quantum fluctuations emerge during the process of decoherence [13]. Similarly, a mechanism for persistent generation of dynamic entanglement even in a hot environment has been considered [14]. These results suggest that decoherence-based quantum algorithms might be used in systems where decoherence is likely to occur. Motivated by these insights, we investigate decoherence in terms of the nature of the environment, demonstrating that specific properties of the environment can give rise to novel quantum phenomena. To quantify this effect, we consider the Caldeira-Leggett model.

The Caldeira-Leggett model explains quantum decoherence through the entanglement of a quantum system and its environment. Decoherence in this model occurs by tracing out the environmental degrees of freedom. Then, it is evident that decoherence depends on the nature of the environment. For instance, anomalous diffusion has been observed in living cells [15,16], which appears due to the high-density gel-like environment that can be characterized by fractal Brownian motion [17–21]. When the spectral density of the environment is expressed as $J(\omega) \propto \omega^s$ (with ω the angular frequency), it is suggested that nonintegral values of s may be relevant for fractal environments [22].

Decoherence can lead to new dynamics in quantum systems. The term "time crystal" originally refers to the self-organized oscillations and rhythms in biological systems [23]. Since 2012, time crystals have been actively studied as dynamical quantum-mechanical ground states [24–31], but whether or not such ground states can be realized is still under investigation. On the other hand, it was shown that time crystal oscillation can emerge in finite-size systems during the process of decoherence [13]. Very recently, time crystalline dynamics was also suggested in fractional Langevin systems [32]. Here, by investigating the nature of the environment in detail, we show that the interaction of a quantum system with its environment can lead to a periodic oscillation between coherence and decoherence, i.e., a periodic oscillation between quantum and classical, which we call the Caldeira-Leggett quantum time crystal.

This article reveals the s dependence of quantum decoherence in the sub-Ohmic regime, where s describes the power-law exponent of the environment spectral function $J(\omega) \propto \omega^s$ (Fig. 1). We extend the Caldeira-Leggett model to the case of sub-Ohmic environments and study the purity p of Gaussian wave packets. Here, p is a measure of quantum coherence, with $p = 1$ for an isolated quantum system, and the value of p decreases as the system becomes more classical due to decoherence. We discover that the smaller s is, the longer the coherence lifetime of the quantum

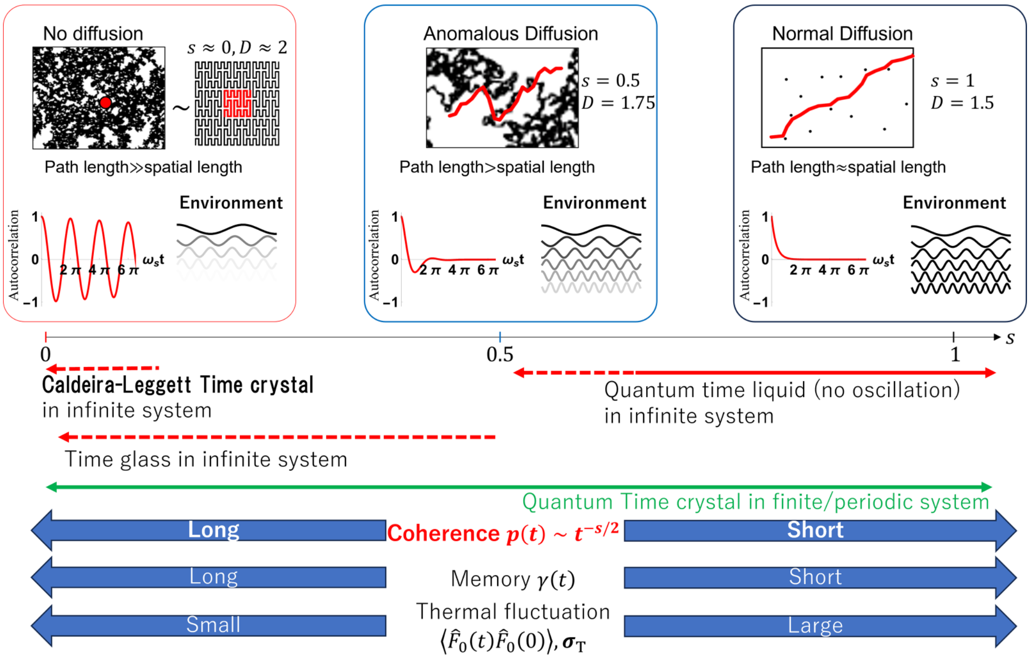

FIG. 1. Scenario of the Caldeira-Leggett quantum time crystal and time glass obtained from this study. The nonintegral value of s represents a fractal environment with a fractal dimension $D = 2 - s/2$. Velocity-dependent dissipation occurs in the Ohmic regime $s \approx 1$, while a particle in the deep sub-Ohmic regime $s \approx 0$ is localized in space due to the high-density fractal environment, leading to a position-dependent oscillatory behavior. We classify dynamical regimes in terms of the autocorrelation function $\dot{G}(t) = \mathrm{Re}\,\langle \hat{x}(t)\hat{x}(0)\rangle \,/\, \langle \hat{x}(0)^2\rangle$ which is directly related to the time evolution of the purity $p(t)$. For $0.5 < s < 1$ we obtain a *time-liquid* regime in which $G(t)$ decays monotonically without revivals. Revival of $\dot{G}(t)$ (and hence revival of coherence) occurs for $0.1 < s < 0.5$ due to the long-lived memory of interaction, which is in turn due to the suppression of high-frequency environment modes. This revival with rapid decay corresponds to the *time-glass* regime. Periodic revival of coherence appears in the deep sub-Ohmic regime $s < 0.1$, which we refer to as the Caldeira-Leggett quantum time crystal regime.

system becomes. Specifically, the purity behaves like $t^{-s/2}$ over the entire sub-Ohmic regime, which originates from the suppression of thermal fluctuations due to fractality. The purity decreases monotonically in the regime $0.5 < s \leqslant 1$, which corresponds to the time-liquid regime. Revival of coherence occurs for $0.1 < s < 0.5$, which corresponds to the time-glass regime. Then, in the deep sub-Ohmic regime with $s < 0.1$, there is a periodic revival of coherence, which we refer to as the Caldeira-Leggett quantum time crystal regime. The revival and oscillatory behavior of the purity can be explained by the dynamics of the environment: we discover that an environment in the sub-Ohmic regime behaves as a gel-like fractal medium with viscosity and elasticity, where the viscosity is associated with the loss tangent $\tan\delta \sim \sin(\pi s/2)$. At $s = 0$, the viscosity of the elastic environment vanishes, and the purity oscillates with a finite frequency without damping.

II. RESULTS

A. Purity of a Gaussian state

The Robertson-Schrödinger uncertainty relation between position $\hat{x}$ and momentum $\hat{p}$ which satisfy $[\hat{x}, \hat{p}] = i\hbar$ is given by [33,34]

$$\sigma_{xx}^2\sigma_{pp}^2 - \sigma_{xp}^4 \geqslant \frac{\hbar^2}{4}, \tag{1}$$

where we have used the notation $\sigma_{uv}(t)^2 = \langle\, \hat{u}(t)\hat{v}(t) + \hat{v}(t)\, \hat{u}(t)\rangle/2 - \langle \hat{u}(t)\rangle\, \langle \hat{v}(t)\rangle$. A weaker form of Eq. (1) is the more famous uncertainty relation $\sigma_{xx}\sigma_{pp} \geqslant \hbar/2$. However, the reason that we consider the stronger form (1) is that it has a clear geometric interpretation. Suppose that the density matrix for a system (S) of interest is given by the Gaussian state

$$\rho_\mathrm{S}(x, y) = \sqrt{\frac{a}{\pi}}\, e^{-a\left(\frac{x+y}{2}\right)^2 - b(y-x)^2 - ic\left(\frac{x+y}{2}\right)(y-x)} \tag{2}$$

with $a = \frac{1}{2\sigma_{xx}^2}$, $b = \frac{\sigma_{xx}^2\sigma_{pp}^2 - \sigma_{xp}^4}{2\sigma_{xx}^2\hbar^2}$, and $c = \frac{\sigma_{xp}^2}{\sigma_{xx}^2\hbar}$. Then, defining the covariance matrix

$$\boldsymbol{\sigma}^2 = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xp}^2 \\ \sigma_{px}^2 & \sigma_{pp}^2 \end{bmatrix}, \tag{3}$$

the Wigner function—the wave function in phase space (position-momentum space)—of this Gaussian state can be written as

$$\begin{aligned} W_\mathrm{S}(x, p) &= \frac{1}{2\pi\hbar}\int dy\, \rho_\mathrm{S}\left(x + \frac{y}{2}, x - \frac{y}{2}\right) e^{-ipy/\hbar} \\ &= \frac{1}{2\pi\sqrt{\det\boldsymbol{\sigma}^2}}\, e^{-\frac{1}{2}Z^T\boldsymbol{\sigma}^{-2}Z}, \end{aligned} \tag{4}$$

where $Z = [x \;\; p]^T$ is the phase-space coordinate. Then, it is clear that

$$D = \sqrt{\det\boldsymbol{\sigma}^2} = \sqrt{\sigma_{xx}^2\sigma_{pp}^2 - \sigma_{xp}^4} \tag{5}$$

is the area of a Gaussian state in phase space, hence Eq. (1) means that the lower bound of the phase-space area is $\hbar/2$.

The purity is a measure of quantum coherence, which has the well-known form in the case of a Gaussian state [35]

$$p(t) = \mathrm{Tr}_{\mathrm{S}}[\hat{\rho}_{\mathrm{S}}(t)^2] = 2\pi\hbar \int dx dp\, W_{\mathrm{S}}(x,p)^2 = \frac{\hbar}{2D(t)}, \tag{6}$$

where $\hat{\rho}_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{E}}\, \hat{\rho}(t)$ is the reduced density matrix of the system, $\hat{\rho}(t)$ is the density matrix for the system plus environment, and Tr_{S} and Tr_{E} are the trace operation with respect to the system and environment degrees of freedom, respectively. For a free Gaussian wave packet with the Hamiltonian $\hat{H} = \hat{p}^2/2M$, one can show that $\sigma_{xx}(t)^2 = \sigma^2 + \hbar^2 t^2/(4M^2\sigma^2)$, $\sigma_{pp}(t)^2 = \hbar^2/(4\sigma^2)$, and $\sigma_{xp}(t)^2 = \hbar^2 t/(4M\sigma^2)$, which saturates the inequality (1), i.e. $\det\boldsymbol{\sigma}(t)^2 = \hbar^2/4$, hence the purity attains its maximum value $p(t) = 1$. On the other hand, the purity decreases if the system is no longer isolated and couples to its environment, where the uncertainty in (1) increases. Therefore, the phase-space area $D(t)$ of a Gaussian state is directly related to the coherence of the system, but the explicit time dependence of $D(t)$ heavily depends on the nature of the environment as we show below.

B. Suppression of decoherence in the sub-Ohmic environment and the Caldeira-Leggett quantum time crystal

The Caldeira-Leggett model is given by the following Hamiltonian [36]

$$\hat{H} = \frac{1}{2M}\hat{p}^2 + \sum_{j=1}^{N} \frac{1}{2}\left[\frac{\hat{P}_j^2}{m_j} + \kappa_j(\hat{R}_j - \hat{x})^2\right], \tag{7}$$

where the interaction between the system S (variables $\hat{x}$ and $\hat{p}$) and its surrounding environment E (variables $\{\hat{R}_j\}$ and $\{\hat{P}_j\}$) is modeled by springs with coefficients κ_j. The equation of motion obtained from the Hamiltonian (7) is the generalized Langevin equation [37]

$$M\ddot{\hat{x}}(t) + M\frac{d}{dt}\int_0^t d\tau\, \gamma(t-\tau)\hat{x}(\tau) = \hat{F}_0(t), \tag{8}$$

which is similar to the Langevin equation that describes Brownian motion under a "stochastic" force $\hat{F}_0(t)$, except that the memory of past interaction with the environment is given by the memory function $\gamma(t)$ which is often assumed to have the following form [22]

$$\gamma(t) = \begin{cases} 2\omega_1\delta(t), & s = 1 \\ \frac{1}{\Gamma(1-s)}\omega_s^{2-s}t^{-s}, & 0 \leqslant s < 1 \end{cases} \tag{9}$$

with $\Gamma(\cdot)$ the gamma function. Equation (9) has been derived using the spectral density $J(\omega) = \frac{\pi}{2}\sum_{j=1}^{N}\omega_j\kappa_j\delta(\omega - \omega_j)$ with the power-law form $J(\omega) \propto \omega^s$ assumed in the continuum limit (see Appendixes A and B for detail). One can see from Eq. (9) that the memory of interactions is lost instantaneously in the Ohmic case ($s = 1$), while the memory of past interactions remains for a longer time in the sub-Ohmic case ($s < 1$). The solution of the system with the assumption Eq. (9) is given by (see Ref. [13] and Appendix B for detail)

$$\hat{x}(t) = \dot{G}(t)\hat{x}(0) + \frac{G(t)}{M}\hat{p}(0) + \frac{1}{M}\int_0^t d\tau\, G(t-\tau)\hat{F}_0(\tau), \tag{10}$$

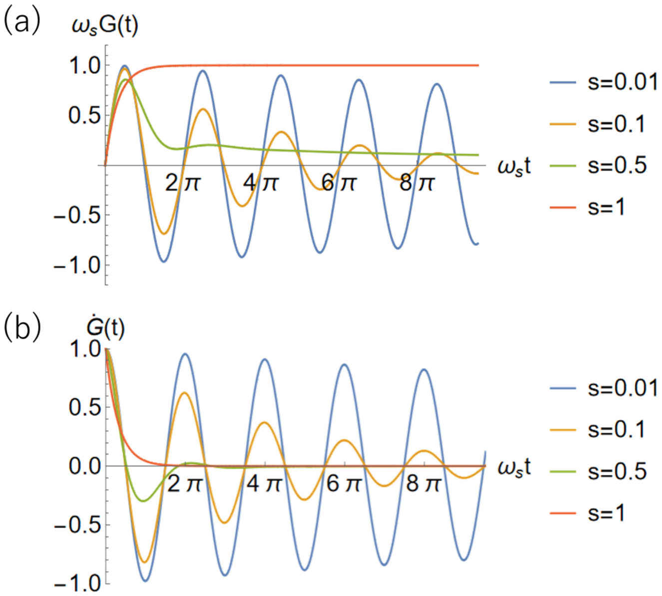

FIG. 2. Fundamental solutions of the system (a) $G(t)$ and (b) $\dot{G}(t)$. The solution at $s = 1$ is that of a particle with friction, while the solution for $s < 0.5$ is similar to that of a damped oscillator, which implies that a sub-Ohmic environment behaves as a viscoelastic medium.

with the fundamental solution given by

$$G(t) = \begin{cases} \frac{1}{\omega_1}(1 - e^{-\omega_1 t}), & s = 1 \\ tE_{2-s,2}[-(\omega_s t)^{2-s}], & 0 < s < 1 \\ \frac{1}{\omega_0}\sin[\omega_0 t], & s = 0 \end{cases}. \tag{11}$$

where $E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}$ is the generalized Mittag-Leffler function, which generalizes the exponential and trigonometric functions [38]. $G(t)$ and $\dot{G}(t)$ are shown in Figs. 2(a) and 2(b), respectively [46].

The solution at $s = 1$ is that of a particle with friction, while the solution for $s < 0.5$ is similar to that of a damped oscillator, the latter being a result of long-lived memory of interaction.

One can show that (see Appendix D)

$$\dot{G}(t) = \mathrm{Re}\frac{\langle\hat{x}(t)\hat{x}(0)\rangle}{\langle x(0)^2\rangle}, \tag{12}$$

so the temporal autocorrelation of the system is given by $\dot{G}(t)$. This autocorrelation rapidly decreases in the regime $0.5 < s \leqslant 1$, reappears in the regime $s < 0.5$, and reappears periodically in the regime $s < 0.1$. These results can be understood as time liquid, time glass, and time crystal. The authors of Refs. [32,39] classify the regime $s < 0.1$ as a time-glass phase. However, in the present work we use the term "time glass" for $s < 0.5$ because the first revival of the autocorrelation already appears at that value.

The Caldeira-Leggett model considers the case $s = 1$, where decoherence is a result of the dissipation of the system's energy into the environment. Therefore, we focus on the sub-Ohmic regime ($s < 1$) and investigate how the resulting oscillatory dynamics influence decoherence. Assuming that the system at $t = 0$ is given by a Gaussian state with width

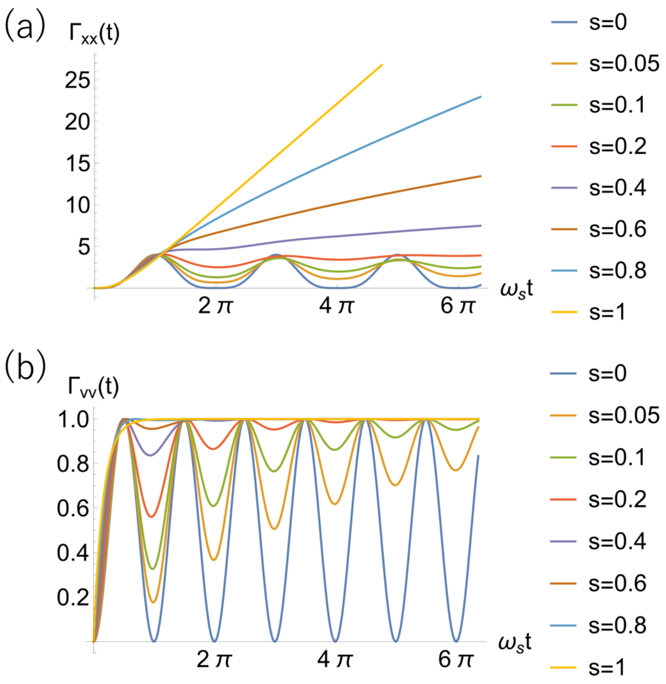

FIG. 3. (a) Plot of $\Gamma_{xx}(t)$ and (b) plot of $\Gamma_{vv}(t)$ with $Mk_{\mathrm{B}}T/\hbar = 1$. $s = 1$ corresponds to the normal Brownian motion with the thermally generated variance increasing as $\Gamma_{xx} \sim t$. $s < 1$ correspond to the fractal Brownian motion with $\Gamma_{xx} \sim t^s$, meaning that thermal fluctuation is reduced in fractal environments.

σ, the time-dependent covariance matrix of the system can be calculated from the solution Eq. (10) and we obtain

$$\boldsymbol{\sigma}(t)^2 = \boldsymbol{\sigma}_Q(t)^2 + \boldsymbol{\sigma}_T(t)^2, \tag{13}$$

where $\boldsymbol{\sigma}_Q(t)^2$ is the quantum covariance matrix that is given by

$$\boldsymbol{\sigma}_Q(t)^2 = \sigma^2 \begin{bmatrix} \dot{G}(t)^2 & M\dot{G}(t)\ddot{G}(t) \\ M\dot{G}(t)\ddot{G}(t) & M^2\ddot{G}(t)^2 \end{bmatrix} + \frac{\hbar^2}{4M^2\sigma^2} \begin{bmatrix} G(t)^2 & MG(t)\dot{G}(t) \\ MG(t)\dot{G}(t) & M^2\dot{G}(t)^2 \end{bmatrix} \tag{14}$$

and $\boldsymbol{\sigma}_T(t)^2$ is the classical thermally generated covariance matrix

$$\boldsymbol{\sigma}_T(t)^2 = \frac{\hbar}{M^2} \begin{bmatrix} \Gamma_{xx}(t) & M\Gamma_{xv}(t) \\ M\Gamma_{vx}(t) & M^2\Gamma_{vv}(t) \end{bmatrix}. \tag{15}$$

In the high-temperature limit $k_{\mathrm{B}}T \gg \hbar\Omega$, where Ω is the ultraviolet cutoff frequency of the environment, we obtain (see Appendix E)

$$\Gamma_{xx}(t) = \frac{k_{\mathrm{B}}TM}{\hbar}\left(2\int_0^t d\tau\, G(\tau) - G(t)^2\right), \tag{16}$$

$$\Gamma_{xv}(t) = \Gamma_{vx}(t) = \frac{k_{\mathrm{B}}TM}{\hbar}[G(t) - G(t)\dot{G}(t)], \tag{17}$$

$$\Gamma_{vv}(t) = \frac{k_{\mathrm{B}}TM}{\hbar}[1 - \dot{G}(t)^2], \tag{18}$$

which are shown in Fig. 3.

For instance, $\Gamma_{xx}(t)$, which is equivalent to the classical thermally generated variance $\langle x(t)^2\rangle_{\mathrm{Classical}}$ of a particle, behaves as $\Gamma_{xx}(t) \sim t$ in the Ohmic case as in the Brownian motion. For the sub-Ohmic case $s < 1$, we observe that $\Gamma_{xx}(t)$ increases as t^s while $\Gamma_{vv}(t)$ is bounded by a constant value. Moreover, for $s < 0.1$, $\Gamma_{xx}(t)$ and $\Gamma_{vv}(t)$ oscillate periodically with angular frequency ω_s. The oscillatory behavior of $\Gamma_{xx}(t)$ in the sub-Ohmic regime was also discovered in previous studies [32,39] in a different context.

Using the following identity for two 2×2 matrices A and B,

$$\det(A + B) = \det(A) + \det(B) + \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB), \tag{19}$$

we obtain

$$\begin{aligned} D(t) &= \sqrt{\det\boldsymbol{\sigma}(t)^2} \\ &= \sqrt{\det\left(\boldsymbol{\sigma}_Q^2\right) + \det\left(\boldsymbol{\sigma}_T^2\right) + \mathrm{tr}\left(\boldsymbol{\sigma}_Q^2\right)\mathrm{tr}\left(\boldsymbol{\sigma}_T^2\right) - \mathrm{tr}\left(\boldsymbol{\sigma}_Q^2\boldsymbol{\sigma}_T^2\right)}. \end{aligned} \tag{20}$$

Therefore, coupling to the environment introduces the thermally generated covariance $\boldsymbol{\sigma}_T^2$ that increases the total phase-space area $D(t)$ of the Gaussian state, thereby lowering its purity.

The result of purity for various s values is shown in Fig. 4(a). First, we observe that the purity decreases as

$$p(t) \sim t^{-s/2}, \tag{21}$$

which originates from the fact that $\Gamma_{xx}(t)$ increases as t^s and $D(t)$ increases as $t^{s/2}$. The purity at $s = 0$ oscillates periodically around a constant value, as shown in Fig. 4(b). This oscillation represents a periodic revival of quantum coherence, namely the Caldeira-Leggett quantum time crystal. The amplitude of oscillation is characterized by the following ratio of characteristic length scales and energy scales:

$$r = \frac{2M\sigma^2 k_{\mathrm{B}}T}{\hbar^2} = 4\pi\frac{\sigma^2}{\lambda_T^2} = \frac{E_{\mathrm{elastic}}E_{\mathrm{thermal}}}{E_{\mathrm{GS}}^2}, \tag{22}$$

where λ_T is the thermal de Broglie wavelength, $E_{\mathrm{elastic}} = M\omega_s^2\sigma^2/2$ is the elastic energy of the system, $E_{\mathrm{thermal}} = k_{\mathrm{B}}T$ is the thermal energy, and $E_{\mathrm{GS}} = \hbar\omega_s/2$ is the ground-state energy of the system. For other values of s, which are shown in Fig. 4(c), the coherence-decoherence oscillation remains with a finite lifetime of the order of $e^{2/s}$ which can be derived from Eq. (21).

C. Fractality and viscoelasticity of the sub-Ohmic environment

The oscillatory behavior of the purity appears due to the long-lived memory of interaction with the environment. Therefore, such oscillations should be understood by analyzing the environment on an equal footing as the system. It was shown recently that the generalized Langevin equation (8) with the assumption Eq. (9) can be rewritten as a fractional differential equation [39]

$$\frac{d^2\hat{x}(t)}{dt^2} + \omega_s^{2-s}\frac{d^s\hat{x}(t)}{dt^s} = \frac{1}{M}\hat{F}_0(t), \tag{23}$$

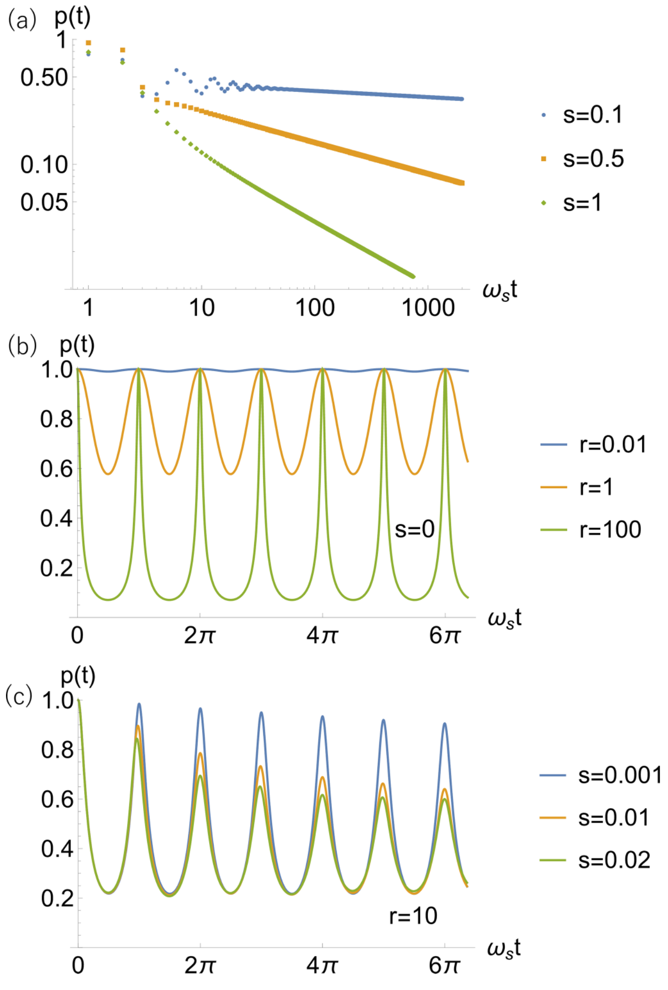

FIG. 4. Purity at the high-temperature limit $k_BT \gg \hbar\Omega$. (a) The purity behaved like $t^{-s/2}$ over the entire sub-Ohmic regime, which originates from the suppression of thermal fluctuations due to fractality. (b) Periodic coherence-decoherence oscillation, namely the Caldeira-Leggett quantum time crystal, is obtained at $s = 0$. (c) For $s < 0.1$, the Caldeira-Leggett crystal has a finite lifetime of the order of $e^{2/s}$.

with the Riemann-Liouville fractional derivative which is defined as [38]

$$\frac{d^s\hat{x}(t)}{dt^s} = \frac{1}{\Gamma(1-s)}\frac{d}{dt}\int_0^t d\tau\ (t-\tau)^{-s}\hat{x}(\tau). \tag{24}$$

Our interpretation of the fractional derivative is that a particle moving in a fractal environment, the fractal dimension of which is larger than the spatial dimension, takes a longer time to move a unit length per unit time [40]. Specifically, the sub-Ohmic parameter s leads to the fractal dimension $D = 2 - s/2$ [41]. A particle in an Ohmic environment moves in a more or less straight path with fluctuations, as shown in Fig. 1. For this reason, interaction with the environment leads to velocity-dependent friction, which is reasonable since the faster the particle moves, the more frequently it interacts with the environmental degrees of freedom, and its energy dissipates faster. On the other hand, a particle at $s = 0$ effectively moves along a two-dimensional space-filling curve, like the Peano curve, and hence localizes in space. In this case, interaction with the environment is independent of the velocity of the particle and depends on its position, leading to an oscillatory potential.

The power-law behavior of the purity can be interpreted as follows. In normal Brownian motion, the classical variance of a particle due to thermal fluctuations increases linearly like $\sigma_{xx} \sim t$. Adding thermal fluctuations to quantum fluctuations increases the determinant of the covariance matrix and lowers the purity, in agreement with the Caldeira-Leggett model [36]. On the other hand, for fractal Brownian motion, the classical variance deviates from linear increase as $\sigma_{xx} \sim t^{2H}$ [42]. If $H < 1/2$, then the increments of the Wiener process are negatively correlated, meaning that future increments tend to be smaller. Assuming that the thermally generated variance is the same as the classical variance in fractal Brownian motion, we obtain $s = 2H$. In other words, the effect of thermal fluctuation is small if the environment is fractal, and the coherence lifetime increases.

The sub-Ohmic environment being fractal means that this environment must be treated as a gel-like system with elasticity and viscosity. In fact, one can use Fourier transform to solve Eq. (23) and obtain the following particular solution of $x(t)$:

$$\hat{x}(t) = \frac{1}{M}\sum_{j=1}^{N} A_j\left(\hat{R}_j(0)\cos[\omega_j t - \delta_j] + \frac{\hat{P}_j(0)}{m_j}\frac{\sin[\omega_j t + \delta]}{\omega_j}\right), \tag{25}$$

with

$$A_j = \frac{1}{\omega_j\sqrt{\left(1 - \omega_s^{2-s}\omega_j^{s-2}\cos\frac{\pi s}{2}\right)^2 + \left(\omega_s^{2-s}\omega_j^{s-2}\sin\frac{\pi s}{2}\right)^2}}, \tag{26}$$

$$\tan\delta_j = \frac{(\omega_s/\omega_j)^{2-s}\sin\frac{\pi s}{2}}{1 - (\omega_s/\omega_j)^{2-s}\cos\frac{\pi s}{2}}. \tag{27}$$

The nonzero value of δ_j means that the system responds to the environmental force with a delay, which indicates the presence of a viscosity. In addition, we obtain $\delta_j = 0$ at $s = 0$, which means that viscosity vanishes and the environment behaves as an elastic medium.

The oscillatory behavior can also be understood by examining the Hamiltonian (7). The system is connected by springs in parallel with the environmental coordinates, so the solution of the system is a linear superposition of oscillation due to each environmental degree of freedom R_j. In the Ohmic regime, the system responds to the system with a delay given by Eq. (27), which is interpreted as the viscosity of the environment, that is, the environment moves together with the system. In the deep sub-Ohmic regime, the system responds immediately to the environment, meaning that the viscosity vanishes and the environment is an elastic medium. In this case, the environment can be considered as being fixed in space and the system oscillates with the angular frequency $\omega_s \approx \sqrt{\gamma(0)}$.

III. DISCUSSION

In this article, we showed that a sub-Ohmic environment should be understood as a gel-like fractal system with viscosity and elasticity. Consequently, the purity of a quantum system coupled to a sub-Ohmic environment behaves as $t^{-s/2}$, meaning that coherence time becomes longer in the sub-Ohmic regime. In the deep sub-Ohmic regime, we obtained a periodic oscillation of the purity, namely the Caldeira-Leggett quantum time crystal. The persistence and revival of quantum coherences have also been reported in the sub-Ohmic spin-boson model (see, for instance, Ref. [43] and references therein), which, however, was not attributed to intrinsic dynamics of the environment. The fact that two very different systems (two-level system and free boson) produce the same qualitative behavior strongly suggests that coherence revival is a universal hallmark of a deep sub-Ohmic environment.

We now compare our results with previous studies on time crystals. First, time crystals are usually discussed as phases that spontaneously break time-translation symmetry in the thermodynamic limit, where an infinitesimal interaction with the environment (the detail of which is irrelevant) results in spontaneous temporal oscillation of a physical observable [24,44]. In the present work we focus on the nature of the environment itself and show that the environment is an active ingredient that controls both the emergence and the lifetime of coherent oscillations. Second, time crystalline dynamics was also suggested in fractional Langevin systems with two environments [32]. However, this study analyzed mean-squared displacements in a semiclassical framework, and quantum coherence was not considered. In contrast, our study reveals that long-lived quantum time crystals with coherence can be obtained by decreasing s. Our results demonstrate that coherence revival is a genuinely quantum effect that is absent in purely classical observables. Third, our work may be extended to other models of time crystals, where the periodic revival of coherence appears as a "superlattice" of a time crystal, which should be discussed in future studies.

Our results indicate that quantum decoherence can be reduced using a gel-like medium, and decoherence-based dynamics appear in the deep sub-Ohmic regime, which has potential application in quantum information and quantum biology. To the best of our knowledge, the physical properties of the environment are currently not actively considered in discussions of decoherence, but decoherence should be reconsidered from the perspective of solid-state properties of the environment, such as elasticity and viscosity.

In particular, the intracellular environment, such as cytosol composed of gel-like substances, contains many mysteries about its nature. This high-density gel-like protein environment can be characterized by its fractal nature, resulting in a sub-Ohmic spectral function [17–21]. In fact, recent studies suggest that the non-Markovianity of this environment, the non-Ohmicity in our case, improves quantum coherence *in vivo* [8,9]. Could living systems have evolved by choosing an environment with smaller s to use quantum effects more efficiently? The use of sub-Ohmicity may be a vital principle to be investigated in future studies.

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All authors have contributed equally.

The authors declare no competing interests.

DATA AVAILABILITY

The data that support the findings of this article are available from the authors upon reasonable request.

APPENDIX A: EQUATIONS OF MOTION

Let $\hat{A}(t) = e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}$ denote the Heisenberg time evolution of an operator $\hat{A}$. Then, the Heisenberg equations of motion obtained from the Hamiltonian (7) are given by

$$\frac{d\hat{x}(t)}{dt} = \frac{1}{M}\hat{p}(t), \quad \frac{d\hat{p}(t)}{dt} = \sum_{j=1}^{N}\kappa_j\hat{R}_j(t) - \sum_{j=1}^{N}\kappa_j\hat{x}(t), \tag{A1}$$

$$\frac{d\hat{R}_j(t)}{dt} = \frac{1}{m_j}\hat{P}_j(t), \quad \frac{d\hat{P}_j(t)}{dt} = -\kappa_j\hat{R}_j(t) + \kappa_j\hat{x}(t). \tag{A2}$$

Suppose that the system and the environment start to interact at $t = 0$, then Eq. (A2) has the following solution:

$$\hat{R}_j(t) = \hat{R}_j\cos[\omega_j t] + \hat{P}_j\frac{\sin[\omega_j t]}{m_j\omega_j} + \omega_j\int_0^t d\tau\,\sin[\omega_j(t-\tau)]\hat{x}(\tau) \tag{A3}$$

where $\omega_j = \sqrt{\kappa_j/m_j}$, and insert Eq. (A3) into Eq. (A1) to obtain

$$M\frac{d^2\hat{x}(t)}{dt^2} = -\sum_{j=1}^{N}\kappa_j\frac{d}{dt}\int_0^t d\tau\,\cos[\omega_j(t-\tau)]\hat{x}(\tau) + \sum_{j=1}^{N}\kappa_j\left(\hat{R}_j\cos[\omega_j t] + \hat{P}_j\frac{\sin[\omega_j t]}{m_j\omega_j}\right) \tag{A4}$$

which can be written as the generalized Langevin equation

$$\frac{d^2\hat{x}(t)}{dt^2} + \frac{d}{dt}\int_0^t d\tau\gamma(t-\tau)\hat{x}(\tau) = \frac{1}{M}\hat{F}_0(t), \tag{A5}$$

where $\gamma(t-\tau)$ is the memory function defined as

$$\gamma(t-\tau) = \sum_{j=1}^{N}\frac{\kappa_j}{M}\cos[\omega_j(t-\tau)], \tag{A6}$$

and $\hat{F}_0(t)$ is given by

$$\hat{F}_0(t) = \sum_{j=1}^{N}\kappa_j\left(\hat{R}_j\cos[\omega_j t] + \hat{P}_j\frac{\sin[\omega_j t]}{m_j\omega_j}\right). \tag{A7}$$

APPENDIX B: SOLUTION

Equation (A5) can be solved using Laplace transform $\tilde{f}(z) = \mathcal{L}_z f = \int_0^\infty e^{-zt}f(t)dt$:

$$\hat{x}(t) = \hat{x}G_2(t) + \frac{\hat{p}}{M}G_1(t) + \frac{1}{M}\int_0^t d\tau G_1(t-\tau)\hat{F}_0(\tau), \tag{B1}$$

where $\tilde{G}_1(z) = \frac{1}{z^2+z\tilde{\gamma}(z)}$ and $\tilde{G}_2(z) = \frac{z}{z^2+z\tilde{\gamma}(z)}$. One needs an explicit form of $\tilde{\gamma}(z)$ to calculate the inverse Laplace transform of $\tilde{G}_1(z)$ and $\tilde{G}_2(z)$, which can be achieved by defining the spectral function

$$J(\omega) = \frac{\pi}{2}\sum_{j=1}^{N}\omega_j\kappa_j\delta(\omega-\omega_j), \tag{B2}$$

such that the memory function can be rewritten as

$$\gamma(t-\tau) = \frac{2}{M\pi}\int_0^\infty d\omega\frac{J(\omega)}{\omega}\cos[\omega(t-\tau)]. \tag{B3}$$

The power-law spectrum $J(\omega) = Mg_s\omega^s$ with an ultraviolet cutoff frequency Ω_U is often assumed in the continuum limit. Then, the memory function can be written as

$$\gamma(t-\tau) = \begin{cases} 2g_1\frac{\sin[\Omega_U(t-\tau)]}{\pi(t-\tau)}, & s=1 \\ \frac{2g_s}{\pi}\,{}_1F\,2\left(\frac{s}{2};\frac{1}{2},1+\frac{s}{2};-\frac{1}{4}(t-\tau)^2\Omega_U^2\right), & 0<s<1 \end{cases} \tag{B4}$$

$$\approx \begin{cases} 2\omega_1\delta(t-\tau), & s=1 \\ \frac{1}{\Gamma(1-s)}\omega_s^{2-s}|t-\tau|^{-s}, & 0<s<1 \end{cases} \tag{B5}$$

where ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is the generalized hypergeometric function, $\Gamma(\cdot)$ is the gamma function, the limit $\Omega_U\to\infty$ is taken in the last line, and we define $\omega_s = (\frac{g_s}{\sin\frac{\pi s}{2}})^{1/(2-s)}$. The divergence of ω_s at $s=0$ is a consequence of infrared divergence, which can be removed by introducing an infrared cutoff frequency Ω_I, yielding $\omega_0^2 = \gamma(0)|_{s=0} = \frac{2g_0}{\pi}\log(\Omega_U/\Omega_I)$ as we explain later. The Laplace transform of the memory function is then given by the approximate value

$$\tilde{\gamma}(z) \approx \begin{cases} \omega_1, & s=1 \\ \omega_s^{2-s}z^{s-1}, & 0<s<1 \end{cases}, \tag{B6}$$

which implies $G_1(t) = G(t)$ and $G_2(t) = \dot{G}(t)$, hence we obtain

$$\hat{x}(t) = \hat{x}\dot{G}(t) + \frac{\hat{p}}{M}G(t) + \frac{1}{M}\int_0^t d\tau G(t-\tau)\hat{F}_0(\tau), \tag{B7}$$

$$\hat{p}(t) = M\hat{x}\ddot{G}(t) + \hat{p}\dot{G}(t) + \int_0^t d\tau\dot{G}(t-\tau)\hat{F}_0(\tau), \tag{B8}$$

where the second equation follows from $G(0)=0$.

APPENDIX C: GENERALIZED LANGEVIN EQUATION AS A FRACTIONAL DIFFERENTIAL EQUATION

One can also insert Eq. (B5) into Eq. (A5) to obtain

$$\frac{d^2\hat{x}(t)}{dt^2} + \omega_1\frac{d\hat{x}(t)}{dt} = \frac{1}{M}\hat{F}_0(t), \quad (s=1) \tag{C1}$$

$$\frac{d^2\hat{x}(t)}{dt^2} + \omega_s^{2-s}\frac{d^s\hat{x}(t)}{dt^s} = \frac{1}{M}\hat{F}_0(t), \quad (0\leqslant s<1) \tag{C2}$$

where $\frac{d^s\hat{x}(t)}{dt^s} = \frac{1}{\Gamma(1-s)}\frac{d}{dt}\int_0^t d\tau(t-\tau)^{-s}\hat{x}(\tau)$ is the Riemann-Liouville fractional derivative.

APPENDIX D: AUTOCORRELATIONS

From the solutions (A3), (B7), and (B8) we can obtain various expectation values and correlations. Assuming that the environment at $t=0$ is given by a Gaussian state with $\langle\hat{R}_j\rangle = \langle\hat{P}_j\rangle = 0$, we obtain $\langle\hat{F}_0(\tau)\rangle = 0$. We also obtain various autocorrelations

$$\langle\hat{x}(t)\hat{x}(t')\rangle = \dot{G}(t)\dot{G}(t')\langle\hat{x}^2\rangle + \frac{\dot{G}(t)G(t')}{M}\langle\hat{x}\hat{p}\rangle + \frac{G(t)\dot{G}(t')}{M}\langle\hat{p}\hat{x}\rangle + \frac{G(t)G(t')}{M^2}\langle\hat{p}^2\rangle + \frac{\hbar}{M^2}\Gamma_{xx}(t,t'),$$

$$\langle\hat{x}(t)\hat{p}(t')\rangle = M\dot{G}(t)\ddot{G}(t')\langle\hat{x}^2\rangle + \dot{G}(t)\dot{G}(t')\langle\hat{x}\hat{p}\rangle + G(t)\ddot{G}(t')\langle\hat{p}\hat{x}\rangle + \frac{G(t)\dot{G}(t')}{M}\langle\hat{p}^2\rangle + \frac{\hbar}{M}\Gamma_{xv}(t,t'),$$

$$\langle\hat{p}(t)\hat{x}(t')\rangle = M\ddot{G}(t)\dot{G}(t')\langle\hat{x}^2\rangle + \ddot{G}(t)G(t')\langle\hat{x}\hat{p}\rangle + \dot{G}(t)\dot{G}(t')\langle\hat{p}\hat{x}\rangle + \frac{\dot{G}(t)G(t')}{M}\langle\hat{p}^2\rangle + \frac{\hbar}{M}\Gamma_{vx}(t,t'),$$

$$\langle\hat{p}(t)\hat{p}(t')\rangle = M^2\ddot{G}(t)\ddot{G}(t')\langle\hat{x}^2\rangle + M\ddot{G}(t)\dot{G}(t')\langle\hat{x}\hat{p}\rangle + M\dot{G}(t)\ddot{G}(t')\langle\hat{p}\hat{x}\rangle + \dot{G}(t)\dot{G}(t')\langle\hat{p}^2\rangle + \hbar\Gamma_{vv}(t,t'),$$

where we define

$$\Gamma_{xx}(t,t') = \frac{1}{\hbar}\int_0^t d\tau\int_0^{t'} d\tau' G(t-\tau)\langle\hat{F}_0(\tau)\hat{F}_0(\tau')\rangle G(t'-\tau'), \tag{D1}$$

$$\Gamma_{xv}(t,t') = \frac{1}{\hbar}\int_0^t d\tau\int_0^{t'} d\tau' G(t-\tau)\langle\hat{F}_0(\tau)\hat{F}_0(\tau')\rangle \dot{G}(t'-\tau'), \tag{D2}$$

$$\Gamma_{vx}(t,t') = \frac{1}{\hbar}\int_0^t d\tau\int_0^{t'} d\tau' \dot{G}(t-\tau)\langle\hat{F}_0(\tau)\hat{F}_0(\tau')\rangle G(t'-\tau'), \tag{D3}$$

$$\Gamma_{vv}(t,t') = \frac{1}{\hbar}\int_0^t d\tau\int_0^{t'} d\tau' \dot{G}(t-\tau)\langle\hat{F}_0(\tau)\hat{F}_0(\tau')\rangle \dot{G}(t'-\tau'). \tag{D4}$$

The correlation $\langle \hat{F}_0(\tau)\hat{F}_0(\tau')\rangle$ can be calculated using standard methods: Defining the creation operators $\hat{a}_j = \sqrt{\frac{m_j\omega_j}{2\hbar}}(\hat{R}_j + \frac{i}{m_j\omega_j}\hat{P}_j)$ and the annihilation operator $\hat{a}_j^\dagger = \sqrt{\frac{m_j\omega_j}{2\hbar}}(\hat{R}_j - \frac{i}{m_j\omega_j}\hat{P}_j)$, one can rewrite the environment Hamiltonian as $\hat{H}_\mathrm{E} = \sum_j \hbar\omega_j(\hat{a}_j^\dagger\hat{a}_j + 1/2)$ and the density matrix of the environment at temperature T is given by $\hat{\rho}_\mathrm{E} = e^{-\beta\hat{H}_\mathrm{E}}/\mathrm{Tr}[e^{-\beta\hat{H}_\mathrm{E}}]$ where $\beta = 1/k_\mathrm{B}T$. Then, using $\hat{F}_0(t) = \sum_j \kappa_j\sqrt{\frac{\hbar}{2m_j\omega_j}}(\hat{a}_j e^{-i\omega_j t} + \hat{a}_h^\dagger e^{i\omega_j t})$, $\langle \hat{a}_j\hat{a}_k\rangle = \langle \hat{a}_j^\dagger\hat{a}_k^\dagger\rangle = 0$, $\langle \hat{a}_j^\dagger\hat{a}_k\rangle = \delta_{jk}\langle n_j\rangle$ and $\langle \hat{a}_j\hat{a}_k^\dagger\rangle = \delta_{jk}(1 + \langle n_j\rangle)$, where $\langle n_j\rangle = 1/(e^{\hbar\beta\omega_j} - 1)$, one can show that

$$\langle \hat{F}_0(\tau)\hat{F}_0(\tau')\rangle = \hbar[\alpha_\mathrm{R}(\tau-\tau') + i\alpha_\mathrm{I}(\tau-\tau')], \tag{D5}$$

where

$$\alpha_\mathrm{R}(\tau-\tau') = \sum_j \frac{\omega_j\kappa_j}{2}\cos[\omega_j(\tau-\tau')]\coth\left[\frac{\hbar\beta\omega_j}{2}\right], \tag{D6}$$

$$\alpha_\mathrm{I}(\tau-\tau') = -\sum_j \frac{\omega_j\kappa_j}{2}\sin[\omega_j(\tau-\tau')], \tag{D7}$$

which can also be written using the spectral function Eq. (B2):

$$\alpha_\mathrm{R}(\tau-\tau') = \frac{1}{\pi}\int_0^\infty J(\omega)\cos[\omega(\tau-\tau')]\coth\left[\frac{\hbar\beta\omega}{2}\right]d\omega, \tag{D8}$$

$$\alpha_\mathrm{I}(\tau-\tau') = -\frac{1}{\pi}\int_0^\infty J(\omega)\sin[\omega(\tau-\tau')]d\omega. \tag{D9}$$

At $t = t'$ the equal-time thermal correlations are defined as follows:

$$\Gamma_{xx}(t) = \Gamma_{xx}(t,t), \quad \Gamma_{xv}(t) = \Gamma_{xv}(t,t), \quad \Gamma_{vx}(t) = \Gamma_{vx}(t,t), \quad \Gamma_{vv}(t) = \Gamma_{vv}(t,t). \tag{D10}$$

APPENDIX E: LAPLACE TRANSFORM OF THERMAL CORRELATIONS

The integrals in Eqs. (D1)–(D4) can be calculated explicitly using double Laplace transform as in Ref. [45]. For instance, we have

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{xx}(t,t')]] = \int_0^\infty dt\int_0^\infty dt'\int_0^t d\tau\int_0^{t'} d\tau' e^{-zt}e^{-z't'}G(t-\tau)\alpha(\tau-\tau')G(t-\tau'). \tag{E1}$$

Using $\int_0^\infty dt\int_0^t d\tau = \int_0^\infty d\tau\int_\tau^\infty dt$ and introducing new variables $u = t-\tau$, $u' = t'-\tau'$, $\sigma = \tau-\tau'$, we obtain

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{xx}(t,t')]] = \int_0^\infty d\tau\int_0^\infty d\tau'\int_0^\infty du\int_0^\infty du' e^{-zu}e^{-z\tau}e^{-z'u'}e^{-z'\tau'}G(u)\alpha(\tau-\tau')G(u') \tag{E2}$$

$$= \tilde{G}(z)\int_0^\infty d\tau\int_0^\infty d\tau' e^{-z\tau}e^{-z'\tau'}\alpha(\tau-\tau')\tilde{G}(z'), \tag{E3}$$

$$= \tilde{G}(z)\int_0^\infty d\tau'\int_0^\infty d\sigma e^{-z\sigma}e^{-(z+z')\tau'}\alpha(\sigma)\tilde{G}(z') \tag{E4}$$

$$+ \tilde{G}(z)\int_0^\infty d\tau'\int_{-\tau'}^0 d\sigma e^{-z\sigma}e^{-(z+z')\tau'}\alpha(\sigma)\tilde{G}(z') \tag{E5}$$

$$= \tilde{G}(z)\frac{\tilde{\alpha}(z) + \tilde{\alpha}(z')}{z+z'}\tilde{G}(z'), \tag{E6}$$

where integration by parts has been used in the last line. Similarly, one can show that

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{x,v}(t,t')]] = \tilde{G}(z)\frac{\tilde{\alpha}(z) + \tilde{\alpha}(z')}{z+z'}\tilde{\dot{G}}(z'), \tag{E7}$$

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{v,x}(t,t')]] = \tilde{\dot{G}}(z)\frac{\tilde{\alpha}(z) + \tilde{\alpha}(z')}{z+z'}\tilde{G}(z'), \tag{E8}$$

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{v,v}(t,t')]] = \tilde{\dot{G}}(z)\frac{\tilde{\alpha}(z) + \tilde{\alpha}(z')}{z+z'}\tilde{\dot{G}}(z'). \tag{E9}$$

Note that the Laplace components $\tilde{\gamma}(z)$ and $\tilde{\alpha}_{\mathrm{R}}(z)$ are related by the fluctuation-dissipation theorem [22]

$$\tilde{\alpha}_{\mathrm{R}}(z) = \frac{M}{2} z \cot\left(\frac{\hbar z}{2k_{\mathrm{B}}T}\right)\tilde{\gamma}(z). \tag{E10}$$

For the high-temperature limit $k_{\mathrm{B}}T \gg \hbar\Omega$ we have

$$\tilde{\alpha}_{\mathrm{R}}(z) = \frac{k_{\mathrm{B}}TM}{\hbar}\tilde{\gamma}(z), \tag{E11}$$

which implies

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{xx}(t,t')]] = \frac{k_{\mathrm{B}}TM}{\hbar}\tilde{G}(z)\frac{\tilde{\gamma}(z)+\tilde{\gamma}(z')}{z+z'}\tilde{G}(z'). \tag{E12}$$

Now, using the relations $-z\tilde{G}(z)\tilde{\gamma}(z) = z\tilde{\dot{G}}(z) - 1$ we obtain

$$\mathcal{L}_z[\mathcal{L}_{z'}[\Gamma_{xx}(t,t')]] = \frac{k_{\mathrm{B}}TM}{\hbar}\left(\frac{z^{-1}\tilde{G}(z') + z'^{-1}\tilde{G}(z)}{z+z'} - \tilde{G}(z)\tilde{G}(z')\right). \tag{E13}$$

The inverse Laplace transform gives

$$\Gamma_{xx}(t,t') = \frac{k_{\mathrm{B}}TM}{\hbar}\left(\int_0^t d\tau\,[G(\tau) - G(\tau - t')\Theta(\tau - t')]\int_0^{t'} d\tau'[G(\tau') - G(\tau' - t)\Theta(\tau' - t)] - G(t)G(t')\right), \tag{E14}$$

$$\Gamma_{xv}(t,t') = \frac{k_{\mathrm{B}}TM}{\hbar}[G(t - t')\Theta(t - t') + G(t') - G(t' - t)\Theta(t' - t) - G(t)\dot{G}(t')], \tag{E15}$$

$$\Gamma_{vx}(t,t') = \frac{k_{\mathrm{B}}TM}{\hbar}[G(t) - G(t - t')\Theta(t - t') + G(t' - t)\Theta(t' - t) - \dot{G}(t)G(t')], \tag{E16}$$

$$\Gamma_{vv}(t,t') = \frac{k_{\mathrm{B}}TM}{\hbar}[\dot{G}(t - t')\Theta(t - t') + \dot{G}(t' - t)\Theta(t' - t) - \dot{G}(t)\dot{G}(t')]. \tag{E17}$$

At $t = t'$ we obtain Eqs. (16)–(18).

APPENDIX F: INFRARED RENORMALIZATION AT $s = 0$

Using the spectral function and the power-law spectrum $J(\omega) = Mg_s\omega^s$ with an ultraviolet cutoff frequency Ω_U, we obtain

$$\gamma(t - \tau) = \frac{2g_s}{\pi}\int_0^{\Omega_U} d\omega\omega^{s-1}\cos[\omega(t-\tau)] \approx \frac{g_s}{\sin\frac{\pi s}{2}}\frac{|t-\tau|^{-s}}{\Gamma(1-s)}. \tag{F1}$$

This integral diverges as $s \to 0$, which is a result of infrared divergence due to zero-frequency modes. Therefore, we introduce an infrared cutoff frequency Ω_I and obtain the following result:

$$\gamma(t) = \frac{2g_s}{\pi}\int_{\Omega_I}^{\Omega_U} d\omega\omega^{s-1}\cos\omega t = \frac{2g_s}{\pi}t^{-s}\mathrm{Re}\{i^{-s}[\gamma(s, i\Omega_U t) - \gamma(s, i\Omega_I t)]\} \tag{F2}$$

where $\gamma(s,a) = \int_a^\infty z^{s-1}e^{-z}$ is the incomplete gamma function, which should be distinguished from the memory function $\gamma(t)$, and at $t = 0$ we obtain

$$\gamma(0) = \frac{2g_s}{\pi}\int_{\Omega_I}^{\Omega_U} d\omega\omega^{s-1} = \frac{2g_s}{\pi}\frac{\Omega_U^s - \Omega_I^s}{s} \quad (s > 0), \tag{F3}$$

$$\gamma(0) = \frac{2g_0}{\pi}\int_{\Omega_I}^{\Omega_U} d\omega\omega^{0-1} = \frac{2g_0}{\pi}\log\frac{\Omega_U}{\Omega_I} \quad (s = 0). \tag{F4}$$

The factor $\log\frac{\Omega_U}{\Omega_I}$ is typically of the order of 10.

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