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Catalysis in quantum information theory

Patryk Lipka-Bartosiko

Department of Applied Physics, University of Geneva, 1211 Geneva, Switzerland

Henrik Wilming

Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany

Nelly H. Y. Ngo

School of Physical and Mathematical Sciences, Nanyang Technological University, 639673 Singapore

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Catalysts open up new reaction pathways that can speed up chemical reactions while not consuming the catalyst. A similar phenomenon has been discovered in quantum information science, where physical transformations become possible by utilizing a quantum degree of freedom that returns to its initial state at the end of the process. In this review, a comprehensive overview of the concept of catalysis in quantum information science is presented and its applications in various physical contexts are discussed.

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I. INTRODUCTION

Puzzles have been around since the dawn of human history and have guided countless discoveries. A good puzzle is easy



FIG. 1. Towers of Hanoi. The puzzle involves k rods that can hold disks of different sizes. The objective is to move a stack of n > 1 disks from the first rod to the second rod, one disk at a time, without placing a larger disk on top of a smaller disk. The puzzle is solvable when k = 3, but it cannot be solved when k = 2.

to formulate but challenging to solve, while its solution can lead to deep insights. This review is devoted to quantum catalysis, a puzzle that has propagated to various areas of quantum information theory. As a warm-up, we first consider another puzzle, known as the towers of Hanoi. This brainteaser helps us to elucidate some of the fundamental concepts of catalysis without having to deal with the complexities of quantum physics.

The towers of Hanoi is a mathematical puzzle that involves k rods and n disks of different sizes. At the beginning, the disks are arranged on a single rod in decreasing order of size, with the smallest disk on top; see Fig. 1. The goal of the puzzle is to find the minimal number of moves needed to transfer the entire stack of disks from one rod to another, given that each disk can be placed only on top of a larger disk.

When the number of disks n is greater than 1, the puzzle requires at minimum k = 3 rods to be feasible, in which case the minimal number of moves needed to transfer the disks from one rod to another is $2^n - 1$ (Petkovic, 2009). Removing one of the three rods makes the puzzle impossible to solve since the set of allowed moves becomes too restrictive. However, adding a fourth rod allows the puzzle to be solved in only six moves, which is less than the minimal number of moves (seven) required in the case of n = 3 (Bousch, 2014). The minimal number of moves for more than four rods remains an open problem.

We can refer to the arrangement of disks on each rod as the *state* of the rod. When a new rod is introduced, the problem becomes more complex due to the increased number of potential states. Crucially, to solve the puzzle the newly added rod must ultimately return to its initial state (i.e., remain empty). In more technical terms, adding extra degrees of freedom expands the configuration space of the problem. The key point to remember is that even though the additional degrees of freedom return to their initial states in the end, expanding the configuration space allows the puzzle to be solved in a manner that was previously unattainable. This phenomenon can be seen as an instance of *catalysis*. Its analog in quantum information science is the main focus of this review.

The towers of Hanoi puzzle may serve as a simplified representation of a physical scenario. Yet, it contains several essential ingredients that we later encounter in this review. First, the possible states of each rod in the puzzle are analogous to the potential states of a physical system (for example, an atom). Second, the puzzle's rules correspond to limitations on how the physical system can be manipulated (for example, due to energy conservation). Third, the puzzle's objective can be viewed as a state-transition problem, which involves determining whether a physical system in one state can be transformed into another (for example, transforming a hot atom into a colder one). The rods in the puzzle represent various physical systems utilized to accomplish the task. Finally, despite its seemingly straightforward description, finding the minimum number of moves required for k > 4 remains an unsolved problem. This demonstrates the complexity and richness of the puzzle.

The laws of physics place fundamental constraints on what is possible, much like the rules of the towers of Hanoi puzzle. For example, conservation laws tell us that certain measurable properties of isolated physical systems remain constant as the system evolves over time. These laws are essential to our understanding of the physical world in that they describe which processes can or cannot occur in nature. Since we cannot globally create or destroy conserved quantities, such as mass energy, we can change them locally only by introducing additional degrees of freedom that store or inject these physical quantities. In the same way, the time-translation symmetry of physical processes means that we cannot create a state that violates time-translation invariance without accessing a system that already breaks it. Clocks are an example of such systems: They use an internal asymmetric flow of information to distinguish a preferred direction of time.

The situation becomes more intricate when quantum theory is considered the fundamental description of nature. When all the involved degrees of freedom are taken into account, the overall dynamics is ultimately unitary. This combined with conversation laws imposes nontrivial constraints that are not immediately evident from conservation laws alone. For instance, consider a probabilistic mixture of spin-up and spin-down of a spin-1/2 particle in a magnetic field. Although this mixture has the same average energy as a superposition of the two states, we cannot transform the mixture into the superposition using unitary transformations. This is true regardless of the fact that both states have the same expected energy, and regardless of whether we have access to a clock; see also Bartlett, Rudolph, and Spekkens (2006).

To give another example, the relativistic principle of locality states that we cannot instantaneously affect spacelike separated systems. Entanglement can be described as the property between such systems that cannot be created using only local actions and classical communication. To overcome the limitations imposed by the relativistic locality, additional resources (such as preshared entanglement) are required. Yet, even if a quantum state ψ contains more entanglement than ϕ , it does not necessarily mean that ψ can be converted into ϕ using local operations and classical communication. In this case, the interplay between operational restrictions gives rise to a rich mathematical structure that requires more refined ways of quantifying entanglement; see Horodecki *et al.* (2009).

Using auxiliary degrees of freedom to lift restrictions imposed by physical laws often comes at a cost: Such degrees of freedom typically lose their ability to lift restrictions, as seen in the previous examples. For instance, a clock used to break time-reversal symmetry ideally produces a series of evenly distributed ticks. However, this process has an intrinsic backaction on the clock, driving it closer to equilibrium with each tick and gradually decreasing its time-keeping potential. To counteract this mechanism, clocks must continuously consume nonequilibrium resources; see Cao *et al.* (2015), Barato and Seifert (2016), Erker *et al.* (2017), and Milburn (2020). Similarly, the use of preshared entanglement in the form of an ancillary quantum state to entangle two distant physical systems results in the loss of entanglement of the ancillary state itself (Lo and Popescu, 2001).

It is surprising that some restrictions imposed by unitarity of quantum mechanics can often be lifted using auxiliary degrees of freedom that do not degrade; i.e., their quantum state remains identical before and after the process. Nevertheless, their presence allows for physical transformations that would otherwise be impossible to accomplish.

This phenomenon, which has been named quantum catalysis, demonstrates that the presence of quantum states opens up new and interesting possibilities. The name *quantum catalysis* is derived from the analogy to chemistry, where catalysts allow for chemical reactions at higher rates by opening up new dynamical pathways with lower activation energies. In a similar manner, quantum catalysts open up new dynamical pathways in the Hilbert space, connecting quantum states that would otherwise be dynamically disconnected due to physical constraints. Understanding when such alternative connections can be formed not only helps us to better understand the fundamental limitations of quantum physics but also provides protocols exploiting quantum resources more efficiently.

To date several reviews discussing quantum catalysis in brevity have been published. Horodecki et al. (2009) mentioned the usage of catalysis in the context of entanglement transformations. Goold et al. (2016), Vinjanampathy and Anders (2016), and Lostaglio (2019) discussed some aspects of catalysis from a thermodynamic point of view. Gour et al. (2015) and Chitambar and Gour (2019) discussed the subject for generic resource theories; see Sec. II.A. What all of these works have in common is that they discussed quantum catalysis only as a supporting concept, i.e., as an extension of existing protocols like resource conversion or work extraction. A similar treatment of quantum catalysis was manifested in numerous prominent works that used catalytic effects as a central proof technique; see Bose, Vedral, and Knight (1999), Groisman, Popescu, and Winter (2005), Bartlett, Rudolph, and Spekkens (2007), Berta, Christandl, and Renner (2011), and Bennett et al. (2014). This approach, however, does not emphasize the role of catalysis as a standalone concept.

This review gathers our current understanding of quantum catalysis and its applications across various fronts of quantum physics. During the preparation of this review, another review on catalysis in the context of quantum resource theories appeared as a preprint (Datta, Kondra *et al.*, 2023). It can be considered complementary reading.

Structure of the review. In the remaining part of this section we establish the basic notation used throughout the review. Section II introduces the basic concept of catalysis as understood in this review in a general manner. This is followed by a discussion of the concept of resource theories and a summary of basic mathematical tools used throughout the review. In Sec. III, we provide a detailed introduction to the various types of catalysis that can arise by relaxing constraints discussed in Sec. II in different ways. We also offer an example to illustrate the different types in detail. Subsequently, Sec. IV explores the known ways to mathematically construct catalyst states. The core of the review is Sec. V, which collects and discusses the applications of catalysis in various physical settings. Section VI collects additional applications and discussions that do not naturally fit into the previous sections. However, this does not imply that we consider them less important or less interesting. Open problems and avenues for future work are mentioned throughout the review. Owing to their diverse nature, it would be difficult to collect them in a conclusions section, and we have intentionally refrained from doing so.

A. Quantum states and channels

This short section sets up the notation and prerequisites used throughout the review. We start by introducing some basic terminology used in quantum mechanics. Physical systems (or systems for short) are denoted in the review by sans-serif font, for example, A, B, and S. Describing the problem may sometimes require the use of multiple such systems, in which case the systems will be addressed by additional subscripts, such as S_1 and S_2 . With every physical system we associate a complex Hilbert space denoted by \mathcal{H}_S for a system S, with $|S| := \dim(\mathcal{H}_S)$ its dimension. For most of the review it is sufficient to consider finite-dimensional Hilbert spaces. The set of bounded linear operators acting on \mathcal{H} is denoted as $\mathcal{L}(\mathcal{H})$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called positive semidefinite if it is self-adjoint and satisfies $\langle x|A|x\rangle \ge 0$ for all $|x\rangle \in \mathcal{H}$. We denote by " \geq " the Löwner partial order; i.e., for two linear operators X and Y the relation $X \ge Y$ means that X - Y is positive semidefinite.

In quantum mechanics the possible states of a system S are described using density operators, i.e., positive semidefinite operators acting on \mathcal{H}_S with unit trace. We collect such operators in a set

$$\mathcal{D}(\mathbf{S}) \coloneqq \{\rho \in \mathcal{L}(\mathcal{H}_{\mathbf{S}}) | \rho \ge 0, \operatorname{tr}[\rho] = 1\}.$$
(1)

To highlight that a density operator ρ corresponds to a specific system **S**, we often write $\rho_{\rm S}$. The evolution (or dynamics) of quantum systems is fundamentally unitary if all involved degrees of freedom are incorporated. This means that a quantum system **S** prepared in a state $\rho_{\rm S}$ evolves as $\rho_{\rm S} \rightarrow U\rho_{\rm S}U^{\dagger}$, where U satisfies $UU^{\dagger} = U^{\dagger}U = 1$. The effective unitary U represents discrete-time dynamics obtained by integrating the Schrödinger equation up to a fixed time (\hbar is set to 1).

Often the system of interest S interacts with other systems such as a thermal environment or a measurement apparatus. In such cases, the effective dynamics on the system of interest may no longer be unitary. To model this, one introduces an environment that encompasses all of the auxiliary degrees of freedom. Owing to their role in the global evolution, in this review we differentiate between two types of environments: an ordinary environment E and a catalytic environment (or catalyst for short) C. The environment E corresponds to all of the degrees of freedom that cannot be accessed or are seen as irrelevant after the evolution. This is usually the case for thermal environments. However, the environment C captures all of the degrees of freedom that cannot, or should not, experience any irreversible backaction from the system. In the remainder of the review, we demonstrate that this distinction is well motivated from both a physical and a mathematical perspective. It is also often more convenient to consider the effective dynamics on SC alone instead of the unitary dynamics on SCE. If the initial state of E is uncorrelated to SC, then the resulting dynamics takes the form of a quantum channel $\mathcal{E}: D(SC) \to D(SC)$, which is a completely positive trace-preserving linear map. In most cases, when the space on which the channel acts is clear from the context, we avoid explicitly writing the domain and image of the channel.

Conversely, any quantum channel acting on SC can be realized with a suitable choice of environment state ρ_E and unitary U (Stinespring, 1955). Given a quantum channel \mathcal{E} , we refer to (ρ_E, U) as a *dilation* of \mathcal{E} . The channel is obtained by averaging over all the degrees of freedom of the environment E. Mathematically this is obtained by applying the partial trace $\operatorname{Tr}_E(\cdot)$ with respect to E, which is a quantum channel $\operatorname{Tr}_E: \mathcal{D}(\operatorname{SCE}) \to \mathcal{D}(\operatorname{SC})$. The dilation (ρ_E, U) is said to implement a quantum channel \mathcal{E} when

$$\mathcal{E}[\rho_{\mathsf{SC}}] = \mathrm{Tr}_{\mathsf{E}}[U(\rho_{\mathsf{SC}} \otimes \rho_{\mathsf{E}})U^{\dagger}]$$
(2)

holds for every $\rho_{SC} \in \mathcal{D}(SC)$. We also call \mathcal{E} the quantum channel induced by (ρ_E, U) . A quantum channel can be physically realized in different ways, with multiple dilations inducing the same channel locally. However, if we restrict to pure states ρ_E , there always exists a minimal Stinespring dilation in which E has minimal dimension. Any other Stinespring dilation with a pure state on a system E' is related to this one via an isometry from E to E'.

II. THE CONCEPT OF CATALYSIS

In this section we formalize the concept of catalysis in a general manner. We then describe the paradigm of resource theories and argue that it provides a convenient set of tools allowing for a systematic study of the catalytic effects. Finally, we summarize the most relevant mathematical tools and techniques that are used throughout this review. We emphasize that this section focuses on generic features of catalysis that are valid regardless of the particular physical setting.

We begin by describing a simple motivating example from the early times of quantum information science to illuminate some essential features of catalysis. This example was first proposed theoretically by Phoenix and Barnett (1993) and Cirac and Zoller (1994), and the first experimental demonstration was reported by Hagley *et al.* (1997). Consider an optical cavity with a field mode F in resonance with the transition frequency of two two-level systems, such as the energy eigenstates $|\downarrow\rangle$ and $|\uparrow\rangle$ of two identical atoms A and B. We assume that we are restricted to performing the following actions: initializing the cavity in the vacuum state $|0\rangle_{\sf F}$, initializing the atoms in one of their two eigenstates, and turning on an energy-preserving interaction Hamiltonian between the atom A (B) and the cavity F for a chosen amount of time *t*.

We now observe how one can make use of the cavity to prepare a maximally entangled Bell state between atoms A and B while returning the cavity to its initial vacuum state. To begin, prepare atom A in the excited state $|\uparrow\rangle_A$, and let it interact with the cavity for a time t_A . If the interaction time is short enough, we can assume that no dissipation occurs and the interaction is described by the Jaynes-Cummings Hamiltonian, which has the property that in the interaction picture $|0\rangle_F|\downarrow\rangle_A$ is invariant, while the states $|1\rangle_F|\downarrow\rangle_A$ and $|0\rangle_F|\uparrow\rangle_A$ experience Rabi oscillations. Here $|1\rangle_F$ is the state of the cavity with one photon. If we choose t_A to correspond to a quarter of a Rabi oscillation, the state of FA after the interaction is (in the interaction picture)

$$U_{\mathsf{F}\mathsf{A}}(t_{\mathsf{A}})|0\rangle_{\mathsf{F}}|\uparrow\rangle_{\mathsf{A}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{F}}|\uparrow\rangle_{\mathsf{A}} - |1\rangle_{\mathsf{F}}|\downarrow\rangle_{\mathsf{A}}).$$
(3)

Now prepare the second atom B in the ground state $|\downarrow\rangle_{B}$, and let it interact with the cavity for a time t_{B} such that $|1\rangle_{F}|\downarrow\rangle_{B}$ experiences half of a Rabi oscillation so that the states $|1\rangle_{F}|\downarrow\rangle_{B} \leftrightarrow |0\rangle_{F}|\uparrow\rangle_{B}$ are interchanged (up to a phase). Since $|0\rangle_{F}|\downarrow\rangle_{B}$ is invariant under the dynamics, the final state of FAB is (up to a global phase)

$$|0\rangle_{\mathsf{F}} \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle_{\mathsf{A}}|\downarrow\rangle_{\mathsf{B}} - |\downarrow\rangle_{\mathsf{A}}|\uparrow\rangle_{\mathsf{B}}). \tag{4}$$

The systems AB end up in a Bell state, while F returns to its initial state and can be used to entangle further pairs of atoms. In the words of Hagley *et al.* (1997), "The field, which starts and ends up in vacuum and remains at the end of the process decorrelated from the atoms, acts as a 'catalyst' for the atomic entanglement." Similar techniques can be used to implement a two-qubit CNOT gate between trapped ions, where the role of the cavity is now played by the center-of-mass mode of the trapped ions. This is the basis for trapped-ion quantum computers as envisioned by Cirac and Zoller (1995). At the same time, it is clear that without the cavity we cannot achieve this.

In comparison with the towers of Hanoi puzzle from the Introduction, the limited actions correspond to the allowed moves in the game and the empty cavity corresponds to an empty rod.

The aforementioned simple example demonstrates that catalytic effects can be observed in generic quantum systems. While it is only a specific instance of catalysis, it contains some of the essential features. For one, catalysis always requires at least two systems: a system of interest S and a catalyst C. In our previous example, the field corresponds to the catalyst C and the atoms to the system S. Given a quantum system S, we introduce a family of quantum channels \mathcal{O} that describes all physical operations that can be implemented under the constraints of a particular physical setting. In the

previous example, the set \mathcal{O} corresponds to all unitary evolutions that can be generated by turning on the interaction between the cavity and one of the atoms for some time. More generally, any physical operation in \mathcal{O} is described by some quantum channel $\mathcal{F} \in \mathcal{O}$. Limitations on \mathcal{O} can originate in many ways, as we see in the following partial list:

- As a consequence of conservation laws.
- Owing to limited resources (for example, a finite power supply).
- As a result of finite control accuracy or complexity (for example, a limited time of evolution).

• Owing to the causality constraints in special relativity. The limitations may be of a fundamental nature or may arise from practical limitations. While most work on catalysis in quantum information theory focuses on fundamental limitations, throughout the review we do not distinguish between the two. Now imagine that our system S is prepared in a state ρ_S and that we would like to bring it to a target state σ_S ,

$$\rho_{\rm S} \longrightarrow \sigma_{\rm S}.$$
(5)

It may then happen that, with the operations that we can implement as described by the set \mathcal{O} , we are unable to perform the transformation. This means that there is no quantum channel $\mathcal{F} \in \mathcal{O}$ such that $\mathcal{F}[\rho_S] = \sigma_S$. However, Eq. (5) may become possible if we can act on additional degrees of freedom represented by a quantum system C. The central idea of catalysis is that this type of activation may be possible even without disturbing the quantum state of the subsystem C, and hence one can reuse C to "catalyze" the same state transition on another system S' prepared in the same state ρ . In the ideal case, given a set of operations \mathcal{O} , we say that a state transition is achievable with a catalyst and denote this by $\rho_S \hookrightarrow \sigma_S$, if there exists a state ω_C on C and a quantum channel $\mathcal{F} \in \mathcal{O}$ such that $\mathcal{F}[\rho_S \otimes \omega_C] = \sigma_S \otimes \omega_C$. The central question is then as follows:

Which state transitions are made possible by catalysis?

In approaching this question, we begin by presenting a simple lemma that serves as a guideline for such considerations. Imagine that we want to implement a quantum channel \mathcal{F} using a unitary U and auxiliary degrees of freedom E prepared in some state $\rho_{\rm E}$, as described in Sec. I.A.

Lemma II.1 (Basic lemma). Consider a finite-dimensional Hilbert space $\mathcal{H}_{SEC} = \mathcal{H}_{SE} \otimes \mathcal{H}_{C}$. Let *U* be a unitary on \mathcal{H}_{SEC} , let $\rho_{SE} = \rho_{S} \otimes \rho_{E} \in \mathcal{D}(SE)$ be a density operator on SE, and let $\omega_{C} \in \mathcal{D}(C)$ be a density operator on C. If

$$U(\rho_{\mathsf{S}} \otimes \rho_{\mathsf{E}} \otimes \omega_{\mathsf{C}})U^{\dagger} = \sigma_{\mathsf{SE}} \otimes \omega_{\mathsf{C}} \tag{6}$$

for a density operator $\sigma_{SE} \in \mathcal{D}(SE)$, then there exists a unitary operator V on \mathcal{H}_{SE} such that $\sigma_{SE} = V \rho_{SE} V^{\dagger}$.

Proof. Owing to the tensor-product structure of the output state, raising Eq. (6) to any power $k \ge 0$ and taking the trace yield $tr[\rho_{SE}^k] = tr[\sigma_{SE}^k]$. This is possible for two matrices only if their eigenvalues (including multiplicities) coincide. Hence, ρ_{SE} and σ_{SE} are related by a unitary transformation.

FIG. 2. Illustration of Lemma II.1. Unitary U uses E to implement a quantum channel \mathcal{F} on SC. If C is catalytic and remains uncorrelated to ES, then the same state transition on S can be realized using a unitary V acting only on ES by implementing a quantum channel \mathcal{E} on S. The channels fulfill $\operatorname{tr}_{C} \circ \mathcal{F}[\rho_{S} \otimes \omega_{C}] = \mathcal{E}[\rho_{S}] = \sigma_{S}$.

Lemma II.1 is illustrated in Fig. 2. It tells us that if we use a Stinespring dilation (U, ρ_E) to implement \mathcal{F} on SC such that the state on C does not change and remains uncorrelated to both S and E, then there exists a unitary operator V on SE alone that leads to the same state transition on SE (and therefore on S). In other words, the state transition $\rho_S \rightarrow \sigma_S$ can already be implemented using E, and hence there is no need to use an additional catalyst C. From this, we can observe the three following essential ways in which one can circumvent the assumptions of the lemma (and therefore enable catalysis):

(1) Catalysis is not exact, which means that

$$\operatorname{Tr}_{S}[\mathcal{F}[\rho_{S} \otimes \omega_{C}]] \neq \omega_{C}.$$

$$\tag{7}$$

Therefore, the state on C changes at least a small amount.

(2) The final state on C is correlated with SE, that is,

$$U(\rho_{\mathsf{S}} \otimes \rho_{\mathsf{E}} \otimes \omega_{\mathsf{C}})U^{\dagger} = \sigma_{\mathsf{SEC}} \neq \sigma_{\mathsf{SE}} \otimes \omega_{\mathsf{C}}, \quad (8)$$

where $\text{Tr}_{\text{SE}}[\sigma_{\text{SEC}}] = \omega_{\text{C}}$. In particular, ω_{C} is not pure. (3) The set of implementable operations \mathcal{O} is sufficiently

(5) The set of implementatic operations C is sufficiently restricted. This means that, while the unitary V in Lemma II.1 exists, the quantum channel it induces on S,

$$\mathcal{E}[\rho_{\mathsf{S}}] = \operatorname{tr}_{\mathsf{E}}[V\rho_{\mathsf{S}} \otimes \rho_{\mathsf{E}}V^{\dagger}], \tag{9}$$

is not an implementable operation, i.e., $\mathcal{E} \notin \mathcal{O}$.

Different ways of relaxing Eq. (6), as captured by the aforementioned items (1) and (2), lead to different types of catalysis; see Sec. III. However, as captured by item (3), different physical settings may lead to distinct sets of implementable operations \mathcal{O} . This in turn generally leads to different answers concerning the central question of catalysis. In recent years, such operational restrictions have been formalized under the framework of *resource theories*. In Sec. II.A we discuss the main concepts of resources theories and describe three paradigmatic examples.

Before we close this section, we discuss the reusability of catalysts and how this is related to correlations that catalysts establish with other systems. As mentioned, a catalyst $\omega_{\rm C}$ that catalyzes a state transition $\rho_{\rm S} \longrightarrow \sigma_{\rm S}$ via $\mathcal{F}[\rho_{\rm S} \otimes \omega_{\rm C}] = \sigma_{\rm S} \otimes \omega_{\rm C}$ can be reused to catalyze a further such state

transition on a fresh copy of the same state (in other words, on a different system S' such that $\rho_{S'} = \rho_S$), and S' is uncorrelated to C prior to its interaction with C. This is because the catalyst by definition is guaranteed to act as a catalyst only for a specific initial state ρ and a specific operation \mathcal{F} . In other words, if we change the initial state from ρ_S to σ_S , the catalyst will not in general remain catalytic,

$$\operatorname{tr}_{\mathsf{S}}[\mathcal{F}[\sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}]] \neq \omega_{\mathsf{C}} \quad \text{if } \sigma_{\mathsf{S}} \neq \rho_{\mathsf{S}}. \tag{10}$$

In this sense a catalyst is always fine-tuned. However, a small error on the preparation leads to a small error on the final states of S and C only. This is further discussed later. Since the main purpose of the catalyst is to change the state of S, there is not much sense in discussing the reusability of the catalyst on the same system. Even so, reusing the catalyst on a different system also has its own drawbacks: As shown in Lemma II.1 the catalyst can become correlated to either S or E in general. If the catalyst is now reused on S', these correlations generally will spread to S'. Whether or not such an uncontrollable spread of correlations is problematic generally depends on the context. Furthermore, if S and S' were already initially correlated, then after the action of \mathcal{F} on SC, the resulting state $\hat{\rho}_{S'C}$ may be correlated, and therefore $tr_{S}[\mathcal{F}[\hat{\rho}_{S'C}]] \neq \omega_{C}$, even if $\hat{\rho}_{S'} = \rho_{S'}$ and $\hat{\rho}_{C} = \omega_{C}$.

Despite the previous discussion, one can easily imagine situations in which it is reasonable to demand that the catalyst also remains invariant for different states on S. For one such situation, see Sec. III.G, where we discuss the closely related notion of state-independent catalysis.

A. Resource theories

We have seen that catalysis is strongly related to the set of operations that obey certain constraints that can be implemented on a physical system. Such allowed operations can be formalized using the concept of a resource theory, which we introduce in this section, first in an abstract way and then by means of three well-studied examples that become important in the remainder of this review. An in-depth review on resource theories was given by Chitambar and Gour (2016).

A resource theory partitions the space of states and operations on physical systems into those that are either "easy" or "difficult" to prepare or implement under given physical constraints. The easy states and operations are typically called free. We continue to use the symbol \mathcal{O} to denote free operations, while we use the symbol \mathcal{S} to denote the set of free states. A resource theory is specified by such a tuple $\mathcal{R} = (\mathcal{S}, \mathcal{O})$. For a fixed system S we denote by $\mathcal{S}(S) = \mathcal{S} \cap \mathcal{D}(S)$ the set of free states corresponding to that system. One usually assumes the consistency condition that free operations always transform free states into free states, i.e.,

$$\rho_{\mathsf{S}} \in \mathcal{S}, \qquad \mathcal{F}_{\mathsf{S}} \in \mathcal{O} \Rightarrow \mathcal{F}_{\mathsf{S}}[\rho_{\mathsf{S}}] \in \mathcal{S}. \tag{11}$$

Any state that is not a free state is interpreted as *resourceful* because it cannot be created using only operations from the set O acting on states in S. However, resourceful states can be used to implement nonfree operations: Given a

resourceful state σ_A on an auxiliary system A and a free operation \mathcal{F}_{SA} on SA, the operation defined by

$$\mathcal{E}_{\mathsf{S}}[\rho_{\mathsf{S}}] = \operatorname{tr}_{\mathsf{A}}[\mathcal{F}_{\mathsf{S}\mathsf{A}}[\rho_{\mathsf{S}} \otimes \sigma_{\mathsf{A}}]] \tag{12}$$

is typically not free, that is, $\mathcal{E}_{S} \notin \mathcal{O}$. To denote when ρ can be converted into σ , we use the standard notation

$$\rho \xrightarrow[\mathcal{O}]{} \sigma \Leftrightarrow \exists \mathcal{F} \in \mathcal{O} \quad \text{such that } \mathcal{F}[\rho] = \sigma.$$
(13)

The relation $\xrightarrow{\mathcal{O}}$ between quantum states is generally only a partial order one, i.e., typically there are pairs of states for which neither $\rho \xrightarrow{\mathcal{O}} \sigma$ nor $\sigma \xrightarrow{\mathcal{O}} \rho$ is true. Given a resource theory \mathcal{R} , the central question is: Which state transitions are made possible using free operations?

In general it is difficult to characterize the partial order induced by a given resource theory. However, it is sometimes possible to specify a set of necessary conditions for the state transition, in terms of so-called resource monotones. A resource monotone f with respect to a resource theory O is a function from quantum states to real numbers such that

$$\rho \xrightarrow{\rho} \sigma \Rightarrow f(\rho) \ge f(\sigma). \tag{14}$$

In other words, since free operations cannot make the state more resourceful, every resource monotone can be seen as measuring some abstract resource. For this reason, resource monotones are sometimes viewed as resource-theoretic analogs of the second law of thermodynamics. A typical example of a resource monotone is the quantum relative entropy with respect to the set of free states (Brandão and Gour, 2015; Berta *et al.*, 2022), i.e.,

$$f(\rho_{\mathsf{S}}) = \inf_{\gamma_{\mathsf{S}} \in \mathcal{S}(\mathsf{S})} D(\rho_{\mathsf{S}} \| \gamma_{\mathsf{S}}), \tag{15}$$

where $D(\rho || \sigma) \coloneqq tr\{\rho[\log(\rho) - \log(\sigma)]\}$ is the relative entropy between two density matrices; see Eq. (31) for a more formal discussion. Sometimes it is possible to find a complete set of resource monotones, that is, a family $\{f_{\alpha}\}$ of resource monotones that collectively characterizes both necessary and sufficient conditions for state transformations. More formally,

$$\rho \xrightarrow{\rho} \sigma \Leftrightarrow f_{\alpha}(\rho) \ge f_{\alpha}(\sigma) \quad \forall \ \alpha. \tag{16}$$

It can be shown that, under some general assumptions about the resource theory \mathcal{R} , any complete set of resource monotones must be infinite (Datta, Ganardi *et al.*, 2023). This may be surprising since, in finite dimensions, it should be clear that a finite number of parameters specify whether a state transformation is possible. Nevertheless, the two statements are compatible because a complete set of monotones allows one to evaluate the possibility of state transitions by simply checking whether none of the monotones increase. It is, however, often possible to determine the possibility of a state transition via more complicated functions of a finite number of monotones.

Most of the statements in this review are formulated for general quantum resource theories. However, in some cases we need to restrict ourselves to a specific types of resource theories that we refer to as *permutation-free* resource theories. In such resource theories permuting subsystems (i.e., physically swapping two subsytems) is allowed for free. In this case permuting subsystems can be considered simply as a relabeling of the subsystems. To our knowledge, the resource theories considered thus far in the literature all fulfill this assumption either fully or for specific subsystems, such as local subsystems in the context of local operations and classical communication (LOCC), which we discuss. We emphasize, however, that permuting subsystems cannot always simply be considered a free relabeling: For example, swapping qubits in a quantum computer is an essential operation that requires nontrivial gates that pick up errors in general. In a fault-tolerant quantum computer, we can expect that swapping *logical qubits* can be done error free using Clifford operations (see Sec. V.F) but can still require significant computational time.

With this general introduction in mind, we turn to some examples.

1. Entanglement

The idea of resource theories originated with entanglement theory in the 1990s (Bennett, Bernstein *et al.*, 1996; Vedral *et al.*, 1997), when it was realized that entanglement shared between multiple parties can be useful for certain tasks, and that it cannot be increased in the absence of quantum communication. The theory of quantum entanglement is now a mature field of study, as reviewed by Horodecki *et al.* (2009), and forms a central part of quantum information theory (Wilde, 2009; Nielsen and Chuang, 2012; Watrous, 2018). The resource theory that captures these restrictions is known as LOCC, denoted as $\mathcal{R}_{LOCC} = (\mathcal{S}_{LOCC}, \mathcal{O}_{LOCC})$. The corresponding set of free operations \mathcal{O}_{LOCC} consists of any protocol that is composed of the following operations performed by the different parties:

- (i) Preparing arbitrary local quantum states.
- (ii) Applying local unitary operations and measurements.
- (iii) Exchanging classical messages between the parties.
- (iv) Discarding physical subsystems.

This choice of free operations encapsulates the idea that preparing quantum states locally and communicating classically is easy, but exchanging (or communicating) quantum states in a coherent manner is difficult. The set of free states S_{LOCC} is simply given by all quantum states that can be prepared using the free operations $\mathcal{O}_{\text{LOCC}}$. All the free states can be prepared using only classical shared randomness. Consequently, they are called classically correlated states (Werner, 1989) or separable states (Horodecki and Horodecki, 1996; Peres, 1996). Any state that is not classically correlated is called entangled.

The amount of entanglement can be quantified using various resource monotones (Vedral *et al.*, 1997; Vidal, 2000; Horodecki *et al.*, 2009), which in this case are usually called entanglement monotones. For example, in the setting with only two parties Alice (A) and Bob (B) the monotone defined in Eq. (15) translates into the relative entropy of entanglement

$$E_{\rm rel}(\rho_{\sf AB}) = \inf_{\gamma_{\sf AB} \in \mathcal{S}_{\rm LOCC}(\sf AB)} D(\rho_{\sf AB} \| \gamma_{\sf AB}). \tag{17}$$

In Sec. V.B we review the role of catalysis in the resource theory of LOCC.

2. Asymmetry

Conservation laws place fundamental restrictions on the possible dynamics on physical systems. To focus our attention, we consider the case when the conserved quantity is the total energy. Suppose that we want to include all participating degrees of freedom in our quantum description such that the dynamics is ultimately unitary. The unitary operator U that describes the dynamics must conserve the total energy. In other words, owing to Noether's theorem U has to commute with the group of time translations $t \mapsto t + s$, which on the Hilbert space is represented by $s \mapsto \exp(-isH)$, with H denoting the total Hamiltonian of the system.

The resource theory of asymmetry with respect to a group G (Janzing and Beth, 2003b; Marvian, 2012), also known as the resource theory of reference frames (Bartlett, Rudolph, and Spekkens, 2007; Gour and Spekkens, 2008; Vaccaro *et al.*, 2008), provides an idealized framework to study the restrictions imposed by such commutation conditions. In $\mathcal{R}_{asym} = (S_{asym}, \mathcal{O}_{asym})$, every system S carries a projective unitary representation of the group G given by $g \mapsto W_S(g)$. The free operations \mathcal{O}_{asym} consist of all quantum channels that are covariant: a quantum channel $\mathcal{F}: \mathcal{D}(S) \to \mathcal{D}(S')$ is covariant with respect to the given representations of the group G if, $\forall g \in G$,

$$\mathcal{F}[W_{\mathsf{S}}(g)\rho_{\mathsf{S}}W_{\mathsf{S}}(g)^{\dagger}] = W_{\mathsf{S}'}(g)\mathcal{F}[\rho_{\mathsf{S}}]W_{\mathsf{S}'}(g)^{\dagger}.$$
 (18)

For two independent systems S_1 and S_2 , we further require that they jointly carry the representation $g \mapsto W_{S_1S_2}(g) = W_{S_1}(g) \otimes W_{S_2}(g)$. The free states S_{asym} are all density matrices that are invariant or symmetric with respect to G, i.e.,

$$W_{\mathsf{S}}(g)\rho_{\mathsf{S}}W_{\mathsf{S}}(g)^{\dagger} = \rho_{\mathsf{S}} \quad \forall \ \rho_{\mathsf{S}} \in \mathcal{S}_{\operatorname{asym}}(\mathsf{S}).$$
 (19)

Thus, the valuable resources in \mathcal{R}_{asym} are all quantum states which are asymmetric with respect to the group G. In the particular \mathcal{R}_{asym} with respect to time translation, the free states are simply stationary states of the time evolution. Moreover, a state is resourceful if and only if it carries coherence between different energies.

In the resource theory of asymmetry, Eq. (15) translates into the relative entropy of asymmetry, also known as the relative entropy of frameness (Vaccaro *et al.*, 2008; Gour, Marvian, and Spekkens, 2009),

$$\mathcal{A}(\rho_{\mathsf{S}}) = \inf_{\gamma_{\mathsf{S}} \in \mathcal{S}_{asym}} D(\rho_{\mathsf{S}} \| \gamma_{\mathsf{S}}) = H(\mathcal{G}[\rho_{\mathsf{S}}]) - H(\rho_{\mathsf{S}}), \quad (20)$$

where $H(\rho_S) = -\text{tr}[\rho_S \log(\rho_S)]$ is the von Neumann entropy (von Neumann, 1932). The rightmost equality holds if the infimum is attained, in which case \mathcal{G} is the group-twirling channel that maps a quantum state to the closest symmetric state. For a compact group, this channel corresponds simply to a group average over its normalized, unique left- and right-invariant Haar measure μ ,

$$\mathcal{G}[\rho_{\mathsf{S}}] = \int W_{\mathsf{S}}(g) \rho W_{\mathsf{S}}(g)^{\dagger} \mathrm{d}\mu(g).$$
(21)

Every covariant quantum channel can be realized by (1) adding an ancillary system E prepared in a symmetric state, (2) applying a covariant unitary (or, more generally, an isometry) Umapping from \mathcal{H}_{SE} to $\mathcal{H}_{S'E}$, and (3) tracing out the ancillary system E (Keyl and Werner, 1999). This leads to a quantum channel of the form

$$\mathcal{F}[\rho_{\mathsf{S}}] = \operatorname{tr}_{\mathsf{E}}[U\rho_{\mathsf{S}} \otimes \rho_{\mathsf{E}}U^{\dagger}], \qquad (22)$$

with the conditions

$$UW_{\mathsf{S}}(g) \otimes W_{\mathsf{E}}(g) = W_{\mathsf{S}'}(g) \otimes W_{\mathsf{E}}(g)U,$$
 (23)

$$W_{\mathsf{E}}(g)\rho_{\mathsf{E}}W_{\mathsf{E}}(g)^{\dagger} = \rho_{\mathsf{E}}.$$
(24)

Moreover, the state ρ_{E} may be chosen to be pure. In Sec. V.D we discuss catalysis in the context of symmetries and conservation laws.

3. Athermality

Our last example is the resource theory of athermality $\mathcal{R}_{atherm} = (\mathcal{S}_{atherm}, \mathcal{O}_{atherm})$, which can be seen as an idealized model of thermodynamics in the quantum realm, see Janzing *et al.* (2000), Brandão *et al.* (2013, 2015), Horodecki and Oppenheim (2013), Gour *et al.* (2015), and Yunger Halpern and Renes (2016). In this resource theory, every system **S** carries a Hamiltonian \hat{H}_{S} and \mathcal{R}_{atherm} is defined with respect to a fixed background temperature $T = \beta^{-1}$ (we set the Boltzmann constant $k_{B} = 1$). The free states are given by thermal (Gibbs) states,

$$\gamma_{\beta}(\hat{H}_{\mathsf{S}}) \coloneqq \frac{e^{-\beta \hat{H}_{\mathsf{S}}}}{Z_{\mathsf{S}}}, \qquad Z_{\mathsf{S}} = \mathrm{Tr}[e^{-\beta \hat{H}_{\mathsf{S}}}]. \tag{25}$$

Thus, the set of free states of a system S consists of a single state, that is, $S_{\text{atherm}}(S) = \{\gamma_{\beta}(\hat{H}_{S})\}$. There are multiple possibilities for choosing the set of quantum channels representing \mathcal{O}_{atherm} . All such channels must at least preserve S_{atherm} , and hence map thermal states to thermal states at the same temperature. Channels with this property are referred to as Gibbs-preserving maps (Faist, Oppenheim, and Renner, 2015). Often it is additionally demanded that the free operations be covariant with respect to the group of time translations (see the previous section). This set of operations was coined enhanced thermal operations (Ćwikliński et al., 2015). The two aforementioned classes of free operations lead to a valid resource theory; however, they lack a clear physical interpretation, and therefore a more restrictive set of free operations known as thermal operations is commonly considered. These are channels that can be written as

$$\mathcal{F}_{\mathsf{S}}[\rho_{\mathsf{S}}] = \operatorname{tr}_{\mathsf{E}}\{U[\rho_{\mathsf{S}} \otimes \gamma_{\beta}(\hat{H}_{\mathsf{E}})]U^{\dagger}\},$$
(26)

where $\gamma_{\beta}(\hat{H}_{\rm E})$ is a Gibbs state of inverse temperature β and the unitary U is strictly energy preserving, i.e., it satisfies $[U, \hat{H}_{\rm S} + \hat{H}_{\rm E}] = 0$. It is known that thermal operations are a strict subset of enhanced thermal operations (Ding, Ding, and Hu, 2021). Major progress in understanding the state-transition conditions for enhanced thermal operations was reported by Gour *et al.* (2018), but no simple characterization is known.

An important resource monotone in the resource theory of athermality is the nonequilibrium free energy

$$F^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) \coloneqq \operatorname{tr}[\rho_{\mathsf{S}}\hat{H}_{\mathsf{S}}] - \frac{1}{\beta}H(\rho_{\mathsf{S}})$$
(27)

$$= \frac{1}{\beta} \{ D[\rho_{\mathsf{S}} \| \gamma_{\beta}(\hat{H}_{\mathsf{S}})] - \log Z_{\mathsf{S}} \}.$$
(28)

Since S_{atherm} consists of only a single state, up to rescaling and a shift by the equilibrium free energy $-\beta \log Z_S$, the nonequilibrium free energy corresponds to the general monotone defined in Eq. (15). When free operations are given by (enhanced) thermal operations, the resource theory of athermality can be seen as the resource theory of asymmetry with respect to time translations, with the additional restrictions imposed on the sets of free states and free operations. In Secs. V.C.3 and V.C.5 we discuss catalysis in the context of the resource theory of athermality.

We have discussed three major examples of resource theories. However, there are many more, such as the resource theory of contextuality (see Sec. V.B.11), non-Gaussianity (see Sec. V.E.4), and stabilizer operations (see Sec. V.F), to name a few that we encounter later in this review.

B. Basic mathematical tools

In this section we introduce some mathematical tools that are often used to describe the properties and the relationships between quantum states, for example, to capture the partial order induced by a resource-theoretic framework. These tools serve as a technical basis to study how such partial orders can change in the presence of catalysts.

1. Distinguishability measures

It is often necessary to measure how easy or difficult it is to distinguish between two quantum states. In the context of this review, this is used for two main purposes: (i) quantifying how close the final state of the system S is to a given target state and (ii) quantifying how close the final state of the catalyst C is to its initial state.

Suppose that we receive two datasets from two independent runs of some experiment, each of which is derived from raw measurement data via the same postprocessing technique. It is then intuitively clear that all the information that distinguishes the two datasets has already been present in the measurement data, i.e., before postprocessing was applied. In other words, postprocessing two datasets in the same way cannot increase their distinguishability. Similarly, no postprocessing, as represented generally by a quantum channel \mathcal{E} , should increase the distinguishability between two quantum states ρ and σ . Therefore, any operationally meaningful measure of distinguishability $d(\cdot, \cdot)$ between two quantum states should satisfy

$$d(\mathcal{E}[\rho], \mathcal{E}[\sigma]) \le d(\rho, \sigma) \tag{29}$$

for any two density operators ρ and σ and any quantum channel \mathcal{E} . Equation (29) is called the data-processing inequality (DPI) and is a central concept in quantum information theory. An important measure that satisfies the DPI is the trace distance $\Delta(\cdot, \cdot)$, i.e.,

$$\Delta(\rho, \sigma) \coloneqq \frac{1}{2} \|\rho - \sigma\|_1, \tag{30}$$

where $\|\cdot\|_1$ is the Schatten 1-norm. When the two density matrices commute, i.e., when $[\rho, \sigma] = 0$, the trace distance reduces to the total variation distance between the two probability vectors formed from the eigenvalues of ρ and σ . Another relevant distinguishablity measure is the Umegaki quantum relative entropy (Umegaki, 1962), which is defined as

$$D(\rho \| \sigma) \coloneqq \operatorname{tr} \{ \rho[\log(\rho) - \log(\sigma)] \}, \tag{31}$$

with $D(\rho \| \sigma) = +\infty$ if the support of ρ is not contained in that of σ . For commuting density operators ρ and σ , the quantum relative entropy reduces to the Kullback-Leibler divergence (Kullback and Leibler, 1951). One can also consider generalizations of relative entropy, so-called quantum Rényi divergences; see the next section and Tomamichel (2016). While the trace distance is a metric, the quantum relative entropy is not. Specifically it neither is symmetric in its arguments nor satisfies the triangle inequality. Nevertheless, it satisfies the data-processing inequality of Eq. (29), and furthermore the quantum relative entropy has a strong operational relevance (Hiai and Petz, 1991): It gives rise to the logarithm of the minimal type-2 error in a quantum hypothesis testing scenario (involving ρ and σ as the null and alternative hypotheses) regularized in the asymptotic limit. The relative entropy can also be used to express the mutual information

$$I(\mathsf{A}:\mathsf{B})_{\rho} = D(\rho_{\mathsf{A}\mathsf{B}} \| \rho_{\mathsf{A}} \otimes \rho_{\mathsf{B}}), \tag{32}$$

which is a measure of the amount of correlations between subsystems A and B.

Another important measure of distinguishability is the fidelity $F(\cdot, \cdot)$ defined as $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$. The classical counterpart of fidelity is known as the Bhattacharyya distance (Bhattacharyya, 1943). As opposed to previous distinguishability measures, the fidelity fulfills a reverse DPI, i.e.,

$$F(\rho, \sigma) \le F(\mathcal{E}[\rho], \mathcal{E}[\sigma]), \tag{33}$$

and is close to unity for states that are similar. Although the fidelity is not a metric, other metrics derived from the fidelity naturally satisfy the data-processing inequality (Gilchrist,

Langford, and Nielsen, 2005) and are therefore valid distinguishability measures.

In the context of a resource theory $\mathcal{R} = (\mathcal{S}, \mathcal{O})$, every distinguishability measure $d(\cdot, \cdot)$ that fulfills the DPI allows a resource monotone f_d to be defined via (Gonda and Spekkens, 2023)

$$f_d(\rho_{\mathsf{S}}) \coloneqq \inf_{\sigma_{\mathsf{S}} \in \mathcal{S}(\mathsf{S})} d(\rho_{\mathsf{S}}, \sigma_{\mathsf{S}}). \tag{34}$$

The monotone $f_d(\rho_S)$ therefore measures how the state ρ_S can be distinguished from the free states S(S) on S, as measured by $d(\cdot, \cdot)$.

2. Entropic quantifiers

Information encoded in physical systems can be conveniently characterized using various entropic quantifiers, most of them having well-established operational interpretations. Perhaps the most well-known entropic quantifier is the Shannon entropy $H(\mathbf{p})$ (Shannon, 1948), which for a probability vector \mathbf{p} is defined as $H(\mathbf{p}) \coloneqq -\sum_i p_i \log(p_i)$, where p_i denotes the *i*th element of the vector \mathbf{p} .¹ Throughout the review we use the logarithm of base 2, i.e., $\log(2) = 1$. The generalization of Shannon entropy to density operators is known as von Neumann entropy (von Neumann, 1932),

$$H(\rho_{\mathsf{S}}) = H[\lambda(\rho)] \equiv -\operatorname{Tr} \rho \log \rho, \qquad (35)$$

where $\lambda(A)$ denotes the vector of eigenvalues of *A*, including multiplicities. Another generalization of the concept of entropy are the α -Rényi entropies (Rényi, 1961), which, for a parameter² $\alpha \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$, are defined as

$$H_{\alpha}(\boldsymbol{p}) \coloneqq \frac{\operatorname{sgn}(\alpha)}{1-\alpha} \log \sum_{i} p_{i}^{\alpha}.$$
 (36)

In Eq. (36), we use the convention that $sgn(0) = 1, 0^0 = 0$, and $0^{\alpha} = +\infty$ for $\alpha < 0$. In particular, $H_{\alpha}(\mathbf{p}) = \infty$ for $\alpha < 0$ if \mathbf{p} has an entry $p_i = 0$, whereas $H_{\alpha}(\mathbf{p})$ for $\alpha > 0$ depends only on the nonzero entries of \mathbf{p} . For $\alpha = \{0, 1, \infty\}$ the associated entropies are defined by continuity in α , i.e.,

$$H_0(\boldsymbol{p}) = \log (\operatorname{rank}(\boldsymbol{p})), \qquad (37)$$

$$H_1(\boldsymbol{p}) = -\sum_i p_i \log(p_i) = H(\boldsymbol{p}), \qquad (38)$$

$$H_{\infty}(\boldsymbol{p}) = -\log(\max\{p_i\}),\tag{39}$$

where rank(p) is defined as the number of nonzero elements of p, so rank[$\lambda(\rho)$] = rank(ρ) for any density operator ρ . The α -Rényi entropy of a density operator ρ is defined as

¹When p corresponds to a probability distribution, we also sometimes use p(i) to denote the *i*th element of p.

²More often the Rényi entropies are defined in the literature for $\alpha \ge 0$ since, for negative values of α , $H_{\alpha}(\mathbf{p})$ tends to infinity when \mathbf{p} is not of full rank. However, this generalization proves to be useful in the context of catalysis as we see later.

 $H_{\alpha}(\rho) = H_{\alpha}[\lambda(\rho)]$ (Wehrl, 1976; Thirring, 1980; Ohya and Petz, 1993), so

$$H(\rho) = \frac{\operatorname{sgn}(\alpha)}{1 - \alpha} \log(\operatorname{tr}[\rho^{\alpha}]).$$
(40)

The Rényi entropies are nonincreasing in α such that, in particular, $H_0(\rho) \ge H_1(\rho) \ge H_{\infty}(\rho)$. All Rényi entropies derive from a family of parent quantities known as Rényi divergences (Rényi, 1961). These quantities generalize the Kullback-Leibler divergence (as we define later in the review); hence, they can also be seen as measures of distinguishability between probability distributions. Rényi divergences for two probability vectors p and q and parameter $\alpha \in \mathbb{R}$ are defined as

$$D_{\alpha}(\boldsymbol{p} \| \boldsymbol{q}) \coloneqq \begin{cases} \frac{\operatorname{sgn}(\alpha)}{\alpha - 1} \log \left[\sum_{i} q_{i} \left(\frac{p_{i}}{q_{i}} \right)^{\alpha} \right] & \text{if } \boldsymbol{p} \ll \boldsymbol{q}, \\ +\infty & \text{otherwise,} \end{cases}$$
(41)

where $p \ll q$ means that $q_i = 0$ implies $p_i = 0$ for all *i*. In the limit $\alpha \rightarrow 1$ we recover the Kullback-Leibler divergence (Kullback and Leibler, 1951), or relative entropy,

$$D(\boldsymbol{p} \| \boldsymbol{q}) = \sum_{i} p_{i} \log\left(\frac{p_{i}}{q_{i}}\right).$$
(42)

Rényi entropies and Rényi divergences are linked via

$$D_{\alpha}(\boldsymbol{p} \| \boldsymbol{\pi}) = \operatorname{sgn}(\alpha) \log(d) - H_{\alpha}(\boldsymbol{p}), \quad (43)$$

where $\pi \propto (1, 1, ..., 1)^{\top}/d$ is the uniform *d*-dimensional probability vector. In contrast to Rényi entropies, there is no unique quantum generalization for Rényi divergences. If $[\rho_{\rm S}, \sigma_{\rm S}] = 0$ (the quasiclassical case), we can simply diagonalize both density matrices in a common eigenbasis and consider $D_{\alpha}[\lambda(\rho_{\rm S})||\lambda(\sigma_{\rm S})]$. Perhaps the simplest and most commonly used candidate for the quantum generalization of Rényi divergences are the Petz-Rényi divergences (Petz, 1986a), which for $\alpha \in [0, 1) \cup (1, \infty)$ are defined as

$$D_{\alpha}(\rho \| \sigma) = \begin{cases} \frac{1}{\alpha - 1} \log(\operatorname{tr}[\rho^{\alpha} \sigma^{1 - \alpha}]) & \text{if } \rho \ll \sigma, \\ +\infty & \text{otherwise,} \end{cases}$$
(44)

where $\rho \ll \sigma$ means that the support of ρ is contained in the support of σ ; i.e., $\langle \psi | \sigma | \psi \rangle = 0$ implies $\langle \psi | \rho | \psi \rangle = 0$ for all vectors $|\psi\rangle$. The Petz-Rényi divergences fulfill the data-processing inequality for $\alpha \in [0, 2]$. Another commonly considered family of quantum Rényi divergences is the minimal (or sandwiched) Rényi divergence (Müller-Lennert *et al.*, 2013), which is defined as

$$\tilde{D}_{\alpha}(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \left\{ \operatorname{tr} \left[\left(\sigma^{(1 - \alpha)/2\alpha} \rho \sigma^{(1 - \alpha)/2\alpha} \right)^{\alpha} \right] \right\}$$
(45)

for $\alpha \in (1/2, 1) \cup (1, \infty)$ if $\rho \ll \sigma$, and $\tilde{D}_{\alpha}(\rho \| \sigma) = \infty$ otherwise.³ The sandwiched Rényi divergence fulfills the dataprocessing inequality (29) for the given range of α and, just as with the Petz-Rényi divergence, coincides with the relative entropy $D(\rho_{\rm S} || \sigma_{\rm S})$ in the limit $\alpha \rightarrow 1$. See Tomamichel (2016) for detailed information about quantum generalizations of Rényi divergences.

3. Majorization

Majorization is a preorder between vectors: Given two vectors p and $q \in \mathbb{R}^d$, we say that p majorizes q if

$$\sum_{i=1}^{k} p_i^{\downarrow} \ge \sum_{i=1}^{k} q_i^{\downarrow} \quad \text{for all } k \in \{1, \dots, d\}$$
(46)

and $\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} q_i$, where p^{\downarrow} and q^{\downarrow} denote vectors ordered nonincreasingly. We use $p \succ q$ to denote that pmajorizes q. For probability vectors, the normalization condition is automatically satisfied, and the partial sums appearing in Eq. (46) can be interpreted as discrete cumulative distribution functions (CDFs) for p^{\downarrow} and q^{\downarrow} . Thus, $p \succ q$ if and only if the CDF of p^{\downarrow} is at each point larger than or equal to the CDF of q^{\downarrow} . Moreover, any deterministic probability vector [for example, $p = (1, 0, ..., 0)^{\top}$] majorizes all probability vectors, while the uniform one $\pi = (1/d, ..., 1/d)^{\top}$ is majorized by all probability vectors of dimension d.

Majorization can be extended to density matrices, in which case it can be seen as a preorder of their spectra. Specifically, we say that $\rho \succ \sigma$ if $\lambda(\rho) \succ \lambda(\sigma)$, where $\lambda(A)$ denotes the vector of eigenvalues of a matrix *A* (including multiplicities). This seemingly simple preorder of density matrices can be used to characterize randomness or uncertainty in states and is tightly linked to the resource theories of (LOCC) entanglement (see Sec. II.A.1) and noisy operations (see Sec. III.H). A particularly useful theorem in the context of majorization is the Schur-Horn theorem, which uses majorization to relate the spectrum of a Hermitian matrix with its diagonal (Schur, 1923; Horn, 1954).

Theorem II.2 (Schur-Horn theorem). Let H be any d-dimensional Hermitian matrix with a vector of eigenvalues $\lambda(H)$. The following statements are equivalent:

(1) There exists a unitary U such that $\lambda' = \text{diag}(UHU^{\dagger})$. (2) $\lambda(H) \succ \lambda'$.

Majorization is an indispensable tool in the theory of statistical comparisons (Blackwell, 1953). Suppose that pand q describe the information encoded in a physical system (such as an energy distribution). We then say that p is more informative than q, in the absence of prior knowledge, when the latter can be obtained from the former by a bistochastic processing. Owing to the Hardy-Littlewood-Pólya theorem, this is equivalent to $p \succ q$ (Hardy, Littlewood, and Pólya, 1952). More generally, when prior knowledge is available in the form of probability distributions p' and q' (for example, thermal distribution of energies), we say that a pair of probability distributions (p, p') is more informative than (q, q') when there exists a stochastic processing that maps p onto q while also mapping p' into q'. When such a processing exists, the first pair is said to relatively majorize the second (Hardy, Littlewood, and Pólya, 1952; Ruch, Schranner, and Seligman, 1978); see also Renes (2016). Finally, we note that several extensions of relative majorization to density matrices

³As is also the case for Petz-Rényi divergences, the condition $\rho \ll \sigma$ is required only for $\alpha > 1$, which guarantees that the resulting quantity is finite.

have been proposed (Buscemi and Gour, 2017; Gour et al., 2018).

An important generalization of majorization is approximate majorization (van der Meer, Ng, and Wehner, 2017; Horodecki, Oppenheim, and Sparaciari, 2018). Let $d(\cdot, \cdot)$ denote some distance measure between vectors. If $p \succ q$ does not hold, one can then still ask whether p majorizes qapproximately. Formally, one asks whether there exists a sufficiently small error $\epsilon > 0$ and a probability vector $\boldsymbol{q}_{\epsilon}$ such that $d(q, q_{\epsilon}) < \epsilon$ and $p \succ q_{\epsilon}$. One can also consider a related problem where the approximation error ϵ is located in the initial state, i.e., the existence of a probability vector p_{ϵ} such that $d(\mathbf{p}, \mathbf{p}_{\epsilon}) < \epsilon$ and $\mathbf{p}_{\epsilon} \succ \mathbf{q}$. The two resulting relations are sometimes called, respectively, postmajorization and premajorization and are known to be equivalent (Chubb, Tomamichel, and Korzekwa, 2018, 2019). One can also ask for the minimal approximation error ϵ such that either $p \succ q_{\epsilon}$ or $p_{\epsilon} \succ q$ holds. If $d(\cdot, \cdot)$ is the total variation distance, Horodecki, Oppenheim, and Sparaciari (2018) proposed an algorithm that not only finds the optimal value of ϵ but also provides explicit constructions of the corresponding optimizers, i.e., the so-called steepest and flattest states. These constructions can be generalized for relative majorization (van der Meer, Ng, and Wehner, 2017) or used to address the majorization preorder in the asymptotic independent identically distributed (i.i.d.) limit (Chubb, Tomamichel, and Korzekwa, 2018; Boes, Ng, and Wilming, 2022), as well as to address approximately catalytic transformations (Ng et al., 2015; Lipka-Bartosik and Skrzypczyk, 2021a).

III. CATALYTIC TYPES

In Sec. II we gave a general outlook on the concept of catalysis in quantum mechanics. We also described some basic mathematical tools that will help us gain a better understanding of the mechanisms behind catalytic effects. Moreover, we saw in Lemma II.1 that catalytic effects can emerge only when at least one of the following conditions is satisfied: (1) the catalyst becomes perturbed, (2) the catalyst develops some correlations with the other degrees of freedom, or (3) the set of allowed operations is appropriately restricted. The different ways in which these conditions can be combined and quantified lead to different types of catalysis, which we summarize in Fig. 3. Each type in principle induces a different set of transformation laws (i.e., what is possible and what is not) in a given resource theory. While the variations between different types of catalysis may seem pedantic at first, we later see that these seemingly technical distinctions can give rise to a fundamentally different physical behavior.

In this section we define and discuss various types of catalysis arising from relaxing conditions (1)–(3) in the context of a general resource theory $\mathcal{R} = (\mathcal{S}, \mathcal{O})$. In Sec. V, we discuss their applications in different physical settings.

A. Strict catalysis

In the strictest formulation of catalysis, the catalyst C must be returned unperturbed and uncorrelated with the main system S at the end of the process. Owing to Lemma II.1, nontrivial catalysis in this scenario is possible only if the set of

	Strict	Arb. Str.	Corr.	Approx.	Emb.
	\hookrightarrow	$\xrightarrow{\text{arb.}}$	$\xrightarrow{\text{corr.}}$	$\xrightarrow{approx.}$	
$\epsilon_{C} > 0$	×	×	×	1	1
$\epsilon_{A} \rightarrow 0$	×	1	×	(✔)	(✔)
I(S:C) > 0	×	×	1	×	1

FIG. 3. Overview of allowed errors and correlations for the catalytic types detailed in Sec. III. The condition specified in the left column is fulfilled for a given type of catalysis if a \checkmark is displayed and not fulfilled if a \bigstar is shown. The errors are defined with respect to a state transformation $\rho_S \otimes \omega_C \otimes \omega_A \rightarrow \sigma_{SCA}$ as $\epsilon_i := d(\omega_i, \sigma_i)$ for $i \in \{C, A\}$. Subsystem A is required only for the formal definition of arbitrarily strict catalysis but also can be incorporated into the categories of approximate catalysis and embezzlement by viewing A as part of C [indicated by (\checkmark)]. I(S:C) denotes the mutual information between subsystems S and C in state σ_{SC} . See Sec. III.C for precise definitions of different catalytic types.

allowed operations is sufficiently restricted. We refer to this category as *strict catalysis*.

Definition III.1 (Strict catalysis). A state transition from $\rho_{\rm S}$ to $\sigma_{\rm S}$ is called strictly catalytic if there exists a free operation $\mathcal{F} \in \mathcal{O}$ and a quantum state $\omega_{\rm C}$ such that $\mathcal{F}[\rho_{\rm S} \otimes \omega_{\rm C}] = \sigma_{\rm S} \otimes \omega_{\rm C}$. We denote a strictly catalytic state transition by

$$\rho_{\mathsf{S}} \underset{\mathcal{O}}{\hookrightarrow} \sigma_{\mathsf{S}}.$$
(47)

In this conservative type, catalysis results from the fact that the set of free operations O is nontrivial; therefore, certain state transitions are forbidden. This allows one to avoid the consequences of Lemma II.1 by relaxing, in particular, condition (3). Even so, one sees how correlations play a critical role: First, despite the fact that the initial and final states of the system S and the catalyst C are not correlated, they have to be correlated during the dynamical process that implements the state transformation. In this sense, building temporary correlations between S and C is what actually enables the state transformation. Second, when the set of free operations \mathcal{O} consists of nonunitary quantum channels (as is the case in the resource theory of athermality), correlations can still build up among S, C, and an environment E, which dilates the original nonunitary quantum channel. These correlations are formally lost upon discarding the environment.

Note that any resource monotone f with respect to O that is additive over tensor products will also be a resource monotone for any strictly catalytic state transformation. Specifically,

$$f(\rho_{\mathsf{S}}) + f(\omega_{\mathsf{C}}) = f(\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}) \ge f(\sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}})$$
$$= f(\sigma_{\mathsf{S}}) + f(\omega_{\mathsf{C}}). \tag{48}$$

Conversely, suppose that \mathcal{O} admits a complete family of resource monotones $f_{\alpha}(\rho_{\rm S}) \ge f_{\alpha}(\sigma_{\rm S})$ for all valid α implies that $\rho_{\rm S} \xrightarrow{\sim} \sigma_{\rm S}$. If the operations admit a single example of

nontrivial strict catalysis, at least one of the monotones f_{α} cannot be additive over tensor products (Fritz, 2017).

The earliest report of strict catalysis in a resourcetheoretic framework was given by Jonathan and Plenio (1999) in the context of entanglement theory. They affirmatively solved a conjecture of S. Popescu by showing for the first time that strict catalysis arises in LOCC; see Sec. V.B.1. This result made clear that the mathematical structure of entanglement is much richer than previously expected. Jonathan and Plenio (1999) started the systematic exploration of catalytic effects in entanglement theory and resource theories more generally. As discussed in our example, a method of generating entangled states using strict catalysis was previously proposed by Phoenix and Barnett (1993) and Cirac and Zoller (1994) and connected to the term catalyst by Hagley et al. (1997). To our knowledge, this is the first explicit appearance of the notion of catalysis in quantum information literature.⁴

One can also consider strictly catalytic transformations that produce the desired state on the system S approximately. In this case, instead of transforming the system into the target state σ_S , one transforms it into σ_S^{ϵ} , which is ϵ close to σ_S . Typically, ϵ is quantified using the trace distance in Eq. (30), but, depending on the situation, other notions of distance could also be used.

In general relatively little is known on how to construct suitable catalyst states. Known constructions for various types of catalysis are discussed in Sec. IV. An extreme case arises with self-catalysis (Duarte, Drumond, and Terra Cunha, 2016), where $\omega_{\rm C} = \rho_{\rm S}$, where a copy of the system catalyzes itself. Finally, one can also consider variations of strict catalysis: there are cases where two state transitions impossible on their own mutually catalyze each other (Feng, Wang, and Xu, 2002). Instead of demanding that the catalyst is returned exactly, one could try to save as much of the resources that are being lost on S in the catalyst C as possible, thereby increasing the resource content of C while facilitating a state conversion on S. This approach is called *supercatalysis* and was studied in the context of entanglement theory by Bandyopadhyay and Roychowdhury (2002). Self-catalysis, supercatalysis, and mutual catalysis remain understudied (particularly outside of LOCC), however, and provide an interesting avenue for further research.

B. Correlating catalysis

In strict catalysis, we allow for correlations between the environment E and the joint system SC, but not between S and C. When condition (2) is considered, a first step toward relaxing the requirements for strict catalysis is to allow for correlations between S and C to persist at the end of the process.

Definition III.2 (Correlating catalysis). A state transition from ρ_{S} to σ_{S} is called *correlating catalytic* if there exists a

free operation $\mathcal{F} \in \mathcal{O}$ and a quantum state ω_{C} such that the state $\sigma_{SC} \coloneqq \mathcal{F}[\rho_{S} \otimes \omega_{C}]$ fulfills

$$\operatorname{tr}_{\mathsf{C}}[\sigma_{\mathsf{SC}}] = \sigma_{\mathsf{S}}, \qquad \operatorname{tr}_{\mathsf{S}}[\sigma_{\mathsf{SC}}] = \omega_{\mathsf{C}}. \tag{49}$$

We denote a correlating-catalytic state transition by

$$\rho_{\mathsf{S}} \stackrel{\text{corr}}{\underset{\mathcal{O}}{\longleftrightarrow}} \sigma_{\mathsf{S}}.$$
 (50)

We see in Sec. V.A that, even when the set of operations \mathcal{O} consists of all possible unitary operations, nontrivial catalytic effects persist. Allowing the catalyst to retain correlations with the system enables finite-size effects in state transitions for many resource theories to be bypassed; see Sec. IV.B. Specifically, the ability to correlate the system with the catalyst radically simplifies the state-transition conditions, so a single monotone is often sufficient to characterize all possible state transformations; see Sec. III.H. If a catalyst **C** remains correlated with some system **S** and is then reused for the same state transition $\rho_{\rm S} \stackrel{\rm corr}{\underset{\mathcal{O}}{\overset{\rm corr}{\sigma}} \sigma_{\rm S}$ on a different system **S**', then the resulting joint state of both systems $\sigma_{\rm S'S}$ will in general be correlated (Vaccaro, Croke, and Barnett, 2018; Boes *et al.*, 2019, 2020),

$$\sigma_{\mathbf{S}'\mathbf{S}} \neq \sigma_{\mathbf{S}} \otimes \sigma_{\mathbf{S}'}.\tag{51}$$

As emphasized in Sec. II, whether such residual correlations are problematic, however, depends on the concrete physical context. In particular, one can envision situations where the ability to bypass finite-size effects overcomes the drawback of residual correlations. In Sec. IV.B we further see that, for a large class of resource theories, the residual correlations can be made arbitrarily small. However, Rubboli and Tomamichel (2022) showed that arbitrarily small residual correlations require arbitrarily large catalysts (as measured by resource content) in a wide class of resource theories. Recent works investigated the interplay between quantum and classical correlations that a catalyst C may develop with an external reference R under quantum channels that have local access only to C. In particular, Lie and Ng (2023) found that any such process must degrade genuinely quantum correlations between subsystems C and R.

Strict catalysis typically imposes strong constraints on state interconvertibility. A key guiding intuition of why correlations between S and C help to overcome these constraints is that, in any resource theory \mathcal{O} that allows for permuting systems, we have for any state $\rho_{S_1S_2}$ that

$$\rho_{\mathsf{S}_1}\mathsf{S}_2 \xrightarrow[\mathcal{O}]{\text{corr}} \rho_{\mathsf{S}_1} \otimes \rho_{\mathsf{S}_2}.$$
 (52)

To see this, choose a copy of S_2 as the catalyst, i.e., take $\omega_{\rm C} = \rho_{\rm S_2}$, and choose the free operation that simply swaps the catalyst C for S₂. Thus, with the help of correlating-catalytic transformations one can freely decorrelate the subsystems. Furthermore, any monotone for correlating-catalytic transformations has to fulfill $f(\rho_{\rm S_1S_2}) \ge f(\rho_{\rm S_1} \otimes \rho_{\rm S_2})$. This often rules out constructions of monotones based on Rényi

⁴The prior example, however, falls outside of the resource theory of LOCC and is therefore only slightly related to the structure of entanglement itself. Indeed, within LOCC it is not possible to generate new entanglement even when using strict catalysis.

divergences because the quantum relative entropy is the unique continuous and superadditive Rényi divergence. In other words,

$$D(\rho_{\mathsf{A}\mathsf{B}} \| \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}) \ge D(\rho_{\mathsf{A}} \| \sigma_{\mathsf{A}}) + D(\rho_{\mathsf{B}} \| \sigma_{\mathsf{B}})$$
(53)

holds for all states ρ_{AB} , while for any D_{α} with $\alpha \neq 1$ counterexamples to the inequality can be found.

Lostaglio, Müller, and Pastena (2015) and Müller and Pastena (2016) were the first to make use of the aforementioned fact, in the context of a slightly different scenario called *marginal-correlating catalysis*. In this situation, a multipartite catalyst is used, and a state transition is possible via marginalcorrelating catalysis when there exists a free operation \mathcal{F} and an initially uncorrelated catalyst

$$\omega_{\mathbf{C}_1\cdots\mathbf{C}_n} = \bigotimes_{i=1}^n \omega_{\mathbf{C}_i} \tag{54}$$

such that $\mathcal{F}[\rho_{S} \otimes \omega_{C_{1}\cdots C_{n}}] = \sigma_{S} \otimes \omega'_{C_{1}\cdots C_{n}}$, where for each i = 1, ..., n we have $\omega'_{C_{i}} = \omega_{C_{i}}$. In other words, instead of allowing for final correlations to persist between system and catalyst, one allows for correlations to exist between different parts of the catalyst.

An immediate question arises as to the relationship between correlating catalysis and marginal-correlating catalysis. If $\rho \stackrel{\text{corr}}{\longrightarrow} \sigma$ using a catalyst ω_{C_1} , and the set \mathcal{O} includes permutations of subsystems, then the state transition is also possible via marginal-correlating catalysis, using the catalyst $\tilde{\omega}_{C_1C_2} \coloneqq \omega_{C_1} \otimes \sigma_{C_2}$. To see this, first use the C_1 part of the catalyst in the same way as for the correlating-catalytic state transition $\rho_{\rm S} \stackrel{\rm corr}{\underset{\mathcal{O}}{\hookrightarrow}} \sigma_{\rm S}$, then substitute the system S for the second part of the catalyst (C_2) . Consequently, the set of marginal-correlating-catalytic state transitions includes the set of correlating-catalytic state transitions. Marginal-correlating catalysis often leads to similar (or even identical) statetransition conditions as correlating catalysis does. However, its physical significance is less clear since in general the catalyst cannot be reused, even when one starts with a fresh copy of the system. Moreover, in the resource theory of asymmetry for time translation, marginal-correlating catalysis can essentially trivialize all state-transition conditions (Takagi and Shiraishi, 2022); see also Sec. V.D.2. This can be seen as a particular form of the general embezzlement phenomenon, which we further discuss in Sec. III.E.

C. Arbitrarily strict catalysis

Thus far in our classification of catalysis, we have not allowed for any errors on the catalyst. However, from a physical point of view, this is likely a stringent restriction: it is practically impossible for any physical system to undergo an evolution and be returned in exactly the same state. Hence, for all practical purposes it should be sufficient if the catalyst can be returned with a small perturbation. This, in view of Lemma II.1, corresponds to relaxing condition (1).

A subtle point that can be easily missed is the distinction between two scenarios in which the error on the catalyst can be defined. In the first scenario, one allows for a small error for a fixed state of the catalyst. In the second scenario, one first fixes the error, then constructs the catalyst (which generally depends on the error). Perhaps surprisingly, these two scenarios are significantly different. In this section we discuss the former, then address the latter in Sec. III.D.

For clarity of presentation we assume that the errors occur on a fixed subsystem of the catalyst. To emphasize this distinction, we denote this part with A (for an auxiliary system).

Definition III.3 (Arbitrarily strict catalysis). A state transition from $\rho_{\rm S}$ to $\sigma_{\rm S}$ is called *arbitrarily strictly catalytic* if there exists a quantum state $\omega_{\rm C}$, a sequence of quantum states $\{\omega_{\rm A}^{(n)}\}_{n=0}^{\infty}$ on a fixed, finite-dimensional quantum system A, and a sequence of free operations $\mathcal{F}^{(n)} \in \mathcal{O}$ such that the following hold:

(1) For each $n \ge 1$ we have

$$\mathcal{F}^{(n)}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}} \otimes \omega_{\mathsf{A}}^{(0)}] = \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}} \otimes \omega_{\mathsf{A}}^{(n)}.$$
 (55)

(2) The state of the auxiliary system A changes an arbitrarily small amount as $n \to \infty$, i.e.,

$$\lim_{n \to \infty} \Delta(\omega_{\mathsf{A}}^{(n)}, \omega_{\mathsf{A}}^{(0)}) = 0.$$
 (56)

We denote arbitrarily strictly catalytic state transitions by

$$\rho_{\mathsf{S}} \stackrel{\text{arb}}{\underset{\mathcal{O}}{\hookrightarrow}} \sigma_{\mathsf{S}}.$$
 (57)

In the example of the resource theory of purity, an alternative but equivalent definition of arbitrarily strict catalysis was given by Gour *et al.* (2015) in their Definition 45. Arbitrarily strict catalysis was first introduced as exact catalysis by Brandão *et al.* (2015). It might be confusing as to why this catalytic type should be called exact given that an error does occur on the catalyst. However, there is a good consensus that a state transition that achieves its target state with arbitrarily good precision should be referred to as being exact if the systems involved are of fixed dimensions; see the discussions of Brandão *et al.* (2015) and Gour *et al.* (2015). Nevertheless, here we have decided to use the term arbitrarily strict catalysis to prevent misunderstanding.

One can wonder when arbitrarily strict catalysis actually differs from strict catalysis. For some resource theories, such as LOCC entanglement theory and athermality, the statetransition conditions can be highly sensitive to the rank of the density matrices. Note, however, that the rank is not a continuous function of the density operator. In particular, any density operator can be approximated arbitrarily well with a full-rank operator. In this sense, arbitrarily strict catalysis often allows such instabilities to be regularized. To see this, consider two density matrices $\rho_{\rm S}$ and $\sigma_{\rm S}$, and let $\sigma_{\rm S}^{\epsilon}$ be a fullrank approximation of $\sigma_{\rm S}$ satisfying $\Delta(\sigma_{\rm S}, \sigma_{\rm S}^{\epsilon}) < \epsilon$. Suppose that $\rho_{\rm S} \underset{O}{\hookrightarrow} \sigma_{\rm S}^{\epsilon}$ holds for any arbitrarily small $\epsilon > 0$ but not for $\epsilon = 0$. Thus, whenever O allows for permuting subsystems, we have $\rho_{\rm S} \underset{O}{\overset{\text{arb}}{\hookrightarrow}} \sigma_{\rm S}$. Specifically, choose $\omega_{\rm A}^{(0)} = \sigma$ and $\omega_{\rm A}^{(n)} =$ σ^{ϵ_n} with $\epsilon_n = 1/n$ for all n > 0 so that

$$\rho_{\mathsf{S}} \otimes \sigma_{\mathsf{A}} \underset{\mathcal{O}}{\hookrightarrow} \sigma_{\mathsf{S}}^{\epsilon_n} \otimes \sigma_{\mathsf{A}} \underset{\mathcal{O}}{\longrightarrow} \sigma_{\mathsf{S}} \otimes \sigma_{\mathsf{A}}^{\epsilon_n}, \tag{58}$$

where in the second step we swapped subsystems A and S. Thus, we see that with the help of arbitrarily strict catalytic transformations one can avoid the instabilities resulting from changing the rank of a density operator.

D. Approximate catalysis

We now discuss the second scenario that corresponds to relaxation of (1) in Lemma II.1. In this case, one fixes the magnitude of the allowed perturbation of the catalyst and then constructs its state. Generally, this leads to dynamical constraints that are less tight than in the case of arbitrarily strict catalysis. In other words, approximate catalysis usually enables state transformations that were previously impossible under arbitrarily strict catalysis. Formally, we state the following definition:

Definition III.4 (Approximate catalysis). A state transition from ρ_{S} to σ_{S} is called *e-approximate catalytic* with respect to the distance measure $d(\cdot, \cdot)$ if there are two quantum states ω_{C} and ω'_{C} and a free operation $\mathcal{F} \in \mathcal{O}$ such that

$$\mathcal{F}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}] = \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}', \qquad d(\omega_{C}', \omega_{C}) \le \epsilon.$$
(59)

We refer to $\epsilon \ge 0$ as the "smoothing parameter" and denote an ϵ -approximate catalytic state transition as

$$\rho_{\mathsf{S}} \stackrel{\text{approx}}{\underset{\mathcal{O}}{\hookrightarrow}} \sigma_{\mathsf{S}}.$$
 (60)

From a physical perspective, approximate catalysis provides a more realistic framework for investigating catalytic effects for reasons given earlier. Another perspective that supports approximate catalysis is that some physical processes, due to fundamental reasons, must change the state of the catalyst (ancilla). For example, a problem often studied in thermodynamics is that of minimizing the energy dissipated to the environment. In this case, nontrivial dynamics on the system can only be achieved when the state of the environment changes. From the perspective of approximate catalysis, this problem amounts to finding an appropriate catalytic environment that suffers minimal backaction from the system; see Sec. V.C.1 for more details.

From a mathematical perspective, approximate catalysis can be viewed as an interpolation between strict catalysis and generic activation phenomena. Activation is a phenomenon demonstrating that quantum systems processed in assistance with another are strictly more useful than systems that are processed independently. The standard example comes from thermodynamics, where the so-called passive states, i.e., states that cannot perform any thermodynamic work, can provide work when processed collectively (Lenard, 1978; Pusz and Woronowicz, 1978; Alicki and Fannes, 2013). Similar examples can be found in entanglement theory, where ancillary states can reveal nonlocal properties of quantum states (Masanes, 2006; Palazuelos, 2012; Cavalcanti et al., 2013; Yamasaki et al., 2022). In this sense, approximate catalysis puts further restrictions on how the activator might be processed, with the aim of preserving the quality of the activator for future transformations. The second extreme case of approximate catalysis is the previously discussed case of arbitrarily strict catalysis; see Sec. III.C. Therefore, for any $\epsilon \ge 0$ we obtain a different set of dynamic constraints, which translates into a different partial order between states. In the language of resource theories this is often referred to as a "regularization" of the underlying partial order. In Sec. IV.B we describe this effect in more detail. We emphasize that even though the possible state transitions depend sensitively on the error ϵ in the chosen distance measure, the realized final state of the system and catalyst in a state transformation is stable to small perturbations to their initial states. See also Vidal, Jonathan, and Nielsen (2000) for an early discussion of the robustness of catalysis in the context of LOCC.

The choice of the distance measure $d(\cdot, \cdot)$ used in Definition III.4 is essential. It not only must quantify the closeness of $\omega'_{\rm C}$ to $\omega_{\rm C}$ but should also preserve its catalytic properties (or resource content). It is perhaps surprising that these two requirements are not always simultaneously satisfied. For example, the trace distance $\Delta(\cdot, \cdot)$ is often used as a default distance measure between two quantum states. This is because of its strong operational meaning in terms of distinguishing states via an optimal measurement (Helstrom, 1969; Nielsen and Chuang, 2012), and its desirable properties as a metric fulfilling the data-processing inequality; see Sec. II.B.1. However, even if an arbitrarily small fixed perturbation $\epsilon > 0$ in trace distance is allowed on the catalyst, in a large class of resource theories any state transition can become possible. This phenomenon is known as embezzlement, and we discuss it in Secs. III.E and IV.C. One way of understanding this is that, although two states can be close to each other in terms of the trace distance, extensive quantities can still grow with the logarithm of the Hilbert-space dimension; see also Sec. VI.B. Therefore, for any given target error ϵ one can always choose a large enough dimension that the change in resource monotone is significant.

To prevent embezzlement, one can consider a stronger restriction, for example, that the deviation in an extensive quantity must be small, for example, $d(\rho, \tilde{\rho}) = |H(\rho) - H(\tilde{\rho})|$. A more stringent requirement would be to ask for the largest deviation in a set of quantities (monotones) $\{f_{\alpha}\}_{\alpha}$, for example,

$$d(\rho, \tilde{\rho}) = \sup_{\alpha} |f_{\alpha}(\rho) - f_{\alpha}(\tilde{\rho})|, \tag{61}$$

or even a linear combination of several such distance measures. As an alternative to changing $d(\cdot, \cdot)$ one can also let the allowed error ϵ depend on different parameters, such as the dimension of the state; see Brandão *et al.* (2015) and Sec. III.H. To summarize, in general the choice of $d(\cdot, \cdot)$ and the allowed error ϵ qualitatively change the landscape of possible state transitions.

When the state $\omega_{\rm C}$ of C changes, its ability to catalyze other transformations generally decreases, and therefore it becomes less useful. Still, if the final state $\omega'_{\rm C}$ of the catalyst is sufficiently close to $\omega_{\rm C}$, it may be possible to reuse it without significantly affecting the transformation on **S**. Concretely consider a free transformation \mathcal{F} that leads to a perturbed

catalyst ω'_{C} satisfying Eq. (59), where $d(\cdot, \cdot)$ satisfies the triangle inequality and the data-processing inequality; see Sec. II.B.1. Reusing the same catalyst on a new copy of the system S and \mathcal{F} leads to a state $\tilde{\sigma}_{S} \coloneqq \text{Tr}_{C}[\mathcal{F}[\rho_{S} \otimes \omega'_{C}]]$, where

$$d(\tilde{\sigma}_{\mathsf{S}}, \sigma_{\mathsf{S}}) \coloneqq d(\operatorname{Tr}_{\mathsf{C}}\mathcal{F}(\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}'), \sigma_{\mathsf{S}})$$
$$\leq d(\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}', \rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}) \leq \epsilon, \qquad (62)$$

which follows from the data-processing and triangle inequalities, with Eq. (59). Suppose that we now want to use the same catalyst *n* times to implement \mathcal{F} (each time with a fresh copy of ρ_{S}). To look at the resulting state, we introduce intermediate states for each step $0 \le i \le n$,

$$\sigma_{\mathsf{S}}^{(i+1)} \coloneqq \mathrm{Tr}_{\mathsf{C}} \mathcal{F}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{(i)}], \tag{63}$$

$$\omega_{\mathsf{C}}^{(i+1)} \coloneqq \mathrm{Tr}_{\mathsf{S}} \mathcal{F}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{(i)}], \tag{64}$$

with identifications $\sigma_{S}^{(1)} \coloneqq \sigma_{S}$ and $\omega_{C}^{(0)} \coloneqq \omega_{C}$. The state of the system and the catalyst after *k* transformations satisfy

$$d(\sigma_{\mathsf{S}}^{(k)}, \sigma_{\mathsf{S}}) \le k\epsilon, \qquad d(\omega_{\mathsf{C}}^{(k)}, \omega_{\mathsf{C}}) \le k\epsilon,$$
 (65)

which is again a consequence of the triangle inequality, the data-processing inequality, and Eq. (59). The catalyst can therefore be used approximately $k = \alpha/\epsilon$ times before it accumulates an error of the size α . Intuitively, in approximate catalysis we try to make ϵ as small as possible so that the catalyst retains its catalyzing properties. From the previous discussion it should be clear that quantifying the perturbation in the state of the catalyst, as well as identifying its usefulness for future applications, is at the core of approximate catalysis.

We close this section with a comment on nomenclature. One can also consider combinations of different types of catalysis that arise from lifting several restrictions at once. For example, by allowing for an error on the catalyst C and correlations between S and C, one can study *correlated approximate catalysis*. In fact, some simply use the term *approximate catalysis* for that setting; see Datta, Kondra *et al.* (2023).

E. Embezzlement

In the previous section, we saw that when the state of the catalyst is allowed to change, it is often possible to unlock transformations that are impossible under (arbitrarily) strict catalysis. This naturally comes at the cost of reducing the usefulness of the catalyst for future transformations. One might argue that in such cases, rather than using C catalytically, one is consuming it as a resource, nonfree state. However, sometimes it is possible for the catalyst to completely lift all of the relevant dynamical constraints while at the same time suffering almost no reduction of its catalytic capabilities. This phenomenon was first reported by van Dam and Hayden (2003), who called it embezzlement.

Embezzlement provides a mechanism for simulating forbidden dynamics on S, using operations that are free on the composite system SC while perturbing C only an arbitrarily small amount in trace distance. Formally, it is defined as follows:

Definition III.5 (Embezzlement). Given a system S and a set of free operations \mathcal{O} , a state ω_{C} is a δ embezzler if for all ρ_{S} and $\sigma_{S} \in \mathcal{D}(S)$ there exists $\mathcal{F} \in \mathcal{O}$ such that

$$\Delta(\mathcal{F}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}], \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}) \le \delta.$$
(66)

An embezzling family is a sequence of δ_n embezzlers $\{\omega_{\mathbf{C}}^{(n)}\}_n$ such that $\delta_n \to 0$ as $n \to \infty$.

At first sight, embezzlement can be viewed as a particular instance of approximate catalysis; see Sec. III.D. However, it was initially surprising that approximate catalysis can fully trivialize arbitrary state transitions when a single catalyst that is not fine-tuned for a particular initial system state is used. The first example of an embezzlement family was introduced in $\mathcal{R}_{\text{LOCC}}$ by van Dam and Hayden (2003); see Sec. V.B.5. In Sec. IV.C, we describe a general construction for embezzlers that works whenever permutations of subsystems are allowed. This construction was first used by Leung, Toner, and Watrous (2013).

Embezzlement results from the fact that trace distance is not sensitive enough to capture the difference in resource content between two quantum states. In other words, when a quantum channel \mathcal{E} transforms one state into another, it could be that there is no optimal quantum measurement that can determine (significantly better than a random guess) whether the resource has been consumed during \mathcal{E} .

We describe a simple example illustrating the aforementioned idea. Consider a wave function of a harmonic oscillator that is spread over many energy levels, such as a coherent state of light with a large photon number. Such states are usually used to describe light emitted from a laser with a fixed frequency; see Sec. V.D.1. Shifting the wave function in the energy space can change the expected energy (photon number) by a fixed amount while at the same time perturbing the state an arbitrarily small amount in terms of trace distance. In



FIG. 4. Embezzlement of energy. Consider a coherent state of light $|\alpha\rangle$ with energy levels $|i\rangle$ whose energy variance is given by $\operatorname{Var}(E) = |\alpha|^2$; see Eq. (174) for details. Left panel: change in the occupation $p_i = \langle i | \alpha | i \rangle$ of the energy level *i* obtained by shifting the state by ΔE , i.e., $|\alpha\rangle \rightarrow |\beta\rangle = S|\alpha\rangle$ with $S = \sum_{i=0}^{\infty} |i + \Delta E\rangle\langle i|$. Right panel: relationship between ΔE and the trace distance $\epsilon := (1/2) ||\alpha - \beta||_1$ for different values of α . A large energy variance allows one to change the average energy significantly while maintaining a large overlap with the initial state.

this case, even when a single photon disappears from the beam of light there is no physical process that can determine with high probability whether this process actually happened; see Fig. 4 for details.

In practice, the aforementioned mechanism can be used to implement coherent unitary operations on the system S that seemingly violate conservation of energy. This is achieved using an interaction between the system S and a laser field C that is globally energy preserving while perturbing the state of the field an arbitrarily small amount. Indeed, as we discuss in Sec. V.D.1, whenever a large-dimensional system C is used to implement a unitary dynamics on S, it can be done in such a way that the state on C is perturbed an arbitrarily small amount. Thus, in a sense embezzlement is ubiquitous in physics and occurs whenever a "macroscopic" quantum system is used to implement a controlled unitary dynamics on a smaller quantum system. In this case, the system C can be seen to act as a macroscopic reference frame for the timetranslation group; see Sec. V.D for more results relating catalysis to quantum reference frames. In Sec. VI.B we discuss the results of Coladangelo and Leung (2019), who showed how embezzlement can lead to interesting mathematical results in quantum theory, namely, that there is a limit to how continuous extensive quantities can be.

F. Infinite-dimensional catalysis

Thus far we have almost exclusively discussed quantum systems described by finite-dimensional Hilbert spaces. While this is natural from the perspective of quantum information theory, physics also requires infinite-dimensional Hilbert spaces. This is exemplified by continuous-variable systems such as quantum harmonic oscillators, or more generally by quantum field theory. It is therefore reasonable to ask how the phenomenon of catalysis changes when we allow for infinitedimensional catalysts.

In this discussion, we keep the system S finite dimensional to highlight the key differences between finite- and infinitedimensional catalysis. For a discussion of catalytic effects in continuous-variable systems and quantum optics, see Sec. V.E. We also restrict our attention to strict and correlating catalysis since embezzlement already shows that allowing for fixed errors (even arbitrarily small ones) on the catalyst trivializes the state-transition problem in infinite dimensions. Furthermore, we also discuss the notion of perfect embezzlement, which sheds light on some fundamental questions about the formulation of quantum theory. It was discovered early on by Daftuar (2004) that there is a strict difference between catalysis with (i) finite-dimensional catalysts, but of arbitrarily large dimension, and (ii) infinite-dimensional catalysts. Specifically, in \mathcal{R}_{LOCC} there are bipartite states $|\psi\rangle$ and $|\phi\rangle$ such that $|\phi\rangle$ is reachable from $|\psi\rangle$ when (ii) is used but not reachable when (i) is used. Moreover, Aubrun and Nechita (2008, 2009) showed that the target state $|\phi\rangle$ cannot be reached even with arbitrarily small error with finite-dimensional catalysts. They also made progress toward characterizing the set of reachable states (of a fixed dimension) using infinite-dimensional catalysts. However, to our knowledge no complete characterization of this set has been given. Since the previously described results are all based on majorization, they immediately transfer to the resource theory of noisy operations; see Sec. III.H.

We close this section by discussing embezzlement using infinite-dimensional systems. Based on the discussion in Sec. III.E one may be tempted to expect that an infinitedimensional catalyst C could be used to perform perfect embezzlement: there are no changes in the catalyst, while any state on a finite-dimensional system S could still be achieved (with an arbitrary accuracy). However, as is typical with infinite-dimensional spaces, subtleties exist that must be carefully addressed. Indeed, Cleve, Liu, and Paulsen (2017) and van Luijk et al. (2024a, 2024b) showed that local unitary perfect embezzlement is impossible in the framework of LOCC if spacelike separated parties are modeled by tensor products of Hilbert spaces, as is common in quantum information theory. In contrast, perfect embezzlement becomes possible in a so-called commuting operator framework; see also Sec. V.B.6. This is common in quantum field theory, where spacelike separated parties are modeled by commuting operators on a single, infinite-dimensional Hilbert space. In this context, Cleve, Liu, and Paulsen (2017) showed that for every finite dimension d there exists a quantum state $|\Omega
angle$ on a separable and infinite-dimensional Hilbert space ${\cal R}$ such that, for every $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, there are unitaries u_{ψ} and v_{ψ} on $\mathbb{C}^d \otimes \mathcal{R}$ and $\mathcal{R} \otimes \mathbb{C}^d$, respectively, that satisfy

$$(u_{\psi} \otimes \mathbb{1})(\mathbb{1} \otimes v_{\psi})|0\rangle \otimes |\Omega\rangle \otimes |0\rangle = |\psi\rangle \otimes |\Omega\rangle \qquad (67)$$

for some fixed $|0\rangle \in \mathbb{C}^d$, and such that $[u_{\psi} \otimes \mathbb{1}, \mathbb{1} \otimes v_{\psi}] = 0$. Equation (67) should be read with an implicit reordering of tensor factors.

The construction by Cleve, Liu, and Paulsen (2017) had the drawback that in general $[u_{\psi} \otimes 1, 1 \otimes v_{\varphi}] \neq 0$ if $|\psi\rangle$ and $|\varphi\rangle$ are different states (perhaps arising because both parties attempt to embezzle different states). This shows that a proper bipartite structure is missing. van Luijk *et al.* (2024a, 2024b) showed that if Minkowski spacetime is partitioned into two wedges L and R, each of which is interpreted as one local system, then all pure states in the vacuum representation of relativistic quantum field theories allow any finite-dimensional entangled state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ to be embezzled to arbitrary precision in the sense that for every $\varepsilon > 0$ there exist local unitaries $u_{\psi,\varepsilon}$ and $v_{\psi,\varepsilon}$ such that

$$\|u_{\psi,\varepsilon}v_{\psi,\varepsilon}|0\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} \otimes |\Omega\rangle - |\psi\rangle \otimes |\Omega\rangle\| < \varepsilon, \quad (68)$$

where $u_{\psi,\varepsilon}$ acts on LA and $v_{\psi,\varepsilon}$ acts on RB, respectively. Importantly, $[u_{\psi,\varepsilon}, v_{\phi,\varepsilon'}] = 0$ for all states $|\psi\rangle$ and $|\phi\rangle$ because they are elements of commuting von Neumann algebras \mathcal{M}_{LA} and \mathcal{M}_{RB} associated with different subsystems. These results are shown by establishing a quantitative connection between embezzlement of entanglement and the classification of von Neumann algebras.

In Sec. V.B.6, we see that the (non)existence of perfect embezzlers is closely related to the fact that only certain kinds of correlations can exist in quantum theory. Moreover, in Sec. V.E.4 we discuss catalysis in the context of Gaussian states and operations on continuous-variable systems, where both the catalyst C and the system S are infinite dimensional. Note, however, that the Gaussian framework is special because states and operations can be fully represented using finite-dimensional matrices.

Finally, we are not aware of studies of perfect embezzlement outside of the pure-state LOCC framework. Still, it is possible to make the following general remark: If the resource theory in question has an additive monotone f, then it must diverge on a perfect embezzler. Consider a transformation $\rho_S \rightarrow \sigma_S$ using a hypothetical perfect embezzler ω_C . Then

$$f(\rho_{\mathsf{S}}) + f(\omega_{\mathsf{C}}) \ge f(\sigma_{\mathsf{S}}) + f(\omega_{\mathsf{C}}) \tag{69}$$

by monotonicity under free operations and additivity. But since we are considering embezzlement, we can choose $\rho_{\rm S}$ and $\sigma_{\rm S}$ such that $f(\rho_{\rm S}) < f(\sigma_{\rm S})$, which leads to a contradiction with Eq. (69) unless $f(\omega_{\rm C}) = \infty$. For example, a perfect embezzler in the resource theory of quantum thermodynamics would require an infinite amount of free energy if it were to exist mathematically.

G. State-independent catalysis

One can consider the possibility that a given catalyst C prepared in a state ω_C remains a catalyst not only for a single state but also for a set of input states. Given a free operation $\mathcal{F}: \mathcal{D}(SC) \to \mathcal{D}(S'C)$ in a resource theory \mathcal{R} and a density operator ω_C , we can define the associated set of catalytic states $\mathcal{C}(\mathcal{F}, \omega_C)$ by

$$\mathcal{C}(\mathcal{F}, \omega_{\mathsf{C}}) \coloneqq \{ \rho_{\mathsf{S}} | \operatorname{tr}_{\mathsf{S}'} [\mathcal{F}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}]] = \omega_{\mathsf{C}} \}.$$
(70)

The set $C(\mathcal{F}, \omega_{C})$ may be empty. The other extreme case occurs when C is a catalyst for any input density matrix. In this case we have the following definition:

Definition III.6 (Catalytic quantum channel). The pair $(\mathcal{F}, \omega_{\mathsf{C}})$ is called a *catalytic quantum channel* if $\mathcal{C}(\mathcal{F}, \omega_{\mathsf{C}}) = \mathcal{D}(\mathsf{S})$.

Despite the natural definition, state-independent catalysis has received comparably little attention thus far. Vidal and Cirac (2002) showed how certain unitary operations could be realized under the constraints imposed by LOCC while using a catalyst that could not be realized otherwise. Furthermore, Lie and Jeong (2021b) considered the special case where $\mathcal{F}[\cdot] =$ $U(\cdot)U^{\dagger}$ is a unitary channel acting on SC. They referred to the effective quantum channel $\mathcal{E}(\cdot) \coloneqq \operatorname{Tr}_{\mathsf{C}}\{U[(\cdot)_{\mathsf{S}} \otimes \omega_{\mathsf{C}}]U^{\dagger}\}$ induced on S by the pair $(U, \omega_{\rm C})$ as a randomness-utilizing quantum channel. To see why this is an adequate description, note that such a randomness-utilizing quanum channel must be doubly stochastic, i.e., it must leave the maximally mixed state invariant. Moreover, it follows that $[U, \mathbb{1}_{S} \otimes \omega_{C}] = 0$. In particular, when $\omega_{\rm C}$ has no degenerate eigenvalues, i.e., all of its eigenspaces are one dimensional, this implies that $U = \sum_{i} U_i \otimes |i\rangle \langle i|_{\mathsf{C}}$, where $|i\rangle_{\mathsf{C}}$ is the eigenbasis of ω_{C} . The resulting dynamics on **S** is then given by a mixed-unitary channel of the form

$$\rho_{\mathsf{S}} \mapsto \sum_{i=1}^{m} p_i U_i \rho_{\mathsf{S}} U_i^{\dagger}, \tag{71}$$

where $p_i = \langle i | \omega_{\mathsf{C}} | i \rangle$.

Channels of the form in Eq. (71) can be interpreted as a nonselective measurement of classical information represented by C and a unitary processing U_i applied to S, conditioned on the measurement outcome *i*. They can also be interpreted as quantum channels that can be reversed by first measuring the environment appearing in the Stinespring dilation and then performing a correcting unitary operation to recover ρ_S (Gregoratti and Werner, 2003).

How to characterize the set of all randomness-utilizing channels is currently an open problem. Lie and Jeong (2021a) showed that they are a strict subset of doubly stochastic quantum channels. Moreover, they also showed that Uinduces a randomness-utilizing quantum channel on **S** for some state $\omega_{\rm C}$ if and only if $U^{T_{\rm S}}$ is also a unitary, where $T_{\rm S}$ is a partial transpose on **S**. In this case, the unitary U also induces a generally different randomness-utilizing channel for the maximally mixed catalyst state $\omega_{\rm C} = \mathbb{1}_{\rm C}/d_{\rm C}$. Moreover, it follows from a result of Haagerup and Musat (2011) that randomness-utilizing quantum channels are a strict superset of mixed-unitary channels (Lie, 2022).

H. Illustrative example: Noisy operations

In this section, we compare the various types of quantum catalysis that were formally defined in Sec. III. We do so by focusing on a simple yet illustrative example of majorization-based resource theories. In such resource theories the state-transition conditions are fully characterized by majorization. The paradigmatic example here is the resource theory of noisy operations (NOs) (Horodecki, Horodecki, and Oppenheim, 2003; Horodecki *et al.*, 2003), also known as the resource theory of purity or informational nonequilibrium (Gour *et al.*, 2015). In this resource theory the set of free operations \mathcal{O}_{NO} consist of all quantum channels that can be implemented with a maximally mixed environment E, i.e.,

$$T \in \mathcal{O}_{\mathrm{NO}} \Rightarrow T(\rho) = \mathrm{Tr}_{\mathsf{E}} \left[U \left(\rho_{\mathsf{S}} \otimes \frac{\mathbb{1}_{\mathsf{E}}}{d_{\mathsf{E}}} \right) U^{\dagger} \right]$$
(72)

for some dimension d_{E} . The Schur-Horn lemma (Theorem II.2) implies that

$$\rho \xrightarrow[\mathcal{O}_{NO}]{} \sigma \Leftrightarrow \lambda(\rho) \succ \lambda(\sigma).$$
(73)

We later summarize families of state-transition conditions that characterize different types of catalytic transformations. We see that these conditions can be formulated in terms of progressively fewer entropic conditions. To illustrate this, Fig. 5 shows quantum states that can be obtained using different types of catalysis when starting from a fixed initial state. Nielsen's theorem (see Theorem V.2) implies that all the results equivalently apply to the case of LOCC operations restricted to pure states. In fact, many of the results were first obtained in this setting.

For the following, note that the Rényi entropies H_{α} are *antimonotone*, as they are Schur concave, i.e.,

$$\lambda(\rho) \succ \lambda(\sigma) \Rightarrow H_{\alpha}(\rho) \le H_{\alpha}(\sigma). \tag{74}$$



FIG. 5. States achievable with noisy operations under different catalytic types. The three colored sets indicate achievable states starting with a fixed state ρ (the turquoise dot) with eigenvalues $\lambda(\rho) = [0.65, 0.2, 0.15]$ under various types of catalytic transformations. In this case ($d_S = 3$) the set of states achievable under strict catalysis coincides with the set of states that are majorized by ρ , that is, all states achievable without using a catalyst. Every state can be obtained to arbitrary precision with embezzlement.

Equation (73) then implies that the Rényi entropies can only increase under free operations.

Strict catalysis. A strictly catalytic state transformation requires a finite-dimensional density operator ω_{C} such that

$$\lambda(\rho_{\mathsf{S}}) \otimes \lambda(\omega_{\mathsf{C}}) \succ \lambda(\sigma_{\mathsf{S}}) \otimes \lambda(\omega_{\mathsf{C}}). \tag{75}$$

Conditions for strict catalysis were first derived by Klimesh (2007) and Turgut (2007) [see also the more recent works by Kribs, Pereira, and Plosker (2013) and Pereira and Plosker (2013, 2015)] and are known as trumping conditions. Here we provide a simplified but still equivalent set of conditions expressed in terms of the Rényi entropies H_{α} , as proposed by Brandão *et al.* (2015). Specifically, unless $\rho_{\rm S}$ and $\sigma_{\rm S}$ are unitarily equivalent, we have

$$\rho_{\mathsf{S}} \underset{\mathcal{O}_{\mathsf{NO}}}{\hookrightarrow} \sigma_{\mathsf{S}} \Leftrightarrow \begin{cases} H_{\alpha}(\rho_{\mathsf{S}}) < H_{\alpha}(\sigma_{\mathsf{S}}) & \forall \ \alpha \in \mathbb{R} \setminus \{0\}, \\ H_{0}(\rho_{\mathsf{S}}) \le H_{0}(\sigma_{\mathsf{S}}). \end{cases}$$
(76)

If an arbitrarily small error on S (but not C) is allowed, then the previous strict inequalities relax to nonstrict ones. Strict catalysis can enable new transformations only if the dimension of the system is large enough, i.e., $d_{\rm S} > 3$ (Jonathan and Plenio, 1999). General bounds on the catalyst (such as its dimension) were obtained by Sanders and Gour (2009) and Grabowecky and Gour (2019).

Arbitrarily strict catalysis. Since the Rényi entropies H_{α} for $\alpha > 0$ are continuous, the conditions in Eq. (76) are stable under small perturbations in the state of the catalyst, which is the case for arbitrarily strict catalysis. The conditions corresponding to $\alpha < 0$, however, may be removed. This can be achieved by introducing a qubit A initially prepared in a pure

state $\omega_A = |\psi\rangle\langle\psi|$ that is returned in a full-rank state ω'_A that is arbitrarily close to $|\psi\rangle\langle\psi|$. Now observe that any pure state has $H_\alpha(\omega_A) = \infty$ for $\alpha < 0$, and at the same time any full-rank state must have finite Rényi entropies, i.e., $H_\alpha(\omega'_A) < \infty$. This simple observation effectively removes all entropic conditions corresponding to $\alpha < 0$ in Eq. (76). Hence, unless ρ_S and σ_S are unitarily equivalent, we have

$$\rho_{\mathsf{S}} \xrightarrow{\text{arb}}_{\mathcal{O}_{\mathsf{NO}}} \sigma_{\mathsf{S}} \Leftrightarrow \begin{cases} H_{\alpha}(\rho_{\mathsf{S}}) < H_{\alpha}(\sigma_{\mathsf{S}}) & \forall \ \alpha > 0, \\ H_{0}(\rho_{\mathsf{S}}) \le H_{0}(\sigma_{\mathsf{S}}). \end{cases}$$
(77)

Correlating catalysis. The Rényi entropies H_0 and H_1 are the only Rényi entropies that are subadditive; i.e., they fulfill $H_{0/1}(\rho_{S_1S_2}) \leq H_{0/1}(\rho_{S_1}) + H_{0/1}(\rho_{S_2})$. It is therefore clear that they cannot decrease under a correlating-catalytic state transformation. Conversely, any monotone under correlating catalysis that is additive over tensor products must also be superadditive; see Sec. III.B. Müller (2018) first showed that the simultaneous increase of both H_0 and $H_1 \equiv H$ is indeed the sole criterion for the existence of a correlating-catalytic transformation in the case of noisy operations. Specifically, if ρ_S and σ_S are not unitarily equivalent, then

$$\rho_{\mathsf{S}} \xrightarrow[\mathcal{O}_{\mathsf{NO}}]{\text{corr}} \sigma_{\mathsf{S}} \Leftrightarrow \begin{cases} H_0(\rho_{\mathsf{S}}) \le H_0(\sigma_{\mathsf{S}}), \\ H(\rho_{\mathsf{S}}) < H(\sigma_{\mathsf{S}}). \end{cases}$$
(78)

Since any state can be approximated (up to an arbitrary accuracy) by a state with a full rank, then if state transformations up to an arbitrarily small error (on S, not C) are considered, the only condition that remains is $H(\rho_S) \leq H(\sigma_S)$. In fact, we see in Sec. V.A that already correlating-catalytic unitary state transformations (i.e., without the environment E) yield the same set of state transitions (Boes *et al.*, 2019; Wilming, 2021, 2022).

Approximate catalysis. As approximate catalysis depends on the distance measure used, no general statement can be made. If a finite error $\epsilon > 0$ in terms of the trace distance is allowed on the catalyst **C**, then any state transition becomes possible; see Sec. IV.C. However, if an error of the order of $1/\log(d_{\rm C})$ is allowed, with $d_{\rm C}$ the dimension of the Hilbert space of the catalyst, then $H(\rho_{\rm S}) < H(\sigma_{\rm S})$ is the only remaining condition for state transitions (Brandão *et al.*, 2015). In other words, we essentially obtain the same statetransition conditions as we do for correlating catalysis. The interplay between the error ϵ on the catalyst and the resulting simplification of the state-transition conditions was analyzed by Ng *et al.* (2015) and (Lipka-Bartosik and Skrzypczyk (2021a).

Embezzlement. Per the definition of embezzlement, any state can be reached to arbitrary accuracy via embezzlement. A general construction of embezzling families that achieve this is given in Sec. IV.C. The first example of an embezzling family $\{\omega_{\rm C}^{(n)}\}$ was presented by van Dam and Hayden (2003) in the context of LOCC. In the context of an NO it takes the form

$$\omega_{\mathsf{C}}^{(n)} = \frac{1}{C_n} \sum_{j=1}^n \frac{1}{j} |j\rangle \langle j|, \qquad C_n = \sum_{j=1}^n \frac{1}{j}, \qquad (79)$$



FIG. 6. Embezzlement in noisy operations. The maximal amount of entropy change ΔH between the state $\omega_{C}^{(n)}$ [see Eq. (79)] and its δ perturbation is shown for different dimensions *n*.

for some orthonormal basis $\{|j\rangle\}$; see also Fig. 6. George and Chitambar (2023) provided a detailed discussion on the optimality of this construction.

Infinite-dimensional catalysis. The same remarks as given in Sec. III.F apply here to the case of pure-state LOCC.

State-independent catalysis. To our knowledge, the set of catalytic quantum channels for noisy operations has not yet been characterized. However, the randomness-utilizing channels from Sec. III.G constitute a subset.

IV. CONSTRUCTING CATALYSTS

The transformation laws for quantum catalysis rarely tell us anything about the state of the catalyst that enables a desired transformation. Therefore, most current results treat catalytic transformations as enigmatic "black boxes." While it is theoretically possible to achieve certain transformations with some catalysts, it is unclear how to identify the suitable catalyst state or determine the actual transformation. This limitation severely hinders the practical applicability of catalysis. In this section, we describe the few existing results on the explicit constructions of catalysts.

A. From multicopy transformations to strict catalysis

In many resource theories [denoted by $\mathcal{R} = (\mathcal{S}, \mathcal{O})$], one commonly adopted assumption is the ability to condition operations on classical randomness. Consider a system A described by a state ω_A diagonal in some fixed basis $|i\rangle_A$. We view this system as classical due to the distinguished basis. Let S be another system in state ρ_S , and let $\{\mathcal{F}_i\}_i$ be a set of free operations on S, i.e., $\mathcal{F}_i \in \mathcal{O}$ for every *i*. The transformation

$$\rho_{\mathsf{S}} \otimes \omega_{\mathsf{A}} \mapsto \sum_{i} \mathcal{F}_{i}[\rho_{\mathsf{S}}] \otimes |i\rangle \langle i| \omega_{\mathsf{A}}|i\rangle \langle i| \tag{80}$$

is then also a free operation on SA. The resource theories presented in Sec. II.A all permit such operations for free. Note that if ω_A is diagonal in the basis of $|i\rangle_A$, then system A retains its marginal density operator (but potentially builds up correlations with S). If the resource theory in question additionally allows classical randomness to be prepared (i.e., any state diagonal in $|i\rangle_A$ of an arbitrary dimension)



FIG. 7. Strict catalysis emulating multicopy transformations. Illustration of the operation implementing the transformation $\rho_{\rm S} \otimes \omega_{\rm C}^{(n)}(\rho, \sigma) \rightarrow \sigma_{\rm S} \otimes \omega_{\rm C}^{(n)}(\rho, \sigma)$. The dark purple dots signify ρ , the lighter green triangles represent σ , and the rows correspond to the states $|i\rangle_{\rm A}$. The free transformation \mathcal{F} is applied to the dashed box, followed by the cyclic permutations indicated by the gray arrows.

for free and allows subsystems to be discarded, then the sets S and O must be convex.

Now consider the situation where a state $\rho_{\rm S}$ cannot be transformed to $\sigma_{\rm S}$ but, for some sufficiently large $n \in \mathbb{N}$, the multicopy state $\rho_{\rm S}^{\otimes n}$ can be transformed to $\sigma_{\rm S}^{\otimes n}$. This can be seen as a form of activation (see Sec. III.D) and was first observed in the context of LOCC by Bandyopadhyay, Roychowdhury, and Sen (2002). A key observation is that if \mathcal{R} allows one to permute identical subsystems and condition operations on classical information, then $\rho_{\rm S}$ can be transformed to $\sigma_{\rm S}$ via strict catalysis,

$$\rho_{\mathsf{S}}^{\otimes n} \xrightarrow{\mathcal{O}} \sigma_{\mathsf{S}}^{\otimes n} \Rightarrow \rho_{\mathsf{S}} \xrightarrow{\mathcal{O}} \sigma_{\mathsf{S}}.$$
(81)

Therefore, strict catalysis can be used to reduce multicopy transformations to catalytic transformations. The converse of this statement is false in general, as demonstrated by Feng, Duan, and Ying (2006); see also Aubrun and Nechita (2009) and Gupta *et al.* (2022).

Equation (81) is shown by providing a general construction for the required catalyst. This construction first appeared in the context of LOCC transformations given by Duan *et al.* (2005a). Consider the density operator

$$\omega_{\mathsf{C}}^{(n)}(\rho,\sigma) \coloneqq \frac{1}{n} \sum_{i=1}^{n} \rho^{\otimes i-1} \otimes \sigma^{\otimes n-i} \otimes |i\rangle \langle i|_{\mathsf{A}} \qquad (82)$$

acting on n - 1 tensor copies of the Hilbert space \mathcal{H}_S and an *n*-dimensional Hilbert space \mathcal{H}_A representing classical information. Thus, $C = S_2 \cdots S_n A$, and we later use the convention $S = S_1$. We can envision the state as corresponding to *n* distinguishable boxes, where the *i*th box contains i - 1 particles in the state ρ and n - i particles in the state σ . The boxes are distinguished by the classical label *i*, which is unknown.

Suppose that we can perform a free transformation $\mathcal{F} \in \mathcal{O}$ such that $\mathcal{F}[\rho^{\otimes n}] = \sigma^{\otimes n}$ for some $n \in \mathbb{N}$. The following protocol then transforms a single copy of ρ_{S} into σ_{S} using $\omega^{(n)}(\rho, \sigma)$ as a strict catalyst (see Fig. 7 for an illustration):

(1) Apply a conditional operation as in Eq. (80). Perform \mathcal{F} on systems $S_1 \cdots S_n$ if A is in state $|n\rangle\langle n|$, and the identity otherwise.

(2) Cyclically permute the quantum part $S_1 \cdots S_n$ of the system SC such that

$$\rho^{\otimes i} \otimes \sigma^{\otimes n-i} \otimes |i\rangle \langle i|_{\mathsf{A}} \to \sigma \otimes \rho^{\otimes i} \otimes \sigma^{\otimes n-i-1} \otimes |i\rangle \langle i|_{\mathsf{A}}.$$

(3) Cyclically relabel the classical register A of C, i.e., map $|i\rangle \rightarrow |i+1\rangle$ for i < n and $|n\rangle \rightarrow |1\rangle$.

As a result, the joint state of the system SC transforms as

$$\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{(n)}(\rho, \sigma) \to \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{(n)}(\rho, \sigma).$$
(83)

The dimension of the catalyst $\omega_{C}^{(n)}(\rho, \sigma)$ increases exponentially with the number of necessary copies *n* for multicopy activation. Furthermore, the catalyst must also be finely tuned with respect to the initial and target states on S. Despite these disadvantages, the construction still serves as a key technical tool in various proofs of catalysis.

B. From asymptotic transformations to correlating catalysis

The main problem addressed in any resource theory is determining whether there exists a free operation $\mathcal{F} \in \mathcal{O}$ such that $\mathcal{F}(\rho) = \sigma$. This is generally a difficult problem that has to be addressed on a case-by-case basis. In general, there exists no systematic way of characterizing which transitions are allowed in a given resource theory. One potential approach to better understand what is achievable within a particular resource theory is to employ reasonable relaxations.

The first relaxation that we discuss is to allow for a small error in the final system state, which is also called smoothing. Specifically, given a fixed error ϵ we allow the channel to produce any state $\sigma_{\epsilon} = \mathcal{F}[\rho]$ that is ϵ close in trace distance to the target state σ . Owing to the definition of trace distance, no observables of σ_{ϵ} differ much from σ . The second relaxation is assuming an i.i.d. distribution; that is, instead of considering a direct transformation from ρ to σ , we ask whether it is possible to convert $\rho^{\otimes n}$ into $\sigma^{\otimes \lfloor rn \rfloor}$ for some r > 0 and a large enough $n \in \mathbb{N}$. One key insight is that for large *n* the multicopy state $\rho^{\otimes n}$, when smoothed, is almost indistinguishable from the maximally mixed state spanned over its typical subspace (Wilde, 2009). Therefore, characterizing state transformations between such states is much easier than solving the corresponding single-shot problem. The standard quantity studied in this smooth asymptotic limit is the conversion rate, which for any set of free operations \mathcal{O} can be defined as

$$R^{n}_{\epsilon}(\rho,\sigma) \coloneqq \sup\{r|\inf_{\mathcal{F}\in\mathcal{O}}\Delta(\mathcal{F}[\rho^{\otimes n}],\sigma^{\otimes \lfloor nr \rfloor}) \leq \epsilon\}.$$
(84)

The rate $R_{\epsilon}^{n}(\rho, \sigma)$ represents approximately how many copies of the target state can be generated per copy of the initial state when using *n* copies of the initial state. Of particular importance is the asymptotic rate R_{ϵ}^{n} in the limit of infinite copies, i.e.,

$$R^{\infty}(\rho,\sigma) = \lim_{n \to \infty} R^n_{\epsilon}(\rho,\sigma).$$
(85)

Importantly, $R^{\infty}(\rho, \sigma)$ does not depend on ϵ as long as $\epsilon > 0$. Note that the asymptotic rate R^{∞}_{ϵ} allows the approximate conversion problem to be fully characterized in the asymptotic limit. Indeed, when $R^{\infty}(\rho, \sigma) > 1$, there exists an $n \in \mathbb{N}$ and a free operation mapping $\rho^{\otimes n}$ onto a state that is ϵ close to $\sigma^{\otimes n}$. Conversely, when $R^{\infty}(\rho, \sigma) < 1$, we are guaranteed that there is no free operation that can ever map *n* copies of ρ onto *n* copies of σ for an arbitrary small error ϵ . In this sense, the partial order of states becomes regularized, or simplified, in the approximate i.i.d. limit. All possible state transformations can now be characterized by a single quantity, the rate, as opposed to the single-shot case, in which they are usually characterized by a set of monotones.

It was first observed by Shiraishi and Sagawa (2021) in the context of thermodynamics that asymptotic state transformations and correlating-catalytic state transformations are closely related using a generalization of the construction employed by Duan *et al.* (2005a) that is presented in Sec. IV.A. Specifically, the following lemma can be proven, which can be seen as a generalized summary of the results of Char *et al.* (2021), Kondra, Datta, and Streltsov (2021), Lipka-Bartosik and Skrzypczyk (2021b), Shiraishi and Sagawa (2021), Wilming (2021), Takagi and Shiraishi (2022), and Datta, Ganardi *et al.* (2023).

Lemma IV.1. Let ρ_S and σ_S be two density operators, and let $\mathcal{R} = (S, \mathcal{O})$ be a resource theory that allows for classical conditioning and permutation of identical subsystems. Then

$$R_{\epsilon}^{n}(\rho_{\mathsf{S}},\sigma_{\mathsf{S}}) \ge 1 \Rightarrow \rho_{\mathsf{S}} \stackrel{\text{corr}}{\underset{\mathcal{O}}{\hookrightarrow}} \sigma_{\mathsf{S}}^{\epsilon}, \tag{86}$$

with $\Delta(\sigma_{\rm S}^{\epsilon}, \sigma_{\rm S}) \leq \epsilon$. In fact, the catalyst $\omega_{\rm C}$ can be chosen such that $\Delta(\eta_{\rm SC}, \sigma_{\rm S} \otimes \omega_{\rm C}) \leq 2\epsilon$, where $\eta_{\rm SC}$ is the final state on SC.

The lemma shows that whenever it is possible to transform n copies of ρ_S into n copies of σ_S with a given accuracy, there also exists a correlating-catalytic transformation that converts a single copy of ρ_S into σ_S with the same accuracy. When the transformation can be performed without error, i.e., when $\epsilon = 0$, the mapping becomes strictly catalytic and no correlations are established between catalyst and system, thus recovering the setting of Sec. IV.A.

Let \mathcal{F} be a free operation such that $\Delta(\mathcal{F}[\rho_{S}^{\otimes n}], \sigma_{S}^{\otimes n}) \leq \epsilon$, which exists whenever $R_{\epsilon}^{n}(\rho_{S}, \sigma_{S}) \geq 1$. To prove Lemma IV.1 by construction, one follows the same strategy as in Sec. IV.A but replaces the states σ_{S}^{n-i} in the definition of $\omega_{C}^{(n)}$ with the reduced state $\operatorname{tr}_{>n-i}[\mathcal{F}[\rho_{S}^{\otimes n}]]$ on the first n-i marginals. The result then follows by making use of the data-processing inequality for the trace distance.

In certain resource theories (such as majorization-based resource theories), it is known that R^{∞} can be directly computed and expressed using an appropriate resource monotone. Let \mathcal{R} be a resource theory whose asymptotic rate is given by

$$R^{\infty}(\rho_{\mathsf{S}}, \sigma_{\mathsf{S}}) = \frac{f(\rho_{\mathsf{S}})}{f(\sigma_{\mathsf{S}})} \quad \text{for any } \epsilon > 0, \tag{87}$$

where f is a monotone of the resource theory. For example, as we see in Sec. V.B.2, f corresponds to the entanglement entropy in the context of pure-state entanglement. Lemma IV.1 then shows that $f(\rho_S) > f(\sigma_S)$ implies that a correlating-catalytic state transformation is possible. To our knowledge, it is currently unknown if the possibility of a correlating-catalytic state transition also implies that the asymptotic rate fulfills $R^{\infty}(\rho_S, \sigma_S) \ge 1$.

The previous result shows not only that correlating-catalytic state transformations are strictly more powerful than the corresponding single-shot free operations but also that they sometimes have a simple mathematical characterization in terms of the asymptotic resource monotone determining the rate R^{∞} . In Sec. V we review and describe applications of the previous results in different physical settings.

C. Constructions for approximate catalysts and embezzlers

In Sec. IV.B we showed that correlating catalysis enables state transitions that are otherwise possible only in an approximate sense in the asymptotic limit. In this section we describe a generic construction for approximate catalysis that allows one to map between arbitrary states whenever the set of free operations O allows identical subsystems to be permuted. We then generalize it to construct an embezzler for state transitions on a fixed finite-dimensional system S. The construction explicitly demonstrates that a fixed error in trace distance is not sensitive enough to prevent embezzlement. We follow Leung, Toner, and Watrous (2013), but the construction is closely related to the construction of strict catalysts to emulate multicopy transformations presented in Sec. IV.A.

Suppose that we want to implement the state transition $\rho_S \rightarrow \sigma_S$ using a catalyst ω_C that is perturbed by at most ϵ in trace distance. Consider the states

$$\omega_{\mathsf{C}} = \frac{1}{n-1} \sum_{k=1}^{n-1} \rho_{\mathsf{S}}^{\otimes k} \otimes \sigma_{\mathsf{S}}^{\otimes n-k}, \tag{88}$$

$$\omega'_{\mathsf{C}} = \frac{1}{n-1} \sum_{k=2}^{n} \rho_{\mathsf{S}}^{\otimes k} \otimes \sigma_{\mathsf{S}}^{\otimes n-k}.$$
(89)

By applying a cyclic permutation of subsystems represented by the unitary U_{π} , we find $U_{\pi}(\rho_{\rm S} \otimes \omega_{\rm C})U_{\pi}^{\dagger} = \sigma_{\rm S} \otimes \omega_{\rm C}'$. Since

$$\omega_{\mathsf{C}}' - \omega_{\mathsf{C}} = \frac{1}{n-1} [\rho_{\mathsf{S}}^{\otimes n} - \rho_{\mathsf{S}} \otimes \sigma_{\mathsf{S}}^{\otimes n-1}], \qquad (90)$$

we find that $\Delta(\omega_{\rm C}, \omega'_{\rm C}) \leq 1/(n-1)$. Thus, for fixed ϵ and sufficiently large *n* any state transition is possible using the given construction for approximate catalysts.

Note that the construction in Eq. (90) currently depends on $\rho_{\rm S}$ and $\sigma_{\rm S}$. However, using it we can construct a universal embezzler for any fixed system S with finite-dimensional Hilbert space: Since the set of density operators on a finite-dimensional Hilbert space is compact, for any $\delta > 0$ there exists a finite collection of density operators $\{\chi_{\rm S}^{(i)}\}_{i=1}^{N}$ such that for any density operator $\rho_{\rm S}$ we have $\Delta(\rho_{\rm S}, \chi_{\rm S}^{(i)}) \leq \delta/4$ for some $1 \leq i \leq N$. The size of this set N depends on δ . For any $1 \leq i, j \leq N$, let $\omega_{\rm C_{ij}}^{(i,j)}$ be the approximate catalyst

implementing the state transition $\chi_{S}^{(i)} \rightarrow \chi_{S}^{(j)}$. The embezzler consisting of N^2 copies C_{ij} with state

$$\omega_{\mathsf{C}}^{(\delta)} = \bigotimes_{i,j=1}^{N} \omega_{\mathsf{C}_{ij}}^{(i,j)} \tag{91}$$

is then a δ embezzler. Suppose that we want to transform ρ_{S} to σ_{S} . We need only to identify $\chi_{S}^{(i)}$ and $\chi_{S}^{(j)}$ such that

$$\Delta(\rho_{\mathsf{S}}, \chi_{\mathsf{S}}^{(i)}) \le \delta/4, \qquad \Delta(\sigma_{\mathsf{S}}, \chi_{\mathsf{S}}^{(j)}) \le \delta/4.$$
(92)

Using $\omega_{C}^{(\delta)}$ as a catalyst, one finds that there is a free operation \mathcal{F} consisting only of permutations involving **S** and the subcatalyst C_{ij} such that

$$\Delta(\mathcal{F}[\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{(\delta)}], \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{(\delta)}) \le \delta.$$
(93)

A related formulation was used by Datta *et al.* (2022) to construct a correlated catalyst for all state transitions in LOCC between pure states where the entanglement entropy is non-increasing. In the context of LOCC, the structure of embezzling families has been studied in much more detail and can be characterized comprehensively; see van Dam and Hayden (2003), Leung and Wang (2014), and Zanoni, Theurer, and Gour (2023).

We close the section with a word of caution on nomenclature: According to our classification of catalysis, the previously constructed states $\omega_{\rm C}$ are approximate catalysts, and we reserve the term universal embezzler for $\omega_{\rm C}^{(\delta)}$ because only the latter allow arbitrary state transitions to be implemented with vanishing error (as $N \to \infty$). Yet, at times the states $\omega_{\rm C}$ have already been referred to as universal embezzlers in the literature, with the idea that the notion of embezzlement refers to the fact that approximate catalysis in terms of a fixed trace distance allows for arbitrary state transitions. In Sec. VI.B we discuss how the aforementioned construction can be used to constrain continuity properties of resource monotones.

D. Numerical construction of catalysts

In this section, we discuss a simple yet useful method for computing the exact state of the catalyst. Specifically, for a fixed quantum channel \mathcal{E}_{SC} and a fixed input state on S, this method determines the corresponding state of the catalyst, namely, the fixed point of the described quantum channel on C. To our knowledge, this approach was first used in the context of catalysis by Boes *et al.* (2020).

Let $\mathcal{E} \in \mathcal{L}(\mathcal{H}_{S} \otimes \mathcal{H}_{C})$ be an arbitrary quantum channel acting on SC, and let $\rho_{S} \in \mathcal{D}(\mathcal{H}_{S})$ be a fixed density operator. The effective channel acting on C can be written as

$$\mathcal{E}_{\mathsf{C}}(\cdot) = \mathrm{Tr}_{\mathsf{S}} \mathcal{E}[\rho_{\mathsf{S}} \otimes (\cdot)_{\mathsf{C}}]. \tag{94}$$

Suppose that we want to implement the correlating-catalytic transformation discussed in Sec. III.B. This means that we have to find a density operator $\omega_{\rm C}$ that satisfies $\mathcal{E}_{\rm C}(\omega_{\rm C}) = \omega_{\rm C}$. This is equivalent to finding the fixed points of the map $\mathcal{E}_{\rm C}$,

which exist due to Brouwer's fixed point theorem (Wolf, 2012). This problem can be formulated as a semidefinite program (SDP), i.e.,

$$\begin{array}{ll} \min_{X} & 0 \\ \text{subject to} & \mathcal{E}_{\mathsf{C}}[X] = X, \\ & X \geq 0, \ \mathrm{Tr}X = 1. \end{array}$$

The solution of Eq. (95) is a positive semidefinite operator X, which can be interpreted as the quantum state of the correlated catalyst, i.e., $\omega_{\rm C} = X$. The main advantage of semidefinite programs stems from the fact that they can be efficiently solved numerically (Boyd, Boyd, and Vandenberghe, 2004), for example, using the modeling language CVX (Grant and Boyd, 2014). Moreover, in certain cases some essential features of the solution can even be inferred analytically; see Napoli et al. (2016), Cavalcanti and Skrzypczyk (2017), and Bavaresco, Murao, and Quintino (2021). As an example in the context of catalysis, note that the output state $\sigma_{\rm S} =$ $\operatorname{Tr}_{\mathbf{C}} \mathcal{E}[\rho_{\mathbf{S}} \otimes \omega_{\mathbf{C}}]$ is, a priori, arbitrary. However, if some additional information about \mathcal{E} is available, it may even be possible to deduce some essential features of $\sigma_{\rm S}$ analytically. Boes et al. (2020) successfully used this technique in the context of work extraction from multipartite systems to show that correlating catalysis allowed the stringent conditions imposed by Jarzynski's inequality to be surpassed.

The formulation in Eq. (95) allows additional constraints on the catalyst state to be added as long as they can be formulated in terms of semidefinite constraints. This freedom can be used to investigate other regimes of catalysis. To see this, consider a more general version of the problem in Eq. (95),

$$\min_{X} F(X)$$
subject to $\|\mathcal{E}_{\mathsf{C}}(X) - X\|_{1} \le \epsilon$, (96)
 $X \ge 0$, $\operatorname{Tr} X = 1$,

where *F* is a linear function of *X* and $\epsilon > 0$. Equation (96) is still an SDP, as can be seen using a standard reformulation of the trace norm $\|\cdot\|_1$ in terms of a semidefinite constraint (Vandenberghe and Boyd, 1996). Equation (96) allows the approximate catalysts discussed in Sec. III.D to be determined by taking F(X) = 0, and fixing $\epsilon > 0$ allows the state of an approximate catalyst to be found.

One can also consider nonlinear functions $F(\cdot)$ as long as they can themselves be expressible through semidefinite programs. An example of such a function is the trace distance, i.e., $\|\cdot\|_1$; see Watrous (2009). Taking F(X) = $\|\mathcal{E}(\rho_S \otimes X_C) - \sigma_S \otimes X_C\|_1$ allows one to determine the state of the catalyst that ends up being the least correlated with the system S. When the SDP problem defined according to this recipe achieves $F(X^*) = 0$, the resulting optimal variable X^* corresponds to the strict catalyst discussed in Sec. III.A.

At this point, we note that from a resource-theoretic perspective the aforementioned method is limited, as it requires \mathcal{E}_{SC} and ρ_S to be fixed. Therefore, it does not allow the catalyst that would enable a given state transformation on the system to be found. However, in many realistic applications the available joint quantum channel \mathcal{E}_{SC} either is

fixed or can be parametrized using a small number of parameters. This happens in experimentally relevant models of light-matter interactions like the Jaynes-Cummings (Jaynes and Cummings, 1963) and Dicke models (Hepp and Lieb, 1973). Moreover, owing to experimental capabilities the set of states $\rho_{\rm S}$ prepared is usually also restricted and efficiently parametrized. In such special cases, the previously described method of determining the catalyst performs reasonably well. This approach was used by de Oliveira Junior *et al.* (2023) to demonstrate the effect of catalytic activation of Wigner negativity (see also Sec. V.E.2) and by Lipka-Bartosik, Perarnau-Llobet, and Brunner (2023) to show a catalytic enhancement in cooling or heating using a single-mode optical cavity.

Finally, we note that, beyond the regime of analytical constructions and convex optimization, the avenue for determining whether $\rho \rightarrow \sigma$ is possible using a certain type of catalyst remains a space for much potential exploration. For example, by training a neural network to learn majorization, machine learning techniques were implemented to see if they could successfully identify whether a transition could be achieved catalytically (Acácio and Duarte, 2022). Similar techniques could be used to learn how to identify a catalyst that activates the desired transformation.

V. APPLICATIONS OF CATALYSIS

A. Unitary quantum mechanics

One of the simplest resource theories that one could imagine is given by unitary quantum mechanics $\mathcal{R}_{OM} =$ $(S_{QM}, \mathcal{O}_{QM})$. The set of free states S_{QM} is empty (every state is resourceful) and the free operations \mathcal{O}_{QM} are all simply unitary transformations. That means that all free operations are reversible and randomness (even classical) is costly. In particular, the resource theory is not convex and the tools of Sec. IV.B do not directly apply. While perhaps less plausible physically at first sight, \mathcal{R}_{OM} serves as an interesting test bed to distinguish among different types of catalysis. First, the law for state transitions is simple: Two states ρ_{S} and σ_{S} on the same Hilbert space can be converted into each other if and only if their eigenvalues (including multiplicities) are identical. Second, Lemma II.1 implies that strict catalysis is useless in this scenario. However, correlating catalysis was shown to enlarge the set of states that can be reached. Specifically, Boes et al. (2019) and Wilming (2021, 2022) showed that the set of achievable states is described by a statement akin to the second law of thermodynamics.

Theorem V.1. Let ρ_S and σ_S be density matrices on a finitedimensional Hilbert space that are not unitarily equivalent. Thus, the following two statements are equivalent:

(1) There exists a finite-dimensional density operator σ_{C} and a unitary operator U on SC such that

$$\operatorname{tr}_{\mathsf{C}}[U\rho_{\mathsf{S}}\otimes\sigma_{\mathsf{C}}U^{\dagger}]=\sigma_{\mathsf{S}},\qquad(97)$$

$$\operatorname{tr}_{\mathsf{S}}[U\rho_{\mathsf{S}}\otimes\sigma_{\mathsf{C}}U^{\dagger}]=\sigma_{\mathsf{C}}.$$
(98)

(2)
$$H(\rho_{\mathsf{S}}) < H(\sigma_{\mathsf{S}})$$
 and $\operatorname{rank}(\rho_{\mathsf{S}}) \leq \operatorname{rank}(\sigma_{\mathsf{S}})$.

The same results hold in the classical case where density matrices are replaced by probability vectors and unitary transformations are replaced by permutations. By continuity of von Neumann entropy, the theorem implies that any state $\sigma_{\rm S}$ can be reached from $\rho_{\rm S}$ to arbitrary accuracy using correlating catalysis if and only if $H(\sigma_S) \ge H(\rho_S)$. Thus, we find a complete operational characterization of von Neumann entropy without any reference to thermodynamics, entanglement, or information theory. Previously, Müller (2018) showed that Theorem V.1 holds if one allows for general mixed-unitary quantum channels instead of reversible unitary channels. Boes et al. (2019) used this to generalize a classic result of Aczél, Forte, and Ng (1974) for Shannon entropy to the quantum case: If a continuous function on density matrices is (a) invariant under unitary transformations, (b) additive over tensor products, and (c) subadditive, then it is given by von Neumann entropy up to rescaling and a dimensiondependent shift of the origin. In particular, if appropriate normalization for maximally mixed states and pure states is required, this uniquely singles out von Neumann entropy.

In passing, we remark that the conditions in Eqs. (97) and (98) also appeared in Deutsch's analysis of quantum mechanics near closed timelike curves (D. Deutsch, 1991), where the condition that the state on C remains the same is a consistency requirement to prevent certain logical paradoxes. Deutsch already observed $H(\sigma_S) \ge H(\rho_S)$ and that some σ_C and σ_S fulfilling Eqs. (97) and (98) always exist once ρ_S and U are specified.

Where does the entropy increase on S come from? It corresponds precisely to the correlations built up between the system S and the catalyst C when measured in terms of mutual information,

$$H(\sigma_{\mathsf{S}}) - H(\rho_{\mathsf{S}}) = I(\mathsf{S}:\mathsf{C})_{U\rho_{\mathsf{S}}\otimes\sigma_{\mathsf{C}}U^{\dagger}}.$$
(99)

How large does the system C need to be? When $\rho_S > \sigma_S$, the catalyst C can be chosen to be a maximally mixed state of dimension at most $\lceil \sqrt{d_S} \rceil$ (Boes *et al.*, 2018), corresponding to a regime where the entropy increase on S can be considered large. However, as $H(\sigma_S) - H(\rho_S)$ becomes small, it can be shown that there exists catalytic state transitions requiring arbitrarily large catalyst dimensions (Boes, Ng, and Wilming, 2022). In particular, if $H(\rho_S) = H(\sigma_S)$, either the two states are unitarily equivalent or there does not exist a finite-dimensional catalyst. It is currently unknown whether an infinite-dimensional catalyst can be used in this case.

In Sec. V.C.6 we discuss some applications of Theorem V.1 in the context of quantum thermodynamics. Gallego *et al.* (2018) further used the previous result of Müller (2018) to show that in principle a single catalyst can be used to bring an arbitrary many many-body systems that are initially in equilibrium permanently out of equilibrium.

B. Entanglement theory

Entanglement is perhaps the most striking manifestation of the nonclassical nature of quantum mechanics (Einstein, Podolsky, and Rosen, 1935). After its first experimental demonstration (Aspect, Dalibard, and Roger, 1982), researchers realized that entanglement could be used as a resource enabling new types of protocols, including new communication tasks (Bennett *et al.*, 1993) and unconditionally secure cryptographic schemes (Bennett and Brassard, 1984; Ekert, 1991).

The most common approach for studying entanglement is the so-called distant-lab paradigm, which was introduced in Sec. II.A.1. Consider the scenario where a multipartite quantum system is distributed to spatially separated parties, who are restricted to act locally on their respective subsystems by performing local quantum operations. A common assumption is that the parties can exchange classical information in order to enhance their measurement strategies. Quantum operations implemented in this manner are known as LOCC. When communication is not allowed, such as in Bell nonlocality, the relevant set of operations involves local operations and shared randomness (LOSR). In this paradigm parties cannot communicate; however, they are allowed to share classical randomness. This captures the natural restrictions encountered in Bell-like experiments and nonlocal games (Buscemi, 2012). In both cases, the set of free states $S_{\text{LOCC/LOSR}}$ is given by classically correlated states. In the bipartite scenario involving Alice (A) and Bob (B), they take the form

$$\rho_{\mathsf{A}\mathsf{B}} = \sum_{i} p_{i} \rho_{\mathsf{A}}^{i} \otimes \rho_{\mathsf{B}}^{i} \tag{100}$$

for some probability distribution p_i . Since the LOSR operation is an LOCC protocol without communication, the LOSR set is a strict subset of LOCC. By definition, neither LOSR nor LOCC operations can create entanglement. Both of these classes have a fairly intuitive physical description; however, they are notoriously difficult to characterize mathematically (Chitambar *et al.*, 2014). In this section, we review the role of catalysis in LOCC (Sec. V.B.1) and LOSR (Sec. V.B.5) transformations.

1. Local operations and classical communication

A physical operation is called a one-way LOCC operation from A to B when it can be implemented by applying arbitrary local quantum operations by Alice (A) and Bob (B), and a single round of classical communication from A to B. More generally, n-LOCC operations involve exchanging n rounds of classical communication. The class of LOCC operations LOCC(A:B) between Alice and Bob is the union of all *n*-LOCC operations. It is known that (n + 1)-LOCC operations are strictly more powerful than *n*-LOCC operations (Chitambar et al., 2014). In general the outcome of an LOCC operation on an initial state ρ consists of classical measurement outcomes x and the associated conditional quantum states σ_x . In other words, an LOCC operation corresponds to a quantum instrument $\{\mathcal{F}_x\}$, where \mathcal{F}_x are completely positive maps such that $\mathcal{F} \coloneqq \sum_x \mathcal{F}_x$ is a quantum channel and $\sigma_x = \mathcal{F}_x[\rho]/p_x$ with $p_x \coloneqq \text{Tr}[\mathcal{F}_x[\rho]]$. We say that an LOCC operation converts ρ to σ (deterministically) if $\sigma_x = \sigma$ for all x. The success probability to convert ρ to σ via the LOCC operation $\{\mathcal{F}_x\}$ is the total probability of the events x that have σ as the outcome,

$$p_{\text{succ}}(\rho \to \sigma | \{\mathcal{F}_x\}) \coloneqq \sum_{x:\sigma_x = \sigma} p_x.$$
 (101)

The optimal success probability $p_{succ}(\rho \rightarrow \sigma)$ to convert ρ into σ via LOCC is obtained by maximizing the success probability over all LOCC instruments $\{\mathcal{F}_x\}$. An easy-to-miss subtlety is the following: LOCC allows subsystems, particularly those storing classical measurement records, to be discarded, thereby effectively averaging over these outcomes. Thus, if $\sigma = \sum_x p_x \sigma_x$ is the average outcome of a given LOCC protocol acting on ρ , then there is a different LOCC protocol (represented by the quantum channel $\mathcal{F} = \sum_x \mathcal{F}_x$) that converts ρ into σ deterministically. If σ is pure, then all σ_x already need to be identical to σ .

When restricted to bipartite pure states, then the most general LOCC transformation requires only one-way communication (Lo and Popescu, 2001). This is because any bipartite pure state admits a Schmidt decomposition that is symmetric (up to local unitaries) under an exchange of parts A and B. Therefore, without loss of generality an LOCC protocol involving pure states can always be executed by a single measurement on one party, followed by a local unitary operation on the second party conditioned on the measurement outcome. Consequently, while characterizing general LOCC transformations for general mixed states is difficult, for bipartite pure states a simple characterization exists. It is known as Nielsen's theorem (Nielsen, 1999).

Theorem V.2 (Nielsen's theorem). State $|\psi\rangle$ can be converted into state $|\phi\rangle$ by means of LOCC if and only if

$$\lambda(\psi_{\mathsf{A}}) \prec \lambda(\phi_{\mathsf{A}}),\tag{102}$$

where $\lambda(\psi_A)$ is the vector of Schmidt coefficients of $|\psi\rangle$, i.e., the eigenvalues of the reduced state $\psi_A := \text{Tr}_B[|\psi\rangle\langle\psi|_{AB}]$.

Nielsen's theorem provides a connection between entanglement transformations and the theory of majorization (Marshall, Olkin, and Arnold, 2011). Moreover, the conditions given in Eq. (102) are easy to check numerically, and therefore provide a powerful tool for determining when one pure bipartite state can be converted into another via LOCC. Nielsen's theorem directly implies that there are incomparable states, in the sense that neither ψ nor ϕ can be directly transformed into another using LOCC. To address this interconversion barrier, Vidal (1999) generalized Nielsen's work by characterizing probabilistic transformations in the LOCC framework.

Theorem V.3 (Vidal's theorem). State $|\psi\rangle$ can be conclusively converted into $|\phi\rangle$ with probability μ by means of LOCC if and only if

$$\lambda(\psi_{\mathsf{A}}) \prec^{\scriptscriptstyle W} \mu \lambda(\phi_{\mathsf{A}}). \tag{103}$$

The symbol \prec^{w} denotes a general form of majorization called weak majorization (Bhatia, 2013) to compare unnormalized distributions. Vidal (1999) also presented an optimal protocol demonstrating that the transformation from $|\psi\rangle$ to $|\phi\rangle$ is always possible probabilistically, with the success probability

$$p_{\text{succ}}(|\psi\rangle \to |\phi\rangle) = \min_{1 \le k \le d} \frac{1 - \mathcal{L}_k(\psi)}{1 - \mathcal{L}_k(\phi)}, \qquad (104)$$

where $\mathcal{L}_k(\psi) \coloneqq \sum_{i=1}^{k-1} \lambda^{\downarrow}(\psi_A)_i$, with $\lambda(\psi_A)_0 \equiv 0$. See Example V.4 for an application.

Example V.4. Consider the following bipartite pure states:

$$\begin{aligned} |\psi\rangle &= \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}|33\rangle, \\ |\phi\rangle &= \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle. \end{aligned}$$
(105)

The eigenvalues of ψ_A and ϕ_A are given by

$$\lambda(\psi_{\mathsf{A}}) = [0.4, 0.4, 0.1, 0.1], \tag{106}$$

$$\lambda(\phi_{\mathsf{A}}) = [0.5, 0.25, 0.25, 0]. \tag{107}$$

Since $\lambda(\psi_A) \neq \lambda(\phi_A)$, Nielsen's theorem precludes the existence of a deterministic LOCC transformation $|\psi\rangle \rightarrow |\phi\rangle$. Still, if one attempts to transform $|\psi\rangle$ into $|\phi\rangle$ probabilistically, Vidal's theorem from Eq. (104) implies that it is possible to achieve it with a probability $p_{\text{succ}}(|\psi\rangle \rightarrow |\phi\rangle) = 4/5$.

It was later observed by Jonathan and Plenio (1999) that a catalytic version of LOCC can be considered where Alice and Bob are allowed to use preshared entanglement in the form of auxiliary bipartite pure states. One can do so in such a way that the auxiliary state is returned exactly in its initial state after the transformation. To demonstrate this, Jonathan and Plenio (1999) proposed a particular example of this type of transformation; see Example V.5 for details.

Example V.5. Consider the two states defined in Eq. (105) and a two-qubit catalyst prepared in the state

$$|\omega\rangle = \sqrt{0.6}|00\rangle + \sqrt{0.4}|11\rangle. \tag{108}$$

Nielsen's theorem applied to the transformation $|\psi\rangle \otimes |\omega\rangle \rightarrow |\phi\rangle \otimes |\omega\rangle$ implies that the transition can be realized deterministically via LOCC.

The example demonstrates that an appropriately chosen catalyst can increase the success probability of a transformation, even from p < 1 to p = 1. This type of transformation was named entanglement-assisted LOCC (ELOCC) by Jonathan and Plenio (1999) and is the first example of strict catalysis in LOCC. They also showed that two bipartite pure states $|\psi\rangle$ and $|\phi\rangle$ that were interconvertible via ELOCC, i.e., each state was convertible to the other, were already equivalent up to local unitaries. In other words, strict catalysis does not arise for the question of interconvertibility between pure states (cf. Sec. VI.A). The mathematical characterization of strict catalysis in the framework of LOCC was later examined by Daftuar and Klimesh (2001). It was discovered that there is no upper bound on the dimension that should be considered. Specifically, for most initial states the set of final states achievable with catalysts of a local dimension $d_{\rm C}$ is strictly larger than the set of states achievable with catalysts of a smaller dimension $(d'_{\rm C} < d_{\rm C})$. Moreover, there are state transitions that cannot be implemented with any finite-dimensional catalyst but can be implemented with an infinite-dimensional catalyst (Daftuar, 2004). This poses serious difficulty in characterizing which state transformations are possible under ELOCC. Furthermore, Daftuar and Klimesh (2001) proved that any entangled state that has at least two nonzero and nonequal Schmidt coefficients can serve as a catalyst for some LOCC transformation. A separable state cannot be useful as a catalyst. However, counterintuitively the maximally entangled state cannot catalyze any transition either, since taking a tensor product with this state would preserve the majorization structure.

In general not much is known about the structure of strict catalysis in LOCC in the case in which the dimension of the catalyst is bounded. An exception to this unsatisfactory state of the art was discussed by Anspach (2001), who derived the necessary and sufficient conditions for the existence of a catalyst when the systems A and B have the local dimension $d_A = d_B = 4$ and the local dimension of the catalyst is $d_C = 2$. Xiaoming, Duan, and Ying (2005) presented a polynomial time algorithm to decide whether a given entanglement transformation can be deterministically catalyzed by a pure bipartite catalyst with a local dimension k.

The first full characterization of strict catalysis in entanglement theory (ELOCC) for a case of unbounded catalyst dimension was given by Turgut (2007)), who used an infinite set of entropic conditions. Independently, Klimesh (2007) proposed a different description of a complete set of inequalities characterizing ELOCC. These two sets of conditions are in fact equivalent; see Kribs, Pereira, and Plosker (2013) and Pereira and Plosker (2013, 2015). Finally, the closure of the set containing all states reachable from a fixed entangled state $|\psi\rangle$ was studied by Kribs, Pereira, and Plosker (2013), who found that the set could be fully described using a generalization of the majorization relation called *power majorization* (Allen, 1988). The mathematical structure of probabilistic entanglement transformations was studied by Feng, Duan, and Ying (2005), who gave a necessary and sufficient condition for the existence of strict catalysts that could increase the success probability of a state transformation; see also the previous work of Feng, Duan, and Ying (2004) and Duan et al. (2005b). Feng, Duan, and Ying (2005) showed that the maximal probability of success p_{succ} for a transformation $|\psi\rangle \otimes |\omega\rangle \rightarrow |\phi\rangle \otimes |\omega\rangle$ optimized over $|\omega\rangle$ depends only on the minimal Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$. Moreover, Feng, Duan, and Ying (2005) also derived necessary and sufficient conditions for a given k-dimensional bipartite pure catalyst state $|\omega_k\rangle$ to enable a given transformation $|\psi\rangle \rightarrow |\phi\rangle$.

One of the main limitations of studying catalysis in LOCC arises from the complete state-transition conditions being known only for bipartite pure states. Eisert and Wilkens (2000) took the first steps in generalizing catalysis to the case of mixed states. Specifically, they found an explicit example of two bipartite mixed states $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\rho \hookrightarrow \sigma$, but there exists no LOCC transformation turning ρ into σ .

2. Entanglement distillation and formation

Suppose that Alice and Bob want to communicate quantum information over a noisy quantum channel. The quality of their communication depends crucially on their ability to maintain a high degree of entanglement (Lloyd, 1997).

Entanglement distillation and formation constitute two fundamental protocols that aim to enhance quantum communication by collectively processing multiple copies of quantum states (Bennett, Bernstein *et al.*, 1996; Dür and Briegel, 2007; Horodecki *et al.*, 2009).

In the paradigmatic setting of entanglement distillation, Alice and Bob share $n \gg 1$ copies of a bipartite state ρ , and the aim is to transform them into *m* copies of the maximally entangled state ϕ via LOCC, where the rate is given by R := m/n (Bennett, Bernstein *et al.*, 1996). Formally, the asymptotic distillation cost $E_D(\rho)$ is defined as the largest rate R^* for which

$$\lim_{n \to \infty} \inf_{\mathcal{E} \in \text{LOCC}(A:B)} \|\mathcal{E}[\rho^{\otimes n}] - \phi_+^{\otimes m}\|_1 = 0.$$
(109)

While $E_D(\rho)$ measures the maximum rate of distilling ϕ from ρ via LOCC, one can also ask about the inverse process: What is the optimal rate of creating *n* copies of ρ from *m* maximally entangled states ϕ_+ . This has been termed *entanglement formation* (Bennett, DiVincenzo *et al.*, 1996), and the associated optimal conversion rate is the asymptotic formation cost $E_F(\rho)$, i.e., the largest rate $R^* = n/m$ for which

$$\lim_{n \to \infty} \inf_{\mathcal{E} \in \text{LOCC}(A:B)} \|\mathcal{E}[\phi_+^{\otimes n}] - \rho^{\otimes n}\|_1 = 0.$$
(110)

Generally $E_D(\rho) \neq E_F(\rho)$, leading to the fundamental irreversibility of entanglement transformations even in the asymptotic limit. However, for bipartite pure states, the two entanglement quantifiers coincide and are equal to the entanglement entropy of the state, i.e., for a pure state ψ_{AB} ,

$$E_D(\psi_{\mathsf{A}\mathsf{B}}) = E_F(\psi_{\mathsf{A}\mathsf{B}}) = H(\psi_{\mathsf{A}}).$$
(111)

Outside of the asymptotic limit, i.e., when $n < \infty$, the quantifiers $E_D(\rho)$ and $E_F(\rho)$ lose their operational significance. Recently it was discovered that this operational significance is recovered when correlating catalysis is allowed. This approach was addressed by Kondra, Datta, and Streltsov (2021), who proved a variant of Theorem V.2 for correlating-catalytic LOCC with bipartite pure states. Specifically, Kondra, Datta, and Streltsov (2021) proved the following theorem, which can be seen as an instance of Lemma IV.1.

Theorem V.6. Given bipartite pure states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$, $H(\psi_A) \ge H(\phi_A)$ if and only if $|\psi\rangle_{AB} \xrightarrow[\mathcal{O}_{LOCC}]{\mathcal{O}} |\phi\rangle_{AB}$ to arbitrary accuracy in terms of trace distance.

More generally, with the help of a correlating-catalytic LOCC operation it is possible to convert any distillable state ρ_{AB} (to arbitrary accuracy) into a pure state ψ_{AB} with entanglement entropy $H(\psi_A) \leq E_D(\rho_{AB})$. Conversely, via correlating-catalytic LOCC it is also possible to create any bipartite quantum state ρ_{AB} from a pure entangled state ψ_{AB} with entanglement entropy $H(\psi_A) \geq E_C(\rho_{AB})$.

Furthermore, Kondra, Datta, and Streltsov (2021) used similar techniques to show that in the tripartite setting, i.e., when three spatially separated parties A, B, and C share a pure quantum state $|\psi\rangle$, the analog of distillable entanglement extended by the use of correlating catalysis is given by $\min[H(\psi_A), H(\psi_B)]$, in close analogy to the asymptotic setting (Smolin, Verstraete, and Winter, 2005).

This result uses a catalyst to convert an asymptotic result into the single-shot setting. Previously Bennett et al. (2000) defined a notion of catalysis directly in the asymptotic setting, i.e., while also asymptotically involving many copies of the catalyst and allowing for an arbitrarily small error. They showed that such transformations can even simulate LOCC transformations that additionally allow for a sublinear amount of quantum communication. Vidal and Cirac (2001) argued that this class of catalytic transformations, nevertheless, cannot be used to distill *bound entanglement*; see Sec. V.B.9. Bound entangled states are entangled states with $E_D(\rho) = 0$, and hence their entanglement cannot be extracted using LOCC. Recently Lami, Regula, and Streltsov (2023) conclusively showed that not even correlating or marginal-correlating catalysis can be used to distill bound entanglement, while Ganardi, Kondra, and Streltsov (2023) showed that the distillable entanglement of distillable states cannot be increased using correlating catalysis.

3. Quantum state merging

Quantum state merging, sometimes called coherent state transfer, is a communication task that enables a known quantum state to be transferred to a distant party that already holds part of the state (Horodecki, Oppenheim, and Winter, 2005, 2006). Specifically, consider three parties: A, B, and Referee (R) who share asymptotically many copies of some pure state $|\psi_{ABR}\rangle$. The goal is to send the A part of the state to B while preserving all correlations with R. It is further assumed that Alice and Bob can perform any LOCC protocol and, on top of that, they may share an arbitrary number of maximally entangled states. As shown by Horodecki, Oppenheim, and Winter (2005), the rate at which maximally entangled states have to be supplied in order to accomplish this process is given by the quantum conditional entropy

$$H(\mathsf{A}|\mathsf{B})_{\mathsf{w}} \coloneqq H(\psi_{\mathsf{A}\mathsf{B}}) - H(\psi_{\mathsf{B}}). \tag{112}$$

In other words, for $H(A|B)_{\psi} > 0$ state merging is possible when singlets are consumed at a rate $H(A|B)_{\psi}$. For $H(A|B)_{\psi} < 0$, state merging not only is possible but also generates entangled pairs at a rate given by $-H(A|B)_{\psi}$. This provides an operational meaning to the quantum conditional entropy. Variants of the state merging task were analyzed, including its single-shot version (Berta, 2008) and an extension to multiple parties (Dutil and Hayden, 2010).

An extension of the state merging protocol allowing correlating catalysis was proposed by Kondra, Datta, and Streltsov (2021). The setup is similar to the single-shot setting discussed by Berta (2008): the parties A, B, and R share a single copy of $|\psi\rangle_{ABR}$ and are allowed to use any entangled state as a correlating catalyst. In this extended protocol, state merging can be performed as long as $H(A|B)_{\psi} > 0$. Conversely, if $H(A|B)_{\psi} < 0$, then catalytic state merging can be performed not only without extra entanglement but also with Alice and Bob gaining an additional pure state with entanglement entropy

 $-H(A|B)_{\psi}$. Both procedures are optimal in the sense that state merging is not possible if a pure state with a smaller entanglement entropy is provided [when $H(A|B)_{\psi} > 0$], and state merging with entanglement gain exceeding $-H(A|B)_{\psi}$ is not possible when $H(A|B)_{\psi} < 0$. This procedure provides an operational interpretation of the quantum conditional entropy in the single-shot regime.

4. Quantum teleportation

Quantum teleportation enables quantum states to be transferred using preshared entanglement and classical communication (Bennett *et al.*, 1993). A and B share an entangled state ρ_{AB} . Alice is additionally given a quantum state $\varphi_{A'}$ that is unknown to both parties. They then attempt to transfer the state on A' from Alice to Bob using a quantum channel $\mathcal{E} \in \text{LOCC}(A'A:B)$ and the entangled state that they share. The goal of the protocol is to simulate a noiseless quantum channel from Alice to Bob, i.e., the identity map $id_{A' \rightarrow B}$. The quality of teleportation is usually quantified using the average fidelity of teleportation introduced by Popescu (1994),

$$\langle F \rangle_{\rho} \coloneqq \max_{\mathcal{E}} \int \langle \varphi |_{\mathsf{B}} \mathrm{tr}_{\mathsf{A}'\mathsf{A}} \mathcal{E}(\varphi_{\mathsf{A}'} \otimes \rho_{\mathsf{A}\mathsf{B}}) | \varphi \rangle_{\mathsf{B}} \mathrm{d}\varphi_{\mathsf{B}}$$
such that $\mathcal{E} \in \mathsf{LOCC}(\mathsf{A}'\mathsf{A}\!:\!\mathsf{B}).$ (113)

The integral in Eq. (113) is computed over a uniform distribution of all pure input states $\varphi_{\rm B} = |\varphi\rangle\langle\varphi|_{\rm B}$ according to a normalized Haar measure $\int d\varphi_{\rm B} = \mathbb{1}_{\rm B}$. It can easily be verified that $0 \le \langle F \rangle_{\rho} \le 1$, where $\langle F \rangle_{\rho} = 1$ corresponds to perfect teleportation that is possible if and only if ρ is maximally entangled. As shown by Horodecki, Horodecki, and Horodecki (1999), the fidelity of teleportation (113) can be conveniently expressed as

$$\langle F \rangle_{\rho} = \frac{f(\rho)d_{\mathsf{A}'} + 1}{d_{\mathsf{A}'} + 1},$$
 (114)

where $f(\rho) \coloneqq \max\{\langle \phi^+|_{AB} \mathcal{E}(\rho_{AB}) | \phi^+ \rangle_{AB} | \mathcal{E} \in \text{LOCC}(A:B)\}$ is called the entanglement fraction, $d_{A'}$ is the dimension of A', and $|\phi^+\rangle_{AB}$ is a maximally entangled state on AB.

The protocol for quantum teleportation has been extended to the case when ancillary entanglement is used catalytically (Lipka-Bartosik and Skrzypczyk, 2021b). In that case, we assume that Alice and Bob, in addition to ρ_{AB} , also have access to a quantum system $C_A C_B$ prepared in $\omega_{C_A C_B}$. This additional system is distributed such that Alice has access only to C_A , and Bob has access only to C_B . Alice is then given an unknown state $\varphi_{A'}$, and the parties perform a protocol $\mathcal{E} \in LOCC(A'AC_A:BC_B)$ acting on both the shared systems and the input. For the protocol to be catalytic, one further demands that the local state of the system $C_A C_B$ remains the same after \mathcal{E} , although it can become correlated with AB. The protocol \mathcal{E} can then be viewed as an instance of correlating catalysis in the LOCC framework that aims to perform teleportation of A'.

If we have the freedom to choose the state of the catalyst $\omega_{C_A C_B}$, we can then define a benchmark that optimizes over all

possible states of the catalyst. This leads to the average fidelity of catalytic teleportation defined as

$$\begin{split} \langle F_{\mathrm{cat}} \rangle_{\rho} &= \max_{\mathcal{E}, \omega} \int \langle \varphi | \mathrm{tr}_{\mathsf{A}'\mathsf{A}\mathsf{C}\mathsf{C}'} \mathcal{E}(\varphi_{\mathsf{A}'} \otimes \rho_{\mathsf{A}\mathsf{B}} \otimes \omega_{\mathsf{C}_{\mathsf{A}}\mathsf{C}_{\mathsf{B}}}) | \varphi \rangle \mathrm{d}\varphi \\ & \text{ such that } \operatorname{tr}_{\mathsf{A}'\mathsf{A}\mathsf{B}} \mathcal{E}(\varphi_{\mathsf{A}'} \otimes \rho_{\mathsf{A}\mathsf{B}} \otimes \omega_{\mathsf{C}_{\mathsf{A}}\mathsf{C}_{\mathsf{B}}}) = \omega_{\mathsf{C}_{\mathsf{A}}\mathsf{C}_{\mathsf{B}}}, \\ & \omega_{\mathsf{C}_{\mathsf{A}}\mathsf{C}_{\mathsf{B}}} \geq 0, \quad \operatorname{Tr}\omega_{\mathsf{C}_{\mathsf{A}}\mathsf{C}_{\mathsf{B}}} = 1, \\ & \mathcal{E} \in \mathsf{LOCC}(\mathsf{A}'\mathsf{A}\mathsf{C}\!:\!\mathsf{B}\mathsf{C}'). \end{split}$$
(115)

The main result of Lipka-Bartosik and Skrzypczyk (2021b) is the following achievable lower bound:

$$\langle F_{\text{cat}} \rangle_{\rho} \ge \frac{f_{\text{reg}}(\rho)d_{\mathsf{A}'} + 1}{d_{\mathsf{A}'} + 1},\tag{116}$$

where $f_{\text{reg}}(\rho)$ is a regularization of entanglement fraction $f_{\text{reg}}(\rho) \coloneqq \lim_{n \to \infty} [f_n(\rho^{\otimes n})/n]$, with $f_n(\sigma)$ given by

$$f_{n}(\sigma) \coloneqq \max_{\mathcal{E}} \sum_{i=1}^{n} \langle \phi^{+} | \operatorname{tr}_{i} \mathcal{E}(\sigma) | \phi^{+} \rangle$$

such that $\mathcal{E} \in \operatorname{LOCC}(\mathsf{A}_{1} \cdots \mathsf{A}_{\mathsf{n}} : \mathsf{B}_{1} \cdots \mathsf{B}_{\mathsf{n}}), \quad (117)$

where $\operatorname{tr}_{i}(\cdot)$ is the partial trace performed over local parties $1 \cdots i - 1, i + 1 \cdots n$. Note that, by taking a suboptimal guess $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \cdots \otimes \mathcal{E}_n$ with $\mathcal{E}_1 = \mathcal{E}_2 = \cdots = \mathcal{E}_n$, we have $f_{\operatorname{reg}}(\rho) \ge f(\rho)$ for all density operators ρ .

The proof of Eq. (116) is constructive; that is, there exists a protocol $T \in \text{LOCC}(A'AC_A:BC_B)$ and a catalyst $\omega_{C_AC_B}$ that achieves the bound from Eq. (116). Moreover, for most pure bipartite states ψ_{AB} we have $f_{\text{reg}}(\psi) > f(\psi)$, implying that $\langle F_{\text{cat}} \rangle_{\psi_{AB}} > \langle F \rangle_{\psi_{AB}}$. As a consequence, the quality of teleportation can be improved when ancillary entanglement is used catalytically.

5. Local operations and shared randomness

We denote the set of all LOSR operations between A and B as LOSR(A:B). Any operation $\mathcal{E} \in LOSR(A:B)$ can be written in Kraus representation as

$$\mathcal{E}[\rho_{\mathsf{A}\mathsf{B}}] = \sum_{i} p_i (A_i \otimes B_i) \rho_{\mathsf{A}\mathsf{B}} (A_i \otimes B_i)^{\dagger}, \qquad (118)$$

where $\{A_i\}$ and $\{B_i\}$ are sets of local Kraus operators and $\{p_i\}_i$ is a probability distribution. In analogy with Nielsen's theorem, transformations between pure bipartite states via LOSR can be characterized using majorization.

Theorem V.7 (Schmid *et al.*, 2020). A bipartite quantum state $|\psi\rangle$ shared between two parties can be converted into $|\phi\rangle$ by means of LOSR if and only if there is a bipartite state $|\xi\rangle$ such that

$$\lambda(\psi_{\mathsf{A}})^{\downarrow} = [\lambda(\phi_{\mathsf{A}}) \otimes \lambda(\xi_{\mathsf{A}'})]^{\downarrow} \tag{119}$$

Theorem V.7 implies that strict catalysis cannot provide an advantage for LOSR between pure bipartite states. Specifically, Schmid *et al.* (2020) showed that if $|\psi\rangle$ cannot

be converted into $|\phi\rangle$ via LOSR, then $|\psi\rangle \otimes |\omega\rangle$ also cannot be converted into $|\phi\rangle \otimes |\omega\rangle$ for any bipartite state $|\omega\rangle$. To see this, note that due to Theorem V.7 $|\psi\rangle \underset{\mathcal{O}_{\text{LOSR}}}{\hookrightarrow} |\phi\rangle$ is possible if and only if there exists a pure bipartite state $|\xi\rangle$ with Schmidt coefficients $\lambda(\xi_{\text{A}'})$ such that

$$[\lambda(\psi_{\mathsf{A}}) \otimes \lambda(\omega_{\mathsf{C}_{\mathsf{A}}})]^{\downarrow} = [\lambda(\phi_{\mathsf{A}}) \otimes \lambda(\omega_{\mathsf{C}_{\mathsf{A}}}) \otimes \lambda(\xi_{\mathsf{A}'})]^{\downarrow}.$$
 (120)

We can read Eq. (120) as $[\mathbf{p} \otimes \mathbf{r}]^{\downarrow} = [\mathbf{q} \otimes \mathbf{r}]^{\downarrow}$ with probability vectors $\mathbf{p} = \lambda(\psi_A)$, $\mathbf{q} = \lambda(\phi_A) \otimes \lambda(\xi_{A'})$, and $\mathbf{r} = \lambda(\omega_{C_A})$. But this is possible only if $\mathbf{p}^{\downarrow} = \mathbf{q}^{\downarrow}$: The largest entry gives $p_1^{\downarrow} r_1^{\downarrow} = q_1^{\downarrow} r_1^{\downarrow}$, so $p_1^{\downarrow} = q_1^{\downarrow}$. We can now remove all equations involving p_1^{\downarrow} . But then we similarly obtain $p_2^{\downarrow} = q_2^{\downarrow}$. Repeating these steps, we get $\mathbf{p}^{\downarrow} = \mathbf{q}^{\downarrow}$, which corresponds to Eq. (119). This means that any LOSR transformation that can be achieved via strict catalysis can also be achieved without using any catalyst, in strong contrast to LOCC. We also see in Sec. V.D.2 that a similar distinction in terms of catalysis arises when considering different ways to formalize the concept of coherence as a resource theory.

This reasoning relies on two crucial assumptions: the tensor-product structure of strictly catalytic transformations and the purity of the catalyst state. Therefore, the proof ceases to hold when applied directly to correlating-catalytic transformations or when one is dealing with mixed states. It is an interesting open question as to whether correlating catalysis might be suitable for the LOSR framework. However, embezzling was studied early on in LOSR: van Dam and Hayden (2003) started with the state-transition conditions of LOCC, i.e., majorization of Schmidt vectors in Eq. (102), and showed that embezzling not only trivializes the state-transition conditions (i.e., by enabling the preparation of a maximally entangled state from a product state for free) but also does so in such a way that the process can be accomplished using only LOSR. All of this is achieved at the cost of only a disturbance $\delta > 0$ induced on the embezzler (in terms of trace distance), which can be made as small as possible by choosing a large enough dimension.

6. (Non)closure of quantum correlations

Bell's theorem (Bell, 1964) states that no local hiddenvariable theory can account for all predictions of quantum theory under the assumption that the hidden variables are statistically independent from the measurement settings. At the core of this theorem is the notion of a Bell experiment: A source distributes two physical systems to distant observers Alice and Bob so that they share a bipartite quantum state described by a density matrix ρ_{AB} . Each observer performs a randomly chosen measurement on their part of the system labeled *x* and *y* and obtains outcomes *a* and *b*. The experiment is characterized by the joint distribution p(ab|xy).

We say that the experiment has a local hidden-variable model if there exists a hidden random variable λ with an associated distribution $p(\lambda)$ and two local response functions $p(a|x, \lambda)$ for Alice and $p(b|y, \lambda)$ for Bob such that p(ab|xy)can be achieved using λ ,

$$p(ab|xy) = \sum_{\lambda} p(\lambda) p(a|x, \lambda) p(b|y, \lambda).$$
(121)

The distribution p(ab|xy) is called nonlocal when it cannot be described in the form of Eq. (121). Every possible action taken by Alice and Bob can be viewed as an LOSR operation, and vice versa (Buscemi, 2012).

Bell nonlocality is often formalized in terms of nonlocal games, which have been extensively studied in computer science, where they are a special instance of interactive proof systems (Cleve et al., 2004). In such games, A and B play against R. The referee R chooses a question $x \in \mathcal{X}$ for Alice and $y \in \mathcal{Y}$ for Bob according to a probability distribution $p(x, y): \mathcal{X} \times \mathcal{Y} \to [0, 1]$, where \mathcal{X} and \mathcal{Y} denote finite sets of questions. Without communicating, Alice (Bob) returns an answer $a \in \mathcal{A}$ ($b \in \mathcal{B}$) from a finite set of possible answers \mathcal{A} (\mathcal{B}) . In quantum mechanics, this corresponds to applying local measurements $\{M_{a|x}\}$ for Alice and $\{N_{b|y}\}$ for Bob. Based on the received answers and according to a prearranged set of rules, the referee decides whether the players win or lose the game. These rules are typically expressed using a function $V: \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, where $V_{ab}^{xy} = 1$ if and only if Alice and Bob win the game when answering a and b for questions x and y. The probability of Alice and Bob winning the game $\mathcal{G} = \{p(x, y), V\}$, maximized over all measurement strategies, is given by

$$p_{\text{guess}}(\mathcal{G}, \rho_{\mathsf{AB}}) = \max \sum_{a, b, x, y} p(x, y) p(a, b|x, y) V_{ab}^{xy}, \quad (122)$$

where $p(a, b|x, y) = \text{tr}[(M_{a|x} \otimes N_{b|y})\rho_{AB}]$. In Eq. (122) we make explicit that we work in the tensor-product framework, where Alice's and Bob's quantum systems are described by local Hilbert spaces \mathcal{H}_A and \mathcal{H}_B , respectively, and the joint Hilbert space is given by their tensor product. In the more general commuting operator framework, it is simply assumed that Alice's and Bob's measurement operators are defined on a global Hilbert space \mathcal{H}_{AB} and that $[M_{a|x}, N_{b|y}] = 0$; see also Sec. III.F. The tensor-product framework and the commuting operator framework are equivalent for finite-dimensional Hilbert spaces.

Bell inequalities provide upper bounds on Eq. (122) with which the players can win the game using the best classical strategy, i.e., when ρ_{AB} is separable. A violation of a Bell inequality indicates that there is a quantum strategy that outperforms the best classical strategy. In other words, Bell's theorem (Bell, 1964) asserts that there are games where players who share entanglement can outperform players who do not. The most well-known example of this is the Clauser-Horne-Shimony-Holt (CHSH) game \mathcal{G}_{CHSH} (Clauser *et al.*, 1969), where Tsirelson's theorem provides an upper bound on the guessing probability $p_{guess}(\mathcal{G}_{CHSH}, \rho_{AB})$ for classical strategies (Cirel'son, 1980). For a more complete account of nonlocal games, see Watrous (2018).

An immediate application of nonlocal games is the task of witnessing the dimension of entanglement. Specifically, a nonlocal game \mathcal{G} can be used as an entanglement dimension witness if a certain $p_{guess}(\mathcal{G}, \rho_{AB})$ can be achieved only with entangled states ρ_{AB} on a Hilbert space with a given minimal

dimension (Brunner *et al.*, 2008). In this spirit, Coladangelo (2020) proposed a nonlocal game $\mathcal{G}_{Coladangelo}$ in which the players' maximum achievable guessing probability increased monotonically with the allowed Hilbert-space dimension. In the following, let

$$p_{\text{Coladangelo}}^* \coloneqq \sup_{\rho_{\text{AB}}} p_{\text{guess}}(\mathcal{G}_{\text{Coladangelo}}, \rho_{\text{AB}}), \quad (123)$$

where the maximization is performed over quantum states ρ_{AB} on Hilbert spaces with a finite (but arbitrarily large) dimension. Coladangelo (2020) showed that $p_{\text{guess}}(\mathcal{G}_{\text{Coladangelo}}, \rho_{\text{AB}}) < p^*_{\text{Coladangelo}}$ within the tensor-product framework for both finite-dimensional and infinitedimensional quantum states $^{5}\,\rho_{\mathsf{AB}}.$ The proof of this statement is based on choosing ρ_{AB} to be an approximate embezzler; see Sec. III.E. Moreover, there it is further shown that a strategy achieving $p^*_{\text{Coladangelo}}$ requires a perfect embezzler, which requires a commuting operator framework; see also Sec. III.F. An approximate embezzler allows Alice and Bob to simulate an approximately coherent strategy, i.e., to perform measurements in a way that retains a high degree of coherence between the parties. High coherence of the process, however, leads directly to a high score in the nonlocal game. The main idea behind using embezzlers in such a coherent state exchange process is described in Example V.8.

Example V.8 (Coherent state exchange). Consider a bipartite pure state $|\phi\rangle_{AB}$ shared between A and B. The goal is to transform $|\phi\rangle_{AB}$ into $|\psi\rangle_{AB}$ under two assumptions: (i) no communication is allowed and (ii) the process must be implemented in a coherent way, that is,

$$\begin{aligned} \alpha |00\rangle_{\mathsf{A}'\mathsf{B}'}|\gamma\rangle_{\mathsf{A}\mathsf{B}} &+ \beta |11\rangle_{\mathsf{A}'\mathsf{B}'}|\phi\rangle_{\mathsf{A}\mathsf{B}} \\ &\to \alpha |00\rangle_{\mathsf{A}'\mathsf{B}'}|\gamma\rangle_{\mathsf{A}\mathsf{B}} + \beta |11\rangle_{\mathsf{A}'\mathsf{B}'}|\psi\rangle_{\mathsf{A}\mathsf{B}}, \end{aligned} \tag{124}$$

where A'B' represent control systems, one held by each player. Nielsen's theorem implies that Eq. (124) is impossible when $\lambda(\psi_A) > \lambda(\phi_A)$. However, this is no longer true when catalysis is involved: using an auxiliary state

$$|\omega_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\phi\rangle^{\otimes i} |\psi\rangle^{\otimes (n-i+1)}$$
(125)

and cyclically permuting local subsystems transform $|\phi\rangle_{AB}|\omega_n\rangle$ into $|\psi\rangle_{AB}|\omega'_n\rangle$, with $\langle\omega'_n\rangle\omega_n = 1 - 1/n$, so the auxiliary state is approximately preserved. This protocol is almost identical to the universal embezzlement construction in Sec. IV.C. The fact that the auxiliary system is approximately catalytic is the key property that ensures that the transformation is done coherently. Controlling the cyclic permutation on the control systems yields

⁵The set of correlations achievable by infinite-dimensional density matrices in the tensor-product framework is included in the closure of the set of correlations achievable by finite-dimensional quantum states (Scholz and Werner, 2008).

$$\begin{split} &\alpha |00\rangle_{\mathsf{A}'\mathsf{B}'}|\gamma\rangle_{\mathsf{A}\mathsf{B}}|\omega_n\rangle + \beta |11\rangle_{\mathsf{A}'\mathsf{B}'}|\phi\rangle_{\mathsf{A}\mathsf{B}}|\omega_n\rangle \\ &\rightarrow \alpha |00\rangle_{\mathsf{A}'\mathsf{B}'}|\gamma\rangle_{\mathsf{A}\mathsf{B}}|\omega_n\rangle + \beta |11\rangle_{\mathsf{A}'\mathsf{B}'}|\psi\rangle_{\mathsf{A}\mathsf{B}}|\omega_n'\rangle, \end{split}$$

and hence the parties can approximately implement the coherent state exchange since $|\omega'_n\rangle \approx |\omega_n\rangle$. If the state $|\omega'_n\rangle$ is far from $|\omega_n\rangle$, then discarding the ancilla effectively decoheres the primary system. Indeed, the requirement (ii) can be satisfied only if the process does not significantly change the auxiliary state. For example, the players cannot swap $|\phi\rangle_{AB}$ with an initially shared copy of $|\psi\rangle_{AB}$ without losing coherence.

From the perspective of the Bell nonlocality, Coladangelo (2020) proposed an embezzlement-based Bell inequality that cannot be maximally violated within the tensor-product framework, but where the limiting value can be achieved arbitrarily well. This shows that the set of quantum correlations from a Bell experiment described in the tensor-product framework is not closed. In fact, the existence of a suitable game was first proved using representation theory for finitely presented groups by Slofstra (2019, 2020), with subsequent alternative proofs given by Dykema, Paulsen, and Prakash (2019) and Musat and Rørdam (2020). In the case of Coladangelo (2020), however, embezzlement-based techniques allowed for a simpler proof using only basic linear algebra and an explicit nonlocal game. Previously Cleve, Liu, and Paulsen (2017) showed a similar result for coherent embezzlement games (Regev and Vidick, 2013). Coladangelo (2020) transparently demonstrated the relationship between the dimension of the used entangled state and the associated score in a nonlocal game. A similar technique involving embezzlement was used to prove that any number of parties can coherently exchange any pure quantum state for another, without communication, given prior shared entanglement (Leung, Toner, and Watrous, 2013).

7. Embezzlement and the reverse coding theorem

The classical channel capacity is the central concept in information theory. It quantifies the maximum rate at which classical data can be transmitted in the limit of many uses of the channel (Cover, 1999). The noisy channel coding theorem is a milestone result that provides a closed formula for the channel capacity (Shannon, 1948). It states that the classical capacity *C* of a memoryless classical communication channel is given by the supremum of mutual information between the input and the output of the channel. Specifically, let *X* be a random variable distributed according to a probability distribution p_X , and let \mathcal{E} be a classical and memoryless channel. The classical capacity of \mathcal{E} is then given by

$$C(\mathcal{E}) = \sup_{p_X} I(X:Y)_{\mathcal{E}[p_X]},$$
(126)

where $Y \sim \mathcal{E}[\mathbf{p}_X]$ denotes the output of the channel and $I(X:Y)_{\mathcal{E}[\mathbf{p}_X]} := H(X) + X(Y) - H(X, Y)$. After the appearance of Shannon's theorem, a closely related classical reverse coding theorem was proven (Bennett *et al.*, 2002). The theorem states that every classical channel \mathcal{E} can be simulated exactly using $C(\mathcal{E})$ bits of classical communication and free

shared randomness between the sender and the receiver. In other words, when given access to shared randomness and local operations, any classical channel \mathcal{N} can be simulated using \mathcal{E} at a rate given by $C(\mathcal{E})/C(\mathcal{N})$. Therefore, every classical channel is completely characterized by its capacity in the limit of many uses of the channel.

The communication problem becomes more complex in quantum theory, where channels are known to exhibit different types of capacities (Holevo, 1998; Shor, 2003; Devetak, 2005). Presumably the most natural analog of classical channel capacity in the quantum regime is the entanglement-assisted classical capacity, C_E . It is defined as the highest rate at which classical information can be transmitted when the sender and receiver share unlimited noiseless entanglement (Bennett *et al.*, 1999). An analog of the classical noisy channel coding theorem was proven by Bennett *et al.* (2002) and provides a closed-form formula relating C_E to the quantum mutual information [see Eq. (32)],

$$C_{E}(\mathcal{E}) = \max_{\rho \in \mathcal{D}(\mathsf{X})} I(\mathsf{Y}:\mathsf{X}')_{(\mathcal{E}\otimes 1)\Phi_{\rho}}, \qquad (127)$$

where \mathcal{E} is a quantum channel from X to Y, Φ_{ρ} is a purification of ρ on XX', and the optimization ranges over all input states ρ on X. Bennett *et al.* (2002) conjectured a quantum version of the reverse coding theorem that was subsequently proven (Bennett *et al.*, 2014).

The quantum reverse coding theorem states that any quantum channel \mathcal{E} can be simulated using $C_E(\mathcal{E})$ bits of classical communication under unlimited shared entanglement. As a consequence, C_E suffices to characterize the quantum channel when preshared entanglement is allowed for free. The reverse coding theorem conveys the idea that entanglement cannot be easily discarded when access to communication is limited. Specifically, when two entangled states $|\phi_1\rangle_{AB}$ and $|\phi_2\rangle_{AB}$ are part of a superposition, i.e., $|\psi\rangle_{ABC} = (|\phi_1\rangle_{AB}|0\rangle_{C} + |\phi_2\rangle_{AB}|1\rangle_{C})/\sqrt{2}$, then it is impossible to retain the entanglement in one branch of the superposition and remove it from the other without either using classical communication or causing decoherence. This can also be understood as a consequence of the fact that changing the *entanglement spread* of an entangled pure quantum state always requires communication (Harrow, 2010).

In quantum information theory, free entanglement usually takes the form of maximally entangled states. However, even if one has an infinite supply of maximally entangled states, the optimal entanglement-assisted capacity C_E from Eq. (127) cannot be reached (Bennett *et al.*, 2014). Specifically, a communication protocol can attain Eq. (127) only if it can be implemented coherently, as in the coherent state exchange protocol in Example V.8. The quantum reverse coding theorem can be achieved using entanglement in the form of embezzlers instead of maximally entangled states. This allows one to overcome the constraints imposed by entanglement spread and ultimately leads to the rate of communication achieving the capacity C_E .

In the previous example, embezzlement was used to generalize a classical information processing task to the quantum case. Recently George, Hsieh, and Chitambar (2023) also used embezzlement to generalize the classical task of distributed source simulation to a fully quantum setting, which they called embezzling source simulation.

8. Cryptography

In many situations, communication protocols require an authentication step to ensure secure communication between the two parties (Alice and Bob). This is necessary to protect against potential security breaches from an impostor (Eve). To authenticate Bob to Alice, the following protocol involving strict catalysis was proposed by Barnum (1999):

- (1) Alice and Bob select a pair of incomparable states such as $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$. Alice then prepares $|\psi\rangle_{AB}$ in her lab.
- (2) They also share a strict catalyst $|\omega\rangle_{A'B'}$ that enables the forbidden transition $|\psi\rangle_{AB} \rightarrow |\phi\rangle_{AB}$.
- (3) Alice sends one part of the state $|\psi\rangle_{AB}$ to Bob.
- (4) Alice and Bob use the catalyst to perform the transition $|\psi\rangle_{AB} \otimes |\omega\rangle_{A'B'} \rightarrow |\phi\rangle_{AB} \otimes |\omega\rangle_{A'B'}$.

(5) Bob sends his part of $|\phi\rangle_{AB}$ to Alice, who measures it. The security of this protocol relies on the incomparability of the states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$. If Eve intercepts the communication and receives the B part of $|\psi\rangle_{AB}$, she will not be able to transform it into $|\phi\rangle_{AB}$ without access to the catalyst.

A second cryptographic scenario involving catalysis was discussed by Boes *et al.* (2018). Here Alice wants to communicate an unknown quantum state ρ_D on a data register D to Bob over a public quantum channel in such a way that no information about ρ_D is revealed to the public. Alice and Bob are assumed to share a number of maximally entangled qubits (ebits). Alice now applies a suitable unitary (independent of ρ_D) to D and her part of the shared ebits with the result that the state on D is maximally mixed.

Afterward she sends D to Bob using a public quantum channel. By applying a suitable unitary, Bob can recover the initial state on D and, moreover, restore the ebits to their initial state. The catalytic ebits act as a secret key that is used to encrypt the quantum data for the transmission over the public quantum channel. For a thorough review on quantum cryptographic schemes, see Gisin *et al.* (2002) and Pirandola *et al.* (2020), as well as the more recent review by Portmann and Renner (2022).

9. Beyond local operations and classical communication

Thus far we have discussed two classes of quantum operations in entanglement theory: LOCC and LOSR. The relevance of these sets of operations is twofold: first, they reflect typical physical restrictions imposed by many basic protocols in quantum information theory (Alber *et al.*) and, second, they characterize operationally entangled states as those that cannot be prepared via LOCC or LOSR (Chitambar *et al.*, 2014).

Entanglement transformations are challenging to study due to the fact that the mathematical structure of LOCC and LOSR is far from being well understood. This motivates us to investigate other operations that are potentially more easily characterized. One such example is positive-partialtranspose-preserving (PPTP) operations (Rains, 1999), which are defined as quantum channels that map the set of states with a positive partial transpose back on itself. It is known that PPTP operations are strictly more powerful than LOCC; for example, they allow bound entangled states to be created from product states (Eggeling et al., 2001); see Rains (2001) and Ishizaka and Plenio (2005) for studies of the properties of PPTP operations. The significance of PPTP operations lies mainly in the fact that they can be efficiently characterized in terms of semidefinite constraints. Since PPTP operations form a strict superset of LOCC operations, one can then define and determine relevant quantifiers under PPTP operations, such as entanglement cost (Audenaert, Plenio, and Eisert, 2003) and distillable entanglement (Ishizaka and Plenio, 2005). This approach allows one to obtain meaningful bounds on the corresponding quantifiers under LOCC. Similarly, one can consider catalytic PPTP operations in order to understand the limitations of catalytic LOCC.

Along these lines, Matthews and Winter (2008) investigated transformations of pure bipartite states under PPTP operations. In particular, they demonstrated that, in analogy with LOCC, strict catalysis also enlarges the set of possible transformations under PPTP operations. In contrast to the LOCC paradigm, even a maximally entangled quantum state can be a useful catalyst under strictly catalytic PPTP operations. Specifically, Matthews and Winter (2008) showed that a PPTP channel between two pure bipartite states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ can exist only if

$$H_{\alpha}[\lambda(\psi_{\mathsf{A}})] \ge H_{\alpha}[\lambda(\phi_{\mathsf{A}})] \quad \text{for } \alpha \in \{\frac{1}{2}, 1, \infty\}.$$
(128)

Matthews and Winter (2008) found necessary and sufficient conditions for transforming a rank-k maximally entangled state into any other pure state via PPTP operations, assisted by a maximally entangled catalyst. That is, they found that a transformation

$$\Phi^{+,k}\rangle_{\mathsf{AB}} \underset{\mathcal{O}_{\mathsf{PPT}}}{\hookrightarrow} |\phi\rangle_{\mathsf{AB}} \tag{129}$$

is possible if and only if the Rényi entropy of the order of $\alpha = 1/2$ strictly decreases,

$$H_{1/2}[\lambda(\Phi_{\mathsf{A}}^{+,k})] > H_{1/2}[\lambda(\phi_{\mathsf{A}})].$$
(130)

Note that in Eq. (130) both the initial system state and the catalyst are maximally entangled. Because of Eq. (130), it is possible to reach states that increase the Shannon entropy of the Schmidt coefficients (i.e., the Rényi entropy of the order of $\alpha = 1$), which is in stark contrast to strictly catalytic LOCC state transformations that can never increase any of the Rényi entropies, or correlating-catalytic LOCC state transformations that never increase the Shannon entropy. As a consequence, catalytic PPTP operations can increase the asymptotic entanglement content of quantum states and therefore loses some of its operational significance when catalysis is allowed.

10. Multipartite entanglement

Thus far our discussion of entanglement has been mainly restricted to scenarios involving two parties (bipartite scenarios). For pure states, these scenarios can be relatively easily studied using majorization due to the Nielsen's Theorem V.2. However, this is not the case for genuine multipartite scenarios, i.e., those involving more than two parties. In that case, we do not have a simple criterion to decide which state transitions are possible. Consequently, studying catalysis in these scenarios is challenging.

One of the rare results for multipartite scenarios was given by Chen et al. (2010). Given an N-partite pure quantum state $|\psi\rangle \in \bigotimes_{i=1}^{N} \mathcal{H}_{i}$, the tensor rank $\mathrm{rk}(\psi)$ of $|\psi\rangle$ is defined as the smallest number of product states $\{\bigotimes_{i=1}^{N} |\psi'\rangle_i\}_{i=1}^{\operatorname{rk}(\psi)}$ whose linear span contains $|\psi\rangle$. For N = 2, the tensor rank reduces to the Schmidt rank; however, for N > 2 computing this quantity is generally nondeterministic polynomial time hard (Johan, 1990). It is known that the tensor rank is a monotone under LOCC operations (Bengtsson and Zyczkowski, 2006). The converse, however, is true only for certain classes of states; in particular, it is true for states that are reversibly interconvertible with respect to the N-partite Greenberger-Horne-Zeilinger (GHZ) state: $|\text{GHZ}_N^d\rangle = (1/\sqrt{d})\sum_{i=1}^d |i\rangle^{\otimes N}$. For such states, the possible state transitions are fully characterized by the tensor rank. Chen et al. (2010) made use of this knowledge to study catalysis in multipartite entanglement. They found examples of nontrivial strict catalysis in the multipartite scenario under probabilistic LOCC. Specifically, they found that strict catalysis allows for an increase of the success probability of a state transition from zero to a strictly positive value. More recently Neven et al. (2021) reported the first examples of strict catalysis under deterministic LOCC in a genuinely multipartite scenario.

An important distinction between bipartite and multipartite scenarios is that the latter leads to multiple inequivalent classes of entanglement: States of one class cannot be converted to states of the other class with a nonzero success probability (Dür, Vidal, and Cirac, 2000). In this context, Ghiu, Bourennane, and Karlsson (2001) showed that strict catalysis does not help one to convert between distinct classes.

11. Contextuality

The notion of contextuality aims to characterize the property of quantum theory that it is impossible to assign predetermined values (hidden variables) to observables such that the functional relationship between compatible observables is conserved (Bell, 1966; Kochen and Specker, 1967). Specifically, let *M* be a set of *n* observables associated with a physical setting. In general, not all of these observables will be jointly measurable, but for any subset $C = \{A_1, ..., A_m\} \subset$ M of jointly measurable observables, called the measurement context, there is an associated probability distribution $p_C(a_1,...,a_m)$ for the outcomes a_j of the measurement of observable A_i . A set of observables is jointly measurable in quantum theory if and only if the observables arise as a coarse graining of a parent observable (Gühne et al., 2023). A noncontextual hidden-variable model for a physical setting is given by a random variable λ with distribution $q(\lambda)$ and an assignment of outcomes a_1, \ldots, a_n to all *n* observables with probability $q(a_1, \ldots, a_n | \lambda)$ such that the following apply:

(1) For each value of λ and every context *C*, the assignment of outcomes preserves the functional relationship

between the observables. That is, if $A_i, A_j \in C$ and $A_i = f(A_j)$ for some function f, then $a_i = f(a_j)$.

(2) The assignment reproduces the distributions p_C ,

$$p_C(a_1, \dots, a_m) = \sum_{\lambda} q_C(a_1, \dots, a_m | \lambda) q(\lambda), \quad (131)$$

where $q_C(a_1, ..., a_m | \lambda)$ is the restriction of $q(\cdot | \lambda)$ to the given measurement context *C*. If the distributions p_C cannot be reproduced by a noncontextual hidden-variable model, the physical setting is called *contextual*.

Bell's theorem and the Kochen-Specker theorem (Kochen and Specker, 1967) showed that quantum theory is contextual for Hilbert-space dimensions $d \ge 3$. It has been found that contextuality is closely related to quantum computational speedups (Raussendorf, 2013; Howard et al., 2014; Bermejo-Vega et al., 2017). This further suggests formulating a resource theory where the free objects correspond to noncontextual hidden-variable models and free operations correspond to consistent wirings of the models. See Abramsky et al. (2019) and Amaral (2019) for two different approaches to formulating a resource theory of contextuality. Karvonen (2021) showed that in the resource-theoretic framework of Abramsky et al. (2019) strict catalysis is not useful, mirroring the case of Bell nonlocality; see Sec. V.B.5. Since the set of free operations considered by Abramsky et al. (2019) is strictly larger than that of Amaral (2019) [see Budroni et al. (2022)], it is not clear whether strict catalysis could be useful in the latter. Furthermore, as far as we know it is currently unknown whether other types of catalysis, such as correlating catalysis, could be useful in contextuality.

C. Thermodynamics

Quantum thermodynamics is an interdisciplinary field that adopts the tools of multiple frameworks like stochastic thermodynamics (Esposito, Harbola, and Mukamel, 2009; Seifert, 2012), open quantum systems (Breuer and Petruccione, 2002; Kosloff, 2013), and quantum information (Goold et al., 2016; Vinjanampathy and Anders, 2016) to extend thermodynamic considerations to the quantum realm. Although quantum thermodynamics often uses different frameworks to make its considerations quantitative, the central questions it raises are universal across complementary frameworks. Most of these questions can be formulated in a unified way by analyzing the closed evolution of the system and environment. To do so, we consider the interaction of a system of interest S prepared in a state ρ_{S} with a thermal environment E in the state $\gamma_{\mathsf{E}} = e^{-\beta \hat{H}_{\mathsf{E}}}/Z_{\mathsf{E}}$ with $Z_{\mathsf{E}} = \operatorname{Tr} e^{-\beta \hat{H}_{\mathsf{E}}}$ by means of a global unitary U. The final state of the composite system SE after the interaction is given by

$$\sigma_{\mathsf{SE}} = U(\rho_{\mathsf{S}} \otimes \gamma_{\mathsf{E}})U^{\dagger}.$$
 (132)

The unitary U encodes all physically plausible types of interactions with respect to their strength (weak or strong coupling), complexity (local or collective), or duration (short or long relative to natural timescales). It also enables us to study time-dependent Hamiltonians and work protocols, and

furthermore makes no assumptions about the structure of E, which can either be macroscopic or have dimensions comparable to those of S. The mapping in Eq. (132) is therefore general and captures most of the effects encountered in thermodynamics.

Despite its generality, Eq. (132) does not explicitly include one of the most ubiquitous assumptions in classical thermodynamics: auxiliary systems are used cyclically in the thermodynamic process. These can be seen as an implicit way to use catalysis in textbook thermodynamics: For example, the operation of a piston when compressing an ideal gas can be seen as a catalytic transformation where the catalyst (piston) assists in transforming the system (ideal gas).

The cyclic auxiliary systems can be modeled by adding a catalytic environment C interacting with both the system S and the thermal environment E. The two environments are fundamentally distinct: While E describes an environment whose properties can freely change, C captures all of the degrees of freedom that must be preserved. We can therefore write the final state of the composite system SCE after an arbitrary unitary interaction U as

$$\sigma_{\mathsf{SCE}} = U(\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}} \otimes \gamma_{\mathsf{E}}) U^{\dagger}, \qquad (133)$$

where we further require that $\text{Tr}_{\mathsf{E}}[\sigma_{\mathsf{SCE}}] \approx \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}$. Furthermore, unless otherwise stated, we denote the local Hamiltonians of the three subsystems as \hat{H}_{A} with $\mathsf{A} \in \{\mathsf{S},\mathsf{E},\mathsf{C}\}$. Catalysts evolving according to Eq. (133) can be used to model various components of thermodynamic processes. Some particular examples include the following:

- *Finite environments*. When the environment is significantly larger than the system, the backaction that it experiences is usually negligible. However, in certain situations (such as when the environment is finite) it is important to quantify how it reacts due to the interaction with the system. This is relevant, for example, when one is interested in minimizing heat dissipation; see Sec. V.C.1. In such cases one can think about the finite thermal environment as an approximate catalyst.
- *Heat engines*. Heat engines are machines that make use of the temperature gradient to generate useful work. The machinery of an engine can be viewed as a catalyst that facilitates the conversion of heat into work while remaining unchanged during the operation. In fact, Clausius's statement of the second law of thermodynamics is formulated for a system undergoing a cyclic process, and therefore also for a catalyst. For details, see Sec. V.C.2.
- *Clocks.* Control operations with respect to time are required for turning on and off a time-dependent interaction term in the Hamiltonian, thus allowing thermodynamic protocols to be run in an autonomous manner. Any irreversible change in the clock system can deteriorate the protocol, and thus minimizing its disturbance is of much importance. For this reason, clock systems are modeled explicitly as catalysts; see Secs. V.C.3 and V.C.5.
- Apparatus. Laboratory equipment usually facilitates experiments by augmenting control or improving

performance without undergoing change themselves (clocks are a special example). This apparatus can be modeled as a catalyst when one wants to avoid using it as a source of energy or entropy; see Secs. V.C.6 and V.E.4.

1. Minimizing dissipation in thermodynamic protocols

Owing to the seminal work of Landauer (1961), it has been recognized that a logically irreversible erasure of information from a memory system leads to an unavoidable increase in the entropy of its environment. Specifically, let S be the memory system prepared as $\rho_{\rm S}$ with Hamiltonian $\hat{H}_{\rm S}$, and let E be its thermal environment with Hamiltonian \hat{H}_{E} . The erasure process is modeled using Eq. (132) and leads to a joint state σ_{SE} . The heat transferred to the environment E is defined as the change in average energy, i.e., $Q \coloneqq \text{Tr}[\hat{H}_{\mathsf{E}}(\sigma_{\mathsf{E}} - \rho_{\mathsf{E}})].$ Erasing information from S is equivalent to transforming it into a deterministic (pure) state, which causes an inevitable increase in the entropy of the environment equal to $\Delta H_{\rm E} :=$ $H(\sigma_{\rm E}) - H(\rho_{\rm E})$ (Bennett, 1982). Owing to the unitarity of the underlying dynamics, the entropy change on the system is given by $\Delta H_{\rm S} := H(\sigma_{\rm S}) - H(\rho_{\rm S}) = -\Delta H_{\rm E}$. The heat Q and $\Delta H_{\rm S}$ are therefore related by

$$\beta Q \ge -\Delta H_{\rm S},\tag{134}$$

which can be seen as a fundamental bound on the minimal amount of heat that has dissipated to the environment (Landauer, 1961). Reeb and Wolf (2014) derived a sharpened equality version of Eq. (134), i.e.,

$$\beta Q = -\Delta H_{\mathsf{S}} + I(\mathsf{S:E})_{\sigma_{\mathsf{SE}}} + D(\sigma_{\mathsf{E}} \| \rho_{\mathsf{E}}), \quad (135)$$

which, due to the non-negativity of mutual information and relative entropy, implies Eq. (134). The quantity $Q_{\text{diss}} \coloneqq Q + \Delta H_S / \beta$ is known as the dissipated heat and captures the irreversible character of the thermodynamic process (Jarzynski, 2011). In the case in which $\Delta H_S \leq 0$, Reeb and Wolf (2014) proved the following lower bound:

$$\beta Q_{\text{diss}} \ge \frac{2(\Delta H_{\text{S}})^2}{\log^2(d_{\text{E}} - 1) + 4}.$$
 (136)

Equation (136) is a strict improvement over the Landauer's bound $Q_{\text{diss}} > 0$ whenever the environment involved in the process has a finite Hilbert-space dimension d_{E} . The bound was further shown to be achievable for specific states ρ_{S} and Hamiltonians \hat{H}_{S} . It is currently not known whether Eq. (136) is tight for general quantum processes. Furthermore, as Eq. (136) results from mathematical properties of the relative entropy, it is not clear whether there exists a physical process that achieves $Q_{\text{diss}} \propto (\log d_{\text{E}})^{-2}$. Specifically, we consider the minimal achievable heat dissipation Q_{diss}^* ,

$$Q_{\rm diss}^* \coloneqq \min_{\hat{H}_{\sf E}, U} Q_{\rm diss}, \tag{137}$$

where $Q = Q(\rho_S, \beta, \hat{H}_S, \hat{H}_E, U)$, and we proceed similarly for ΔH_S . The best known protocols that aim to minimize Q^*_{diss} were proposed by Reeb and Wolf (2014) and further analyzed by Skrzypczyk, Short, and Popescu (2014) and Bäumer *et al.* (2019). In all of these cases heat dissipation decreases linearly with $\log d_{\mathsf{E}}$, i.e.,

$$\beta Q_{\rm diss}^* = \mathcal{O}\left(\frac{1}{\log d_{\sf E}}\right). \tag{138}$$

We now describe a protocol that achieves this scaling. Consider E to be a system composed of $n = \log d_{\rm E} / \log d_{\rm S}$ subsystems, ${\sf E} = {\sf E}_1 {\sf E}_2 \cdots {\sf E}_n$, where each ${\sf E}_i$ has the same dimension $d_{\sf S}$ as the system of interest S and its own Hamiltonian $\hat{H}_{{\sf E}_i}$. The global unitary, denoted U_{π} , is a sequential swap between S and each ${\sf E}_i$, leading to the overall action

$$U_{\pi}|i_{0}\rangle_{\mathsf{S}}|i_{1}\rangle_{\mathsf{E}_{1}}|i_{2}\rangle_{\mathsf{E}_{2}}\cdots|i_{n}\rangle_{\mathsf{E}_{n}}$$

= $|i_{n}\rangle_{\mathsf{S}}|i_{0}\rangle_{\mathsf{E}_{1}}|i_{1}\rangle_{\mathsf{E}_{2}}\cdots|i_{n-1}\rangle_{\mathsf{E}_{n}}.$ (139)

This is the same unitary used in describing the relationship between correlating-catalytic and multicopy transformations (see Sec. IV.B) and in the construction of universal approximate catalysts (see Sec. IV.C). To further perform the minimization in Eq. (137), one optimizes over local Hamiltonians \hat{H}_{E_i} , which yields the scaling in Eq. (138). In fact, for any thermal environment composed of noninteracting subsystems the scaling of Q_{diss} is at most linear in $\log d_{E}$ (Reeb and Wolf, 2014).

The aforementioned protocol for minimizing Q_{diss}^* can alternatively be viewed as the problem of finding the leastdisturbed approximate catalyst where the thermal environment itself is treated as the approximate catalyst or embezzler. This follows from the correspondence $\gamma_{\mathsf{E}} \propto \log \hat{H}_{\mathsf{E}}$, which makes optimizing Eq. (137) over \hat{H}_{E} equivalent to optimizing it over all possible density operators. To demonstrate this, we consider a thermal environment whose Hamiltonian \hat{H}_{E} is chosen such that its thermal state coincides with the universal approximate catalyst described in Sec. IV.C. Specifically, choose $\hat{H}_{\mathsf{E}} = -(1/\beta) \log \rho_{\mathsf{E}}$, where ρ_{E} is given by

$$\rho_{\mathsf{E}} = \frac{1}{n-1} \sum_{i=1}^{n-1} \rho_{\mathsf{S}}^{\otimes i} \otimes \sigma_{\mathsf{S}}^{\otimes (n-i)}.$$
 (140)

This means that $\gamma_{\mathsf{E}} = \rho_{\mathsf{E}}$ and furthermore $d_{\mathsf{E}} = d_{\mathsf{S}}^n$. Applying the unitary U_{π} from Eq. (139) to $\rho_{\mathsf{S}} \otimes \rho_{\mathsf{E}}$ gives

$$\sigma_{\mathsf{SE}} = U_{\pi}(\rho_{\mathsf{S}} \otimes \rho_{\mathsf{E}}) U_{\pi}^{\dagger} = \sigma_{\mathsf{S}} \otimes \sigma_{\mathsf{E}}, \qquad (141)$$

with $\Delta(\rho_{\mathsf{E}}, \sigma_{\mathsf{E}}) \leq 1/(n-1)$, as in the analysis leading to Eq. (90). For generic states ρ_{E} and σ_{E} of the aforementioned structure, numerical analysis shows that $D(\sigma_{\mathsf{E}}||\rho_{\mathsf{E}})$ scales as $O(1/\log d_{\mathsf{E}})$, as in Eq. (138). This brings us to the conclusion that processes in which heat dissipation obeys this scaling are actually performing thermal embezzlement on the thermal environment. This connection between heat dissipation and thermal embezzlement, to our knowledge, has not yet been appreciated in the literature. Furthermore, it is an interesting open question as to whether one can use stricter notions of catalysis to engineer better, i.e., less dissipative, thermodynamic protocols.

An alternative approach for lowering the heat dissipation $Q_{\rm diss}$ was proposed by Henao and Uzdin (2023). They included an additional system C (the catalyst) such that the joint system SCE evolved according to Eq. (133). The unitary U is then composed of two steps, i.e., $U = V_{SCE}(V_{SE} \otimes \mathbb{1}_{C})$. That is, one first implements a unitary V_{SE} on the system and the thermal environment, obtaining $\sigma_{SE} = V_{SE}(\rho_S \otimes \gamma_E) V_{SE}^{\dagger}$. One then applies a second unitary V_{SCE} to the joint system SCE, leading to the final global state $\sigma'_{\sf SCE} =$ $V_{\rm SCE}(\sigma_{\rm SE}\otimes\omega_{\rm C})V_{\rm SCE}^{\dagger}$. The second interaction (which involves the catalyst) is introduced to mitigate the heat $Q_{\rm diss}$ dissipated to the thermal environment during the first interaction. Henao and Uzdin (2023) showed that, for any correlated state σ_{SF} obtained in the first step of the protocol, i.e., if $I(S:E)_{\sigma} > 0$, there always exists a unitary V_{SCE} that can reduce the local entropy of the environment without changing the local states of the system and the catalyst, that is, $\sigma'_{S} = \sigma_{S}$ and $\sigma'_{C} = \sigma_{C}$. As a consequence, the final dissipated heat Q'_{diss} computed for the state σ'_{SCE} is lower than Q_{diss} . Based on this observation, Henao and Uzdin (2023) argued that the use of a catalyst allows heat dissipation to be mitigated in the process of information erasure (i.e., when $\Delta H_{\rm S} < 0$). This reduction in heat dissipation can be understood as a consequence of broadcasting correlations from SE to the joint system SCE that lowers the entropy of the environment, i.e., $H(\sigma'_{\mathsf{F}}) - H(\sigma_{\mathsf{E}}) < 0.$

We conclude this section by mentioning another situation in which the thermal environment is formally treated as a strict catalyst. In the theory of open quantum systems, the Born-Markov approximation gives rise to a Markovian master equation for the dynamics on the thermal environment E (Breuer and Petruccione, 2002). Moreover, when combined with a rotating wave approximation, it implies that the steady state of the dynamics on E is a thermal state. The Born-Markov approximation is often stated to assume that the thermal environment remains thermal at all times and does not build up correlations with the system S, i.e., that the evolution from Eq. (132) can be approximated as

$$U(\rho_{\mathsf{S}} \otimes \gamma_{\mathsf{E}})U^{\dagger} \approx \sigma_{\mathsf{S}} \otimes \gamma_{\mathsf{E}}.$$
 (142)

From the point of view of this review it can hence be seen as a form of catalysis. We emphasize, however, that the validity of the approximation in fact requires only (a) that the correlations between the subsystems S and E are negligible as measured by the interaction and (b) that the two-time correlation functions of the operators of the interaction term on the thermal environment match the thermal ones with high accuracy. These requirements may already be fulfilled for a thermal environment that is only locally thermal, for example, in the sense of the eigenstate thermalization hypothesis; see J. M. Deutsch (1991), Srednicki (1994), Polkovnikov *et al.* (2011), D'Alessio *et al.* (2016), and Gogolin and Eisert (2016). Strict catalysis is therefore not actually required for the approximation to hold. However, the fact that the thermal environment has to remain locally thermal can be seen as a

form of approximate catalysis of the resource content (the local thermality) of the bath.

2. Work extraction

Consider the general map from Eq. (132) specified to the case in which the thermal environment is not included in the global dynamics, i.e., $U = U_{S}(t) \otimes \mathbb{1}_{E}$. To do so, consider a quantum system S that evolves under a Hamiltonian $\hat{H}_{S}(t) = \hat{H}_{S} + \hat{V}(t)$, where $\hat{V}(t)$ is a time-dependent external potential. From control theory we know that any interaction $\hat{V}(t)$ can be formally written using a unitary $U_{\rm S} = \mathcal{T} \exp\left[-(i/\hbar)\int_0^t \hat{H}_{\rm S}(s) \mathrm{d}s\right]$, where \mathcal{T} is the timeordering operator. We consider a cyclic process during which the system S performs work on an external source. A thermodynamic process is called cyclic when the system is coupled at time t = 0 to an external source of work and fully decouples at time t = T, i.e., V(0) = V(T) = 0. Since the system generally does not return to its initial state at time T, this allows one to perform thermodynamic work on the external source. The maximal amount of work that can be extracted under a cyclic process is known as the ergotropy and can be written as (Allahverdyan, Balian, and Nieuwenhuizen, 2004)

$$W_{\mathcal{U}}(\rho_{\mathsf{S}}) \coloneqq \max_{U \in \mathcal{U}_{\mathsf{S}}} \operatorname{Tr}[\hat{H}_{\mathsf{S}}(\rho_{\mathsf{S}} - \sigma_{\mathsf{S}})],$$
(143)

where $\sigma_{\rm S} \coloneqq U\rho_{\rm S} U^{\dagger}$ and $\mathcal{U}_{\rm S}$ stands for the set of all unitary operators on $\mathcal{D}(S)$. We also identify $\rho_{\rm S} \equiv \rho_{\rm S}(0)$. Quantum states from which no work can be extracted using unitary operations are called passive states. The concept of passivity is fundamental in thermodynamics: for instance, it singles out Gibbs thermal states as the only passive states that remain passive when one takes an arbitrary many uncorrelated copies (Lenard, 1978; Pusz and Woronowicz, 1978).

The previous scenario can be extended to the case in which an additional system (a catalyst) is used to increase the amount of work extracted in the process. Note that due to the basic lemma of catalysis (see Lemma II.1), no strict catalyst exists that can be used to increase the ergotropy in Eq. (143). However, this is no longer the case for the other, less stringent notions of catalysis. To see this, we now consider an extension of ergotropy to the case of correlating catalysis. In that case, one can define the correlating-catalytic ergotropy as

$$W_{\mathcal{U}}^{\text{CC}}(\rho_{\mathsf{S}}) \coloneqq \max_{U \in \mathcal{U}_{\mathsf{SC}}} \max_{\omega_{\mathsf{C}}} \operatorname{Tr}[\hat{H}_{\mathsf{S}}(\rho_{\mathsf{S}} - \sigma_{\mathsf{S}})], \quad (144)$$

where $\sigma_{SC} = U(\rho_S \otimes \omega_C)U^{\dagger}$ and $\sigma_C = \omega_C$. Sparaciari, Jennings, and Oppenheim (2017) first observed that passive states undergoing unitary evolutions can be activated when a suitable catalyst is used. In particular, Sparaciari, Jennings, and Oppenheim (2017) showed for three-level systems and Lipka-Bartosik and Skrzypczyk (2021b) and Wilming (2021) showed in general that

$$W_{\mathcal{U}}^{\rm CC}(\rho_{\rm S}) = {\rm Tr}\{\hat{H}_{\rm S}[\rho - \gamma_{\beta^*}(\hat{H}_{\rm S})]\},\tag{145}$$

where $\gamma_{\beta^*}(\hat{H}_S)$ is the unique Gibbs state at inverse temperature β^* satisfying $H(\rho_S) = H[\gamma_{\beta^*}(\hat{H}_S)]$. What one observes here is

that the catalyst allows the main system to evolve in a way that does not keep its spectrum invariant, which allows it to reach a Gibbs state with the same entropy. This final state is optimal in the sense that, among all states of the same entropy, $\gamma_{\beta^*}(\hat{H}_S)$ always achieves the minimum of the average energy.

The previous results were further generalized by Lipka-Bartosik and Skrzypczyk (2021b) to general resource theories $\mathcal{R} = (\mathcal{S}, \mathcal{O})$ that satisfy certain reasonable assumptions.⁶ More concretely, one can consider two much different notions of generalized ergotropy, one being achievable under correlating-catalytic state transformations, that is,

$$W_{\mathcal{O}}^{\rm CC}(\rho_{\rm S}) \coloneqq \max_{\mathcal{F} \in \mathcal{O}} \max_{\omega_{\rm C}} \operatorname{Tr}[\hat{H}_{\rm S}(\rho_{\rm S} - \sigma_{\rm S})] \qquad (146)$$

such that
$$\sigma_{SC} = \mathcal{F}(\rho_S \otimes \omega_C), \quad \sigma_C = \omega_C, \quad (147)$$

and the other being the asymptotic rate of extractable work $W^{\infty}_{\mathcal{O}}(\rho_{\mathsf{S}}) \coloneqq \lim_{n \to \infty} W_{\mathcal{O}}(\rho^{\otimes n})/n$, which is computed using the total Hamiltonian of *n* independent copies of the quantum system S, that is, $\hat{H}^{n}_{\mathsf{S}} \coloneqq \sum_{i=1}^{n} \hat{H}_{\mathsf{S}_{i}}$, with $\hat{H}_{\mathsf{S}_{i}} \equiv \mathbb{1}_{/\mathsf{S}_{i}} \otimes \hat{H}_{\mathsf{S}_{i}}$.

It then follows from the partial-order regularization using correlating catalysis described in Sec. IV.B that $W_{\mathcal{O}}^{CC}(\rho_{S}) = W_{\mathcal{O}}^{\infty}(\rho_{S})$. In other words, the use of an appropriate catalyst allows exactly the same amount of work to be extracted from a passive state as in the limit of asymptotically many copies on average (i.e., per copy). In fact, even though $W_{\mathcal{O}}^{CC}$ and $W_{\mathcal{O}}^{\infty}$ have been defined according to the Hamiltonian of the system \hat{H}_{S} , the techniques used to prove this fact are also applicable to other observables, like particle number or the overlap with a fixed quantum state. In this sense, correlating catalysis allows passive states to activate with respect to arbitrary observables as long as the corresponding asymptotic rate is larger than the single-copy one (Lipka-Bartosik and Skrzypczyk, 2021b).

At this point, one notes that the ergotropy $W_{\mathcal{U}}$ defined in Eq. (143) is not the sole existing quantification of extractable work in thermodynamics. While it quantifies thermodynamic work as the difference in average energy induced in the main system, this energy difference is not stored explicitly in another physical system (a battery). In fact, owing to the conservation of energy, the surplus energy must implicitly be released into some external environment that in general could be difficult to access for future use.

The natural question that arises is whether the energy extracted from a quantum system can be stored in a way that is fully usable in the future. Gallego, Eisert, and Wilming (2016) argued from operational principles that one should always model this "battery" or "work-storage device" explicitly in a manner that is compatible with the intended future use of the energy. In a similar spirit, explicit battery models are often used to define work on a single-shot level in the context of the resource theory of athermality; see Brandão *et al.* (2015) and Faist *et al.* (2015). Here storing an amount of energy *E* corresponds to preparing an appropriate battery system in an

⁶The result holds for any resource theory $\mathcal{R} = (\mathcal{S}, \mathcal{O})$ in which the free operations \mathcal{O} allow identical subsystems to be permuted and operations to be conditioned on classical information; see Sec. IV.A.

energy eigenstate of energy *E*, thereby modeling a deterministic energy storage. For more details, see Sec. V.C.5.

3. Thermal operations

We now consider the system of interest S interacting with both the thermal (E) and the catalytic (C) environments according to Eq. (133). Assume furthermore that there are no external sources of work such that the total energy is conserved and the global unitary U satisfies $[U, \hat{H}_{S} + \hat{H}_{E} + \hat{H}_{C}] = 0$. This scenario is the main subject of the resource theory of athermality,⁷ which we introduced in Sec. II.A.3.

The first steps for explicitly studying catalysis in the resource theory of athermality were taken by Brandão *et al.* (2015). They raised the question of whether it is possible to characterize all possible state transformations on the system of interest S when both the catalytic and thermal environments can be chosen arbitrarily (i.e., up to their respective constraints). They started with the state-transition conditions for thermal operations (thermomajorization) introduced in Sec. II.A.3 and showed that they can be relaxed by strict catalysis into a set of monotonic entropic conditions known as the generalized second laws of quantum thermodynamics.

More concretely for energy-incoherent states that fulfill $[\rho_S, \hat{H}_S] = 0$, they identified a family of monotones

$$F_{\alpha}^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) \coloneqq F_{\mathrm{eq}}^{\beta}(\hat{H}_{\mathsf{S}}) + \frac{1}{\beta} D_{\alpha}(\rho_{\mathsf{S}} \| \gamma_{\beta}(\hat{H}_{\mathsf{S}})), \qquad (148)$$

where $D_{\alpha}(\cdot \| \cdot)$ are the classical Rényi divergences, $F_{eq}^{\beta}(H) \coloneqq -(1/\beta) \log Z_{\beta}(H)$ is the equilibrium Helmholz free energy, and $Z_{\beta}(H) \coloneqq \operatorname{Tr} e^{-\beta H}$ is the partition function. When the initial state ρ_{S} and the target state σ_{S} are energy incoherent, the necessary and sufficient conditions for a strictly catalytic state transition $\rho_{S} \underset{TO}{\hookrightarrow} \sigma_{S}$ are then given by

$$F_{\alpha}^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) > F_{\alpha}^{\beta}(\sigma_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) \quad \text{for } \alpha \in \mathbb{R} \setminus \{0\}.$$
(149)

By relaxing the notion of catalysis from strict to arbitrarily strict catalysis (recall Sec. III.C), all conditions with $\alpha < 0$ can be dropped. Refined alternative statements and proofs of this result were since done by Rethinasamy and Wilde (2020) and Gour and Tomamichel (2021). The fully quantum case where $\rho_{\rm S}$ or $\sigma_{\rm S}$ is energy coherent is still open, even in the case in which no catalyst is used. In the noncatalytic case several necessary monotones are known (Brandão *et al.*, 2015; Lostaglio, Jennings, and Rudolph, 2015; Lostaglio *et al.*, 2015), but the minimal sufficient set of complete monotones remains undetermined, with the exception of when the system S is a qubit (Ćwikliński *et al.*, 2015). For energy-incoherent states in the i.i.d. limit, the generalized free energies from Eq. (148) for all $\alpha \in \mathbb{R}$ converge to a single quantity, namely, the nonequilibrium free energy [see also Eq. (27)]

$$F^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) = \lim_{\alpha \to 1} F^{\beta}_{\alpha}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}).$$
(150)

As a consequence, both thermomajorization conditions and the generalized second laws converge to a single inequality, namely, the monotonicity of the nonequilibrium free energy. This convergence is consistent with our understanding from macroscopic thermodynamics, strengthening the role of thermal operations as a physical description of the microscopic regime.

Historically the resource theory of thermodynamics served as a natural starting ground for considering different relaxations to the notion of catalysis. Brandão *et al.* (2015) envisioned that catalysts describe the apparatus used to control the thermodynamic process, such as a clock (a reference frame) that keeps track of the duration in which an interaction Hamiltonian is switched on; see Sec. V.D.1.

At that time the existing research on quantum reference frames highlighted the central issue of their inevitable degradation, i.e., that the state of the reference frame does not return exactly to its original state whenever information about the main system is inferred from the reference frame (Poulin and Yard, 2007), as opposed to the ideal setting (Bartlett, Rudolph, and Spekkens, 2007; Ahmadi, Jennings, and Rudolph, 2010). This was part of the original motivation to investigate the robustness of the state-transition conditions (149) under small errors in the catalyst. It was realized that earlier results by van Dam and Hayden (2003) and Leung and Wang (2014) implied the emergence of embezzlement in catalytic thermal operations, resulting in the breakdown of the generalized second laws of thermodynamics (even the macroscopic second law singled out in the i.i.d regime). Indeed, the construction of Sec. IV.C also applies to thermal operations. This undesirable effect is unphysical and indicates that more care needs to be taken when allowing for an error in the catalyst.

Brandão et al. (2015) then identified three different regimes of catalysis corresponding to (1) embezzling, (2) approximate catalysis where the remaining monotone is the nonequilibrium free energy, and (3) strict catalysis. In particular, the second regime can be seen as the intersection of approximate and correlating catalysis (which used the same catalyst construction as discussed in Secs. IV.A and IV.C). In this regime Brandão et al. (2015) assumed that the disturbance of the catalyst in trace distance decreases with the size of the catalyst. This allows one to show that the nonequilibrium free energy F^{β} is the only relevant monotone that fully characterizes state transitions under this relaxation of catalysis. The problem of thermal embezzlement was later addressed by Ng et al. (2015), who derived the minimal achievable error as a function of the catalyst dimension. Ng et al. (2015) also showed that additional physical constraints on the catalyst, such as an upper bound on the average energy, allowed for the recovery of various monotones.

⁷The resource theory of athermality was originally introduced with a number of additional technical assumptions about the spectrum of the thermal environment (i.e., its Hamiltonian H_B); see Janzing *et al.* (2000), Horodecki and Oppenheim (2013), and Brandão *et al.* (2015). However, one can equivalently consider all possible Hamiltonians for the thermal environment and then show that no advantage is obtained in terms of the possible state transformation.

A conceptually different functioning of the catalyst was discussed by Woods and Horodecki (2023), who analyzed the problem of implementing energy-preserving unitaries for thermal operations in an autonomous manner, i.e., via a time-independent global Hamiltonian; see also Malabarba, Short, and Kammerlander (2015), Silva et al. (2016), Erker et al. (2017), and Woods, Silva, and Oppenheim (2019) and Sec. V.D. To achieve this, they explicitly modeled a quantum clock that provides the necessary timing information, and consequently degrades in the process. In the language of this review such systems can be viewed as approximate catalysts. Woods and Horodecki (2023) started by showing that some amount of backaction on the clock system was inevitable unless the clock had an unphysical Hamiltonian (such as those without a ground state). They proceeded to bound the maximal error (in trace distance) that can be induced on the clock, denoted by $\varepsilon_{\rm emb}$, and showed that such a backaction would necessarily affect the accuracy of the implemented state transition on the system. In that way, they derived an upper bound on the resolution error $\varepsilon_{\rm res}$ for achieving the desired transformation on the system, i.e., the minimum achievable trace distance between a final system state with the original target state. The upper bound on ε_{res} in particular depends on the dimension of the system, the catalyst used, and the clock considered, as well as the allowed error ε_{emb} . The main result of Woods and Horodecki (2023) was that the two errors $\varepsilon_{\rm res}$ and $\varepsilon_{\rm emb}$ can vanish together even when the clock system has a physical Hamiltonian (what they referred to as a quasi-ideal clock). This result implied that all catalytic thermal operations can be (1) implemented in a fully autonomous manner with an inexact catalyst where (2) the state-transition conditions in Eq. (148) remain robust.

While strict catalysis was first investigated in the context of entanglement transformations under LOCC, the concept of correlating catalysis is much better founded in the framework of thermal operations. This is because the majorization relation, via Nielsen's theorem, fully describes state transitions only in the case of pure bipartite systems. In contrast, its thermodynamic analog, relative majorization, also characterizes state transitions under thermal operations between mixed states. This implies that it can be used to address the extensions of exact catalysis, for example, by allowing for residual correlations between S and C (correlating catalysis) or within different parts of C (marginal-correlating catalysis), even after tracing out the environment E. Lostaglio, Müller, and Pastena (2015) and Müller and Pastena (2016) constructed an explicit multipartite catalyst such that, by allowing for final correlations to be created between the partitions, it enabled the by passing of generalized second laws from Eq. (149) for all α except $\alpha = 0$ and 1. The guiding intuition behind this work is the fact that the relative entropy D_1 is the unique quantity out of the entire family of Rényi divergences D_{α} , which is superadditive (with the exception of D_0 , which is unstable under perturbations of the state). Subsequently Gallego, Eisert, and Wilming (2016) argued from operational principles that one should consider the notion of correlating catalysis. Wilming, Gallego, and Eisert (2017) showed that both correlating catalysis and marginal-correlating catalysis single out the nonequilibrium free energy F^{β} as the unique continuous and additive monotone. A natural question that emerged from these considerations concerns whether monotonicity of the nonequilibrium free energy alone could be sufficient for the convertability via correlating catalysis. In other words, it was conjectured that (when allowing for an arbitrarily small error on the final state, which eliminates the constraint arising from F_{Ω}^{β})

$$F^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) \ge F^{\beta}(\sigma_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) \Rightarrow \rho_{\mathsf{S}} \stackrel{\text{corr}}{\underset{\mathrm{TO}}{\hookrightarrow}} \sigma_{\mathsf{S}}.$$
(151)

For energy-incoherent states, Eq. (151) was proven by Müller (2018) and Rethinasamy and Wilde (2020). It is wrong for general coherent states since thermal operations cannot build up coherences. This can be circumvented by enlarging the set of free operations from thermal operations to Gibbs-preserving (GP) operations (Faist, Oppenheim, and Renner, 2015). Using the construction discussed in Sec. IV.B, Shiraishi and Sagawa (2021) showed that Eq. (151) is true for general quantum states if the thermal operation (TO) is replaced by a GP operation. These results show that correlating catalysis can lift the infinite family of second laws (149) to just the monotonicity of the standard nonequilibrium free energy. Note that in these results the catalyst can remain correlated to the system of interest to an arbitrarily small degree provided that its dimension is large enough.

As we conclude this section, we note that identifying an appropriate state of a catalyst for thermal operations is generally a challenging task. To date there has been no comprehensive method that can determine the state of the catalyst required by a given state transformation. This issue was partially addressed by Lipka-Bartosik and Skrzypczyk (2021a). Specifically, for a given pair of energy-incoherent states $\rho_{\rm S}$ and $\sigma_{\rm S}$ that satisfy Eq. (149) for $\alpha \ge 0$, they observed that a quantum state with randomly distributed occupations in the energy basis can, with high probability, serve as an approximate catalyst for the transition $\rho_{\rm S} \xrightarrow{\rm TO} \sigma_{\rm S}$. This success probability furthermore increases for catalysts with a larger dimension. Moreover, Lipka-Bartosik and Skrzypczyk (2021a) claimed to formally prove that, for any pair of energy-incoherent states satisfying the second laws of Eq. (149) for $\alpha \ge 0$, any energy-incoherent quantum state can act as an approximate catalyst for the transition as long as enough copies of the catalyst are available. However, it was later discovered that the proof of this statement has a gap (i.e., Lemma 4 therein is not valid), which leaves this claim unresolved.

4. Thermal operations with limited control

Thermal operations capture generic energy-preserving interactions between a system and its surroundings. However, sometimes it can be appealing to study specific classes of thermal operations that can admit more straightforward experimental implementations, or better reflect actual control capabilities. For this reason, two main classes of thermal processes have been studied in recent years. The first class is known as elementary thermal operations (ETOs) (Lostaglio, Alhambra, and Perry, 2018), which are systembath interactions that can be decomposed into sequences of two-level operations on the system. The second class is called Markovian thermal processes (MTPs) (Lostaglio and Korzekwa, 2022), which is a hybrid approach that combines Markovian master equations with resource-theoretic formulations to describe memoryless system dynamics. While these two classes of operations are formally different from each other, for energy-incoherent initial states, it was shown by Lostaglio and Korzekwa (2022) that the set of states reachable via MTPs is a subset for that of ETOs.

Strictly catalytic versions of the aforementioned classes have been studied (Son and Ng, 2022). The first point of observation is that in both ETOs and MTPs, while a lot of freedom is still given in preparing and using thermal Gibbs states, such states are technically no longer free states in the usual resource-theoretic sense. Take ETOs, for example: For each two-level operation on the system of interest, one is allowed to append any Gibbs state and turn on the system-bath interaction; however, before proceeding to the next step this Gibbs state is traced out and replaced by a fresh copy of another Gibbs state. For this reason, in contrast to thermal operations, even a catalyst in the Gibbs state can be useful (Korzekwa and Lostaglio, 2022; Son and Ng, 2023). As for the second point, since the gap between the ETO or MTPs and the TOs exist due to a restricted form of Markovianity of the former, intuitively this gap should be closed when a proper memory system is allowed (for example, in the form of a catalyst in a Gibbs state).

The previous idea was formalized and examined in two studies using different approaches. Czartowski, de Oliveira Junior, and Korzekwa (2023) developed a protocol using MTPs, combined with an explicit modeling of a memory acting as a strict catalyst. They showed that, in the infinite temperature limit, MTPs can approximately reach all states that are reachable by the TOs, with the precision increasing with the size of the catalyst. They also conjectured that the same holds at finite temperatures. Son and Ng (2023) took a different approach by focusing on ETOs. They showed that for any TO it is possible to decompose the global energyconserving unitary into a sequence of two-level unitaries acting on the system up to an arbitrary precision using a Suzuki-Trotter expansion. They then proved that any TO can be implemented with arbitrary precision using ETOs with a strict catalyst prepared in a Gibbs state. For energy-incoherent initial states, they also showed that any final state achievable via strictly catalytic TOs is also achievable via strictly catalytic ETOs and MTPs.

5. Work extraction with explicit batteries

The concept of work is not easy to define in the formalism of resource theories. This is because the resource-theoretic approach requires us to explicitly model the physical mechanism of storing and supplying thermodynamic work. Recall that in Sec. V.C.2 work was modeled via an external potential that performs work on the system. This approach is not sufficient in the resource-theoretic mindset where all external resources must be accounted for explicitly. The usual approach of extending the scenario from Eq. (133) is to add an explicit battery system B that supplies and stores the thermodynamic work associated with the process. With this in mind, Brandão *et al.* (2015) modeled the battery system B using a two-level system with an energy gap *w* initialized in a pure energy eigenstate $|0\rangle\langle 0|_{B}$. They defined deterministic work as the maximum value of *w* for which the following state transition is possible:

$$\rho_{\mathsf{S}} \otimes |0\rangle \langle 0|_B \xrightarrow[\mathrm{TO}]{arb} \sigma_{\mathsf{S}} \otimes |1\rangle \langle 1|_B,$$
(152)

where a positive value of *w* corresponds to extracting work from the system, while a negative value corresponds to supplying work for the transformation. When the generalized second laws (149) in the aforementioned state transition is used, it can be found that the optimal value of *w* that allows the system to be taken from one energy-incoherent state ρ_S to another one σ_S is given by

$$W_{\text{dist}} = \inf_{a \ge 0} [F_{\alpha}^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) - F_{\alpha}^{\beta}(\sigma_{\mathsf{S}}, \hat{H}_{\mathsf{S}})], \qquad (153)$$

which is also known as the work distance (Brandão *et al.*, 2015). The immediate question arises as to whether the exact form of W_{dist} has to depend strongly on the battery model. Brandão *et al.* (2015) showed that W_{dist} can be defined more generally as the ability of extracting and storing work in a pure (i.e., zero entropy) state. They illustrated that by considering another battery model, namely, the purity battery (Bennett, 1982; Faist *et al.*, 2015), one arrives at Eq. (153) when applying Landauer's erasure to relate the purity of the battery to thermodynamic work.

Building on these results, Woods, Ng, and Wehner (2019) studied the scenario in which catalysts are used as controls or machines undergoing a cyclic process in the presence of two baths at different inverse temperatures β_c and β_h . More specifically, they considered a setting where the hot bath acts as the background that defines the set of catalytic thermal operation to be TO(β_h), and the other bath is of a finite size, initialized in the Gibbs state of inverse temperature β_c with its Hamiltonian H_S .

Casting the problem of work extraction into a question of battery state preparation allowed Woods, Ng, and Wehner (2019) to establish a generic framework that provides systematic tools for investigating the quality of the extracted energy. They characterized numerous types of extracted work according to how the entropy of the battery compared with the extracted average energy. The first type, which they called perfect work, includes the situation in Eq. (152), where there is a strictly zero increase in entropy of the battery state. A slight relaxation of this condition, called near-perfect work, allows for an increase in entropy of the battery as long as it is arbitrarily small compared to the amount of extracted work. Woods, Ng, and Wehner (2019) analyzed the impact of the generalized free energies (148) on such a heat engine setting and concluded that they place fundamental limitations on the maximal efficiency η for the case of a single qubit of the system, where

$$\eta = \frac{W_{\text{ext}}}{\Delta H},\tag{154}$$

with W_{ext} extracted, near-perfect work, and ΔH the amount of energy change in the hot bath. In particular, the existence of

generalized free energies emerging as monotones apart from the well-known nonequilibrium free energy implied that for certain qubit Hamiltonians the maximum achievable efficiency fell strictly below the Carnot limit $\eta_{\rm C} = 1 - \beta_{\rm h} / \beta_{\rm c}$. Furthermore, it was shown that allowing for correlations to build up between the system and the catalyst (i.e., relaxing the notion of catalysis from strict to correlated) cannot be used to bypass this limitation. This seems to contradict the insight from Sec. IV.A, which states that correlating catalysis completely regularizes the single-shot partial order into the asymptotic one. However, note that Woods, Ng, and Wehner (2019) derived the optimal efficiency of work extraction in the limit of vanishing extractable work (the Carnot efficiency limit). In this limit the correlations have to vanish sufficiently quickly relative to the extracted work in order to achieve the Carnot efficiency. In particular, they have to vanish quicker than the scaling we discussed in Sec. IV.A when one analyzes the single-shot regularization. These additional requirements imply that the single-shot effects, as captured by the generalized second laws, may still persist even when correlations between the system and the catalyst are considered.

As discussed in Sec. V.C.3, correlating catalysis can lift the second laws to essentially just the monotonicity of the free energy. Therefore, when a correlating catalyst is used, the amount of extractable work [in the sense of the battery model from Eq. (152)] can be made arbitrarily close to

$$W = F^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) - F^{\beta}(\sigma_{\mathsf{S}}, \hat{H}_{\mathsf{S}}), \qquad (155)$$

which can be significantly larger than W_{dist} . This is true either for incoherent states when considering thermal operations or for arbitrary states when considering the more general Gibbspreserving maps as free operations.

Another perspective on catalysis in thermodynamics was unveiled in the context of fluctuation theorems (Jarzynski, 1997; Crooks, 1999; Tasaki, 2000), where the notion of work is defined by the two-point measurement scheme, namely, the difference between energy measurement outcomes before and after a thermodynamic process,

$$W = E_f - E_i. \tag{156}$$

More specifically, fluctuation theorems rule out the possibility of extracting a macroscopic (i.e., scaling extensively with the number of copies of the system) amount of work with nonnegligible probability in the case of unitary evolution of a system prepared in a Gibbs state. Boes *et al.* (2020) proposed a protocol that uses correlating catalysis in thermal operations to extract a macroscopic amount of work with a nonnegligible probability in such as way that the averaged second law still holds. To achieve macroscopic work, the logarithm of the dimension of the catalyst and its free energy must also grow linearly with the number of system copies. The general results of Rubboli and Tomamichel (2022) applied to this scenario imply that this extensive scaling of the catalyst's dimension and free energy is necessary.

Moving away from resource-theoretic considerations, Rodriguez *et al.* (2023) considered the charging of quantum batteries in a concrete physical model where a quantum battery (harmonic oscillator) is charged by coupling it to a harmonic oscillator driven by a classical laser field. They showed that adding an intermediate coupler can enhance the energy transfer from charger to battery while at the same time removing the requirement to fine-tune the laser frequency to the involved coupling strengths. At the same time the intermediate coupler itself stores almost no energy and hence effectively works as a catalyst.

6. Cooling

The efficient cooling of quantum systems has been a central topic in quantum thermodynamics, given both its fundamental importance stemming from the third law of thermodynamics (Nernst, 1907) and its significance for quantum technologies (Bloch, Dalibard, and Zwerger, 2008; Langen, Geiger, and Schmiedmayer, 2015).

We once again consider the situation described by Eq. (133) in the special case of thermal operations, where the unitary Uconserves global energy. The problem of cooling down a quantum system S prepared in some quantum state ρ_S can be cast as a state-transition problem, i.e.,

$$\rho_{\mathsf{S}} \stackrel{\text{arb}}{\underset{\mathcal{O}_{\mathrm{TO}(\beta)}}{\longleftrightarrow}} \sigma_{\mathsf{S}},\tag{157}$$

where if one sets $\sigma_{\rm S} = \gamma_{\beta'}(\hat{H}_{\rm S})$, then the goal of cooling is defined as achieving a large inverse temperature β' . Moreover, the assumption that $\sigma_{\rm S}$ must necessarily be thermal can be dropped; generic measures for cooling can also be used, such as simply maximizing the overlap of $\sigma_{\rm S}$ with the system ground state or minimizing its average energy.

To understand the ultimate limits of cooling under the framework of thermal operations, one can apply the generalized second laws from Eq. (149) to the aforementioned state transition and study the amount of nonequilibrium resources required to perform this task. This approach was taken by Wilming and Gallego (2017), who showed that, while general state transitions are governed by a continuous family of monotones indexed by $\alpha \in \mathbb{R}$, in the context of cooling down to $T' := (\beta')^{-1} \to 0$ only one monotone, called the vacancy, remains relevant,

$$\mathcal{V}_{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) \coloneqq D(\gamma_{\beta}(\hat{H}_{\mathsf{S}}) \| \rho_{\mathsf{S}}). \tag{158}$$

The vacancy is related to the (Petz-)Rényi divergences by

$$\frac{\partial}{\partial_{\alpha}}\Big|_{\alpha=0} D_{\alpha}(\rho_{\mathsf{S}} \| \gamma_{\beta}(\hat{H}_{\mathsf{S}})) = V_{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}).$$
(159)

Since $\gamma_{\beta}(\hat{H}_{S})$ is of full rank, the vacancy diverges for states ρ_{S} which do not have full rank (in particular, for states approaching zero temperature). Using a resourceful state ρ_{R} with Hamiltonian \hat{H}_{R} , one can cool an initially thermal system S to a generic target state σ_{S} only if

$$\mathcal{V}_{\beta}(\rho_{\mathsf{R}}, \hat{H}_{\mathsf{R}}) \ge \mathcal{V}_{\beta}(\sigma_{\mathsf{S}}, \hat{H}_{\mathsf{S}}). \tag{160}$$

Furthermore, Eq. (160) is achievable up to a correction factor that vanishes when σ_S is a thermal state of a temperature $T_S \rightarrow 0$, meaning that this inequality fully characterizes the fundamental limits of cooling. Similar results were previously established by Janzing *et al.* (2000) for the special case where the target system is composed of qubits, and assuming that the resource ρ_R has an i.i.d. structure. In this respect, Wilming and Gallego (2017) established the third law of thermodynamics in the fully single-shot regime by allowing strict catalysis. Wilming and Gallego (2017) also discussed the robustness of Eq. (160) under approximate catalysis with respect to errors in the catalyst measured via a vacancy change. In particular, closeness of the catalyst to its original state in terms of the vacancy change is a sufficiently strong measure to maintain the robustness of Eq. (160).

The advantages of correlating catalysis can also be seen in cooling. These advantages were discussed by Boes *et al.* (2019), especially for the special case where the system Hamiltonian is irrelevant. In such situations, cooling usually then refers to the process of preparing states of high purity. We saw in Sec. V.A that correlating catalysis within unitary quantum mechanics allows for a system S to go from ρ_S to σ_S as long as $H(\rho_S) > H(\sigma_S)$. Boes *et al.* (2019) observed that this result could be applied to cooling in an interesting way. To illustrate this, consider S to contain two uncorrelated systems (in general, states of full rank) $\rho_S = \rho_{S1} \otimes \rho_{S2}$ such that both $H(\rho_{S1}), H(\rho_{S2}) < \log d_S/4$ and $d_{S1} = d_{S2} = \sqrt{d_S}$. One can then use a unitary transformation involving a correlating catalyst to bring the system to any final state of the form

$$\sigma_{\mathsf{S}} = \sigma_{\mathsf{S}1} \otimes \frac{\mathbb{1}}{d_{\mathsf{S}2}},\tag{161}$$

where σ_{S1} has to be a state of full rank but can have an arbitrarily small amount of entropy. In particular, σ_{S1} can be arbitrarily close to a pure state. Similar cooling rates were obtained using protocols based on the idea of data compression (Schulman and Vazirani, 1999; Boykin *et al.*, 2002; Dahlsten *et al.*, 2011) for the special case of qubits where, instead of a catalyst, many identical copies of the system were used. For example, we can take $\rho_{S}^{\otimes n}$ qubits and roughly distill *Rn* many close-to-pure qubits in the asymptotic limit, where

$$R \approx 1 - H(\rho_{\rm S}) \tag{162}$$

and we use $\log(2) = 1$. In contrast, with the help of a correlating catalyst (albeit probably with high dimension), one can perform the distillation and reach the optimal rate using as few as two copies of the system. Boes *et al.* (2019) cautioned that the end result, while similar to the asymptotic limit in terms of rates, has a subtle difference: a repeated use of the catalyst creates correlations among all of the final cooled states. Therefore, if the catalytic cooling protocol is repeated *n* times with the same catalyst, the final state on the resulting *n* copies of S1 is not given by $\sigma_{S1}^{\otimes n}$.

Moving toward a more practical mindset, Henao and Uzdin (2021) proposed explicit protocols for catalytic cooling for systems of low dimension. This work relates the capacity for cooling to the passitivity of nonequilibrium states when joined with a bath or a catalyst. This setting has also been studied in the context of ergotropy (Sparaciari, Jennings, and Oppenheim, 2017). The protocols designed by Henao and Uzdin (2021) are particularly interesting because of their use of catalysts of small dimension (a qubit) and relatively simple operations (up to three-body interactions).

The starting point of the investigation of Henao and Uzdin (2021) was as follows: assuming that $\rho_{\rm S} = \sum_i p_{\rm S}(i) |i_{\rm S}\rangle \langle i_{\rm S}|$ is a passive state with respect to the Hamiltonian $\hat{H}_{\rm S}$ (and hence expressed diagonally in its energy eigenbasis $\{|i_{\rm S}\rangle\}_i$), under what conditions does the bipartite state $\rho_{\rm S} \otimes \rho_{\rm B}$ remain passive with respect to $\hat{H}_{\rm S}$? Here $\rho_{\rm B}$ represents a thermal state or bath that is used in the process of cooling. Henao and Uzdin (2021) showed that the passitivity of such a bipartite state is determined fully by

$$\frac{p_{\mathsf{S}}(i)}{p_{\mathsf{S}}(i+1)} \ge \frac{p_{\mathsf{B}}(1)}{p_{\mathsf{B}}(d_{\mathsf{B}})} \quad \text{for all } i, \tag{163}$$

where $p_{B}(i)$ are the descendingly ordered eigenvalues of the state ρ_{B} . From this analysis, one can observe that if ρ_{B} is set to be a Gibbs state with temperature *T*, then the higher *T* is, the smaller the rhs of Eq. (163); in particular, it converges to 1 in the infinite temperature limit. Conversely, the left-hand side is always greater than or equal to 1 due to the passitivity of ρ_{S} . In summary, Eq. (163) will eventually be satisfied for some high enough temperature *T*, making $\rho_{S} \otimes \rho_{B}$ passive with respect to \hat{H}_{S} and thus disallowing further cooling of S.

Equation (163) spells out the limitations of achievable cooling by means of unitary operations, having access only to a thermal reservoir. Henao and Uzdin (2021) then proceeded to show that an extension of Eq. (163) holds for an *ancilla*assisted cooling process: in other words, passivity including a general ancilla $\rho_{\rm C}$ is again fully determined by

$$\frac{p_{\mathsf{S}}(i)}{p_{\mathsf{S}}(i+1)} \ge \frac{p_{\mathsf{B}}(1)}{p_{\mathsf{B}}(d_{\mathsf{B}})} \frac{p_{\mathsf{C}}(1)}{p_{\mathsf{C}}(d_{\mathsf{C}})} \quad \text{for all } i, \qquad (164)$$

where $p_{\rm C}(1)$ and $p_{\rm C}(d_{\rm C})$ are the maximum and minimum eigenvalues of $\rho_{\rm C}$, respectively. If we have a cooling protocol that preserves the ancilla, we also have a catalytic cooling process. Hence, Henao and Uzdin (2021) focused on a particular subset of cooling processes that can be decomposed in a two-step process: the first step corresponds to identifying a unitary $U_{\rm cool}$ that decreases the average energy of system C whenever Eq. (164) is satisfied. This process U_{cool} in general alters $\rho_{\rm C}$ and hence is subsequently followed up by a restoring unitary, i.e., a unitary $V_{\rm res}$ that brings the local reduced state of the catalyst back to $\rho_{\rm C}$. In particular, Henao and Uzdin (2021) showed that $U_{\rm cool}$ and $V_{\rm res}$ can be chosen such that they act on orthogonal subspaces. More importantly they showed that the question of whether an ancilla can truly be recovered is fully characterized: namely, by constructing an explicit V_{res} that consists of a series of partial two-level swaps and determining whether each of these swaps satisfies a simple condition that can be checked algorithmically.

D. Asymmetry and coherence

In Sec. II.A we introduced the resource theory of asymmetry \mathcal{R}_{asym} , which models the physical constraints arising from conservation laws and the corresponding symmetries present in the system. To recap, the resource theory of asymmetry deals with scenarios where every physical system **S** carries a unitary representation $W_{\rm S}$ of some group G [for example, G = SO(3) corresponds to rotations in threedimensional Euclidean space \mathbb{R}^3], and the allowed dynamics has to be covariant with respect to this group. If G is a connected Lie group, then the representations of its generators can be physically interpreted as conserved quantities, as any covariant unitary channel has to leave them invariant. This can be seen as an expression of Noether's theorem. Indeed, if $W_{\rm S}[\exp(h)] = \exp(-i\hat{H}_{\rm S})$ for a generator h in the Lie algebra of G and $\mathcal{F}[(\cdot)] = U(\cdot)U^{\dagger}$ is a covariant unitary quantum channel on S, then

$$\mathcal{F}^*[\hat{H}_{\mathsf{S}}] = U^{\dagger} \hat{H}_{\mathsf{S}} U = \hat{H}_{\mathsf{S}}.$$
 (165)

For example, if $G = \mathbb{R}$ is the group of time translations, then \hat{H}_{S} can be identified with the Hamiltonian of the system and Eq. (165) expresses energy conservation. A quantum system **S** is then said to be resourceful if its state ρ_{S} is asymmetric with respect to the group representation, meaning that there exists at least one $g \in G$ such that $W_{S}(g)\rho_{S}W_{S}(g)^{\dagger} \neq \rho_{S}$.

Closely related to the resource theory of asymmetry is the resource theory of coherence \mathcal{R}_{coh} , which was introduced by Baumgratz, Cramer, and Plenio (2014) and Levi and Mintert (2014) and reviewed by Streltsov, Adesso, and Plenio (2017). From the point of view of the resource theory of coherence, the degree of coherence of a quantum state measures the magnitude of the off-diagonal terms in a fixed basis $\mathbb{H} = \{|i\rangle\}$. For example, a qubit in the state $(|0\rangle + |1\rangle)/\sqrt{2}$ is highly coherent with respect to the basis $\mathbb{A} = \{|0\rangle, |1\rangle\}$ but completely incoherent with respect to the basis $\mathbb{B} = \{|+\rangle, |-\rangle\}$. From the point of view of the resource theory of asymmetry, we can consider a representation of the group $G = \mathbb{R}$ of time translations induced by a nondegenerate Hamiltonian \hat{H}_{S} with eigenbasis \mathbb{H} via $t \mapsto \exp(-i\hat{H}_{S}t)$. A quantum state is then said to be coherent if it is asymmetric with respect to the group representation. Consequently, in this case the set of free states in \mathcal{R}_{coh} coincides with the set of free states in \mathcal{R}_{asvm} .

Despite the aforementioned similiarity of \mathcal{R}_{coh} and \mathcal{R}_{asym} for the case $G = \mathbb{R}$, a crucial distinction arises in the allowed free operations. While in \mathcal{R}_{asym} only covariant operations are allowed, in \mathcal{R}_{coh} the so-called incoherent operations (Baumgratz, Cramer, and Plenio, 2014; Levi and Mintert, 2014; Streltsov, Adesso, and Plenio, 2017) are allowed, which is a strictly larger set than the corresponding covariant operations (Marvian, Spekkens, and Zanardi, 2016). Specifically, in the resource theory of asymmetry the set of free operations depends on the concrete representation of time translations (induced by the aforementioned Hamiltonian \hat{H}_S), and not simply on the basis H. This distinction has significant consequences for catalysis, as we review in Sec. V.D.2. For critical comparisons of different approaches to establishing a physically consistent resource theory of coherence, see Chitambar and Gour (2016), Marvian and Spekkens (2016), and Marvian, Spekkens, and Zanardi (2016).

Quantitative measures of coherence and asymmetry can be defined in multiple ways. The most common one is perhaps the relative entropy of asymmetry defined in Sec. II.A.2. In the special case of coherence with respect to a basis $\mathbb{H} = \{|i\rangle\}$ it reduces to the relative entropy of coherence given by (Aberg, 2006)

$$\mathcal{A}_{\mathbb{H}}(\rho) = H(\mathcal{G}_{\mathbb{H}}[\rho]) - H(\rho), \tag{166}$$

where the twirling channel is given by $\mathcal{G}_{\mathbb{H}} = \sum_{i} |i\rangle \langle i|\rho |i\rangle \langle i|$. The relative entropy of coherence is monotonic under incoherent operations, satisfies several desireable properties of a proper coherence measure (Baumgratz, Cramer, and Plenio, 2014), and has a plausible operational interpretation as the maximal distillable coherence from the state (Winter and Yang, 2016).

1. Apparent violations of conservation laws

Coherence (or asymmetry) present in a quantum state can be viewed as a resource that enables implementing coherent operations on quantum systems. Specifically, in order to bring an atom S from the ground state $|g\rangle_{S}$ to the excited state $|e\rangle_{S}$ using time-translation covariant (energy-conserving) operations, we need an auxiliary system E from which the missing energy can be taken. However, if we wanted to turn S into a coherent superposition of the ground and excited states, i.e., $(|g\rangle_{S} + |e\rangle_{S})/\sqrt{2}$, then providing the necessary energy is not enough to perform this transformation. Under the constraint of energy conservation, the unitarity of quantum mechanics forbids this type of operation if the atom and the energy reservoir E have initially definite energies. Specifically, there does not exist a unitary operator U for which

$$U|g\rangle_{\mathsf{S}} \otimes |E\rangle_{\mathsf{E}} = \frac{1}{\sqrt{2}}(|e\rangle_{\mathsf{S}} + |g\rangle_{\mathsf{S}}) \otimes |\psi\rangle_{\mathsf{E}} \quad (167)$$

and, at the same time, $[U, \hat{H}_{S} + \hat{H}_{E}] = 0$, where $|E\rangle_{E}$ is an eigenstate of \hat{H}_{E} and $|\psi\rangle_{\mathsf{E}}$ is arbitrary. This problem can be circumvented by introducing a reservoir of coherence, i.e., a large quantum system prepared in coherent superposition of many energy levels. In the case of equally spaced energy levels, such a reservoir of coherence constitutes a phase reference and is usually achieved with the help of electromagnetic fields such as laser beams (Mandel and Wolf, 1995) and radio-frequency fields (Vandersypen and Chuang, 2005). The coherence between orthogonal states of the field is a resource that enables different energies to be mixed in a coherent, rather than probabilistic, way. Similarly, instead of energy preservation, we can consider the more general notion of covariance with respect to a group G as the constraint (such as spatial rotations). In this case the coherence reservoir needs to be replaced by a more general quantum reference frame for the respective group; see the review by Bartlett, Rudolph, and Spekkens (2007) and references therein.

It has been observed that coherence reservoirs are useful in the context of thermodynamics (Janzing and Beth, 2003a; Janzing, 2006; Vaccaro *et al.*, 2008; Lostaglio, Jennings, and Rudolph, 2015; Lostaglio *et al.*, 2015; Malabarba, Short, and Kammerlander, 2015; Mitchison *et al.*, 2015; Korzekwa *et al.*, 2016; Woods and Horodecki, 2023), where a coherence reservoir is often understood as a clock that provides timing information. For example, the nonequilibrium free energy of a quantum system in a state ρ_S and Hamiltonian \hat{H}_S [see Eq. (27)] naturally splits into an incoherent and a coherent part,

$$F^{\beta}(\rho_{\mathsf{S}}, \hat{H}_{\mathsf{S}}) = F_{\beta}(\mathcal{G}_{\mathbb{H}}[\rho_{\mathsf{S}}], \hat{H}_{\mathsf{S}}) + \frac{1}{\beta}\mathcal{A}_{\mathbb{H}}(\rho_{\mathsf{S}}), \quad (168)$$

with $\mathcal{A}_{\mathbb{H}}(\rho_{S})$ as in Eq. (166) for the eigenbasis \mathbb{H} of \hat{H}_{S} . Without the coherence reservoir only the incoherent part, i.e., $F^{\beta}(\mathcal{G}_{\mathbb{H}}[\rho_{S}], \hat{H}_{S})$, can be extracted as work (Janzing, 2006). A coherence reservoir can thus be seen as a thermodynamic resource. Similarly, general quantum reference frames are thermodynamic resources for other conserved quantities such as angular momentum or spin.

A "good" quantum reference frame C can be used in such a way that its state changes only minimally while at the same time lifting all conservation laws. Indeed, it was proven first by Tajima, Shiraishi, and Saito (2018, 2020) [and later independently by Chiribella, Yang, and Renner (2021) and Yang, Renner, and Chiribella (2022)] for the case of coherence that whenever a quantum reference frame C in a pure state can be used to approximately implement a unitary dynamics on S to high precision using covariant dynamics on SC, it can be done in such a way that the state on C changes only slightly. van Luijk, Werner, and Wilming (2023) treated the same problem for the resource theory of asymmetry with respect to arbitrary groups. See also Tajima, Shiraishi, and Saito (2018, 2020), Takagi and Tajima (2020), Tajima and Saito (2021), and Tajima, Takagi, and Kuramochi (2022) for trade-off relations between the precision of the implemented unitary and the required resources on the quantum reference frame.

The underlying reason for these results is that the approximately coherent dynamic on S requires that almost no information about the state on S flows to the environment, since otherwise S would necessarily become entangled to the environment. This is the information-disturbance trade-off in quantum mechanics (Fuchs and Peres, 1996; Fuchs, 1998; Kretschmann, Schlingemann, and Werner, 2008). To use the terminology that we introduced in Sec. III.D, the coherence reservoir essentially acts as an approximate catalyst. Moreover, if it can be used to implement arbitrary state transformations with high precision, it is even an embezzler: By Stinespring's theorem, any quantum channel on S can be formally implemented to high precision using unitaries acting on an extended Hilbert space of systems ES, where E is an auxiliary environment with dimension d_s^2 . Thus, if C can be used to implement arbitrary unitaries on SE to high precision, it is an embezzler. We now present a simple example illustrating these considerations.

Suppose that we want to implement some quantum channel \mathcal{E} on a two-level system S with Hamiltonian $\hat{H}_{S} = \omega |1\rangle \langle 1|_{S}$, where $|0\rangle_{S}$ and $|1\rangle_{S}$ denote its ground and excited states, respectively. Moreover, suppose that we are only able to apply energy-conserving unitaries but have access to a harmonic oscillator C with matching frequency ω (in resonance with S)

and energy eigenstates $|n\rangle_{C}$. We now provide a construction that manages to apply \mathcal{E} to arbitrary accuracy on S if the initial state on the oscillator is sufficiently coherent. This discussion follows Åberg (2014); see also Messinger *et al.* (2020). Consider a family of subspaces \mathcal{H}_n spanned by the "logical" states $|\bar{0}\rangle_n$, $|\bar{1}\rangle_n$ defined for all $n \ge 1$ by

$$|\bar{0}\rangle_n \coloneqq |0\rangle_{\mathsf{S}}|n\rangle_{\mathsf{C}}, \qquad |\bar{1}\rangle_n \coloneqq |1\rangle_{\mathsf{S}}|n-1\rangle_{\mathsf{C}}. \tag{169}$$

Any unitary acting separately on subspaces \mathcal{H}_n is energy preserving (note that $|0\rangle_{S}|0\rangle_{C}$ must be an eigenstate of the unitary). However, within each subspace \mathcal{H}_n the unitary is unconstrained. We now further introduce an auxiliary system E with basis $\{|\alpha\rangle_{E}\}$ that serves as the dilating system in the Stinespring dilation of \mathcal{E} with unitary V on ES. For simplicity we assume that it has a trivial Hamiltonian (or we have access to only one energy subspace of a larger system). To implement V, and thereby \mathcal{E} , we define an energy-preserving unitary U on ESC by its matrix elements via

$$\langle \alpha |_{\mathsf{E}} \langle \bar{k} |_{n} U | \beta \rangle_{\mathsf{E}} | \bar{l} \rangle_{n} \coloneqq \langle \alpha |_{\mathsf{E}} \langle k |_{\mathsf{S}} V | \beta \rangle_{\mathsf{E}} | 1 \rangle l_{\mathsf{S}}$$
(170)

for all *n* and $\langle \alpha |_{\mathsf{E}} \langle 0 |_{\mathsf{S}} \langle 0 |_{\mathsf{C}} U | \beta \rangle_{\mathsf{E}} | 0 \rangle_{\mathsf{S}} | 0 \rangle_{\mathsf{C}} = \delta_{\alpha\beta}$. In other words, *U* acts as *V* on subspaces $\mathcal{H}_{\mathsf{E}} \otimes \mathcal{H}_n$ and trivially on $\mathcal{H}_{\mathsf{E}} \otimes \operatorname{span}\{|0\rangle_{\mathsf{S}}|0\rangle_{\mathsf{C}}\}$.

We now write a general pure product state on ESC as

$$\begin{aligned} |\chi\rangle_{\mathsf{E}}|\psi\rangle_{\mathsf{S}}|\phi\rangle_{\mathsf{C}} &= |\chi\rangle_{\mathsf{E}}(\alpha|0\rangle_{\mathsf{S}} + \beta|1\rangle_{\mathsf{S}}) \left(\sum_{n=0}^{\infty} c_{n}|n\rangle_{\mathsf{C}}\right) \\ &= \sum_{n=1}^{\infty} c_{n} \left(\alpha|\chi\rangle_{\mathsf{E}}|\bar{0}\rangle_{n} + \beta \frac{c_{n-1}}{c_{n}}|\chi\rangle_{\mathsf{E}}|\bar{1}\rangle_{n}\right) \\ &+ c_{0}\alpha|\chi\rangle_{\mathsf{E}}|0\rangle_{\mathsf{S}}|0\rangle_{\mathsf{C}}. \end{aligned}$$
(171)

If $c_{n-1} \approx c_n$ for many values of *n* (which requires each c_n to be small) we then get

$$U|\chi\rangle_{\mathsf{E}}|\psi\rangle_{\mathsf{S}}|\phi\rangle_{\mathsf{C}} \approx (V|\chi\rangle_{\mathsf{E}}|\psi\rangle_{\mathsf{S}})|\phi\rangle_{\mathsf{C}}.$$
 (172)

In other words, if the state on C is spread smoothly over many energy levels, then we can implement the desired unitary on SE to an arbitrary accuracy. Concretely we can choose a coherent superposition of M energy levels occupied starting from the energy level $|n_0\rangle_{\rm C}$, that is, $c_n = 1/\sqrt{M}$, where $n_0 + M \ge n \ge n_0 > 0$, and $c_n = 0$ otherwise. The fidelity between the two sides of Eq. (172) is then given by

$$|\langle \chi|_{\mathsf{E}} \langle \psi|_{\mathsf{S}} \langle \phi|_{\mathsf{C}} U^{\dagger}(V|\chi\rangle_{\mathsf{E}}|\psi\rangle_{\mathsf{S}}) |\phi\rangle_{\mathsf{C}}|^{2} \ge 1 - \frac{2}{M}.$$
(173)

Similarly, we can choose the initial state on C to be a coherent state

$$|\alpha\rangle_{\mathsf{C}} \coloneqq e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_{\mathsf{C}}$$
(174)

and obtain an error that decreases with the mean photon number $|\alpha|^2$, which is also the variance of the photon number.

We emphasize that this effect is made possible because the state of a large quantum system may change an arbitrarily small amount, as measured by the norm on vectors in the Hilbert space, despite the fact that the mean of some observable changes by a finite amount (here the energy). In this way, a finite amount of energy can be transferred coherently to S while perturbing the state on C an arbitrarily small amount. In this sense, conservation laws can apparently be violated. This is closely related to the continuity properties of extensive quantities described in Sec. VI.B.

The seminal paper by Åberg (2014) further showed that if the flat state of Eq. (173) is used, then despite the fact that the state on C is changed, it can be reused a finite number of times n_0 to implement exactly the same quantum channel on n_0 uncorrelated copies of **S** (while also making use of n_0 copies of E; see also Secs. II and III.B). In this special sense the coherence stored in C is catalytic [or *repeatable*, as named by Korzekwa et al. (2016)], while its quantum state is not. Åberg (2014) referred to this phenomenon as catalytic coherence. However, as emphasized by Vaccaro, Croke, and Barnett (2018) and Messinger et al. (2020), the resulting state on the n_0 copies of **S** is not uncorrelated. Thus, this procedure does not implement the quantum channel $\mathcal{E}^{\otimes n_0}$. The reason for this is that the system C mediates correlations that are being built up between all the copies of ES, so their final joint state is not given by $(V\rho_{\mathsf{ES}}V^{\dagger})^{\otimes n_0}$; see also Sec. V.C.6.

2. Correlations and broadcasting of quantum information

The no-cloning theorem (Park, 1970; Dieks, 1982; Wootters and Zurek, 1982) states that it is impossible to prepare exact, uncorrelated copies of an unknown quantum state: There is no quantum channel \mathcal{E} such that $\mathcal{E}[\rho_S] = \rho_S \otimes \rho_S$ for all states ρ_S on a fixed system S. More generally the nobroadcasting theorem (Barnum *et al.*, 1996) implies that there is no quantum channel \mathcal{E} from S to two copies of S such that $\operatorname{tr}_{S_i}[\mathcal{E}[\rho_S]] = \rho_S$ for i = 1, 2. That is, it is impossible to convert an unknown quantum state ρ_S to one where both marginals coincide with ρ_S (broadcasting ρ_S), but the two marginal states are possibly correlated. (Indeed, if this were possible, one could clone pure quantum states.) In fact, this feat is not possible even when \mathcal{E} is required to implement broadcasting for a pair of noncommuting quantum states (instead of all quantum states).

One can ask whether, instead of broadcasting the full quantum state, one could broadcast only some aspect of it, such as some amount of its coherence or, more generally, its asymmetry with respect to a group G. For that, in the language of the resource theory of asymmetry, we ask whether it is possible to implement the state transition

$$\rho_{\mathsf{C}} \underset{\mathcal{O}_{\mathrm{asym}}}{\longrightarrow} \sigma_{\mathsf{C}_1 \mathsf{C}_2} \tag{175}$$

such that $\sigma_{C_1} = \rho_C$ and $\sigma_{C_2} \notin S_{asym}$, i.e., the final state on subsystem C_2 is not symmetric with respect to *G*. One can observe that if this were possible, then there would have to be a covariant quantum channel \mathcal{F} on C_1C_2 , and a symmetric state ρ_{C_2} such that

$$\mathcal{F}[\rho_{\mathsf{C}} \otimes \rho_{\mathsf{C}_2}] = \sigma_{\mathsf{C}_1\mathsf{C}_2}.\tag{176}$$

Indeed, if \mathcal{F}_1 were the covariant quantum channel implementing Eq. (175), then the covariant quantum channel $\mathcal{F} = \mathcal{F}_1 \circ tr_{C_2}$ would work for any state ρ_{C_2} . But, in this case we can see that $\mathbf{C} = \mathbf{C}_1$ acts as a catalyst that becomes correlated to $\mathbf{S} = \mathbf{C}_2$. Therefore, we can equivalently ask whether there is a correlating-catalytic transformation of the form

$$\rho_{\mathsf{S}} \stackrel{\mathrm{corr}}{\underset{\mathcal{O}_{\mathrm{asym}}}{\longleftrightarrow}} \sigma_{\mathsf{S}},\tag{177}$$

where $\rho_{\rm S} \in S_{\rm asym}$ and $\sigma_{\rm S} \notin S_{\rm asym}$. We refer to the aforementioned task as broadcasting of asymmetry: the possibility to use an asymmetric quantum state (a quantum reference frame) in a catalytic manner to transform a symmetric state into an asymmetric state. Similarly, we call the special case of strict catalysis cloning of asymmetry,

$$\rho_{\mathsf{S}} \underset{\mathcal{O}_{\mathrm{asym}}}{\longleftrightarrow} \sigma_{\mathsf{S}}.$$
 (178)

It has been established that broadcasting and cloning of asymmetry are both impossible in the case where G is a connected Lie group (Lostaglio and Müller, 2019; Marvian and Spekkens, 2019); see also Janzing and Beth (2003b). In particular, broadcasting of coherence is impossible. With this fact, it is tempting to think that the power of correlating catalysis is the same as that of strict catalysis in the resource theory of asymmetry of a connected Lie group. However, this is not true: there are states $\rho_{S}, \sigma_{S} \notin S_{asym}$ such that Eq. (177) is possible while Eq. (178) is not (Ding, Hu, and Fan, 2021). That is, when acting on asymmetric states, it is generally useful to build up correlations between the system and the catalyst. Moreover, when one allows for marginal-correlating catalysis (see Sec. III.B), catalysts can essentially lift all restrictions on state transitions in the case of coherence (Takagi and Shiraishi, 2022). Kondra, Ganardi, and Streltsov (2023) and Shiraishi and Takagi (2023)) showed that essentially the same statement is true for correlating catalysis while one still allows for arbitrary small correlations between system and catalyst. Thus, as long as an arbitrary small amount of coherence is present in the initial state, correlating catalysis can lift all restrictions imposed by demanding covariance under time translation. In other words, an arbitrary small amount of initial coherence can completely circumvent the no-broadcasting theorem for quantum coherence. It is an open problem to generalize the result to noncommutative Lie groups.

In fact, establishing correlations between the catalyst and some other degrees of freedom is not only useful but necessary. van Luijk, Werner, and Wilming (2023) showed that if a system C acts as a useful catalyst for a state transition on S in the resource theory of asymmetry for a connected Lie group (such as in the case of coherence), then it must necessarily become correlated either to S or to the environmental degrees of freedom E that dilate the covariant quantum channel on SC to covariant unitary dynamics on ESC, just as in Lemma II.1. (In fact, the same conclusion can be derived for the case of thermal operations; see Sec. II.A.3.) In particular, this implies that a useful catalyst can never be in a pure state, a result previously shown by Ding, Hu, and Fan (2021) for the special case of coherence, and by Marvian and Spekkens (2013) for the case where the states of interest on S are pure but the group in question is an arbitrary compact connected Lie group.

This situation is distinctly different than the resource theory of coherence, where the full class of incoherent operations is allowed, which renders pure catalysts useful and makes it possible to characterize catalytic state transitions on pure states using Rényi entropies, as in the case of LOCC (Bu, Singh, and Wu, 2016), because the statetransition conditions are characterized by majorization (Du, Bai, and Guo, 2015). In the context of incoherent operations Liu and Zhou (2020) and (Xing and Yang (2020) studied the strictly catalytic transformations between mixed states. Chen et al. (2019) studied the one-shot distillation of coherence using catalysts in this framework, related the distillable coherence to the dimension of the catalyst, and showed that coherence can be embezzled. Somewhat surprisingly Lami, Regula, and Streltsov (2023) showed that neither the asymptotic coherence cost nor the distillable coherence under incoherent operations change when strict, correlating, or marginal-correlating catalysts are allowed.

Rubboli and Tomamichel (2022) provided tools to lower bound the dimension of correlating catalysts in certain limiting cases in this setting. The distinction in terms of catalysis with pure states between the resource theory of asymmetry and the resource theory of coherence hence mirrors the one between LOSR and LOCC in the context of entanglement; see Sec. V.B.5.

We emphasize that the restriction to connected groups (relevant for the constraints imposed by conservation laws) is necessary for the aforementioned results to hold. As shown by Marvian and Spekkens (2013), if G is a finite group and ρ_S and σ_S are arbitrary states on the system S with a unitary representation $W_S(g)$ of G, then there exists a system C with a unitary representation $W_C(g)$ of G and a pure state $|\phi\rangle\langle\phi|_C$ together with a covariant quantum channel \mathcal{F} such that

$$\mathcal{F}[\rho_{\mathsf{S}} \otimes |\phi\rangle \langle \phi|_{\mathsf{C}}] = \sigma_{\mathsf{S}} \otimes |\phi\rangle \langle \phi|_{\mathsf{C}}. \tag{179}$$

In other words, strict catalysis with pure catalysts can lift all constraints on possible state transitions. This is possible because the state $|\phi\rangle_{\rm C}$ can be chosen to be a perfect quantum reference frame for the group *G*, i.e., a state satisfying $\langle \phi | W_{\rm C}(g) | \phi \rangle = 0$ for all $g \neq 1$. The channel \mathcal{F} can then be defined as

$$\mathcal{F}[X] = \sum_{g \in G} \operatorname{tr}[(\mathbb{1} \otimes W_{\mathsf{C}}(g) | \phi \rangle \langle \phi |_{\mathsf{C}} W_{\mathsf{C}}(g)^{\dagger}) X] \times \cdots \\ \times W_{\mathsf{S}}(g) \rho_{\mathsf{S}}' W_{\mathsf{S}}(g)^{\dagger} \otimes W_{\mathsf{C}}(g) | \phi \rangle \langle \phi |_{\mathsf{C}} W_{\mathsf{C}}(g)^{\dagger}.$$
(180)

A similar construction can be used to implement any unitary transformation V on S (or ES) perfectly and not just approximately as in Sec. V.D.1.

3. Locality of interactions and conservation laws

Locality of interactions is a fundamental property of physical systems. In the case of short-range interactions, it implies a finite speed of propagation of information, as highlighted by the Lieb-Robinson bound (Lieb and Robinson, 1972). In this sense locality puts restrictions on the short-time dynamics and implies certain unitary evolutions. Nevertheless, any unitary time evolution of a composite physical system can be implemented with a time-dependent Hamiltonian using only local interactions as long as it is allowed to evolve for a sufficiently long time. This is the essence of a fundamental result in quantum computing that states that unitary transformation on a composite system can be represented (to arbitrary accuracy) by a quantum circuit consisting only of two local unitary transformations, i.e., unitary transformations acting nontrivially on at most two subsystems (Deutsch, Barenco, and Ekert, 1995; DiVincenzo, 1995; Lloyd, 1995).

As discussed throughout this section, the presence of symmetries in physical systems also puts constraints on their time evolution, as any realizable unitary has to respect the associated symmetry. In view of this fact, a natural question to ask is whether all symmetric unitaries on a composite system can be generated using local symmetric unitaries, in analogy with the aforementioned universality result that holds for general evolutions. This question was posed and subsequently answered in the negative by Marvian (2022): The universality of interactions is no longer valid in the presence of continuous symmetries such as U(1) or time-translation covariance (energy conservation). Generic symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries. In other words, a global unitary that obeys a certain symmetry in general cannot be decomposed as a combination of two local unitaries, where each local unitary obeys the corresponding symmetry constraint. This implies that in the presence of locality symmetries of the Hamiltonian impose extra constraints on the time evolution of the system that are not captured by Noether's theorem.

In certain cases the aforementioned no-go theorem can be circumvented using auxiliary systems. Such systems have to be prepared in a fixed state and return to their initial states at the end of the process. In the terminology of this review this is an instance of strict catalysis. Specifically, Marvian (2022) discussed the case of energy conservation and demonstrated that the constraints resulting from the interplay of locality and energy conservation can be circumvented provided that the composite system is allowed to interact with a qubit catalyst. In this case the use of a catalytic ancilla allows the locality constraint to be lifted even in the presence of symmetries. In this sense, the catalytic system manifestly opens up new dynamical pathways that could not previously be reached. A similar result was shown by Marvian, Liu, and Hulse (2022) for the case of rotationally symmetric dynamics and a catalyst of two qubits; see also Marvian, Liu, and Hulse (2021).

E. Continuous-variable systems and quantum optics

In this section, we shift our attention to continuous-variable systems (Lloyd and Braunstein, 1999; Braunstein and van

Loock, 2005). Specifically, we consider an *n*-mode bosonic quantum system S_n with continuous degrees of freedom. A finite set of *n* continuous degrees of freedom can be represented by *n* pairs of Hermitian operators \hat{x}_i and \hat{p}_i fufilling the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}\mathbb{1}$. We also define the bosonic annihilation and creation operators a_i and a_i^{\dagger} via $a_i \coloneqq (\hat{x}_i + i\hat{p}_i)/\sqrt{2}$. Following the usual nomenclature of quantum optics, we refer to the labels *i* here as modes since they typically correspond to modes of the electromagnetic field.

Of special importance in quantum optics are so-called quadratic Hamiltonians, i.e., Hamiltonians that can be expressed as polynomials of the order of 2 in the canonical operators. A standard example of such a Hamiltonian is the *n*-mode quantum harmonic oscillator, i.e., $\hat{H}_{HO} =$ $(1/2)\sum_{i=1}^{n} (\hat{p}_{i}^{2} + \omega_{i}^{2}\hat{x}_{i}^{2}) = \sum_{i=1}^{n} \omega_{i}(a_{i}^{\dagger}a_{i} + 1/2).$ Quadratic Hamiltonians are common and provide a consistent approximation of quantum dynamics in many experimentally relevant situations, for example, ion traps (Paul, 1990; Bruzewicz et al., 2019), optomechanical systems (Hänsch and Schawlow, 1975; Stenholm, 1986), nanomechanical oscillators (Aspelmeyer, Kippenberg, and Marquardt, 2014), and many other systems (Bogoliubov, 1947; Itzykson and Zuber, 2012). Unitaries that can be implemented using quadratic Hamiltonians are known as Gaussian unitaries.

A common tool used to represent quantum states of continuous-variable systems is the Wigner function. This is a quasiprobability distribution that assigns a value to each point in phase space, thus allowing one to visualize quantum states and observables in a manner that is similar to classical probability distributions in the classical phase space. The Wigner function W(x, p) of a quantum state ρ of a single continuous variable is defined as

$$W(x,p) = \frac{2}{\pi} \int_{\mathbb{R}} e^{i2px'} \langle x - x' | \rho | x + x' \rangle \mathrm{d}x'.$$
(181)

Equation (181) is normalized as $\int_{\mathbb{R}^2} W(x, p) dx dy = 1$, and its integral over one canonical coordinate gives the probability of measuring the conjugate coordinate, for example, $\int_{\mathbb{R}} W(x, p) dp = \langle x | \rho | x \rangle$. The Wigner function can be defined in a similar manner for multimode systems.

An important family of states encountered in quantum optics are Gaussian states, whose Wigner function is a Gaussian. A typical example is the coherent states $|\alpha\rangle$, which are eigenstates of the annihilation operator a and whose wave function in phase space is a Gaussian centered at the complex number α . Any Gaussian state has a non-negative Wigner function. Therefore, a Wigner function that is negative somewhere necessarily corresponds to a non-Gaussian state. An important example is Fock states. For a single mode they are the energy eigenstates $|k\rangle$ of the quantum harmonic oscillator \hat{H}_{HO} and generally correspond to states with a fixed number of photons. In fact, all pure quantum states with a non-negative Wigner function correspond to Gaussian states (Hudson, 1974; Soto and Claverie, 1983). In other words, all non-Gaussian pure states exhibit Wigner negativity. This is not true for mixed states, as indicated by numerous examples (Walschaers, 2021). While Gaussian states on multiple modes can exhibit genuine quantum phenomena such as Bell nonlocality (García-Patrón *et al.*, 2004; Nha and Carmichael, 2004), they are in general easier to prepare than non-Gaussian quantum states (Braunstein and van Loock, 2005; Weedbrook *et al.*, 2012). Certain quantum phenomena such as entanglement and coherence can result in negative values of the Wigner function in some regions of phase space, indicating genuine nonclassical features of the quantum state (Kenfack and Życzkowski, 2004; Spekkens, 2008).

Of special importance in quantum optics are Gaussian operations, which map Gaussian states to Gaussian states. They are usually defined as quantum channels that can be implemented using linear quantum optics, which also makes them relatively easy to manipulate experimentally (Weedbrook *et al.*, 2012). A particular example is the previously mentioned Gaussian unitaries. Gagatsos, Oreshkov, and Cerf (2013) considered the utility of a beam splitter in creating bipartite pure entangled states and examined the majorization structure of the corresponding Schmidt coefficients generated. In this way, they revealed sets of incomparable states via LOCC that can be activated by catalysis.

The experimental feasibility of Gaussian operations comes naturally with the complementary weakness that Gaussian operations can simulate only a small subset of all quantum channels. Since any Gaussian operation acting on Gaussian states can be simulated efficiently on a classical computer (Mari and Eisert, 2012), Wigner negativity is also closely related to computational speedups; see Bartlett *et al.* (2002), Galvão (2005), and Veitch *et al.* (2012).

This mirrors the situation for Clifford operations on discrete systems; see Sec. V.F. To implement more complex quantum operations, one therefore has to use non-Gaussian states, thereby motivating us to investigate potential tools to prepare non-Gaussian states.

1. Multiphoton catalysis

When only Gaussian operations are available, such as in setups involving linear optical systems, non-Gaussian quantum states become valuable resources (Kok et al., 2007; Slussarenko and Pryde, 2019). For this reason, one might wonder whether such expensive states can be prepared only once, and then reused multiple times in a specific task. In other words, one could ask whether non-Gaussian states can serve as useful catalysts. This approach was investigated by Lvovsky and Mlynek (2002), who considered a phenomenon that they termed "quantum optical catalysis" and that was further generalized to "multiphoton catalysis" in later works (Scheel et al., 2003; Bartley et al., 2012; Xu, 2015; Hu, Liao, and Zubairy, 2017); see also Birrittella, Baz, and Gerry (2018), Zhou et al. (2018), and Zhang et al. (2021). The basic idea behind multiphoton catalysis is that one can build a linear optical process acting on two modes (S and C) that takes a coherent Gaussian state $|\alpha\rangle_{\rm S}$ in one mode and a Fock state $|k\rangle_{\rm C}$ of k photons in the second mode and outputs a nonclassical state of light contingent upon measuring k photons in the output mode C of the experiment. While interesting from an experimental point of view, such a process cannot be seen as catalytic as described in this review for two reasons: (i) The density operator describing the measured output port **C** is not given by $|k\rangle\langle k|_{C}$, so there is a nonzero probability that $l \neq k$ photons are measured instead, and (ii) even if k photons are measured on **C**, the measurement is typically destructive, so the photons cannot be reused for further processes. These two issues can be resolved when the k-photon state $|k\rangle_{C}$ is replaced by a two-level system in resonance with the coherent state of light $|\alpha\rangle_{S}$. In Sec. V.E.2 we describe such an alternative method of preparing nonclassical states of light based on generic light-matter interactions.

2. Activation of nonclassicality

We now describe another approach for preparing nonclassical states of light. The analysis presented here is based on the work of de Oliveira Junior *et al.* (2023). Consider a two-level system interacting with a single mode of light trapped in an optical cavity. Specifically, let **S** denote the electromagnetic field in the cavity with a bosonic annihilation operator *a*, and let **C** be a two-level system with energy levels $|g\rangle$ and $|e\rangle$ and the raising and lowering operators $\sigma_+ = |e\rangle\langle g| = \sigma_-^{\dagger}$. These two systems interact via the Jaynes-Cummings Hamiltonian (Jaynes and Cummings, 1963; Larson and Mavrogordatos, 2021), which reads

$$\hat{H}_{\rm SC} = \gamma_{\rm S} a^{\dagger} a + \gamma_{\rm C} |e\rangle \langle e| + \hat{H}_{\rm int}$$
(182)

$$\coloneqq \hat{H}_{\mathsf{S}} + \hat{H}_{\mathsf{C}} + \hat{H}_{\mathsf{int}},\tag{183}$$

where $\hat{H}_{int} \coloneqq g(\sigma_+ a + \sigma_- a^{\dagger})$. The quantity γ_S is the angular frequency of the mode and γ_C is the transition frequency of the two-level system. We assume that the two-level system is driven on resonance, meaning that $\gamma_S = \gamma_C = \gamma$. This ensures that the unitary evolution $U(t) \coloneqq e^{-i\hat{H}_{SC}t}$ generated by \hat{H}_{SC} conserves the total energy of both systems when it is considered noninteracting: $[U(t), \hat{H}_S + \hat{H}_C] = 0$.

We assume that the electromagnetic field S is initialized in a coherent state $\rho_{\rm S} = |\alpha\rangle\langle\alpha|_{\rm S}$, where $|\alpha\rangle$ is as defined in Eq. (174). Suppose that the atom C starts in a state $\omega_{\rm C}(\tau)$ that is the solution of the operator equation

$$\omega_{\mathsf{C}}(\tau) = \mathrm{Tr}_{\mathsf{S}}[U(\tau)\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}(\tau)U(\tau)^{\dagger}], \qquad (184)$$

where $\tau \in \mathbb{R}$. In other words, the atom is prepared in a state chosen such that at time $t = \tau$ it returns to its initial state.

The strength of the interaction between the cavity and the atom is specified by the coupling constant g of the interaction Hamiltonian \hat{H}_{int} . From now on, we focus on the state of the cavity and analyze the following two scenarios: (a) when there is no interaction between the cavity and the atom (g = 0) and (b) when the systems interact, i.e., $g \neq 0$. We are interested here in investigating the nonclassical properties of the state of the electromagnetic cavity at time $t = \tau$. For that reason we analyze the Wigner function W(x, p) of the mode S.

Consider the case in which g = 0. In this scenario $U(t) = U_{S}(t) \otimes U_{C}(t)$, and both systems evolve independently. The Wigner function of the cavity remains positive at all



FIG. 8. The Wigner function of the mode S at time $t = \tau$. Left panel: the evolved state of the cavity in the case of no coupling (g = 0). Right panel: the evolved state in the case of nonzero coupling (g > 0). Nonclassicality is clearly generated, while the process is (correlated) catalytic. The evolution was computed with the parameters $\alpha = (1 + i)/2$, $\gamma = 2\pi$, $\tau = 6$, and $g = 0.05\gamma$.

times because $U_{S}(t)$ is a Gaussian unitary that cannot produce Wigner negativity (Weedbrook *et al.*, 2012).

When $q \neq 0$, the field S and the atom C exchange energy over time. By construction, after a time $t = \tau$ the atom returns to its initial state $[\omega_{\rm C}(\tau) = \omega_{\rm C}(0)]$. However, the same is not true for the state of the electromagnetic field, which in general can evolve into a state with a negative Wigner function, as shown in Fig. 8. This is achieved by starting with a coherent state for which the Wigner function is everywhere positive, while the atom returns exactly to its initial state. This means that it cannot be responsible for delivering any resources (such as energy) to the cavity. If we could decouple the atom from the field and have it interact with another matching mode in the same initial coherent state for time τ , then the second mode would end up in the same state with the same negative Wigner function. Note that in this procedure the second mode generally ends up being correlated with the first one.

3. Continuous-variable quantum computation

An effect related to the one in Sec. IV.E.2 was previously discussed by Lau and Plenio (2016) and Lau and Plenio (2017) in the context of quantum computation. Suppose that one has access to a qubit C (for example, an atom) and is able to do the following:

- (a) Implement coherent rotations of the form $\exp(i\theta X_{\rm C})$.
- (b) Implement a controlled interaction of the qubit with each of a set of optical modes a_i in the form $U_i = \exp[i(\pi/2)(\mathbb{1}_{\mathsf{C}} Z_{\mathsf{C}})a_i^{\dagger}a_i]$.
- (c) Implement passive linear mode transformations, i.e., beam splitters and phase shifters.

Here X and Z refer to the corresponding Pauli matrices. Lau and Plenio (2016, 2017) showed that under these conditions it is possible to implement universal quantum computation using the qubit C in a strictly catalytic way. Specifically, up to initialization and final measurements the qubit remains in the state $|+\rangle_{C}$ throughout the computation. Note that this scheme is possible for a large variety of encodings of the logical states of computation into the quantum states of the optical modes without having to change the implemented operations. See Sec. V.F for a further discussion of catalysis in the context of computation.

4. Gaussian thermal operations

Gaussian systems, owing to their experimental feasibility, provide a platform for investigating thermodynamic properties of quantum systems. Applications of the Gaussian tool kit to thermodynamic problems involve, among many others, work extraction (Brown, Friis, and Huber, 2016; Singh *et al.*, 2019; Francica *et al.*, 2020), battery charging (Friis and Huber, 2018), optimization of thermodynamic protocols (Mehboudi and Miller, 2022), and quantification of the information gained from a continuously monitored system (Belenchia *et al.*, 2020).

In this section we discuss the thermodynamic analog of catalysis within the Gaussian setting. Sometimes it is reasonable to impose additional limitations on the allowed Gaussian operations, such as when one is interested in the thermodynamic cost of implementing specific operations. Along these lines, Serafini et al. (2020) and Narasimhachar et al. (2021) introduced a class of quantum channels that they termed Gaussian thermal operations (GTOs). These channels form a subset of thermal operations (see Sec. II.A.3) restricted to the case in which the Hamiltonians \hat{H}_{S} and \hat{H}_{B} are quadratic and the systems S and B interact via a Gaussian energy-preserving unitary. Serafini et al. (2020) and Narasimhachar et al. (2021) formulated GTOs in terms of maps acting on covariance matrices, which is a standard representation in the Gaussian regime, and determined the necessary and sufficient conditions for the existance of a GTO state transformation between Gaussian states. Subsequently Yadin et al. (2022) investigated two types of catalysis within the class of GTOs: strict and correlating catalysis, which Yadin et al. (2022) referred to as strong and weak catalysis, respectively. They found that strict catalysis provides no advantage over noncatalytic GTOs, mirroring the no-go results for LOSR entanglement (see Sec. V.B.5), contextuality (see Sec. V.B.11), and the resource theory of asymmetry (see Sec. V.D.2). Specifically, they showed that all possible state transitions under strictly catalytic GTOs on the mode S can be described as thermalizations toward the bath mode B. which can be easily achieved without the help of any catalyst mode C. However, when the system and the catalyst form a single-mode Gaussian state, Yadin et al. (2022) found that a correlating catalyst can significantly enlarge the set of achievable states. Specifically, they discussed the power of correlating catalysis in terms of their ability to concentrate thermodynamic resources in a subset of modes. Without a catalyst or with a strict catalyst, the Gaussian resources in each mode need to be monotonic under GTOs, leading to severe restrictions. However, a correlating catalyst allows thermodynamic resources to move from one mode to another, under the condition that the total thermodynamic resource across different modes is monotonic. Finally, in the case of multiple modes they found explicit necessary conditions on state transformations, which they subsequently expressed using the majorization relation.

F. Reversible computation and quantum computation

We now turn to the topic of computation and discuss catalysis in the context of classical reversible computation and fault-tolerant quantum computation. In a seminal work, Landauer (1961) showed that reversible (one-to-one) logical operations such as NOT can be performed without heat dissipation. However, irreversible (many-to-one) operations such as erasure always dissipate heat proportionally to the number of bits of information lost; see Sec. V.C.1. This leads to the question of whether thermodynamically irreversible processes are needed to perform arbitrary computations.

It turns out that any classical computation (i.e., one that can be realized using a Turing machine) can be performed in a logically reversible manner. This is possible either by saving the entire history of the process in ancillary systems (Bennett, 1973) or by embedding the irreversible mapping in a more complex (and reversible) mapping (Fredkin and Toffoli, 1982). Therefore, in principle any classical computation can be performed with arbitrarily little heat dissipation.

To implement a logically irreversible function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ taking *n* bits as input, one makes use of *m* auxiliary bits initialized to 0. One then constructs a reversible (invertible) function $R_f: \{0, 1\}^{n+m} \rightarrow \{0, 1\}^{n+m}$ such that, for any $x \in \{0, 1\}^n$,

$$R_f(x, 0, ..., 0) = f(x), g(x),$$
(185)

where f(x) denotes concatenation of bit strings and g(x) corresponds to some "garbage" bit string on n + m - 1 bits. It turns out that it is always possible to construct such a reversible function R_f if sufficiently many auxiliary bits are available. To see this, note that a classical bit can always be reversibly copied to an empty register via the CNOT gate. Therefore, by appending an additional bit to the string in Eq. (185), one can implement f reversibly by (i) copying the result of the computation f(x) into the appended bit and (*ii*) undoing R_f on the first n + m bits. This leads to the process

$$x, 0 \cdots 0, 0 \xrightarrow{R_f} f(x), \qquad g(x), 0 \xrightarrow{\text{CNOT}} f(x), g(x), f(x)$$
 (186)

$$\xrightarrow{R_f^{-1}} x, 0 \cdots 0, f(x).$$
(187)

This approach is known as *uncomputation* (Bennett *et al.*, 1997). Instead of a string of 0s, the auxiliary bits could be initialized in any other fixed state y (but the function R_f depends on y). From the resource-theoretic perspective, we can view reversible computations as a restricted class of free operations \mathcal{O}_{rev} and the m auxiliary bits in a fixed state y as a catalyst. In this context, the catalyst implements the reversible catalytic transformation

$$x, 0 \underset{\mathcal{O}_{\text{rev}}}{\longrightarrow} x, f(x).$$
(188)

If the input x is not deterministic, then the auxiliary bits remain perfectly catalytic and uncorrelated to the remaining bits. However, in this case the data bit (corresponding to the output of f) and the original input register become correlated. The amount of correlations between a random input and the output of a function f can be seen as a measure for how irreversible f is: For a reversible function the input can be reconstructed from the output; hence, the input and output are perfectly correlated. Conversely, for the constant function the output is independent of the input.

We now turn our attention to quantum computation in the circuit model (Nielsen and Chuang, 2012). Since this is a reversible model for computation, most of our previous remarks on reversible computation are applicable to quantum computation as well. However, owing to the no-cloning theorem quantum data cannot be simply copied, which leads to subtleties regarding uncomputation (Bennett *et al.*, 1997).

Even more interesting effects related to catalysis appear due to the need for quantum error correction: Since quantum computation is inherently fragile to errors, it requires faulttolerance techniques, in particular, quantum error correcting codes. The most common approach to quantum error correction uses so-called stabilizer codes (Gottesman, 1996, 1997; Calderbank *et al.*, 1997): Let \mathcal{P}_n be the group of operators that can be written as a "word" of single-qubit Pauli operators on a set of *n* qubits, possibly with an additional phase ± 1 or $\pm i$. We refer to any $P \in \mathcal{P}_n$ as a Pauli operator. Consider a set of m independent,8 mutually commuting, and Hermitian Pauli operators $S = \{S_j\}_{j=1}^m$ with $S_j \neq -1$. Their common eigenspace corresponding to eigenvalue +1 has the dimension 2^{n-m} and is called a stabilizer subspace. The vectors in a stabilizer subspace represent the logical states of a stabilizer code that encodes k = n - m qubits. Unitary operators that map Pauli operators to Pauli operators are called Clifford operators, and they are generated by the set of unitary operator $\{H, S, CNOT\}$. Here *H* is the Hadamard gate, *S* is the phase gate, and CNOT is the controlled-NOT gate. They typically correspond to unitary operations that can be implemented directly on the logical states represented by stabilizer states. However, Clifford unitaries are insufficient to implement universal quantum computation. In fact, any circuit consisting of Clifford unitaries acting on stabilizer states and followed by measurements of Pauli operators on some qubits can be efficiently simulated on a classical computer (Gottesman, 1999; Aaronson and Gottesman, 2004; Veitch et al., 2012). The aforementioned formulation closely resembles the situation for Gaussian states discussed in Sec. V.E.4; see also Gross (2006) and Mari and Eisert (2012). This situation naturally leads to a resource-theoretic approach where stabilizer states are the free states and Clifford unitaries combined with measurements of Pauli operators and classical feed-forward correspond to free operations. As expected, the operations so defined map stabilizer states to stabilizer states (Veitch et al., 2014). We refer to this set of free operations simply as "Clifford operations" or "stabilizer operations" \mathcal{O}_{stab} . Resource states corresponding to nonstabilizer states can be used to implement non-Clifford unitaries such as the singlequbit T gate, which is defined as

$$T = \begin{pmatrix} 1 & 0\\ 0 & e^{i\pi/4} \end{pmatrix} \tag{189}$$

in the $\{|0\rangle, |1\rangle\}$ basis. One way to implement a *T* gate is via the technique known as gate teleportation (also known as state injection) (Gottesman and Chuang, 1999). Specifically, by first preparing a suitable nonstabilizer state $|T\rangle$, one can then use Clifford operations to simulate a *T* gate. The pure states required to implement a *T* gate have been named *magic states* and can be distilled using Clifford operations from noisy magic states (Knill, 2004; Bravyi and Kitaev, 2005). Examples of single-qubit magic states are the eigenstates of the Clifford operator *H*, that is,

$$H|H_0\rangle = |H_0\rangle, \qquad H|H_1\rangle = -|H_1\rangle,$$

as well as the state $|T\rangle \coloneqq T|+\rangle$. A binary vector $\boldsymbol{v} =$ $(v_1, ..., v_n)$ further specifies the magic state $H_{\boldsymbol{v}} = \bigotimes_{i=1}^n$ $|H_{v_i}\rangle$ on *n* qubits. Several resource measures have been introduced to quantify the amount of magic contained in a quantum state. These measures can be used to bound the classical simulation cost of quantum circuits (Boykin et al., 2002; Veitch et al., 2012, 2014; Brandão et al., 2015; Pashayan, Wallman, and Bartlett, 2015; Bravyi and Gosset, 2016; Howard and Campbell, 2017; Ahmadi et al., 2018; Bravyi et al., 2019; Seddon and Campbell, 2019; Wang, Wilde, and Su, 2019, 2020; Raussendorf et al., 2020). In particular, the classical simulation cost of a quantum circuit acting on stabilizer states scales exponentially with the number of T gates (Aaronson and Gottesman, 2004). Hence, if we implement a quantum circuit using non-Clifford unitaries via gate teleportation, its classical simulation cost will scale with the number of required magic states.

It is therefore natural to ask how one can reduce the number of required resource states to implement certain non-Clifford unitaries. Since different non-Clifford gates require different resource states, it is useful to know whether catalysis can be used to convert certain resource states into others more efficiently. This would reduce the number of magic states that need to be distilled in order to implement a given quantum circuit. We summarize two notable results in this direction. An early result was from Campbell (2011), who first showed that catalysis indeed happens in the resource theory of magic states. Specifically, for a state $|\psi\rangle = (|H_{000}\rangle + |H_{111}\rangle)/\sqrt{2}$ they presented a Clifford circuit implementing the strictly catalytic transformation

$$|\psi\rangle|H_0\rangle \to |H_0\rangle|H_0\rangle.$$
 (190)

Note that $|\psi\rangle$ cannot by itself be converted into $|H_0\rangle$ via Clifford operations; see Campbell (2011) for details. Furthermore, they showed that strictly catalytic transformations are also possible when $|\psi\rangle$ is replaced by a mixed state. More recently Gidney (2018) focused on the *T* count of the adder circuit *A*, which is characterized by its action on the *n*-qubit computational basis,

$$A|i\rangle|j\rangle \to |i\rangle|i+j \bmod 2^n\rangle, \tag{191}$$

⁸Independence means that none of the operators S_j can be expressed as a product of the remaining operators $S \setminus \{S_j\}$.



FIG. 9. Example of magic state catalysis. The circuit identity shows that $|T\rangle$ catalyzes the transition $|CS\rangle \rightarrow |T\rangle$.

which is a critical subroutine in quantum Fourier transform. Using catalysis, Gidney (2018) introduced a circuit decomposition that reduced the number of *T* counts from 8n + O(1) in previous constructions to 4n + O(1).

Building on the previously mentioned investigations, Beverland *et al.* (2020) provided a more systematic study of catalysis for magic states and generalized previous constructions from Selinger (2013), Gidney (2018), and Gidney and Fowler (2019). In particular, they observed that catalysis can be used to provide lower bounds on the implementation cost of general unitaries. Furthermore, they showed that for a large class of state conversions some degree of catalysis was necessary. Beverland *et al.* (2020) then proceeded to characterize a broad class of magic state conversions that can be achieved via strict catalysis, and, in particular, demonstrated that catalysts can increase multicopy conversion rates to and from certain magic states.

We close this section with a further example of catalysis from Beverland *et al.* (2020). While the s gate acting as $|0\rangle \mapsto$ $|0\rangle, |1\rangle \mapsto i|1\rangle$ is a Clifford gate, the controlled s gate, denoted as CS, is not. It can be nevertheless implemented using the resource state $|CS\rangle = CS|+\rangle|+\rangle$. A no-go theorem used by Beverland *et al.* (2020) implied that $|CS\rangle$ cannot be converted to $|T\rangle = T|+\rangle$ using Clifford operations. Consider, however, the circuit equality shown in Fig. 9: The circuit on the right corresponds to $|CS\rangle|T\rangle$ and Clifford operations. Together they yield $T \otimes T$ corresponding to two $|T\rangle$ states when they are applied to $|+\rangle|+\rangle$. Therefore, $|T\rangle$ catalyzes the otherwise impossible transition $|CS\rangle \rightarrow |T\rangle$: A single $|T\rangle$ state together with *n* copies of $|CS\rangle$ can be converted to n + 1 copies of $|T\rangle$ using stabilizer operations.

VI. FURTHER TOPICS AND RELATED AREAS

In this section we discuss several topics that are related to catalyis but that either did not fit into the previous sections or do not precisely fit into the definition of catalysis that we use in the rest of the review.

A. Catalytic transformations of dichotomies

A central task in statistics is to infer a parameter $\theta \in \Theta$ from experimental samples. The experiment can described by a θ -dependent family of probability distributions $\{p_{\theta}\}$ that is typically assumed to be known. In the simplest case of binary experiments $\Theta = \{0, 1\}$, the task amounts to distinguishing between two hypotheses. The two hypotheses are represented by a pair (p_0, p_1) of probability distributions, which is often called a *dichotomy*. A dichotomy (p_0, p_1) is said to be more informative than a second dichotomy (q_0, q_1) if the latter arises from the former by stochastic processing (Hardy, Littlewood, and Pólya, 1952; Blackwell, 1953; Le Cam, 1996; Cohen, Kempermann, and Zbaganu, 1998). We also say that the first dichotomy relatively majorizes the second when there exists a stochastic map *T* such that $Tp_{\theta} = q_{\theta}$ (Hardy, Littlewood, and Pólya, 1952).

In quantum mechanics, the objects to be compared are *quantum dichotomies*, denoted by (ρ, σ) for two density operators ρ and σ . We say that the dichotomy (ρ_0, ρ_1) is more informative than (σ_0, σ_1) if there exists a quantum channel \mathcal{E} such that $\sigma_{\theta} = \mathcal{E}(\rho_{\theta})$, and we say that they are equivalent if there also exists a quantum channel \mathcal{R} such that $\rho_{\theta} = \mathcal{R}(\sigma_{\theta})$ (Petz, 1986b, 1988; Ohya and Petz, 1993; Shmaya, 2005; Jenčová and Petz, 2006; Chefles, 2009). When the two density operators forming a quantum dichotomy commute, they can be simultaneously diagonalized, and thus can be treated classically. When the two dichotomies do not commute, the inference task becomes genuinely quantum and is notoriously harder to characterize.

Some considered a generalization of the problem of comparing dichotomies using a dichotomy that acts as a catalyst. Specifically, Rethinasamy and Wilde (2020) introduced the notion of catalytic transformations between dichotomies of probability vectors (which they termed catalytic relative majorization). In this task, the catalyst consists of a pair (c_0, c_1) of probability vectors c_{θ} . The catalyst is then used in conjunction with the original input pair (p_0, p_1) to generate the output (q_0, q_1) by means of a stochastic matrix T acting as

$$T(\boldsymbol{p}_{\theta} \otimes \boldsymbol{c}_{\theta}) = \boldsymbol{q}_{\theta} \otimes \boldsymbol{c}_{\theta}. \tag{192}$$

In the language of this review, this corresponds to strict catalysis. However, in complete analogy to Eq. (192), one can consider here other types of catalysis such as correlating catalysis. While the physical interpretation of such a catalytic transformation is less clear, the problem is well defined and interesting from a mathematical perspective.

The main result of Brandão *et al.* (2015), which was discussed in Sec. V.C.3, can be formally translated into the language of dichotomies by interpreting the thermal state as one of the probability distributions in a dichotomy, thereby providing necessary and sufficient conditions for arbitrarily strict catalytic transformations of dichotomies in terms of Rényi divergences. Mu *et al.* (2021) independently obtained these results for generic dichotomies. Farooq *et al.* (2023) generalized the results from dichotomies to the case where Θ is a general finite set and allowing for an arbitrarily small error on the system (but not the catalyst). Rethinasamy and Wilde (2020) obtained the necessary and sufficient conditions for transforming dichotomies via correlating catalysis, showing that the Kullback-Leibler divergence is the essentially unique relevant monotone.

The problem of catalytically transforming generic quantum dichotomies is more difficult. Shiraishi and Sagawa (2021) showed that whenever $D(\rho_0 || \rho_1) \ge D(\sigma_0 || \sigma_1)$ there is a correlating-catalytic state transition from (ρ_0, ρ_1) to $(\sigma_0^{\epsilon}, \sigma_1)$ when allowing for an arbitrarily small error on the system (but not the catalyst).

One can also ask whether catalysis is relevant for the equivalence of dichotomies. For example, could it be true that (ρ_0, ρ_1) and (σ_0, σ_1) are not equivalent, while $(\rho_0 \otimes \omega_0, \rho_1 \otimes \omega_1)$ is equivalent to $(\sigma_0 \otimes \omega_0, \sigma_1 \otimes \omega_1)$? Galke, van Luijk, and Wilming (2023) showed that this form of nontrivial strict catalysis does not occur for classical dichotomies (or commuting quantum dichotomies) and conjectured that the same holds for the quantum case. In particular, this shows that there is no catalysis for interconvertibility of incoherent quantum states via thermal operations (cf. Sec. V.C.3), thus mirroring the situation for pure-state LOCC; see Sec. V.B.1.

B. Continuity of extensive quantities and embezzlement

We have emphasized throughout that embezzlement can be seen as arising from the fact that trace distance can change an arbitrarily small amount while a resource measure changes by a finite amount. In other words, it is a consequence of the lack of continuity of resource measures. More generally we now discuss how approximate catalysis can be used in a simple way to give bounds on how continuous extensive resource measures can be. Let $\mathcal{D}(\star)$ denote the set of density matrices of any finite dimension, i.e., $\mathcal{D}(\star) = \bigcup_d \mathcal{D}(\mathcal{H}_d)$. Given an arbitrary function f from density matrices to real numbers, we provide the following definition:

Definition VI.1 (Asymptotic continuity) (Donald, Horodecki, and Rudolph, 2002). Let $f: \mathcal{D}(\star) \to \mathbb{R}_+$. We say that f is asymptotically continuous if there is a Lipshitz constant K such that, for any d and $\rho, \sigma \in \mathcal{D}(\mathcal{H}_d)$,

$$|f(\rho) - f(\sigma)| \le K \|\rho - \sigma\|_1 \log d + \eta(\|\rho - \sigma\|_1), \quad (193)$$

where η does not depend on d and satisfies $\eta(0) = 0$.

A standard example of an asymptotically continuous quantity is the von Neumann entropy. In this case Eq. (193) reduces to the well-known Fannes-Audenaert inequality (Fannes, 1973; Audenaert, 2007). In the case of von Neumann entropy, the property of asymptotic continuity often leads to asymptotic continuity of many other entanglement measures. This is often a key step in deriving expressions or bounds for the asymptotic rates in quantum information processing tasks; see Horodecki (1998) and Terhal *et al.* (2002) for examples.

Following Coladangelo and Leung (2019), the aforementioned definition of continuity can be generalized to capture the maximal change in the function f with respect to the dimension of the density operator that takes the role of its argument. We now consider the following definition:

Definition VI.2 ("More than asymptotic" continuity). Let $f: \mathcal{D}(\star) \to \mathbb{R}_+$. We say that f is more than asymptotically continuous if there are $\alpha < 1$ and a Lipshitz constant K such that, for any d and $\rho, \sigma \in \mathcal{D}(\mathcal{H}_d)$,

$$|f(\rho) - f(\sigma)| \le K \|\rho - \sigma\|_1 (\log d)^{\alpha} + \eta(\|\rho - \sigma\|_1), \qquad (194)$$

where η does not depend on d and satisfies $\eta(0) = 0$.

We now characterize three properties of reasonable resource measures. We say that f is nonconstant if there are two density matrices ρ and σ such that $f(\rho) \neq f(\sigma)$. For tensor-product Hilbert spaces corresponding to composite quantum systems, a function f is said to be permutationally invariant if it does not depend on the particular labeling of the subsystems. More precisely, for any $n \in \mathbb{N}$ and any density operator ρ on $\mathsf{R} =$ $R_1 \otimes R_2 \cdots R_n$ and any permutation $\pi \in S_n$, it holds that $f(U_{\pi}\rho U'_{\pi}) = f(\rho)$, where U_{π} is the unitary that permutes registers R_i according to permutation π . Finally, a function f is additive over tensor products if $f(\rho \otimes \sigma) = f(\rho) + f(\sigma)$ holds for any density matrices ρ and σ . These three characteristics are natural properties of resource measures. For example, the total magnetization number of a collection of identical spins and many entanglement measures (for example, entanglement entropy) satisfy them naturally. In this context, Coladangelo and Leung (2019) proved that any function fthat is nonconstant, permutation invariant, and additive can never be more than asymptotically continuous according to Definition VI.2. This is a direct consequence of the general construction in Sec. IV.C that substantiates the relationship between embezzlement and asymptotic continuity.

We now summarize the proof of Coladangelo and Leung (2019). For two density matrices $\rho_{\rm S}$ and $\sigma_{\rm S}$ such that $c = |f(\rho_{\rm S}) - f(\sigma_{\rm S})| > 0$, consider the approximate catalyst states $\omega_{\rm C}$ and $\omega'_{\rm C}$ on n-1 copies of S from Sec. IV.C. They fulfill $\|\omega_{\rm C} - \omega'_{\rm C}\|_1 \le 2/(n-1)$ and

$$U_{\pi}(\rho_{\mathsf{S}} \otimes \omega_{\mathsf{C}}) U_{\pi}^{\dagger} = \sigma_{\mathsf{S}} \otimes \omega_{\mathsf{C}}^{\prime}, \qquad (195)$$

where U_{π} is the unitary implementing a cyclic permutation of all subsystems. The permutation invariance property of fimplies that $f(\rho_{\rm S} \otimes \omega_{\rm C}) = f(\sigma_{\rm S} \otimes \omega'_{\rm C})$. Since f is furthermore additive, we have

$$c = |f(\rho_{\mathsf{S}}) - f(\sigma_{\mathsf{S}})| = |f(\omega_{\mathsf{C}}') - f(\omega_{\mathsf{C}})|.$$
(196)

We now assume, by contradiction, that f is more than asymptotically continuous. There are then a constant K and $\alpha < 1$ such that

$$|f(\omega_{\mathsf{C}}') - f(\omega_{\mathsf{C}})| \le K ||\omega_{\mathsf{C}} - \omega_{\mathsf{C}}'||_{1} (n \log d)^{\alpha} + \eta(||\omega_{\mathsf{C}} - \omega_{\mathsf{C}}'||_{1})$$
$$\le 2K \frac{n^{\alpha}}{n-1} (\log d)^{\alpha}.$$
(197)

Since $\alpha < 1$ by assumption, the rhs tends to zero as *n* goes to infinity, which contradicts our initial assumption that c > 0.

C. Quantum channel catalysis

Throughout we have focused on catalysis on the level of quantum states. But one can also consider catalysis on the level of quantum channels instead of quantum states. Suppose that we can implement a channel \mathcal{E} on a system **C** that is not a free operation: $\mathcal{E} \notin \mathcal{O}$. There may be free operations $\mathcal{F}_{SC \to SC}^{(enc)}, \mathcal{F}_{S'}$ and $\mathcal{F}_{S'C \to SC}^{(dec)}$ such that

$$\mathcal{F}_{\mathsf{S'C}\to\mathsf{SC}}^{(\mathrm{dec})}\circ(\mathcal{F}_{\mathsf{S'}}\otimes\mathcal{E}_{\mathsf{C}})\circ\mathcal{F}_{\mathsf{SC}\to\mathsf{S'C}}^{(\mathrm{enc})}=\mathcal{E}_{\mathsf{S}}\otimes\mathcal{E}_{\mathsf{C}},\qquad(198)$$

and where \mathcal{E}_{S} is not a free operation. In Eq. (198) we allowed the *encoding* map $\mathcal{F}_{SC \to S'C}^{(enc)}$ and the *decoding* map $\mathcal{F}_{S'C \to SC}^{(dec)}$ to map between different systems S'C and SC. The quantum channel \mathcal{E}_{C} then catalyzes the nonfree quantum channel \mathcal{E}_{S} . A simple example is given when S' = S, \mathcal{E}_{C} simply prepares a fixed state ω_{C} , \mathcal{F}_{S} prepares a fixed free state ρ_{S} , and the encoding map is trivial. Equation (198) then reduces to $\mathcal{F}[\rho_{\rm S} \otimes \omega_{\rm C}] = \sigma_{\rm S} \otimes \omega_{\rm C}$, with $\sigma_{\rm S} = \mathcal{E}_{\rm S}[\rho_{\rm S}]$; i.e., we recover strict catalysis for quantum state transitions.

An example of similar behavior was discussed by Brun, Devetak, and Hsieh (2006) and Brun, Devetak, and Hsieh (2014) in the context of quantum error correction: An ideal quantum channel $\mathcal{E}_{C} = Id_{C}$ can be used to catalyze a noisy quantum channel into an ideal quantum channel using an entanglement-assisted quantum error correcting code. Note, however, that the unassisted quantum capacity of a noisy channel together with a noiseless channel is simply the sum of the two (Bennett, DiVincenzo *et al.*, 1996). Therefore, this form of channel catalysis cannot be used to effectively increase the rate of the noisy channel beyond its usual capacity. Systematic studies of quantum channel catalysis in various resource theories provide an interesting opportunity for further research.

D. Catalytic decoupling and resource erasure

It is well known that it is impossible to construct a quantum channel that achieves perfect cloning of unknown quantum states on a given system S; see also Sec. V.D.2. Similarly, it is impossible to construct a quantum channel \mathcal{E} on a bipartite sytems S_1S_2 that removes the correlations for arbitrary quantum states $\rho_{S_1S_2}$ in the sense that

$$\mathcal{E}[\rho_{\mathsf{S}_1\mathsf{S}_2}] = \rho_{\mathsf{S}_1} \otimes \rho_{\mathsf{S}_2} \quad \forall \ \rho_{\mathsf{S}_1\mathsf{S}_2} \quad (\text{impossible}). \tag{199}$$

Decoupling refers to a process where one tries to remove the correlations between a system A from a system E. Decoupling protocols play an important role in quantum information theory (Groisman, Popescu, and Winter, 2005; Horodecki, Oppenheim, and Winter, 2005; Hayden *et al.*, 2008; Abeyesinghe *et al.*, 2009; Dupuis, 2010; Dupuis *et al.*, 2014; Majenz *et al.*, 2017; Berta *et al.*, 2018; Li and Yao, 2021) and are closely related to quantum state merging protocols; see Sec. V.B.3. The amount of resources that one has to invest to achieve decoupling can be seen as a measure for the correlations between A and E (Groisman, Popescu, and Winter, 2005). One approach to decoupling is to try to unitarily concentrate all correlations between A and E into a subsystem A₂ of A by way of a unitary U_A acting only on A, and then to trace out A₂,

$$\operatorname{tr}_{\mathsf{A}_{2}}[U_{\mathsf{A}}\rho_{\mathsf{A}\mathsf{E}}U_{\mathsf{A}}^{\dagger}] \approx \omega_{\mathsf{A}_{1}} \otimes \omega_{\mathsf{E}}.$$
 (200)

The Hilbert-space dimension of the subsystem A_2 required to achieve the aforementioned items with a given precision can then be interpreted as a measure for the amount of correlations between A and E (Dupuis, 2010). Majenz *et al.* (2017) defined catalytic decoupling by introducing an additional system C in a state σ_C that was initially uncorrelated to AE. In other words, we have $AC = A_1A_2$ and

$$\operatorname{tr}_{\mathsf{A}_2}[U_{\mathsf{A}\mathsf{C}}\rho_{\mathsf{A}\mathsf{E}} \otimes \sigma_{\mathsf{C}} U_{\mathsf{A}\mathsf{C}}^{\dagger}] \approx \omega_{\mathsf{A}_1} \otimes \omega_{\mathsf{E}}.$$
(201)

The Hilbert-space dimension d_{A_2} of the subsystem A_2 that one has to trace out is at most as large as in the case of standard decoupling. In fact, Majenz *et al.* (2017) showed that $\log(d_{A_2})$ precisely gives an operational meaning to the smooth maxmutual information. Moreover, they showed that one can choose $A_1 = AC_1$ (i.e., A_1 corresponds to the input system A and part of the catalyst C) and $\omega_A = \text{tr}_{C_1}[\omega_{A_1}] = \rho_A$. That is, the marginal on A does not change. Note, however, that the full catalyst itself is not returned in the same state. Additionally, the part C₂ that is traced out must in general build up strong correlations to E. In particular, the catalyst may not immediately be reused to help decouple a further system from E.

A task closely related to decoupling is the task of erasing resources from a quantum state using random unitary operations from the set of free operations in a resource theory. Specifically, one can then ask for the minimal amount of classical randomness that is required to return a resourceful state close to the set of free states. If one further allows for an approximate, correlated catalyst in a free state during the process, then the smooth max-relative entropy with respect to the free states quantifies the amount of randomness required to achieve resource erasure (Anshu, Hsieh, and Jain, 2018; Berta and Majenz, 2018).

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