Colloquium: Hydrodynamics and holography of charge density wave phases

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(published 4 January 2023)

The Colloquium reviews recent progress in the effective description of strongly correlated phases of matter with spontaneously broken translations, such as charge density waves or Wigner crystals. In real materials, disorder is inevitable and pins the Goldstones of broken translations. The Colloquium describes how pinning can be incorporated into the effective field theory at low energies without making any assumptions on the presence of boost symmetry. The essential role played by gauge-gravity duality models in establishing these effective field theories with only approximate symmetries is reviewed. The Colloquium closes with a discussion on the relevance of these models to the phenomenology of dc and ac transport in strongly correlated strange and bad metals, such as high-temperature superconductors.

DOI: 10.1103/RevModPhys.95.011001

CONTENTS

I. INTRODUCTION

Strongly correlated states of matter present a serious theoretical challenge, as perturbation theory typically fails to describe them. High-critical-temperature (T_c) superconductors (Bednorz and Müller, 1986) constitute an archetypal example and have resisted theoretical efforts to account for their phenomenology since their discovery (Keimer *et al.*, 2015; Alexandradinata *et al.*, 2020). The absence of long-lived quasiparticles, as reported in photoemission experiments, and their unconventional transport properties are two signatures of their incompatibility with the Fermi liquid paradigm (Lifshitz and Pitaevskii, 1980). The Hubbard model (Arovas *et al.*, 2022), marginal Fermi liquid theory (Varma *et al.*, 1989), and various field theories with a large (infinite) number of degrees of freedom (Sachdev, 2011) aided by random interactions (Chowdhury *et al.*, 2021) all provide some degree of insight into this problem.

Progress in understanding the physics at play in these systems has been complicated by the variety of phases that appear to be competing in (or working in concert with) different regions of the phase diagram characterized by temperature, doping, magnetic field, pressure, etc. Hydrodynamics and effective field theory (EFT) methods (Kovtun, 2012; Nicolis *et al.*, 2015; Liu and Glorioso, 2018) offer a complementary avenue by eschewing the microscopic details of strongly correlated systems, as done in the cases of graphene (Lucas and Fong, 2018) and bad metals (Hartnoll, 2015). The cost is that the analysis is limited to low energies, late times, and long distances and breaks down at low temperatures (particularly in the vicinity of any quantum critical point where fluctuation effects cannot be neglected). The effective frameworks also take as input a number of parameters that are constrained by various consistency requirements but whose values can be computed only within a microscopic completion.

Gauge-gravity duality (also referred to as holography) maps a strongly coupled, large N matrix model (where N is the rank of the gauge group) to classical Einstein gravity coupled to a set of matter fields (Maldacena, 1998). The application of this set of techniques to strongly correlated condensed matter systems has been intensively pursued in the past 15 years

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(Zaanen *et al.*, 2015; Hartnoll, Lucas, and Sachdev, 2018). The original duality (Maldacena, 1998) relates a specific gauge theory to a specific string theory so that in principle microscopic degrees of freedom on both sides of the duality can be matched (in practice, this can be technically involved). A more common approach in applied gauge-gravity duality is the bottom-up one, where the dual field theory is not known precisely, nor is it clear whether the classical gravity dual can be promoted to a full quantum gravity. Assuming, though, that such a dual field theory exists and within the range of validity of the low-energy classical gravity theory, the equation of state and transport coefficients of its classical saddle point can be computed.

In applying these tools, identifying the right set of symmetries is paramount, as this will dictate the starting point of an effective approach. This does so by determining the set of hydrodynamic conservation equations governing the low-energy dynamics of the system in one case, or by acting as a guiding principle to write the appropriate bulk action in the second.

The aim of this Colloquium is to describe recent progress in effective hydrodynamic and holographic theories of phases with spontaneously broken translations, motivated by the ubiquity of such phases in the phase diagram of strongly correlated electron materials, in particular, cuprate- or iron-based high-T_c superconductors, kagome materials, organic conductors, transition metals, dichalcogenides, etc. While translational "spin-charge stripe" order was long anticipated on theoretical grounds to play an important role in underdoped cuprates and other doped Mott insulators (Machida, 1989; Poilblanc and Rice, 1989; Zaanen and Gunnarsson, 1989; Kivelson, Fradkin, and Emery, 1998; Mross and Senthil, 2012a, 2012b; Beekman, Nissinen, Wu, Liu *et al.*, 2017) and was experimentally confirmed subsequently in most families of underdoped cuprate materials (Frano *et al.*, 2020) as well as in numerical studies of the Hubbard model (Huang *et al.*, 2017; Zheng *et al.*, 2017), recent experiments suggest that charge density fluctuations (Kivelson *et al.*, 2003) and short-range charge density wave order are actually found across the phase diagram (Peng *et al.*, 2018; Arpaia *et al.*, 2019; Lin *et al.*, 2020; Arpaia and Ghiringhelli, 2021; Kawasaki *et al.*, 2021; S. Lee *et al.*, 2021; W. S. Lee *et al.*, 2021; Ma *et al.*, 2021; Miao *et al.*, 2021; Tam *et al.*, 2022).[1] These observations prompted a number of investigations of the impact of such fluctuating charge order on transport and spectroscopic experiments (Caprara *et al.*, 2017; Delacrétaz *et al.*, 2017a, 2017b; Amoretti *et al.*, 2019b; Delacrétaz, Goutéraux, and Ziogas, 2021; Seibold *et al.*, 2021).

In seeking to apply effective field theory methods to this problem, one is inevitably confronted with the impact of disorder and other sources of explicit translation symmetry breaking on the dynamics of the charge density wave, leading to the phenomenon of pinning (Grüner, 1988). When the explicit breaking is weak, the symmetry rules that usually tightly constrain effective field theories are relaxed and it becomes more arduous to develop a consistent double expansion, in powers of the strength of the explicit breaking and of the effective field theory cutoff. On the other hand, gauge-gravity duality allows one to model such phases from first principles and acts as a testing arena for effective field theories with approximate symmetries.

Sections II and III give a review of hydrodynamics and of holographic methods. In Sec. IV, we then describe recent progress in incorporating background strain into the hydrodynamics of spontaneously broken translation phases, without assuming any particular boost symmetry, and expand on verifications of this theory using various holographic models. Next in Sec. V we turn to the physics of pseudospontaneous translation symmetry breaking in hydrodynamics and in holography and discuss phenomenological implications. We also comment on the role of topological defects and magnetic fields.

II. HYDRODYNAMICS

Hydrodynamics (Kadanoff and Martin, 1963; Forster, 1975; Chaikin and Lubensky, 2000) is based on symmetries and the conservation equations that derive from them. Symmetries and their spontaneous breaking provide a natural route to classifying states of matter, formalized by Landau's theory of second-order phase transitions (Landau and Lifshitz, 2013). This is an example of effective field theory, valid around the critical temperature at which the phase transition occurs, where the relevant degrees of freedom are only the order parameter and its fluctuations.

Hydrodynamics and its extensions to nonliquid states of matter (such as elasticity theory) (Chaikin and Lubensky, 2000) constitute another class of effective field theories that describe the long-distance, late-time dynamics of the system. Microscopic degrees of freedom are integrated out in this limit and are reorganized into fast and slow degrees of freedom. Fast degrees of freedom equilibrate on time and length scales that are short compared to the local equilibration scales, which are typically set by the temperature of the system. Slow degrees of freedom are protected by symmetries and need to be retained in the effective field theory: they are the conserved densities of the system, such as energy, charge, and momentum. Their evolution is described by conservation laws descending from the previously mentioned symmetries. They cannot decay locally and are transported away on scales that are much larger than the local equilibration scales to other regions of the system by hydrodynamic modes, such as sound and diffusion.

More concretely, the equations of motion for the conserved densities n^a take the form[2]

$$\dot{n}^a(t,x) + \nabla \cdot j^a(t,x) = 0. \tag{1}$$

Upper dots stand for time derivatives ($\dot{} \equiv \partial/\partial t$). For a fluid with a conserved U(1) charge, the n^a's are the set of energy ε, momentum π^i, and charge n densities. The spatial currents

[1]A recent numerical study of the Hubbard model also reported fluctuating stripes across the phase diagram (Huang *et al.*, 2022).

[2]To simplify the notation, we are not including spatial indices for the moment; see Sec. IV.B for details.

$j^a = \{j^i_\varepsilon, \tau^{ij}, j^i\}$ are generally not slow operators.[3] They decay locally in the thermal bath of conserved densities, and therefore their expectation values over hydrodynamic timescales are tied to their overlap with the conserved densities via local expansions in terms of the densities and external sources[4]:

$$\langle j_a \rangle = \alpha^{(0)}_{ab} \langle n^b \rangle + \alpha^{(1)}_{ab} \nabla \langle n^b \rangle + \cdots . \qquad (2)$$

The angular brackets in Eq. (2) denote a thermal average. $\alpha^{(0),(1)}_{ab}$ are transport coefficient matrices, order by order in the gradient expansion, with dots denoting higher-order terms. Which of these coefficients are nonzero depends on the details of the system and the symmetry-breaking pattern. The underlying reason why such expansions are possible is related to the central assumption to hydrodynamics: all microscopic, high-energy modes relax on short scales of the order of the thermalization time or length and can be integrated out. At longer scales, only hydrodynamic fields are retained, and they are the sole source of nonanalyticities in the retarded Green's functions. In other words, in the hydrodynamic regime, the retarded Green's functions contain only the gapless hydrodynamic poles.

In this Colloquium, we limit ourselves to expansions to first order in gradients. We also ignore the effects of fluctuations (De Schepper, Van Beyeren, and Ernst, 1974; Forster, Nelson, and Stephen, 1977), which generally spoil the analyticity of retarded Green's functions and of the dispersion relations of the hydrodynamic modes beyond first order in gradient terms. In gauge-gravity duality, these fluctuations are suppressed by the $N \to +\infty$ limit (Kovtun and Yaffe, 2003).

Upon inserting Eq. (2) into Eq. (1), these become evolution equations for the vacuum expectation values (VEVs) of the conserved densities, which can now be solved. Taking a spatial Fourier transform and dropping the angular brackets for convenience, we obtain a set of dynamical equations given by

$$\dot{n}_a(t,q) + M_{ab}(q) n^b(t,q) = 0. \qquad (3)$$

By construction, the matrix $M_{ab}(q) = M_1 q + M_2 q^2 + \cdots$ has a local expansion in powers of the wave vector q, with each term suppressed by the cutoff length of hydrodynamics $\ell_{\rm th}$.

[3]In special cases they can be. For example, in a system invariant under Galilean boosts, the corresponding Ward identity gives an operator equation between the charge current and momentum operators, so the charge current is a slow operator in this case. In a relativistic system, the Lorentz boost Ward identity equates the momentum and energy current operators, so the energy current becomes a slow operator as well.

[4]Throughout this Colloquium, we work in a hydrodynamic frame where the time components of the conserved currents match the microscopic conserved densities, and dissipative corrections enter only with spatial gradients. Any time derivative correction can be traded for spatial derivatives using the equations of motion and constitutive relations at lower order in derivatives.

We now compute the retarded Green's functions of the system. As usual, this implies turning on a time-dependent deformation of the Hamiltonian

$$H_o \mapsto H(t) = H_o - \int d^d x n^a(t,x) \delta\mu_{e,a}(t,x) \qquad (4)$$

(with d the number of spatial dimensions), upon which the equations of motion become (Kadanoff and Martin, 1963; Chaikin and Lubensky, 2000)

$$\dot{n}^a(t,q) + M^a_b(q)[n^b(t,q) - \chi^b_c \delta\mu^c_e] = 0. \qquad (5)$$

In Eq. (5) χ is the matrix of static susceptibilities obtained by functional differentiation of the equilibrium free energy as

$$\chi_{ab}(x - x') = -\frac{\delta^2 W[\mu_e]}{\delta\mu^a_e(x) \delta\mu^b_e(x')}, \qquad (6)$$

where $W = -T \log {\rm Tr} e^{-\beta H}$. This matrix encodes the linear response of the system to static perturbations $\delta\mu_e(x)$. In the static limit, from Eq. (5) $n^a = \chi^a_b \delta\mu^b_e$, and thus χ_{ab} is simply the matrix of thermodynamic derivatives. It should be positive definite in order for the system to be locally thermodynamically stable.

Taking a Laplace transform of Eq. (5) [see Kovtun (2012) for more details] leads to the retarded Green's functions

$$G^R_{ab}(\omega,q) \equiv \frac{\delta n^a(\omega,q)}{\delta\mu^b_e(\omega,q)} = -(i\omega - M)^{-1} M\chi, \qquad (7)$$

where ω is the frequency. The hydrodynamic poles of the system are found by solving $\det(-i\omega + M) = 0$. As a point of reference, in the case of a single conserved U(1) the constitutive relation for the spatial current compatible with invariance under parity and time reversal and with external sources turned on is

$$j^i = -D_n(\nabla^i n - \chi_{nn} \nabla^i \delta\mu_e) + \cdots, \qquad i = 1, \ldots, d, \qquad (8)$$

leading to a quadratically dispersing, diffusive mode $\omega = -iD_n q^2 + \cdots$. The diffusivity can be measured using the following Kubo formula:

$$D_n = \frac{1}{\chi_{nn}} \lim_{\omega\to 0} \lim_{q\to 0} \frac{\omega}{q^2} {\rm Im} G^R_{nn}(\omega,q). \qquad (9)$$

Instead, in the longitudinal sector a neutral fluid would have two linearly dispersing sound modes $\omega = \pm c_s q - i(\Gamma/2) q^2$, where the longitudinal sound velocity is determined by the various static susceptibilities and the sound attenuation Γ is determined by first-order-in-gradients dissipative corrections to the constitutive relation of the energy current and stress tensor; see Chaikin and Lubensky (2000).

Hydrodynamics gives access only to gapless poles with a vanishing dispersion relation at zero wave vector $\omega(q = 0) = 0$, the hydrodynamic modes. Nonhydrodynamic gapped modes of the system cannot reliably be included in the hydrodynamics in general, except in certain special circumstances, for instance, when the gap is generated by breaking weakly one of the symmetries of the system (Davison and Goutéraux, 2015b; Grozdanov, Lucas, and Poovuttikul, 2019). One of the goals of this Colloquium is to explain how to incorporate such weakly gapped degrees of freedom in the low-energy effective field theory. Generic gapped modes that do not fall under the previous category typically signal the breakdown of the effective field theory description (Withers, 2018; Grozdanov *et al.*, 2019) and can be accounted for only by supplementing hydrodynamics with a microscopic completion.[5]

III. HOLOGRAPHIC METHODS

The hydrodynamic construction outlined in Sec. II can be systematically carried out order by order in the gradient expansion. The procedure quickly becomes intractable analytically due to the proliferation of terms to be considered (Grozdanov and Kaplis, 2016). The equation of state and each transport coefficient needs to be measured experimentally or computed in a microscopic model.

Most microscopic models, nevertheless, face serious difficulties whenever the system under investigation is strongly interacting, made up of a large number of constituents, placed at finite chemical potential, or placed at finite temperature, or when its real-time dynamics is considered. Under these circumstances, the AdS/CFT correspondence[6] provides a self-consistent framework to attack these problems and guide new interdisciplinary explorations. Holography posits a duality between a large class of quantum field theories with a gauge group of dimension N and higher-dimensional gravitational theories; see Aharony *et al.* (2000), Ammon and Erdmenger (2015), Nastase (2015), Natsuume (2015), Zaanen *et al.* (2015), and Baggioli (2019) for details. The duality was discovered in the context of string theory (Gubser, Klebanov, and Polyakov, 1998; Maldacena, 1998; Witten, 1998), which provides a precise formulation of the conjecture between a supersymmetric gauge theory [$\mathcal{N}=4$ super Yang-Mills theory with gauge group $\mathrm{SU}(N)$] and a string theory (type-IIB string theory on $\mathrm{AdS}_5\times S^5$), now widely accepted as proven. The simplifying limit of classical gravity without extended objects corresponds to considering a dual field theory in the regime of strong coupling and in the large N limit[7] and is known as the bottom-up approach. Bottom-up holographic methods have been applied in several directions, such as quantum chromodynamics (QCD) and heavy ion collisions (Casalderrey-Solana *et al.*, 2014; Berges *et al.*, 2021), condensed matter many-body systems, and quantum information (Zaanen *et al.*, 2015; Rangamani and Takayanagi, 2017; Hartnoll, Lucas, and Sachdev, 2018; Liu and Sonner, 2020a, 2020b).

From a formal point of view, the duality is built on the identification of the field theory generating functional W with the gravitational on-shell path integral. The field theory operators and sources are given by the specific coefficients of the asymptotic expansion of dynamical fields living in the curved, higher-dimensional bulk spacetime. Thermal, finite density states in the dual field theory are captured by gravitational charged black hole solutions in the bulk, with the field theory temperature given by the Hawking temperature at the event horizon and the chemical potential given by the boundary value of the bulk gauge field. From this gravitational background, all thermodynamic quantities can be computed, as well as the static susceptibilities. Linear perturbations of the gravitational solution together with opportune boundary conditions for the bulk fields (Son and Starinets, 2002) yield the real-time, space-dependent retarded Green's functions, the poles of which are given by the quasinormal modes of the black hole solution. This linear analysis also gives access to all linear transport coefficients through the appropriate Kubo formulas. This way one can obtain the dispersion relations of the low-energy excitations in the dual field theory as well as those of the nonhydrodynamic modes of the system, going far beyond the hydrodynamic regime (Kovtun and Starinets, 2005; Berti, Cardoso, and Starinets, 2009). Holographic results have been successfully matched to the predictions of charged, relativistic linearized hydrodynamics (Policastro, Son, and Starinets, 2002a, 2002b; Baier *et al.*, 2008; Erdmenger *et al.*, 2009; Banerjee *et al.*, 2011).[8] Besides providing a concrete test bed for hydrodynamics, holography is a microscopically complete framework that allows one to compute all transport coefficients, from which important lessons on strongly coupled dynamics can be extracted. A case in point is the noted viscosity-entropy-ratio bound (Policastro, Son, and Starinets, 2001; Cremonini, 2011).

As in the case of hydrodynamics, and more often in general effective field theories, bottom-up holography is based and built on symmetries as guiding principles. Using the well-established holographic dictionary, one maps local symmetries in the bulk onto global symmetries of the boundary field theory. Any combination of explicit or spontaneous symmetry breaking can be considered. Explicit breaking corresponds to the presence of a source in the dual field theory that appears in Ward identities and spoils conservation equations; spontaneous breaking, on the contrary, is characterized by the appearance of a nontrivial vacuum expectation value for an operator (the condensate) that breaks a subset of the symmetries of the action from which it is derived (Beekman, Rademaker, and Wezel, 2019). Finally, the pseudospontaneous regime appears when a small (as quantified

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[5]When the gradient expansion can be systematically computed in a microscopically complete framework, the dispersion relation of gapped modes can be obtained by resumming the hydrodynamic series (Withers, 2018).

[6]The acronyms stand for anti–de Sitter spacetime (Gibbons, 2000) and conformal field theory (Ginsparg, 1990), the two end points of the original string-inspired holographic duality. We ignore more general situations in which the UV fixed point of the dual field theory is not a Lorentz-invariant conformal field theory.

[7]Here N is the rank of the dual gauge field theory ('t Hooft, 2002). In the absence of a precisely identified dual field theory, this limit has to be understood as a large number of degrees of freedom. See Zaanen *et al.* (2015) for a discussion on this point.

[8]The full nonlinear structure of the hydrodynamic theory can also be derived from the gravitational equations using the fluid-gravity correspondence (Rangamani, 2009).

more precisely later) source is added on top of a purely spontaneous state (Weinberg, 1972; Burgess, 2000). From the bulk point of view, this distinction is encoded in the asymptotic behavior of the field responsible for the symmetry breaking close to the boundary of the AdS spacetime (Skenderis, 2002). The corresponding boundary Ward identities can be computed directly from the bulk as well; see Argurio *et al.* (2016) for the simplest case of a global U(1) symmetry.

The holographic description of broken-symmetry, strongly coupled phases of matter with an eye toward condensed matter was initiated by Hartnoll, Herzog, and Horowitz (2008a, 2008b) and Gubser (2008), who considered the spontaneous breaking of a global U(1) symmetry: a superfluid state. Superfluid hydrodynamics correctly predicts the low-energy dynamics of holographic superfluids (Herzog, Kovtun, and Son, 2009; Sonner and Withers, 2010; Bhattacharya, Bhattacharyya, and Minwalla, 2011; Herzog *et al.*, 2011; Bhattacharya *et al.*, 2014; Arean *et al.*, 2021).

Holographic lattices explicitly breaking translations were constructed numerically a few years later by Horowitz, Santos, and Tong (2012a, 2012b).[9] While the significance of explicit translation symmetry breaking was recognized early on in the holographic community (Hartnoll *et al.*, 2007; Hartnoll and Herzog, 2008), full holographic realizations had to tackle with the technical challenge related to solving inhomogeneous space-dependent Einstein's equations (Dias, Santos, and Way, 2016; Andrade, 2017; Krikun, 2018b). A change of paradigm happened with the discovery of the so-called homogeneous models, holographic setups in which translations are broken but the background metric and the dual stress tensor remain homogeneous (regardless of the spatial coordinates). This property is due to the existence of specific global structures that mix with spacetime symmetries, leading to a simplification in the computations of physical observables. The homogeneous models fall into different classes: (I) de Rham–Gabadadze–Tolley (dRGT) massive gravity theory (Vegh, 2013),[10] (II) "axion" models (Andrade and Withers, 2014; Donos and Gauntlett, 2014b; Goutéraux, 2014b; Taylor and Woodhead, 2014; Baggioli and Pujolas, 2015; Baggioli, Cisterna, and Pallikaris, 2021) [see Baggioli *et al.* (2021) for a review], (III) Q-lattices (Donos and Gauntlett, 2014a), (IV) higher-form models (Grozdanov and Poovuttikul, 2018),[11] (V) helical lattices (Nakamura, Ooguri, and Park, 2010; Iizuka *et al.*, 2012; Donos and Hartnoll, 2013; Donos, Goutéraux, and Kiritsis, 2014), and (VI) "solidon" models (Esposito *et al.*, 2017). Irrespective of the specific holographic model employed, in the regime of weak explicit breaking, the low-energy dynamics matches well with the field theory expectations for a metallic phase with slowly relaxing momentum (Hartnoll *et al.*, 2007; Hartnoll and Herzog, 2008; Davison, 2013; Davison and Goutéraux, 2015b; Lucas, Sachdev, and Schalm, 2014; Lucas and Sachdev, 2015). A common feature of all holographic models is that the graviton acquires a mass (Alberte *et al.*, 2016; Vegh, 2013; Blake, Tong, and Vegh, 2014).[12]

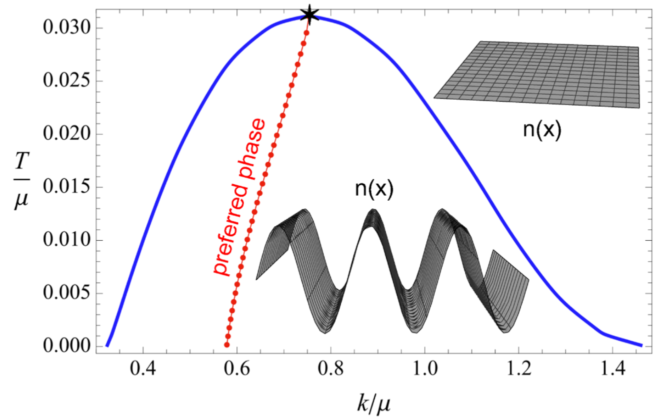

FIG. 1. The typical curve describing the onset of the instability toward a holographic phase breaking translations spontaneously. Insets: spatial profiles of the charge density n in the broken and normal phases. Adapted from Cai *et al.*, 2017.

Closely following the global U(1) case, instabilities toward holographic spatially modulated phases breaking translations spontaneously were investigated as well (Nakamura, Ooguri, and Park, 2010; Donos and Gauntlett, 2011b; Donos, Gauntlett, and Pantelidou, 2012; Alsup *et al.*, 2013; Cremonini and Sinkovics, 2014). In those studies, one looked for a spatially modulated, normalizable bulk mode (i.e., one without a source at the boundary) in the translation-invariant, homogeneous bulk geometry. The outcome of this analysis is an instability curve displaying the maximum temperature at which the mode can be found versus the wave vector of the modulation (see Fig. 1), i.e., the onset of the instability. The apex of this curve gives the thermodynamically preferred wave vector with the highest critical temperature, below which fully backreacted, spatially modulated phase breaking translations can spontaneously be expected to be found. As such, when the preferred trajectory within the instability curve is followed (red bullets in Fig. 1), these phases are true global minima of the thermodynamic free energy (Donos and Gauntlett, 2013b). The breaking of parity and time reversal through Chern-Simons couplings in the bulk and/or an external magnetic field originally played an important role to generate the instabilities but is not always necessary

[9] See Chesler, Lucas, and Sachdev (2014), Donos and Gauntlett (2015), Langley, Vanacore, and Phillips (2015), and Rangamani, Rozali, and Smyth (2015) for further numerical constructions of holographic lattices.

[10] dRGT corresponds to a precise fine-tuned choice of more general Lorentz-violating massive gravity theories built in terms of Stückelberg fields (Dubovsky, 2004). This fine tuning is not necessary around Lorentz-violating vacua. In this sense, the axion models in item (II) are not different from the dRGT model, which ultimately represents only an infinitesimal subclass of them. See Alberte *et al.* (2016) for a more detailed discussion of this point in the context of holographic models.

[11] Before being analyzed from a holographic point of view, static black hole solutions including matter in the form of free scalar and p-form fields were constructed by Bardoux, Caldarelli, and Charmousis (2012).

[12] See Zaanen, Balm, and Beekman (2021) for a discussion on the connections between elasticity theory and massive gravity.

(Donos and Gauntlett, 2013a). The original works focused on inhomogeneous instabilities, but helical phases proved easier to construct at first (Donos and Gauntlett, 2011a, 2012a, 2012b). Backreacted inhomogeneous phases spatially modulated along one direction were constructed by Donos (2013), Rozali *et al.* (2013), and Withers (2013), who bore on the expertise developed to construct explicit holographic lattices. Generalizations to two-dimensional, checkerboard, or triangular patterns were given by Donos and Gauntlett (2016) and Withers (2014), with the triangular lattice providing the minimum free energy state. These phases all include circulating current loops together with spontaneous parity breaking, which is reminiscent of the loop current order proposed to underlie the pseudogap phase of underdoped cuprate high-T_c superconductors (Varma, 1999). This phenomenology is a direct consequence of the bulk Chern-Simons term.

Probe brane constructions can also display spatially modulated instabilities (Jokela, Jarvinen, and Lippert, 2013, 2014, 2017a). Being top-down models descending from specific string theory realizations, they have the advantage of offering a more precise field theory interpretation. On the other hand, it is not clear how one should interpret the spontaneous spatial modulation of charge and current densities, since in these setups momentum and temperature fluctuations are frozen.

Phases in which the breaking of translations and of a global U(1) are intertwined are of interest to model pair density wave phases (Fradkin, Kivelson, and Tranquada, 2015), which are thought to play an important role in the phase diagram of underdoped cuprate high-T_c superconductors. They have been argued to be the mother phase from which daughter charge density wave and superconducting phases emerge. Holographic realizations of these phases were given by Cai *et al.* (2017) and Cremonini, Li, and Ren (2017a, 2017b). These constructions rely on a combination of bulk Chern-Simons terms and the introduction of Stückelberg scalars, which give rise to pair density wave phases where the condensate is spatially modulated with a zero average and periodicity that is twice that of the charge density wave.[13]

Natural next steps were to combine all of these strands together by considering holographic phase breaking translations pseudospontaneously and how their low-energy dynamics match to field theory expectations. The purpose of this Colloquium is to summarize the progress in these directions over the last few years. These developments came about from the intersection between various pieces of work: the spontaneous incorporation of the physics of explicit symmetry breaking in the hydrodynamics of phase breaking translations (Delacrétaz *et al.*, 2017a, 2017b; Armas, Jain, and Lier, 2021; Delacrétaz, Goutéraux, and Ziogas, 2021); the construction of simpler homogeneous holographic models for (pseudo)spontaneously breaking translations (Andrade and Krikun, 2016; Amoretti *et al.*, 2017, 2018b, 2019b; Alberte, Ammon, Baggioli *et al.*, 2018; Alberte, Ammon, Jiménez-Alba *et al.*, 2018; Andrade, Baggioli *et al.*, 2018; Ammon, Baggioli, and Jiménez-Alba, 2019; Ammon *et al.*, 2019; Donos *et al.*, 2020), which provided a far more tractable platform to compare with hydrodynamic predictions; a thorough analysis of how background strain and external sources enter into the hydrodynamic theory (Armas and Jain, 2020a, 2020b); and a subsequent comparison with holographic constructions (Ammon *et al.*, 2020).

IV. HYDRODYNAMICS OF PHASES WITH BROKEN TRANSLATIONS

Continuous, global symmetries can be spontaneously broken in the ground state; see Beekman, Rademaker, and Wezel (2019) for an introduction. Formally, this means that the ground state is invariant under a smaller set of symmetries than the Hamiltonian of the system. A corresponding number of gapless modes, the Goldstone bosons, appear in the spectrum. In the simplest case of an internal symmetry and when the broken generators commute, the number of Goldstones is given by the number of broken generators. In other cases, such as that of spacetime symmetries (Low and Manohar, 2002), the counting rule is more complicated; see Watanabe (2020) for a review.

Hydrodynamics can be advantageously extended to systems with spontaneously broken symmetries, like superfluid helium or crystalline solids (Chaikin and Lubensky, 2000). The set of slow degrees of freedom is enlarged to include the Goldstone modes, the dynamical evolution of which is described by so-called Josephson equations (historically, the Josephson equation has described the phase difference in a superconductor in the presence of an external voltage).

In this Colloquium, we focus on cases of broken translations, such as crystalline solids, charge density waves, and Wigner crystals. We incorporate the effects of background strain, which proves important to match to existing holographic studies. Moreover, strain or pressure is also a common experimental tool in the investigation of broken translation phases in strongly correlated materials and has a strong effect on the onset of the charge density wave and superconducting phases (Hicks *et al.*, 2014; Kim *et al.*, 2018). We also do not assume any particular boost symmetry.[14]

A. Thermodynamics

For simplicity, we assume that the system is two dimensional ($d = 2$), that it is isotropic, and that translations are spontaneously broken in all spatial directions: extensions to anisotropic or higher-dimensional crystals are conceptually straightforward but technically tedious due to more complicated tensor structures and a larger number of transport coefficients. Here we do not consider the coupling to background sources, which can be realized along the lines followed by Armas and Jain (2020a, 2020b).

[13]See also Baggioli and Frangi (2022) for a homogeneous model realizing the concomitant spontaneous breaking of translations together with a global U(1) symmetry, a supersolid phase.

[14]In this Colloquium, we do not consider equilibrium states with a background fluid velocity and thus work only at linear order in the fluid velocity. See de Boer *et al.* (2018a, 2018b, 2020), Novak, Sonner, and Withers (2020), Poovuttikul and Sybesma (2020), and Armas and Jain (2021) for fully nonlinear treatments of fluid hydrodynamics without boosts.

Since spatial translations are spontaneously broken in all directions, we expect as many Goldstone modes as there are broken translations.[15] The Goldstone modes of broken translations are often called phonons and are related to the displacements of the underlying crystal structure, which we denote as u^i. They transform nonlinearly under spatial translations $x^i \to x^i + a^i$ as

$$u^i \to u^i + a^i. \tag{10}$$

The free energy of the system must be invariant under these shifts, and therefore the displacements can appear therein only with derivatives. From a Lagrangian perspective,[16] we define the nonlinear, Lagrange strain tensor u_{ij} as

$$dx'^2 - dx^2 \equiv 2u_{ij}dx^i dx^j. \tag{11}$$

In Eq. (11) $x'(x) = x + u(x)$ is the new location of the point originally at x after u is deformed by a small displacement. Writing that $dx' = x'(x+dx) - x'(x)$, the nonlinear strain is then given in terms of the displacements as (Chaikin and Lubensky, 2000)

$$u_{ij} = \nabla_{(i}u_{j)} + \tfrac{1}{2}\nabla_i u_k \nabla_j u^k. \tag{12}$$

The free energy depends only on u_{ij}. Lowercase latin indices $i, j, \ldots$ are raised and lowered with the Kronecker delta δ_{ij} and run over spatial dimensions.

The elastic part of the equilibrium free energy density of the system is then, by isotropy,

$$f_{\rm el} = \frac{B_o(X,Y)}{2}X^2 + G_o(X,Y)Y, \tag{13}$$

where we have defined $X \equiv u_i^i$ and $Y \equiv u_{ij}u^{ij} - (1/2)(u_i^i)$ and suppressed the dependence on temperature and chemical potential for now.[17] The first parameter X corresponds to a purely volumetric deformation $\Delta V/V = X$, while the second parameter Y corresponds to a deviatoric deformation that modifies the shape of the material but not its total volume. Accordingly, the coefficients B_o and G_o are the bare nonlinear bulk and the shear moduli, respectively. If translations are broken in one dimension only, there is only a bulk modulus. Both quantities are nonlinear functions of the deformation parameters X and Y and, in what follows, of the temperature and chemical potential as well.

Since we want to compute the linear response of the system to external perturbations, the first step is to determine the static susceptibilities. To this end, we expand the static free energy to quadratic order in fluctuations using[18]

$$u_i = mx^i + (1+m)\delta\phi_i(x), \qquad i = \{x, y\}, \tag{14}$$

where m is a real parameter and we have assumed for simplicity that the linear perturbations $\delta\phi_i$ depend on only one spatial dimension.

The first term mx^i in Eq. (14) can be thought as the additional displacement from the would-be static equilibrium configuration $m = 0$ to the actual configuration with isotropic background strain[19] $m \neq 0$:

$$X = u_l^l = u_o + \mathcal{O}(\nabla), \qquad u_o \equiv m(m+2). \tag{15}$$

For this state, $Y = \mathcal{O}(\nabla^2)$. In this configuration, the equilibrium free energy density $f_{\rm el}$ and the bare elastic moduli B_o and G_o are functions of u_o. As we soon see, $\delta\phi_i$ are the Goldstone modes of the system around the configuration with background strain u_o.

Plugging Eq. (14) into the elastic free energy (13) and expanding to quadratic order in fluctuations, we obtain

$$f_{\rm el} = \frac{u_o^2}{2}B_o - p_{\rm el}\lambda_\parallel + \frac{G}{2}(\lambda_\perp)^2 + \frac{1}{2}(B+G)(\lambda_\parallel)^2, \tag{16}$$

where it is convenient to define the longitudinal and transverse Goldstone modes $\lambda_\parallel = \nabla\cdot\delta\phi$ and $\lambda_\perp = \nabla\times\delta\phi$. In the presence of nonzero background strain u_o, the free energy has a term that is linear in $\lambda_\parallel$ that defines the background elastic pressure $p_{\rm el} = -(1+u_o)\partial_{u_o}(u_o^2 B_o)/2$. It is manifest that the configuration with zero strain minimizes the free energy, so states with finite background strain must be sourced by nontrivial boundary conditions. The bulk and shear moduli also pick up the new contributions

$$B \equiv \tfrac{1}{2}(1+u_o)^2\partial_{u_o}^2(u_o^2 B_o),$$
$$G \equiv (1+u_o)^2 G_o + \frac{u_o^2}{2}(1+u_o)^2\partial_Y B_o - p_{\rm el}. \tag{17}$$

In the limit of zero strain, we recover $B(u_o = 0) = B_o$ and $G(u_o = 0) = G_o$.

The free energy in Eq. (16) now displays a linear term in $\delta\phi_x$ with the coefficient[20] $p_{\rm el}$. This linear term implies that the

[15]Rotations are also spontaneously broken but do not have independent Goldstone modes, per the Goldstone counting theorems for broken spacetime symmetries (Low and Manohar, 2002). The underlying reason is that translations and rotations are not independent local transformations.

[16]This assumption is important only at the nonlinear level since all different definitions agree with each other at the linear one (Ogden, 1985).

[17]One could equivalently define a generic function $f_{\rm el}(X,Y)$. Our parametrization makes the linear limit $X, Y \to 0$ and the limit of zero background strain clearer. In $d > 2$, this function would depend on d independent scalars (Nicolis, Penco, and Rosen, 2014; Esposito *et al.*, 2017).

[18]This maps to the formulation of Armas and Jain (2020b) as follows: $u^i \leftrightarrow \Phi^I - x^I$, $u^{ij} \leftrightarrow h^{IJ} - \delta^{IJ}$, and $m = 1/\alpha - 1$.

[19]Anisotropic strains can easily be considered by allowing for additional off-diagonal terms of the type $u_i \sim x_j$; see Baggioli, Castillo, and Pujolas (2020).

[20]This term also appears in the relativistic treatment given by Armas and Jain (2020b). Here we generate it by expanding around the state with background strain in Eq. (14). Armas and Jain (2020b) directly introduced $p_{\rm el}$ as a force contribution to the free energy. They assumed that the reference contribution did not minimize the free energy. After matching the conventions, both approaches gave the same results.

system is not in mechanical equilibrium whenever m is nonzero: a background strain is applied to the system through nontrivial boundary conditions, and the resulting sum of external forces does not vanish. Instead, in mechanical equilibrium there is no background displacement, $m = 0$, and $p_{\rm el} = 0$. The standard treatment of elasticity theory [see Chaikin and Lubensky (2000)] assumes that the reference configuration, the choice of which is arbitrary, corresponds with the state of mechanical equilibrium. Nevertheless, this new term might be relevant for many experiments that study phases with broken translations under the application of pressure. It appears as well in the viscoelastic description of prestrained materials (Birch, 1938; Biot, 1940, 1965; Hayes and Rivlin, 1969; Destrade and Saccomandi, 2007; Destrade, Ogden, and Saccomandi, 2009; Berjamin and Pascalis, 2021), and it was recently considered in the relativistic viscoelastic framework given by Armas and Jain (2020b) under the name of crystal or lattice pressure.

Turning on background external sources $f \mapsto f - s_\parallel \lambda_\parallel - s_\perp \lambda_\perp$, we obtain the Goldstone static susceptibilities after integrating out the Goldstone modes[21]:

$$\chi_{\lambda_\parallel \lambda_\parallel} \equiv -\frac{\partial^{(2)} f_{\rm el}}{\partial {s_\parallel}^2} = \frac{1}{B+G},$$
$$\chi_{\lambda_\perp \lambda_\perp} \equiv -\frac{\partial^{(2)} f_{\rm el}}{\partial {s_\perp}^2} = \frac{1}{G}. \tag{18}$$

$B + G$ and G should both be positive definite in order for the phase to be locally thermodynamically stable, which follows from the usual requirement that the determinant of the Hessian of the free energy be positive definite. We later see that this ensures that sound modes will have a positive velocity squared. Through Eq. (17), we observe that both the bulk and shear moduli have a nontrivial dependence on background strain. Varying the background strain may lead to thermodynamic instabilities, signaled by divergences in the static susceptibilities (18) when G or $B + G$ changes sign. Determining whether these instabilities are actually present requires one to know their functional dependence on u_o and is beyond the effective field theory approach.

By their properties under parity transformations $x \mapsto -x$, we also expect the longitudinal phonon $\lambda_\parallel$ to couple to entropy and charge. To this end, we include temperature and chemical potential dependence in the bare moduli $B_o(u_i^i, T, \mu)$ and $G_o(u_i^i, T, \mu)$ in Eq. (13). Linearizing around Eq. (14) together with $\{T, \mu\} = \{T_o, \mu_o\} + \{\delta T, \delta\mu\}$ allows one to identify the off-diagonal susceptibilities

$$\chi_{n\lambda_\parallel} \equiv -\frac{\partial^{(2)} f_{\rm el}}{\partial s_\parallel \partial \mu} = \frac{\partial_\mu p_{\rm el}}{B+G},$$
$$\chi_{s\lambda_\parallel} \equiv -\frac{\partial^{(2)} f_{\rm el}}{\partial s_\parallel \partial T} = \frac{\partial_T p_{\rm el}}{B+G}. \tag{19}$$

[21]This leads to $s_\perp = G\lambda_\perp$ and $s_\parallel = (G+B)\lambda_\parallel$.

They are nonzero even in the absence of background strain and correspond physically to the chemical and thermal expansion of the system under strain.

The full longitudinal static susceptibility matrix reads

$$\chi_{o,\parallel} = \begin{pmatrix} \chi_{nn} & \chi_{n\varepsilon} & 0 & \chi_{n\lambda_\parallel} \\ \chi_{n\varepsilon} & \chi_{\varepsilon\varepsilon} & 0 & \chi_{\varepsilon\lambda_\parallel} \\ 0 & 0 & \chi_{\pi\pi} & 0 \\ \chi_{n\lambda_\parallel} & \chi_{\varepsilon\lambda_\parallel} & 0 & \chi_{\lambda_\parallel\lambda_\parallel} \end{pmatrix}. \tag{20}$$

The equality of off-diagonal components follows from invariance under PT symmetry.

In the transverse sector, the susceptibility matrix $\chi_{o,\perp}$ is diagonal with the two nonzero elements $\chi_{\lambda_\perp\lambda_\perp}$, which is given by Eq. (18), and $\chi_{\pi_\perp\pi_\perp} = \chi_{\pi\pi}$, by isotropy.

B. Dynamics

We are now ready to state the equations that govern the dynamics of the system in the hydrodynamic regime. Assuming rotation, translation, and U(1) symmetry, these are the conservation of energy, charge, and momentum density

$$\dot\varepsilon + \nabla \cdot j_\varepsilon = 0, \quad \dot n + \nabla \cdot j = 0, \quad \dot\pi^i + \nabla_j \tau^{ji} = 0, \tag{21}$$

together with the Josephson equation for the dynamic evolution of the Goldstone modes,

$$\frac{d}{dt} u^i = -v^i + \cdots. \tag{22}$$

In Eq. (22) v^i is the velocity field conjugate to the momentum π^i, $d/dt \equiv \partial_t + v^i \nabla_i$ stands for the material derivative, and the dots represent dissipative corrections to this relation.

We can derive the nondissipative terms in Eq. (22) in the following way. The Goldstone fields are canonically conjugate to the momentum density, i.e., the conserved charge that generates the broken symmetry,

$$i[\pi^i(x), u^j(x')] = -\delta^{(2)}(x - x')(\delta^{ij} + \nabla^i u^j). \tag{23}$$

We then deform the Hamiltonian using an external velocity source $H_o \mapsto H = H_o - \int d^2x \pi_i v_e^i$ and use the Schrödinger equation to compute the time evolution of the displacement,

$$\dot u^i = i[H, u^i] = v_e^i + v_e^j \nabla_j u^i. \tag{24}$$

Since u^i must be time independent in thermodynamic equilibrium ($v^i = v_e^i$), this means that the Josephson relation must take the form

$$\dot u^i = (v_e^j - v^j)(\delta_j^i + \nabla_j u^i) + \tilde u^i, \tag{25}$$

in agreement with Eq. (5). Taking a divergence or a curl of Eq. (25) with sources off leads to Eq. (22). We have allowed for a possible dissipative correction $\tilde u^i$.

In our thermodynamic ensemble, the first law of thermodynamics is

$$df = -sdT - nd\mu + h^{ij} d(\nabla_i u_j), \tag{26}$$

where $h^{ij} \equiv \partial f / \partial(\nabla_i u_j)$ is

$$h^{ij} = \left(XB_o + \frac{X^2}{2} \partial_X B_o + Y \partial_X G_o \right) X^{ij} + \left(2G_o + Y \partial_Y G_o + \frac{X^2}{2} \partial_Y B_o \right) Y^{ij}, \tag{27}$$

with

$$X^{ij} = \frac{\partial X}{\partial \nabla_i u_j} = \delta^{ij} + \nabla^i u^j,$$
$$Y^{ij} = \frac{\partial Y}{\partial \nabla_i u_j} = 2(u^{ij} + u^{ik} \nabla_k u^j) - XX^{ij}. \tag{28}$$

Using the condition that the entropy density must be conserved $\dot{s} + \nabla_i(s v^i + \tilde{j}_q^i / T) = 0$ in the absence of dissipative (gradient) corrections, the ideal constitutive relations are found to be

$$j_\varepsilon^i = (\varepsilon + p) v^i + h^{ij} v_j + h^{il} v^j \nabla_j u_l + \tilde{j}_\varepsilon^i, \tag{29}$$

$$\tau^{ij} = p \delta^{ij} + h^{ij} + h^{il} \nabla^j u^l + v^i \pi^j + \tilde{\tau}^{ij}, \tag{30}$$

$$j^i = n v^i + \tilde{j}^i. \tag{31}$$

In Eqs. (29)–(31) p is the thermodynamic pressure, which verifies that $p = -f = -\varepsilon + sT + n\mu + v_k \pi^k$. It is straightforward to verify that the stress tensor τ^{ji} is symmetric by substituting the expression for h^{ij} in terms of u^{ij} into Eq. (29). $\tilde{j}_q^i$, $\tilde{j}_\varepsilon^i$, $\tilde{\tau}^{ji}$, and $\tilde{j}^i$ all stand for dissipative corrections that are at least first order in gradients.

The form of dissipative corrections are determined by a well-known algorithm. We start by allowing all possible terms that are spatial derivatives of the fields (the conserved densities and the Goldstone modes) consistent with the symmetries; for instance, we do not allow terms that violate parity. We then require that these terms do not lead to nonlocalities in the equations of motion. Finally, we verify that the entropy current is positive definite while also imposing Onsager relations. The outcome of this procedure, which we detail in the Appendix, leads to the following constitutive relations:

$$\tilde{j}^i = -\sigma_o^{ij} \nabla_j \mu - \alpha_o^{ij} \nabla_j T - \tfrac{1}{2} \xi_\mu^{ij} \nabla^k h_{kj},$$
$$\frac{\tilde{j}_q^i}{T} = -\alpha_o^{ij} \nabla_j \mu - \frac{\bar{\kappa}_o^{ij}}{T} \nabla_j T - \frac{1}{2} \xi_T^{ij} \nabla^k h_{kj},$$
$$\tilde{\tau}^{ij} = -\eta^{ijkl} \nabla_{(k} v_{l)},$$
$$\tilde{u}^i = \xi_\mu^{ij} \nabla_j \mu + \xi_T^{ij} \nabla_j T + \xi_h^{ij} \nabla^k h_{kj},$$
$$\tilde{j}_\varepsilon^i = \tilde{j}_q^i + \mu \tilde{j}^i - h_{ij} \tilde{u}^j + v_j \tilde{\tau}^{ij}. \tag{32}$$

In the absence of background strain and to linear order in the fluid velocity, all the transport matrices would have a trivial index structure and depend on temperature and chemical potential only, for instance, $\sigma_o^{ij} = \sigma_{(o)}(T, \mu) \delta^{ij}$ or $\eta^{ijkl} \nabla_{(k} v_{l)} = -\eta \sigma^{ij} - (2/d) \zeta \partial_k v^k \delta^{ij}$, where we have defined the shear rate tensor $\sigma_{ij} = \nabla_{(i} v_{j)} - (2/d) \nabla_k v^k \delta_{ij}$.

In the presence of background strain, the strain tensor u_{ij} provides an independent rank-2 tensor. This gives rise to new terms in the transport matrices that all take the form[22] $\sigma_o^{ij} = \sigma_{(o)}(T, \mu, u_o) \delta^{ij} + \sigma_{(u)}(T, \mu, u_o) u^{ij}$ in $d = 2$, with some arbitrary dependence on u_o (since $X = u_o$ and $Y = 0$), when evaluated on the background (14). There is more freedom in the viscosity rank-4 tensor, which takes the general form

$$\eta_{ijkl} = 2\eta^{(0)} (\delta_{ik} \delta_{jl} - \tfrac{1}{2} \delta_{ij} \delta_{kl}) + \zeta^{(0)} \delta_{ij} \delta_{kl} + 2\eta^{(u)} (\delta_{ik} u_{jl} - \tfrac{1}{2} \delta_{ij} u_{kl} - \tfrac{1}{2} u_{ij} \delta_{kl} + \tfrac{1}{4} u_m^m \delta_{ij} \delta_{kl}) + 2(\zeta^{(u)} + \bar{\zeta}^{(u)}) \delta_{ij} u_{\langle kl \rangle} + 2(\zeta^{(u)} - \bar{\zeta}^{(u)}) u_{\langle ij \rangle} \delta_{kl}, \tag{33}$$

where the angular brackets stand for the transverse, traceless part of the tensor.

C. Linear response

With the constitutive relations in hand, we can now investigate the linear response of the system about the equilibrium state (denoted with an o subscript). Making use of the underlying translation invariance of the system to decompose the linear perturbations in the plane waves, we take $n^a = n_o^a + \delta n^a e^{-i\omega t + iqx}$ and $\mu^a = \mu_o^a + \delta \mu^a e^{-i\omega t + iqx}$, where n^a are the various conserved densities and μ^a are their conjugate sources. We do not consider a background fluid velocity in this Colloquium.

We start with the transverse sector. In contrast to the fluid case discussed at the end of Sec. II [see also Kovtun (2012)], the transverse Goldstone field mixes with the transverse momentum to form a pair of sound modes propagating in opposite directions:

$$\omega = \pm q \sqrt{\frac{G}{\chi_{\pi\pi}}} - \frac{i}{2} \left(\frac{\eta}{\chi_{\pi\pi}} + G\xi \right) q^2 + O(q^3). \tag{34}$$

Equation (34) is the noted shear sound mode of crystalline solids. Its velocity is real provided that the matrix of static susceptibility is positive definite, which implies that $G > 0$ and $\chi_{\pi\pi} > 0$. The sound attenuation receives two contributions

$$\eta \equiv \eta^{(0)} + \frac{u_o}{2} \eta^{(u)}, \qquad \xi \equiv \frac{1}{1 + u_0} \left(\xi_h^{(0)} + \frac{u_0}{2} \xi_h^{(u)} \right). \tag{35}$$

In Eq. (35), as in the longitudinal sector, we find that the effect of the extra terms in the constitutive relations due to background strain can be hidden away in a redefinition of the transport coefficients contributing to the linear response. This is advantageous, as it means that there is no proliferation of transport coefficients. For instance, the shear Kubo formula that usually measures the shear viscosity for a fluid becomes

[22]In $d > 2$, additional tensor structures such as $u_k^i u^{kj}$ would enter. To map to the formulation of Armas and Jain (2020b), higher-order terms in their strain u^{IJ} need to be considered.

$$\eta^{(0)} + \frac{u_o}{2}\eta^{(u)} \equiv \eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{\tau^{xy}\tau^{xy}}(\omega, q=0), \tag{36}$$

and it is the linear combination (35) that appears, not the individual transport coefficients $\eta^{(0)}$ and $\eta^{(u)}$.

There is a similar Kubo formula for ξ_μ,

$$\xi_\mu = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{j\dot{u}}(\omega, q=0), \tag{37}$$

and for ξ,

$$\xi = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{\dot{u}\,\dot{u}}(\omega, q=0), \tag{38}$$

which as we later see defines the Goldstone diffusivity.

In the longitudinal sector, there are four modes: two sound modes propagating in opposite directions and two diffusive modes. Their expressions are in general quite complicated, so we report them only for a neutral, relativistic system (in which one of the diffusive modes disappears),[23]

$$\begin{aligned}
&\omega = \pm c_\parallel q - \frac{i}{2}\Gamma_\parallel q^2, \qquad \omega = -iD_\parallel q^2,\\
&{c_\parallel}^2 = \frac{B+G}{\chi_{\pi\pi}} + \frac{T^2(s_o - \chi_{\mathrm{sh}_\parallel})^2}{\chi_{\pi\pi}\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon}},\\
&\Gamma_\parallel = \frac{\eta+\zeta}{\chi_{\pi\pi}} + \frac{\xi\chi_{\pi\pi}}{{c_\parallel}^2}\left({c_\parallel}^2 - \frac{T(s_o - \chi_{\mathrm{sh}_\parallel})}{\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon}}\right)^2,\\
&D_\parallel = \frac{(B+G)\chi_{\pi\pi}\xi}{{c_\parallel}^2\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon}}.
\end{aligned} \tag{39}$$

In Eq. (39) $\chi_{\mathrm{sh}_\parallel} = \partial s/\partial\lambda_\parallel = -\partial p_{\mathrm{el}}/\partial T_0$, and similarly $\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon} = c_v T_0$ (with c_v the heat capacity), which is computed fixing $\lambda_\parallel$. As in the transverse sector, only certain linear combinations of transport coefficients appear (such as ξ instead of both $\xi_h^{(0)}$ and $\xi_h^{(u)}$). After matching conventions, these expressions agree with Armas and Jain (2020a, 2020b). The modes can be worked out in full generality (absence of boost symmetry, finite density, nonzero background strain) but become rather complicated. The appearance of instabilities related to a change of sign of $B+G$ is manifest in the expression for $D_\parallel$, as the corresponding purely imaginary mode would then cross to the upper half complex frequency plane.

At low temperatures, we expect the sound modes to be carried mostly by the Goldstone field and the longitudinal momentum density, while the diffusive mode corresponds to thermal diffusion. At temperatures close to the critical temperature, the sound modes are carried by thermal and momentum fluctuations, while the diffusive mode is carried predominantly by the Goldstone mode.[24]

[23]See the Appendix for details on how to take this limit. See also Armas and Jain (2020a) for the complete expressions at finite density.

[24]This could be verified explicitly by deriving the fluctuation eigenvectors of the hydrodynamics equations and also those from the holographic computation of quasinormal modes, as Arean *et al.* (2021) did for a superfluid.

D. Holography

Elastic properties in holographic models with broken translations have been investigated for some time (Alberte, Baggioli, and Pujolas, 2016; Alberte *et al.*, 2016). While hydrodynamics for Galilean-invariant phases with broken translations is an old subject (Martin, Parodi, and Pershan, 1972; Enz, 1974; Fleming and Cohen, 1976; Chaikin and Lubensky, 2000), it was revisited by Delacrétaz *et al.* (2017b), who incorporated phase relaxation by defects and pinning by disorder in a hydrodynamic framework without assuming Galilean invariance. This led to a flurry of activity in the holographic community intent on verifying the match between the holographic and hydrodynamic approaches (Alberte, Ammon, Jiménez-Alba *et al.*, 2018; Amoretti *et al.*, 2018a, 2019a; Ammon *et al.*, 2019, 2020; Baggioli and Grieninger, 2019; Armas and Jain, 2020b, 2020a; Baggioli, Grieninger, and Li, 2020). Ultimately, this lead to a consistent hydrodynamic construction with nonzero background strain and with coupling to external sources (Armas and Jain, 2020a, 2020b).

The presence of an isotropic background strain (equivalently, a background elastic pressure p_{el}) is a common feature of homogeneous holographic models based on massive gravity or Q-lattices,[25] which at an operational level can be directly observed by identifying an extra contribution p_{el} to the relativistic momentum susceptibility $\chi_{\pi\pi} = \varepsilon + p - p_{\mathrm{el}}$. This implies that the states considered in these models are not global (or even local) minima of the holographic thermodynamic free energy (Donos and Gauntlett, 2013b, 2016) (when the free energy is minimized, $p_{\mathrm{el}} = 0$). In spite of this, they are locally thermodynamically stable, with a positive definite static susceptibility matrix. Accordingly, they do not have poles in the upper half complex frequency plane. The low-energy dynamics of these holographic models is also precisely given by the effective theory developed in Secs. IV.A–IV.C (Ammon *et al.*, 2020).

Helical homogeneous or inhomogeneous models do not require background strain, as the free energy can be minimized nontrivially as a function of the modulation wave vector (Donos and Gauntlett, 2011a, 2012a, 2012b, 2016; Donos, 2013; Rozali *et al.*, 2013; Withers, 2013, 2014), but are technically more challenging to work with.

The underlying conformal invariance of holographic models places a number of constraints on the equation of state and transport coefficients since the stress-energy tensor is now traceless ($T^\mu_\mu = 0$). For instance, the transverse and longitudinal speeds of sound obey the following simple relation (Esposito *et al.*, 2017; Armas and Jain, 2020b):

[25]The reason for this is clear: in the bulk, the fluctuations of the massless scalars are dual to the Goldstone modes (Ammon *et al.*, 2019; Amoretti *et al.*, 2019a; Donos and Pantelidou, 2019), and their background ansatz is chosen to be $\phi^i(r, x^i) = m x^i$ in order for homogeneity to be preserved. Their action is a bulk version of the EFT elastic free energy (13) (Nicolis, Penco, and Rosen, 2014; Nicolis *et al.*, 2015), and the bulk ansatz can be directly compared to Eq. (14), leading to a nonzero background strain. If the bulk parameter m is set to zero, translations are no longer broken in any way.

$$
{c_\parallel}^2 = \frac{1}{d-1} + 2\frac{d-2}{d-1}{c_\perp}^2. \tag{40}
$$

Increasing the background strain gives additional contributions to the effective elastic moduli through Eq. (17). Depending on the specific functional dependence on strain, this may lead to thermodynamic instabilities if the effective elastic moduli vanish [this leads in turn to a divergence of the corresponding susceptibilities (18)]. These thermodynamic instabilities have dynamical counterparts, as the transverse sound velocity [Eq. (34)] becomes complex or the longitudinal diffusive mode $D_\parallel$ [Eq. (39)] crosses to the upper half plane. The conjectured end point of this instability is the nucleation of topological defects, which relaxes the background strain and probably leads to plastic behavior and the failure of the rigidity of the system.[26] Viscoelasticity with background strain (or, equivalently, stress) was discussed in multiple engineering-oriented works (Birch, 1938; Biot, 1940, 1965; Hayes and Rivlin, 1969; Destrade and Saccomandi, 2007; Destrade, Ogden, and Saccomandi, 2009; Berjamin and Pascalis, 2021). The onset of instability in the dispersion relation of low-energy modes was experimentally observed by Clatterbuck *et al.* (2003) and Isaacs and Marianetti (2014) and recently reformulated in the context of relativistic effective field theories (Alberte *et al.*, 2019; Pan *et al.*, 2022). These instabilities have not yet been investigated by holographic methods, although Baggioli, Castillo, and Pujolas (2020) took the first steps with a pure shear strain.

In holographic systems, the black hole horizon provides a large bath of $\mathcal{O}(N^2)$ degrees of freedom. It follows that the Goldstone mode can relax into this bath. This is embodied by a modernized version of the "membrane paradigm" (Thorne, Price, and MacDonald, 1986; Damour and Lilley, 2008), whereby transport coefficients characterizing linear response are expressed in terms of the background solution evaluated on the black hole horizon through the construction of radially conserved bulk fluxes (Iqbal and Liu, 2009; Donos and Gauntlett, 2014c, 2015).

This was used to compute the linear, relativistic transport coefficients (32) in holographic models of spontaneously broken translations that are either homogeneous (Amoretti *et al.*, 2018a, 2019a) or inhomogeneous (Donos *et al.*, 2018; Goutéraux, Jokela, and Pönni, 2018). In homogeneous holographic models, it is well understood how to encode quantum critical infrared fixed points with broken translations (Donos, Goutéraux, and Kiritsis, 2014; Goutéraux, 2014b). Near such critical phases, it was observed (Amoretti *et al.*, 2019a) that some of the transport coefficients are not independent and saturate a bound originating from the positivity of the entropy production [Eq. (A9)],

$$
\xi_\mu = -\left(\frac{\mu}{\chi_{\pi j_q}}\right)\sigma_o, \qquad \xi = \left(\frac{\mu}{\chi_{\pi j_q}}\right)^2\sigma_o, \tag{41}
$$

where for relativistic phases $\chi_{\pi j_q} = s_o T_o - p_{\mathrm{el}}$. Effectively, the Goldstone relaxation processes are governed by the incoherent (i.e., without momentum drag) diffusivity σ_o, which also controls the thermal diffusivity with open circuit boundary conditions (Davison, Gentle, and Goutéraux, 2019a; Davison, Goutéraux, and Hartnoll, 2015).

This can be understood as arising from the dominance of the following effective interaction between the momentum and the heat current j_q^i in the infrared Hamiltonian:

$$
\Delta H = \frac{1}{\chi_{\pi j_q}} \int d^d x \pi_i j_q^i. \tag{42}
$$

This in turn implies that

$$
\dot{u}^i = i[H, u^i] = \frac{j_q^i}{\chi_{\pi j_q}}. \tag{43}
$$

Plugging Eq. (43) into the Kubo formulas for ξ_μ and ξ [Eqs. (37) and (38)] and evaluating them leads to Eq. (41).

One may wonder why the specific coupling (42) appears rather than some arbitrary linear combination of the electric and heat currents. It is plausible that this is an artifact of the homogeneous holographic Q-lattice and massive gravity models, where the heat current plays a distinguished role in relaxation processes (Blake, 2015; Donos *et al.*, 2020). Whether this remains true in homogeneous helical models (Donos and Gauntlett, 2012a; Andrade, Baggioli *et al.*, 2018) or in inhomogeneous models (Donos, 2013; Rozali *et al.*, 2013; Withers, 2013) is an open question, although the recent numerical results of Andrade and Krikun (2022) tend to indicate a negative answer. The difference between these models is the Chern-Simons bulk term and the associated breaking of parity, as well as the absence of background strain. A complete match between the hydrodynamics of Secs. IV.A–IV.C and those models is also yet to appear.[27]

E. Emergent higher-form symmetries and topological defects

Ordinary, zero-form symmetries [such as a global U(1)] give rise to conserved, one-form currents (for instance, $\nabla_\mu J^\mu = 0$ in relativistic notation). The associated conserved charges are pointlike objects. Gaiotto *et al.* (2015) pointed out the existence of more general symmetries associated with differential forms of a higher rank. A prototypical example is the U(1) of electromagnetism in four spacetime dimensions. There the Bianchi identity can be reformulated as the conservation equation of a magnetic U(1) symmetry by Hodge dualizing the Maxwell field strength $J^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The charge $Q = \int_\Sigma \star J$ counts the number of magnetic lines across a codimension-2 surface Σ, and its associated

[26]In this sense, this mechanism shares several commonalities with the Landau instability for superfluids triggered by a background superfluid velocity (Lifshitz and Pitaevskii, 1980) and has already been observed in bottom-up holographic models (Amado *et al.*, 2014; Lan *et al.*, 2020).

[27]Probe brane models can also display spontaneous breaking of translations (Jokela, Jarvinen, and Lippert, 2014, 2017a). Because of the presence of an additional long-lived mode (Karch, Son, and Starinets, 2009; Nickel and Son, 2011; Davison and Starinets, 2012; Chen and Lucas, 2017), the hydrodynamics presented does not apply directly to these holographic models.

conserved current is now a two-form $\nabla_\mu J^{\mu\nu} = 0$. Among various applications, this provides a starting point for a consistent formulation of magnetohydrodynamics (Grozdanov, Hofman, and Iqbal, 2017).

A similar treatment can be applied to phases with a spontaneously broken global U(1) symmetry (superfluids). Keeping to relativistic notation, the absence of topological defects (vortices) implies that derivatives commute, $\nabla_{[\mu}\nabla_{\nu]}\phi = 0$, where ϕ here is the superfluid phase. In $2+1$ dimensions, defining $J^{\mu\nu} = \epsilon^{\mu\nu\rho}\nabla_\rho\phi$ leads to an emergent conservation equation for the higher-form symmetry $\mathrm{U}(1)_w$ associated with the conservation of winding planes $\nabla_\mu J^{\mu\nu} = 0$ (Delacrétaz, Hofman, and Mathys, 2020). This emergent symmetry is broken when the theory is coupled to a background gauge field for the microscopic U(1). Indeed, the background gauge field appears as a mixed 't Hooft anomaly on the right-hand side of the conservation of $\mathrm{U}(1)_w$, $\nabla_\mu J^{\mu\nu} = -q\epsilon^{\nu\kappa\lambda}F_{\kappa\lambda}$, where $F_{\kappa\lambda} = \nabla_{[\kappa}A_{\lambda]}$ and the anomaly coefficient q is the charge of the condensate.

Emergent symmetries are often anomalous, and their higher-form generalizations are no exception (Gaiotto *et al.*, 2015; Landry, 2021). Such anomalies give rise to anomaly matching conditions, which put strong constraints on the hydrodynamic gradient expansion. Ultimately, for superfluids they are responsible for the emergence of second sound, whose velocity is proportional to the anomaly coefficient (Delacrétaz, Hofman, and Mathys, 2020), and give rise to dissipationless transport (Else and Senthil, 2021). Preliminary investigations of the higher-form symmetry formulation[28] of phases with spontaneously broken translations were conducted by Armas and Jain (2020b) and Grozdanov, Lucas, and Poovuttikul (2019) but do not include any mixed 't Hooft anomaly. Whether the anomaly-based mechanism for sound modes and dissipationless transport also operates for spacetime symmetries remains to be determined.

In the condensed phase, the winding operators $W_{ij} = \int d^2x \nabla_i u_j$ are conserved and measure elastic deformations of the crystal or density wave. They lead to undamped propagation of uniform bulk and shear strains, for example,[29]

$$\eta(\omega) \equiv \frac{i}{\omega} G^R_{\tau^{xy}\tau^{xy}}(\omega, q=0) = \eta + G\frac{i}{\omega}. \tag{44}$$

This infinite dc "shear conductivity" is the analog of dissipationless charge transport in superfluids.

At finite temperatures, bound pairs of defects or antidefects (dislocations or disclinations) nucleate. Above the Berezinskii-Kosterlitz-Thouless (BKT) temperature, thermal fluctuations lead to their unbinding and they become mobile: the BKT phase transition (Kosterlitz and Thouless, 1973; Halperin and Nelson, 1978; Nelson and Halperin, 1979; Zippelius, Halperin, and Nelson, 1980).[30] Mobile defects relax the windings, and the corresponding emergent symmetry $\mathrm{U}(1)_w$ is explicitly broken.[31] This leads to relaxation of the longitudinal and transverse phonons,

$$\dot{\lambda}_{\parallel,\perp} = -\Omega_{\parallel,\perp}\lambda_{\parallel,\perp} + \cdots. \tag{45}$$

Equation (45) is valid when the anisotropic rates $\Omega_\parallel$ and $\Omega_\perp$ are small, close to the BKT phase transition. The phase relaxation rates are set by the viscosities of the normal phase and the density of free defects n_f, such as $\Omega_\perp \sim n_f/\eta_{\mathrm{normal}}$ (Zippelius, Halperin, and Nelson, 1980).[32] "Climb" motion of dislocations is usually suppressed compared to "glide," i.e., $\Omega_\parallel \ll \Omega_\perp$. In the language of higher-form symmetries, the emergent higher-form symmetry counting winding planes is broken by irrelevant operators (the defects) (Delacrétaz, Hofman, and Mathys, 2020). When Eq. (44) is evaluated again, the viscosities of the condensed phase are finite but large: $\eta(\omega=0) = \eta + G/\Omega_\perp$.

V. PSEUDOSPONTANEOUS BREAKING OF TRANSLATIONS

The total momentum of the full system is always conserved due to the translation invariance of the ambient spacetime in which the crystal lives. Thus, the emergent continuous translation symmetry at long distances in crystalline solids cannot be explicitly broken. In systems at finite density such as metals, the conduction electrons (or more generally the charge carriers at strong coupling) can be considered in some regimes (typically, low enough temperatures) to be weakly coupled to lattice degrees of freedom and other sources of inelastic scattering. The electron momentum then becomes approximately conserved, with an emergent electronic translation symmetry in the infrared broken by irrelevant operators (such as umklapp). Disorder gives rise to elastic scattering and to a residual zero-temperature resistivity, and thus should be weak in order for momentum to remain approximately conserved. In an electronic charge density wave or Wigner crystal phase, electronic translations are spontaneously broken and give rise to a spatially modulated electronic density of states; see Grüner (1988) for a review. New Goldstone degrees of freedom emerge, called phasons, or sometimes phonons by an abuse of terminology (not to be confused with the phonons of the underlying lattice).

It then becomes interesting to study how the previously mentioned weak explicit breaking affects the dynamics of

[28]This differs from the dual formulations of Beekman, Nissinen, Wu, Liu *et al.* (2017) and Beekman, Nissinen, Wu, and Zaanen (2017).

[29]Notice the difference with the Kubo formula in Eq. (36), in which the divergent $1/\omega$ term would not appear.

[30]See Kivelson, Fradkin, and Emery (1998), Mross and Senthil (2012a, 2012b), Beekman, Nissinen, Wu, Liu *et al.* (2017), and Beekman, Nissinen, Wu, and Zaanen (2017) for the quantum case.

[31]The explicit breaking of higher-form symmetries was considered using effective field theory methods by Baggioli, Landry, and Zaccone (2022).

[32]See Delacrétaz *et al.* (2017b) for a memory matrix calculation of these rates.

the Goldstone modes. These acquire both a small mass[33] q_o and a damping Ω, leading to a nonzero real and imaginary part in their $q=0$ dispersion relation, respectively. Phenomenologically, the spontaneous, spatially modulated phase is no longer free to slide and is pinned at a frequency $\omega_o \sim c_s q_o$ that is proportional to the mass of the Goldstone mode, which now has a finite correlation length. Correspondingly, there is a gap in the real part of the frequency-dependent conductivity with a peak at a frequency $\omega \sim \omega_o$, representing the energy cost to depin the density wave.[34]

The pinning of charge density waves is an old subject (Fukuyama and Lee, 1978a; Lee, Rice, and Anderson, 1993) and was confirmed in many experiments on quasi-one-dimensional materials (Grüner, 1988). It was revived in recent years, motivated by a combination of mounting experimental evidence on the role of charge density wave phases or fluctuations across the phase diagram of cuprate high-T_c superconductors (Peng *et al.*, 2018; Arpaia *et al.*, 2019; Lin *et al.*, 2020; Kawasaki *et al.*, 2021; S. Lee *et al.*, 2021; W. S. Lee *et al.*, 2021; Ma *et al.*, 2021; Miao *et al.*, 2021; Tam *et al.*, 2022) [see Arpaia and Ghiringhelli (2021) for a review], theoretical developments on the application of hydrodynamics and related effective field theoretic descriptions of transport to strongly correlated electronic materials (Hartnoll, 2015; Lucas and Sachdev, 2015; Levitov and Falkovich, 2016; Zaanen, 2019), and the development of holographic methods for phases with broken translations.

Following the initial work of Delacrétaz *et al.* (2017b), who incorporated pinning by explicit breaking of translations and damping by defects into a hydrodynamic framework, a number of groups set out to investigate these phases using holographic methods. The original expectation was that these systems would display a pinning frequency ω_o and a momentum relaxation rate Γ but no phase relaxation rate Ω, as none of these holographic models included mobile elastic defects.[35] It then initially came as a surprise when it was recognized that they exhibited a finite phase relaxation rate governed by the pseudo-Goldstone mass and diffusivity $\Omega = G q_o^2 \xi$ (Amoretti *et al.*, 2019b; Donos *et al.*, 2019), with further confirmations given by Ammon, Baggioli, and Jiménez-Alba (2019), Andrade and Krikun (2019), Baggioli and Grieninger (2019), Amoretti, Areán *et al.* (2020), Andrade, Baggioli, and Krikun (2020), and Donos *et al.* (2020).

Note that the assumption of hydrodynamics is not necessary to the existence of a phase relaxation rate $\Omega \sim \omega_o^2$ in the presence of weakly broken translations. A memory matrix approach [see Forster (1975) and Hartnoll, Lucas, and Sachdev (2018) for reviews] suffices (Delacrétaz *et al.*, 2017b). Where hydrodynamics enters is in the determination of the relevant memory matrix element in terms of a diffusive transport coefficient ξ. This belongs to the same class of hydrodynamic relaxation mechanisms giving rise to flux-flow resistance in phase-relaxed superconductors (Bardeen and Stephen, 1965; Davison, Richard *et al.*, 2016) or minimal viscosity scenarios for cuprate strange metals (Davison, Schalm, and Zaanen, 2014; Zaanen, 2019). As we soon elaborate upon, Ω captures the contribution of ungapped excitations to the dc resistivity.

The main theoretical achievement of this collective effort is the construction of a hydrodynamic theory of pseudospontaneously broken translations (Armas, Jain, and Lier, 2021; Delacrétaz, Goutéraux, and Ziogas, 2021), which explains the previous observations and which we now describe. For simplicity, we consider states without background strain throughout this section, but this can be incorporated straightforwardly (Armas, Jain, and Lier, 2021).

A. Hydrodynamics

When translations are weakly broken explicitly, the free energy at quadratic order in fluctuations now includes the following mass term for the Goldstone modes[36]:

$$\delta f^{(2)} = \frac{B+G}{2}(\nabla^i \delta\phi_i)^2 + \frac{G}{2}(\nabla \times \delta\phi)^2 + \frac{G q_o^2}{2}\delta\phi_i \delta\phi^i, \tag{46}$$

which shifts the unpinned static susceptibility matrices $\chi_{o,\parallel}$ and $\chi_{o,\perp}$ as

$$\chi_o^{-1} \mapsto \chi^{-1} = \chi_o^{-1} + \Delta\chi^{-1}, \tag{47}$$

where $\Delta\chi^{-1}$ is a matrix whose only nonzero elements are $(\Delta\chi^{-1})_{\lambda_\parallel \lambda_\parallel} = (\Delta\chi^{-1})_{\lambda_\perp \lambda_\perp} = G q_o^2/q^2$. As a result, the static susceptibility matrix χ becomes nonlocal.

The charge and energy conservation equations in Eqs. (21) remain unchanged. On the other hand, since translations are broken explicitly, momentum is no longer conserved:

$$\dot{\pi}^i + \nabla_j \tau^{ji} = -\Gamma \pi^i - G q_o^2 \delta\phi^i. \tag{48}$$

The Γ term in Eq. (48) is allowed on general grounds and captures momentum relaxation, while the second term encodes the effects of the mass of the Goldstone mode and can be derived by computing $\dot{\pi}^i = i[H, \pi^i]$, including a mass deformation (46) in the Hamiltonian H and using the commutator (23).

The constitutive relations and the Josephson equation can all contain terms linear in ϕ^i without any spatial gradient since the shift symmetry is broken. These terms are constrained by locality and Onsager relations. After imposing these constraints, the constitutive relations and the Josephson equation read

[33]By a similar mechanism that leads to the Gell-Mann–Oakes–Renner (GMOR) relation (Gell-Mann, Oakes, and Renner, 1968) for pion masses in QCD.

[34]If disorder or lattice effects are strong, the density wave is strongly pinned and locked at impurity sites.

[35]Though see Andrade, Krikun *et al.* (2018) and Krikun (2018a) for a holographic construction of phases with static discommensurations.

[36]The mass term can be thought to originate from expanding a $\cos u_i$ deformation of the Hamiltonian of the system to quadratic order in fluctuations, so the u_i are still compact scalars.

$$
\begin{aligned}
j^i &= -\sigma_o \nabla^i \mu - \alpha_o \nabla^i T + \xi_\mu h^i, \\
\frac{\tilde{j}_q^i}{T} &= -\alpha_o \nabla^i \mu - \frac{\bar{\kappa}_o}{T} \nabla^i T + \xi_T h^i, \\
\tau^{ij} &= -\eta \sigma^{ij} - \zeta \nabla \cdot v \delta^{ij}, \\
\dot{\phi}^i &= v^i + \xi_\mu \nabla^i \mu + \xi_T \nabla^i T - \xi h^i,
\end{aligned} \tag{49}
$$

where $h^i = \partial f / \partial \phi^i = G q_o^2 \delta\phi^i - \nabla_j h^{ji}$ and where in the absence of background strain the transport coefficients are no longer matrices. These dissipative corrections ensure that the equations of motion remain local (Delacrétaz, Goutéraux, and Ziogas, 2021) and that the divergence of the entropy current is positive (Armas, Jain, and Lier, 2021). Translating the h^i terms to fields ϕ^i generates new relaxation terms in the constitutive relations and Josephson equations proportional to q_o^2 and various dissipative transport coefficients: ξ_μ, ξ_T, and ξ. For instance, the Josephson equations take the form

$$
\dot{\delta\phi}^i = -\Omega \delta\phi^i + O(\nabla^i), \tag{50}
$$

where the damping term Ω,

$$
\Omega = G q_o^2 \xi, \tag{51}
$$

is universally determined by the Goldstone mass and ξ. The parameter ξ is a diffusive transport coefficient of the translation-invariant theory that enters into the attenuation of sound and diffusive modes of Sec. IV and encodes dissipation of the Goldstone mode in the thermal bath over long distances.

In the framework of effective hydrodynamic theories, Eq. (51) is a direct consequence of locality (Delacrétaz, Goutéraux, and Ziogas, 2021) or the second law of thermodynamics (Armas, Jain, and Lier, 2021) with external sources on.[37]

The damping term Ω is allowed on general grounds, since the shift symmetry of the Goldstone modes is broken by the explicit breaking of translations, without having to assume a hydrodynamic regime. A memory matrix analysis (Delacrétaz *et al.*, 2017b) shows that it is given by the following Kubo formula:

$$
\Omega = G q_o^2 \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G^R_{\partial_t \phi^i \partial_t \phi^i}(\omega, k = 0). \tag{52}
$$

In this approach, the retarded Green's function on the right-hand side of Eq. (52)should be evaluated in the purely spontaneous theory. Using the hydrodynamic form of the retarded Green's function gives back Eq. (51).

In the presence of explicit breaking, Ω captures the relaxation of the pseudo-Goldstone mode in the surrounding bath of thermal excitations. Andrade, Baggioli, and Krikun (2020) showed that in the absence of a gap the time-dependent Ginzburg-Landau equation gives a good account of the dynamics of these systems near T_c. For one-dimensional systems with quasiperfect nesting of the modulation wave vector and gapping of the Fermi surface, the charge density wave formation is described by the Peierls instability (Grüner, 1988). The gap equation is BCS-like, and the density of uncondensed electrons is exponentially suppressed at low temperatures. In this case, there are few thermal excitations that the pseudo-Goldstone mode can relax into and we expect the damping Ω to be suppressed, which explains why it has not been discussed in previous literature (Grüner, 1988). In other words, in the absence of a thermal bath the Goldstone mode is gapped and cannot "leak" to arbitrarily low energies.

Pinning also introduces new relaxation parameters in the constitutive relations for the currents

$$
j^i = n v^i + \Omega_n \delta\phi^i + O(\nabla), \qquad \frac{j_q^i}{T} = s v^i + \Omega_s \delta\phi^i + O(\nabla), \tag{53}
$$

with

$$
\Omega_n = G q_o^2 \xi_\mu, \qquad \Omega_s = G q_o^2 \xi_T. \tag{54}
$$

With translations explicitly weakly broken, the quasinormal modes of the system have both an imaginary and a real gap

$$
\omega_\pm = \pm \sqrt{\frac{G}{\chi_{\pi\pi}}} q_o - \frac{i}{2} (\Gamma + G q_o^2 \xi) + O(q^2, g^3). \tag{55}
$$

In Eq. (55), we have assumed the scaling $q_o \sim g$ and $\Gamma \sim g^2$, where g is the source of the microscopic operator breaking translations explicitly. This assumption can be lifted, and the dispersion relation then takes a more complicated form. Equation (55) makes manifest the damped oscillator behavior of the system, with a pinning frequency $\omega_o \equiv q_o \sqrt{G/\chi_{\pi\pi}}$, and two contributions to the damping rate: $G q_o^2 \xi$ takes a universal form in terms of the parameters of the effective field theory, while Γ does not. The only gapless modes left are two diffusive modes transporting charge and thermal fluctuations. Their expressions, as well as the leading q dependence of the gapped modes, can easily be computed with Eq. (49) in hand, but their expressions are not particularly illuminating, and we leave it to the interested reader to write them down.

Armas, Jain, and Lier (2021) reported extra coefficients when coupling to external sources. Since these terms originate from extra freedom in how currents are coupled to external sources when symmetries are explicitly broken, they appear only in the numerator of retarded Green's functions and do not affect the poles. In particular, they do not affect Eqs. (51) and (54). It is also not currently known how they affect the electric conductivity, which is our primary focus in Sec. V.B. For simplicity, we omit these terms here and refer the interested reader to Armas, Jain, and Lier (2021) for details. This is justified to some extent by the fact that these terms are either absent from or can be redefined away in the holographic models with pseudospontaneous breaking investigated thus far (Ammon, Baggioli, and Jiménez-Alba, 2019;

[37]Analogous relations apply for other symmetry-broken phases, such as superfluids (Ammon *et al.*, 2021; Armas, Jain, and Lier, 2021; Delacrétaz, Goutéraux, and Ziogas, 2021), QCD in the chiral limit (Grossi *et al.*, 2020; Grossi *et al.*, 2021), nematic phases, (anti)ferromagnets (Delacrétaz, Goutéraux, and Ziogas, 2021), and quasicrystals (Baggioli, 2020; Baggioli and Landry, 2020).

Amoretti *et al.*, 2019b; Donos and Pantelidou, 2019; Donos *et al.*, 2019, 2020; Ammon *et al.*, 2021).

B. Charge transport in pinned crystals

In a translation-invariant system at nonzero density, the electric conductivity is infinite in the dc limit $\sigma_{\rm dc} \equiv \sigma(\omega = 0)$. This is because at nonzero density the electric current, which is a fast mode, overlaps with the electronic momentum density, which is conserved. This is manifested in a nonzero cross susceptibility χ_{JP} between the charge and momentum operators. Hence, the electric current cannot relax, which manifests itself as a divergence of the zero frequency conductivity. This can be proven rigorously on general grounds using the memory matrix formalism; see Hartnoll, Lucas, and Sachdev (2018). This continues to be true when translations are spontaneously broken (such as for an electronic charge density wave in a clean system). Using the hydrodynamics equations of Secs. IV.A–IV.C, the conductivity can be obtained from the Ward identity for charge conservation,[38]

$$\sigma(\omega) \equiv \frac{i}{\omega} G^R_{jj}(\omega, q = 0) = \frac{i}{\omega} \lim_{q\to 0} \frac{\omega^2}{q^2} G^R_{nn}(\omega, q), \quad (56)$$

and is found to be

$$\sigma(\omega) = \sigma_o + \frac{n_o^2}{\chi_{\pi\pi}} \frac{i}{\omega}. \quad (57)$$

The $\omega = 0$ pole in the imaginary part is physical and cannot be removed by contact terms. As discussed, its residue is directly proportional to the off-diagonal susceptibility $\chi_{JP} = n_o$, which is identified as the charge density of the system. It gives rise to a delta function in the real part through Kramers-Krönig relations. There is also a finite contribution to the real part, captured by the transport coefficient σ_o. It is always nonzero except in a Galilean-invariant system, where it vanishes as a consequence of the Ward identity for Galilean boosts, $j^i = \pi^i$ (where for simplicity we set the electric charge and particle mass to unity in this formula). Intuitively, it is the contribution to electric transport of "incoherent" processes (meaning those that do not give rise to dissipationless current) (Davison, Goutéraux, and Hartnoll, 2015). It has no equivalent in a simple quasiparticle picture, which is intrisically Galilean invariant. It can be generated in Boltzmann transport by including terms breaking Galilean invariance; see Huang and Lucas (2021). It would also be present in a translation-invariant fluid without Galilean boosts, and there it transports fluctuations of entropy per unit charge $\delta(n/s)$ diffusively (Hartnoll, Lucas, and Sachdev, 2018). (When translations are spontaneously broken, the eigenmode is more complicated due to the coupling to the longitudinal Goldstone mode.)

When translations are explicitly broken, the electronic momentum is no longer conserved. In the regime where it relaxes slowly enough to be kept in the effective field theory as a light mode, the conductivity is strongly modified. It is helpful to first consider the case without spontaneous breaking (Hartnoll *et al.*, 2007). The only relaxation parameter is the momentum relaxation rate Γ, and the electric conductivity becomes[39]

$$\sigma(\omega) = \sigma_o + \frac{n_o^2}{\chi_{\pi\pi}} \frac{1}{\Gamma - i\omega} + \mathcal{O}(\Gamma^0). \quad (58)$$

The $\omega = 0$ pole is now located at $\omega = -i\Gamma$ and is identified with slowly relaxing momentum. In real space, we expect $\langle \pi^i(t)\rangle \sim \pi_0^i e^{-\Gamma t}$. The real part of the conductivity shows a sharp peak centered at zero frequency (the Drude peak), of width Γ and weight $n_o^2/\chi_{\pi\pi}\Gamma$. In the weakly relaxing regime $\Gamma \ll \Lambda$ (with Λ the thermalization scale), the dc conductivity $\sigma_{\rm dc} = \sigma_o + n_o^2/\chi_{\pi\pi}\Gamma \simeq n_o^2/\chi_{\pi\pi}\Gamma$ is large and completely dominated by this "Drude" contribution. The system is a hydrodynamic metal where the electronic momentum relaxes by inelastic scattering off impurities or by umklapp processes.

By contrast, when translations are pseudospontaneously broken, the frequency-dependent conductivity becomes

$$\sigma(\omega) = \sigma_o + \frac{(n_o^2/\chi_{\pi\pi})(i\omega - \Omega) + 2n_o\Omega_n + (\Omega_n^2/\omega_o^2)(\Gamma - i\omega)}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_o^2}. \quad (59)$$

Compared to the case without a spontaneous breaking of translations, we observe new contributions to inelastic scattering that are proportional to q_o^2 and contained in the ω_o, Ω, and Ω_n terms. The line shape interpolates between a Lorentzian centered at ω_o when Ω and Ω_n can be neglected [matching previous hydrodynamic treatments of the collective zero mode (Grüner, 1988)] and a Drude-like peak centered at $\omega = 0$ when the damping rates become more important, as illustrated in Fig. 2; for the original argument see Delacrétaz *et al.* (2017a). In the work of Grüner (1988) and experimental references therein, the focus was on low temperatures. It would be interesting to examine Eq. (59) to experimental data at higher temperatures, where ungapped degrees of freedom become non-negligible.

The dc conductivity

$$\sigma_{\rm dc} = \sigma_o + \frac{(n_0^2/\chi_{\pi\pi})\Omega - 2n_o\Omega_n - (\Omega_n^2/\omega_o^2)\Gamma}{\Gamma\Omega + \omega_o^2} \quad (60)$$

is nonvanishing due to the nonzero symmetry-breaking terms Ω and Ω_n and to the "non-Galilean" transport coefficient σ_o. Previous hydrodynamic treatments [see Grüner (1988)] usually assume the Galilean limit, where the coefficients σ_o, ξ_μ, and consequently Ω_n would be zero, but did not account for Ω

[38] While in hydrodynamics the continuity equation is a dynamical equation for the time evolution of VEVs of operators, the Ward identity is a consequence of the U(1) symmetry and is more fundamental. It is an operator equation that can be used in Green's functions.

[39] Equation (58) is really correct only to order $\mathcal{O}(1/\Gamma)$. Generally, susceptibilities will receive $\mathcal{O}(\Gamma)$ corrections that need to be included in order to consistently describe the dc conductivity to order $\mathcal{O}(\Gamma^0)$ (Davison and Goutéraux, 2015a).

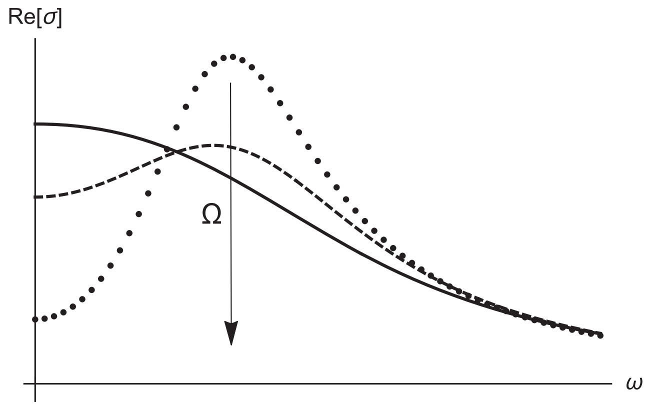

FIG. 2. Schematic representation of the ac conductivity (59) in the Galilean limit when Ω is dialed from large (solid line) to small (bullets), keeping all other parameters fixed. The transfer of spectral weight from the zero frequency Drude-like peak to the off-axis peak is evident.

in the dynamics of the collective mode. Here in the Galilean limit (setting $n_o = ne$ and $\chi_{\pi\pi} = mn$, where e is the electron unit charge, n is the density, and m is the mass), the resistivity $\rho \equiv 1/\sigma\ (\omega = 0)$ is

$$\rho_{\text{Galilean}} = \frac{m}{ne^2}\Gamma + \frac{1}{(ne)^2\xi}. \tag{61}$$

In other words, we do not expect a translation-broken phase such as a charge density wave to be necessarily insulating: the inelastic scattering of the Goldstone mode into the bath of thermal excitations provides a conduction channel. In electronic charge density wave materials, such as those reviewed by Grüner (1988), the Fermi surface may only be partially gapped in the charge density wave phase. The second term in Eq. (61) captures the inelastic scattering of the Goldstone mode into these uncondensed electrons. Instead, if the Fermi surface is fully gapped, $\Omega = 0$ and the collective mode does not contribute to dc transport, as found by Grüner (1988). In other words, in the absence of gapless thermal excitations, there is a finite energy cost to make the density wave slide.

The resistivity (61) takes a Drude-like form with a "transport scattering rate"

$$\frac{1}{\tau_{\text{tr}}} = \Gamma + \frac{1}{(mn)\xi}. \tag{62}$$

In general, this picture is misleading, as there is no single pole located at $\omega = -i/\tau_{\text{tr}}$ as in the Drude model (58). Rather, both poles in Eq. (55) give important contributions to the line shape. When the Goldstone damping rate is large compared to Γ and ω_o, the poles are located at

$$\omega_- = -i\left(\Omega - \frac{\omega_o^2}{\Omega} + \cdots\right), \qquad \omega_+ = -i\left(\Gamma + \frac{\omega_o^2}{\Omega} + \cdots\right), \tag{63}$$

where the ellipses denote subleading $1/\Omega$ corrections. The ω_- pole recedes deeply into the lower half plane and drops out of the effective theory, while the ω_+ pole remains long-lived. Accordingly, the ac conductivity becomes Drude-like, as along the blue solid line in Fig. 2. Amoretti *et al.* (2019b) determined that this process takes place at low temperatures.

Above T_c, we also expect to recover a Drude-like conductivity, as in Eq. (58), that is dominated by a single pole $\omega \simeq -i\Gamma$. Andrade, Baggioli, and Krikun (2020) showed that this occurs through a vanishing of the residue of the ω_- pole, while Ω remains finite through the phase transition. Above T_c, there is no condensate but Ω captures the fluctuations of the order parameter.

Moving away once again from the Galilean limit, the dc conductivity (60) no longer depends on the explicit symmetry-breaking parameter q_o after inserting Eqs. (51) and (54). At low temperatures, inelastic scattering off impurities is expected to dominate the momentum relaxation rate and does not contribute any temperature dependence $\Gamma \sim T^0$. The primary temperature dependence of the resistivity then originates from incoherent scattering processes encapsulated in the diffusive transport coefficients σ_o, ξ_μ, and ξ, in sharp contrast to metallic phases. In a metal, extrinsic processes dominate the resistivity through the scattering rate ($\rho_{\text{dc}} \sim \Gamma$), while when translations are pseudospontaneously broken, intrinsic ones dominate.

C. Holography

Pseudospontaneous breaking of translation symmetry has been implemented in several holographic models in recent years (Ling *et al.*, 2014; Amoretti *et al.*, 2017; 2019b; Jokela, Jarvinen, and Lippert, 2017b; Alberte, Ammon, Baggioli *et al.*, 2018; Andrade, Baggioli *et al.*, 2018; Andrade, Krikun *et al.*, 2018; Ammon, Baggioli, and Jiménez-Alba, 2019; Andrade and Krikun, 2019; Baggioli and Grieninger, 2019; Donos and Pantelidou, 2019; Li and Wu, 2019; Amoretti, Areán *et al.*, 2020; Andrade, Baggioli, and Krikun, 2020; Donos *et al.*, 2020). Regardless of the concrete model at hand, this limit is always achieved by introducing on top of the purely spontaneous state a small space-dependent source for a boundary operator that is therefore responsible for the explicit translation symmetry breaking.[40]

This body of work firmly established the validity of the previously presented hydrodynamic theory of pseudophonons,[41] and more specifically of Eq. (51). Recent works (Armas, Jain, and Lier, 2021; Delacrétaz, Goutéraux, and Ziogas, 2021) further confirmed that this relation is not an artifact either of the homogeneity of the holographic models used or of the large N limit inherent in the holographic approach. Various works also established a GMOR-like relation between the mass of the pseudo-Goldstone mode, the condensate, and the source of explicit breaking (Andrade, Baggioli *et al.*, 2018; Ammon, Baggioli, and Jiménez-Alba, 2019; Amoretti *et al.*, 2019b; Andrade and Krikun, 2019;

[40]In some of the examples, this boundary operator is the same one that breaks translations spontaneously.

[41]For the reasons mentioned in footnote 27, the hydrodynamics of probe brane setups are of a different nature.

Baggioli and Grieninger, 2019; Li and Wu, 2019; Andrade, Baggioli, and Krikun, 2020; Wang and Li, 2021).[42]

Holographic models can easily account for phases that are either insulating, where the resistivity diverges toward low temperatures (Andrade, Baggioli *et al.*, 2018; Andrade, Krikun *et al.*, 2018), or metallic (Amoretti *et al.*, 2019b), with a vanishing resistivity at low temperatures. The former case is in some respects more similar to conventional charge density wave systems, in the sense that a gap forms and the damping rate Ω does not make a large contribution to the dc conductivity, as evidenced by the negligible value of the dc conductivity compared to the height of the off-axis peak in the ac conductivity (Andrade and Krikun, 2019). An important difference is that the gap is algebraic, and the resistivity diverges like a power law. In the helical, homogeneous setup of Andrade and Krikun (2019), this scaling is rooted in the critical behavior of the infrared geometry in the near-horizon, near-extremal limit. Indeed, as is well known in holographic models, such critical geometries leave a strong imprint on the scaling of transport observables at low temperatures (Donos, Goutéraux, and Kiritsis, 2014; Goutéraux, 2014b). It is surprising then that the resistivity continues to scale in the inhomogeneous construction of Andrade, Krikun *et al.* (2018) even though there is no evidence thus far of scaling behavior in the geometry. A better understanding of this result remains unobtained.

In the metallic case, an inverse transfer of spectral weight is observed (Amoretti *et al.*, 2019b) as the off-axis peak in the ac conductivity smoothly interpolates back to a Drude-like peak at zero frequency upon lowering the temperature, as depicted in Fig. 2. This is accompanied by a nontrivial motion of the poles in the lower half complex frequency plane. At low enough temperature, the poles are once again purely imaginary and the width of the Drude-like peak is controlled by the pole closest to the real axis. Its partner quickly recedes down the axis and becomes incoherent. Whether this behavior can be reproduced in a more realistic, inhomogeneous state is not known. Nonetheless, it bears resemblance to what is observed experimentally in cuprate high-T_c superconductors and many other strongly correlated materials, as described in Sec. VII.

Given that $\Omega = G q_o^2 \xi$ and $\Omega_n = G q_o^2 \xi_\mu$, the same effective interaction as we described in Sec. IV.D around Eq. (42) operates near homogeneous holographic quantum critical phases with pseudospontaneously broken translations. This further implies that the low-temperature resistivity is controlled by a single diffusive transport coefficient σ_o of the clean state, with subleading contributions from explicit symmetry breaking (assuming disorder and umklapp processes to be irrelevant and/or to contribute no significant temperature dependence to the momentum relaxation rate), $\rho_{\rm dc} \simeq (sT/\mu n_o)^2/\sigma_o + \mu\Gamma/n_o$. As the transport coefficient σ_o can be computed in terms of data at the black hole horizon, it is sensitive to the scaling properties of the low-temperature critical phase, and hence so is the resistivity. This does not suffice to explain the results of Andrade, Krikun *et al.* (2018) but resonates with the scaling form of the low-temperature resistivity uncovered there.

Recently Andrade and Krikun (2022) numerically investigated the thermoelectric ac conductivities in helical and inhomogeneous models and found that the numerical data can be well fitted to the hydrodynamic formulas. Their fit allowed them to determine the transport coefficients σ_o, ξ_μ, and ξ, which they found do not obey Eqs. (41). The models that they used break parity due to the presence of Chern-Simons terms in the bulk, and their ground states had different critical properties than homogeneous Q-lattices as well as no background strain. The essential feature giving rise to Eqs. (41) remains an open question, especially since similar relations appear to hold in experimental realizations of Wigner solids; see Sec. VI.

In underdoped cuprates, the Hall (the dc electric transverse response in a magnetic field) and Seebeck coefficients (the dc electric response to a temperature gradient) change sign at low temperatures (Badoux *et al.*, 2016; Collignon *et al.*, 2021), which is usually attributed to the reconstruction from a large, holelike Fermi surface to small, electronlike pockets (Doiron-Leyraud *et al.*, 2007; Vignolle *et al.*, 2008; Yelland *et al.*, 2008) due to the formation of a charge density wave (CDW). An important outcome of the analysis given by Andrade and Krikun (2022) is that the Seebeck coefficient changes sign at low temperatures, without the presence of a Fermi surface or a reconstruction thereof.

When the spontaneous spatially modulated structure is coupled to an explicit lattice, one would expect their periodicities to become commensurate for a sufficiently large lattice strength. This phenomenon is beyond homogeneous constructions (Andrade and Krikun, 2016). Instead, more realistic inhomogeneous constructions display commensurability effects (Andrade and Krikun, 2017). The black hole horizon is strongly spatially modulated by the spontaneous structure, which is weak in the ultraviolet near the boundary (since it is not sourced) but important in the infrared. The explicit lattice is strong in the ultraviolet but irrelevant (weak) in the infrared. The commensurability that develops between these two structures is a reflection of a strong UV-IR mixing upon increasing the UV lattice strength and turns the system into a Mott insulator (Andrade, Krikun *et al.*, 2018), albeit with an algebraic rather than exponential gap and reminiscent of underdoped cuprates.

VI. MAGNETIC FIELDS

External magnetic fields are a valuable experimental probe in the study of strongly correlated electronic phases of matter. They are particularly important in the context of two-dimensional systems in which they produce new physical phenomena (Chen, 2005). Moreover, the interplay between translational symmetry breaking and the presence of an external magnetic field results in a complex structure of low-energy excitations, including the appearance of a type-II Goldstone boson with quadratic dispersion $\mathrm{Re}(\omega) \sim k^2$: the magnetophonon. This mode arises from the hybridization of the original longitudinal and transverse phonons into a gapless magnetophonon and a gapped

[42]The GMOR relation itself was shown to hold in holographic QCD models [see Erlich *et al.* (2005)] and in holographic models where a U(1) global symmetry is pseudospontaneously broken (Argurio *et al.*, 2016).

magnetoplasmon, which is now allowed because of the time-reversal symmetry breaking induced by the magnetic field (Watanabe and Murayama, 2014). As a consequence, the original Goldstone modes are no longer independent: $[\phi_i, \phi_j] \neq 0$. Following the Watanabe-Brauner argument (Watanabe and Brauner, 2011; Watanabe and Murayama, 2012; Hidaka, 2013), the number of Goldstone modes is reduced and their dispersion converted into a quadratic type.

Early accounts of the dynamics of two-dimensional pinned charge density waves in the presence of an external magnetic field were given by Fukuyama and Lee (1978b) and Normand, Littlewood, and Millis (1992). Their hydrodynamics were recently revisited by Delacrétaz *et al.* (2019), Amoretti, Arean, Brattan, and Magnoli (2021), and Delacrétaz, Goutéraux, and Ziogas (2021). In the presence of pinning and a magnetic field, new relations of the type of Eq. (51) arise.

Delacrétaz *et al.* (2019) considered the match between the hydrodynamic ac conductivity and an experimental measurement in GaAs heterojunctions (Chen, 2005; Chen *et al.*, 2006, 2007), in which Wigner crystallization occurred at large enough magnetic fields in between quantum Hall plateaux.[43] The conductivity is characterized by the peak frequency ω_{pk}, the magnetophonon damping rate Ω, and an extra asymmetry parameter a compared to the case without a magnetic field. Topological defects and pinning are expected to give independent contributions to the magnetophonon damping rate Ω. These rates can be computed using the memory matrix formalism. For a defect-dominated phase the ratio $\Omega/\omega_{\mathrm{pk}} a = 2$, while for a disorder-dominated phase where the magnetophonon predominantly relaxes into the electric current the ratio $\Omega/\omega_{\mathrm{pk}} a = 1$. These values seem to account well for fits to the experimental results at low temperatures or with strong magnetic fields. In the disorder-dominated case, the relaxation mechanism into a hydrodynamic current is reminiscent of the analogous mechanism in holographic systems discussed in Secs. IV.D and V.C.

From the holographic perspective, the introduction of an external magnetic field in homogeneous models with broken translations was considered by Baggioli, Grieninger, and Li (2020), Amoretti, Arean, Brattan, and Martinoia (2021), and Donos, Pantelidou, and Ziogas (2021). A full holographic calculation of all linear transport coefficients combined with matching to the hydrodynamic dispersion relation for the modes has not yet been performed.

VII. TRANSPORT IN STRANGE METALS AND PSEUDOSPONTANEOUS BREAKING OF TRANSLATIONS

Can the physics of pseudospontaneous breaking of translations shed light on the phenomenology of high-T_c superconductors, in particular, on their strange metallic phase? Transport experiments famously measure a resistivity linear in temperature (Gurvitch and Fiory, 1987) that extends for optimally doped samples from above room temperature to the lowest temperatures experimentally available when a magnetic field suppresses superconductivity. This observation brings two important puzzles. The absence of resistivity saturation at high temperatures violates the Mott-Ioffe-Regel bound (Gunnarsson, Calandra, and Han, 2003; Hussey, Takenaka, and Takagi, 2004) and precludes any notion of quasiparticle-based transport, calling for other descriptions of transport in systems with short-lived excitations (Hartnoll, 2015). Charge transport in conventional metals with long-lived quasiparticles is often analyzed with the Drude model. Applying this framework to the resistivity of strange metals identifies a "Planckian" scattering rate (Bruin *et al.*, 2013), which on theoretical grounds can be argued to be the shortest relaxation timescale consistent with Heisenberg's uncertainty principle (Zaanen, 2004; Sachdev, 2011); see Hartnoll and Mackenzie (2021) for a recent review on Planckian dissipation in metals and bounds on transport.

At low temperatures, the persistence of a T-linear component to the resistivity over a range of dopings (Cooper *et al.*, 2009; Hussey *et al.*, 2011, 2013; Legros *et al.*, 2019; Putzke *et al.*, 2021) clashes both with Fermi liquid predictions of a T^2 resistivity, which is only fully recovered beyond the superconducting dome for overdoped samples, and with conventional expectations of transport in the vicinity of a quantum critical point (Sachdev, 2011), where quantum critical behavior is not expected outside the quantum critical cone.

The slope of the T-linear resistivity appears to be of the same order of magnitude across different materials (Legros *et al.*, 2019), which hints at a universal mechanism underpinning this phenomenon. Extrapolations of the resistivity to zero temperature show that the disorder of the sample does not play an important role, with values of the residual resistivity sometimes varying over an order of magnitude or more across materials. Further evidence of disorder independence comes from ion-irradiation experiments (Rullier-Albenque *et al.*, 1995, 1997; 2000; Rullier-Albenque, Alloul, and Tourbot, 2003), which show that resistivity curves simply shift upward when disorder is increased without any change in the slope of the T-linear component.

Transport experiments also report a T^2 cotangent of the Hall angle (Chien, Wang, and Ong, 1991) and a magnetoresistance linear in the magnetic field at large field over a range of dopings (Hayes *et al.*, 2016; Giraldo-Gallo *et al.*, 2018; Ayres *et al.*, 2021). This is once again at odds with quasiparticle-based transport and the Boltzmann equation, which predicts that the resistivity and the Hall angle are controlled by the same transport timescale, and that the magnetoresistance is quadratic in field. Instead, the different temperature dependencies of the resistivity and Hall angle are often interpreted in a two-timescale scenario (Anderson, 1991; Coleman, Schofield, and Tsvelik, 1996a, 1996b). More generally there is some experimental support for two sectors contributing to transport, one coherent and the other incoherent (Licciardello *et al.*, 2019; Ayres *et al.*, 2021; Culo *et al.*, 2021). Transport experiments in overdoped cuprates are often analyzed using the Boltzmann equation. Angle-dependent magnetoresistance experiments allow one to infer the quasiparticle scattering

[43] One may question whether these systems are truly in a hydrodynamic regime. Here we observe that in fact a memory matrix analysis is enough to establish the expressions for the ac conductivities (Delacrétaz *et al.*, 2017b), so the hydrodynamic assumption is not in the narrow sense needed; it is required only that translations are explicitly weakly broken.

rates [with qualitatively different results for different materials (Abdel-Jawad *et al.*, 2006; Grissonnanche *et al.*, 2021)] but do not always allow one to reproduce in-plane transport experiments (Ayres *et al.*, 2021). Thus, even on the overdoped side, the validity of Boltzmann transport is not entirely obvious.

Turning to optics, the ac in-plane conductivity in the strange metal regime above the temperature at which superconductivity sets in is Drude-like, with a peak centered at zero frequency and a width of the order of T. At higher temperatures, a number of compounds reveal a transfer of spectral weight and the zero frequency peak moves off axis to a nonzero frequency (Hussey, Takenaka, and Takagi, 2004; Delacrétaz *et al.*, 2017a). The ac conductivity also features an infrared contribution (Uchida *et al.*, 1991; Quijada *et al.*, 1995, 1999) extending beyond the peak, which scales as $|\sigma(\omega)| \sim \omega^{-2/3}$ (van der Marel *et al.*, 2003; Hwang, Timusk, and Gu, 2007). This is weaker than the expected Drude scaling $|\sigma(\omega)| \sim \omega^{-1}$. This resonates with the two-component analysis of transport. However, fits to optics data typically assume that the dc conductivity solely originates from the Drude component, ascribing a frequency dependence to the infrared component that vanishes as $\omega \to 0$. Recently van Heumen *et al.* (2022) vindicated this picture in a careful study of optics across a range of dopings in single-layer bismuth strontium calcium copper oxide. From a theoretical perspective, there is tension between assuming a gapless, scaling contribution decaying as some power of frequency for frequencies $\omega \gtrsim T$ but that would not produce a corresponding decaying power of temperature in the regime $\omega \lesssim T$, as ω/T scaling would dictate and as seems to hold well experimentally (Michon *et al.*, 2022; van Heumen *et al.*, 2022). It would be interesting to investigate to what extent this constraint in fitting optics can be relaxed and cross-referenced to dc transport data.

These experimental facts pose an immediate conundrum when one attempts to interpret them in the framework of a metal with slowly relaxing momentum. The ac conductivity at not too high temperatures suggests that a Drude analysis may work but fails to account for the appearance of an off-axis peak at higher temperatures or for the infrared non-Drude contribution. The ac conductivity of a slowly relaxing metal is given in Eq. (58). If momentum relaxes weakly, Γ is small compared to some parameter determining the scale at which other degrees of freedom start to be important, usually temperature. But this theoretical assumption contradicts the experimental observation that $\Gamma \sim \mathcal{O}(T)$. The dc conductivity should be dominated by the "coherent" contribution from the Drude peak $\sigma_{\rm coh} \sim n_o^2/\chi_{\pi\pi}\Gamma + \mathcal{O}(\Gamma^0)$. Γ strongly depends on disorder strength (Hartnoll, Lucas, and Sachdev, 2018), which contradicts the experiments where disorder is varied by ion irradiation that was just referred to.

While experimentally difficult to establish, the notion of coherent and incoherent charge transport in a slowly relaxing metal is easy to understand from a theoretical standpoint. All that is required is to give up Galilean invariance, which imposes that the electric current is equal to the momentum density, thereby killing any incoherent contribution to transport. Doing so, new processes are allowed that conduct charge but do not drag momentum and are encapsulated in the appearance of the transport coefficient σ_o in the dc conductivity (58). These processes naturally appear in hydrodynamics (Davison, Goutéraux, and Hartnoll, 2015), memory matrix approaches (Lucas and Sachdev, 2015; Hartnoll, Lucas, and Sachdev, 2018), and holographic models (Davison and Goutéraux, 2015a).

Relaxing Galilean invariance is not enough, though, as in a metal with slowly relaxing momentum such incoherent processes inevitably give contributions to transport (of the order of Γ^0) that are subleading compared to the coherent contribution (of the order of $1/\Gamma$). There are several avenues one can think of to suppress the coherent contribution to transport.

(i) Suppress the Drude weight through some emergent particle-hole symmetry that would effectively set $n_o = 0$.
(ii) Assume strong explicit breaking of translations.
(iii) More radically, require that $\chi_{\pi\pi} \to +\infty$ (Else and Senthil, 2021).
(iv) Short-circuit the large contribution from slowly relaxing momentum by assuming that translations are spontaneously broken (Delacrétaz *et al.*, 2017a).

Strange metals arise in doped Mott insulators, which leads one to disregard (i) (in contrast to the example of graphene near the charge neutrality point). The ability to synthesize clean samples with a low residual resistivity (Giraldo-Gallo *et al.*, 2018) also works against (ii). (iii) was recently considered by Else and Senthil (2021), who argued that strange metals arise in the vicinity of an ordered phase where the order parameter has the same symmetries as the loop currents (Varma, 1999, 2006) and that this would lead to the divergence of all susceptibilities in the same symmetry sector. Note that holographic checkerboards (Withers, 2014; Donos and Gauntlett, 2016; Cai *et al.*, 2017) naturally feature such current loops intertwined with translation symmetry breaking thanks to the bulk Chern-Simons terms. Here we note that fluctuations of the loop current order parameter have been put forward as the origin of the T-linear resistivity in the strange metallic phase (Varma, 2020), as fermions scattering off them have a marginal Fermi-liquid-like self-energy (Varma *et al.*, 1989). The Sachdev-Ye-Kitaev model (Chowdhury *et al.*, 2021), where a large number of species N of fermions is introduced together with random interactions, provides a consistent theoretical framework realizing the marginal Fermi liquid self-energy. The flavor randomness and large N limit make the computation of transport properties tractable. In its simplest incarnation, the T-linear term in the resistivity is perturbatively small; see also previous works on non-Fermi liquids without flavor randomness (Hartnoll *et al.*, 2014; Patel and Sachdev, 2014). Recently random (in flavor and real space) Yukawa-type couplings to a gapless boson[44] were shown to give rise to a T-linear term that is $\mathcal{O}(1)$ in the strength of spatial disorder (Patel *et al.*, 2022). The gapless boson represents the fluctuations of an order parameter at zero or nonzero wave vector. This suggests that the interplay between disorder and order parameter fluctuations might play

[44]Theories of non-Fermi liquids where fermions couple to a gapless boson have a long history; see Lee (2018) for a review.

an important role in understanding strange metals. Whether this T-linear component can arise over a range of dopings remains an open question.

We now consider (iv) how pseudospontaneous translation symmetry breaking may shed light on transport in strange metals. Further motivation for this was found in recent x-ray scattering reports of charge density fluctuations across the phase diagram (Peng *et al.*, 2018; Arpaia *et al.*, 2019; Lin *et al.*, 2020; Arpaia and Ghiringhelli, 2021; Kawasaki *et al.*, 2021; S. Lee *et al.*, 2021; W. S. Lee *et al.*, 2021; Ma *et al.*, 2021; Miao *et al.*, 2021; Tam *et al.*, 2022), rather than restricted to the underdoped regime as previous experiments suggested (Keimer *et al.*, 2015). Besides charge density fluctuations at high temperatures, long-range, or at least longer-ranged than their underdoped counterparts, CDWs have now been reported on three different materials (Peng *et al.*, 2018; Miao *et al.*, 2021; Tam *et al.*, 2022). Thus, the strange metallic phase at optimal doping appears to be the only region where static CDW order has not yet been discovered. The ubiquitousness of high-temperature charge modulations is backed up by numerical (determinant quantum Monte Carlo) studies of the Hubbard model that also report intertwined charge and spin stripes at optimal doping and in the overdoped regime (Huang *et al.*, 2022). Theoretical arguments on the impact of fluctuating charge density wave order on strange metal transport were given by Caprara *et al.* (2017), Delacrétaz *et al.* (2017a, 2017b), Amoretti *et al.* (2019b), Delacrétaz, Goutéraux, and Ziogas (2021), and Seibold *et al.* (2021); see also Kivelson, Fradkin, and Emery (1998), Taillefer (2010), Mross and Senthil (2012a, 2012b), where the emphasis is more on the underdoped range.

We first discuss the ac conductivity in a pinned crystal [Eq. (59)]. It is straightforward to see that the frequency dependence deriving from Eq. (59) interpolates between a Drude-like peak centered at $\omega = 0$, if pinning q_o is sufficiently weak compared to the typical frequency scales set by Γ and Ω, and an off-axis peak once pinning becomes stronger. A precise inequality can be derived from Eq. (59), asking when all maxima in $\mathrm{Re}\sigma(\omega)$ are for $\omega = 0$ or complex frequencies. This is as far as effective approaches can take us since to determine how the frequency dependence of the conductivity varies in any given system requires a microscopic calculation or an experimental measurement. In gauge-gravity duality models, the peak can remain off axis at all temperatures in the ordered phase (Andrade, Baggioli *et al.*, 2018; Andrade, Krikun *et al.*, 2018; Andrade and Krikun, 2019; Andrade, Baggioli, and Krikun, 2020), or interpolate between being on axis and off axis (Amoretti *et al.*, 2019b; Donos and Pantelidou, 2019). In spectroscopic experiments, whether an off-axis peak develops at high temperatures seems material dependent; materials where this behavior is seen have been compiled by Delacrétaz *et al.* (2017a). In $YBa_2Cu_4O_8$, the ac conductivity interpolates from Drude-like to an off-axis peak upon Zn disordering (Basov, Dabrowski, and Timusk, 1998). This is in qualitative agreement with charge transport in the pseudospontaneous regime since stronger disorder will lead to an increase in the pseudo-Goldstone mass q_o and in the pinning frequency ω_o. It would be interesting to better understand the effects of Zn doping on pinning charge density wave fluctuations in scattering experiments (Suchaneck *et al.*, 2010; Guguchia *et al.*, 2017; Lozano *et al.*, 2021), especially in light of the results given by Arpaia *et al.* (2019).

Turning now to dc transport, by looking at the dc conductivity of a pinned crystal [Eq. (60)] alone it is hard to disentangle the individual contributions of various scattering processes.[45] This said, we can distinguish between two types of processes.

- First are extrinsic processes encapsulated in the momentum relaxation rate Γ. It is through this relaxation coefficient that disorder or umklapp processes feed in the dc conductivity. Their scaling is expected to be sensitive to irrelevant deformations and to the details of the disorder distribution, leading to scattering rates $\Gamma_{\mathrm{ext}} \sim T(g/T^{\Delta_g})^2 \ll T$ (Hartnoll and Hofman, 2012; Lucas, Sachdev, and Schalm, 2014; Davison, Gentle, and Goutéraux, 2019b). For this reason, it is unlikely that they are the origin of the T-linear resistivity.
- Second are intrinsic processes, coming from dissipation into the bath of thermal, critical excitations, that are encapsulated in transport coefficients such as σ_o, ξ_μ, and ξ.

  These are much stronger candidates as the source of T-linear resistivity. Gauge-gravity duality allows one to easily calculate these transport coefficients and verify that their temperature dependence indeed reflects the scaling properties of the underlying critical phase (Davison, Goutéraux, and Hartnoll, 2015; Davison, Gentle, and Goutéraux, 2019a, 2019b). These results have inspired scaling theories to explain transport data in cuprates, such as those given by Hartnoll and Karch (2015) and Karch, Limtragool, and Phillips (2016). A crucial extra ingredient compared to previous attempts at a scaling theory (Phillips and Chamon, 2005) is the introduction of anomalous scaling dimensions for the charge density at the critical point (Goutéraux, 2014a, 2014b; Karch, 2014; La Nave, Limtragool, and Phillips, 2019).[46]

As emphasized in Sec. V.B, introducing pseudospontaneous breaking of translations short-circuits the extrinsic contribution to the resistivity, which is now $\rho_{\mathrm{dc}} \sim \mathcal{O}(\Gamma^0)$ rather than $\mathcal{O}(1/\Gamma)$ in a metal. The order $O(\Gamma^0)$ terms are determined by σ_o, ξ_μ, and ξ, are intrinsic, and are dominant against

[45] See Amoretti, Meinero *et al.* (2020) for an attempt at fitting the hydrodynamic theory of pinned charge density waves in a magnetic field to magnetotransport data in a cuprate. While this analysis has the merit of fitting a consistent set of data on a single material, the set of data used does not allow one to unambiguously determine all the parameters in the effective theory.

[46] Holographic models combining explicit breaking of translations and these new scaling laws met with difficulties (Davison, Schalm, and Zaanen, 2014; Blake and Donos, 2015; Amoretti *et al.*, 2016; Blauvelt *et al.*, 2018), including matching all scaling laws and/or suppressing the coherent, extrinsic contribution to the conductivity from momentum without resorting to strong explicit breaking. This task is made harder by the experimental hurdle of producing thermoelectric transport data displaying clean scaling laws over sufficiently large ranges of temperature. How such scaling theories extend to pseudospontaneously broken translations has not been investigated.

the extrinsic $O(\Gamma)$ terms. From Eqs. (59) and (60), it is evident that they contribute both to the coherent (the peak) and to the incoherent (the infrared band) parts of the conductivity. This gives one further motivation to revisit the two-component analysis of ac conductivity data, which customarily assumes that the infrared band does not contribute to the dc conductivity.

Heavily overdoped, nonsuperconducting cuprates feature a purely T^2 resistivity, while the T-linear component turns on at the onset of superconductivity, turning gradually stronger until the critical doping where the resistivity is purely T linear. CDW order has not been reported for nonsuperconducting samples and thus far does not extend all the way to the edge of the superconducting dome for all superconducting overdoped samples (Tam *et al.*, 2022). Plots of the derivative of the resistivity with respect to temperature in overdoped $La_{2-x}Sr_xCuO_4$ and Tl2201 show a gradual change of slope below about 250 K from a high-temperature, T-linear incoherent bad metallic behavior to a low-temperature $T + T^2$ behavior (Hussey *et al.*, 2011, 2013; Putzke *et al.*, 2021) for all overdoped samples, including those where x-ray experiments do not find a static CDW (Tam *et al.*, 2022). Whether the change in the resistivity slope can be more precisely connected to the onset of charge density fluctuations and static CDW order at lower temperatures remains to be determined.

Bearing this caveat in mind, we can assume that the temperature dependence of $\Gamma \sim \gamma_0 + \gamma_2 T^2 + \cdots$, originating from disorder (the zero-temperature residual resistivity) and umklapp (the Fermi-liquid-like behavior recovered outside the superconducting dome). On the other hand, intrinsic processes controlling the CDW contribution relaxation to the resistivity might be responsible for the disorder-independent, T-linear component at low temperatures. The magnitude of this contribution is naturally proportional to the elastic modulus and would be expected to become stronger as temperature is decreased and the CDW order sets in, which is consistent with the increase in the onset temperature of the linear component as doping decreases.

Why should those intrinsic processes carry a T-linear dependence? This is a difficult question barring a concrete microscopic model of cuprates. The holography-inspired scaling theories alluded to previously give one possible answer but have not yet been extended to the pseudospontaneous case.

An alternative relies on theoretical arguments by which diffusivities D in strongly correlated systems tend to saturate a Planckian bound (Hartnoll, 2015),

$$D \gtrsim v^2 \tau_{\rm Pl}, \qquad \tau_{\rm Pl} = \frac{\hbar}{k_B T}. \tag{64}$$

In Eq. (64) v is the characteristic velocity, which is sometimes argued to be the Fermi velocity, the Lieb-Robinson velocity, or the butterfly velocity; see Blake (2016). Through this general mechanism, which is applied to the diffusive transport coefficients σ_o, ξ_μ, and ξ, we may expect various disorder-independent, T-linear contributions to the resistivity, split between the coherent and incoherent terms. This resonates with the analysis of the magnetoresistance data given by Ayres *et al.* (2021), who found it necessary to include a T-linear component in both coherent and incoherent contributions.

The diffusivity σ_o is directly related to the thermal diffusivity (Davison, Gentle, and Goutéraux, 2019b). Energy diffusion is likely to be universal in a critical phase. Indeed, measurements of this observable in hole-doped (Zhang *et al.*, 2017) and electron-doped cuprates (Zhang *et al.*, 2019), as well as in crystalline insulators (Behnia and Kapitulnik, 2019; Mousatov and Hartnoll, 2020). All suggest that the thermal diffusivity in these materials is close to a Planckian bound.

One may legitimately wonder why the same ought to hold for the Goldstone diffusive coefficient ξ. The Goldstone modes are weakly coupled in the low-energy effective field theory (Son, 2002; Nicolis, Penco, and Rosen, 2014; Nicolis *et al.*, 2015), meaning that it does not naturally follow that they relax on Planckian scales (the attenuation of superfluid phonons being a case in point). On the other hand, in Secs. IV.D, V.C, and VI, we have highlighted a dissipation mechanism in hydrodynamic currents at play both in holographic systems and in 2D electron gases hosting Wigner crystal phases. This mechanism links the Goldstone diffusivity ξ to the thermal diffusivity, which itself is likely to be close to a Planckian bound in a strongly correlated system.

At the critical doping, the resistivity is purely T linear with an $O(1)$ coefficient. At this doping, quasiparticles are completely lost due to strong correlations (He *et al.*, 2018), thus vindicating the applicability of quantum bounds on transport of the kind of Eq. (64). In the absence of quasiparticles, the Goldstone sound velocity is a plausible candidate to enter into the bound, in which case the factors of the elastic moduli cancel out from the resistivity, yielding an $O(1)$ prefactor for the T-linear resistivity.

On the other hand, if the strange metal regime near critical doping is related to a kind of CDW critical point dominated by charge density fluctuations, then fluctuations of the amplitude of the order parameter ought to be included in the effective description, not just its phase (Hohenberg and Halperin, 1977) [see Grossi *et al.* (2021) for a recent application to QCD in the chiral limit], bringing us back to the arguments developed by Patel *et al.* (2022) for the origin of the T-linear resistivity at critical doping. Holography will also certainly be a valuable tool to construct such EFTs augmented with order parameter fluctuations (Herzog, 2010; Donos and Pantelidou, 2022).

Pseudospontaneous breaking of translations thus appears to be a promising avenue to understand various features of strange and bad metals. While it is difficult to be more conclusive at this stage, further analyses of experimental data, revolving around the influence of disorder on charge density fluctuations, a systematic analysis of charge, heat, and magnetotransport data on the same compound, and a refinement of the two-component analysis of optics data may give further support to this hypothesis or disprove it.

ACKNOWLEDGMENTS

We thank Aristomenis Donos for collaborating at an early stage of this work and for several fruitful discussions on the topics of this Colloquium. We thank Luca Delacrétaz, Sean Hartnoll, Akash Jain, Stephen Kivelson, John Tranquada, Erik van Heumen, and Jan Zaanen for numerous inspiring discussions on the topic of phases with spontaneously broken translations over the years. We also thank all our collaborators for our common work, which provided part of the basis for this Colloquium. We also thank Sean Hartnoll, Akash Jain, Matti Järvinen, Alexander Krikun, Li Li, and Vaios Ziogas for pointing out various misprints and providing helpful feedback on a previous version of this manuscript. M. B. acknowledges the support of the Shanghai Municipal Science and Technology Major Project (Grant No. 2019SHZDZX01). M. B. also acknowledges the sponsorship from the Yangyang Development Fund. The work of B. G. is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 758759).

APPENDIX: POSITIVITY OF ENTROPY PRODUCTION

In this appendix, we give more details on the steps leading to the Lorentz-invariant constitutive relations (32). Using Eq. (31) together with the first law of thermodynamics

$$T ds = d\varepsilon - \mu dn - v_i d\pi^i - h^{ij} d(\nabla_i u_j) \tag{A1}$$

as well as the equations of motion, the divergence of the entropy current is found to be

$$T\dot{s} + T\nabla_i\left(\frac{j_q^i}{T}\right) = \tilde{u}^j(K_j + \nabla^i h_{ij}) - \tilde{j}_q^i \frac{\nabla_i T}{T} - \tilde{j}^i \nabla_i \mu - \tilde{\tau}^{ij}\nabla_i v_j, \tag{A2}$$

with

$$j_q^i = T s v^i + \tilde{j}_q^i, \qquad \tilde{j}_q^i = \tilde{j}_\varepsilon^i - \mu \tilde{j}^i + h_{ij}\tilde{u}^j - v_j \tilde{\tau}^{ij}. \tag{A3}$$

In Eq. (A3) we have turned on an external source for u^i, $f_{\rm el} \mapsto f_{\rm el} - K_i u^i$, which we take to be first order in gradients $K_i \sim \mathcal{O}(\nabla)$.

The right-hand side of Eq. (A3) must be positive so that entropy is not destroyed by dissipative processes. This constrains the constitutive relations to take the following form[47]:

$$\begin{aligned}
\tilde{j}^i &= -\sigma_o^{ij}\nabla_j\mu - \alpha_o^{ij}\nabla_j T - \gamma_\mu^{ij}(K_j + \nabla^k h_{kj}),\\
\frac{\tilde{j}_q^i}{T} &= -\bar{\alpha}_o^{ij}\nabla_j\mu - \frac{\bar{\kappa}_o^{ij}}{T}\nabla_j T - \gamma_T^{ij}(K_j + \nabla^k h_{kj}),\\
\tilde{\tau}^{ij} &= -\eta^{ijkl}\nabla_{(k}v_{l)},\\
\tilde{u}^i &= \xi_\mu^{ij}\nabla_j\mu + \xi_T^{ij}\nabla_j T + \xi_h^{ij}(K_j + \nabla^k h_{kj}).
\end{aligned} \tag{A4}$$

[47]The ideal equations of motion are used to remove all time derivatives in the constitutive relations, and we choose a frame such that the conserved densities are not corrected at first order in gradients. See Kovtun (2012, 2019) for discussions on the role of frames in relativistic hydrodynamics and de Boer *et al.* (2018a, 2018b), de Boer *et al.* (2020), Novak, Sonner, and Withers, 2020, Poovuttikul and Sybesma (2020), and Armas and Jain (2021) for discussions on hydrodynamics without boosts.

Turning on the external source K_j is necessary to remove terms like $\nabla_j h_k^k$, which otherwise would appear to be allowed. In the main text and in the remainder of this appendix, we turn off the external sources.

The Onsager relations can be imposed either on the matrix of the retarded Green's function

$$S[G^R(\omega, -q)]^T = G^R(\omega, q)S \tag{A5}$$

or, as is often simpler, directly on the $M\chi$ matrix

$$S[M(-q)\chi]^T = M(q)\chi S, \tag{A6}$$

where M is defined from the equations of motion and S is the matrix of time-reversal eigenvalues of the corresponding fields $(n, \varepsilon, \pi_\parallel, \lambda_\parallel, \pi_\perp, \lambda_\perp)$. Here $S = \mathrm{diag}(1, 1, -1, 1, -1, 1)$.

The $M \cdot \chi$ matrix reads

$$M\chi = \begin{pmatrix}
\sigma_0 q^2 & \alpha_0 q^2 & iqn & \gamma_\mu q^2 & 0 & 0\\
\bar{\alpha}_0 q^2 & (\bar{\kappa}_0/T)q^2 & iqs & \gamma_T q^2 & 0 & 0\\
iqn & iqs & (\zeta+\eta)q^2 & -iq & 0 & 0\\
\xi_\mu q^2 & \xi_T q^2 & -iq & \xi q^2 & 0 & 0\\
0 & 0 & 0 & 0 & \eta q^2 & -iq\\
0 & 0 & 0 & 0 & -iq & \xi q^2
\end{pmatrix}. \tag{A7}$$

The Onsager relations further fix

$$\gamma_\mu = \xi_\mu, \qquad \gamma_T = \xi_T. \tag{A8}$$

Recall that all the transport coefficient matrices and tensors are decomposed as $\sigma_o^{ij} = \sigma_{(o)}\delta^{ij} + \sigma_{(u)}u^{ij}$, and the final coefficient appearing in Eq. (A7) is a linear combination of $\sigma_{(o)}$ and $\sigma_{(u)}$such as $\sigma_o = \sigma_{(o)} + (u_o/2)\sigma_{(u)}$.

At linearized level, it is enough for us to impose positivity of Eq. (A7), but in general one should instead require the quadratic form on the right-hand side of Eq. (A2) to be positive definite. Positivity of Eq. (A7) follows if all eigenvalues are positive, which in turn is equivalent to all principal minors of this matrix being positive. The following constraints are sufficient to that effect:

$$\begin{gathered}
\sigma_o, \bar{\kappa}_o, \eta, \zeta+\eta \geq 0,\\
\sigma_o\bar{\kappa}_o \geq T\alpha_o^2, \qquad \sigma_o\xi \geq \xi_\mu^2, \qquad \bar{\kappa}_o\xi \geq T\xi_T^2.
\end{gathered} \tag{A9}$$

The Lorentz boost Ward identity implies that $j_\varepsilon^i = \pi^i$. At the ideal level, using Eq. (29) this fixes

$$\chi_{\pi\pi} = \varepsilon + p - p_{\rm el}, \tag{A10}$$

while at first order in gradients, from Eqs. (A4) and (A8), the following relations between the longitudinal transport coefficients

$$T\xi_T + \mu\xi_\mu - p_{\rm el}\xi = 0,$$
$$T\alpha_o + \mu\sigma_o - p_{\rm el}\xi_\mu = 0,$$
$$\mu\alpha_o + \bar{\kappa}_o - p_{\rm el}\xi_T = 0, \qquad \text{(A11)}$$

or in matrix form

$$\alpha_o^{ij} + \frac{\mu}{T}\sigma_o^{ij} + \frac{1}{T}h_{ik}\xi_\mu^{kj} = 0,$$
$$\kappa_o^{ij} + \frac{\mu}{T}\alpha_o^{ij} + \frac{1}{T}h_{ik}\xi_T^{kj} = 0,$$
$$\xi_T^{ij} + \frac{\mu}{T}\xi_\mu^{ij} + \frac{1}{T}h_{ik}\xi_h^{kj} = 0. \qquad \text{(A12)}$$

The constitutive relations then become

$$\tilde{j}^i = -T\sigma_o^{ij}\nabla_j\frac{\mu}{T} - \gamma_\mu^{ij}\nabla^k h_{kj},$$
$$\frac{\tilde{j}_q^i}{T} = (\mu\sigma_o^{ij} + h^{il}\xi_\mu^{lj})\nabla_j\frac{\mu}{T} - (\mu\xi_\mu^{ij} + h^i{}_l\xi_h^{lj})\nabla^k\frac{h_{kj}}{T},$$
$$\tilde{\tau}^{ij} = -\eta^{ijkl}\nabla_{(k}v_{l)},$$
$$\tilde{u}^i = T\xi_\mu^{ij}\nabla_j\frac{\mu}{T} + \xi_h^{ij}\nabla^k\frac{h_{kj}}{T}. \qquad \text{(A13)}$$

In the Galilean limit, the Galilean boost Ward identity enforces $j^i \propto \pi^i$ and instead

$$\sigma_o^{ij} = 0, \qquad \alpha_o^{ij} = 0, \qquad \xi_\mu^{ij} = 0. \qquad \text{(A14)}$$

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