Colloquium: Geometric phases of light: Insights from fiber bundle theory

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Geometric phases are ubiquitous in physics; they act as memories of the transformation of a physical system. In optics, the most prominent examples are the Pancharatnam-Berry phase and the spin-redirection phase. Recent technological advances in phase and polarization structuring have led to the discovery of additional geometric phases of light. The underlying mechanism for all of these is provided by fiber bundle theory. This Colloquium reviews how fiber bundle theory not only sheds light on the origin of geometric phases of light but also lays the foundations for the exploration of high-dimensional state spaces, with implications for topological photonics and quantum communications.

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I. INTRODUCTION

Phase is a curious protagonist in the land of physics. It bears no physical significance when a single wave is considered, yet becomes crucially important when several waves are involved, thereby causing spectacular effects such as interference. In 1984, Sir Michael Berry established that the wave function of a quantum system can gain a phase of geometric nature in addition to the dynamic phase naturally acquired over time (Berry, 1984). This discovery impacted various areas of physics, such as condensed matter, nuclear, plasma, and optical physics (Wilczek and Zee, 1984; Wilczek and Shapere, 1989). Although geometric phases may appear as mere theoretical curiosities, they have led to a myriad of applications and in optics are now at the basis of wave front shaping technologies (Cohen *et al.*, 2019; Jisha, Nolte, and Alberucci, 2021). Their importance for exotic surface effects, including superconductivity and topological insulators such as the quantum Hall effect, was honored in the Nobel Prize awarded to David Thouless, Duncan Haldane, and Michael Kosterlitz for research on topological phases of matter.

A matrix-based formalism can be used to determine whether a system will acquire a geometric phase (O'Neil and Courtial, 2000); however, this approach gives little insight in regard to the origin of the phenomenon. Fiber bundle theory provides a deeper understanding of geometric phases: it links a phase to a state transformation based on geometrical considerations. This mathematical framework was developed in the first half of the 20th century and turned out, to everyone's surprise, to provide an excellent description of gauge fields, including electromagnetic fields (Yang, 2014). It became the universal language of geometric phases almost immediately, even before Berry had time to publish his seminal work (Simon, 1983), and played a key role in the generalization of Berry's phase to nonadiabatic and noncyclic systems (Aharonov and Anandan, 1987; Samuel and Bhandari, 1988).

A plethora of geometric phases have been witnessed in optics (Samuel and Bhandari, 1988; Vinitskiĭ *et al.*, 1990; Bhandari, 1997). Well-known examples include the Pancharatnam-Berry phase (Pancharatnam, 1956), born from polarization transformations, and the spin-redirection phase (Rytov, 1938; Vladimirskiy, 1941), which arises when light is taken along a nonplanar trajectory. Recent decades have seen significant technological advances in the control of phase and polarization structured light, with an ever-expanding repertoire of higher order spatial modes and complex vector light fields. These developments have revealed new geometric phases of light, caused by the transformation of spatial transverse modes (Van Enk, 1993; Galvez *et al.*, 2003; Calvo, 2005; Galvez and O'Connell, 2005; Gutiérrez-Cuevas *et al.*, 2020), and of general vectorial fields (Milman and Mosseri, 2003; Milione *et al.*, 2011, 2012; Liu *et al.*, 2017). However, with just a handful of exceptions (Bouchiat and Gibbons, 1988; Bliokh, 2009; Cohen *et al.*, 2019) these phases are rarely linked to fiber bundles, causing key concepts such as connection and curvature to be surrounded by an aura of mathematical mystery.

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In this Colloquium, we illustrate how fiber bundle theory can bring about a deeper understanding of geometric phases. We do not expect the reader to have prior knowledge in the area of fiber bundle theory and introduce a few key concepts. We show that the geometric phases recently observed in structured light beams are based mostly on two-dimensional subspaces of a much larger state space, and that fiber bundle theory could guide the exploration of the entire state space. Establishing a firmer link between geometric phases of light and fiber bundle theory could highlight interdisciplinary research opportunities and stimulate new discoveries. The experimental simplicity and versatility of optical systems could even allow us to test concepts of fiber bundle theory itself. We start our discussion by recalling how geometric phases differ from their dynamic counterparts using a simple interferometric construction.

II. GEOMETRIC VERSUS DYNAMIC PHASE IN A NUTSHELL

Interferometry is a precious tool for measuring the phase difference between two beams of light. In a Mach-Zehnder interferometer, the phase difference $\Delta\phi$ between the two beams exiting the interferometer is null if the arms of the interferometer are of equal optical path length; see Fig. 1(a). The phase naturally acquired over time as the beam propagates is called the dynamical phase (ϕ_d). Increasing the optical path length of one of the arms (by introducing a piece of glass, for instance) will create an excess of dynamical phase $\phi_d + \phi'_d$ in this arm such that $\Delta\phi = \phi'_d$ [see Fig. 1(b)], thereby modifying interference and leading to a difference in the interferometer output.

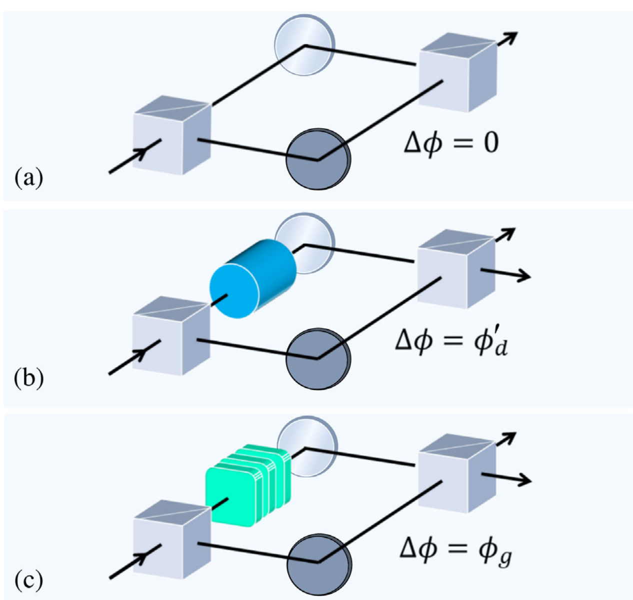

FIG. 1. Phase measurements with Mach-Zehnder interferometers. (a) Balanced interferometer with arms of equal optical path length. (b) Introducing a dynamic phase by changing the optical path length with a piece of glass in one arm. (c) Introducing a geometric phase by performing a succession of state transformations in one arm.

It is also possible to obtain a finite phase difference even if the arms are of equal optical path length, by imposing a series of state transformation to one of the beams; see Fig. 1(c) (Aharonov and Anandan, 1987). These transformations will cause the beam propagating through this arm to acquire a phase ϕ_g, solely dependent on the path formed in the state space, in addition to the dynamical phase acquired upon propagation, such that at the exit of the interferometer $\Delta\phi = \phi_g$. The phase ϕ_g is said to be geometric. In Sec. III, we show how a succession of polarization transformations can create such a geometric phase, which is called the Pancharatnam-Berry phase.

III. THE PANCHARATNAM-BERRY PHASE

The Pancharatnam-Berry (PB) phase is one of the most ubiquous geometric phases of light (Vinitskiĭ *et al.*, 1990; Bhandari, 1997; De Zela, 2012; Lee *et al.*, 2017). It was discovered by Pancharatnam in 1956 upon generalization of the notion of interference for partially orthogonal polarized beams (Pancharatnam, 1956) and was identified as a geometric phase by Ramaseshan and Nityananda in 1986 (Ramaseshan and Nityananda, 1986). This led Berry to provide a quantum interpretation of this phenomenon, causing his name to be linked to this phase along with Pancharatnam's (Berry, 1987b).

A. Experimental realization

The PB phase is generated by changing the polarization state of a beam of light propagating along a fixed direction. In practice, a sequence of polarization transformations can be realized using several retarders, which would correspond to the optical elements in Fig. 1(c). For simplicity, we assume that the retarders do not change the optical path length. If the beam of light is initially horizontally polarized (state 1), we can use a quarter wave plate (QWP) to convert the beam into a circularly polarized state (state 2), use a second QWP to return the polarization state to linear (state 3), rotated, however, by 45° with respect to the horizontal, then employ a suitably oriented half wave plate to restore the polarization direction to horizontal (state 4). The sequence of polarization transformation is illustrated in Fig. 2(a). As previously stated, a geometric phase is dependent on the path formed in the state space. To determine whether our sequence of state

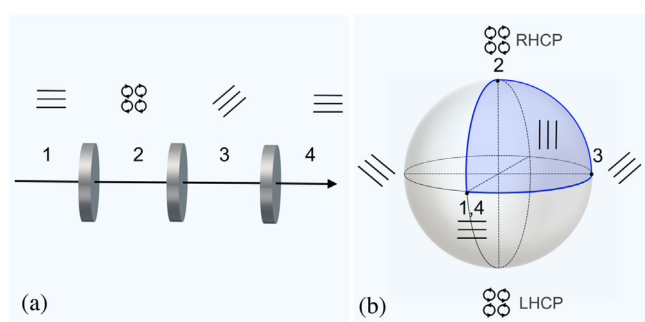

FIG. 2. Sequence of polarization transformations. (a) Practical realization. (b) Geometric interpretation: a closed path is traced on the Poincaré sphere.

transformation will generate a geometric phase, we therefore need to turn to geometric considerations.

B. Geometric interpretation

Realizing a sequence of unitary polarization transformation can be visualized as a path on the Poincaré sphere. The Poincaré sphere is the state space of purely polarized light, meaning that each point on the sphere represents a pure polarization state. By convention, the poles represent circularly polarized light, the equator stands for linearly polarized light, and the hemispheres represent right and left elliptically polarized light; see Fig. 2(b). All states on the sphere can be conveniently obtained from a linear superposition of diametrically opposed states. The path corresponding to the polarization transformation shown in Fig. 2(a) is drawn in Fig. 2(b), where successive polarization states have been linked using geodesics.

In optics, it is common practice to calculate the PB phase, which we denote as ϕ_g, directly from the solid angle Ω_{PS} enclosed by the path formed on the Poincaré sphere, shown in light blue (shaded area) in Fig. 2 (Pancharatnam, 1956; Samuel and Bhandari, 1988), using the simple relation

$$\phi_g = -\tfrac{1}{2}\Omega_{\mathrm{PS}}. \tag{1}$$

If the sequence of polarization transformations is associated with a vanishing solid angle, no PB phase will be generated. Equation (1) provides a straightforward manner to calculate the PB phase; however, the relationship between the phase and a path formed on the state space is far from obvious. Indeed, in physics states are defined up to a phase factor, meaning that two-state vectors $|\psi\rangle$ and $\exp(i\phi)|\psi\rangle$, where $\phi \in [0, 2\pi[$, are considered to be physically equivalent. A path traced on the Poincaré sphere thus does not directly provide information on the evolution of the phase of the system. An additional structure capable of tracking this evolution is needed, and this is where fiber bundle theory comes into play.

IV. THE ORIGIN OF GEOMETRIC PHASES

In what follows, we introduce some fundamentals of fiber bundle theory and show how geometric phases are interpreted in terms of fiber bundles. We then examine the PB phase from this new perspective.

A. Fiber bundle theory: A universal model

In the landscape of mathematics, fiber bundle theory sits at the crossroads of differential geometry, topology, and connection theory. It was developed independently from physics in the first half of the 20th century (Hopf, 1931; Seifert, 1933; Whitney, 1935; Feldbau, 1939; Ehresmann, 1949; Serre, 1951; Steenrod, 1951). The overlap with physics became evident only in retrospect, when Dirac's theory on magnetic monopoles (Dirac, 1931) was examined from a geometric perspective, showing that Dirac had described a fiber bundle (Lubkin, 1963; Wu and Yang, 1975). Wu and Yang went further by demonstrating that fiber bundle theory is the natural language of gauge theory. They summarized this idea in a

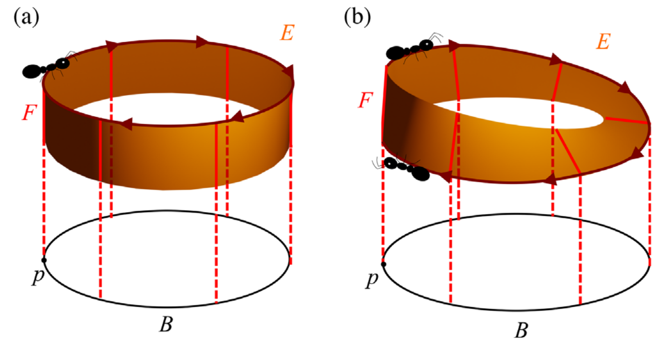

FIG. 3. Illustration of (a) a trivial and (b) a nontrivial fiber bundle. The base spaces B are identical circles, and several fibers are indicated as solid line segments embedded in the total space E [the cylinder in (a) and Möbius strip in (b)].

table showing how the two theories describe the same concept using different terminologies (Wu and Yang, 1975). At that point, fiber bundle theory ceased to be an abstract framework and became suitable for the description of physical reality. This discovery profoundly influenced the physics and mathematics communities during the late 20th century, as detailed in the overview provided by Boi (2004).

We illustrate the basic idea of fiber bundles in Fig. 3 with the example of a cylinder and a Möbius strip. A fiber bundle is constructed from a topological space B called the base space [which for both Figs. 3(a) and 3(b) is a circle]. Above each point $p \in B$ is a space called the fiber F (shown as a line segment, defined by its two end points), which is linked to the base space by a projection map (indicated as dashed lines). At the core of fiber bundle theory is the idea that locally, within a close neighborhood of points $p \in B$, the total space of the fiber bundle is the direct product of the fiber space and the base space: $E \approx B \otimes F$. For the cylinder this is true globally, making it a trivial fiber bundle. The topological and geometrical properties of the fiber bundle may, however, prevent us from obtaining from a consistent global mapping. This is the case for the twisted Möbius strip, as indicated by the ant, which can move from one end of the line segment F to the other when traversing along the Möbius band. The fiber bundle is then said to be nontrivial (Batterman, 2003). A fiber bundle thus contains two types of information: local and global. Physical phenomena are often studied from a local (infinitesimal) perspective; fiber bundle theory invites us to take a step back and look for global properties that may also affect our system.

To complete the definition of a fiber bundle, one may also specify the structure group G acting on the bundle. In the case of the Möbius strip, $G = \pm 1$, where the element -1 acts on the fiber by sending an element from the top to the bottom of the fiber (Batterman, 2003).

Fiber bundles relevant in optics and quantum optics tend to operate in larger state spaces, e.g., with fibers keeping track of phase evolutions. While it is perfectly feasible to describe optics in terms of complex electromagnetic fields, and quantum optics in terms of quantum states, fiber bundle theory offers a supplementary geometric interpretation, transcending specific applications and potentially allowing us to

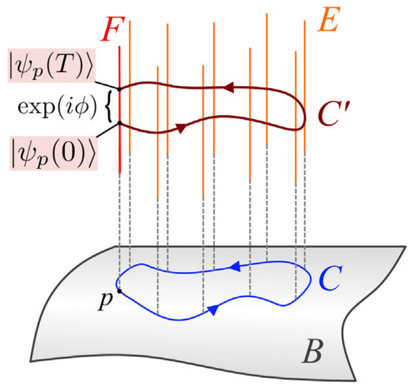

FIG. 4. Illustration of Aharonov and Anadan's fiber bundle. A closed path C, starting and ending at point $p \in B$, is lifted into the total space E. The beginning and end points of C' lie on the same fiber F_p and are related by a phase factor in Aharonov and Anadan's fiber bundle.

develop a more intuitive understanding of the underlying phenomena.

The concept of a fiber bundle linking a phase to a state transformation was introduced by Aharonov and Anandan (1987). Here the base space B of the fiber bundle is the complex projective Hilbert space (as illustrated in Fig. 4), which we call the state space. The fiber above each state consists of all the normalized state vectors capable of representing that state, namely, $\exp(i\phi)|\psi\rangle$, where $\phi \in [0, 2\pi[$. The structure group is the unitary group U(1), and the total space E is the Hilbert space. This fiber bundle is a principal bundle, meaning that the fiber is the structure group.

As mentioned earlier, a state transformation can be visualized as a path in the state space. If we assume that the transformation is cyclic, meaning that the state transforms back to the initial state at the end of the transformation, it forms a closed path C in the state space, shown by the curve in the base space B in Fig. 4. Knowledge of this path alone does not include any phase information. To record phase information, the path C is "lifted" to form a path C' in E. Since many curves C' can project down onto the same curve C, there are many ways to realize this lift. Some rules must be provided; this is the role of the connection $\mathcal{A}$. The connection decomposes the tangent space of the bundle into vertical and horizontal components, specifying the direction along the fibers and "perpendicular" to the fibers, respectively. The curve C is then lifted along the horizontal direction (along the fiber in Fig. 4). This allows us to compare (connect) points on different fibers. The splitting of the space into its horizontal and vertical components hence ties the connection to a particular set of coordinates.

Lifting a closed path in B will often result in an open path in E. If we assume that the beginning and end points of the path lie on the same fiber, they are linked by a simple phase factor $\exp(i\phi)$ [a U(1) transformation] called the "holonomy of the connection on the fiber bundle" (Nakahara, 1990). This phase factor indicates that the wave function has failed to come back to itself at the end of the transformation. Explicitly, for a cyclic evolution of period T, $\psi_{\mathrm{p}}(T) = \exp(i\phi)\psi_{\mathrm{p}}(0)$.

In the classical realm, holonomies can take the form of a rotation, such as the one allowing a cat falling from an upside-down position to land on its paws (Montgomery, 1993) or the rotation of the oscillation plane of Foucault's pendulum, after a day has elapsed (Hannay, 1985; von Bergmann and von Bergmann, 2007).

Aharonov and Anandan identified the connection $\mathcal{A}^{\mathrm{AA}}$ that yields the geometric phase $\exp(i\phi) = \exp(i\phi_g)$ as its holonomy. To do so, they defined the geometric phase as the difference between the total phase and the dynamic phase (Aharonov and Anandan, 1987; Zwanziger, Koenig, and Pines, 1990). Not all evolutions are cyclic. Samuel and Bhandari (1988) showed that the path formed in the projective space can be simply closed using the shortest geodesic, a process that does not affect the geometric phase (Benedict and Fehér, 1989). The geometric phase can then be calculated using the following connection of Aharonov and Anandan:

$$\phi_g = \oint_C \mathcal{A}^{\mathrm{AA}} = i \oint_C \langle \tilde{\psi} | d | \tilde{\psi} \rangle, \tag{2}$$

where d is an exterior differential operator and $|\tilde{\psi}\rangle$ is a basis vector field, also known as the section or gauge [an explicit derivation was given by Bohm *et al.* (2003)]. Equation (2) can be used if $\mathcal{A}^{\mathrm{AA}}$ is uniquely defined over the region of the state space covered by the path C. In practice, several $\mathcal{A}^{\mathrm{AA}}$ may coexist due to the geometry of the Hilbert space and of the projective Hilbert space (Urbantke, 1991). One may then prefer the following expression, which is obtained using the Stokes theorem:

$$\phi_g = \int_S d\mathcal{A}^{\mathrm{AA}} = \int_S \mathcal{V}^{\mathrm{AA}}, \tag{3}$$

where S is the surface in the state space enclosed by C and $\mathcal{V}^{\mathrm{AA}}$ is the curvature of the connection. Unlike the connection $\mathcal{A}^{\mathrm{AA}}$, the curvature $\mathcal{V}^{\mathrm{AA}}$ is well defined everywhere. It measures the dependence of the phase holonomy on the path formed in the projective Hilbert space. The geometric phase owes its name to this path dependence, and one may say that it is the curvature of the state space that gives birth to geometric phases (Anandan, 1988).

It is possible to witness a phase holonomy even if the curvature vanishes when the path C cannot be shrunk to a point. This typically happens if the path encloses a topological defect. The phase holonomy becomes a signature of the defect and is insensible to the shape of the path, and hence it is called a "topological phase" (Lyre, 2014).

A notable fact about the fiber bundle interpretation of geometric phases proposed by Aharonov and Anandan is that it remains valid regardless of the dimension of the state space. We will return to this after providing a fiber bundle interpretation of the PB phase.

B. From Poincaré to Hopf

In the case of the PB phase, the relevant state space is the Poincaré sphere. The sphere representation is specific to two-dimensional systems, and studying its construction reveals the associated fiber bundle. When fully polarized light propagates

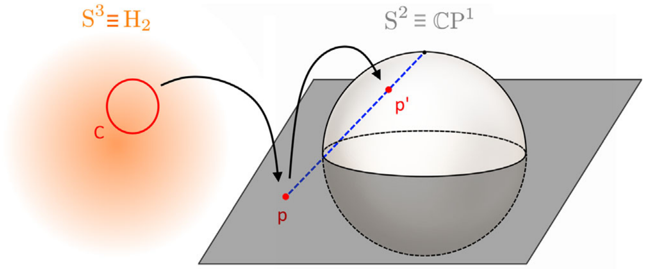

FIG. 5. Schematic of the Hopf map. A set of equivalent state vectors representing a pure quantum state (the circle in the Hilbert space H_2) is mapped onto a point in $\mathbb{R}^2(+\infty)$, and from there onto a point in the complex projective Hilbert space $\mathbb{C}P^1$ via an inverse stereographic projection (dotted line). We have chosen the north pole as the projection point.

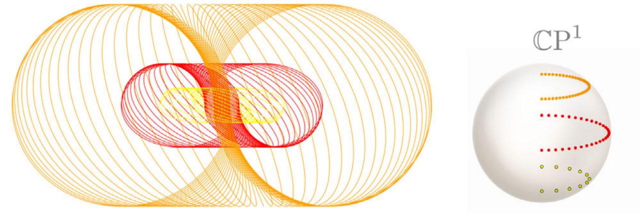

FIG. 6. Fiber structure of the Hopf fibration in $\mathbb{R}^3$. Each fiber is a circle. The set of fibers linked to the red (dark gray), orange (gray), and yellow (light gray) points on the $\mathbb{C}P^1$ sphere form three tori.

along a fixed direction, say, z, it becomes analogous to a two-state (qubit) system,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{4}$$

where $|0\rangle$ and $|1\rangle$ are the eigenstates of the Pauli spin operator σ_z, and α and β are complex parameters with $|\alpha|^2 + |\beta|^2 = 1$ to ensure normalization. The state vector $|\psi\rangle$ lives in the two-dimensional Hilbert space, which is denoted by H_2. This space is our total space E, which can be pictured as a hypersphere S^3 embedded in $\mathbb{R}^4$, represented in orange (the left shaded area) in Fig. 5(a). Sugic *et al.* (2021) named this space the *optical hypersphere*.

In optics, a pure state $|\psi\rangle$ can be identified only up to a phase factor $\exp(i\phi)$, and in quantum theory the set of state vectors $\exp(i\phi)|\psi\rangle$ describe the same physical state: the probability of a measurement of the system given by Born's rule for $|\psi\rangle$ and $\exp(i\phi)|\psi\rangle$ is 1. To account for this, in the projective Hilbert space all states $\exp(i\phi)|\psi\rangle$, where $\phi \in [0, 2\pi[$, represent the same quantum state.

This set of equivalent state vectors form a fiber, which can be pictured as a circle S^1 parametrized by ϕ (C in Fig. 5). For a two-state system, the state space is the projective Hilbert space $\mathbb{C}P^1$, which is an ordinary sphere known to mathematicians as S^2. The state space is obtained by mapping each quantum state (circle) in the total space onto a point on the sphere. This mapping is performed using the Hopf map, which maps a circle onto a point $\mathbf{p}$ in a plane $\mathbb{R}^2(+\infty)$, then maps this point onto a point $\mathbf{p}'$ on the sphere via an inverse stereographic projection, as illustrated in Fig. 5(a) (Mosseri and Dandoloff, 2001). This is how the Poincaré sphere, and all spheres representing two-state systems, is constructed.

The PB phase then corresponds to the holonomy of the connection $\mathcal{A}^{\mathrm{AA}}$ on a fiber bundle where the base space is $\mathbb{C}P^1$ (the Poincaré sphere), a fiber is a set of equivalent state vectors, the group structure is U(1), and the total space is H_2. This fiber bundle is known as the Hopf fibration, and it is capable of describing all two-state systems, not just polarization. As such, it is often encountered in physics where it describes magnetic monopoles, two-dimensional harmonic oscillators, Taub–Newman-Unti-Tamburino space (relevant in the framework of general relativity), and twistors (Urbantke, 1991). Because it involves spaces embedded in different dimensions, the Hopf bundle is difficult to visualize; however, performing a direct stereographic map from S^3 to $\mathbb{R}^3$ will make its fiber structure apparent (Mosseri and Ribeiro, 2007). A schematic is provided in Fig. 6.

It is not possible to assign a single connection $\mathcal{A}^{\mathrm{AA}}$ to the entire Poincaré sphere. Indeed, if we introduce polar coordinates θ, ϕ, we may define a connection $\mathcal{A}$ as follows using the basis $|\tilde{\psi}\rangle$ (Bouchiat and Gibbons, 1988; Kataevskaya and Kundikova, 1995):

$$|\tilde{\psi}\rangle = (\cos(\theta/2), e^{-i\phi}\sin(\theta/2)), \quad \mathcal{A} = \tfrac{1}{2}(1-\cos\theta)d\phi. \tag{5}$$

$\mathcal{A}$ is defined everywhere except at $\theta = 0$ (the north pole). To cover the entire sphere we may introduce another basis $|\tilde{\psi}'\rangle$ and a second connection $\mathcal{A}'$:

$$|\tilde{\psi}'\rangle = (e^{i\phi}\cos(\theta/2), \sin(\theta/2)), \quad \mathcal{A}' = \tfrac{1}{2}(-1-\cos\theta)d\phi. \tag{6}$$

$\mathcal{A}'$ is defined everywhere but at $\theta = \pi$ (the south pole). The singular points $\theta = \pi$ and $\theta = 0$ correspond to Dirac string singularities (Dirac, 1931; Yang, 1996); they can be moved around the sphere by choosing a different basis but cannot be removed. The sphere is thus divided into two overlapping regions, with each having a different connection. In the overlapping region, the connections are related using a phase transformation (Urbantke, 1991). In this case, it is preferable to calculate the PB phase using the Stokes theorem:

$$\phi_g = \frac{1}{2}\oint_C (\pm 1 - \cos\theta)d\phi = \frac{1}{2}\int_S \sin\theta d\theta d\phi = \frac{1}{2}\Omega_S, \tag{7}$$

where Ω_S is the solid angle enclosed by the path C formed in clockwise fashion on the Poincaré sphere. We have recovered Eq. (1) using fiber bundle theory and have detailed how the PB phase arises from a state transformation. Equation (1) is valid only because the state space can be represented as a sphere, which is true for all two-state systems. Equation (7) shows that the curvature on the Hopf fibration does not vanish and confirms that the PB phase depends on the path traced in the state space. It is truly a "geometric" phase.

PB phases are at the core of state-of-the-art wave front shaping technologies (Bomzon *et al.*, 2002; Kim *et al.*, 2015; Radwell *et al.*, 2016; Dorrah *et al.*, 2022). Q plates (Marrucci, Manzo, and Paparo, 2006), in particular, rely on PB phases to impart a helical phase profile to a beam of light and have been commercialized under the name of vortex retarders. The underlying principle of these devices is that a spatially variant PB phase profile can be obtained by realizing spatially resolved polarization transformations. Conversely, exotic polarization distributions such as the ones witnessed when multiple beams of light are interfered with (Galvez *et al.*, 2012; Cardano *et al.*, 2013) or obtained upon tight focusing (Bauer *et al.*, 2015) may contain interesting geometric phase profiles.

In this section we have directed our attention to fully polarized light for didactic purposes. Note, however, that geometric phases can also arise from the transformation of partially polarized light. In this case, the state space becomes the Poincaré ball to include points inside the sphere (Sjöqvist *et al.*, 2000; Ericsson *et al.*, 2003). The geometric phase can then be obtained by purifying the state (Milman, 2006). The Poincaré ball naturally incorporates some hyperbolic geometry (Ungar, 2002), whose relevance with regard to special relativity was highlighted by Samuel and Sinha (1997) and Lévay (2004).

V. EXPLORING HIGH-DIMENSIONAL STATE SPACES

The state space of an n-state system where $n > 2$ can no longer be represented as an ordinary sphere (Bengtsson and Zyczkowski, 2006). Such spaces have recently become accessible in optics through spatial transverse modes, strongly focused light, and general vectorial light. In the following we review how geometric phases are currently calculated for these spaces and discuss how fiber bundle theory could lay the foundation for the exploration of high-dimensional state spaces. Our first encounter with a high-dimensional state space stems from the study of polarized beams of light with a spatially varying propagation direction.

A. The spin-redirection phase

The spin-redirection phase is a geometric phase that arises when polarized light is taken along a nonplanar trajectory. It was first witnessed in inhomogeneous media (Rytov, 1938; Vladimirskiy, 1941) and in optical fibers (Ross, 1984; Chiao and Wu, 1986), in which case it is the result of an adiabatic transformation, meaning that a photon that is initially in an eigenstate of the spin operator, aligned with the direction of the wave vector, will remain in this eigenstate at all times. In other words, its helicity does not change upon propagation. At the time, it was believed that the cycling of the parameters driving the adiabatic transformation determines the existence of geometric phases. The geometric phases were calculated from the path traversed in time formed in the space of parameters, in our case the sphere of directions of the wave vector $\boldsymbol{R}(t) = \mathbf{k}(t)/k$. The fiber bundle linking a phase to a parameter transformation was introduced by Simon (1983). The adiabatic geometric phase can then be calculated from the connection on Simon's fiber bundle. The adiabatic geometric phase $\phi_{g,a}$ acquired by a photon when the direction of the wave vector is cycled reads

$$\phi_{g,a} = -\sigma\Omega_k(\mathcal{C}), \tag{8}$$

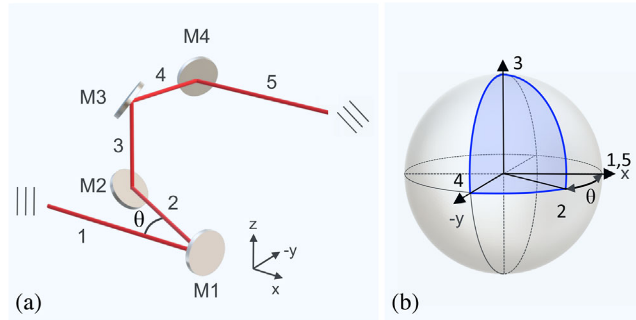

FIG. 7. Generation of SR phases. (a) Linearly polarized beam taken along a nonplanar trajectory. The mirrors M2 and M3 form a beam elevator. At the end of the trajectory, the polarization axis is rotated. (b) Path formed on the sphere of the spin directions.

where $\Omega_k(\mathcal{C})$ is the solid angle subtended by the path formed on the sphere of directions $\mathbf{k}$ and the helicity σ denotes the projection of the spin onto $\mathbf{k}$, which takes values $\sigma = 1$ for left-handed and $\sigma = -1$ for right-handed circularly polarized light. Here $\phi_{g,a}$ is analogous to a well-known adiabatic phase, namely, the Berry phase obtained from the evolution of a spin particle interacting with a time-varying magnetic field $\mathbf{B}(t)$ of constant amplitude, where the directions of the magnetic field $\mathbf{R}(t) = \mathbf{B}(t)/B$ are the parameters (Berry, 1984).

The geometric phase $\phi_{g,a}$ produces a characteristic effect: when $\mathbf{k}$ recovers its initial orientation, the polarization axis of linearly and elliptically polarized light is rotated. This rotation can be understood in terms of circular birefringence: The left and right circularly polarized light components of the beam acquire opposite geometric phases (Tomita and Chiao, 1986).

It soon appeared that this rotation can also be observed when light is redirected using a sequence of mirrors (Berry, 1987a; Kitano, Yabuzaki, and Ogawa, 1987). However, in this case the transformation is nonadiabatic because mirror reflections reverse the helicity. An attempt was made to continue using the parameter space to calculate the geometric phases, but it became clear that this description had reached its limit: it had to rely on modified wave vectors and had to account for occasional π phase shifts (Kitano, Yabuzaki, and Ogawa, 1987). At a similar time, Aharonov and Anandan (1987) changed their emphasis from the parameter space to the state space. They showed that adiabaticity is not a necessary condition for the existence of geometric phase, but instead that it is the state evolution that matters. Similarly, for the case of a spin particle in a magnetic field, Anandan (1992) considered the evolution of the spin instead of the evolution of the magnetic field direction (parameter), thereby lifting the adiabatic requirement. For spin-$1/2$ particles, the sphere of the spin directions is the Bloch sphere. Inspired by this work, Chiao *et al.* (1988) shifted the emphasis from the evolution of the direction of the wave vector to the evolution of the spin vector $\mathbf{S}$ of the photons. The geometric phase, now called the spin-redirection (SR) phase, became

$$\phi_{g,a} = -\sigma\Omega_{\mathrm{SR}}(C), \tag{9}$$

where $\Omega_{\mathrm{SR}}(C)$ is the familiar solid angle formed on the sphere representing the directions of the spin $\mathbf{S}$ of the photon in real space. Figure 7 illustrates how a beam of light can be taken along a nonplanar trajectory using a succession of mirrors and shows the respective path traced on the sphere of the spin directions of the photons.

Note, however, that, unlike the Bloch sphere of spin-$1/2$ particles, which incorporates information on the direction of the spin in real space and identifies all pure states, the sphere of the spin directions of the photons is not a state space. The state space identifying all pure polarization states was presented earlier: it is the Poincaré sphere. However, the Poincaré sphere is built on the assumption that polarization characterizes the oscillation of a two-dimensional electric field contained in the plane transverse to a constant propagation direction. If the propagation direction varies, so does the orientation of the transverse plane spanning the polarization.

When the propagation direction of a beam of light is varied, the electric field becomes a three-component vector $\mathbf{E} = (E_x, E_y, E_z)$ in the laboratory frame. The normalized state vector $|\psi\rangle$ representing the system then corresponds to a rotated three-component spinor, reflecting the spin-1 nature of the photons (Berry, 1987b; Hannay, 1998). We are thus dealing with a three-state system, of state space $\mathbb{C}P^2$, which is no longer an ordinary sphere. State spaces become difficult to visualize as their dimension increases, and so does picturing the evolution of the state in that space. Majorana (1932) provided an elegant way to circumvent this difficulty. Majorana was studying the behavior of a spin system of arbitrary angular momentum $\mathbf{j}$ in the presence of a magnetic field when he realized that varying the direction and magnitude of the magnetic field amounts to rotating $\mathbf{j}$. After the rotation, a system that was originally in an eigenstate finds itself in a superposition of $2j+1$ states. The $\mathbf{j}$-spin problem thus becomes equivalent to relating angular momentum states associated with different directions in space (Schwinger, 1977). This is equivalent to the problem we encounter when we compare polarization along a varying propagation direction. Majorana continued by representing a spin $\mathbf{j}$ state as a constellation of $2j$ points on an ordinary sphere. Each point, called a star, represents the direction of a spin-$1/2$ angular momentum (Bloch and Rabi, 1945). From a geometric perspective, what Majorana really did was write an n-dimensional state space $\mathbb{C}P^n$ as an unordered product of $n-\mathbb{C}P^1$, the space of all unordered sets of n points on a sphere.

In 1998, Hannay used the Majorana representation to visualize 3D polarized light as two stars on a sphere; see Fig. 8 (Hannay, 1998). In his work, he managed to relate Majorana's mathematical construct to a concept of a polarization ellipse and its orientation direction in 3D space, which is familiar to all researchers working in optics. Specifically, he showed that the foci of the polarization ellipse are given by the projection of the stars onto the plane perpendicular to the bisector of their angle. He also deduced the geometric phase associated with the transformation of 3D polarized light from the circuits traced by the two stars.

Nonparaxial fields, for which the electric field component along the propagation direction is non-negligible, have attracted increasing attention in the past decades by virtue of their capacity to mix the spin and orbital angular momentum content of the beam (Barnett and Allen, 1994; Bliokh *et al.*, 2010; Ma *et al.*, 2016). This attention renewed interest in their geometric phases and brought the Majorana representation back into the spotlight (Bliokh, Alonso, and Dennis, 2019; Alonso, 2020).

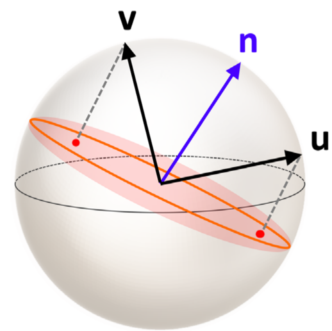

FIG. 8. Hannay's representation of 3D polarized light. The stars correspond to the tips of the vectors $\mathbf{v}$ and $\mathbf{u}$. The polarization ellipse is represented in orange (solid gray circle), and n is aligned in the propagation direction. The foci of the ellipse correspond to the projection of the stars onto the plane orthogonal to their bisector.

While picturing the state evolution is certainly helpful, we showed in Sec. IV that all we really need to calculate the geometric phase is the connection on the relevant fiber bundle. For a spin-1 system, the base space is the state space $\mathbb{C}P^2$, the total space is the Hilbert space H_3, and the fiber is $\mathrm{U}(1)$. A good description of this fiber bundle was provided by Bouchiat and Gibbons (1988). In this case, the set of pure states is characterized by four parameters consisting of three Euler angles θ, φ, and α giving the orientation in space of the principal axis of the polarization vector, and an extra parameter defining the shape of the ellipse δ. The geometric phase of 3D polarized light reads

$$\phi_g = \oint_C \mathcal{A} = \oint_C [\sin\delta\cos\theta d\varphi + (\sin\delta - 1)d\alpha]. \quad (10)$$

Hannay (1998) recognized that Eq. (10) is equivalent to the one found using the Majorana representation. The set of coordinates on $\mathbb{C}P^2$ contains some singularities, like the ones that we identified at the poles on the Poincaré sphere (Bouchiat and Gibbons, 1988). It would be interesting to determine whether this has physical consequences. A clear geometric interpretation of the limiting cases, where the geometric phase becomes the Pancharatnam-Berry phase or the spin-redirection phases, would also be useful. It has been suggested that, in the context of general relativity, $\mathbb{C}P^2$ can be regarded as a half pseudoparticle surrounded by a cosmological event horizon, and that it shares properties of the Yang-Mills instanton (Gibbons and Pope, 1978). We ask whether investigating the phase holonomies of 3D polarized light could be exploited to study these systems.

Turning paraxial light into a 3D field is relatively straightforward. One can use a high numerical aperture or rely on scattering (Bliokh *et al.*, 2011). Measuring the entire electric field, however, is a highly challenging task. Note, however, that it is now possible to access high-dimensional state space

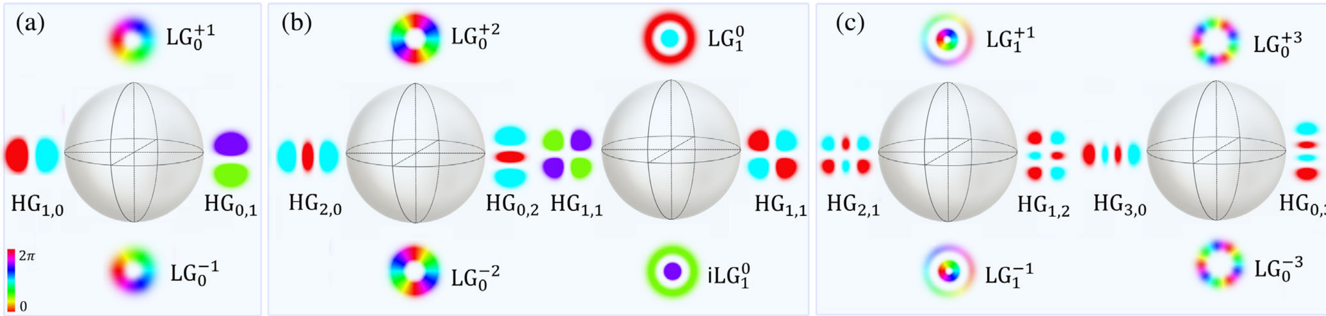

FIG. 9. Spheres of spatial transverse modes following the convention of Habraken and Nienhuis (2010), with all modes on one sphere linked by optical mode converters and rotators. (a) The sphere of first order modes is the direct analog of the Poincaré sphere. (b), (c) Second and third order modes represented on two spheres each.

without breaking the paraxiality by structuring light in its spatial degree of freedom.

B. Geometric phases of spatial transverse modes

Optical modes are characterized not only by their polarization but also by their spatial profile, determining both phase and intensity distribution across the beam (Forbes, Oliveira, and Dennis, 2021). While polarization is usually limited to a two-dimensional state space, there is an infinite number of orthogonal spatial modes, with Hermite-Gaussian ($HG_{n,m}$) and Laguerre-Gaussian (LG_p^ℓ) modes providing possible basis sets in Cartesian and polar coordinates, respectively.

A spatial transverse mode of order $N = n + m = 2p + |\ell|$ may be represented by a normalized vector $|\psi\rangle$, which may refer to a coherent state as approximated by a classical light beam, or the wave function of a photon. The state vector then lives in a Hilbert space of dimension $N + 1$, and the state space is $\mathbb{C}P^N$ (Allen, Courtial, and Padgett, 1999).

For $N = 1$, $|\psi\rangle$ is a two-state system of the form of Eq. (4), where $|0\rangle$ and $|1\rangle$ correspond to any two linear dependent orthogonal modes, we can choose the LG_0^1 and LG_0^{-1} modes or the $HG_{0,1}$ and $HG_{1,0}$ modes. As in all two-state systems, this state space can be pictured as an ordinary sphere, the so-called sphere of first order modes shown in Fig. 9(a) (Van Enk, 1993; Agarwal, 1999; Padgett and Courtial, 1999). By convention, the poles represent the modes $LG_0^{\pm 1}$ and the equator corresponds to first order HG modes of varying alignment. All diametrically opposed modes form a suitable orthogonal basis system, from which all modes on the sphere can be obtained as a linear superposition. A path C can be formed on the sphere using a sequence of mode-preserving optical elements, like a pair of Dove prisms acting as a mode rotator or a pair of cylindrical lenses acting as a mode convertor (Beijersbergen *et al.*, 1993). The geometric phase associated with the transformation of the first order modes reads (Van Enk, 1993; Galvez *et al.*, 2003)

$$\phi_{g,N=1} = -\tfrac{1}{2}\Omega(C), \tag{11}$$

where $\Omega(C)$ is the solid angle enclosed by the path formed on the sphere, which is analogous to the PB phase. This is not surprising, since the underlying geometry is the same. The phase $\phi_{g,N=1}$ can be interpreted as the holonomy of the connection on the Hopf fibration, where the base space corresponds to the sphere of the first order modes. This interpretation, to our knowledge, has not yet been made explicit in the literature.

For $N > 1$, the dimension of the state space $|\psi\rangle$ grows. Second order modes, for example, need to be expressed in terms of not two but three fundamental modes, where LG_0^{+2}, LG_1^0, and LG_0^{-2} form a complete basis; third order modes require four fundamental modes, where LG_0^{+3}, LG_1^{+1}, LG_1^{-1}, and LG_0^{-3} form a complete basis, etc. Thus far geometric phases have been calculated on two-dimensional subspaces of these high-dimensional state spaces that are represented as spheres. Habraken and Nienhuis (2010) used a number of $(N+1)/2$ spheres to represent modes of odd mode order N and a number of $(N+2)/2$ spheres in order to represent modes of even mode order. In practice, this means that both second and third order modes will be represented using two ordinary spheres; see Figs. 9(b) and 9(c).

However, unlike for first and third order modes, not all the poles of the spheres of second order modes carry orbital angular momentum, indeed, one sphere presents the LG_1^0 mode and the mode (iLG_1^0) at the poles. This is a general feature of even modes. Note also that the modes at the equator of the spheres no longer correspond to the linear superposition of the poles, as would be the case for generalized Poincaré spheres: for the first sphere of second order modes, we would expect to see $HG_{1,1}$ rather than $HG_{0,2}$ modes at the equator. This reflects the choice of Habraken and Nienhuis (2010) to obtain all the modes on these spheres by performing a mode-preserving transformation on the modes at the poles, which can easily be realized in the laboratory using astigmatic mode converters (to move along lines of constant longitude) and image rotators (to move along lines of constant latitude). The geometric phase obtained from a cyclic mode-preserving transformation, which effectively forms a path on these subdimensional state spaces, is then calculated using (Calvo, 2005)

$$\phi_{g,N} = -\tfrac{1}{2}\ell\Omega, \tag{12}$$

where Ω is the solid angle formed on the sphere describing the transformation. When a path is formed on a sphere on which all modes carry the same amount of orbital angular momentum, like the second sphere of second order modes, no geometric phase is generated (Galvez and O'Connell, 2005). This indicates that geometric phases are mediated by a variation of orbital angular momentum, in the same way that polarization transformations that generate a PB phase involve variation of the spin angular momentum (Tiwari, 1992; Van Enk, 1993; McWilliam *et al.*, 2022). While the sphere-based representation is useful, as it directly relates to transformations that are easily realizable in the laboratory, it is not suitable to describe generic transformations in the state space of higher order modes.

Interpreting spatial transverse modes in terms of a fiber bundle would allow us to explore geometric phases over the entire state space, not just two-dimensional subspaces. For a spatial transverse mode of the order of N, the relevant fiber bundle would be the so-called tautological line bundle, with the same base space $\mathbb{C}P^N$, total space H_{N+1}, and $\mathrm{U}(1)$ as the fiber. It would be interesting to determine, at least theoretically, whether transformations over extended portions of the state space lead to the discovery of new geometric or topological phases, with possible applications to topological photonics and quantum communication. In tandem with such fundamental discussions, we may develop experimental techniques that can realize general forms of mode transformations, thereby leading to the expansion of the spatial mode shaping toolbox.

In reality, the exploration of high order spaces in optics has already begun. Indeed, a Majorana representation of structured Gaussian beams was introduced in 2020, revealing that geometric phases born from a cyclic model transformation of generalized structured Gaussian beams can be discrete (Gutiérrez-Cuevas *et al.*, 2020). Following Hannay's observation about the Majorana represention, we expect this result to be confirmed by fiber bundle theory. Investigations are still at an early stage, and other geometric and topological phases may still be awaiting discovery. Note also that more general mode solutions of the paraxial wave equation have received increasing attention in recent years and promise to expand the horizon of geometric phases even further (Alonso and Dennis, 2017; Dennis and Alonso, 2017, 2019). Knotted beams, for which the locus of phase singularities form linked and knotted threats upon propagation (Berry and Dennis, 2001; Leach *et al.*, 2004), may also uncover interesting geometric phases.

Thus far we have considered the spatial and polarization degree of freedom of light independently. We now study vector light fields where they become nonseparable and discuss the implications with regard to their geometric phases.

C. Geometric phases of general vectorial fields

Combining the polarization and spatial degrees of freedom of light amounts to building a bipartite system, where the Hilbert space of the system corresponds to the tensor product of the individual spaces $H_{\mathrm{pol}} \otimes H_{\mathrm{spa}}$. For simplicity, we consider only first order transverse modes, in which case we are dealing with a two-qubit system (Souza *et al.*, 2007). Homogeneously polarized light is described using a product state, separable into a qubit that describes the polarization, and one for the spatial mode. Light with nonhomogeneous polarization instead is nonseparable in these distinct degrees of freedom (Souza *et al.*, 2007). Well-known examples of

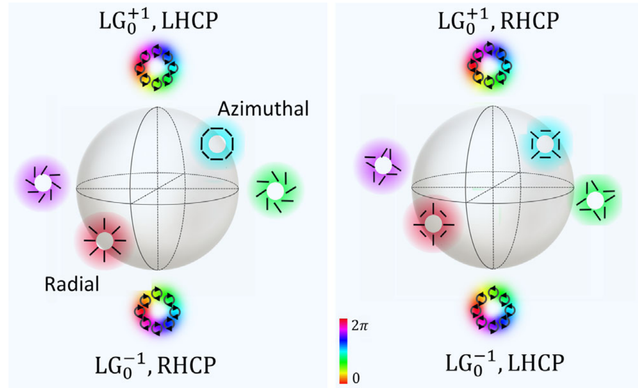

FIG. 10. Spheres of first order polarized modes. In the left sphere, $\sigma = -\ell$ at the poles and the modes on the equator are corotating vector modes. For the sphere on the right, $\sigma = \ell$ and the modes on the equator are counterrotating modes.

nonseparable modes are radially and azimuthally polarized modes of the form $\mathrm{LG}_p^1\sigma_\pm + \mathrm{LG}_p^{-1}\sigma_\mp$, where $\sigma_\pm$ represents left and right circular polarized light, respectively (Zhan, 2009; Otte, Alpmann, and Denz, 2016; Liu *et al.*, 2018; Selyem *et al.*, 2019). These modes have received increasing attention, as they can be focused on tighter spots than their uniformly polarized counterparts (Youngworth and Brown, 2000). General vector beams built from first order modes are usually represented using two Poincaré-like S^2 spheres, shown in Fig. 10, where the poles correspond to uniformly circularly polarized vortex modes, of helicity $\sigma = \pm 1$ and carrying an optical vortex of topological charge $\ell = \pm 1$. The states on the equator correspond to corotating modes, such as radial and azimuthal modes, and counterrotating modes (Holleczek *et al.*, 2011; Milione *et al.*, 2011). The geometric phase associated with the transformation of these modes is then calculated as follows from the solid angle Ω formed on the relevant sphere (Milione *et al.*, 2011):

$$\phi_g = \pm \tfrac{1}{2}(\ell + \sigma)\Omega. \tag{13}$$

The total geometric phase is thus linked to the total angular momentum of the beam $\ell + \sigma$. This was experimentally verified by Milione *et al.* (2012), whose combination of a half wave plate and an astigmatic mode converter realized the mode transformation.

Again, the space of pure states of a two-qubit system $\mathbb{C}P^3$ is not an ordinary sphere. However, depending on the degree of separability of the states, the associated substate space may take a more recognizable form (Bengtsson and Zyczkowski, 2006). Separable states, for instance, form a $\mathbb{C}P^1 \otimes \mathbb{C}P^1$ subspace, called Segre embedding (Bengtsson and Zyczkowski, 2006). There is a curious correspondence between the geometry of arbitrary separable states and the fiber bundle with base space S^4, fiber S^3, and total space S^7 (Mosseri and Dandoloff, 2001). This fibration is a generalization of the Hopf fibration and is normally used to describe quaternions, but it has also been used to study the geometric phases of two-qubit systems (Lévay, 2004). The phase associated with the cyclic evolution of a maximally entangled state is purely topological (Milman and Mosseri, 2003; Lévay, 2004). The topological phase arising under the cyclic transformation of maximally nonseparable optical modes was measured by Souza *et al.* (2007), Souza, Huguenin, and Khoury (2014), and Matoso *et al.* (2019). It would be useful to study whether the tautological line bundle over $\mathbb{C}P^3$ yields similar results, considering the fact that the Hopf fibration was not originally intended for the description of complex fields and does not generalize to arbitrary dimensions.

In this section, we have considered vector modes built from first order modes, but a more general description would include arbitrary vector fields based on spatial modes of higher order. The sphere-based representation presented in Fig. 10 then needs to be expanded by allowing the LG beams at the poles to be of different topological charges ℓ and m. The associated geometric phase then reads (Yi *et al.*, 2015)

$$\phi_g = -\frac{\ell - (m + 2\sigma)}{4}\Omega, \tag{14}$$

where Ω is the solid angle formed on the modified sphere under consideration. This phase was measured by Liu *et al.* (2017), who used two identical q plates. Interpreting this phase in terms of a fiber bundle is certainly possible but would be pure speculation without first addressing the questions raised by two-qubit systems.

VI. SUMMARY AND PERSPECTIVES

Fiber bundle theory presents a rigorous treatment for the understanding of phases. It sheds light on the origin of the solid angle law linking a geometric phase to the path formed on a generalized Poincaré sphere representing the mode space when the mode is transformed. These spaces, however, often represent only two-dimensional subspaces of a high-dimensional state space. They cannot be represented by a sphere and are difficult to visualize. They may, however, present geometric and topologic features giving birth to interesting geometric and topologic phases, which are undetectable in two-dimensional subspace descriptions. Majorana-based representations are slowly emerging. They are capable of providing an accurate expression for geometric phases in high-dimensional state spaces while also providing a clear visual interpretation.

At a more fundamental level, tautological line bundles should be used to calculate these geometric phases; the only ingredient needed is the connection on these bundles. Research on general vectorial modes raises the question of how nonseparability can be accounted for using fiber bundles, and whether this causes measurable effects. With this Colloquium we hope to encourage collaborations between the optics and mathematics communities, as we believe that higher order structured Gaussian modes and vector modes may allow for the exploration of new concepts.

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