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Lepton flavor symmetries

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A general classification of flavor symmetries is provided according to their interplay with the proper Poincaré and gauge groups and to their linear or nonlinear action in field space. The focus is on the lepton sector, and the different types of symmetries describing neutrino masses and the lepton mixing matrix are reviewed. Several illustrative examples are presented for each type of symmetry, and specific strengths and limitations are discussed.

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CONTENTS

I. INTRODUCTION

The replica of fermion families and their masses and intergenerational properties constitute one of the most interesting mysteries of particle physics. While gauge symmetry strongly restricts matter interactions mediated by spin 1 particles, it leaves essentially unconstrained scalar-fermion interactions responsible for fermion masses and mixing angles. In the flavor sector of the standard model (SM) there are as many independent parameters as the number of charged fermion masses and quark mixing parameters. The toll is raised to 22 if we include, in a general low-energy description, neutrino masses and lepton mixing parameters. We are facing a puzzle with many known pieces that we are still unable to put together in a coherent picture. The discovery of neutrino oscillations has brought great hopes for the solution of this puzzle. Neutrinos are extremely light, calling for a different origin of their masses, potentially related to new undiscovered properties of particle interactions. Moreover atmospheric and solar neutrino oscillations require large lepton mixing angles, a completely unexpected feature, clashing against the properties of the quark sector. As we review here, many of these properties have been determined to good precision and there are excellent prospects for future improvements aimed at pinning down the few unknown aspects. Nevertheless, while neutrino data stimulated a great deal of theoretical activity, they also heightened the mystery of fermion masses, in that no compelling underlying principle to describe this aspect of elementary particles has uniquely emerged thus far. Neutrinos and charged leptons possess special features that are the focus of this review, although any description applicable to this sector alone should be viewed as only a partial answer to the general problem of fermion masses.

The observation and study of neutrino oscillations have established that neutrinos are massive. Two independent squared mass differences and three lepton mixing angles have been determined with an accuracy approaching the percent level, moving the entire field into a precision era. Most of the experimental results can be coherently interpreted in the context of three light active neutrinos and CPT invariance. Experiments sensitive to solar, atmospheric, reactor, and accelerator neutrinos provide a consistent picture supported by many redundant tests. In Table I we report the results of recent fits to the oscillation parameters. Notation and conventions are those of the *Review of Particle Physics* by the Particle Data Group (PDG) (Tanabashi *et al.*, 2018) unless otherwise stated. The mixing pattern in the lepton sector appears to be totally different than that in the quark sector, with two large mixing angles and a third one similar in size to the Cabibbo angle.

Global analyses start to be sensitive both to the mass ordering and to the Dirac CP-violating phase δ. A preference for normal mass ordering (NO) over inverted mass ordering (IO) is emerging from the data, at the level of about 3σ. The best fit value for the Dirac CP-violating phase is $\delta \approx (1.2-1.3)\pi$ for NO, but uncertainties are large and CP conservation is still allowed within 2σ.

Dedicated experiments have been planned to determine the mass ordering and δ. Mass ordering measurements with an individual significance of more than 3σ could be realized with several different technologies and methods, exploiting atmospheric (KM3NeT/ORCA, PINGU, INO), reactor (JUNO), and accelerator (DUNE, Hyper-K) neutrinos. DUNE and Hyper-K have planned sensitivities to CP violation higher than 5σ for most of the allowed range, even though a precise determination of δ around the maximal value would be challenging.

TABLE I. Best fit values and 1σ errors of the three-flavor oscillation parameters in the global analysis given by Esteban *et al.* (2019). The results include data on atmospheric neutrinos provided by the Super-Kamiokande Collaboration. There is a difference of $\Delta\chi^2(\mathrm{IO}-\mathrm{NO}) = 10.4$ between inverted ordering (IO) and normal ordering (NO). Note that $\Delta m^2_{3\ell} = \Delta m^2_{31} > 0$ for NO and $\Delta m^2_{3\ell} = \Delta m^2_{32} < 0$ for IO. For other recent global analysis, see De Salas *et al.* (2018), Gariazzo *et al.* (2018), and Capozzi *et al.* (2019).

	Normal ordering	Inverted ordering
$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.310^{+0.013}_{-0.012}$
$\sin^2\theta_{23}$	$0.563^{+0.018}_{-0.024}$	$0.565^{+0.017}_{-0.022}$
$\sin^2\theta_{13}$	$0.02237^{+0.00066}_{-0.00065}$	$0.02259^{+0.00065}_{-0.00065}$
δ/π	$1.23^{+0.22}_{-0.16}$	$1.57^{+0.13}_{-0.14}$
$\Delta m^2_{21}/10^{-5}$ eV^2	$7.39^{+0.21}_{-0.20}$	$7.39^{+0.21}_{-0.20}$
$\Delta m^2_{3\ell}/10^{-3}$ eV^2	$2.528^{+0.029}_{-0.031}$	$-2.510^{+0.030}_{-0.031}$

The absolute neutrino mass scale is still unknown, although it is well constrained by both laboratory and cosmo logical observations. The current laboratory limit $m_\nu = \sqrt{\sum_i |U_{ei}|^2 m_i^2} < 1.1$ eV (90% C.L.), recently set by the KATRIN experiment (Aker *et al.*, 2019), is expected to be further improved in the future. At present cosmology provides the most stringent bound on the sum of neutrino masses, $\sum_i m_i < 0.12 - 0.68$ eV, although it is subject to uncertainties inherent to the adopted cosmological model: the number of free parameters used to fit observations and the actual set of data included in the analysis (Tanabashi *et al.*, 2018). Upper bounds on neutrino masses become weaker when the data are analyzed in the context of extended cosmological models or when a conservative set of data is used, but they are not considerably weaker. These bounds are expected to improve significantly in the future thanks to recently planned experiments. If the Λ cold dark matter model of the Universe is confirmed, and if neutrinos have standard properties, nonvanishing neutrino masses should be detected at the level of at least 3σ (Tanabashi *et al.*, 2018).

The impressive suppression of neutrino masses is peculiar, even compared to that of the lightest charged fermions. Not only is the electron mass suppressed by only a factor $\mathcal{O}(10^5)$ but the latter suppression follows from the interfamily hierarchy displayed by charged fermion masses, with subsequent families separated by only about 2 orders of magnitude. Conversely, all three neutrino families are separated from the electroweak scale by at least 11 orders of magnitude. The striking size of neutrino masses might be related to the possibility that the total lepton number L is violated, although this is certainly not the only possible explanation. Violation of the individual lepton numbers has been established, but we still do not know whether or not L is violated in nature. The experimental clarification of this central aspect might shed light on the possible origin of flavor. Indeed, from the theory viewpoint the simplest explanation of the smallness of neutrino masses is in terms of the violation of L at a large scale, possibly not far from the grand unified scale.

Experimentally, the most promising L-violating transition is the neutrinoless double beta ($0\nu\beta\beta$) decay. If interpreted in the context of three light Majorana neutrinos, present experiments allow researchers to set an upper bound on $|m_{ee}| = |\sum_i U_{ei}^2 m_i|$, a combination of neutrino masses, mixing angles, and Majorana phases. Despite the uncertainties due to the lack of knowledge of absolute masses and Majorana phases, $|m_{ee}|$ can be constrained by neutrino oscillation data alone and, at least in the case of IO, the allowed region is getting closer to the range explored by current $0\nu\beta\beta$ decay experiments. In Table II we report on some recent experimental results. We refer the interested reader to reviews given by Päs and Rodejohann (2015), Dell'Oro *et al.* (2016), Vergados, Ejiri, and Šimkovic (2016), and Dolinski, Poon, and Rodejohann (2019).

Few experimental anomalies require additional observational support or a coherent theoretical interpretation. These include (i) the so-called reactor anomaly (Mention *et al.*,

TABLE II. Lower bound on $T^{0\nu}_{1/2}$ (90% C.L.) and upper bound on $|m_{ee}|$ from GERDA (Agostini *et al.*, 2018), CUORE (Alduino *et al.*, 2018), KamLAND-Zen (Gando *et al.*, 2016), and EXO-200 (Anton *et al.*, 2019). The quoted range reflects the uncertainty in the nuclear matrix elements required to translate the half-life $T^{0\nu}_{1/2}$ into $|m_{ee}|$.

Isotope	Lower bound on $T^{0\nu}_{1/2}$ (yr)	Upper bound on $\lvert m_{ee}\rvert$ (meV)	Collaboration
^{76}Ge	8.0×10^{25}	$120-260$	GERDA
^{130}Te	1.5×10^{25}	$110-520$	CUORE
^{136}Xe	1.07×10^{26}	$61-165$	KamLAND-Zen
^{136}Xe	3.5×10^{25}	$93-286$	EXO-200

2011), i.e., evidence for the disappearance of electron antineutrinos in short baseline experiments; (ii) the gallium anomaly (Abdurashitov *et al.*, 1999, 2006; Kaether *et al.*, 2010), i.e., the observed deficit in the gallium radioactive source experiments; and (iii) indications of $\nu_\mu \to \nu_e$ conversion from the LSND (Aguilar-Arevalo *et al.*, 2001) and MiniBoone (Aguilar-Arevalo *et al.*, 2018) experiments. Taken at face value, these effects do not fit the standard framework with three light neutrinos, and explanations invoking a fourth sterile neutrino have been adopted. Even in such an extended scheme the anomalies do not find a coherent interpretation, due to the tensions between appearance and disappearance data (Dentler *et al.*, 2018), which indicates either the need for a less minimal framework or the invalidation of some of the experiments. While the discovery of a sterile neutrino would represent a major result of the current experimental activity and a nontrivial challenge for its interpretation in the context of the flavor puzzle, here we assume a low-energy framework with three light active neutrinos and CPT invariance. New states are not excluded but are assumed to be heavy, allowing for an effective description of current experiments where only the light degrees of freedom take action.

There are few theoretical tools allowing for a quantitative and predictive description of neutrino mass and mixing parameters. The focus of this review is on flavor symmetries, one of the most appealing options, given the role that symmetries have played in accounting for the properties of fundamental interactions. The idea that relations among mass parameters can be enforced by symmetries is an old one. The most predictive case is represented by exact symmetries, a prototype of which is gauge invariance in quantum electrodynamics, guaranteed only if the photon is massless. Exact symmetries do not apply to fermion masses and mixing angles. For example, SM Yukawa couplings break the large non-Abelian global symmetry of quark gauge interactions, down to the baryon number and to the global hypercharge transformations, which provide no restrictions to mass parameters. The lepton sector follows a similar fate and a realistic description of fermion masses should necessarily rely on approximate symmetries. As a consequence, breaking terms are crucial for determining the correct pattern of masses and mixing angles. Moreover, in interesting cases flavor symmetries are realized far from the exact phase, with symmetry breaking effects playing a leading role. This feature makes it difficult to single out a baseline model or a unique candidate for the flavor group.

For these reasons a large part of this review is devoted to a general discussion of symmetries and symmetry breaking that is independent of their specific realization in model building. We provide a general classification of flavor symmetries that is compatible with a local, gauge-invariant, and relativistic quantum field theory. We distinguish between symmetries acting linearly or nonlinearly in field space. In particular, dealing with the nonlinear case, we go beyond the well-established Callan-Coleman-Wess-Zumino formalism (Callan *et al.*, 1969; Coleman, Wess, and Zumino, 1969), which does not cover the relevant case of discrete symmetries. We offer a more general description that accommodates all cases of interest. We also distinguish symmetries commuting with the Poincaré and gauge groups from those that do not. The latter choice includes CP-like flavor symmetries, which have received significant attention in recent years, especially in connection with discrete symmetry groups. This classification, meant to cover not only the lepton sector but also the entire fermion area, is particularly relevant to clearly identify the uncharted directions from the previously explored ones. Moreover, in our view it should not be viewed as a formal mathematical exercise since it reflects important physical aspects of the symmetries in question. For example, CP-like flavor symmetries are especially efficient in constraining physical phases. Symmetries whose action is nonlinear can potentially enhance the predictive power of the model, as they are able to relate operators of different dimensionality.

We also examine how symmetry breaking can be efficiently described through the use of spurions, allowing one to capture the cases of both explicit and spontaneous breaking. We discuss how predictions about the mixing matrix can be viewed as a solution to a problem of vacuum alignment. When the vacuum arises from the minimization of an energy density functional, general results are encoded in the space of invariants of the theory and in the structure of its boundaries. We provide, for the first time in the context of flavor symmetries, a concise review of this important topic, where the problem of symmetry breaking finds its most natural mathematical formulation. The rest of our review is devoted to summarizing the state of the art in model building, organized according to our general classification of flavor symmetries. Aware that this part can easily become obsolete in a short time, we emphasize the general features of model building, limiting the discussion of specific models to a few examples in each category. We also comment on the possibility of extending each type of symmetries from the lepton sector to the quark one. The number of possibilities offered to model building is large, and many of them have already been surveyed in excellent reviews (Altarelli and Feruglio, 2010; Ishimori *et al.*, 2010; Smirnov, 2011; King and Luhn, 2013; King *et al.*, 2014; King, 2017; Petcov, 2018; Xing, 2019).

Flavor symmetries do not exhaust all possible quantitative approaches to the flavor puzzle. For example, mass and mixing low-energy parameters can satisfy fixed-point relations, originating from the renormalization group flow of generic input parameters defined at a high-energy scale. Infrared stable fixed points of the renormalization group equations for Yukawa couplings and fermion masses were studied long ago. In the lepton sector, no acceptable relations among the mixing angles have been found in the CP-conserving regime (Chankowski and Pokorski, 2002), while in the CP-violating regime the only viable constraint (Casas *et al.*, 2000) requires a strong degeneracy between the closest neutrino masses.

Another possibility is offered by the mechanism of radiative mass generation, when a combination of mass parameters that accidentally vanishes at the classical level gets a nonvanishing calculable contribution at higher orders of perturbation theory. In particular, it has been suggested that the lightness of neutrinos might arise in this context from loop suppression factors. States running in the internal lines of the loop can be sufficiently light to be probed at existing facilities, which is at variance with the typically heavy states of the seesaw mechanism. The new states can also lead to lepton flavor violation, which will potentially be observable at current or future high-intensity facilities. We comment on such a possibility when discussing the mechanism for neutrino masses.

This review consists of seven sections. After recalling the possible origin of neutrino masses in Sec. II, in Sec. III we present a general classification of flavor symmetries and discuss general aspects of symmetry breaking. Sections IV–VI provide a more specific description and several illustrative examples of the type of symmetries classified in Sec. III. Finally, in Sec. VII we summarize our thoughts on the subject. There are many related topics that we only briefly mention or have deliberately left out of this work. This list is long and includes extension to the quark sector within grand unified theories or string theory, realization in the context of extra dimensions, relation to lepton flavor violation searches and leptogenesis, and mathematical aspects such as group theory. We refer the interested reader to the aforementioned literature.

II. ORIGIN OF NEUTRINO MASSES

A. Neutrino masses and the standard model

Neutrinos are massless in the standard model according to its usual definition as a renormalizable theory involving left-handed neutrinos only. While such a prediction is certainly at odds with everything we have learned about neutrinos in recent decades and represents an incontrovertible reason to extend the SM, it can at the same time be considered a success of the SM since it offers a basis for the understanding of the peculiar smallness of neutrino masses.

The SM gauge structure is indeed crucial in forbidding neutrinos from getting a mass. In the effective theory below the electroweak scale, with $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{em}}$ the gauge group, both the charged fermions and the neutrinos are allowed to get a mass. Therefore, the peculiar size of neutrino masses is not addressed by the gauge structure in this case.

The neutrino mass term allowed in the $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{em}}$ theory is of Majorana type, and as such it violates the total lepton number. The fact that such a mass term is not generated by the SM completion can therefore be seen as a consequence of the accidental conservation of lepton number in the SM (or from direct inspection: no renormalizable interaction gives rise to neutrino masses after electroweak symmetry breaking, as we have not introduced, so far, right-handed neutrinos).

Accidental symmetries are not imposed by hand; they just happen to be global symmetries of the most general renormalizable Lagrangian invariant under the given gauge transformations. The SM turns out to have four independent accidental symmetries, associated with the conservation of baryon number B and of the three individual lepton numbers L_i. The total lepton number $L = \sum_i L_i$ is therefore also accidentally conserved. As we later discuss, the SM accidental symmetries are a residual subgroup of the $\mathrm{U}(3)^5 \times \mathrm{U}(1)_H$ global symmetry that the SM acquires when its Yukawa couplings are set to zero, which in turn underlies the idea of flavor symmetries.

The emergence of lepton number as an accidental symmetry is one of the notable features of the SM. On the one hand, it predicts the suppression of lepton-number-violating processes in nature (thus providing a zeroth order approximation for the smallness of Majorana neutrino masses $m_\nu = 0$). On the other hand, since lepton number is not postulated to be a fundamental symmetry, small lepton-number-violating effects are not forbidden. This is welcome, as a small (but conceptually and practically important) breaking of lepton and baryon number takes place even within the SM because of nonperturbative effects ('t Hooft, 1976a, 1976b). Moreover, it is welcome because it leaves room for a small breaking of lepton number and, in particular, for small Majorana neutrino masses, originating from possible UV completions of the SM. Grand unified theories (GUTs) explicitly break lepton and baryon number and are therefore not compatible with enforcement of the conservation of lepton number by hand.

B. Origin of neutrino masses: Standard framework

Section II.A laid the groundwork for a basic understanding of the origin and size of neutrino masses. Such an understanding is based on the sole hypothesis that the new ingredients needed to be added to the SM to account for neutrino masses, whatever they are, lie at a scale significantly larger than the electroweak scale.

If that is the case, effective field theory (EFT) ensures that it is possible to account for the effect (including neutrino masses) of such new ingredients at lower scales by adding to the SM Lagrangian additional nonrenormalizable, or "effective," operators. The nonrenormalizable Lagrangian one obtains is called the SM effective field theory (SMEFT).

The effective operators are suppressed by powers of the scale of the new physics generating them, the "cutoff" Λ. The perturbative validity of the theory is limited to energies well below the cutoff. There the impact of an effective operators is suppressed by a factor of $(E/\Lambda)^{D-4}$, where D is the dimension of the operator in energy. Therefore, the most relevant operators are in principle the lowest-dimensional ones. In the $E \ll \Lambda$ regime, the theory can be renormalized with a finite number of counterterms order by order in an expansion in the operator dimension.

The effective operators contain SM fields only and simply need to obey the SM gauge invariance, so no actual knowledge of the physics originating from them is required to account for its low-energy effect.

The single lowest dimensional operator allowed in the SMEFT, the $D = 5$ Weinberg operator (Weinberg, 1979)

$$\frac{c_{ij}}{2\Lambda}(l_i H)(l_j H), \tag{2.1}$$

is precisely what is needed to account for neutrino masses. In Eq. (2.1), l_i, $i = 1, 2, 3$, are the lepton doublets and H is the Higgs doublet. There and below, $\mathrm{SU}(2)_L$-invariant contractions of the doublet indices are understood because of the 2×2 antisymmetric tensor in Eq. (2.1). The splitting of the coefficient into a dimensionless numerator c_{ij} and a dimensionful denominator Λ is arbitrary. Λ is supposed to represent the scale of the new degrees of freedom whose virtual exchange gives rise to the operator, and c_{ij} is intended to group the coupling, mixings, and loop factors involved, which are not supposed to be larger than $\mathcal{O}(1)$ in a perturbative regime and $\mathcal{O}(4\pi)$ in a nonperturbative one.

The origin of the operator in Eq. (2.1) must be associated with lepton-number-violating physics, as the operator itself breaks the lepton number by two units. It also breaks $B - L$, an important ingredient for high-scale baryogenesis (Kuzmin, Rubakov, and Shaposhnikov, 1985). After electroweak symmetry breaking, the operator gives rise to a neutrino Majorana mass term in the form

$$\frac{m_{ij}}{2}\nu_i \nu_j, \tag{2.2}$$

with

$$m_{ij} = c_{ij}\frac{v^2}{\Lambda}, \tag{2.3}$$

where v is the electroweak scale, with $v = |\langle H \rangle| \approx 174$ GeV.

The peculiarity of neutrino masses is now accounted for by their differing dependence on the electroweak scale. While charged fermion masses are linear in v, neutrino masses turn out to be quadratic in v and thus suppressed by a factor of v/Λ with respect to the former. Their suppression is attributed to the heaviness of the scale Λ at which lepton number is violated. If m_h is the heaviest neutrino mass and c_h is the heaviest eigenvalue of the matrix c_{ij}, we have

$$\Lambda \approx 0.5 \times 10^{15}\ \mathrm{GeV} c_h \left(\frac{0.05\ \mathrm{eV}}{m_h}\right). \tag{2.4}$$

The scale Λ of the new physics associated with the neutrino masses can be as large as 10^{15} GeV, hence hinting at a possible connection with GUT physics, or much smaller if the couplings λ_{UV} on which c_h depends are small. As c_h usually depends quadratically on λ_{UV}, UV couplings of the order of 10^{-2} are sufficient to bring Λ down to 10^{11} GeV.

While the Weinberg operator is the lowest-dimensional, and therefore in principle the most relevant, effective operator giving rise to neutrino masses, higher order operators may become relevant if the former turns out to be suppressed. On the other hand, higher order operators contributing to neutrino masses simply contain additional pairs of conjugated Higgs fields. Therefore, any symmetry suppressing the Weinberg operator would also suppress those higher order operators. Barring an accidental suppression of the former, the latter hardly have a chance to dominate. The situation changes in extensions of the SM Higgs sector by a singlet and/or a second doublet. Then it is possible to define symmetries forbidding the $D = 5$ operator, but not higher order ones (Babu, Nandi, and Tavartkiladze, 2009; Bonnet *et al.*, 2009; Gogoladze, Okada, and Shafi, 2009). In such cases, neutrino masses turn out to be suppressed by higher powers of v/Λ, which lowers the needed scale of Λ. Higher order operators can also involve new fields that do not get a vacuum expectation value (VEV) and still contribute to neutrino masses, if the new field lines close into a loop. If the new fields are heavy and integrated out, this possibility can still be accounted for in terms of the $D = 5$ Weinberg operator (see the following comment on its radiative origin).

1. The case for right-handed neutrinos

While the previously mentioned framework offers a simple and compelling understanding of the size of neutrino masses, it relies on the absence of a "right-handed" counterpart of the SM neutrinos. All the left-handed charged fermions contained in the quark and lepton doublets $q_i = (u_i, d_i)^T$, $l_i = (\nu_i, l_i)^T$ have $\mathrm{SU}(2)_L$ singlet partners[1] u_i^c, d_i^c, e_i^c leading to Dirac masses through the Yukawa interactions $\lambda_{ij}^U u_i^c q_j H + \lambda_{ij}^D d_i^c q_j H^* + \lambda_{ij}^E e_i^c l_j H^* + \mathrm{H.c.}$, so it is not unnatural to postulate that the neutrinos ν_i should also be accompanied by an $\mathrm{SU}(2)_L$ singlet partner ν_i^c, leading to a neutrino Dirac mass term through the Yukawa interaction

$$\lambda_{ij}^N \nu_i^c l_j H + \mathrm{H.c.} \tag{2.5}$$

If so, what would make neutrino masses peculiar?

Note that the existence of the singlet neutrinos ν_i^c is predicted in a number of extensions of the SM providing an understanding for the SM gauge quantum numbers, and thus further motivated. This is the case of extensions based on the left-right symmetric gauge group G_{LR}=$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$, the Pati-Salam group $G_{\mathrm{PS}} = \mathrm{SU}(4)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, or the grand unification group SO(10).

The special size of neutrino masses can be accounted for even in the presence of singlet partners for the neutrinos as well, as such singlet neutrinos carry their own peculiarity. For them to give rise to a neutrino mass term through gauge-invariant Yukawa interactions, the fields ν_i^c should be singlets under the entire SM group.[2] The previously mentioned SM extensions also predict them to be SM singlets. Therefore, the neutrino singlets would be the only fermions allowed to have an explicit, gauge-invariant (and lepton-number-violating) mass term

[1]The superscript c in f^c denotes the charge conjugate of the right-handed component of f in the Dirac spinor formalism, or a left-handed field independent of f in the Weyl spinor formalism.

[2]If the field ν^c is allowed to have more than one component, it could alternatively be a $\mathrm{SU}(2)_L$ triplet. The argument that follows would still go through, as it is based on ν^c being the only fermion in a real representation of the SM group, with all the others belonging to a fully chiral representation.

$$\frac{M_{ij}}{2}\nu_i^c\nu_j^c + \text{H.c.} \tag{2.6}$$

Such a mass term has no ties to the electroweak scale, as it survives in the limit in which the electroweak scale vanishes. Hence, there is no reason why it could not be much heavier than the electroweak scale. If that is the case, the singlet neutrinos simply represent a specific and prototypical realization of the previously discussed framework: new degrees of freedom lying at a scale significantly larger than the electroweak scale. It must therefore be possible to account for their effect at the electroweak scale and below in terms of effective operators. Indeed, integrating them out [as reviewed by Altarelli and Feruglio (2004)] precisely generates the Weinberg operator with, in matrix notation,

$$\frac{c}{\Lambda} = -\lambda_N^T M^{-1}\lambda_N, \tag{2.7}$$

where λ_N and M are the parameters in Eqs. (2.5) and (2.6), respectively. The light neutrino masses end up being given by the seesaw formula (Minkowski, 1977; Gell-Mann, Ramond, and Slansky, 1979; Yanagida, 1979; Glashow, 1980; Mohapatra and Senjanovic, 1980)

$$m_\nu = -m_D^T M^{-1} m_D, \tag{2.8}$$

where $m_D = \lambda_N v$ is a Dirac-like neutrino mass term. The advantage of the EFT derivation over the diagonalization of the 6×6 matrix of the $\nu_i + \nu_i^c$ system is that it allows one to organize the computation of potentially large, log-enhanced radiative corrections to the seesaw formula by means of the renormalization group equations. The coefficient of the Weinberg operator is calculated from Eq. (2.7) at the singlet neutrino scale, and the Weinberg operator is subsequently run down to the electroweak scale. Within the SM, gauge interactions and quark Yukawa interactions only affect (at one loop) the overall neutrino mass scale, while flavor-dependent effects from lepton Yukawa interactions are negligible. Sizable flavor corrections can arise in two Higgs doublet schemes in the large $\tan\beta$ regime in the presence of an "unstable" (Domcke and Romanino, 2016) neutrino mass approximate degeneracy; see Chankowski and Pokorski (2002). If the heavy neutrinos are hierarchical, threshold effects associated with their sequential decoupling may also be important.

2. Tree-level origin of the Weinberg operator

We have seen that neutrino singlets, unless unexpectedly light, represent a specific realization of the general situation in which the new physics needed to account for neutrino masses lies at a scale significantly higher than the electroweak scale. We then wonder what is the most general form of the heavy new physics giving rise to the Weinberg operator. A simple and complete answer is found in the assumption that the Weinberg operator is generated at the tree level. In such a case, the virtual heavy states can have only three types of SM quantum numbers, corresponding to type-I, type-II (Magg and Wetterich, 1980; Lazarides, Shafi, and Wetterich, 1981; Mohapatra and Senjanovic, 1981),[3] and type-III (Foot *et al.*, 1989) seesaw. We now list them using the notation (r_3, r_2, y) for the SM gauge quantum numbers, where r_3 is the $SU(3)_c$ representation, r_2 is the $SU(2)_L$ representation, and y is the value of the hypercharge (in units in which the SM Higgs sector has $y = 1/2$).

Type I.—The virtual messengers are fermions ν^c with SM quantum numbers (1,1,0); i.e., they are SM singlets. This is essentially the case discussed earlier, with the only variation that the number n of singlet neutrinos is not bound to be 3. To reproduce both the atmospheric and solar squared mass differences, $n \geq 2$ is needed. The relevant high-scale Lagrangian is given by Eqs. (2.5) and (2.6) as

$$-\mathcal{L}_{\text{I}} = \lambda_{kj}^N \nu_k^c l_j H + \frac{M_{kh}}{2}\nu_k^c\nu_h^c + \text{H.c.}, \tag{2.9}$$

where the number of singlet neutrinos is now n, the Yukawa coupling λ_N is an $n\times 3$ matrix, and the mass term M is an $n\times n$ symmetric matrix. The effective Weinberg operator and the neutrino masses are again given by

$$\frac{c}{\Lambda} = -\lambda_N^T M^{-1}\lambda_N, \qquad m_\nu = -m_D^T M^{-1} m_D. \tag{2.10}$$

Type II.—The virtual messengers are complex scalars Δ_k, $k = 1,\ldots,n$, with SM quantum numbers (1,3,1); i.e., they are $SU(2)_L$ triplets with hypercharge $Y = 1$. The relevant high-scale Lagrangian is

$$\begin{aligned} -\mathcal{L}_{\text{II}} = {} & \tfrac{1}{2}(y_{ijk} l_i \sigma_a l_j \Delta_k^a + \mu_k H\sigma_a H \Delta_k^{a*} + \text{H.c.}) \\ & + M_{kh}^2 \Delta_k^{a*}\Delta_h^a, \end{aligned} \tag{2.11}$$

where the mass matrix M^2 is now Hermitian and Δ^a, $a = 1,2,3$, are the components of the triplets Δ. Integrating them out gives rise to the Weinberg operator and neutrino masses, with

$$\frac{c_{ij}}{\Lambda} = -y_{ijh}(M^2)_{hk}^{-1}\mu_k, \qquad (m_\nu)_{ij} = -v^2 y_{ijh}(M^2)_{hk}^{-1}\mu_k. \tag{2.12}$$

The role of the cutoff Λ is now played by the combination M^2/μ, where μ^2 can be expected to be of the same order as M^2. Unlike in the type-I and type-III cases, one triplet is in principle sufficient to reproduce both the atmospheric and solar squared mass differences.

Type III.—This case is similar to type I, but the messengers are now $SU(2)_L$ triplets. That is, they are fermions T_k, $k = 1,\ldots,n$, with SM quantum

[3] Schechter and Valle (1980, 1982) noted that a scalar triplet VEV directly contributes to neutrino masses, with no seesaw suppression by the triplet mass.

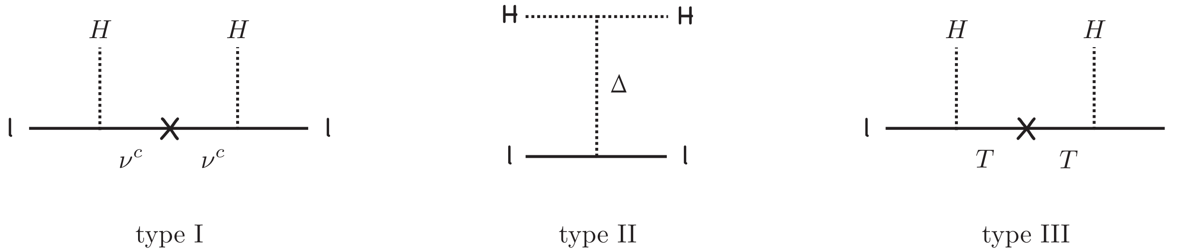

FIG. 1. Diagrammatic representation of the three types of seesaw mechanisms. They all give rise to the Weinberg effective operator in Eq. (2.1) once the intermediate states are integrated out. The crosses denote lepton-number-violating mass insertions.

numbers (1,3,0), and again $n \geq 2$. The relevant high-scale Lagrangian is

$$-\mathcal{L}_{\rm III} = \lambda^T_{ij} T^a_i l_j \sigma_a H + \frac{M_{ij}}{2} T^a_i T^a_j + \text{H.c.}, \qquad (2.13)$$

where T^a, $a = 1, 2, 3$, are the components of the triplets T. Integrating them out generates the Weinberg operator and neutrino masses, with

$$\frac{c}{\Lambda} = -\lambda_T^T M^{-1} \lambda_T, \qquad m_\nu = -m_T^T M^{-1} m_T, \qquad (2.14)$$

where $m_T = v\lambda_T$.

A simple analysis based on gauge invariance shows that the tree-level diagrams in Fig. 1, which correspond to the three previously mentioned seesaw Lagrangians, are the only possible ones (Ma, 1998). A complex scalar with quantum numbers (1,1,1) cannot play a role at the tree level, as it couples to the antisymmetric combination of $l_i l_j$ but not to hh.

3. Radiative origin of the Weinberg operator

While a tree-level origin of the Weinberg operator is the most appealing option [and the only one with unbroken supersymmetry (Megrelidze and Tavartkiladze, 2017)], the possibility of a radiative origin is not excluded; see (Cai *et al.* (2017) for a recent review. Depending on the specific field content of the UV theory, a tree-level origin may not be available, while the Weinberg operator can arise through quantum corrections at the loop level. The topologies of the corresponding Feynman diagrams have been classified up to two-loop order (Babu and Leung, 2001; de Gouvea and Jenkins, 2008; Bonnet *et al.*, 2012; Angel, Rodd, and Volkas, 2013; Aristizabal Sierra *et al.*, 2015) and require at least two new multiplets to play the role of intermediate states (Law and McDonald, 2014). Once those states are integrated out, within an effective-theory approach, the Weiberg operator is not generated at the tree level. Other lepton-number-violating operators are, however, and they give rise to the Weinberg one through loops involving SM interactions and fields. The new states cannot be far from the electroweak scale, and the suppression of the neutrino masses compared to the latter is at least partially accounted for by the loop factor $[1/(16\pi^2)]^\ell$, where ℓ is the loop order at which the diagram arises, if ℓ is sufficiently large.

Such models may be characterized by a possibly interesting phenomenology at colliders and in charged-lepton flavor violation (CLFV) experiments, although their aesthetic appeal does not match the tree-level seesaw one. On the one hand, the suppression of neutrino masses is better accounted for when ℓ is relatively large. On the other hand, the increase of ℓ leads to a rapid increase of the number of diagrams. The structure and field content of the model is not as constrained as in the tree-level case. On the contrary, a plethora of possibilities are available. Finally, the model parameters often need to be fine-tuned to cope with the present bounds on CLFV and reproduce neutrino masses and mixings. For further information on such class of models, see the reviews given by Boucenna, Morisi, and Valle (2014) and Sugiyama (2015).

C. Lower scale origin of neutrino masses

As we have seen, effective field theory provides a simple and compelling understanding of the origin and peculiar smallness of neutrino masses, under the sole hypothesis that the new degrees of freedom needed to account for non-vanishing neutrino masses lie significantly above the electroweak scale. Neutrino masses, on the other hand, can also originate well below the electroweak scale. Dirac neutrinos are the prototypical example. The SM neutrinos in such a case get a purely Dirac mass from Yukawa couplings to otherwise massless singlet neutrinos [$M = 0$ in Eq. (2.6)]. While the standard framework unavoidably leads to lepton-number-violating Majorana neutrino masses, Dirac neutrinos conserve lepton number, which offers an opportunity to experimentally probe the origin of neutrino masses.

Before ending up with $M = 0$ and purely Dirac neutrinos, we consider the intermediate possibility that M does not vanish but is not significantly larger than the electroweak scale, so the SMEFT approach used in Sec. II.B does not apply.

If the singlet neutrino masses are not far from the electroweak scale, they can play a role in future collider phenomenology (Antusch and Fischer, 2015; Deppisch, Bhupal Dev, and Pilaftsis, 2015). As those masses can in principle be as large as the Planck scale, their proximity to the electroweak scale, about 15 orders of magnitude smaller, would represent a nontrivial accident.

In the presence of a single family, a singlet neutrino mass $M \sim \text{TeV}$ requires a neutrino Yukawa coupling as small as

$$\lambda_N \sim 1.3 \times 10^{-6} \left(\frac{m_\nu}{0.05\ \text{eV}} \frac{M}{\text{TeV}} \right)^{1/2}. \qquad (2.15)$$

The smallness of the neutrino masses is accounted for by the smallness of λ, and such a small coupling would make collider effects hardly observable.

With three families, though, larger Yukawa couplings are allowed if cancellations take place in the seesaw formula. Nonaccidental cancellations can be forced by appropriate symmetries, such as the lepton number itself, allowing the large Yukawa couplings while forbidding the neutrino masses (Kersten and Smirnov, 2007; Xing, 2009), and can involve additional singlets (Mohapatra, 1986; Mohapatra and Valle, 1986; Akhmedov *et al.*, 1996a, 1996b; Barr, 2004; Malinsky, Romao, and Valle, 2005; Barr and Dorsner, 2006; Ibarra, Molinaro, and Petcov, 2010). The larger Yukawa couplings then have a chance to be probed at colliders. Such symmetric couplings are no longer directly related to the origin of neutrino masses (and their size), which in this case is instead associated with the symmetry breaking parameters (and their smallness).

The collider prospects are richer when interactions besides those directly related to neutrino masses provide additional production or detection channels. This is the case when the heavy states feel gauge interactions. For example, the SM singlet neutrinos can be charged under extensions of the SM group containing an $\mathrm{SU}(2)_R$ factor (Keung and Senjanovic, 1983; Nemevsek *et al.*, 2011; Das *et al.*, 2012). Even sticking to the SM group, the components of Δ and T (in type-II and type-III seesaws, respectively) charged under the SM can enrich the collider phenomenology (Akeroyd and Aoki, 2005; Han *et al.*, 2007; del Aguila and Aguilar-Saavedra, 2009).

The collider bounds on the charged component of Δ and T prevent the type-II and type-III seesaws from being extrapolated below the electroweak scale (barring an unnatural splitting among neutral and charged components). On the other hand, the singlet neutrino mass in the type-I seesaw can be arbitrarily small, or zero, as previously argued.

In the intermediate regime in which the singlet neutrino masses are lighter than the electroweak scale but significantly larger than the energy of the relevant neutrino processes, it is still possible to integrate out the singlet neutrinos. As the SM group is badly broken in such a regime, it is appropriate in this case to start from the $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{em}}$ invariant Lagrangian (Altarelli and Feruglio, 1999b)

$$m^D_{ij}\nu^c_i\nu_j + \frac{M_{ij}}{2}\nu^c_i\nu^c_j + \mathrm{H.c.}, \tag{2.16}$$

which still leads to the seesaw formula in Eq. (2.8).

Otherwise, if the singlet neutrinos are light enough to be produced or not far from that, a full treatment of the neutrino sector, including the sterile states and their mixing with the active ones, is necessary. In such a regime, the size of neutrino masses requires the relevant parameters to be particularly small. For example, singlet neutrinos in the eV range [a motivated possibility; see Giunti and Lasserre (2019) for a review] require the Yukawa couplings λ_N and the singlet masses M in Eqs. (2.5) and (2.6) to be as small as

$$\lambda_N \lesssim 10^{-11}, \qquad M \lesssim 10^{-18} M_{\mathrm{Pl}} \tag{2.17}$$

(and imply a mild fine-tuning, keeping Dirac and Majorana neutrino masses within 1 or 2 orders of magnitude).

Finally, if the Majorana mass term M is even smaller than the Dirac mass term, solar neutrino experiments force M to be well below the heavier active neutrino mass range (de Gouvea, Huang, and Jenkins, 2009), and we approach the Dirac neutrino limit, in which $M = 0$. In such a limit, lepton number is conserved in the neutrino sector, and the only role of the sterile fields is to pair to the active ones in the Dirac mass term. The corresponding degrees of freedom can hardly be observed, as their production and detection with an energy E is suppressed by a factor of m_ν/E.

In the cases considered here, the size of the neutrino masses is accounted for by the often striking smallness of the Lagrangian parameters. While such a smallness may seem quite *ad hoc*, ideas are available to account for it. The suppression of the Majorana mass term can be associated with the approximate or exact conservation of lepton number. This comes at the price of giving up one of the successes of the SM, as the approximate conservation of lepton number observed in nature would no longer be accounted for by accidental symmetries. Lepton number needs to be enforced as a symmetry by hand, with the drawbacks discussed in Sec. II.A. The smallness of the Yukawa couplings can instead be given a dynamical origin. Small, nonzero couplings can arise through the spontaneous breaking of a symmetry forbidding them (Chikashige, Mohapatra, and Peccei, 1981; Gelmini and Roncadelli, 1981; Georgi, Glashow, and Nussinov, 1981; Chacko *et al.*, 2004; Chen, de Gouvea, and Dobrescu, 2007; Gu *et al.*, 2009), or from a more fundamental theory living in more than four dimensions (Dienes, Dudas, and Gherghetta, 1999; Dvali and Yu. Smirnov, 1999; Mohapatra, Nandi, and Perez-Lorenzana, 1999; Barbieri, Creminelli, and Strumia, 2000; Grossman and Neubert, 2000; Lukas *et al.*, 2000, 2001; Arkani-Hamed *et al.*, 2001; Gonzalez-Garcia and Nir, 2003).

III. SYMMETRIES: GENERAL CONSIDERATIONS

A. The flavor puzzle

Having reviewed possible origins of the neutrino masses and their overall scale, we now come to the main subject of this review: the origin, if any, of the pattern of lepton masses and mixings, i.e., of the flavor structure of the lepton mass matrices, which is part of the so-called SM flavor puzzle.

The flavor puzzle in the SM, here extended to include a source of neutrino masses, has two aspects. The first one is the existence of three fermion families replicating the same set of gauge quantum numbers. Or, equivalently, the invariance of the SM gauge Lagrangian under a global $\mathrm{U}(3)^5$ global symmetry, where each U(3) factor mixes the three families of fermions with identical gauge quantum numbers: q_i, u^c_i, d^c_i, l_i, e^c_i, $i = 1, 2, 3$. The Higgs Lagrangian is invariant under a further $\mathrm{U}(1)_H$ rephasing of the Higgs doublet field. Thus, $G_{\max} \equiv \mathrm{U}(3)^5 \times \mathrm{U}(1)_H$ is the maximal group of global SM field transformations commuting with the actions of the Poincaré and gauge groups. It includes the hypercharge global transformations. In SM extensions, $G_{\max}$ can be larger if the matter field content is extended (singlet neutrinos, for example, or additional Higgs fields), or smaller if the gauge group is extended.

If the source of neutrino masses is neglected, $\mathrm{U}(3)^5 \times \mathrm{U}(1)_H$ is explicitly broken by the SM Yukawa interactions with the four SM accidental symmetries [the U(1) transformations associated with the individual lepton numbers L_e, L_μ, L_τ and the total baryon number B] and to the hypercharge global transformations. The accidental symmetries are anomalous unless they are combinations of $B - L_e/3$, $B - L_\mu/3$, and $B - L_\tau/3$. If neutrino masses are accounted for at the weak scale by the Weinberg operator, the three individual lepton numbers are also broken and only B survives at the perturbative level, although it is anomalous. If neutrino masses are of Dirac type, i.e., they are accounted for at the weak scale by Yukawa couplings to otherwise massless right-handed neutrinos, both B and L survive from an initial $G_{\max} = \mathrm{U}(3)^6 \times \mathrm{U}(1)_H$, and only the $B - L$ combination is nonanomalous.

The second aspect of the flavor puzzle is the peculiar pattern of fermion masses and mixings originating from the explicit breaking of $\mathrm{U}(3)^5 \times \mathrm{U}(1)_H$. The masses of the three families of charged fermion masses turn out to be hierarchical and the quark mixing is small. Lepton mixing is instead large and at least two neutrino masses are separated by less than an order of magnitude.

The two aspects of the flavor puzzle may be related. The fact that the flavor Lagrangian breaks an underlying $\mathrm{U}(3)^5 \times \mathrm{U}(1)_H$ symmetry, manifest in the gauge Lagrangian, may suggest that it originates from the spontaneous breaking of the previously mentioned group or of one of its subgroups $G \subseteq \mathrm{U}(3)^5 \times \mathrm{U}(1)_H$. This is the idea underlying theories based on flavor symmetries (Froggatt and Nielsen, 1979), where G is called the flavor group. The action of G is traditionally assumed to be linear and to commute with gauge and Poincaré transformations. On the other hand, new avenues evading such an assumption have recently been considered. Correspondingly, denoting by ψ_i a generic set of matter fields, we now consider three types of symmetries.

(1) The action of G is linear (and thus unitary in order to preserve canonically normalized kinetic terms) and commutes with gauge and proper Poincaré transformations

$$g \in G\colon\ \psi_i(x) \to U_\psi(g)_{ij}\psi_j(x). \tag{3.1}$$

In such a case, G is a subgroup of $G_{\max}$ and $U_\psi(g)$ is a unitary representation of G. Such a standard framework is reviewed next and in Sec. IV.

(2) The action of G is linear, but it does not commute with proper Poincaré and/or gauge transformations. The case in which flavor and Poincaré transformations do not commute leads to symmetries of the form $G = G_f \rtimes CP$, where G_f is a subgroup of $G_{\max}$ as in the previous case

$$g \in G_f\colon \psi_i(x) \to U_\psi(g)_{ij}\psi_j(x), \quad \psi_i(x) \xrightarrow{CP} X_{ij}\psi_j^*(x). \tag{3.2}$$

Here $U_\psi(g)_{ij}$ and X_{ij} are unitary representations of G_f and CP, respectively. This scenario is reviewed next and in Sec. V. The case in which G commutes with Poincaré but does not commute with gauge transformations has received less attention thus far (Reig *et al.*, 2017).

(3) The action of G is nonlinear: it commutes with the gauge group and with proper Poincaré transformations. G is not necessarily a subgroup of $G_{\max}$. In the realization that we consider, the framework includes an additional scalar sector, typically consisting of fields τ singlet under the gauge group.

$$g \in G\colon \tau \to f_g(\tau), \qquad \psi_i(x) \to U_\psi(g;\tau)_{ij}\psi_j(x), \tag{3.3}$$

where $f_g(\tau)$ and $U_\psi(g;\tau)_{ij}$ describe the nonlinear realization of G on τ and $\psi_i(x)$, respectively. This case will be reviewed next and in Sec. VI.

A fourth possibility, also discussed in Sec. VI, arises by combining cases (2) and (3).

B. Flavor symmetry group and representation

We first consider flavor models based on a flavor group G whose action on fields is linear and commutes with Poincaré and gauge transformations. G then acts on the flavor indices of each set of fields ψ_i sharing the same Lorentz and gauge quantum numbers

$$g \in G\colon\ \psi_i(x) \to U_\psi(g)_{ij}\psi_j(x). \tag{3.4}$$

The representation $U_\psi(g)$ is unitary, as the kinetic terms are assumed to be canonically normalized. Moreover, gauge fields must be invariant under G, and the action of G on the full set of matter fields can be assumed to be faithful without loss of generality. Therefore, G can be identified with a subgroup of the unitary internal transformations. More precisely, $G_{\max} \subseteq \prod_r U(n_r)$, where n_r is the number of identical copies of each irreducible representations r of the Poincaré and gauge groups on matter fields. The Lagrangian is assumed to be invariant under the action of G, and this constrains its flavor structure. The symmetry may be spontaneously broken by a set of scalar fields ϕ called "flavons," or explicitly broken.

Different types of flavor groups can be considered: G can be a Lie group or a discrete group, Abelian or non-Abelian, simple or nonsimple, it can be assumed to be a symmetry or arise accidentally (Ferretti, King, and Romanino, 2006), or it can act rigidly on the fields or be gauged. In the case of gauge groups, proper care should be taken of anomalies, possibly canceling them by adding an appropriate heavy field content. Most often, the scale at which G is spontaneously broken is taken to be significantly higher than the weak scale. As a consequence, the flavons are bound to be SM singlets (they can, however, transform nontrivially under extensions of the SM group).

Flavor symmetry breaking at the electroweak scale or below faces a number of challenges. If G is gauged, constraints from flavor-changing neutral current (FCNC) processes set a lower bound on the mass of the corresponding gauge bosons, and therefore also on the breaking scale. If G is a nonanomalous global Lie group, its spontaneous breaking gives rise to massless Goldstone bosons, which must be then sufficiently weakly coupled to SM fields. This is the case if the coupling is mediated by sufficiently heavy degrees of freedom or, in the effective-theory description, if they couple through nonrenormalizable interactions suppressed by a sufficiently heavy scale. The heavy fields mediating flavor breaking can themselves be a source of FCNC. The scale at which G is broken is then again also bound to be correspondingly large. The same argument applies if G is anomalous, unless would-be Goldstone bosons (and therefore the flavor breaking scale) are heavy enough. In the case of the spontaneous breaking of finite groups, a further constraint comes from the need to avoid domain walls (Riva, 2010; Antusch and Nolde, 2013; Chigusa and Nakayama, 2019). Still, relatively low scales of flavor breaking can be achieved even in the case of gauged models (Grinstein, Redi, and Villadoro, 2010). The possibility that the flavor symmetry is broken together with the electroweak symmetry by means of Higgs doublets has also been considered (Grimus and Lavoura, 2003; Ma, 2007a; Morisi and Peinado, 2009; Morisi *et al.*, 2011). Here we consider the safest case, in which the breaking of the flavor symmetry is due to SM singlets above the electroweak scale.

C. Exact flavor symmetries

We first dismiss the possibility that the flavor symmetry is exact. This is also important because it shows that no overall exact unbroken subgroup can survive the breaking of the flavor symmetry. To begin with, we consider the effective description of neutrino masses through the Weinberg operator. The flavor group acts in the lepton sector through two unitary representations of $g \in G$, one on the lepton doublets l_i and one on the lepton singlets e_i^c,

$$\begin{aligned} l_i &\to U_l(g)_{ij} l_j, \\ e_i^c &\to U_e(g)_{ij} e_j^c. \end{aligned} \tag{3.5}$$

The Higgs field could in principle also transform under G, but its transformation can, without loss of generality, be reabsorbed in those of l_i and e_i^c, and we therefore neglect it.[4]

If the flavor symmetry is not broken, the invariance of the lepton flavor Lagrangian

$$\lambda^E_{0ij} e_i^c l_j H^* + \frac{c_{0ij}}{2\Lambda}(l_i H)(l_j H) \tag{3.6}$$

constrains the couplings λ^E_{0ij} and c_{0ij}, or equivalently the charged fermion and neutrino mass matrices M_E^0 and m_ν^0, as follows:

$$\begin{aligned} M_E^0 &= U_e(g)^T M_E^0 U_l(g), \\ m_\nu^0 &= U_l(g)^T m_\nu^0 U_l(g), \end{aligned} \tag{3.7}$$

for any $g \in G$. The superscript 0 stresses that the lepton couplings and mass matrices are assumed here to be exactly symmetric under G. It turns out that the previously mentioned constraints can lead to fully viable mass matrices (i.e., can be associated with three nonvanishing charged-lepton masses, three nondegenerate neutrinos, and three nonvanishing mixing angles) only if the representation on the lepton doublets is trivial $U_l(g) = \pm\mathbf{1}$. The representation on the e_i^c fields must also be trivial and identical to the one on the lepton doublets. In other words, the only accidental symmetry of the SM Lagrangian augmented by the Weinberg operator is $\mathbf{Z}_2$. The argument is simple and is best formulated in the charged-lepton mass basis, in which M_E^0 is diagonal and positive. The charged-lepton masses relegate G to a subgroup of $\mathrm{U}(1)_e \times \mathrm{U}(1)_\mu \times \mathrm{U}(1)_\tau$, the three lepton number U(1)'s: as U_l and U_{e^c} must commute with $(M_E^0)^2$, which is nondegenerate, U_l and U_{e^c} must both be diagonal matrices of phases; as M_E^0 is nonsingular, Eq. (3.7) forces $U_{e^c} = U_l^*$. The Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix further reduces G to a subgroup of the total lepton number U(1): inserting $m_\nu^0 = U^*(m_\nu^0)_{\mathrm{diag}} U^\dagger$, where U is the PMNS matrix, in Eq. (3.7), we see that the combination $U^\dagger U_l U$ commutes with $[(m_\nu^0)_{\mathrm{diag}}]^2$ and must also be a diagonal matrix of phases. Since all elements of the PMNS matrix are nonvanishing, this means that U_l is just an overall phase, i.e., G acts as a subgroup of the total lepton number. Finally, the Majorana nature of the neutrino operator allows only the $\mathbf{Z}_2$ subgroup, as can be shown by substituting $U_l = e^{i\phi}\mathbf{1}$ into Eq. (3.7). Needless to say, a trivial representation such as $U_l(g) = U_{e^c}(g) = \pm\mathbf{1}$ does not constrain lepton masses and mixings at all, as any M_E and m_ν would satisfy Eq. (3.7). An accurate nontrivial description of lepton flavor thus requires a spontaneously broken flavor symmetry. Moreover, the flavor symmetry should be fully broken. No residual nontrivial subgroup should survive the breaking, except possibly the previously mentioned trivial $\mathbf{Z}_2$. The same conclusion holds if the flavor symmetry constrains the renormalizable theory from which the Weinberg operator originates, provided that the heavy fields stay heavy in the exactly symmetric limit. This is because Eq. (3.7) still holds, as a consequence of the invariance of the full theory.

This assumes a high-scale origin of neutrino masses. In the paradigmatic caveat of Dirac neutrinos masses originating from Yukawa couplings to three right-handed neutrinos, the analysis is different but the conclusion is the same. The only possible exact flavor symmetry in the lepton sector is in this case the total lepton number U(1) or one of its subgroups. Such a flavor group would not constrain lepton masses and mixings at all, as any form of the lepton mass matrices would be allowed.

Finally, these considerations extend to the quark sector. The only allowed exact symmetry is in that case the total baryon number. The latter, however, does not provide any constraints on the quark mass matrices.

D. Symmetry breaking

Having to abandon the idea that lepton masses and mixing angles can be inferred from an exact flavor symmetry, the usefulness of the entire approach relies substantially on the knowledge of breaking effects. In general we can distinguish

[4] This is not necessarily true in extensions of the SM Higgs sector with two or more Higgs fields.

between an explicit breaking, where the nature of the breaking terms is unrelated to the dynamics of the system, and a spontaneous breaking originating from the noninvariance of the vacuum state. Typically the spontaneous breaking offers better chances in terms of predictability, especially if some dynamical requirement, like the minimization of the energy density of the system, is invoked to select the vacuum of the theory. There are, however, exceptions to this general trend. In addition, the case of explicit breaking can retain some predictability if breaking terms are not completely arbitrary. Actually, to some extent the two cases can be described within the same formalism. Consider, for example, the charged-lepton Yukawa coupling $\lambda^E_{ij} e^c_i l_j H^* + \text{H.c.}$ and assume that the singlets e^c and the doublets l transform according to unitary representations r_{e^c} and r_l of the flavor group G. It is useful to write the Yukawa coupling in the form

$$\lambda^E_{ij} e^c_i l_j H^* = \sum_{I\alpha} S^I_\alpha (\Gamma^I_{ij\alpha} e^c_i l_j) H^*, \tag{3.8}$$

where the combinations $(\Gamma^I_{ij\alpha} e^c_i l_j)$ $(\alpha = 1, \dots, d_I)$ transform in the irreducible representations r_I (of dimension d_I) of the group G occurring in the decomposition of the tensor product $r_{e^c} \otimes r_l$. In the case of N_f fermion generations we have the constraint $\sum_I d_I = N_f^2$, and $\Gamma^I_{ij\alpha}$ are Clebsch-Gordan coefficients. The Yukawa interaction can be seen as an invariant of the flavor group, provided that S^I_α are interpreted as spurions transforming in the conjugate representation $\bar{r}_I$. Arbitrary Yukawa couplings λ^E_{ij} are traded by arbitrary spurions S^I_α, and at this stage we see no benefit. However, in model building we can complement the previously mentioned decomposition with some additional assumptions about the set of allowed spurion representations, their size, and their relative orientation in flavor space and thus gather information on the pattern of λ^E_{ij}, through the relation $\lambda^E_{ij} = \sum_{I\alpha} S^I_\alpha \Gamma^I_{ij\alpha}$.

In general the model is specified by the gauge group G_g and the flavor group G together with the field content, which includes matter fields, spurions, and their representations under G_g and G. To cover the general case where the fields S^I_α in Eq. (3.8) are functions of the fundamental G multiplet $S^I_\alpha = S^I_\alpha(\varphi)$, we denote the set of allowed spurions by φ. In the context of flavor symmetries such spurions are simply the flavons. They transform under a possibly reducible representation r_φ of the group G. A common, but not mandatory, choice is to assume that spurions φ are singlets under the gauge group. The Yukawa couplings $\lambda^E_{ij}(\varphi)$ become functions of the spurions φ constrained by the flavor symmetry. If they can be expanded in powers of φ, they assume the following form:

$$\lambda^E_{ij}(\varphi) = \lambda^E_{0ij} + \lambda^{E\alpha}_{1ij}\varphi_\alpha + \lambda^{E\alpha\beta}_{2ij}\varphi_\alpha\varphi_\beta + \cdots, \tag{3.9}$$

and the corresponding interactions are given by

$$e^c \lambda^E(\varphi) l H^* = (e^c l)_1 H^* + (e^c l \varphi)_1 H^* + (e^c l \varphi\varphi)_1 H^* + \cdots, \tag{3.10}$$

where flavor indices are understood and $(\cdot)_1$ stands for a G-invariant combination $(e^c l)_1 = e^c_i \lambda^E_{0ij} l_j$, $(e^c l \varphi)_1 = e^c_i \lambda^{E\alpha}_{1ij}\varphi_\alpha l_j$, etc. This type of description is equally good for both nondynamical spurions and new dynamical degrees of freedom described by the fields φ. In the first case we reproduce an explicit breaking of G, while in the second case the breaking is spontaneous, as it is related to the VEV of φ. In the previous description φ are dimensionless. Fields with canonical dimensions are easily recovered by the replacement $\varphi \to \varphi_{\text{CD}}/\Lambda$, where Λ stands for a new physical scale related to flavor dynamics. Then the expansion of Eq. (3.10) contains operators of growing dimensionality providing, in the spirit of an EFT, a low-energy description of the flavor sector valid at energy scales much lower than Λ. The scale Λ controlling the spurion expansion does not necessarily coincide with that introduced in Eq. (2.1), which breaks the lepton number L. Operators of high dimensions can be helpful to describe light fermions if the expansion parameter $\langle\varphi\rangle$ is sufficiently small.

TABLE III. Representation of leptons, Higgs, and spurions under $G = \text{SU}(2) \times \text{U}(1)$ $(a = 1, 2)$.

G	e^c_3	e^c_a	l_3	l_a	H	φ_1	φ_2
$\text{SU}(2)\times\text{U}(1)$	(1,0)	(2,1)	(1,1)	(2,1)	(1,0)	(1,−1)	(2,−1)

As an example (Linster and Ziegler, 2018) we take $G = \text{U}(2) \sim \text{SU}(2) \otimes \text{U}(1)$ and let the lepton fields transform as in Table III. The product $r_{e^c} \otimes r_l$ decomposes as $(1,1) \oplus (2,1) \oplus (2,2) \oplus (1,2) \oplus (3,2)$. The corresponding combinations $(\Gamma^I_{ij\alpha} e^c_i l_j)$ are given in Table IV. The elements of a generic Yukawa coupling λ^E_{ij} are classified as $\lambda^E_{33} \sim (1,-1)$, $\lambda^E_{3a} \sim (2,-1)$, $\lambda^E_{a3} \sim (2,-2)$, $(\lambda^E_{12} - \lambda^E_{21})/\sqrt{2} \sim (1,-2)$, and $(\lambda^E_{11}, (\lambda^E_{12} + \lambda^E_{21})/\sqrt{2}, \lambda^E_{22}) \sim (3,-2)$. In the absence of any indication about the type, size, and orientation of the spurions, this decomposition brings no useful information. We now assume that the only allowed spurions are φ_1 and φ_2, transforming as $\varphi_1 \sim (1,-1)$ and $\varphi_2 \sim (2,-1)$ under $\text{SU}(2) \times \text{U}(1)$, invariant under the gauge group, and with the VEV orientation $\langle\varphi_2\rangle^T = (\langle\varphi_{21}\rangle, 0)$. The choice of this direction in flavor space is not restrictive if the spurions describe vacuum configurations of dynamical fields, since options related by G transformations lead to equivalent physical systems. In this case, if we consider terms linear in spurions only, the only nonvanishing entries of λ^E_{ij} are $\lambda^E_{33} = \langle\varphi_1\rangle$ and $\lambda^E_{32} = \langle\varphi_{21}\rangle$. To fill the matrix λ^E_{ij} we need terms of higher order. To second order we get

$$\lambda^E = \begin{pmatrix} 0 & a\langle\varphi_1\rangle^2 & 0 \\ -a\langle\varphi_1\rangle^2 & b\langle\varphi_{21}\rangle^2 & c\langle\varphi_1\rangle\langle\varphi_{21}\rangle \\ 0 & \langle\varphi_{21}\rangle & \langle\varphi_1\rangle \end{pmatrix}, \tag{3.11}$$

where the coefficients a, b, and c are parameters related to independent invariant combinations. The vanishing entries of λ^E can be filled by invariants of higher order. An assumption about the relative size of $\langle\varphi_1\rangle$ and $\langle\varphi_{21}\rangle$ can further shape the pattern of λ^E.

The setup we have illustrated is based on an effective description of the flavon interactions with the SM fields and is sufficient for most of our purposes. We now discuss the

TABLE IV. Combinations $(\Gamma^l_{ij\alpha} e^c_i l_j)$ and their transformation properties under $G = \mathrm{SU}(2)\times\mathrm{U}(1)$ ($a = 1, 2$).

G	$e^c_3 l_3$	$e^c_3 l_a$	$e^c_a l_3$	$(e^c_1 l_2 - e^c_2 l_1)/\sqrt{2}$	$(e^c_1 l_1, (e^c_1 l_2 + e^c_2 l_1)/\sqrt{2}, e^c_2 l_2)$
SU(2) × U(1)	(1,1)	(2,1)	(2,2)	(1,2)	(3,2)

possible UV origin of such a setup. This parallels the discussion of the UV origin of the Weinberg operator in Sec. II.B.2.

Consider for simplicity a $D = 5$ operator involving a single flavon, in the form

$$\frac{c_{ij}}{\Lambda}\varphi f^c_i f_j H. \tag{3.12}$$

The latter contributes to the Yukawa interaction $\lambda_{ij} f^c_i f_j H$ for the charged leptons and neutrinos $f = l$, $f^c = e^c$, ν^c (or for the quarks $f = q$, $f^c = d^c, u^c$). As for the Weinberg one, there are only three possible UV renormalizable origins of the operator in Eq. (3.12). They correspond to the exchange of heavy vectorlike messengers with the same SM quantum numbers as f, f^c, or H. We now consider the following exchange of n vectorlike messengers with the quantum numbers of f: $F_\alpha + \bar{F}_\alpha$, $\alpha = 1, \ldots, n$. The renormalizable Lagrangian contains

$$-\mathcal{L}_F = \eta_{\alpha i}\bar{F}_\alpha f_i \varphi + y_{i\alpha} f^c_i F_\alpha H + M_{\alpha\beta}\bar{F}_\alpha F_\beta + \text{H.c.}, \tag{3.13}$$

where the couplings are constrained by the flavor symmetry. Integrating out the F, $\bar{F}$ fields generates the operator in Eq. (3.12), with [cf. Eq. (2.7)]

$$\frac{c}{\Lambda} = -y M^{-1}\eta. \tag{3.14}$$

Note that in the presence of a single family of messengers the Yukawa couplings generated by Eq. (3.13) have rank 1: $\lambda_{ij} = -y_i\eta_j(\langle\varphi\rangle/M)$. The first two charged fermion families vanish in this limit and can be generated by subleading effects involving heavier messengers. This way hierarchical charged fermion masses (and a viable mixing pattern for quarks and leptons) can be accounted for without imposing any flavor symmetries (Ferretti, King, and Romanino, 2006). At the same time a $\mathrm{U}(2)_{f^c}\times\mathrm{U}(2)_f$ symmetry arises accidentally in the limit in which additional contributions to the Yukawa interactions from heavier messengers are neglected.

1. Vacuum alignment

Lepton mixing angles and phases can be determined only when both the neutrino and the charged-lepton sectors have been specified. For instance, when the lepton number L is violated, at low energy the relevant Lagrangian is

$$e^c\lambda^E(\varphi) l H^* + \frac{1}{2\Lambda}(lH)c(\varphi)(lH) + \text{H.c.}, \tag{3.15}$$

where the matrices λ^E and c are now functions of the fields[5] φ and the Lagrangian is invariant under the group G. The mixing matrix is given by

$$U = U_e^\dagger U_\nu, \tag{3.16}$$

where U_e and U_ν are the unitary matrices that diagonalize the combination $\lambda^{E\dagger}\lambda^E$ and c, respectively,

$$U_e^\dagger\lambda^{E\dagger}\lambda^E U_e = (\hat{\lambda}^E)^2, \qquad U_\nu^T c U_\nu = \hat{c}. \tag{3.17}$$

Here $\hat{\lambda}^E$ and $\hat{c}$ are non-negative diagonal matrices and their eigenvalues have been properly ordered, which also accounts for the type of neutrino mass spectrum. After suitable rephasing of the combination $U_e^\dagger U_\nu$, we can put the mixing matrix in a conventional form, for instance, the one used by the PDG, and read the physical parameters. The latter follow necessarily from the interplay of both neutrinos and charged leptons.

Such a trivial observation has important implications on model building. Since both $\lambda^E(\varphi)$ and $c(\varphi)$ depend on φ, a realistic pattern of lepton masses and mixing angles can be achieved only if the VEVs of the fields φ have the right size and orientation in flavor space. If these fields are dynamical, the problem of deriving the desired VEV from the minimization of the energy density is called the vacuum alignment problem. Although the group G is completely broken in the low-energy regime, it might be that the charged-lepton sector and the neutrino sector separately possess an exact or approximate residual symmetry under subgroups G_e and G_ν, respectively. Actually, this scenario has been extensively studied in the context of discrete flavor symmetries to predict or constrain the lepton mixing angles. This special case of vacuum alignment can be implemented by separating φ into two sets $\varphi = (\varphi_e, \varphi_\nu)$ such that λ^E and c depend mainly on φ_e and φ_ν, respectively. The desired residual symmetries are obtained if the VEV of φ_e is invariant under G_e and that of φ_ν is invariant under G_ν. This possibility is discussed in greater detail in Sec. IV.B.1.

This discussion already shows the advantages and limitations of the considered setup. The perspective that fermion masses and mixing angles are determined by some dynamical principle is interesting and makes contact with more fundamental theories like string theory, where in principle Yukawa couplings are calculable functions of a set of fields describing the vacuum configuration. A drawback of the approach is

[5]If $c(\varphi)$ originate from the exchange of heavy degrees of freedom whose mass depends on φ, it might be singular as φ vanish, and a series expansion like the one in Eq. (3.9) might not be possible.

exhibited by Eqs. (3.9) and (3.10). If a realistic description of fermion masses and mixing angles requires the presence of several terms in the expansion, a large number of free parameters might be required, to the detriment of predictability. The predictions can also be affected by the uncertainty related to the entire tower of higher-dimensional operators, unless the expansion parameters $\langle\varphi\rangle$ are small. Moreover, if we insist in deriving the appropriate pattern of a VEV for the fields φ from the minimization of the energy density, the solution of the vacuum alignment problem might require complicated constructions, with many auxiliary fields that do not play any role in shaping λ^E and c and additional symmetries to forbid unwanted terms in the scalar potential. To avoid or reduce the complexity of the vacuum alignment problem, we can give up the possibility that symmetry breaking is dynamically determined. This is a frequent option in models realized in the presence of extra dimensions, where the symmetry breaking can be achieved through an appropriate set of boundary conditions. Examples of this type of breaking for models of neutrino masses were given by Csaki *et al.* (2008), Kobayashi, Omura, and Yoshioka (2008), Hagedorn and Serone (2011), and Hagedorn and Serone (2012).

It is worth noting that this formalism is covariant under a general change of basis in the field space, provided that both the charged-lepton and neutrino sectors are consistently addressed. Let the group G act, in the original basis, as

$$\psi \to U_\psi(g)\psi, \qquad \psi = (e^c, l, \varphi), \tag{3.18}$$

with $U_\psi(g)$ unitary matrices depending on the generic element g of the group. If we perform an arbitrary change of basis described by the following set of unitary matrices Ω_ψ:

$$\psi \to \psi' = \Omega_\psi \psi, \tag{3.19}$$

we end up with new matrices $(\lambda^E)' = \Omega_{e^c}^T \lambda^E \Omega_l$ and $c' = \Omega_l^T c \Omega_l$ in our Lagrangian. The matrices that diagonalize $(\lambda^{E\dagger}\lambda^E)'$ and c' are now $U_{e^c}{}' = \Omega_l^\dagger U_{e^c}$ and $U'_\nu = \Omega_l^\dagger U_\nu$. All the physical parameters are unchanged. In the new basis the group G acts as

$$\psi' \to U'_\psi(g)\psi', \qquad U'_\psi(g) = \Omega_\psi U_\psi(g) \Omega_\psi^\dagger. \tag{3.20}$$

A feature that is not captured by the previous formalism is the possibility that the flavor symmetry is nonlinearly realized. In this case the various terms of the expansion in Eq. (3.10) are not expected to be individually invariant under G transformation, as occurred previously as a result of assuming linear unitary representations. This means that the coefficients λ^E_{0ij},$\lambda^{E\alpha}_{1ij}$, $\lambda^{E\alpha\beta}_{2ij}, \ldots$ might all be related to provide a Yukawa interaction invariant under the group G. This case might present the advantage of requiring fewer free parameters, thereby making it more predictive.

2. Kinetic terms

In general the breaking of the flavor symmetry affects not only the Yukawa interactions, as in Eqs. (3.10) and (3.15), but also the kinetic terms, leading to additional contributions to mass and mixing parameters. The kinetic terms read

$$i\bar{e}^c \bar{\sigma}^\mu K^{e^c}(\varphi)\partial_\mu e^c + i\bar{l}\bar{\sigma}^\mu K^l(\varphi)\partial_\mu l + \cdots, \tag{3.21}$$

where the ellipsis stands for terms including $\partial_\mu K^f(\varphi)$ $(f = e^c, l)$, required by a Hermitian Lagrangian, and $K^f(\varphi)$ are positive-definite Hermitian matrices in flavor space depending on the flavon fields, here assumed to be real. In effective field theories and in linearly realized flavor symmetries, $K^f(\varphi)$ can be expanded in powers of φ. Assuming a choice of basis where $K^f(0) = \mathbb{1}$, we have

$$K^f(\varphi) = \mathbb{1} + K_1^{f\alpha}\varphi_\alpha + K_2^{f\alpha\beta}\varphi_\alpha\varphi_\beta + \cdots, \tag{3.22}$$

where $K_p^{f\alpha_1,\ldots,\alpha_p}$ are numerical matrices constrained by the requirement of G invariance. When flavons acquire a VEV, canonical kinetic terms are recovered through the transformations

$$f \to \left(\mathbb{1} - \tfrac{1}{2}K_1^{f\alpha}\varphi_\alpha + \cdots\right) f \qquad (f = e^c, l), \tag{3.23}$$

and the Yukawa interactions are modified accordingly. For instance the charged-lepton Yukawa couplings become

$$\lambda^E(\varphi) \to \lambda^E(\varphi) - \tfrac{1}{2}K_1^{e^c\alpha}\varphi_\alpha\lambda^E(\varphi) - \tfrac{1}{2}\lambda^E(\varphi)K_1^{l\alpha}\varphi_\alpha + \cdots. \tag{3.24}$$

The consequences of such a change differ depending upon whether we are dealing with a supersymmetric or a nonsupersymmetric theory. In a nonsupersymmetric theory, the transformation (3.24) merely results in a redefinition of the parameters of the Yukawa matrix $\lambda^E(\varphi)$, since $\lambda^E(\varphi)$ exhausts all the polynomial invariants depending on the flavons φ and describing charged-lepton Yukawa couplings. In the supersymmetric case, $\lambda^E(\Phi)$ are holomorphic functions of chiral multiplets Φ, while in the kinetic terms we should distinguish between holomorphic and antiholomorphic variables. The function $K^f(\Phi, \Phi^\dagger)$ depends on both of them

$$K^f(\Phi, \Phi^\dagger) = \mathbb{1} + K_1^{f\alpha}\Phi_\alpha + K_1^{f\alpha\dagger}\Phi_\alpha^\dagger + \cdots. \tag{3.25}$$

The transformation (3.24) becomes

$$\begin{aligned}\lambda^E(\Phi) \to \lambda^E(\Phi) &- \tfrac{1}{2}[K_1^{e^c\alpha}\Phi_\alpha + K_1^{e^c\alpha\dagger}\Phi_\alpha^\dagger]\lambda^E(\Phi) \\ &- \tfrac{1}{2}\lambda^E(\Phi)[K_1^{l\alpha}\Phi_\alpha + K_1^{l\alpha\dagger}\Phi_\alpha^\dagger]\cdots,\end{aligned} \tag{3.26}$$

which induces a nonholomorphic dependence of the physical Yukawa couplings on the flavons. In general this entails additional parameters to the description of masses, mixing angles, and phases. Such effects were analyzed by Dudas, Pokorski, and Savoy (1995), Binetruy, Lavignac, and Ramond (1996), Dudas, Pokorski, and Savoy (1996), Dreiner and Thormeier (2004), Jack, Jones, and Wild (2004), and Dreiner, Murayama, and Thormeier (2005) for Abelian flavor symmetries, by King and Peddie (2004) and Ross, Velasco-Sevilla, and Vives (2004) for non-Abelian continuous flavor symmetries, by Hamaguchi, Kakizaki, and Yamaguchi (2003) and Chen *et al.* (2012, 2013) for non-Abelian discrete flavor symmetries, and by Chen, Ramos-Sanchez, and Ratz (2019) for modular flavor symmetries. Kakizaki and Yamaguchi (2003) exploited such a contribution to explain the hierarchy between the top and the other quark masses. Kawamura (2019) explored a scenario where the flavor group G remains unbroken in Yukawa interactions and the breaking is due entirely to kinetic terms. A model-independent discussion for linearly realized flavor symmetries in the supersymmetric case was given by Espinosa and Ibarra (2004). For degenerate neutrinos, the impact of the kinetic term is especially relevant due to the strong dependence of the mixing angles on new contributions. For hierarchical neutrinos the Kähler potential is expected to provide a contribution to the mixing of the same order of the contribution from the superpotential. Such an effect could be important, for instance, to explain the deviations from maximality, possibly enforced by the superpotential, of the solar and atmospheric mixings. In either case the kinetic terms bring additional free parameters, to the detriment of predictability.

3. The space of invariants

There are general features of the vacuum alignment problem that can be discussed in terms of the symmetry G and the representation assigned to the fields φ without reference to the explicit form of the energy density functional. Consider a Lagrangian $\mathcal{L}(\varphi)$ that is invariant under the action of a group G, depending on a set of scalar fields φ, transforming in a representation r_φ of the group. The fields φ live in a vector space $\mathcal{M}$, the field space, whose dimension is d_φ, which is the dimension of r_φ. In nonlinear theories, $\mathcal{M}$ can be a manifold. If the theory is G invariant, two distinct points in $\mathcal{M}$ related by a G transformation lead to the same predictions for any physical observable. In particular, in any of two such points the system has the same residual symmetry, or little group, up to a conjugation. Thus, the field space offers a redundant description of the physical system that can be simplified by studying the orbits of the group; i.e., the set of points in the field space $\mathcal{M}$ that are related by group transformations. The union of orbits having isomorphic little groups forms a stratum. The full field space $\mathcal{M}$ is partitioned into several strata. For instance, the origin of $\mathcal{M}$ belongs to the stratum of type G, since for $\varphi = 0$ the symmetry is unbroken. Most of the field space $\mathcal{M}$ is made of orbits having a minimal little group; i.e., the symmetry G is broken down to its minimum possible subgroup, which is unique, up to conjugation. This subset of $\mathcal{M}$ is called the principal stratum.

A useful tool is the orbit space $\mathcal{M}_I$. $\mathcal{M}_I$ can be parametrized by the values of invariants, which are constant on the orbits. It is sufficient to consider invariants $I(\varphi)$ that are polynomials in the components of the multiplet φ. The ring of invariant polynomials is infinite, but it is generated by a finite number of invariants $\gamma_\alpha(\varphi)$, which means that any invariant polynomial can be written as a polynomial in γ_α. The invariants γ_α might be related by a number of algebraic relations, or syzygies, $\mathcal{Z}_S(\gamma) = 0$. The space $\mathcal{M}_I$ is spanned by the values of the invariants γ_α of the theory. An entire orbit of $\mathcal{M}$ is mapped into a single point of $\mathcal{M}_I$, which completely characterizes the physical properties of the system, including its symmetry breaking pattern. The crucial property of $\mathcal{M}_I$ is that while $\mathcal{M}$ has no boundaries, $\mathcal{M}_I$ has boundaries that describe the possible breaking chains of the group. The tools that allow one to characterize the orbit space $\mathcal{M}_I$ are the Jacobian matrix (Cabibbo and Maiani, 1970)

$$J \equiv \frac{\partial \gamma}{\partial \varphi}, \tag{3.27}$$

and the so-called $\mathcal{P}$ matrix

$$\mathcal{P} = JJ^T. \tag{3.28}$$

The space $\mathcal{M}_I$ is identified using the requirements that (i) γ belongs to the surface $\mathcal{Z}_S(\gamma) = 0$ and (ii) the matrix $\mathcal{P}$ is positive semidefinite, resulting in a set of inequalities involving the invariants γ_α (Abud and Sartori, 1981, 1983; Procesi and Schwarz, 1985; Talamini, 2006).

As an example, consider the group $G = \mathrm{SU}(3)$ and the real scalar fields $\varphi = \varphi_a \lambda^a$, transforming in the adjoint representation of the group, where λ^a ($a = 1, \ldots, 8$) are the Gell-Mann matrices. As independent invariants in $\mathcal{M}_I(\mathrm{SU}(3))$ we can take $\gamma_1 = \mathrm{tr}(\varphi^2)$ and $\gamma_2 = \det(\varphi)$. The $\mathcal{P}$ matrix is

$$\mathcal{P} = \begin{pmatrix} 8\gamma_1 & 12\gamma_2 \\ 12\gamma_2 & \gamma_1^2/3 \end{pmatrix}, \tag{3.29}$$

and it is positive semidefinite under the conditions $\gamma_1 \geq 0$ and $\det(\mathcal{P}) = 8\gamma_1^3/3 - 144\gamma_2^2 \geq 0$. These inequalities define the space of invariants $\mathcal{M}_I(\mathrm{SU}(3))$, which is spanned by $\gamma_{1,2}$. The space $\mathcal{M}_I(\mathrm{SU}(3))$ is bidimensional and its interior corresponds to the point satisfying $\gamma_1^3 - 54\gamma_2^2 > 0$ and $\gamma_1 > 0$. In any point of the interior the matrices J and $\mathcal{P}$ have rank 2 and the group SU(3) is broken down to a subgroup isomorphic to $\mathrm{U}(1) \times \mathrm{U}(1)$. The one-dimensional boundary is defined by $\gamma_1^3 - 54\gamma_2^2 = 0$ and $\gamma_1 > 0$ and consists of the two branches $\gamma_2 = \pm\sqrt{\gamma_1^3/54}$. Here the matrices J and $\mathcal{P}$ have rank 1 and the group SU(3) is broken down to its subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$. Finally, the two branches meet at $\gamma_1 = 0$, which is a zero-dimensional boundary where J and $\mathcal{P}$ have rank 0 and the group SU(3) is unbroken.

It can be shown that such a decomposition of $\mathcal{M}_I$ is completely general. The boundaries of $\mathcal{M}_I$ can be found by studying the rank of J. In the interior of $\mathcal{M}_I$ the matrix J has maximum rank $r_{\max}$. In this region G is broken down to the smallest residual symmetry group $G_{\min}$. On the boundaries $\mathcal{P}$ has some vanishing eigenvalue and the rank of J is reduced. If the dimension of $\mathcal{M}_I$ is d, in general we have $(d-1)$-dimensional boundaries where $\mathtt{rank}(J) = r_{\max} - 1$. Along these boundaries G is broken down to groups containing $G_{\min}$. These boundaries meet along $(d-2)$-dimensional spaces, where $\mathtt{rank}(J) = r_{\max} - 2$. Here the residual symmetry further increases until the one-dimensional boundaries meet in a point where the entire group G is preserved.

This consideration can be useful when looking for the extrema of a generic smooth function $V(\varphi)$ that is invariant

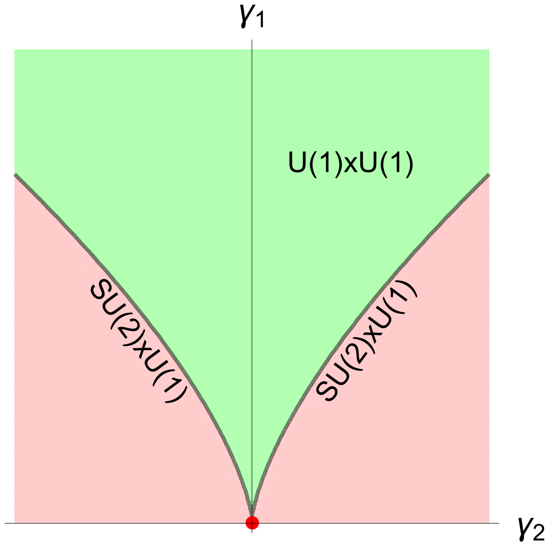

FIG. 2. Space of invariants for $G = \mathrm{SU}(3)$ and φ in the real adjoint representation. The green region is the interior, which is defined by $\gamma_1^3 - 54\gamma_2^2 > 0$ and $\gamma_1 > 0$. The red point, where the full SU(3) symmetry is unbroken, is the intersection of the one-dimensional boundaries.

under G. Such a function depends on φ through the invariants $\gamma_\alpha(\varphi)$ and the extrema lay on orbits of the group. The extrema of $V(\varphi)$ are defined by

$$\frac{\partial V}{\partial \varphi_i} = \frac{\partial V}{\partial \gamma_\alpha}\frac{\partial \gamma_\alpha}{\partial \varphi_i} = \frac{\partial V}{\partial \gamma_\alpha} J_{\alpha i} = 0. \tag{3.30}$$

Consider the previous example with $G = \mathrm{SU}(3)$ (Michel and Radicati, 1973). Along the orbits of the principal stratum, mapped in the interior of $\mathcal{M}_I$, J has rank 2 and the derivatives $(\partial V/\partial \gamma_1, \partial V/\partial \gamma_2)$ should satisfy

$$\frac{\partial V}{\partial \gamma_1} = \frac{\partial V}{\partial \gamma_2} = 0 \qquad (\gamma_1^3 - 54\gamma_2^2 > 0, \gamma_1 > 0). \tag{3.31}$$

Here SU(3) is broken down to the smallest residual symmetry $\mathrm{U}(1) \times \mathrm{U}(1)$. Along the orbits satisfying $\gamma_1^3 - 54\gamma_2^2 = 0$ and $\gamma_1 > 0$, providing the one-dimensional boundary of $\mathcal{M}_I$, J has rank 1 and Eq. (3.30) is solved by requiring $(\partial V/\partial \gamma_1, \partial V/\partial \gamma_2)$ to be one eigenvector of J^T corresponding to the vanishing eigenvalue. This condition reads

$$J_{11}\frac{\partial V}{\partial \gamma_1} + J_{21}\frac{\partial V}{\partial \gamma_2} = 0 \qquad (\gamma_1^3 - 54\gamma_2^2 = 0, \gamma_1 > 0). \tag{3.32}$$

Here the group SU(3) is broken down to its maximal subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$. Finally, the orbit $\gamma_1 = 0$ corresponds to a vanishing J. There are no further conditions on the derivatives $(\partial V/\partial \gamma_1, \partial V/\partial \gamma_2)$ and the symmetry is unbroken. The possible breaking patterns in the γ_2-γ_1 plane are summarized in Fig. 2.

From this example we see that the extrema along the boundaries of $\mathcal{M}_I$ are more natural than the extrema in the interior since they require fewer conditions on the scalar potential V. The extremum where G is unbroken is always present, regardless of the specific form of the G-invariant function V. The corresponding orbit is isolated; that is, in a sufficiently small neighborhood we find no other orbits with the same little group. Any such orbit is always an extremum, regardless of the form of V (Michel, 1971; Michel and Radicati, 1971).

Moreover, if the extremum is subject to the condition that φ is nonvanishing and bound to a compact manifold, V always has extrema having a maximal little group (Michel, 1971; Michel and Radicati, 1971). To reduce the vector space V where the flavons φ live to a compact space, we need to minimize first with respect to the overall normalization of the flavon fields. An assumption is then needed on the scalar potential: given any direction in the flavon space, the overall normalization has a nonzero, symmetry breaking, local minimum; such minima form at least one smooth submanifold M (and hence compact and invariant) in V. Michel's theorem can now be applied. The little groups found on M are the same as the ones in V, except for G itself, which is found in V $(\varphi = 0)$ but not in M (flavor singlets can be neglected without loss of generality). This is welcome, as the trivial minimum $\varphi = 0$ is not relevant here. The extrema of V guaranteed by the theorem are then those corresponding to the maximal little groups of M, i.e., to the little groups in V not contained in any larger little group but G itself. As an example, consider the previously given SU(3) example. The renormalizable scalar potential is given by

$$V = \mu_1^2\gamma_1 + \mu_2\gamma_2 + \lambda\gamma_1^2. \tag{3.33}$$

The condition for the flavor group to be broken in any direction in flavor space is $\mu_1^2 < 0$. Under such a condition, a critical point corresponding to the breaking of SU(3) to the maximal little group $\mathrm{SU}(2) \times \mathrm{U}(1)$ is guaranteed to exist. This is not the case if $\mu_1^2 > 0$.

Extrema on orbits of the principal stratum might be compatible only with specific forms of V. For instance, in the example of Eq. (3.33), extrema with little group $\mathrm{U}(1) \times \mathrm{U}(1)$ are allowed only if $\mu_2 = 0$. For a nonvanishing μ_2, the only allowed little groups of the extrema are SU(3) or $\mathrm{SU}(2) \times \mathrm{U}(1)$. A limitation of this approach is that, without further inputs, we do not know whether the extrema are maxima or minima or saddle points of V.

E. The role of *CP*

In Secs. III.B–III.D, we considered flavor groups commuting with the proper Poincaré group and with gauge transformations. We now relax this hypothesis. We argue that, under mild hypotheses, paritylike transformations are the only possible alternative. Indeed, by the Coleman-Mandula theorem (Coleman and Mandula, 1967), any symmetry of the scattering matrix should provide an automorphism of the Poincaré algebra. Up to Poincaré transformations, i.e., changes of reference frame, and dilatations, which require the theory to be conformally invariant in the symmetric limit, there are only two independent nontrivial automorphisms: parity and time reversal. The action of both on the Poincaré algebra is involutive: it squares to the identity. Dilatations are only allowed if the theory is scale invariant to begin with, which is not a case that we are interested in. Because of the *CPT* theorem, it suffices to consider paritylike automorphisms.

Consider now the following action of such symmetry on the entire collection of matter fields, bosonic and fermionic, including conjugates, denoted by Φ_i:

$$\Phi_i(x) \to X_{ij}\Phi_j^\dagger(x_P), \tag{3.34}$$

where $(x_P)_\mu = x^\mu$. It follows that left-handed Weyl spinors f_a transform into right-handed ones $f_a \to X_{ab}\bar{f}_b$.[6] While the action of this symmetry on the Poincaré algebra is involutive, it does not have to be involutive on the fields $\Phi_i(x)$, and in general XX^* corresponds to a standard flavor transformation that is not necessarily equal to the identity. Additional conditions hold in a gauge theory, where a gauge group G_g acts on the fields Φ_i through its unitary representation $\rho_{ij}(g)$. For the paritylike transformation to be consistent, equivalent field configurations (related by gauge transformations) should be transformed by the paritylike action into equivalent field transformations. Moreover, the gauge interactions should be invariant. The two previous requirements lead to the following two consistency conditions (Grimus and Rebelo, 1997).

- There must exist an automorphism $g \in G_g \to g' \in G_g$ such that

$$X\rho(g)^* X^{-1} = \rho(g'). \tag{3.35}$$

- The paritylike transformation must transform the gauge fields $A^\mu(x) = A^\mu_a(x)t_a$, where t_a are the gauge group generators as

$$A^\mu(x) \to A'_\mu(x_P), \tag{3.36}$$

  where $t_a \to t'_a$ is the generator automorphism induced by $g \to g'$.

The existence of a paritylike transformation inverting the sign of commuting gauge charges is guaranteed (Grimus and Rebelo, 1997) in any gauge theory. This is by definition a CP transformation. On the other hand, a parity transformation commuting with gauge transformations can exist only if the fermions are not chiral, as is well known. Types of interplay with gauge invariance other that the ones defining P and CP are in principle also possible.

Under a CP transformation gauge interactions are automatically invariant, which is not necessarily the case for Yukawa interactions. Indeed, when we turn off the Yukawa couplings of the SM, the theory also becomes invariant under CP transformations, whose action in flavor space is usually assumed to be trivial and thus irrelevant as flavor symmetry. However, generalizations of this action are possible (Ecker, Grimus, and Neufeld, 1987; Neufeld, Grimus, and Ecker, 1988). We consider a theory with a "conventional" (commuting with Poincaré and gauge) *global* flavor symmetry group G_f.[7] If G_f includes all flavor transformation leaving the theory invariant, a meaningful action of CP is guaranteed only for special choices of the flavor group and/or its representations. Indeed, in the presence of a global symmetry G_f, CP transformations should satisfy a set of consistency conditions (Feruglio, Hagedorn, and Ziegler, 2013; Holthausen, Lindner, and Schmidt, 2013a) similar to the one in Eq. (3.35). In such a theory the transformations of the fermion fields f read

$$f \to U(g)f, \qquad f \to X_{CP}\bar{f}, \tag{3.37}$$

where $U(g)$ is a unitary representation of G_f, g is a generic element of G_f, and X_{CP} a unitary matrix representing the action of CP in flavor space. Under the combination of a CP transformation followed by a G_f transformation and an inverse CP transformation, the theory remains invariant. This implies that for each $g \in G_f$ an element $g' \in G_f$ should exist such that

$$X_{CP}U^*(g)X_{CP}^{-1} = U(g'). \tag{3.38}$$

The map $g' = u(g)$, implicitly defined by the previous relation, is an automorphism of the group G_f, since it reshuffles the elements of G_f while preserving the composition law. Moreover, since CP relates particles and antiparticles, the function $g' = u(g)$ should map each representation r of the group G_f onto its conjugate $\bar{r}$. We call such an automorphism a *complex conjugation*. In general, a given group G_f can possess automorphisms other than complex conjugations. When G_f is a continuous semisimple group with an appropriate choice of basis in field space, the constraint (3.38) can always be solved by $X_{CP} = \mathbb{1}$ (Grimus and Rebelo, 1997). Moreover, up to compositions with a transformation of the group G_f, $X_{CP} = \mathbb{1}$ is essentially the most general solution of Eq. (3.38). A single exception is provided by the groups SO(2N) ($N \neq 4$), which admit independent solutions.

The major difference with respect to the case of continuous gauge symmetries is that, if G_f is a discrete group, complex conjugations are not guaranteed to exist. It is useful to distinguish between inner automorphisms of G_f that can be cast in the form $u(g) = hgh^{-1}$ $(h \in G_f)$ and outer automorphisms, which do not allow such a description. The inner automorphisms map each representation of G_f onto an equivalent one, while outer automorphisms can permute the representations. Thus, inner automorphisms can describe solutions of Eq. (3.38) only if the flavor group representation is vectorlike. If it is chiral, the automorphism solving Eq. (3.38) should necessarily be a complex conjugation of the outer type (Holthausen, Lindner, and Schmidt, 2013a). It follows that discrete groups G_f can be divided into two classes (Chen *et al.*, 2014). Those not possessing outer complex conjugations are called type-I groups. Theories having this type of flavor symmetry in general do not allow a consistent definition of CP, at least for a generic field content. An example of a type-I group is $\Delta(27)$. To define CP in such theories, we should restrict the field content to a suitable subset of the available representations, on which an

[6] More precisely, the full CP transformation on Weyl spinors reads $f_a \to X_{ab}(\epsilon f_b^\dagger)$.

[7] Recent reviews on the combination of global and CP symmetries were given by Trautner (2016, 2017) and Chen and Ratz (2019).

automorphism of the group acts as a complex conjugation. Type-II groups possess outer complex conjugation. Theories invariant under such groups admit a consistent definition of CP. Examples of type-II groups are $S_{3,4}$, $A_{4,5}$, and T'. Depending on the choice of the input parameters, these theories can be CP invariant or not, exactly as happens for the SM, that admits a consistent action of CP but is CP invariant only for special values of the parameters.

A CP transformation is involutive, up to inner automorphisms (Nishi, 2013). This can be seen by applying Eq. (3.38) twice, which gives

$$X_{CP}X_{CP}^{*}U(g)X_{CP}^{-1*}X_{CP}^{-1} = U(u^2(g)), \qquad u^2(g) \equiv u(u(g)). \tag{3.39}$$

Since $X_{CP}X_{CP}^{*}$ represents the action of some element s of G_f, we have

$$U(s)U(g)U(s)^{-1} = U(u^2(g)), \tag{3.40}$$

which implies that $u^2(g) = sgs^{-1}$ is an inner automorphism. The relation $u(s) = s$ also follows. If s^n is the identity for some integer n, which is always true for finite groups, it follows that $(X_{CP}X_{CP}^{*})^n = \mathbb{1}$.

Finally, if X_{CP} is a complex conjugation solving the constraint (3.38), then $X'_{CP} = U(h)X_{CP}$ is also for any fixed element h of the group G_f. The action of X'_{CP} differs from that of X_{CP}. For example, we might have a canonical $X_{CP} = \mathbb{1}$ and a generalized X'_{CP} acting in a nontrivial way. It is important to stress that X'_{CP} and X_{CP} set the same constraint on the theory, since X'_{CP} is the combination of X_{CP} with a symmetry transformation. Nevertheless, when considering the breaking of the full flavor symmetry group, it can be useful to exploit generalized CP transformation to classify the available breaking chains and their features. Combining a flavor group G_f with CP results in the group $G = G_f \rtimes CP$ if $CP^2 = 1$. In general, requiring invariance under G sets additional restrictions among parameters with respect only to enforcing G_f. Physical phases can be constrained or predicted, as discussed in Sec. V.

F. Nonlinear flavor symmetries

The action of the flavor group G on the matter multiplets can also be nonlinear. A natural realization of this scenario involves the introduction of a set of real scalar fields φ^α, neutral under the SM gauge group, living in a manifold $\mathcal{M}$ equipped with the metric $g_{\alpha\beta}(\varphi)$. Many SM extensions predict the existence of new scalar degrees of freedom. For instance, in string theory components of the metric tensor describing size and shape of the compactified space are scalar in four dimensions. In this context φ^α play the role of flavons. Terms with two derivatives read

$$\mathcal{L}_\varphi = \tfrac{1}{2} g_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial_\mu\varphi^\beta. \tag{3.41}$$

Under a reparametrization of $\mathcal{M}$, $\varphi^\alpha \to f^\alpha(\varphi)$, the metric transforms as

$$g_{\alpha\beta}(\varphi) \to \tilde{g}_{\alpha\beta}(\varphi) = \frac{\partial f^\gamma}{\partial\varphi^\alpha} g_{\gamma\delta}(f(\varphi)) \frac{\partial f^\delta}{\partial\varphi^\beta}, \tag{3.42}$$

and the Lagrangian becomes

$$\mathcal{L}_\varphi \to \tilde{\mathcal{L}}_\varphi = \tfrac{1}{2}\tilde{g}_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial_\mu\varphi^\beta. \tag{3.43}$$

The isometries are reparametrizations leaving invariant the metric and hence the Lagrangian

$$\tilde{g}_{\alpha\beta}(\varphi) = g_{\alpha\beta}(\varphi), \qquad \tilde{\mathcal{L}}_\varphi = \mathcal{L}_\varphi. \tag{3.44}$$

They form the isometry group G_I of $\mathcal{M}$. The flavor group G is identified with a subgroup of G_I. This framework defines a nonlinear σ model invariant under G_I, to which matter fields of the SM are coupled. For simplicity we consider the SM fermions, collectively denoted by ψ^i, in the limit where gauge interactions are turned off. A minimal coupling comprises

$$\mathcal{L}_\psi = i h_{ij}(\varphi)\bar{\psi}^i\bar{\sigma}^\mu\partial_\mu\psi^j + k_{ij\alpha}(\varphi)\bar{\psi}^i\bar{\sigma}^\mu\psi^j\partial_\mu\varphi^\alpha + \text{H.c.} \tag{3.45}$$

Under a reparametrization of $\mathcal{M}$, the fermions transform as $\psi^i \to \chi^i(\varphi,\psi) = \xi^i_j(\varphi)\psi^j + \cdots$, where the ellipisis stands for possible contributions of higher order in ψ. Here we consider fermion transformations nonlinear in φ but linear in ψ, which is the easiest way to guarantee that the transformed fields have the same gauge quantum numbers as the original ones. Hence, a generic reparametrization reads

$$\varphi^\alpha \to f^\alpha(\varphi), \qquad \psi^i \to \xi^i_j(\varphi)\psi^j. \tag{3.46}$$

Group properties are guaranteed by the relations

$$\begin{aligned} &\varphi \xrightarrow{g_1} f_{g_1}(\varphi) \xrightarrow{g_2} f_{g_1}(f_{g_2}(\varphi)) = f_{g_1g_2}(\varphi), \\ &\psi \xrightarrow{g_1} \xi_{g_1}(\varphi)\psi \xrightarrow{g_2} \xi_{g_1}(f_{g_2}(\varphi))\xi_{g_2}(\varphi)\psi = \xi_{g_1g_2}(\varphi)\psi, \end{aligned} \tag{3.47}$$

and

$$f_e(\varphi) = \varphi, \qquad \xi_e(\varphi) = \mathbb{1}. \tag{3.48}$$

Under Eq. (3.46) the metric $h_{ij}(\varphi)$ and the connection $k^i_{j\alpha}(\varphi) \equiv h^{il}(\varphi)k_{lj\alpha}(\varphi)$ transform as[8]

$$\begin{aligned} &h_{ij}(\varphi) \to \xi^{k*}_i h_{kl}(f(\varphi))\xi^l_j, \\ &k^i_{j\alpha}(\varphi) \to (\xi^{-1})^i_m k^m_{l\beta}(f(\varphi))\xi^l_j \frac{\partial f^\beta}{\partial\varphi^\alpha} + i(\xi^{-1})^i_l \frac{\partial \xi^l_j}{\partial\varphi^\alpha}. \end{aligned} \tag{3.49}$$

If the transformation of Eq. (3.46) is an isometry, the metric and connection are required to be invariant. From Eq. (3.49) we understand the role of the connection $k^i_{j\alpha}(\varphi)$: even when the isometry of the scalar manifold $\mathcal{M}$ is realized by global transformations on φ^α, the fermion transformations are always local due to the explicit space-time dependence of the

[8]Indices are lowered and raised by the metric $h_{ij}(\varphi)$ and the inverse metric $h^{ij}(\varphi)$, respectively.

functions $\xi^i_j(\varphi)$. The two terms in Eq. (3.45) can be combined into a covariant derivative

$$(D_\mu\psi)^i \equiv [\delta^i_j\partial_\mu - ik^i_{j\alpha}(\varphi)\partial_\mu\varphi^\alpha]\psi^j, \tag{3.50}$$

which under an isometry transforms as the fermions ψ^i

$$(D_\mu\psi)^i \to \xi^i_j(\varphi)(D_\mu\psi)^j. \tag{3.51}$$

In the case treated in Sec. VI the isometries act on the fermion fields in the following way:

$$\psi^i \to \left[\det\left(\frac{\partial f}{\partial\varphi}\right)\right]^{-k/2}\rho^i_j\psi^j, \tag{3.52}$$

where k is a real number called the *weight* and ρ is a φ-independent unitary representation of a compact coset G/H, where $G \subseteq G_I$ and H is a normal subgroup of G. A beneficial property of the transformation (3.52) is that it manifestly provides a nonlinear realization of G. Indeed, considering two subsequent isometries we have

$$\begin{aligned}\psi^i &\xrightarrow{g_1}\left[\det\left(\frac{\partial f_{g_1}}{\partial\varphi}\right)\right]^{-k/2}(\rho_{g_1})^i_j\psi^j\\ &\xrightarrow{g_2}\left[\det\left(\frac{\partial f_{g_1}}{\partial\varphi}\right)\right]^{-k/2}_{\varphi\to f_{g_2}(\varphi)}\left[\det\left(\frac{\partial f_{g_2}}{\partial\varphi}\right)\right]^{-k/2}(\rho_{g_1})^i_k(\rho_{g_2})^k_j\psi^j\\ &=\left[\det\left(\frac{\partial f_{g_1g_2}}{\partial\varphi}\right)\right]^{-k/2}(\rho_{g_1g_2})^i_j\psi^j,\end{aligned} \tag{3.53}$$

and the group composition property is guaranteed. Invariance of the metric $h_{ij}(\varphi)$ under the isometry (3.52) requires that

$$h_{ij}(f(\varphi)) = \left[\det\left(\frac{\partial f}{\partial\varphi}\right)\right]^{k}\rho^m_i h_{mn}(\varphi)(\rho^\dagger)^n_j. \tag{3.54}$$

Equation (3.52) can be generalized by allowing different pairs (k,ρ) for distinct irreducible representations $\psi_{(I)}$ of the gauge group

$$\psi^i_{(I)} \to \left[\det\left(\frac{\partial f}{\partial\varphi}\right)\right]^{-k_I/2}\rho_{(I)}{}^i_j\psi^j_{(I)}. \tag{3.55}$$

Invariance of a fermion bilinear[9]

$$\mathcal{L}_Y = \lambda(\varphi)_{ij}\psi^i_{(I_1)}\psi^j_{(I_2)} + \text{H.c.} \tag{3.56}$$

requires a Yukawa coupling $\lambda(\varphi)_{ij}$ satisfying

$$\lambda(f(\varphi))_{ij} = \left[\det\left(\frac{\partial f}{\partial\varphi}\right)\right]^{(k_{I_1}+k_{I_2})/2}[\rho_{(I_1)}]^{k*}_i\lambda(\varphi)_{kl}[\rho^\dagger_{(I_2)}]^l_j. \tag{3.57}$$

[9]For notational convenience we set the Higgs multiplet H to 1, which can be easily reintroduced in our expressions. In addition, H can undergo a transformation of the type of Eq. (3.55).

The overall Lagrangian

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}g_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial_\mu\varphi^\beta + h_{ij}(\varphi)\bar\psi^i\bar\sigma^\mu D_\mu\psi^j\\ &\quad + \lambda(\varphi)_{ij}\psi^i_{(I_1)}\psi^j_{(I_2)} + \text{H.c.}\end{aligned} \tag{3.58}$$

is invariant under the nonlinearly realized flavor symmetry:

$$\varphi^\alpha \to f^\alpha(\varphi), \qquad \psi^i_{(I)} \to \left[\det\left(\frac{\partial f}{\partial\varphi}\right)\right]^{-k_I/2}\rho_{(I)}{}^i_j\psi^j_{(I)}. \tag{3.59}$$

Notice that this formalism, unlike the Callan-Coleman-Wess-Zumino construction (Callan *et al.*, 1969; Coleman, Wess, and Zumino, 1969), covers both the case of a global flavor symmetry and that of a discrete one. The purpose of this approach is to select G, G/H, $\rho_{(I)}$, and $k_{(I)}$ so as to constrain as much as possible the function $\lambda(\varphi)$. In an ideal case, the functional dependence of $\lambda(\varphi)$ on φ is completely determined up to an overall constant and all dimensionless parameters such as mass ratios, mixing angles, and physical phases are all fixed functions of φ, which provides a highly constrained system of predictions. Thus far this program has been explored in the context of a supersymmetric σ model where the flavor group G is the modular group $\mathrm{SL}(2,Z)$ contained in $G_I = \mathrm{SL}(2,R)$ and G/H is a finite modular group.

IV. STANDARD FLAVOR SYMMETRIES

We now consider specific flavor symmetry models. We begin with the "standard" case in which the flavor symmetry commutes with the gauge and Poincaré transformations, in the context of the standard framework discussed in Sec. II.B, in which the origin of neutrino masses lies at scales higher than the electroweak scale. We consider flavor symmetries constraining the effective electroweak scale Lagrangian containing the Weinberg operator in Eq. (2.1) and also flavor symmetries constraining its possible renormalizable high-scale origins (and comment on the equivalence of the two approaches). We also classify models according to whether the symmetry breaking affects the flavor observables mildly or prominently.

We saw in Sec. III.D that a viable flavor symmetry must be broken by a set of flavon or spurion fields ϕ, transforming under a representation U_ϕ of G. The lepton couplings and mass matrices then acquire a dependence on ϕ, $M_E = M_E(\phi)$, $m_\nu = m_\nu(\phi)$. Because the full Lagrangian is assumed to be invariant under G, the mass matrices satisfy

$$\begin{aligned}M_E(\phi) &= U_e(g)^T M_E(U_\phi(g)\phi)U_l(g),\\ m_\nu(\phi) &= U_l(g)^T m_\nu(U_\phi(g)\phi)U_l(g)\end{aligned} \tag{4.1}$$

for any $g \in G$.

It is often but not always the case that the functions $M_E(\phi)$ and $m_\nu = m_\nu(\phi)$ are continuous for $\phi \to 0$, and they admit an expansion in the flavons and their conjugates around their symmetric forms $M^0_E = M(0)$, $m^0_\nu = m_\nu(0)$ [which satisfy Eq. (3.7)]. Flavor symmetry models can be either in the "perturbative" regime in which the symmetry breaking terms provide a moderate correction to the flavor observables or in the "leading order breaking" regime in which symmetry breaking is necessary even for a leading order understanding of the flavor observables. The latter is the case when the neutrino or the charged-lepton mass matrix vanishes in the symmetric limit. In Sec. IV.A, we consider the first possibility. The leading order breaking case is discussed in Sec. IV.B.

A. Perturbative breaking: Mild corrections to flavor observables

We saw in Sec. III.C that the symmetric forms M_E^0, m_ν^0 of the lepton mass matrices cannot provide a nontrivial, accurate description of lepton masses and mixings. It is possible, however, that they provide an approximate description. This is how nonexact symmetries of nature have often emerged. Pions, for example, are close to an isospin symmetric limit in which the charged and neutral pion masses and couplings are equal. Analogously, one wonders whether lepton flavor observables are close to the symmetric predictions of a flavor theory. If this is the case, we can say that the understanding of the leading order pattern of lepton flavor lies in the flavor symmetry itself, and symmetry breaking effects provide only a moderate correction to the observables needed for their accurate description.

1. Flavor symmetries at low scales

We first consider the case in which neutrino masses are fully described by the Weinberg operator and the flavor symmetry operates on the Lagrangian in Eq. (3.6). In such a case, the flavor symmetry constrains the lepton mass matrices as in Eqs. (3.7) and a complete study of the perturbative option is possible. In fact, given a mass and mixing pattern considered to be a viable leading order approximation, the full set of flavor groups and representations leading to that pattern in the symmetric limit can be characterized in terms of the structure of the decomposition of U_l, U_{e^c} into irreducible components; namely, in terms of the type (real, complex, or pseudoreal), dimension, and equivalence of the irreducible components. To be conservative, all the symmetric predictions in which the following three conditions apply are deemed to be viable:

(i) The PMNS matrix is not fully undetermined.

(ii) Both the θ_{23} and θ_{12} angles are allowed to be nonvanishing.

(iii) The nonvanishing charged-lepton masses are not forced to be degenerate.

The flavor symmetry models compatible with these requirements are then those whose representations on the SM leptons have one of the six decompositions listed in Table V (Reyimuaji and Romanino, 2018).

Note that only Abelian representations are allowed (with the only possible exception of a non-Abelian two-dimensional representation on e_1^c, e_2^c when $m_e = m_\mu = 0$). Either neutrinos have an inverted hierarchy or the neutrino mass matrix is completely unconstrained. This is because there are only two possibilities for the representation on lepton doublets. When $U_l \sim 1 + 1 + 1$, at most the three lepton doublets can transform by an overall sign under G. The neutrino mass matrix is then completely unconstrained, and any neutrino masses and mixings are possible. The flavor symmetry is useless in the neutrino sector, where it leads to anarchy (Altarelli, Feruglio, and Masina, 2003; Hall, Murayama, and Weiner, 2000; Haba and Murayama, 2001; Hirsch and King, 2001; de Gouvea and Murayama, 2003) (it may still be useful to explain the charged-lepton mass hierarchy). When $U_l \sim \mathbf{1} + \mathbf{1} + \bar{\mathbf{1}}$, the neutrino masses are in the form $(0aa)$ in the symmetric limit, which is close to the inverted hierarchical spectrum. Therefore, if this hint for normal hierarchy was transformed into evidence, we would conclude that no flavor symmetry can provide a nontrivial approximate understanding of lepton flavor in the symmetric limit. Symmetry breaking effects would then play a leading role in determining at least some of the flavor observables.

In all cases, no precise prediction on any of the lepton observables can be obtained (except possibly $\theta_{13} = 0$, which is not precise on the experimental side), as the representation U_l on lepton doublets is always found to be Abelian and because of the unknown $\mathcal{O}(1)$ factors involved in each matrix element. Note that the one-dimensional representations are always Abelian and that Abelian groups have only one-dimensional irreducible representations. On the other hand, the one-dimensional representations in Table V can also

TABLE V. Classification of flavor groups and representations leading to an approximately viable prediction in the symmetric limit. The Weinberg operator is assumed to describe neutrino masses. The decompositions of the representation on the charged-lepton doublets and singlets l_i and e_i^c into irreducible components is shown in the first two columns. The notation shows the dimension and type (boldface = complex, roman = real) of the representation. Identical symbols are associated with equivalent representations, while $\bar{\mathbf{1}}$ is the complex conjugate of 1. r denotes a generic, possibly reducible, representation. The predicted charged-lepton and neutrino mass patterns are shown in the third and fourth columns. The fifth column shows the type of neutrino mass hierarchy [normal or inverted hierarchical (NH or IH)]. The last column specifies whether or not the PMNS matrix contains a zero and, if so, in which position. In the second line the 13 entry can either vanish or not depending on an unknown "12" rotation determined by the symmetry breaking effect. On the last line, the position of the zero depends on the relative sizes of A, B, and C. In the cases corresponding to the last four rows, the hierarchy of charged-lepton masses is not explained by the flavor model and is accounted for by a hierarchy among the free parameters A, B, and C.

U_l	U_{e^c}	$(m_\tau m_\mu m_e)$	$(m_3 m_2 m_1)$	ν hierarchy	PMNS zeros
$1\ \ 1\ \ 1$	$1\ \ r \not\supseteq 1\ \ 1$	$(A00)$	(abc)	NH or IH	none
$\mathbf{1}\ \ \mathbf{1}\ \ \bar{\mathbf{1}}$	$\bar{\mathbf{1}}\ \ r \not\supseteq \mathbf{1}\ \ \bar{\mathbf{1}}$	$(A00)$	$(0aa)$	IH	none (**13**)
$1\ \ 1\ \ 1$	$1\ \ 1\ \ r \neq 1$	$(AB0)$	(abc)	NH or IH	none
$\mathbf{1}\ \ \mathbf{1}\ \ \bar{\mathbf{1}}$	$\bar{\mathbf{1}}\ \ \bar{\mathbf{1}}\ \ r \neq \mathbf{1}$	$(AB0)$	$(0aa)$	IH	**13**
$1\ \ 1\ \ 1$	$1\ \ 1\ \ 1$	(ABC)	(abc)	NH or IH	none
$\mathbf{1}\ \ \mathbf{1}\ \ \bar{\mathbf{1}}$	$\bar{\mathbf{1}}\ \ \bar{\mathbf{1}}\ \ \mathbf{1}$	(ABC)	$(0aa)$	IH	**13**, 23, 33

belong to non-Abelian groups. In the case in which $G = \mathrm{U}(1)$, the one-dimensional representations are specified by their charges under U(1).

Examples of flavor models corresponding to the nontrivial examples in Table V have long been known. As mentioned, the three cases corresponding to the trivial representation $U_l \sim 1 + 1 + 1$ correspond to anarchical neutrinos. No special prediction is obtained, but the mixing angles and neutrino mass ratios are all expected to be $\mathcal{O}(1)$. Indeed, the neutrino spectrum does not need mass ratios smaller than a factor of 1/5–6, and the smallest mixing element is $|U_{e3}| \sim 1/7$. Moreover, they can arise from moderately small Yukawa couplings in the context of the seesaw, as the neutrino Yukawa couplings are squared in the seesaw formula. The size of $|U_{e3}|$ had an upper bound only, when anarchy was first considered. The measurement of a value not far from that bound corroborated the proposal (Altarelli *et al.*, 2012; de Gouvea and Murayama, 2015). The three cases have different U_{e^c} values. The use of a nontrivial representation on the e^c fields can forbid the electron and the muon masses in the symmetric limit and can therefore be used to account for the hierarchy of charged-lepton masses even in the presence of anarchical neutrinos.

As for the three nonanarchical cases, they require continuous or discrete groups with a complex one-dimensional representation "1" and a representation on the lepton doublets decomposing as $\mathbf{1} + \mathbf{1} + \bar{\mathbf{1}}$. A simple choice is $G = U(1)$ with charges $(q_1^l, q_2^l, q_3^l) = (-1, 1, 1)$ on the three lepton doublets. In all cases, the neutrino mass matrix is of the form

$$m_\nu = \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} + \text{corrections}, \tag{4.2}$$

where the corrections are provided by symmetry breaking effects. Depending on whether or not U_{e^c} matches U_l, the lighter charged-lepton masses may or may not vanish in the symmetric limit, thus providing a rationale for their hierarchy. One obtains in fact

$$M_E^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B & A \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{pmatrix}, \quad \begin{pmatrix} E & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{pmatrix} \tag{4.3}$$

in the three cases of Table V before switching on symmetry breaking effects.

In the U(1) example, the last pattern of Eq. (4.3) can be reproduced by choosing opposing charges $(q_1^{e^c}, q_2^{e^c}, q_3^{e^c}) = (1, -1, -1)$ for the three e^c fields. This corresponds to a U(1) symmetry with charge $L_\tau + L_\mu - L_e$ (Barbieri *et al.*, 1998); similar symmetries were considered by Zeldovich (1952), Konopinski and Mahmoud (1953), and Petcov (1982). None of the charged-lepton hierarchies $m_e \ll m_\mu \ll m_\tau$ are accounted for. Moreover, the PMNS matrix contains a zero in the symmetric limit that should be identified with the U_{13}, but it can appear in the 12 or 33 position, depending on which charged-lepton family ends up being lighter. To get rid of both drawbacks, one can depart from $L_\tau + L_\mu - L_e$ by using different representations $U_{e^c} \neq U_l^*$, forcing $m_e = 0$ and possibly $m_\mu = 0$ in the symmetric limit. In all cases, the solar mixing angle is maximal in the symmetric limit and requires significant corrections from symmetry breaking; see Sec. IV.B.3.

A normal hierarchy can be obtained in an important class of models that does not appear in Table V, in which the neutrino mass matrix is of the form (Barbieri *et al.*, 1998; Grossman, Nir, and Shadmi, 1998; Irges, Lavignac, and Ramond, 1998)

$$m_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & b \\ 0 & b & a \end{pmatrix} + \text{corrections}. \tag{4.4}$$

Such a texture is obtained if $U_l = 1 + 1 + \mathbf{1}$. We use roman and boldface fonts to denote real and complex representations, respectively (see Table V). This texture is sometimes called "semianarchy," as the 23 block of the neutrino mass matrix (corresponding to the trivial representation $1 + 1$), but not the entire one, is now unconstrained. In the symmetric limit, the solar angle is predicted to vanish ($\theta_{12} = 0$), a prediction that is far from the observation. For the previously mentioned texture to be phenomenologically viable, the correction to θ_{12} from symmetry breaking effects cannot be mild. On the contrary, it must be fully responsible for the observed value of θ_{12}. Therefore, such models belong to the class considered in Sec. IV.B. On the other hand, the $\mathcal{O}(1)$ correction to θ_{12} does not need the symmetry breaking effects to be large. This is because of another drawback of the texture in Eq. (4.4): it does not account for the observed suppression $\Delta m_{12}^2 \ll |\Delta m_{23}^2|$. The latter needs an accidental cancellation in the determinant $ac - b^2$, which should vanish up to $\mathcal{O}(m_2/m_3)$ corrections. Once such a mild accident is accepted, subleading $\mathcal{O}(m_2/m_3)$ symmetry breaking effects are sufficient to generate a $\mathcal{O}(1)$ solar angle.

Note that predictions based on accidental relations may be unstable with respect to renormalization group equations (Chankowski and Pokorski, 2002) or generic corrections (Marzocca and Romanino, 2014; Domcke and Romanino, 2016). An apparently accidental suppression can be accounted for in the seesaw context, as we discuss later.

The results in Table V can be extended to the quark sector. The constraints one obtains there are independent of those previously discussed. However, in the context of grand unified theories, unified quarks and leptons cannot be treated separately, as they must be subject to the same flavor representation. This leads to additional constraints. For example, in minimal SU(5) unification, only the anarchical cases in Table V turn out to be allowable.

2. Flavor symmetries at high scales

The previous conclusions were based on the assumption that the flavor symmetry constrains the effective electroweak scale Lagrangian containing the Weinberg operator. The latter, however, presumably represents the low-scale remnant of a more fundamental higher scale renormalizable Lagrangian. One then wonders whether the conclusions summarized in Table V would still hold if the flavor symmetry were assumed to constrain the higher scale Lagrangian. In particular, one wonders whether the choice between anarchy and inverse hierarchy is still necessary if the symmetric predictions are required to be viable. This is part of a more general issue concerning the results obtained in the symmetric limit. Are the predictions obtained when the flavor symmetry acts on a high-scale Lagrangian equivalent to those obtained when the same symmetry constrains the corresponding effective Lagrangian? The answer is no. On the other hand, the converse is true: given a flavor symmetry constraining the effective Lagrangian, it is always possible to extend its action to a high-scale Lagrangian providing the same predictions. Therefore, while the low-scale effective flavor theory does not capture all the features of the high-scale one, an appropriate high-scale realization always captures the features of the low-scale effective one.

There are two reasons why the high-scale predictions might not coincide with the low-scale ones (Reyimuaji and Romanino, 2020). The most obvious is that the mass of some of the high-scale fields vanishes when the symmetry is exact. This happens if the flavor group representation on the high-scale fields is not vectorlike. In such a case, the heavy fields cannot be integrated out (as some of them are massless) before symmetry breaking effects have been switched on. Once the breaking effects are added, all the high-scale fields acquire a mass, including those whose mass vanished in the symmetric limit. The latter get a mass from subleading symmetry breaking effects. Therefore, their mass is expected to be lighter, and as a consequence their exchange dominates the effective Lagrangian and neutrino masses. In standard seesaw language, this corresponds to the so-called single or sequential right-handed neutrino dominance (Barbieri *et al.*, 1998; King, 1998; Altarelli and Feruglio, 1999a; King, 1999, 2000; Antusch and King, 2004b), which arise also in the context of non-Abelian models (King, 2005).

Even in the cases in which all the relevant heavy fields stay heavy when the symmetry is exact, the high- and low-scale predictions can differ. Consider for definiteness a type-I seesaw Lagrangian (with an arbitrary number of singlet neutrinos), and assume that the singlet neutrinos are non-singular in the limit in which the flavor symmetry is exact. It turns out that there is a precise condition under which the high- and low-scale predictions of the flavor symmetry are equivalent: this is the case if and only if the vectorlike part[10] of the representation on the lepton doublets is contained in the representation on the neutrino singlets.

We consider these two possibilities in turn. We start with the case in which the mass of some of the high-scale fields vanishes when the symmetry is exact, in the context of a type-I seesaw model. The possible equivalence of the high- and low-scale approaches in the symmetric limit can still be investigated when the limit $m_\nu(\phi)$ for $\phi \to 0$ exists and is finite.[11]

[10]By *vectorlike part*, we mean the maximal subrepresentation that is vectorlike, i.e., made of real representations, pairs of complex conjugated representations, or pairs of equivalent pseudoreal representations.

[11]In some cases, the analysis can be extended to the cases in which the limit diverges by normalizing the neutrino mass matrix to the largest entry when taking the limit.

In some cases, the two descriptions can still be equivalent. Consider, for example, U(1) seesaw models in which the flavons have charges with definite sign (negative, for example) and the leptons have non-negative charges. We also invoke supersymmetry to prevent a positively charged flavon to be mimicked by a conjugated flavon. In such a case, the high- and low-scale descriptions are equivalent in the symmetric limit, regardless of whether some of the right-handed neutrinos are massless in that limit. Consider, for example, the case of a single flavon with VEV θ (in terms of the cutoff scale) with charge -1, and let $q_i^l \geq 0$, $q_i^{\nu^c} \geq 0$ be the lepton doublet and singlet neutrino charges $i = 1, 2, 3$. Then in the broken phase, the low-scale flavor theory predicts that

$$(m_\nu^{\mathrm{LS}})_{ij} = c_{ij}^{\mathrm{LS}} \theta^{q_i^l + q_j^l}, \tag{4.5}$$

where c is a generic, unknown 3×3 dimensionful matrix. In the high-scale theory, we have instead $(m_D)_{ij} = (c_D)_{ij} \theta^{q_i^{\nu^c} + q_j^l}$, $M_{ij} = C_{ij} \theta^{q_i^{\nu^c} + q_j^{\nu^c}}$ for the Dirac and singlet Majorana mass matrices, respectively. Therefore, the light neutrino mass matrix is

$$(m_\nu^{\mathrm{HS}})_{ij} = c_{ij}^{\mathrm{HS}} \theta^{q_i^l + q_j^l}, \tag{4.6}$$

where $c^{\mathrm{HS}} = -c_D^T C^{-1} c_D$ is also a generic, unknown 3×3 matrix. Therefore, the high- and low-scale definitions of the flavor theories are equivalent.

On the other hand, the two descriptions can be inequivalent. Suppose that the lepton doublets and singlet neutrinos have charges $(q_1^l, q_2^l, q_3^l) = (0, 1, 1)$ and $(q_1^{\nu^c}, q_2^{\nu^c}, q_3^{\nu^c}) = (0, 0, -1)$ under a U(1) model. Then, in the unbroken limit, the low- and high-scale versions of the same U(1) model provide quite different results

$$m_\nu^{\mathrm{LS}} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad m_\nu^{\mathrm{HS}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & b \\ 0 & b & a \end{pmatrix}, \quad \text{with} \quad ac - b^2 = 0. \tag{4.7}$$

The high-scale result follows from the following forms of the unbroken Dirac and Majorana matrices:

$$m_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B & A \end{pmatrix}, \qquad M = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.8}$$

In the "almost unbroken" limit, the seesaw is dominated by the exchange of ν_3^c, the only one taken into account in Eq. (4.7). This is the previously mentioned single right-handed dominance mechanism in its most classical realization, which now accounts for the apparently accidental suppression of the determinant $ab - c^2$ needed in Eq. (4.4).

As mentioned, there is a second case in which the high- and low-scale formulations of the same flavor model are certainly inequivalent, even when all the right-handed neutrinos are allowed to be massive in the unbroken limit, i.e., even when the representation of G on them is vectorlike. This is the case if the vectorlike part of the representation on the lepton doublets is not contained in the representation on the neutrino singlets (Reyimuaji and Romanino, 2020). We illustrate the latter possibility with an example (Altarelli and Feruglio, 1999a). Suppose that the lepton doublets and singlet neutrinos have charges $(q_1^l, q_2^l, q_3^l) = (n, 0, 0)$ and $(q_1^{\nu^c}, q_2^{\nu^c}, q_3^{\nu^c}) = (1, -1, 0)$ under U(1), with $n \neq \pm 1, 0$. Then the unbroken Dirac, singlet, and light neutrino matrices are

$$m_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B & A \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix},$$

$$m_\nu^{\mathrm{HS}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & b \\ 0 & b & a \end{pmatrix}, \quad \text{with } ac - b^2 = 0, \tag{4.9}$$

as $a = A^2/\alpha$, $b = AB/\alpha$, and $c = B^2/\alpha$. Note that the vanishing of the determinant is obtained as a consequence of the seesaw mechanism without the need to invoke the presence of a lighter singlet neutrino. In the symmetric limit, the model predicts large θ_{23}, $m_1 = m_2 = 0$, and θ_{12} undefined. Enforcing the same flavor symmetry in the low-scale effective theory gives, on the other hand, Eq. (4.4), with no condition on the determinant. Therefore, the unbroken predictions are different: m_2 now does not vanish and $\theta_{12} = 0$.

As discussed, most instances of the perturbative breaking of flavor symmetries discussed in this section are associated with models having Abelian symmetries. These have been widely studied in the first wave of model building following the measurement of a large atmospheric angle. Additional examples and further details were given in earlier reviews.

B. Non-Abelian models and leading order breaking

In the case of leading order (LO) breaking of the flavor symmetry, the symmetry breaking effects cannot be disregarded even for a leading order understanding of lepton flavor. This happens when the unbroken limit is not a good approximation. According to the conservative definition used in Sec. IV.A, this is the case when (i) the PMNS matrix is fully undetermined or (ii) either the θ_{23} or θ_{12} angle is forced to vanish or (iii) two of the charged-lepton masses are forced to be degenerate and do not vanish in the unbroken limit. Correspondingly, there are three possible ways out from the results in Table V.

Violating condition (iii) is not appealing. Charged-lepton masses are hierarchical. Therefore, models with degenerate charged leptons in the unbroken limit require a fine-tuned symmetry breaking contribution. We disregard such a possibility.

If θ_{23} or θ_{12} vanishes in the unbroken limit [case (ii)], the symmetry breaking corrections must be sizable enough to strongly modify the symmetric prediction. This is a concrete possibility whose realization does not require large symmetry breaking corrections. As previously discussed, subleading corrections may be sufficient, in the presence of the mild accident needed to account for the m_2/m_3 hierarchy [which can arise naturally in the seesaw context; see Eqs. (4.7) and (4.9)]. This possibility was widely considered and discussed in Sec. IV.A.

We are left with the possibility that the PMNS is fully undetermined in the symmetric limit [case (i)]. This section deals mainly with such a possibility, which arises when either $m_\nu^0 = 0$ or $m_E^0 = 0$ (the suffix 0 denotes the symmetric limit).

Predictability is an independent motivation to consider models leading to $m_E^0 = 0$, as we now discuss. This may seem paradoxical, as the PMNS matrix is completely undetermined in such a case, which is the poorest possible prediction. In fact, the predictions one gets in such cases have little to do with the symmetry itself and much to do with the details of symmetry breaking.

To see how a predictive model may lead to $m_E^0 = 0$, we first note that non-Abelian flavor groups are welcome because they provide precise predictions. The predictive power of Abelian models is limited by the fact that they admit only $d = 1$ irreducible representations (here and to follow d denotes the dimension of the representation). As a consequence, each flavor matrix entry corresponds to an independent invariant Lagrangian operator (see Sec. III.D), with an independent, unknown dimensionless coupling. In the spirit of flavor models, aiming at providing a dynamical explanation of hierarchies, such couplings can be assumed to be $\mathcal{O}(1)$. This means, however, that predictions are typically plagued by $\mathcal{O}(1)$ uncertainties [barring predictions associated with texture zeros (Björkeroth *et al.*, 2019)]. In the charged fermion sector, which is characterized by significant hierarchies, a prediction of up to an $\mathcal{O}(1)$ factor is significant. But in the neutrino sector, where most flavor parameters turn out to be themselves $\mathcal{O}(1)$, a prediction up to an $\mathcal{O}(1)$ factor is less exciting. To avoid systematic $\mathcal{O}(1)$ uncertainties and attempt significant predictions in the neutrino sector, $d > 1$ irreducible representations are then needed. The latter allow one to correlate different matrix entries through symmetry transformations. From this point of view, the highest predictive power is achieved, in principle, when all three neutrinos, i.e., the three lepton doublets, belong to a single $d = 3$ irreducible representation.

We can now appreciate the connection with $m_E^0 = 0$: the matrix m_E^0 is forced to vanish if the lepton doublets belong to a $d = 3$ irreducible representation U_l of the flavor group in order to avoid having three degenerate, massive charged leptons in the unbroken limit. To prove the last statement, we note that Eq. (3.7) implies that $U_l(m_E^{0\dagger} m_E^0) = (m_E^{0\dagger} m_E^0) U_l$. Since U_l is assumed to be irreducible, $(m_E^{0\dagger} m_E^0) = \alpha \mathbf{1}$ by Schur's lemma. For the charged leptons to not be massive and degenerate, we need $\alpha = 0$, i.e., we need m_E to be forced to vanish in the unbroken limit. As a corollary, non-Abelian models with $m_E^0 \neq 0$ require the lepton doublets to transform as doublet + singlet under the flavor group.

Non-Abelian symmetries can be continuous or discrete. Before reviewing the case of finite non-Abelian groups, we discuss some examples of continuous ones. Continuous (Lie) group models share some of the features of the discrete ones, which is discussed in greater detail in the following sections. In particular, they can lead to precise predictions for some mixing parameters, with substantial help from the scalar potential, arranging proper VEV alignments. In practice, this is most often the case in models in which the three families of lepton doublets belong to a single irreducible $d = 3$ representation of the flavor group.

Simple Lie groups with irreducible representations of dimension $d \leq 3$ are $SU(2) \sim SO(3)$, $SU(3)$. The simple factors can be combined, with U(1) factors as well, in larger groups. First, consider the simplest possibilities, with the only possible addition of a U(1) factor. The group SU(2) is indeed often combined with a U(1) factor suppressing the light charged fermion families into $U(2) = SU(2) \times U(1)$ (Barbieri, Dvali, and Hall, 1996; Barbieri, Hall, and Romanino, 1997; Barbieri *et al.*, 1997). Neutrino masses and mixings can also be accounted for (Raby, 2003; Linster and Ziegler, 2018); see also Sec. III.D. The SO(3) case can lead to tri-bimaximal mixing (discussed later) within what was called "constrained sequential dominance" (King, 2005; King and Malinsky, 2006), can originate from gauge-family unification in an SO(18) grand unified theory (Reig, Valle, and Wilczek, 2018), and can underlie A_4 models (Bazzocchi, Morisi, and Picariello, 2008; Berger and Grossman, 2010; Grossman and Ng, 2015). The SU(3) group is more "democratic" than SO(3). The action of SO(3) in terms of real matrices singles out a real vector subspace in the three family complex flavor space. Moreover, SU(3) is, up to a U(1) factor, the maximal flavor group for fermions with given quantum numbers. In fact, in the case of grand unified SO(10) models, $G_{\max} = SU(3) \times U(1)$. As SU(3) [U(3)] typically forces the Yukawa couplings to vanish in the symmetric limit, it must be strongly broken by the top Yukawa coupling to a weakly broken SU(2) [U(2)]. Maximal atmospheric and substantial solar mixing can be obtained together with hierarchical charged fermions (King and Ross, 2003; Ross, Velasco-Sevilla, and Vives, 2004; Antusch, King, and Malinsky, 2008; Bazzocchi *et al.*, 2009). Tri-bimaximal mixing can also be achieved consistently with SO(10) (de Medeiros Varzielas and Ross, 2006; de Anda and King, 2018). This is not as easy as it is with finite group models, where the flavor quantum numbers are often different within a single family.

An example of a less minimal, and in fact almost maximal, flavor group is provided by $G = SU(3)^5 \times SO(3)$. The $SU(3)^5$ term is [neglecting U(1) factors] the maximal SM flavor group; see Sec. III.A. If the SM field content is supplemented by three singlet neutrinos ν_i^c, $i = 1, 2, 3$, and G is required to allow a flavor-universal Majorana mass term in the form $M\nu_i^c\nu_i^c/2$, the maximal flavor group also contains an SO(3) factor acting on the ν_i^c fields. The Yukawa couplings are assumed to arise as VEVs of flavons transforming as $Y_U \sim 3_{u^c} \times 3_q$, $Y_D \sim 3_{d^c} \times 3_q$, $Y_N \sim 3_{\nu^c} \times 3_l$, and $Y_E \sim 3_{e^c} \times 3_l$ under G, in the spirit of minimal flavor violation (D'Ambrosio *et al.*, 2002), extended to the neutrino sector (Cirigliano *et al.*, 2005; Alonso *et al.*, 2012, 2013). The structure of the Yukawa couplings then depends on the scalar potential that they minimize. The techniques introduced in Sec. III.D.3 can be used to study which values of Y can arise as critical points (Alonso *et al.*, 2011; Espinosa, Fong, and Nardi, 2013).

1. Discrete non-Abelian symmetries and the sequestering assumption

Discrete non-Abelian groups can provide precise predictions for lepton mixing.[12] Their study gained considerable momentum when the measured value of the solar angle was found in agreement with the prediction of the tri-bimaximal (TB) mixing pattern (Harrison, Perkins, and Scott, 2002; Harrison and Scott, 2002b, 2003) $\sin^2\theta_{12} = 1/3$. This pattern also corresponds to a maximal atmospheric angle $\sin^2\theta_{23} = 1/2$ and to $\theta_{13} = 0$. The TB pattern, in turn, is predicted by flavor models based on relatively simple discrete groups. The θ_{13} angle ended up to be larger than predicted in most of the early models. However, the tools and ideas developed in this context are still useful and widely used.

We saw in Sec. III.C that G must be completely broken (up to an irrelevant $\mathbf{Z}_2$) by the full (including breaking effects) lepton mass matrices M_E, m_ν. On the other hand, M_E and m_ν might separately be invariant under nontrivial subgroups G_e, $G_\nu \subseteq G$. A popular model-building strategy relies on the following nontrivial assumption: the subgroups G_e, G_ν are nontrivial and rigidly fix, up to phases, the charged-lepton and neutrino mass bases. As the PMNS matrix is nothing but a measure of the misalignment between the two mass bases, the previous requirement unambiguously determines the PMNS matrix in terms of G_e, G_ν. Since G must eventually be completely broken (up to an overall sign change of the lepton fields), their intersection must be trivial ($G_e \cap G_\nu \subseteq \mathbf{Z}_2$), where $\mathbf{Z}_2$ acts as an overall sign change.

The assumption is nontrivial because G_e and G_ν could well be trivial. In other words, both M_E and m_ν could individually break G completely so that G_e, G_ν would not carry any information on the PMNS matrix. Another possibility, illustrated in Sec. IV.B.4, is that G_e and G_ν are nontrivial but do not fully determine the mass eigenstates. Therefore, while most easily handled and interpreted, the results obtained within the "rigid PMNS" assumption do not exhaust all model-building possibilities associated with discrete groups.

As a consequence of G_e and G_ν rigidly fixing the mass bases, it is possible to choose a basis in flavor space for the l_i and e_i^c fields in which the invariance of M_E and m_ν forces them to be in the form

$$M_E = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \qquad m_\nu = U_0^* \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} U_0^\dagger, \tag{4.10}$$

with unconstrained complex diagonal entries and fixed U_0. The PMNS matrix is then determined up to phases and permutations $U = P_e U_0 P_\nu \Psi$, where Ψ is a diagonal matrix of Majorana phases and P_e, P_ν are permutation matrices

[12]For a "physicist-oriented" review of discrete group theory see Altarelli and Feruglio (2010), Ishimori *et al.* (2010), Ramond (2010), and Grimus and Ludl (2012).

arising because the definition of the PMNS matrix assumes lepton masses to have a specific ordering.

Equation (4.10) illustrates three general features of models relying on the previous assumption: Majorana phases are not constrained, the PMNS matrix is predicted up to permutations of its rows and columns (and only one of the possible forms is usually suitable), and neutrino and charged-lepton masses are unconstrained. In particular, the charged-lepton mass hierarchies are not accounted for. As a remedy to the last drawback, this approach can be complemented by adding an additional, possibly Abelian, group factor $G_{\rm FN}$, taking care of the charged-lepton hierarchy. The breaking of $G_{\rm FN}$ is perturbative, and it is arranged in such a way that the first two charged-lepton families get suppressed through a standard Froggatt-Nielsen (FN) mechanism (Froggatt and Nielsen, 1979).

The spontaneous breaking of G is achieved as usual through the VEV of flavon fields ϕ, breaking G completely. This setup can be implemented if (i) there exist subsets ϕ_e, ϕ_ν (not necessarily disjoint) of the full set of flavons breaking G to G_e and G_ν, respectively, and (ii) only ϕ_e (ϕ_ν) enters M_E (m_ν). We therefore refer to such an assumption as the "sequestering" approximation.

The sequestering can hardly be exact: no ordinary flavor symmetry can prevent ϕ_e and ϕ_ν from contaminating both parts of the Lagrangian. It can, however, happen to hold at some order in a perturbative expansion in the number of flavons. In other words, sequestering is "accidental," in the same sense that the lepton and baryon numbers are accidental in the SM. To see that the flavor symmetry cannot prevent contamination, we consider for simplicity the case in which neutrino masses are accounted for by the Weinberg operator. Suppose that only ϕ_e (ϕ_ν) enters M_E (m_ν) so that neutrino and charged fermion masses follow from the invariant Lagrangian

$$\mathcal{L}_{\rm seq} = f(\phi_\nu)_{ij}(l_i H)(l_j H) + g(\phi_e)_{ij} e^c_i l_j H^* \tag{4.11}$$

when $\phi_{\nu,e} \to \langle \phi_{\nu,e} \rangle$. The dependence on the flavons is often simple, but in order to be general, we consider generic (say, polynomial) functions f and g. The invariance of the Lagrangian requires that

$$f(\phi_\nu) = U_l^T f(U_{\phi_\nu}\phi_\nu)U_l, \qquad g(\phi_e) = U_{e^c}^T g(U_{\phi_e}\phi_e)U_l, \tag{4.12}$$

where U_{ϕ_ν} and U_{ϕ_e} are the representations of G on the flavons ϕ_ν and ϕ_e, respectively. It is then easy to see that terms breaking the sequestering assumption are allowed. As an example, terms such as

$$\begin{aligned}\mathcal{L}' = {}& [f(\phi_\nu)g^\dagger(\phi_e)g(\phi_e)]_{ij}(l_i H)(l_j H) \\ & + [g(\phi_e)f^\dagger(\phi_\nu)f(\phi_\nu)]_{ij} e^c_i l_j H^* \end{aligned} \tag{4.13}$$

are allowed and can spoil the invariance of m_ν, M_E under G_ν, G_e. Therefore, no symmetry argument can prevent the sequestering to be spoiled at higher orders in the flavon expansion.[13] In particular, if the typical size of symmetry breaking corrections in the neutrino sector is ϵ, the sequestering-breaking corrections in the charged-lepton sector can be expected to be at least $\mathcal{O}(\epsilon^2)$ and vice versa.

Even if generically present, mixed $\phi_\nu - \phi_e$ corrections to Eq. (4.11) can be negligible. In such a case, it must be possible to account for the exact values of lepton flavor observables in the limit of exact sequestering. In Sec. IV.B.2 we review this class of models,[14] assuming that G_e and G_ν rigidly determine the mass bases, while models in which non-negligible corrections are needed in order to fit the data are considered in Sec. IV.B.3. In Sec. IV.B.4, we consider the case in which G_e and G_ν loosely determine the lepton mass bases.

2. Exact sequestering, rigid PMNS

We consider the possibility that the corrections to sequestering are negligible so that lepton flavor is accounted for, within the present experimental accuracy, by the Lagrangian in Eq. (4.11). The VEVs of the flavons ϕ_ν and ϕ_e break G to the G_ν and G_e subgroups, respectively, under which the full m_ν and M_E are invariant. The subgroups G_ν and G_e are assumed to unambiguously (up to phases) identify the neutrino and charged-lepton mass eigenstate directions in flavor space. In this context, a nonvanishing θ_{13} must be obtained directly from the misalignment of G_ν and G_e. Simple groups such as A_4 and S_4 leading to $\theta_{13} = 0$ are considered in Sec. IV.B.3.

The forms of M_E and m_ν are subject to general constraints. By using a flavor basis in which M_E or m_ν is diagonal and assuming that all neutrinos are massive, we see that $G_e \subseteq \mathrm{U}(1)_e \times \mathrm{U}(1)_\mu \times \mathrm{U}(1)_\tau$ and $G_\nu \subseteq \mathbf{Z}_2^3$, where one of the $\mathbf{Z}_2$ in an overall sign change and is therefore irrelevant. On the other hand, for the mass basis to be rigidly identified by the residual groups and assuming that the residual groups are finite, we need G_e to contain either $\mathbf{Z}_n$, with n a prime number and $n \geq 3$, or $\mathbf{Z}_2^2$.[15] On the neutrino side, we need $G_\nu \supseteq \mathbf{Z}_2^2$. Therefore, we conclude that

[13]Any further symmetry added to take care of the sequestering can be included in G so that the argument would still hold. In the case of supersymmetric models, the holomorphicity of the superpotential prevents the corrections in Eq. (4.13) from arising within the superpotential. On the other hand, they can still arise in the Kähler potential and propagate to the flavor Lagrangian once the Kähler is brought into its canonical form; see Sec. III.D.

[14]Such models are also called "direct" (King and Luhn, 2009b).

[15]To prove this result, we first observe that G_e must contain at least three elements; otherwise the charged-lepton mass basis would not be fully determined. Given a $z \in G_e$, $z \neq 1$, there exists a minimum $n \in \mathbb{N}$ such that $z^n = 1$. If $n \geq 3$, the result is proven (if $n = p \times q$ is not prime, one uses recursively that $\mathbb{Z}^{p\times q}$ contains both $\mathbb{Z}^p$ and $\mathbb{Z}^q$). If $n \leq 2$, then $z^2 = 1$. We call $w \neq 1$, z a third element of G. Again we must have $w^n = 1$ for a minimum $n \in \mathbb{N}$. If $n \geq 3$, the statement is proven. Otherwise, $w^2 = 1$, and G_e contains two $\mathbb{Z}_2$. Moroever, since w and z belong to an Abelian subgroup of $\mathrm{U}(1)_e \times \mathrm{U}(1)_\mu \times \mathrm{U}(1)_\tau$, w and z must commute, and so also the two $\mathbb{Z}_2$. Therefore, in the case $G \supseteq \mathbf{Z}_2 \times \mathbf{Z}_2$, and the statement is proven. Analogously one shows that G_ν must contain $\mathbf{Z}_2 \times \mathbf{Z}_2$.

$$\mathbf{Z}_n (n \geq 3 \text{ prime}) \text{ or } \mathbf{Z}_2^2 \subseteq G_e \subseteq \mathrm{U}(1)_e \times \mathrm{U}(1)_\mu \times \mathrm{U}(1)_\tau$$
$$\mathbf{Z}_2^2 \subseteq G_\nu \subseteq \mathbf{Z}_2^3 \quad \text{(nonzero neutrino masses)}. \tag{4.14}$$

Neutrino data are compatible with one vanishing neutrino mass. If one neutrino is massless, the constraint on G_ν becomes

$$\mathbf{Z}_n \times \mathbf{Z}_2 (n \geq 3 \text{ prime}) \subseteq G_\nu \subseteq \mathrm{U}(1) \times \mathbf{Z}_2^2 \quad \text{(one vanishing neutrino mass)}. \tag{4.15}$$

If one neutrino is massless, there is then more freedom in the choice of G_ν, which is otherwise constrained to be the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (up to a third, irrelevant $\mathbb{Z}_2$).[16]

A systematic analysis of the phenomenologically viable PMNS matrices that can be obtained in this context has been carried out under the assumption that all neutrinos are massive and that the group G is finite (Fonseca and Grimus, 2016). The only possible viable PMNS matrices are in a "trimaximal" form (TM_2; see Sec. IV.B.3), with $|U_{e2}|^2 = |U_{\mu 2}|^2 = |U_{\tau 2}|^2 = 1/3$, which predicts $\sin^2\theta_{12} \geq 1/3$. More precisely $[(|U|^2)_{ij} \equiv |U_{ij}|^2]$,

$$|U|^2 = \frac{1}{3}\begin{pmatrix} 1+\mathrm{Re}(\sigma) & 1 & 1-\mathrm{Re}(\#1)\sigma \\ 1+\mathrm{Re}(\omega\sigma) & 1 & 1-\mathrm{Re}(\omega\sigma) \\ 1+\mathrm{Re}(\omega^2\sigma) & 1 & 1-\mathrm{Re}(\omega^2\sigma) \end{pmatrix}, \tag{4.16}$$

where $\sigma = \exp(2i\pi p/n)$ is a root of unity and $\omega = \exp(2\pi i/3)$. The integers p and n can be taken as coprime, in which case the minimal discrete group leading to a PMNS matrix in the previously mentioned form is

- $\Delta(6m^2)$, where $3m$ is the least common multiple of 6 and n if 9 does not divide n;
- $(\mathbb{Z}_m \times \mathbb{Z}_{m/3}) \rtimes S_3$, where m is the least common multiple of 2 and n if 9 divides n.

The definition of these groups were given by Ishimori *et al.* (2010). Equation (4.16) determines the absolute values of the PMNS entries. The Majorana phases are not constrained, as discussed. The Dirac phase is instead fixed and predicted to be trivial ($\sin\delta = 0$) in all viable cases, which also predict a non-negligible deviation from maximal θ_{23}. For a given choice of σ (and hence of the group), Eq. (4.16) corresponds to one of the 36 possible permutations of rows and columns that can in principle arise.

One of the first attempts at achieving $\theta_{13} \neq 0$ directly from the interplay of G_ν and G_e used the $\Delta(96)$ group [$m = 4$, $n = 12$, $\sigma = \exp(i\pi/6)$] (de Adelhart Toorop, Feruglio, and Hagedorn, 2011; de Medeiros Varzielas and Ross, 2012; Ding, 2012; King, Luhn, and Stuart, 2013) but overshot the experimental value of θ_{13}. Experimentally viable possibilities were considered by Holthausen, Lim, and Lindner (2013), King, Neder, and Stuart (2013), Hagedorn, Meroni, and Vitale (2014), and Talbert (2014). The smallest viable $\Delta(6m^2)$ group corresponds to $m = 22$ ($n = 11, 22, 33, 66$) and is of the order of 2904, while the smallest viable $(\mathbb{Z}_m \times \mathbb{Z}_{m/3}) \rtimes S_3$ corresponds to $m = 18$ ($n = 9, 18$) and is of the order of 648. Such groups are more cumbersome than the ones originally proposed to account for the neutrino mixing pattern. Note that a dynamical mechanism to spontaneously break G and preserve an accurate sequestering also needs to be exhibited.

As mentioned, neutrino data are compatible with a single neutrino being massless. If that is the case, the rules of the game allow G_ν to be larger than the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the structure of the flavor group to be different. In all models studied thus far, non-negligible corrections to the leading order (exact sequestering) results are needed in order to obtain a phenomenologically viable model (Joshipura and Patel, 2013, 2014a; King and Ludl, 2016).

3. Approximate sequestering and rigid PMNS

In this section, we still assume that G_ν and G_e rigidly determine the lepton mass eigenvectors up to phases, and therefore also the PMNS matrix. However, we allow the PMNS matrix thus obtained to be only a leading order approximation of the measured one, and we rely on subleading corrections for an accurate agreement.

Before discussing their origin, we illustrate some possible leading order forms of the PMNS matrix and the size of the needed corrections. Before the measurement of θ_{13}, model-building efforts were based mainly on three forms of the PMNS matrix, all associated with simple discrete flavor symmetries. They all correspond to maximal θ_{23} and vanishing θ_{13} and differ only by the value of the solar angle θ_{12}.

Tri-bimaximal (TB): $\sin^2\theta_{12} = 1/3$, $\sin^2\theta_{23} = 1/2$, $\sin^2\theta_{13} = 0$ (Harrison, Perkins, and Scott, 2002; Harrison and Scott, 2002b, 2003).

Bimaximal (BM): $\sin^2\theta_{12} = 1/2$, $\sin^2\theta_{23} = 1/2$, $\sin^2\theta_{13} = 0$ (Barger *et al.*, 1998; Fukugita, Tanimoto, and Yanagida, 1998).

Golden ratio (GR): $\tan^2\theta_{12} = 1/\phi$ or $\cos\theta_{12} = \phi/2$, $\phi = (1+\sqrt{5})/2$ (golden ratio), $\sin^2\theta_{23} = 1/2$, $\sin^2\theta_{13} = 0$ (Datta, Ling, and Ramond, 2003; Kajiyama, Raidal, and Strumia, 2007; Rodejohann, 2009).

In all three cases, the PMNS matrix, up to external phases, is of the form

$$U = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{4.17}$$

with different values of θ_{12}, as previously specified.

We compare the predictions with the experimental values. The present 1σ ranges of the neutrino mixing angles as obtained from global fits (see Table I) are $\sin\theta_{12} = 0.56 \pm 0.01$, $\sin\theta_{23} = 0.75 \pm 0.02$, $\sin\theta_{13} = 0.150 \pm 0.002$, while the predictions obtained in the previously mentioned schemes are

[16] Another opportunity to enlarge G_ν arises with Dirac neutrinos (Esmaili and Smirnov, 2015).

$(\sin\theta_{12})_{\rm TB} = 0.58$, $(\sin\theta_{12})_{\rm BM} = 0.71$, $(\sin\theta_{12})_{\rm GR} = 0.59$ or 0.62, $(\sin\theta_{23})_{\rm all} = 0.71$, and $(\sin^2\theta_{13})_{\rm all} = 0$.

Most encouraging is the TB prediction for θ_{12}, which is in close agreement with the precise experimental determination. Parametrizing the corrections to Eq. (4.17) in a power series in $\lambda_C = 0.22$ (the Cabibbo angle, an expansion parameter borrowed from the quark sector), we see that the agreement is so precise that only corrections $\mathcal{O}(\lambda_C^{2\div 3})$ or less are allowed. This provided a considerable boost to models accounting for TB mixing, at a time when the θ_{13} angle was still unknown. The experiment now shows that θ_{13} departs from zero by $\mathcal{O}(\lambda_C)$. If that is the expected size of corrections to Eq. (4.17), the success of the TB prediction for θ_{12} should be considered accidental. Within the same $\mathcal{O}(\lambda_C)$ accuracy, the measured value of θ_{12} is compatible with the BM prediction $\theta_{12} = \pi/4$. It has in fact been observed that the empirical relation $\theta_{12} + \lambda_C \approx \pi/4$ ("quark-lepton complementarity") (Minakata and Smirnov, 2004; Raidal, 2004; Datta, Everett, and Ramond, 2005; Everett, 2006; Schmidt and Smirnov, 2006) approximately holds. The size of the corrections hinted at by the value of θ_{13} in this class of models partly jeopardizes the predictivity motivation.

We now focus on the TB scheme and illustrate the model-building logic underlying it. This also serves as an illustration of the ideas and techniques underlying more involved models. The tri-bimaximal form of the PMNS matrix is, up to external phases,

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.18}$$

The form of the PMNS matrix determines the relative orientation of G_e and G_ν in G, the commutation relations of the corresponding elements in G, and consequently the minimal structure of G. The procedure to find the minimal G (when it exists: only specific forms of the PMNS originate from finite groups) is simple. First, we need to specify G_ν and G_e. For G_ν the choice is essentially unique, as we assume here that all three neutrinos are massive: $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$. We call u and s the nontrivial elements of the two $\mathbb{Z}_2$. In a neutrino mass basis, their representation on the lepton doublets is

$$U_l^\nu = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S_l^\nu = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.19}$$

The choice of G_e is not unique; see Eq. (4.14). The smallest (in terms of number of elements) option is $\mathbb{Z}_3$. We call t one of its nontrivial elements. Without loss of generality, its representation on the l and e^c fields in a charged-lepton mass basis is

$$T_l^e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad T_{e^c}^e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{4.20}$$

where $\omega = \exp(2\pi i/3)$. Therefore, with the present choice of G_ν, G_e, the full group G must contain the identity, the three elements u, s, t, and all of their products. Under the assumption that the representation on the leptons is faithful, the group elements can be identified with their representations on the lepton doublets U_l, S_l, T_l. We need, however, to write them in the same basis. Choose, for example, a charged-lepton mass basis. Then T_l is given by Eq. (4.20), while U_l and S_l must be rotated from the neutrino basis used in Eq. (4.19). The rotation is given by the PMNS matrix U (beware of the abuse of the notation "U"): $U_l^e = U U_l^\nu U^\dagger$, $S_l^e = U S_l^\nu U^\dagger$. Here is where the chosen form of U enters. In the TB case, $U = \Psi U_{\rm TB} \Phi$, where Ψ and Φ are diagonal matrices of phases. With a proper choice of the phases of the charged leptons, $\Psi = \mathbf{1}$, while Φ cancels in the products so that $U_l^e = U_{\rm TB} U_l^\nu U_{\rm TB}^\dagger$, $S_l^e = U_{\rm TB} S_l^\nu U_{\rm TB}^\dagger$. All in all,

$$U_l^e = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_l^e = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix},$$
$$T_l^e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{4.21}$$

By taking all possible products of the three matrices in Eq. (4.21), it is easy to show that the group G generated by them is finite, contains 24 distinct elements, and is isomorphic to S_4, the permutation group of four elements.

The S_4 group has two $d = 3$, one $d = 2$, and two $d = 1$ irreducible representations, denoted by 3_1, 3_2, 2, 1_1, 1_2. The 3_1 representation is defined by S, T, U in Eq. (4.21) and the 3_2 has the opposite U. The 2 representation has

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.22}$$

and the 1_1, 1_2 representations have $S = T = 1$ and $U = \pm 1$, respectively.

There is motivation to pursue the S_4 option for TB mixing, and it has been widely studied (Mohapatra, Parida, and Rajasekaran, 2004; Cai and Yu, 2006; Hagedorn, Lindner, and Mohapatra, 2006; Ma, 2006; Zhang, 2007; Bazzocchi and Morisi, 2009; Bazzocchi, Merlo, and Morisi, 2009a, 2009b; Dutta, Mimura, and Mohapatra, 2009; Ishimori, Shimizu, and Tanimoto, 2009; Ding, 2010; Dutta, Mimura, and Mohapatra, 2010; Hagedorn, King, and Luhn, 2010; Meloni, 2010; Bhupal Dev, Mohapatra, and Severson, 2011; Ishimori and Kobayashi, 2011; Morisi, Patel, and Peinado, 2011; Bhupal Dev *et al.*, 2012; Smirnov and Xu, 2018); however, its simplest implementation requires a nontrivial fine-tuning to reproduce hierarchical charged leptons, as we now show.[17] To implement the S_4 symmetry, we first need to assign the lepton fields l_i and e_i^c to the S_4 representations. Equation (4.21) assigns the l_i fields to 3_1. The representation on the e_i^c fields should be such that $T^e_{e^c}$ is given by Eq. (4.20), which in turn requires them to form one of the following four representations: 3_1, 3_2, $2+1_1$, or $2+1_2$. The l and e^c fields must then couple to a combination of flavon fields, with a T-preserving VEV, in an S_4-invariant Yukawa interaction. All possible combinations lead to a diagonal charged-lepton mass matrix with at least two diagonal elements of equal size (and possibly different sign). To obtain hierarchical and nonvanishing charged-lepton masses, a fine-tuning of independent contributions to those diagonal entries must then be invoked. The argument is based on the assumption that T in not broken in the charged-lepton sector (so that its mass basis is rigidly determined) at leading order. The possibility that T is broken is considered in Sec. IV.B.4.

This fine-tuning can be avoided if S_4 arises accidentally in models based on A_4 (Ma and Rajasekaran, 2001; Babu, Ma, and Valle, 2003; Hirsch *et al.*, 2004; Ma, 2004a, 2004b; Altarelli and Feruglio, 2005; Chen, Frigerio, and Ma, 2005; Hirsch *et al.*, 2005; Ma, 2005a, 2005b; Zee, 2005; Adhikary *et al.*, 2006; Altarelli and Feruglio, 2006; He, Keum, and Volkas, 2006; Ma, Sawanaka, and Tanimoto, 2006; Altarelli, Feruglio, and Lin, 2007; Hirsch *et al.*, 2007; Lavoura and Kuhbock, 2007; Ma, 2007b; Morisi, Picariello, and Torrente-Lujan, 2007; Yin, 2007; Adhikary and Ghosal, 2008; Altarelli, Feruglio, and Hagedorn, 2008; Bazzocchi, Frigerio, and Morisi, 2008; Bazzocchi, Kaneko, and Morisi, 2008; Grimus and Kuhbock, 2008; Hirsch, Morisi, and Valle, 2008; Honda and Tanimoto, 2008; Altarelli and Meloni, 2009; Ciafaloni *et al.*, 2009; Lin, 2009a, 2009b; Morisi, 2009; del Aguila, Carmona, and Santiago, 2010; Kadosh and Pallante, 2010; Antusch, King, and Spinrath, 2011; BenTov, He, and Zee, 2012; Gupta, Joshipura, and Patel, 2012; Forero *et al.*, 2013; Holthausen, Lindner, and Schmidt, 2013b; Morisi *et al.*, 2013; Chu and Smirnov, 2016), its subgroup of even permutations. The last has 12 elements and is generated by S and T only. The flavor symmetry extends to S_4 if the Lagrangian (at some order in the flavon expansion) turns out to be accidentally invariant under the U generator. This option is appealing for a number of reasons: the A_4 group is even more minimal than S_4; the invariance of the Lagrangian under the U transformation is accidental, which allows corrections to TB mixing; and, as mentioned, no fine-tuning is required in order to obtain hierarchical, nonvanishing charged leptons. Both A_4 and S_4 can arise from continuous non-Abelian groups (Bazzocchi, Morisi, and Picariello, 2008; Bazzocchi *et al.*, 2009; Berger and Grossman, 2010; Grossman and Ng, 2015) and can be related to compactification in models with two extra dimensions (Altarelli, Feruglio, and Lin, 2007; Kobayashi, Omura, and Yoshioka, 2008) and to the modular group (Altarelli and Feruglio, 2006); see also Sec. VI.

We see how to implement these ideas in a concrete model based on A_4 (Altarelli and Feruglio, 2005, 2006). We first need to specify the A_4 representation on the lepton fields l_i and e_i^c. The A_4 group has one $d=3$ and three $d=1$ irreducible representations, denoted by 3, 1, 1′, 1″. The 3 representation is defined by S, T in Eq. (4.21) and the 1, 1′, 1″ representations are defined by $S=1$ and $T=1$, ω, ω^2, respectively. Equation (4.21) assigns the l_i fields to 3. The representation on the e_i^c fields should be such that $T^e_{e^c}$ is given by Eq. (4.20). Hence, either $e^c\sim 3$ or $e^c\sim 1+1'+1''$. The first option is not welcome, as it allows the charged lepton to get degenerate, nonvanishing, leading order masses. To avoid it, one chooses $e^c\sim 1+1'+1''$.

We now need to couple the leptons to flavons in such a way that G_e and G_ν are preserved at leading order by M_E and m_ν. In the A_4 case, G_ν is generated by S and G_e by T. $G_\nu=\mathbb{Z}_2$ alone is not sufficient to determine the neutrino mass basis up to phases, but it gets help from the U transformation, under which m_ν turns out to be accidentally invariant. To break G to G_e, S must be broken, but T must not. This can be achieved only by using a flavon triplet φ_T, as 1, 1′, 1″ are all invariant under S. The index T refers to the invariance under T, which forces $\langle\varphi_T\rangle=\epsilon_T(1,0,0)^T$. A_4 invariance allows φ_T to couple to $e^c l$ at the linear level. The most general charged-lepton Yukawa Lagrangian at leading order in the flavon expansion is then

$$\mathcal{L}_e^{(1)}=\lambda_1 e_1^c(\varphi_T l)_1 H^*+\lambda_2 e_2^c(\varphi_T l)_{1'}H^*+\lambda_3 e_3^c(\varphi_T l)_{1''}H^*, \tag{4.23}$$

where φ_T is dimensionless, i.e., normalized to some cutoff scale Λ, and $()_{1,1',1''}$ denote the triplet contractions transforming as $1,1',1''$ under A_4. More precisely, if a and b transform as 3, $(ab)_1=a_1b_1+a_2b_3+a_3b_2$, $(ab)_{1'}=a_3b_3+a_1b_2+a_2b_1$, $(ab)_{1''}=a_2b_2+a_1b_3+a_3b_1$ in the basis specified by Eq. (4.21).

Before switching to the neutrino sector, we comment on the previous result. The Lagrangian in Eq. (4.23) generates a diagonal charged-lepton mass matrix as desired, with $(M_E)_{ii}=\lambda_i\epsilon_T v$, where v is the Higgs VEV. The identification of the three families with the e, μ, τ mass eigenstates depends on the relative size of the diagonal entries and might require field permutations. The mass hierarchy can be reproduced, without fine-tuning, by an appropriate choice of the λ_i's, but it is not explained. To account for it and get rid of the permutation ambiguity, an additional $\mathrm{U}(1)_{\mathrm{FN}}$ factor can be added to the flavor group. The latter is assumed to broken by a flavon VEV $\langle\varphi_{\mathrm{FN}}\rangle=\epsilon\ll 1$. By a proper choice of their charge under $\mathrm{U}(1)_{\mathrm{FN}}$, the individual monomials in Eq. (4.23) can be forced to contain different powers of φ_{FN}. The corresponding diagonal masses then get suppressed by different powers of ϵ.

Alternatively, the role of φ_{FN} can be played by the A_4 flavons themselves (Altarelli and Meloni, 2009; Lin, 2009b).

[17]The fine-tuning is associated with the underlying μ-τ symmetry (Fukuyama and Nishiura, 1997; Mohapatra and Nussinov, 1999; Balaji, Grimus, and Schwetz, 2001; Lam, 2001; Ma and Raidal, 2001; Ma, 2002; Grimus and Lavoura, 2013; Xing and Zhao, 2016) corresponding to the U generator of S_4. While the need of fine-tuning in the context of the μ-τ symmetry was pointed out long ago (Kitabayashi and Yasue, 2003), the general argument provided here holds in S_4 independently of the viable choice of the lepton and flavon representations.

Suppose that $\langle\varphi_T\rangle = \epsilon_T(0,1,0)^T$. Such a VEV breaks T, and in fact the entire A_4 and S_4. Moreover, $\langle(\varphi_T^2)_3\rangle = \epsilon_T^2(0,0,1)^T$ and $\langle(\varphi_T^3)_3\rangle = \epsilon_T^3(1,0,0)^T$, where the superscript 3 denotes the component transforming as the 3 of A_4. Therefore, multiple insertions of $\langle\varphi_T\rangle$ are associated with different families and are increasingly suppressed by higher powers of ϵ_T: the A_4 flavon φ effectively plays the role of a FN flavon.

We now come to the neutrino mass matrix and consider for simplicity its description in terms of the Weinberg operator. We first note that A_4 allows an invariant term $(lHlH)_1/\Lambda_L$ corresponding to three degenerate neutrinos. This invariant term needs to be of a similar size as the symmetry breaking terms, which may be expected to be suppressed, if a perturbative flavon expansion is to be meaningful. The invariant term can be correspondingly suppressed by forcing it to break *ad hoc* symmetries. With this in mind, we allow the "invariant" and symmetry breaking contributions to the neutrino mass matrix to be comparable.

To break G down to G_ν, T must be broken, but S must not. T can in principle be broken by a 3, $1'$, or $1''$ representation. To have accidental invariance under U for generic values of the Lagrangian parameters, G should be broken by a flavon triplet φ_S, where the subscript S refers to the invariance under S, which forces $\langle\varphi_S\rangle = \epsilon_S(1,1,1)^T$. A_4 invariance allows φ_S to couple to the Weinberg operator at the linear level. The most general neutrino Lagrangian at the linear order in the flavon expansion is then

$$\mathcal{L}_\nu^{(1)} = \epsilon\frac{(lHlH)_1}{2\Lambda_L} + \varphi_S\frac{(lHlH)_{3S}}{2\Lambda_L}, \tag{4.24}$$

where, in the previously used notation, the symmetric contraction of the lepton indices into a triplet is $(ab)_{3S} = (2a_1b_1 - a_2b_3 - a_3b_2, 2a_2b_2 - a_3b_1 - a_1b_3, 2a_3b_3 - a_1b_2 - a_2b_1)$. The corresponding neutrino mass matrix is

$$m_\nu^{(1)} = \begin{pmatrix} a+2b & -b & -b \\ -b & 2b & a-b \\ -b & a-b & 2b \end{pmatrix},$$
$$a = \epsilon\frac{v^2}{\Lambda_L}, \qquad b = \epsilon_S\frac{v^2}{\Lambda_L}. \tag{4.25}$$

The matrix $m_\nu^{(1)}$ is accidentally invariant under U, as desired. Moreover, together with M_E, it leads to TB mixing. As $m_\nu^{(1)}$ is not the most general matrix invariant under S and U, the relation $m_3e^{i\alpha_{31}} = m_1 - 2m_2e^{i\alpha_{21}}$ holds among the neutrino masses and the Majorana phases α_{21}, α_{31} [defined as in Eq. (4.37)].[18] In the context of seesaw models, an analogous relation holds for the inverse masses.

[18]Which of the three eigenvalues $|3b+a|$, $|a|$, $|3b-a|$ are identified with m_1, m_2, m_3 in their standard ordering depends on their relative sizes. For the TB form of the PMNS matrix not to be spoiled by permutations of its columns, the identification should give $m_1 = |3b+a|$, $m_2 = |a|$, $m_3 = |3b-a|$. The relation among masses and phases is an example of mass sum rules (King, Merle, and Stuart, 2013; Gehrlein, Merle, and Spinrath, 2015).

We have seen that TB mixing can be obtained from the Lagrangian $\mathcal{L}_e^{(1)} + \mathcal{L}_\nu^{(1)}$. Crucial to this result is the fact that the Lagrangian is in the form in Eq. (4.11), with $\phi_e = \varphi_T$ entering only the charged-lepton mass matrix and $\phi_\nu = \varphi_S$ entering only the neutrino mass matrix. To enforce such a sequestering, φ_T and φ_S must be given different quantum numbers under an additional group factor. For example, one can add a $\mathbf{Z}_3$ factor, under which φ_T and e^cl are invariant, while φ_S and $lHlH$ transform nontrivially in conjugated representations. This way the Lagrangian is forced to be of the form $\mathcal{L}_e^{(1)} + \mathcal{L}_\nu^{(1)}$ at the leading order in the flavon expansion.

A complete model must also account for the specific alignment of the VEVs, $\varphi_T \propto (1,0,0)$, $\varphi_S \propto (1,1,1)$, assumed earlier. Indeed, the TB prediction crucially depends on such an alignment, more than from the flavor group itself or the choice of the flavon fields. In other words, what actually underlies the TB prediction is the flavon potential determining the flavon VEVs. It can be shown (Altarelli and Feruglio, 2006) that the needed alignment can be naturally obtained in supersymmetric models.

We have illustrated how TB mixing can be obtained from an A_4 flavor group (supplemented with additional symmetry factors and a proper flavon potential) at the leading order in a flavon expansion. Other finite groups aside from A_4 and S_4 can lead to TB mixing, such as $\mathrm{PSL}_2(7)$ (Luhn, Nasri, and Ramond, 2007a; King and Luhn, 2009a, 2010; Ferreira *et al.*, 2012; Chen, Prez, and Ramond, 2015), $\Delta(27)$ (Luhn, Nasri, and Ramond, 2007b; de Medeiros Varzielas, King, and Ross, 2007; Grimus and Lavoura, 2008; Ma, 2008; Björkeroth *et al.*, 2016, 2017), $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ (Luhn, Nasri, and Ramond, 2007c; Hagedorn, Schmidt, and Smirnov, 2009; Cao *et al.*, 2011; Vien and Long, 2014; Bonilla *et al.*, 2015; Cárcamo Hernández and Martinez, 2016), and $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$ (Hartmann and Zee, 2011; Kajiyama and Okada, 2011; Hartmann, 2012; Pérez *et al.*, 2019). Other mixing schemes can be obtained closely following the outlined model-building lines for S_4 and A_4. For example, BM mixing can be obtained from S_4 (Altarelli, Feruglio, and Merlo, 2009; Meloni, 2011) and GR schemes can be obtained from A_5 (Kajiyama, Raidal, and Strumia, 2007; Everett and Stuart, 2009; Feruglio and Paris, 2011; Hernandez and Smirnov, 2012; Gehrlein, Oppermann *et al.*, 2015; Gehrlein, Petcov *et al.*, 2015).

a. Origin of the corrections to approximate sequestering

The simplest non-Abelian finite group models lead to TB, BM, or GR forms of the PMNS matrix and therefore need to be corrected in order to account for $\theta_{13} \neq 0$. Such corrections are also needed in models based on higher order finite groups leading to a $\theta_{13} \neq 0$, but they are still not in agreement with the experimental value. The previous predictions were obtained at LO in the flavon expansion, at which the Lagrangian has the form of Eq. (4.11), supplemented by a flavon potential providing the necessary alignment of ϕ_e and ϕ_ν. The corrections are associated with higher order terms and can affect the LO predictions by either (i) spoiling the sequestering or (ii) spoiling the alignment mechanism provided by the leading order potential. Such corrections are usually G invariant, but they can also be noninvariant because

(i) part of G arises at LO as an accidental symmetry or (ii) the group G is not gauged and the corrections are of a gravitational nature. For the latter case to be phenomenologically relevant, the cutoff scale Λ characterizing the operator expansion should be sufficiently close to the gravity cutoff. While the form of the corrections is model dependent, a few model-independent considerations can be made.

b. Size of the corrections

The range of the corrections is important to assess whether they can lead to viable predictions and how much they spoil the predictability of the model.

The corrections to the LO predictions are associated with higher orders in the flavon expansion. There are two expansion parameters associated with the typical size of the VEVs of the ϕ_e and ϕ_ν flavons: $\epsilon_e \sim \langle \phi_e \rangle$, $\epsilon_\nu \sim \langle \phi_\nu \rangle$ (note that the flavons are dimensionless here, i.e., normalized to some cutoff scale Λ). We expect corrections to the neutrino and charged-lepton mass matrices to be at least as large as $\mathcal{O}(\epsilon_e^2)$ and $\mathcal{O}(\epsilon_\nu^2)$, respectively, as discussed in Sec. IV.B.1.

The ranges of ϵ_e, ϵ_ν are often loosely constrained at LO. We first focus on ϵ_e and consider the form of $\mathcal{L}_e^{(1)}$ in Eq. (4.23), where $\phi_e \equiv \varphi_T$ and $\epsilon_e \equiv \epsilon_T$. At LO, the tau lepton mass is given by $m_\tau = \lambda_\tau \epsilon_e v$, where λ_τ is the largest among the three couplings in Eq. (4.23). The product $\lambda_\tau \epsilon_e$ is fixed by the tau mass, but ϵ_e is allowed to vary in quite a broad range $10^{-2} \approx m_\tau / v \lesssim \epsilon_e \lesssim 1$. The upper bound is required for the perturbative expansion to be meaningful, and the lower bound corresponds to a coupling λ_τ in the perturbative regime $\lambda_\tau \lesssim 1$. The result still holds if the charged-lepton mass hierarchy is accounted for by an independent suppression factor $\epsilon_{\rm FN}$ associated with an Abelian U(1) factor. In the latter case, $m_{e_i} = \lambda_i \epsilon_{\rm FN}^{n_i} \epsilon_e v$, where n_i is an Abelian charge.

The size of ϵ_ν may be even less constrained. We consider the Lagrangian in Eq. (4.24), where $\phi_\nu \equiv \varphi_S$ and $\epsilon_\nu \equiv \epsilon_S$. Equation (4.25) shows that b is bound to be of the order of the light neutrino masses, but a small ϵ_ν is allowed provided that Λ_L and ϵ are correspondingly small. For $\Lambda_L \gtrsim$ TeV and normal hierarchy, one gets $10^{-12} \lesssim \epsilon_\nu \lesssim 1$.

The ranges for ϵ_e and ϵ_ν are broad enough to allow the next-to-leading order corrections to be negligible or substantial, in either M_e or m_ν, or in both, and in general to allow the expansion parameters and LO corrections to have different sizes in the charged-lepton and neutrino sectors. Such qualitative considerations can be refined or modified in a number of ways. For example, the mass matrices can be nonhomogeneous in ϵ_e, ϵ_ν. This is the case if the A_4 flavons play the role of FN flavons and $\epsilon_{\rm FN} = \epsilon_e$ (Altarelli and Meloni, 2009; Lin, 2009b). In such a case, the size of ϵ_e is determined by the charged-lepton mass ratios. Moreover, additional constraints on the expansion parameters can arise in models accounting for leptogenesis (Mohapatra and Nasri, 2005; Mohapatra, Nasri, and Yu, 2005; Jenkins and Manohar, 2008; Bertuzzo *et al.*, 2009; Branco *et al.*, 2009; Hagedorn, Molinaro, and Petcov, 2009; Lin, 2009a; Riva, 2010; Aristizabal Sierra, Bazzocchi, and de Medeiros Varzielas, 2012; Gehrlein, Petcov *et al.*, 2015).

c. Structure of the corrections

The PMNS matrix gets contributions from both the neutrino and charged-lepton sectors ($U = U_e^\dagger U_\nu$), as in Eq. (3.16). Corrections to the LO form of the PMNS matrix can be due to corrections to M_E (affecting U_e) and to m_ν (affecting U_ν). A special case arises when only one of the two corrections is significant.

First consider the case in which the corrections come from the charged-lepton sector (Giunti and Tanimoto, 2002a, 2002b; Antusch and King, 2004a; Altarelli, Feruglio, and Masina, 2004; Frampton, Petcov, and Rodejohann, 2004; Romanino, 2004; Antusch and King, 2005; King, 2005; Masina, 2006). This can happen if ϵ_e is on the lower side of its range so that the $\mathcal{O}(\epsilon_e^2)$ corrections to m_ν are negligible.

The charged-lepton mass matrix is diagonal at LO due to G_e invariance. Therefore, the leading order form of the PMNS matrix (TB, BM, GR, or otherwise) is $U^0 = U_\nu^0$, where U_ν^0 diagonalizes the LO form of m_ν. At higher orders, M_E is nondiagonal and m_ν is unaffected. Thus, the PMNS matrix gets a correction from the charged-lepton sector $U = U_e^\dagger U^0$.

This observation is not constraining *per se*: any PMNS matrix U can now be obtained by choosing an appropriate $U_e = U_\nu^0 U^\dagger$. The study of charged-lepton corrections is useful when U_e has a nongeneric, motivated pattern. This is indeed often the case, as U_e is in turn constrained by the hierarchy of charged-lepton masses, if the latter is to be stable under small perturbations (Marzocca and Romanino, 2014). If M_{31}^E is not unexpectedly large, $|M_{31}^E|/m_\tau \ll \sin\theta_{13}$, U_e is approximately of the form

$$U_e = R_{23}^T(\theta_{23}^e) R_{12}^T(\theta_{12}^e) \tag{4.26}$$

up to external phases, where $R_{ij}(\theta)$ is a 2×2 rotation by an angle θ in the ij block and the transpose is conventional. In all cases illustrated in Sec. IV.B.3, $\theta_{13} = 0$ in U_0; hence U_ν^0 is of the same form

$$U_\nu^0 = R_{23}(\theta_{23}^\nu) R_{12}(\theta_{12}^\nu) \tag{4.27}$$

up to external phases. The θ_{13} angle then originates purely from the interplay of 23 and 12 rotations, and the PMNS matrix is given by

$$U = R_{12}(\theta_{12}^e) \Phi R_{23}(\theta_{23}') R_{12}(\theta_{12}^\nu) \tag{4.28}$$

up to external phases, where $\Phi = \mathrm{diag}[1, \exp(-i\delta'), 1]$. In Eq. (4.28), θ_{12}^ν corresponds to the LO prediction for θ_{12} and is fixed by the model ($\sin\theta_{12}^\nu = 1/\sqrt{3}$ and $1/\sqrt{2}$ in the TB and BM schemes, respectively). The precise relations between the parametrization in Eq. (4.28) and the standard one was given by Marzocca *et al.* (2013). In a first approximation, $\theta_{23} = \theta_{23}'$ and $\delta = \delta'$ up to the $\mathcal{O}(s_{13}^2)$ and $\mathcal{O}(s_{13})$ corrections, respectively. Moreover, $\sin\theta_{13} = \sin\theta_{12}^e \sin\theta_{23}$ and $\sin\theta_{12} = \sin\theta_{12}^\nu(1 + \sin\theta_{12}^e \cot\theta_{12}^\nu \cos\theta_{23} \cos\delta)$ up to the $\mathcal{O}(s_{13}^2)$ and $\mathcal{O}(s_{13})$ corrections, respectively.

The relation $\sin\theta_{13} = \sin\theta_{12}^e \sin\theta_{23}$ allows one to determine the size of the charged-lepton angle θ_{12}, which turns out to be close to the Cabibbo angle, for $\theta_{23} = \pi/4$ (Minakata and

Smirnov, 2004; Raidal, 2004; Datta, Everett, and Ramond, 2005; Everett, 2006). Motivating such an empirical relation within GUT models, while at the same time accounting for the m_μ/m_s and m_e/m_d ratios, is not straightforward (Antusch and Maurer, 2011; King, 2012; Antusch *et al.*, 2013; Marzocca *et al.*, 2011). Both the deviation of θ_{13} from zero and the deviation of θ_{12} from the LO prediction θ_{12}^ν are determined by θ_{12}^e, and they are therefore expected to be of the same size. Indeed, one gets

$$\theta_{12} = \theta_{12}^\nu + \theta_{13}\cot\theta_{23}\cos\delta + \mathcal{O}(\theta_{13}^2). \tag{4.29}$$

Equation (4.29) is sometimes called the "solar sum rule." It allows one to predict the CP-violating phase δ for a given LO prediction θ_{12}^ν. A solution for δ can be found for TB, GR, and other LO θ_{12} predictions not too far from the measured values.

Actually, as the measured value of θ_{13} is not so small, θ_{12} is expected to deviate quite significantly from its LO prediction $\delta\sin\theta_{12}/\sin\theta_{12} \sim 0.15\cos\delta$. In this context, the success of the TB prediction, corresponding to $\delta\sin\theta_{12}/\sin\theta_{12} \lesssim 0.03$, looks somewhat accidental. Indeed, sizable CP violation, i.e., small $\cos\delta$, is predicted to be necessary in order to accommodate TB mixing in this context (Marzocca *et al.*, 2013). On the other hand, a measurement of a small $\cos\delta$ would restore the success of the θ_{12} prediction.

One can wonder whether the charged-lepton effect on θ_{12} is large enough to account for the observed significant deviation from the BM prediction $\theta_{12} = \pi/4$. The correction in Eq. (4.29) falls short of providing the necessary deviation (Ballett *et al.*, 2014; Girardi, Petcov, and Titov, 2015a, 2015b). Further corrections, pushing θ_{12} in the desired range, can be obtained if U_e is not in the form in Eq. (4.28). This can be the case if M_{31}^E is relatively large. A sizable M_{31}^E may, however, generate sizable contributions to the electron and muon masses that need fine-tuned cancellations, unless the charged-lepton mass matrix has special structures (Marzocca and Romanino, 2014). Such a sizable M_{31}^E can also be used within asymmetric textures to correct the TB prediction, while leading to a prediction for the CP-phase δ in agreement with the present hints (Rahat, Ramond, and Xu, 2018; Pérez *et al.*, 2019).

Sizable corrections to θ_{12} from the neutrino sector are more constrained if the neutrino masses are inverted hierarchically. In such a case, a maximal θ_{12} can be easily obtained from pseudo-Dirac structures in the neutrino mass matrix, which in turn naturally arise within both non-Abelian groups (as in the BM case; see Sec. IV.B.3) and Abelian groups [as in Eq. (4.2)]. In this context, the needed correction to θ_{12}, if arising in the neutrino sector, tends to destabilize the $|\Delta m_{12}^2/\Delta m_{23}^2| \ll 1$ hierarchy, thus leading to fine-tuning (Domcke and Romanino, 2016). To avoid that, the bulk of the corrections to $\theta_{12} = \pi/4$ should come from the charged-lepton sector.

4. Nonrigid determination of the PMNS matrix

The discussion in this section has been based thus far on the assumption that G_e and G_ν, the subgroups of G preserved by M_E and m_ν, are nontrivial, and they rigidly determine the charged-lepton and neutrino mass bases up to phases. Such an assumption allows one to unambiguously determine the PMNS matrix directly from G_e and G_ν. While such an approach is powerful and predictive, the assumption on which it relies is nontrivial. The subgroups G_e and G_ν can well be trivial, in which case they would not lead to the identification of any mass eigenstate. An intermediate possibility is that G_e and G_ν are nontrivial, but they identify the mass basis only partially. We review such a possibility here.[19] To realize it, sequestering is still needed, as G_ν and G_e still need to be different, with a trivial intersection.

The case in which G_ν does not fully determine the neutrino mass basis, while G_e does, has been widely considered (Ge, Dicus, and Repko, 2011, 2012; Hernandez and Smirnov, 2013a). In such a case, the only potentially interesting possibility is $G_\nu = \mathbb{Z}_2$. The residual symmetries now determine the PMNS matrix up to a 2×2 rotation and a phase (and Majorana phases and permutations, as before) $U = U_0 U_{ij}(\theta,\phi)$, where

$$U_{23}(\theta,\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta e^{-i\phi} \\ 0 & -\sin\theta e^{i\phi} & \cos\theta \end{pmatrix}, \qquad \begin{matrix} \theta \in [0,\pi/2], \\ \phi \in [0,2\pi]. \end{matrix} \tag{4.30}$$

Analogously, $U_{12}(\theta,\phi)$ and $U_{13}(\theta,\phi)$ have the 2×2 rotation embedded in the 12 and 13 blocks, respectively.

If G_ν is a subgroup of $\mathbb{Z}_2\times\mathbb{Z}_2$ rigidly determining the neutrino mass basis, as in Secs. IV.B.2 and IV.B.3, and U_0 is the PMNS matrix obtained when $\mathbb{Z}_2\times\mathbb{Z}_2$ is unbroken, the block on which the 2×2 rotation $U_{ij}(\theta,\phi)$ acts depends on which of the three $\mathbb{Z}_2$ subgroups of $\mathbb{Z}_2\times\mathbb{Z}_2$ survives. If U, S are defined as in Eq. (4.19), the three subgroups are generated by U, S, US. Correspondingly, the PMNS matrix is given by

$$U = U_0 U_{ij}(\theta,\phi), \qquad \text{where} \quad ij = \begin{cases} 12 & \text{if } \mathbb{Z}_2 \text{ is generated by } U, \\ 13 & \text{if } \mathbb{Z}_2 \text{ is generated by } S, \\ 23 & \text{if } \mathbb{Z}_2 \text{ is generated by } US. \end{cases} \tag{4.31}$$

Taking into account the diagonal Majorana phases Ψ and the possible permutations P_e, P_ν of the lepton mass bases, one obtains $U \to P_e U P_\nu \Psi$ in Eq. (4.31).

In practice, Eq. (4.31) means that it is possible to loosen the rigid predictions illustrated in Sec. IV.B.2 and IV.B.3 by breaking G to a $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_2\times\mathbb{Z}_2$ in the neutrino sector. Such a possibility is welcome in the models discussed in Sec. IV.B.3, where the θ_{13} prediction obtained in the rigid case needs to be corrected. The correction is provided by the U_{ij} rotation. For the rotation to affect θ_{13}, it should act in either the 13 or the 23 block. In the 13 case (S preserving), the second column of U_0 appears to be identical in U. In the 23 case (US preserving), the first column of U_0 appears to be identical in U.

We apply these ideas to models leading, in the rigid limit, to TB mixing. We focus on the simple option reviewed in

[19] Such models are sometimes called "semidirect."

TABLE VI. Predictions of the TM_1 and TM_2 (i.e., trimaximal) mixing patterns as a function of the parameters $\theta \in [0, \pi/2]$ and $\phi, \alpha, \beta \in [0, 2\pi]$.

Pattern	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	δ	α_{21}	α_{31}
TM_1	$\frac{1}{2}(1-\cos\phi\frac{2\sqrt{6}\sin 2\theta}{5+\cos 2\theta})$	$\frac{\sin^2\theta}{3}$	$\frac{\cos^2\theta}{2+\cos^2\theta}$	$\arg(2e^{-i\phi}-3e^{i\phi}\frac{\cos^2\theta}{\sin^2\theta})$	α	$\beta+2\delta$
TM_2	$\frac{1}{2}(1+\cos\phi\frac{\sqrt{3}\sin 2\theta}{2+\cos 2\theta})$	$\frac{2}{3}\sin^2\theta$	$\frac{1}{2+\cos 2\theta}$	$\arg(e^{-i\phi}-3e^{i\phi}\frac{\cos^2\theta}{\sin^2\theta})$	α	$\beta+2\delta$

Sec. IV.B.3, with $G = S_4$ arising accidentally at LO from an A_4-symmetric sequestered Lagrangian. Note that the accidental S_4 invariance arises because no flavon in the $1'$, $1''$ representations is used to break A_4, in which case G_e is generated by T, G_ν is generated by U and S in Eq. (4.21), and $U = U_{\mathrm{TB}}$ at LO (up to external phases and assuming that lepton masses are correctly ordered).

To reduce G_ν to $\mathbb{Z}_2$ and preserve S, it is then sufficient to introduce flavons φ', φ'' in $1'$, $1''$ representations of A_4 (King and Luhn, 2011; Ma and Wegman, 2011; Shimizu, Tanimoto, and Watanabe, 2011; Cooper, King, and Luhn, 2012). Since we still want T to be preserved by M_E, φ' and φ'' should be sequestered in the neutrino part of the LO Lagrangian. Another possibility is that the role of φ', φ'' is played by the $1'$, $1''$ components of φ_S^2. In such a case, the accidental symmetry breaking, i.e., the corrections to TB mixing, is suppressed by only one power of $\langle\varphi_S\rangle \sim \epsilon_S$ (Lin, 2010) [compare this with the case in which U is not accidental and the corrections are expected to be at $\mathcal{O}(\epsilon_T^2)$]. In both cases [breaking by φ', φ'' or by $(\varphi_S^2)_{1'}$, $(\varphi_S^2)_{1''}$], the Lagrangian is no longer accidentally invariant under U and G_ν is generated by S. The PMNS matrix is then in the form of Eq. (4.31), with $ij = 13$,

$$U_{\mathrm{TM}_2} = U_{\mathrm{TB}} U_{13}(\theta,\phi)\Psi = \begin{pmatrix} \sqrt{\frac{2}{3}}c_\theta & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}s_\theta e^{-i\phi} \\ -\frac{c_\theta}{\sqrt{6}}+\frac{s_\theta}{\sqrt{2}}e^{i\phi} & \frac{1}{\sqrt{3}} & -\frac{c_\theta}{\sqrt{2}}-\frac{s_\theta}{\sqrt{6}}e^{-i\phi} \\ -\frac{c_\theta}{\sqrt{6}}+\frac{s_\theta}{\sqrt{2}}e^{i\phi} & \frac{1}{\sqrt{3}} & \frac{c_\theta}{\sqrt{2}}-\frac{s_\theta}{\sqrt{6}}e^{-i\phi} \end{pmatrix}\Psi, \tag{4.32}$$

where we have now explicitly included the diagonal matrix of Majorana phases Ψ. This form of the PMNS matrix merits a few comments. A nonvanishing θ_{13} has been induced by the rotation θ. With θ as a free parameter, any value of $\sin\theta_{13} \le (2/3)^{1/2}$ can be obtained. The size of $\sin\theta_{13}$ is controlled by $\langle\varphi'\rangle$, $\langle\varphi''\rangle$, and its relative smallness can be accounted for in terms of a mild suppression of $\langle\varphi'\rangle$, $\langle\varphi''\rangle$ [or by the extra ϵ_S insertion if $\varphi' \sim (\varphi_S^2)_{1'}$, $\varphi'' \sim (\varphi_S^2)_{1''}$]. A CP-violating phase is also generated ($\delta \approx \phi$). The solar angle is larger than its TB prediction $\sin\theta_{12} \ge 1/\sqrt{3}$, but only by a $\mathcal{O}(\sin^2\theta_{13})$ amount. The maximal θ_{23} prediction is also modified, at the $\mathcal{O}(\sin\theta_{13})$ order. The precise expression of the PMNS parameters in terms of θ, ϕ is given in Table VI. With four parameters expressed in terms of two, Eqs. (4.32) lead to two predictions ("sum rules") (Grimus and Lavoura, 2008)

$$1 = 3\cos^2\theta_{13}\sin^2\theta_{12}, \qquad \cos\delta = \frac{\cos 2\theta_{13}\cot 2\theta_{23}}{\sin\theta_{13}\sqrt{2-3\sin^2\theta_{13}}}. \tag{4.33}$$

The first relation of Eq. (4.33) is in relatively good agreement with present data, with the central value of the rhs ≈ 0.91 and tension at the 2σ level. In the second relation of Eq. (4.33) the absence of CP violation ($\cos\delta = \pm 1$) would require θ_{23} to be significantly nonmaximal at the boundary of its 3σ range. As θ_{23} approaches $\pi/4$, δ approaches $\pm\pi/2$.

The second column of U_{TM_2} coincides with that of U_{TB} and corresponds to a neutrino $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$ with equal components in ν_e, ν_μ, ν_τ. Such a pattern is called trimaximal mixing (Haba, Watanabe, and Yoshioka, 2006; He and Zee, 2007, 2011; Grimus and Lavoura, 2008; Albright and Rodejohann, 2009; Albright, Dueck, and Rodejohann, 2010; Ishimori *et al.*, 2011). We adhere to a common convention by denoting the form of the PMNS matrix in Eq. (4.32) as "TM_2," in order to distinguish it from the form "TM_1" obtained by combining U_{TB} with a U_{23} rotation. The subscripts 1 and 2 refer to the U_{TB} column unaffected by the rotation. Strictly speaking, only when the second column is unchanged (TM_2) do we actually have trimaximal mixing.

The form TM_1 of the PMNS matrix is obtained from rigid TB models when the residual $\mathbb{Z}_2$ is generated by US (de Medeiros Varzielas and Lavoura, 2013; Grimus, 2013; Luhn, 2013). As US is not part of A_4, such a possibility requires larger flavor groups. The S_4 group is viable from this point of view. The PMNS matrix is of the form of Eq. (4.31), with $ij = 23$,

$$U_{\mathrm{TM}_1} = U_{\mathrm{TB}} U_{23}(\theta,\phi)\Psi = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{c_\theta}{\sqrt{3}} & \frac{s_\theta}{\sqrt{3}}e^{-i\phi} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}}+\frac{s_\theta}{\sqrt{2}}e^{i\phi} & -\frac{c_\theta}{\sqrt{2}}+\frac{s_\theta}{\sqrt{3}}e^{-i\phi} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}}-\frac{s_\theta}{\sqrt{2}}e^{i\phi} & \frac{c_\theta}{\sqrt{2}}+\frac{s_\theta}{\sqrt{3}}e^{-i\phi} \end{pmatrix}\Psi. \tag{4.34}$$

The solar angle is smaller than the successful TB prediction this time ($\sin\theta_{12} \le 1/\sqrt{3}$), but only by a $\mathcal{O}(\sin^2\theta_{13})$ amount. The first column of U_{TM_1} coincides with that of U_{TB}. The expression of the PMNS parameters in terms of θ, ϕ is given in Table VI and leads to two predictions (sum rules) (Albright and Rodejohann, 2009)

$$2 = 3\cos^2\theta_{13}\cos^2\theta_{12}, \qquad \cos\delta = -\frac{(1-5\sin^2\theta_{13})\cot 2\theta_{23}}{2\sqrt{2}\sin\theta_{13}\sqrt{1-3\sin^2\theta_{13}}}. \tag{4.35}$$

The first relation of Eq. (4.35) is in good agreement with the present data, well within 1σ, with the central value of the rhs being ≈ 2.0. The second relation of Eq.(4.35) shows that CP invariance ($\cos\delta = \pm 1$) is not compatible with the present 3σ range for θ_{23}. As θ_{23} approaches $\pi/4$, δ approaches $\pm\pi/2$.

Up to external phases, U_{TM_1} (U_{TM_2}) is the most general unitary matrix with the first (second) column, as in U_{TB}.

As discussed, the Majorana phases in Ψ are unconstrained in this setup. On the other hand, we see in Sec. V that flavor symmetries not commuting with the Poincaré group may constrain them. A general parametrization of Ψ that is useful in Sec. V is

$$\Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha/2} & 0 \\ 0 & 0 & e^{i(\beta/2+\phi)} \end{pmatrix}. \tag{4.36}$$

Note that the Majorana phases are sometimes defined through the following parametrization of the PMNS matrix

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}s^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}s^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}. \tag{4.37}$$

The relation between their parametrizations in terms of α, β and in terms of α_{21}, α_{31} is shown in Table VI.

We have illustrated the possibility of loosening the predictions of rigid models by reducing G_ν in such a way that the neutrino mass basis is only partially determined by G_ν. Analogously, one can consider the possibility that the charged-lepton mass basis is only partially determined by G_e. In such a case, G_e does not necessarily need to contain $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ [cf. Eq. (4.14)]; it is sufficient that it contains $\mathbb{Z}_2$. The possibility that $G_e = 1$ may also be viable. While in the latter case G_e would not constrain the charged-lepton mass matrix at all, a hierarchical structure may be enforced by an additional group factor, playing the role of a FN symmetry, or by organizing the couplings of the flavons in a specific way.

If G_e is loosened, a rigid prediction U_0 is modified by a unitary transformation appearing on the left side of U_0, mixing its rows. We are thus in the presence of charged-lepton corrections to the PMNS matrix, as in Sec. IV.B.3. A contribution to θ_{13} can again be induced. If the charged leptons end up having a hierarchical structure, as they should, such corrections are typically too small to fully account for θ_{13}. Note that the charged-lepton mass hierarchy can now be achieved in S_4 without fine-tuning since the T generator can be broken; see the discussion in Sec. IV.B.3. Even if the charged-lepton contribution to θ_{13} is subleading, it can still be useful when U_0 corresponds to a nonvanishing θ_{13} not too far from its experimental range. The small corrections from the charged-lepton sector can then be sufficient to bring θ_{13} into the experimental range. An example is $\Delta(96)$ (de Adelhart Toorop, Feruglio, and Hagedorn, 2011; Ding, 2012; King, Luhn, and Stuart, 2013). The PMNS matrix is in the latter case in the so-called bi-trimaximal form, a special case of TM_2 mixing corresponding to $\sqrt{2/3}\sin\theta = (1 - 1/\sqrt{3})/2$. Another possibility is $G = \mathrm{PSL}(2,7)$ (Hernandez and Smirnov, 2013a), in which a good fit of the mixing angles can be obtained for near-maximal CP violation, $\delta \sim \pi/2$, or $\delta \sim 3\pi/2$.

5. Extension to quarks

The approach followed thus far aims at understanding lepton flavor observables. On the other hand, a complete theory of flavor should account for the quark sector as well. The extension of the ideas discussed in this section to the quark sector is not straightforward.

One of the main features of the lepton models considered is that all the charged-lepton masses vanish in the symmetric limit, because a $d = 3$ irreducible representation is used for the lepton doublets. Such a setting is not suitable for the up quark sector, which is characterized by a top Yukawa coupling $\lambda_t = \mathcal{O}(1)$. The size of λ_t suggests that the latter is invariant, at least under the flavor group G considered in the lepton sector.[20] Hence, the up quark mass matrix does not vanish in the G-symmetric limit. An invariant λ_t requires both the third family quark doublet q_3 and up quark singlet t^c to be in conjugated $d = 1$ representations of the entire G. This requirement naturally leads to models in which both of the lighter Yukawa couplings are forced to be small because they are not invariant, in contrast to models based on sequestering that do not constrain the values of the Yukawa couplings *per se*.

The different strategies needed in the quark and lepton sectors are not necessarily in conflict. Quarks and leptons can be constrained by different, independent factors of the flavor group, broken by two independent sets of flavons, effectively leading to separate models in the two sectors. It is worthy to combine those strategies, however. As mentioned in Sec. IV.B.3, a FN-type continuous symmetry suppressing light Yukawa couplings can operate in the charged-lepton sector, in combination with a discrete one. Moreover, the two strategies can be combined even more effectively within the discrete groups setup by using discrete groups such as the double tetrahedral group T' (Frampton and Kephart, 1995). As a subgroup of SU(2) with doublet representations, T' contains the necessary ingredients to account for the $(2+1)$ quark structure along the lines of U(2) models (Aranda, Carone, and Lebed, 2000a, 2000b). On the other hand, as T' contains the representations of A_4, it also contains the ingredients necessary to reproduce the lepton observables along the lines of A_4 models (Aranda, 2007; Chen and Mahanthappa, 2007; Feruglio *et al.*, 2007; Frampton and Kephart, 2007; Ding, 2008; Frampton and Matsuzaki, 2009; Aranda *et al.*, 2010; Everett and Stuart, 2011; Carone, Chaurasia, and Vasquez,

[20] A large $\lambda_t = \mathcal{O}(1)$ might arise from the breaking of a larger group $\tilde{G}$. In such a case, the corresponding flavon VEV needs to be close to the cutoff scale, $\langle\phi_t\rangle \sim \Lambda$, and G should be identified with a subgroup of $\tilde{G}$ unbroken by $\langle\phi_t\rangle$. What follows still holds if considered in reference to G.

2017; Carone and Merchand, 2019). In the previous example, the quark mixing is correlated to the quark mass hierarchy. One can wonder whether the same residual subgroup techniques introduced to predict the lepton mixing matrix could be extended to the quark sector. This is possible but not straightforward. The residual subgroups should determine the relative orientations of the up and down quark mass bases. The small quark mixing angles then require a flavor group large enough to contain closely aligned subgroups (Lam, 2007; Blum, Hagedorn, and Lindner, 2008; Araki *et al.*, 2013; Holthausen and Lim, 2013; Yao and Ding, 2015; de Medeiros Varzielas, Rasmussen, and Talbert, 2017; Li, Lu, and Ding, 2018).

Once the flavor symmetry is extended to the quark sector, one can aim at a model compatible with gauge unification. In such a case, the flavor structures of the quark and lepton sectors are necessarily coupled, in a way dictated by the unified group. In grand unified theories such as SU(5), one family of SM fermions is unified into a $5_i + \overline{10}_i$ of SU(5): $5_i \sim (l_i, d_i^c)$, $\overline{10}_i \sim (e_i^c, q_i, u_i^c)$. As the flavor group is assumed here to commute with the gauge group, the flavor quantum numbers of SM fields belonging to the same irreducible SU(5) representation should be the same. Since q_3 and t^c both belong to $\overline{10}_3$, q_3 and t^c should be in a real $d = 1$ representation of G; i.e., they should be invariant up to a sign change. As τ^c is also unified with q_3 and t^c, it should also be in a real $d = 1$ representation. This is not compatible with the A_4 and S_4 settings in the form illustrated in Sec. IV.B.3, which require τ^c to belong to a complex representation. A nonstandard A_4 realization can be achieved with one extra dimension (Altarelli, Feruglio, and Hagedorn, 2008). Unified flavor models were reviewed by King (2017).

6. Outlook

The model-building avenues explored in this section are based on the interplay of two distinct subgroups G_ν and G_e of G. The group-theoretical construction and the structure of G crucially depend on the choice of G_ν and G_e and of their relative orientation. We consider both cases in which the subgroups fully or partially determine the flavor directions corresponding to the lepton mass eigenstates. In all cases, though, m_ν and M_E are by definition invariant under G_ν and G_e.

The model-building options are far from exhausted, even within finite non-Abelian group models. For example, there is no reason why G_ν and G_e should be nontrivial and fully, or partially, determine the lepton mass bases. Another nontrivial and nonessential assumption has to do with the forms of m_ν and M_E, the matrices invariant under G_ν and G_e. The constraints on G_ν and G_e in Eqs. (4.14) and (4.15) assume that m_ν and M_E provide nonvanishing, nondegenerate masses for all the leptons, with the only possible exception of the lightest neutrino. This is not really necessary. In early models, m_ν and M_E could be identified with the exact mass matrices (the PMNS matrix was still compatible with being exactly in TB form). On the other hand, this is not in line either with the generic theoretical expectation of higher order corrections to sequestering or with the experimental determination of the mixing parameters (except in the cases discussed in Sec. IV.B.2). Thus, the matrices m_ν and M_E allowed by G_ν and G_e should not be identified with the exact mass matrices in this context. They are approximations, expected to be corrected by higher order effects, in some cases as large as $\mathcal{O}(\lambda_C)$. The mass eigenvalues are then also expected to be corrected, as the mixing angles are, and there is no reason to demand that m_ν and M_E provide nonvanishing, nondegenerate masses for all leptons. In fact, they could correspond equally well with $m_e = 0$ or $m_e = m_\mu = 0$ or $m_{\nu_1} = m_{\nu_2}$. The corrections to such patterns necessary to obtain viable lepton masses are smaller than those commonly assumed to affect the mixing angles. Such a possibility was considered in connection with partially degenerate neutrinos (Hernandez and Smirnov, 2013b; Joshipura and Patel, 2014b). In principle, any mass pattern that can be considered sufficiently close to the observed one could be considered as well, in the spirit of the discussion in Sec. IV.A.

This shows that the program based on linear, Lorentz-scalar, discrete non-Abelian flavor groups has not been fully explored. Still, it is fair to say that such a program has only partially fulfilled initial expectations. The approach focuses on mixing angles. The predictability potential of the simplest models, one of their stronger motivations, has been frustrated by the experimental determination of the θ_{13} angle that once again challenged theoretical prejudice. Two opposite strategies can be pursued to accommodate the value of θ_{13}, both leading to a certain loss of predictability. On the one hand, one can stick to relatively simple models, at the price of accepting relatively large corrections, which reduce predictability. On the other hand, one can aim at more involved models with predictions close to the experiment, at the price of scanning a dense landscape of models. The significance of the prediction is then reduced by the correspondingly dense number of alternatives available. On the model-building side, the predictions are associated not really with the flavor group but rather to the symmetry breaking effects (ultimately to the detailed structure of the potential determining the VEV alignments), and to a set of auxiliary symmetries and quantum numbers needed to arrange the proper set of couplings in the Lagrangian. On the other hand, the theoretical landscape is still broad, as argued, and we hope that its exploration will provide new insights.

V. *CP*-LIKE FLAVOR SYMMETRIES

The main purpose of including CP transformations in the flavor symmetry group is to constrain Majorana phases. Dedicated reviews on this topic were given by King *et al.* (2014), King (2015, 2017), Neder (2015), Hagedorn (2017), Coloma and Pascoli (2018), and Petcov (2018). In a theory invariant under both a flavor symmetry group G_f and CP, besides Eq. (4.1), the following constraint holds for the lepton mass matrices[21]

[21]In the presence of a single Higgs, a possible phase in its CP transformation can be reabsorbed in the transformation of the lepton fields.

$$
\begin{aligned}
M_E^*(\phi) &= X_{e^c}^T M_E(X_\phi \phi^*) X_l, \\
m_\nu^*(\phi) &= X_l^T m_\nu(X_\phi \phi^*) X_l,
\end{aligned} \tag{5.1}
$$

where we denote with X_f $(f = e^c, l, \phi)$ unitary matrices describing the action of CP on the field f and we assume Majorana neutrinos. In such a theory CP can be broken only spontaneously and the conditions that realize the breaking are as follows:

(i) $X_\phi \phi^* \neq \phi$ on the vacuum.
(ii) No other consistent CP transformation leaving both the theory and the vacuum invariant exists.

A. Sequestering and residual symmetries

As in the case of a flavor symmetry commuting with the proper Poincaré group, to some extent it is possible to analyze the predictions of the theory without referring to an explicit realization, relying on the residual symmetries associated with the charged-lepton sector and the neutrino sector (if any). Provided that M_E and m_ν depend on two separate sets of flavons ϕ_E and ϕ_ν, we can contemplate independent residual symmetries for the two sectors

$$
U_\phi(g_E)\phi_E = \phi_E, \qquad U_\phi(g_\nu)\phi_\nu = \phi_\nu, \tag{5.2}
$$

where g_E and g_ν run in different subgroups of G_f and these relations hold in the vacuum. To constrain Majorana phases we should further assume that CP is conserved in the neutrino sector

$$
X_\phi \phi_\nu^* = \phi_\nu. \tag{5.3}
$$

By combining Eqs. (4.1) and (5.1), we end up with the relations

$$
\begin{aligned}
(M_E^\dagger M_E) &= U_l(g_E)^\dagger (M_E^\dagger M_E) U_l(g_E), \\
m_\nu &= U_l(g_\nu)^T m_\nu U_l(g_\nu), \qquad m_\nu^* = X_l^T m_\nu X_l,
\end{aligned} \tag{5.4}
$$

which constrain the lepton mixing angles and both Dirac and Majorana phases at the same time.

This program has been carried out in the context of discrete flavor symmetry groups G_f. A variety of cases arises from the different possible assignments of the residual symmetries. Assuming three generations, in the neutrino sector the most general group leaving neutrino masses unconstrained is the Klein group $Z_2 \times Z_2$. To avoid mass degeneracies,[22] the matrix X_l is required to be symmetric ($X_l^T = X_l$) (Feruglio, Hagedorn, and Ziegler, 2013). Since X_l is also unitary, this automatically implies $CP^2 = 1$.[23] To guarantee that the action of CP on lepton electroweak doublets is always represented by a symmetric matrix, X_l is required to commute with the four elements of the Klein group. Given the antilinear action of CP, commutation is expressed through relations of the type

$$
X_l U(g_K)^* = U(g_K) X_l, \tag{5.5}
$$

where g_K stands for an element of the Klein group. Indeed, $X_l' = U(g_K) X_l$ represents another CP transformation and Eq. (5.5) implies that the matrix X_l' is symmetric. It also follows that four CP transformations can be selected as residual symmetries of the neutrino sector. Conversely, given these four allowed CP transformations, the Klein group can be fully reconstructed (Chen, Li, and Ding, 2015; Everett and Stuart, 2017). Usually the group G_e consists of a direct product of cyclic symmetries $Z_{m_1} \times \cdots \times Z_{m_p}$, such that all the charged leptons are distinguished by their different transformation properties. Among the residual symmetries of the charged-lepton sector there can also be an accidental CP symmetry that is independent of the one acting in the neutrino sector.

B. Parameter counting

The freedom in the definition of the model gives rise to many cases and, depending on the specific set of assumptions, the PMNS matrix is determined up to a number of continuous free parameters (listed in Table VII). These parameters arise as follows. The invariance under CP provides, in a suitable basis, a reality condition on the neutrino mass matrix, which can be parametrized in terms of three masses and three angles. An additional $Z_2 \times Z_2$ symmetry fully determines these angles, while a single parity Z_2 leaves one angle unconstrained. The three angles remain free parameters if the only residual symmetry of the neutrino sector is CP. In the charged-lepton sector the choice $G_e = Z_{m_1} \times \cdots \times Z_{m_p}$, when all leptons have different transformation properties, leaves no free parameters beyond masses. One free angle originates from $G_e = Z_2 \times CP$ and one angle and one phase from $G_e = Z_2$. Adding the parameters of the two sectors reproduces Table VII. This approach leaves lepton masses unconstrained, and the PMNS matrix is always determined up to permutations of rows and columns. Moreover, the intrinsic parity of

TABLE VII. Number of continuous free parameters describing the lepton mixing matrix $U(\theta_{ij}, \delta, \alpha_{21}, \alpha_{31})$ (Chen, Li, and Ding, 2015; Lu and Ding, 2017). The first and second columns list the residual symmetries G_e and G_ν of the charged-lepton sector and the neutrino sector, respectively. The cyclic symmetry $Z_{m_1} \times \cdots \times Z_{m_p}$ is assumed to distinguish the charged leptons by their different transformation properties. The residual symmetry $Z_2 \times Z_2 \times CP$ is also equivalent to the one generated by the four allowed CP transformations of the neutrino sector; see the text. On the fourth line, CP and CP' are in general independent CP transformations.

G_e	G_ν	Parameters
$Z_{m_1} \times \cdots \times Z_{m_p}$	$Z_2 \times Z_2 \times CP$	0
$Z_{m_1} \times \cdots \times Z_{m_p}$	$Z_2 \times CP$	1
$Z_2 \times CP$	$Z_2 \times Z_2 \times CP$	1
$Z_2 \times CP'$	$Z_2 \times CP$	2
$Z_{m_1} \times \cdots \times Z_{m_p}$	CP	3
Z_2	$Z_2 \times CP$	3

[22] Degeneracies in the neutrino mass spectrum in this context were analyzed by Joshipura and Patel (2018).

[23] In Sec. III.E we saw that $(X_l^* X_l)^n = 1$ holds for a finite group.

TABLE VIII. Specific mixing pattern in four of the five independent cases arising from S_4 and CP invariance, broken down to Z_3 in the charged-lepton sector and to $Z_2 \times CP$ in the neutrino sector $G = S_4 \rtimes CP$ (Feruglio, Hagedorn, and Ziegler, 2013) as a function of the parameters $\theta \in [0, \pi/2]$.

Model	Pattern	$\lvert\sin\phi\rvert$	$\sin\alpha$	$\sin\beta$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\lvert\sin\delta\rvert$	$\sin\alpha_{21}$	$\sin\alpha_{31}$
I	TM_2	1	0	0	$\frac{1}{2}$	$\frac{2}{3}\sin^2\theta$	$\frac{1}{2+\cos 2\theta}$	1	0	0
II	TM_2	0	0	0	$\frac{1}{2}\left(1 \pm \frac{\sqrt{3}\sin 2\theta}{2+\cos 2\theta}\right)$	$\frac{2}{3}\sin^2\theta$	$\frac{1}{2+\cos 2\theta}$	0	0	0
IV	TM_1	1	0	0	$\frac{1}{2}$	$\frac{\sin^2\theta}{3}$	$\frac{\cos^2\theta}{2+\cos^2\theta}$	1	0	0
V	TM_1	0	0	0	$\frac{1}{2}\left(1 \mp \frac{2\sqrt{6}\sin 2\theta}{5+\cos 2\theta}\right)$	$\frac{\sin^2\theta}{3}$	$\frac{\cos^2\theta}{2+\cos^2\theta}$	0	0	0

neutrinos, that is, the relative sign of their masses, cannot be established. As a result the physical phases are fixed modulo π.

C. Examples

1. $\mu-\tau$ reflection symmetry

A simple example is provided by the so-called $\mu-\tau$ reflection symmetry (Harrison and Scott, 2002a, 2002b; Grimus and Lavoura, 2004; Harrison and Scott, 2004).[24] In the basis where the charged-lepton mass matrix is diagonal and ordered from smaller to bigger masses, the CP transformation acting on neutrinos is specified by

$$X_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.6}$$

and the constraint $m_\nu^* = X_l^T m_\nu X_l$ implies the relations $\sin\theta_{23} = 1/\sqrt{2}$, $\sin\theta_{12}\cos\theta_{12}\sin\theta_{13}\cos\delta = 0$, and $\sin\alpha_{21} = \sin\alpha_{31} = 0$. Data require $\sin\theta_{12}\cos\theta_{12}\sin\theta_{13} \neq 0$ and this scheme predicts a maximal Dirac CP phase, $\lvert\sin\delta\rvert = 1$.

2. $G = S_4 \rtimes CP$

If we assume $G_e = Z_{m_1} \times \cdots \times Z_{m_p}$ and $G_\nu = Z_2 \times CP$ in the neutrino sector (see the second row of Table VII), the PMNS matrix depends on a continuous parameter. An example is provided by $G = S_4 \rtimes CP$, $G_e = Z_3$ (Feruglio, Hagedorn, and Ziegler, 2013). Because of the different embedding of the Z_2 subgroup in S_4, there are five inequivalent choices of $Z_2 \times CP$ transformations that leave the neutrino sector invariant. Four of them, labeled I, II, IV, and V, reproduce particular cases of the so-called trimaximal mixing pattern.

Models I and II reproduce U_{TM_2}, while models IV and V give rise to U_{TM_1}, with ϕ, α, and β (see Sec. IV.B.4) quantized and while assuming only the values shown in Table VIII. Models I and IV predict a maximal atmospheric mixing angle, maximal Dirac CP violation, and trivial CP Majorana phases and provide two realizations of the $\mu-\tau$ reflection symmetry having an additional prediction. Models II and V predict no lepton CP violation of the Dirac or Majorana type. Equations (4.33) apply to model I (II) with $\lvert\cos\delta\rvert = 0$ (1). A general property of TM_2 is $\sin^2\theta_{12} > 1/3$. By letting $\sin^2\theta_{13}$ vary in its 3σ allowed range, the first relation in Eqs. (4.33) predicts that $\sin^2\theta_{12} = 0.340 - 0.342$, which is presently allowed within 3σ but out of the 2σ range. In model II, $\tan^2\delta = 0$ and the 3σ allowed range of $\sin^2\theta_{13}$ results in $\sin^2\theta_{23} = \{0.388-0.398\} \cup \{0.602-0.611\}$. The prediction falling in the first octant is excluded at 3σ, whereas the one falling in the second octant is allowed at 2σ. A vanishing $\sin\delta$ is disfavored by the current data, but it is not excluded at 3σ. Equations (4.35) for model IV (V) require that $\lvert\cos\delta\rvert = 0$ (1). In TM_1 we always have $\sin^2\theta_{12} < 1/3$. By letting $\sin^2\theta_{13}$ vary in its 3σ allowed range, the first relation of Eqs. (4.35) predicts that $\sin^2\theta_{12} = 0.316 - 0.319$, which is in good agreement with the present data. Model V is ruled out since the second relation of Eqs. (4.35) with $\tan^2\delta = 0$ leads to values of $\sin^2\theta_{23}$ that are excluded by the data. The quoted ranges have been derived from the results of the global fit given by Esteban *et al.* (2019). The group $G = A_4 \rtimes CP$ leads to the TM_2 mixing pattern shown in Table VI, with $\lvert\sin\phi\rvert = 0$ or $\lvert\sin\phi\rvert = 1$ and $\sin\alpha = \sin\beta = 0$ employed as in models I and II (Ding, King, and Stuart, 2013; Feruglio, Hagedorn, and Ziegler, 2013; Li, Lu, and Ding, 2016; Nishi, 2016). Explicit models were constructed for this case by Ding, King, and Stuart (2013) and Li, Lu, and Ding (2016). Starting from $G = S_4 \rtimes CP$, the models given by Ding *et al.* (2013) and Feruglio, Hagedorn, and Ziegler (2014) reproduce a nearly TM_2 mixing pattern, while those given by Li and Ding (2014) come close to the TM_1 scheme. Other examples of models within $G = S_4 \rtimes CP$ were given by Luhn (2013), Li and Ding (2015a), Penedo, Petcov, and Titov (2017), and Ding, King, and Li (2019).

3. $\Delta(3n^2)$ and $\Delta(6n^2)$

The groups A_4 and S_4 are particular cases of the series $\Delta(3n^2)$ and $\Delta(6n^2)$, respectively, realized with $n = 2$. General results for the entire series were given by Ding, King, and Neder (2014), Hagedorn, Meroni, and Molinaro (2015), Ding and King (2016), de Medeiros Varzielas *et al.* (2017), and Joshipura (2018). For $G = \Delta(3n^2) \rtimes CP$ broken into $G_e = Z_3$ and $G_\nu = Z_2 \times CP$ the mixing pattern is still of TM_2 type and depends on a continuous parameter. When $G = \Delta(6n^2) \rtimes CP$ is broken into $G_e = Z_3$ and $G_\nu = Z_2 \times CP$, more complex mixing patterns arise beyond the trimaximal one. The particular cases $G = \Delta(48) \rtimes CP$ and $G = \Delta(96) \rtimes CP$ were also comprehensively studied by Ding and Zhou

[24]See Zhou (2014), Mohapatra and Nishi (2015), Joshipura and Patel (2015), Rodejohann and Xu (2017), Zhao (2017), Nishi, Sánchez-Vega, and Souza Silva (2018), and Sinha, Roy, and Ghosal (2019) for more recent applications related to the topic of this section.

TABLE IX. Results for lepton mixing parameters from $G_f = \Delta(384)$, $m = 4$, and several CP transformations $X(s)$ (Hagedorn, Meroni, and Molinaro, 2015). The continuous parameter θ has been optimized to reproduce $\sin^2\theta_{13}$.

s	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin\delta$	$\lvert\sin\alpha\rvert = \lvert\sin\beta\rvert$
$s = 1$	0.0220	0.318	0.579	0.936	$1/\sqrt{2}$
	0.0220	0.318	0.421	−0.936	$1/\sqrt{2}$
$s = 2$	0.0216	0.319	0.645	−0.739	1
$s = 4$	0.0220	0.318	0.5	∓ 1	0

(2014, 2015) and Ding and King (2014), respectively. As an example of an interesting mixing pattern, we show in Table IX the predictions of a case arising in $G = \Delta(384) \rtimes CP$ when $G_e = Z_3$ and $G_\nu = Z_2 \times CP$ are chosen. In addition to one real continuous parameter θ, they depend on two discrete parameters m and s, specifying the embedding of the Z_2 and CP transformations, respectively, within $\Delta(384)$. Good agreement with the mixing angles is obtained if $|\sin\delta|$ is large and $m = 4$. In this case the bound $|\sin\delta| > 0.71$ holds. This mixing pattern is of the TM_1 type. For $s = 1$ and $s = 2$ the parameter $|m_{ee}|$ relevant for neutrinoless double beta decay has a nontrivial lower bound, whereas for $s = 4$ both Majorana phases are trivial and a cancellation cannot be avoided for normal ordering of the neutrino masses. Apart from the constraints on CP phases the lepton mixing angles are strongly restricted, which further sharpens the prediction of $|m_{ee}|$.

4. Other examples

A remnant CP symmetry in combination with texture zeros was examined by Barreiros, Felipe, and Joaquim (2019). In the case of $G = A_5 \rtimes CP$, one-parameter families of PMNS matrices have been studied (Ballett, Pascoli, and Turner, 2015; Di Iura, Hagedorn, and Meloni, 2015; Li and Ding, 2015b; Di Iura, López-Ibánez, and Meloni, 2018; López-Ibáñez *et al.*, 2019), and they typically have trivial or maximal Dirac CP phase and trivial Majorana phases. This study was generalized by Turner (2015) to include lepton mixing matrices depending on three parameters. Other groups that have been combined with CP invariance include T' (Girardi *et al.*, 2014), $\Delta(27)$ (Nishi, 2013), the series $D^{(1)}_{9n,3n}$ (Li, Yao, and Ding, 2016), $\Sigma(36\times 3)$ (Rong, 2017), and $\mathrm{PSL}_2(7)$ (Rong, 2019). Variants of this setup exploiting a generalized CP symmetry were considered by Girardi, Petcov, and Titov (2016), Chen *et al.* (2018), and Ding, King, and Li (2018); . Yao and Ding (2016) performed a scan of all groups of an order less than 2000 while assuming that either $(G_e, G_\nu) = (Z_{m_1} \times \cdots \times Z_{m_p}, Z_2 \times CP)$ or $(G_e, G_\nu) = (Z_2 \times CP', Z_2 \times Z_2 \times CP)$, with physical quantities depending on one continuous real parameter. The lepton mixing matrices in good agreement with the data fall into eight different categories up to possible row and column permutations. These viable mixing patterns can be reproduced starting from the discrete flavor groups $\Delta(6n^2)$, $D^{(1)}_{9n,3n}$, A_5, and $\mathrm{PSL}_2(7)$ combined with CP symmetry. Most of them are of the TM_2 type or a deformation thereof. Exceptions are those related to the survival symmetries $(G_e, G_\nu) = (Z_2 \times CP', Z_2 \times Z_2 \times CP)$ or those derivable from $A_5 \rtimes CP$.

If we assume that $G_e = Z_{m_1} \times \cdots \times Z_{m_p}$ and $G_\nu = Z_2 \times Z_2 \times CP$ in the neutrino sector, we potentially end up with the most predictive scenario, as shown in the first row of Table VII. In this case, after specifying the embedding of the residual groups $Z_{m_1} \times \cdots \times Z_{m_p}$ and $Z_2 \times Z_2$ in the full flavor group G, the PMNS matrix is fully determined, up to permutations of rows and columns. However, as shown by King and Neder (2014) and Chen, Yao, and Ding (2015), in this case the only viable PMNS matrix can only be of trimaximal TM_2 type with trivial δ, $\alpha_{31} = 0$, and α_{21} a rational multiple of π. Equations (4.33) with $\tan\delta = 0$ and the relative comments apply. The inverse problem of determining the most general residual CP symmetry of the neutrino sector compatible with the present data was studied by Everett, Garon, and Stuart (2015) and Everett and Stuart (2017) and, assuming tri-bimaximal mixing, by Chen *et al.* (2019a, 2019b). For a generic PMNS matrix, however, it is not guaranteed that the residual symmetries of the neutrino and charged-lepton sectors fit into a finite group.

The possibility of exploiting invariance under CP to predict or constrain physical phases find a natural application in the context of leptogenesis. This aspect was analyzed by Chen, Ding, and King (2016), Hagedorn and Molinaro (2017), Li and Ding (2017), Hagedorn *et al.* (2018), and Samanta, Sinha, and Ghosal (2018).

5. Extension to quarks

Flavor symmetries embedding CP have been also applied to the more general problem of simultaneously describing quarks and lepton masses. Indeed, taking quarks into account is unavoidable. While the latter could in principle be invariant under the action of a standard flavor group operating on the lepton sector, a CP symmetry must transform all fermion fields. Its spontaneous breaking in the quark sector must also be assured to reproduce the observed CP violation in the Cabibbo-Kobayashi-Maskawa matrix.

Several difficulties arise when one tries to extend flavor symmetries embedding CP to the quark sector. Most of them are common to the general framework of discrete symmetries and not due to the specific inclusion of CP. As we have seen, the approach based on selective residual symmetries makes predictions not about masses but rather only about angles. In this context the correlation between quark masses and mixing angles suggested by data and supported by Abelian symmetries is lost. Quark mass hierarchies are typically reproduced with the help of parameters poorly related to the mixing and CP properties. Moreover, to simultaneously describe both lepton and quark mixing angles, flavor groups of large order are generally required. Indeed, the small misalignment between up and down quarks calls for sufficiently close residual symmetries in the two sectors, which usually occurs if the group G_f has a large number of densely distributed subgroups. For example, when quark and lepton electroweak doublets are assigned to irreducible triplets of G_f, groups as large as $\Delta(294)$ (Li, Lu, and Ding, 2018; Lu and Ding, 2018) or $\Delta(384)$ (Hagedorn and König, 2018a, 2018b) are needed.

Apart from aesthetic considerations, implementing the desired symmetry breaking pattern in a concrete model requires a large number of flavon representations. This in turn generates a serious alignment problem, implying that additional cyclic symmetries or selection rules have to be invoked to get only the desired interaction terms. Explicit examples of these constructions have been realized via a stepwise breaking of $G_f = \Delta(384)$ combined with CP, where charged fermion mass hierarchies are reproduced through operators with different numbers of flavons (Hagedorn and König, 2018a). These examples also show that a direct embedding in GUT is problematic, since matter and flavon representations do not fit GUT multiplets.

To reduce the order of the group while preserving predictability about phases of the mixing matrices, the use of dihedral groups in combination with CP has been suggested. This approach takes up the old observation that dihedral groups are suitable to accommodate quark mixing angles (Lam, 2007; Blum, Hagedorn, and Lindner, 2008). Dihedral groups do not possess three-dimensional irreducible representations and quarks and lepton electroweak doublets are assigned to singlets and doublets of the flavor group. By choosing $G_f = D_{14}$ and including CP, quark and lepton mixing angles and phases can both be accommodated by adjusting two continuous free parameters in each sector (Lu and Ding, 2019). It would be desirable to show that the symmetry breaking pattern invoked in this analysis can be effectively realized within a concrete model.

D. Outlook

The embedding of CP in the flavor symmetry provides a valuable complement to the setup dealing with ordinary flavor groups and fully commuting with the proper Poincaré transformations. In such more restricted framework lepton mixing angles, Dirac and Majorana phases can all be predicted simultaneously in terms of a single continuous real parameter in the most realistic and predictive cases. Many explicit models support the viability of such an approach, with similar disadvantages affecting models dealing with ordinary flavor groups: a complicated symmetry breaking sector, additional auxiliary symmetries and fields to trigger the desired pattern of symmetry breaking, and a limited prediction accuracy due to higher-dimensional operators.

VI. NONLINEARLY REALIZED FLAVOR SYMMETRIES

A. The modular group $\bar{\Gamma}$

Nonlinearly realized flavor symmetries have been considered in the context of $\mathcal{N} = 1$ supersymmetric theories by adopting as flavor group the modular group $\bar{\Gamma}$. The idea that modular invariance can play a central role in describing Yukawa couplings is an old one, and it has been naturally realized in the context of string theory (Dixon *et al.*, 1987; Hamidi and Vafa, 1987; Lauer, Mas, and Nilles, 1989, 1991; Erler, Jungnickel, and Lauer, 1992), in D-brane compactification (Cremades, Ibanez, and Marchesano, 2003; Blumenhagen *et al.*, 2005; Abel and Goodsell, 2007; Blumenhagen *et al.*, 2007; Marchesano, 2007; Antoniadis, Kumar, and Panda, 2009; Kobayashi, Nagamoto, and Uemura, 2017), in magnetized extra dimensions (Cremades, Ibanez, and Marchesano, 2004; Abe *et al.*, 2009; Kobayashi, Nagamoto *et al.*, 2018), and in orbifold compactification (Ibáñez, 1986; Casas, Gomez, and Munoz, 1993; Lebedev, 2001; Kobayashi and Lebedev, 2003). Modular invariance was also incorporated in early flavor models (Binetruy and Dudas, 1995; Brax and Chemtob, 1995; Dudas, 1996; Dudas, Pokorski, and Savoy, 1996; Leontaris and Tracas, 1998). A step forward has been taken with the observation that it can be implemented in a bottom-up perspective relying on the group transformation properties of the building blocks of the theory (Feruglio, 2019).

In $\mathcal{N} = 1$ supersymmetric theories, the field τ, called the modulus, is a chiral supermultiplet whose scalar component is restricted to $\mathcal{H}$, the upper half of the complex plane. Under $\bar{\Gamma}$ it transforms as

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{6.1}$$

with a, b, c, and d integers and $ad - bc = 1$. The group $\bar{\Gamma}$ is discrete, infinite, and noncompact. It has the following presentation in terms of the two generators S and T:

$$\tau \xrightarrow{S} -\frac{1}{\tau}, \qquad \tau \xrightarrow{T} \tau + 1 \tag{6.2}$$

satisfying

$$S^2 = (ST)^3 = \mathbb{1}. \tag{6.3}$$

The modular group is ubiquitous in string theory. It is the invariance group of a lattice Λ defined in the complex plane $\mathbb{C}$. Two lattices Λ and Λ' with basis (e_1, e_2) and (e'_1, e'_2), such that $\mathrm{Im}(e_1/e_2)$ and $\mathrm{Im}(e'_1/e'_2)$ are both positive, coincide if and only if

$$\begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{6.4}$$

with a, b, c, and d integers and $ad - bc = 1$. A frequently considered compactification of two extra dimensions gives rise to a torus, defined by the quotient $\mathbb{C}/\Lambda$ modulo rotations and scale transformations, which allow one to chose the basis of Λ of the form $(\tau, 1)$ $[\mathrm{Im}(\tau) > 0]$. It follows that two tori defined by τ and $\gamma\tau$ coincide; see Fig. 3. From this viewpoint $\bar{\Gamma}$ can be thought of as a gauge symmetry. With a gauge choice it

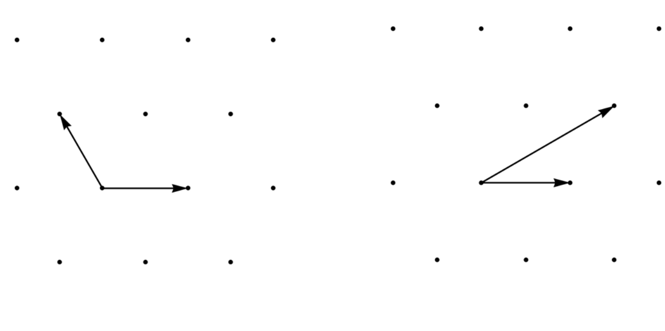

FIG. 3. Two equivalent lattices with basis $(\tau, 1)$ and $(\tau + 2, 1)$.

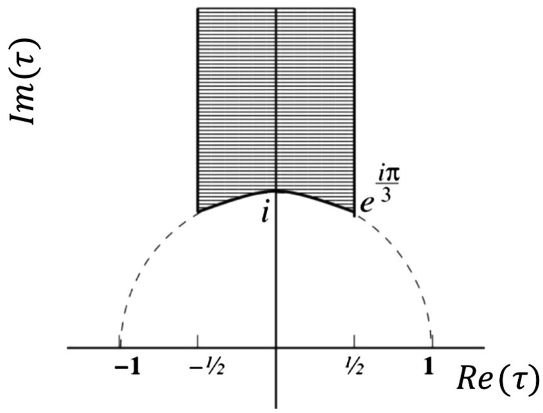

FIG. 4. Fundamental region $\mathcal{F}$: a connected region of $\mathcal{H}$ such that each point of $\mathcal{H}$ can be mapped onto $\mathcal{F}$ by a $\bar{\Gamma}$ transformation, but no two points in the interior of $\mathcal{F}$ are related under $\bar{\Gamma}$.

is always possible to restrict τ to a fundamental region, a representative of which is shown in Fig. 4.

B. Modular invariant supersymmetric theories

We can define the action of $\bar{\Gamma}$ on a set of matter chiral multiplets $\phi^{(I)}$ by specifying a compact quotient of $\bar{\Gamma}$. A series of compact groups can be constructed by taking the quotient of $\bar{\Gamma}$ by a principal congruence subgroup $\bar{\Gamma}(N)$ with elements obeying $a, d = 1(\mathrm{mod}\ N)$ and $b, c = 0(\mathrm{mod}\ N)$, with N a natural number called the *level*. $\bar{\Gamma}(N)$ are normal subgroups of $\bar{\Gamma}$ of finite index, so the quotients $\Gamma_N = \bar{\Gamma}/\bar{\Gamma}(N)$ are finite groups admitting finite-dimensional unitary representations. For the first few levels, they are isomorphic to permutation groups: $\Gamma_2 = S_3$, $\Gamma_3 = A_4$, $\Gamma_4 = S_4$, and $\Gamma_5 = A_5$. We have $\partial(\gamma\tau)/\partial\tau = (c\tau + d)^{-2}$, and under the modular group the matter fields $\phi^{(I)}$ transform as (Ferrara, Lust, and Theisen, 1989; Ferrara *et al.*, 1989)

$$\phi^{(I)} \to (c\tau + d)^{k_I}\rho^{(I)}(\gamma)\phi^{(I)}. \tag{6.5}$$

Equation (6.5) is completely defined by the weight k_I, the level N, and the unitary representation $\rho^{(I)}(\gamma)$ of Γ_N. We also recall that modular forms of level N and weight k are holomorphic functions $Y(\tau)$ of the modulus satisfying

$$Y(\gamma\tau) = (c\tau + d)^k Y(\tau) \tag{6.6}$$

for any $\gamma \in \bar{\Gamma}(N)$. They form a linear space $\mathcal{M}_k(\Gamma(N))$ of finite dimension $d_k(\Gamma(N))$ (Gunning, 1962). Under the full modular group $\bar{\Gamma}$ a basis $Y(\tau)$ of $\mathcal{M}_k(\Gamma(N))$ transforms as $Y(\gamma\tau) = (c\tau + d)^k\rho(\gamma)Y(\tau)$, with $\rho(\gamma)$ a unitary, possibly reducible representation of Γ_N.

Turning off gauge interactions, the action $\mathcal{S}$ of an $\mathcal{N} = 1$ global supersymmetric theory depending on the modulus τ and a set of supermultiplets ϕ, comprising matter fields $\phi^{(I)}$ of the same level N and possibly different weights k_I, reads

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} K(\tau, \phi, \bar{\tau}, \bar{\phi}) + \int d^4x d^2\theta w(\tau, \phi) + \int d^4x d^2\bar{\theta}\, \bar{w}(\bar{\tau}, \bar{\phi}), \tag{6.7}$$

where K and w are the Kähler potential and the superpotential, respectively. Invariance under the transformations of Eqs. (6.1) and (6.5) requires a modular invariant superpotential and a Kähler potential modular invariant up to Kähler transformations

$$K(\tau, \phi, \bar{\tau}, \bar{\phi}) \to K(\tau, \phi, \bar{\tau}, \bar{\phi}) + f(\tau, \phi) + \bar{f}(\bar{\tau}, \bar{\phi}), \quad w(\tau, \phi) \to w(\tau, \phi). \tag{6.8}$$

Equation (6.8) is easily satisfied by minimal forms of the Kähler potential, with an example

$$K(\tau, \phi, \bar{\tau}, \bar{\phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{k_I} |\phi^{(I)}|^2, \tag{6.9}$$

where h is a positive constant. Conversely, the requirement of modular invariance severely restricts the superpotential $w(\tau, \phi)$. Consider the following expansion of $w(\tau, \phi)$ in power series of the supermultiplets $\phi^{(I)}$:

$$w(\tau, \phi) = \sum_n Y_{I_1,\ldots,I_n}(\tau)\phi^{(I_1)} \cdots \phi^{(I_n)}. \tag{6.10}$$

For the nth order term to be modular invariant, the functions $Y_{I_1,\ldots,I_n}(\tau)$ should be holomorphic functions of τ transforming as

$$Y_{I_1,\ldots,I_n}(\gamma\tau) = (c\tau + d)^{k_Y(n)}\rho(\gamma)Y_{I_1,\ldots,I_n}(\tau), \tag{6.11}$$

with the weight $k_Y(n)$ and the unitary representation ρ such that the following are true:

(1) The weight $k_Y(n)$ should compensate for the overall weight of the product $\phi^{(I_1)} \cdots \phi^{(I_n)}$ so that

$$k_Y(n) + k_{I_1} + \cdots + k_{I_n} = 0. \tag{6.12}$$

(2) The product $\rho \times \rho^{I_1} \times \cdots \times \rho^{I_n}$ contains an invariant singlet.

The holomorphic functions $Y_{I_1,\ldots,I_n}(\tau)$ of Eq. (6.11) are modular forms of level N and weight $k = k_Y(n)$. This property sharply constrains the allowed Yukawa couplings, to the point of completely determining in some cases the corresponding mass matrix as a function of τ up to a single overall constant.

As an example, choose $N = 3$ and consider three copies of lepton doublets l and one Higgs supermultiplet H_u transforming, respectively, as irreducible triplets of $\Gamma_3 = A_4$ with weight -1 and as a singlet of Γ_3 with zero weight. Assuming neutrino masses described entirely by the Weinberg operator, the relevant superpotential reads

$$w_\nu = \frac{1}{2\Lambda}(l_i H) Y_{ij}(\tau)(l_j H), \tag{6.13}$$

where the holomorphic functions $Y_{ij}(\tau)$ should be modular forms of level 3, weight +2 transforming as one of the multiplets in the decomposition $(3\times 3)_{\rm SYM} = 1 + 1' + 1'' + 3$. The space $\mathcal{M}_2(\Gamma(3))$ is spanned by three linearly independent modular forms $Y_i(\tau)$ $(i = 1,2,3)$, transforming as a 3 under Γ_3 such that

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right],\\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right],\\
Y_3(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right],
\end{aligned} \tag{6.14}$$

where $\eta(\tau)$ is the Dedekind eta function defined in the upper complex plane as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) q \equiv e^{i2\pi\tau}. \tag{6.15}$$

It follows that w_ν consists of a unique modular invariant combination and is fully determined up to an overall constant. In a suitable basis the neutrino mass matrix reads

$$m_\nu = m_0\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau)\\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau)\\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau)\end{pmatrix}. \tag{6.16}$$

As long as supersymmetry is unbroken there are no corrections coming from higher-dimensional holomorphic operators. The matrix m_ν in Eq. (6.16) is exact and all the terms in the expansion in powers of τ are completely determined.

Nonvanishing modular forms transforming under Γ_N require even integer non-negative weights (Gunning, 1962). Modular forms of vanishing weight are constant, that is, τ independent. Modular forms for the first few levels N have been explicitly constructed and the combinations transforming as irreducible representations of Γ_N have been identified for the first few weights. The results for $\Gamma_2 \approx S_3$ (Kobayashi, Tanaka, and Tatsuishi, 2018), $\Gamma_3 \approx A_4$ (Feruglio, 2019), $\Gamma_4 \approx S_4$ (Penedo and Petcov, 2019), $\Gamma_5 \approx A_5$ (Ding, King, and Liu, 2019b; Novichkov *et al.*, 2019b), and $\Gamma_7 \approx \Sigma(168)$ (Ding *et al.*, 2020) are summarized in Table X. Modular forms of generic integer weights were discussed by Liu and Ding (2019), together with their application to neutrino mass models. They have been shown to form representations of the homogeneous finite modular groups Γ'_N, double coverings of Γ_N.

C. Modular invariance and *CP*

The action of *CP* on τ is uniquely determined up to modular transformations (Dent, 2001a, 2001b; Baur *et al.*, 2019a, 2019b; Novichkov *et al.*, 2019a)

$$\tau \xrightarrow{CP} -\tau^*. \tag{6.17}$$

Such a law corresponds to the following outer automorphism of $\bar{\Gamma}$:

$$S \xrightarrow{CP} S, \qquad T \xrightarrow{CP} T^{-1}. \tag{6.18}$$

By choosing a suitable basis for the generators S and T, where both are described by symmetric matrices in any representation of Γ_N, the action of *CP* on matter multiplets ϕ reduces to the canonical one

$$\phi \xrightarrow{CP} X_{CP}\phi^*, \qquad X_{CP} = \mathbb{1}. \tag{6.19}$$

In this basis the requirement of *CP* invariance amounts to restricting all the Lagrangian parameters to real values. In such a theory *CP* invariance can only be spontaneously broken.

TABLE X. Dimension of $\mathcal{M}_k(\Gamma(N))$ and decomposition of multiplets of modular forms in representations of the finite modular group Γ_N for the first few levels and weights. Modular forms of higher weight can be obtained from polynomials of modular forms of lower weight. Partial knowledge is available for modular forms of weight 2 for levels 8 and 16 (Kobayashi and Tamba, 2019).

	$d_k(\Gamma(N))$	$k=2$	$k=4$	$k\geq 6$
$\Gamma_2 \approx S_3$	$k/2+1$	2	$1+2$	$\cdots$
$\Gamma_3 \approx A_4$	$k+1$	3	$1+1'+3$	$\cdots$
$\Gamma_4 \approx S_4$	$2k+1$	$2+3'$	$1+2+3+3'$	$\cdots$
$\Gamma_5 \approx A_5$	$5k+1$	$3+3'+5$	$1+3+3'+4+5+5$	$\cdots$
$\Gamma_7 \approx \Sigma(168)$	$14k-2$	$3+7+8+8'$	$1+3+6+6'+7+7'+8+8'+8''$	$\cdots$

The values of τ preserving CP lie along the imaginary τ axis or along the border of the fundamental region shown in Fig. 4, where $-\tau^* = \tau$, up to a modular transformation.

D. Modular invariance and standard flavor symmetries

It is worth mentioning that in the low-energy theory arising from string theory compactification the flavor group generally comprises both modular transformations and ordinary transformations acting linearly on matter fields. The consistent combination of the two types of transformations was analyzed by Nilles, Ramos-Sánchez, and Vaudrevange (2020a, 2020b). The ordinary linear transformations belong to a group G, leave the modulus τ invariant, and act on the fields $\phi^{(I)}$ through a unitary matrix $U^{(I)}(g)$ as follows:

$$\tau \to \tau, \qquad \phi^{(I)} \to U^{(I)}(g)\phi^{(I)}. \tag{6.20}$$

The two sets of transformations with Eqs. (6.1) and (6.5) and Eq. (6.20) should obey the consistency condition

$$\rho^{(I)}(\gamma)U^{(I)}(g)\rho^{(I)}(\gamma^{-1}) = U^{(I)}(g') \tag{6.21}$$

for some element $g' \in G$. It follows that G is a normal subgroup of the overall flavor group G_{ecl}, called eclectic by Nilles, Ramos-Sánchez, and Vaudrevange, generated by both ordinary and modular transformations. At the same time the modular transformations define an automorphism of G that, in the nontrivial cases, is of the outer type. This construction allows for a unified description of standard, nonlinear and CP-like transformations. Not all groups G can be embedded in such a framework, which may open new possibilities in model building.

E. Modular invariance and local supersymmetry

This setup can easily be extended to the case of $\mathcal{N} = 1$ local supersymmetry, where Kähler potential and superpotential are not independent functions since the theory depends on the combination

$$\mathcal{G}(\tau,\phi,\bar\tau,\bar\phi) = K(\tau,\phi,\bar\tau,\bar\phi) + \log w(\tau,\phi) + \log \bar w(\bar\tau,\bar\phi). \tag{6.22}$$

The modular invariance of the theory can be realized in two ways (Ferrara *et al.*, 1989). Either $K(\tau,\phi,\bar\tau,\bar\phi)$ and $w(\tau,\phi)$ are separately modular invariant or the transformation of $K(\tau,\phi,\bar\tau,\bar\phi)$ under the modular group is compensated for by that of $w(\tau,\phi)$. An example of the second possibility is given by the Kahler potential of Eq. (6.9), with the superpotential $w(\tau,\phi)$ transforming as

$$w(\tau,\phi) \to e^{i\alpha(\gamma)}(c\tau + d)^{-h} w(\tau,\phi). \tag{6.23}$$

In the expansion (6.10) the Yukawa couplings $Y_{I_1,\ldots,I_n}(\tau)$ should have weight $k_Y(n)$ such that $k_Y(n) + k_{I_1} + \cdots + k_{I_n} = -h$ and such that the representation $\rho(\gamma)$ is subject to requirement 2 in Eq. (6.12). When we have $k_{I_1} + \cdots + k_{I_n} = -h$, we get $k_Y(n) = 0$ and the functions $Y_{I_1,\ldots,I_n}(\tau)$ are τ-independent constants. This occurs for supermultiplets belonging to the untwisted sector in the orbifold compactification of the heterotic string.

F. Models

Models of lepton masses and mixing angles have been constructed for levels 2–5, following two different approaches depending on whether the charged-lepton mass matrix depends only on τ, as the neutrino one, or on a separate set of flavons. In either case the VEV of τ is usually treated as an additional parameter and scanned in order to maximize agreement with the data. We show here an example for each possibility. In both examples neutrino masses arise from the type-I seesaw mechanism and, after integrating out the right-handed neutrinos N^c, the low-energy superpotential reads

$$w = -E^{cT}\mathcal{Y}_e H_d L - \frac{1}{2\Lambda}(H_u L)^T(\mathcal{Y}_\nu^T \mathcal{C}^{-1}\mathcal{Y}_\nu)(H_u L). \tag{6.24}$$

An example of the first possibility is the model given by Novichkov *et al.* (2019a, 2019c) that is realized at level 4, with the particle content displayed in Table XI.

The matrices $\mathcal{Y}_e$, $\mathcal{Y}_\nu$, and $\mathcal{C}$ are given by

$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_3 & \alpha Y_5 & \alpha Y_4 \\ \beta(Y_1Y_4 - Y_2Y_5) & \beta(Y_1Y_3 - Y_2Y_4) & \beta(Y_1Y_5 - Y_2Y_3) \\ \gamma(Y_1Y_4 + Y_2Y_5) & \gamma(Y_1Y_3 + Y_2Y_4) & \gamma(Y_1Y_5 + Y_2Y_3) \end{pmatrix}, \tag{6.25}$$

$$\mathcal{Y}_\nu = g\left[\begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix}\right],$$
$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6.26}$$

where $Y_{1,2}$ and $Y_{3,4,5}$ are the five independent modular forms of weight 2 and level 4. They transform as 2 and $3'$ under $\Gamma_4 \approx S_4$, respectively. Invariance under CP implies that g/g' is real. Charged-lepton masses can be correctly reproduced by adjusting α, β, and γ. The remaining Lagrangian parameters are an overall scale and g/g'. The VEV of τ is treated as an additional free parameter.

An example of the second possibility is the model given by Criado and Feruglio (2018) that is realized at level 3, with the particle content displayed in Table XII.

TABLE XI. Chiral supermultiplets, transformation properties, and weights of the model given by Novichkov *et al.* (2019a, 2019c).

	(E_1^c, E_2^c, E_3^c)	N^c	L	H_d	H_u
$SU(2)_L \times U(1)_Y$	$(1,+1)$	$(1,0)$	$(2,-1/2)$	$(2,-1/2)$	$(2,+1/2)$
$\Gamma_4 \approx S_4$	$(1',1,1')$	$3'$	3	1	1
k_I	$(0,-2,-2)$	0	-2	0	0

TABLE XII. Chiral supermultiplets, transformation properties, and weights of the model given by Criado and Feruglio (2018).

	(E_1^c, E_2^c, E_3^c)	N^c	L	H_d	H_u	φ
$SU(2)_L \times U(1)_Y$	$(1, +1)$	(1,0)	$(2, -1/2)$	$(2, -1/2)$	$(2, +1/2)$	(1,0)
$\Gamma_3 \approx A_4$	$(1, 1'', 1')$	3	3	1	1	3
k_I	-4	-1	$+1$	0	0	$+3$

The matrices $\mathcal{Y}_e$, $\mathcal{Y}_\nu$ and $\mathcal{C}$ are given by

$$\mathcal{Y}_e = \begin{pmatrix} a\varphi_1 & a\varphi_3 & a\varphi_2 \\ b\varphi_2 & b\varphi_1 & b\varphi_3 \\ c\varphi_3 & c\varphi_2 & c\varphi_1 \end{pmatrix}, \qquad \mathcal{Y}_\nu = y_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\mathcal{C} = \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}. \tag{6.27}$$

Beyond the parameters a, b, and c, which control charged-lepton masses, the low-energy Lagrangian depends on a single parameter, the overall scale y_0^2/Λ. Additional parameters are provided by the VEVs of τ and the flavon φ, which are assumed to be aligned along the $(1, 0, \mathrm{Re}(\varphi_3))$ direction. The results of the two models are collected in Table XIII.

In both models the mass ordering is normal. The atmospheric mixing angle is close to maximal but predicted to lie in the first octant. CP is broken spontaneously by the VEV of τ and both Dirac and Majorana phases are predicted. In addition, the absolute value of neutrino masses and the combination relevant to $0\nu\beta\beta$ are predicted. The lightest neutrino has a mass close to 0.01 eV, resulting in a relatively large $|m_{ee}| \approx 10$ meV for a normally ordered mass spectrum.

Several other models of lepton masses and mixing angles have been built at level 2 (Kobayashi, Tanaka, and Tatsuishi, 2018; Kobayashi *et al.*, 2019d), level 3 (Criado and Feruglio, 2018; Kobayashi, Omoto *et al.*, 2018; Ding, King, and Liu, 2019a; Ding *et al.*, 2019; Feruglio, 2019; Nomura and Okada, 2019; Novichkov, Petcov, and Tanimoto, 2019), level 4 (Criado, Feruglio, and King, 2019; King and Zhou, 2019; Novichkov *et al.*, 2019c; Penedo and Petcov, 2019; Liu, Yao, and Ding, 2020; Novichkov, Penedo, and Petcov, 2020), and level 5 (Criado, Feruglio, and King, 2019; Ding, King, and Liu, 2019b; Novichkov *et al.*, 2019b). The higher the level N, the more solutions that are found in $\mathcal{H}$, corresponding to physically distinct sets of predictions in good agreement with data. Most of the solutions predicting NO prefer a nearly degenerate spectrum with $m_1 > 10$ meV and $|m_{ee}|$ on the high side of the allowed range. This is shown in Fig. 5.

A common feature of all proposed models is the minimal form of the Kähler potential [Eq. (6.9)]. While this is the simplest choice, it is not the most general one compatible with modular invariance. The symmetry of the Kähler potential K of Eq. (6.9) is bigger than the modular one. Indeed, K is invariant under transformation of $SL(2, \mathbb{R})$ and the modulus τ parametrizes the coset $SL(2, \mathbb{R})/SO(2)$. Such a continuous symmetry is broken by the superpotential down to the modular group. In a bottom-up approach there is no reason to exclude from the Kähler potential K terms that are invariant only under the discrete modular group. In particular, a candidate modification of the Kähler potential (6.9) is an additive contribution depending explicitly on both the matter supermultiplets and the modular forms $Y(\tau)$ (Feruglio, 2019). The power counting controlling the size of these contributions is unknown, but examples in the string theory context suggest that, in the parameter region $\mathrm{Im}(\tau) = \mathcal{O}(1)$ that is the one of interest in neutrino physics, they might be of similar importance as K in Eq. (6.9). Indeed, these types of corrections were analyzed by Chen, Ramos-Sanchez, and Ratz (2019), who showed that the new parameters appearing in the Kähler potential considerably reduce the predictability of the approach. Currently, the problem of better controlling the Kähler potential remains unresolved.

An interesting question concerns the dynamical determination of the VEV of τ. It has been conjectured (Cvetic *et al.*, 1991; Kobayashi *et al.*, 2019c) that extrema of modular invariant scalar potentials of $\mathcal{N} = 1$ supergravity theories lie on the imaginary τ axis or along the border of the fundamental region $\mathcal{F}$ of Fig. 4. This is precisely the region where CP is unbroken if the theory is CP invariant. In concrete models a small deviation from the border of $\mathcal{F}$ suffices to allow for sizable CP-violating effects. For instance in the model given by Novichkov *et al.* (2019a, 2019c) the value of τ that maximizes agreement with the data is

TABLE XIII. Results of the $N = 4$ model given by Novichkov *et al.* (2019a, 2019c) for $\tau = 0.099\,22 + i1.0578$ and $g/g' = -0.020\,93$ and of the $N = 3$ model given by Criado and Feruglio (2018), for $\tau = -0.2005 + i1.0578$ and $\varphi = (1, 0, 0.117)$.

N	$r \equiv \lvert\Delta m^2_{\mathrm{sol}}/\Delta m^2_{\mathrm{atm}}\rvert$	$\sin^2\theta_{12}$	$\sin^2\theta_{13}$	$\sin^2\theta_{23}$	δ/π
4	0.0298	0.305	0.0214	0.486	1.641
3	0.0299	0.306	0.0211	0.459	1.438

N	α_{21}/π	α_{31}/π	m_1 (meV)	m_2 (meV)	m_3 (meV)	$\lvert m_{ee}\rvert$ (meV)
4	0.346	1.254	12.1	14.8	51.4	12.0
3	1.704	1.201	10.9	13.9	51.1	10.4

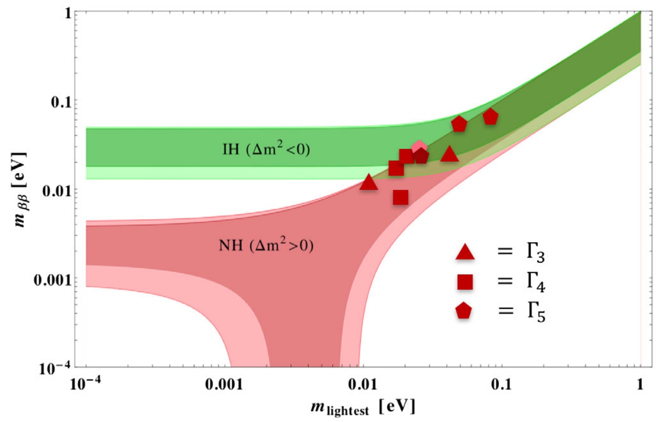

FIG. 5. Regions allowed in the $(m_{\text{lightest}}, |m_{ee}|)$ plane for normal ordering (red) and inverted ordering (green), and predictions of modular invariant models at levels 3–5.

$0.099\,22 + i1.0578$. An attempt to dynamically determine the VEV of τ was given by Kobayashi *et al.* (2019a), who worked in the context of supergravity. At the minima of the scalar potential the energy density is negative, and some *ad hoc* mechanism should be invoked to reproduce the correct cosmological constant. This is confirmed by the analysis given by Gonzalo, Ibáñez, and Uranga (2019), who found no minima with positive energy density. Corrections from supersymmetry breaking have been shown to be negligible (Criado and Feruglio, 2018) provided that there is a sufficient gap between the sparticle masses and the messenger scale. The modulus-electron interactions can be directly tested in neutrino oscillations provided that the modulus is extremely light (Ding and Feruglio, 2020). In such a case scalar nonstandard neutrino interactions can affect lepton mass matrices and produce deviations in oscillation patterns in media with a sufficiently large electron number density.

G. Extension to quarks

The possibility of extending modular invariance to the quark sector was investigated by Kobayashi *et al.* (2019d), Okada and Tanimoto (2019a, 2019b, 2020) and, in a GUT context, by de Anda, King, and Perdomo (2018) and Kobayashi *et al.* (2019b). Description of the quark sector alone seems to require a large number of parameters, often larger than the number of observables. Having many parameters at our disposal, it is not surprising that a unified description of leptons and quarks, adopting the same value of τ to simultaneously describe the two sectors, can be achieved. One of the major obstacle toward the realization of a more economical model is the fact that each charged fermion mass requires an independent parameter. In its present realization, modular invariance seems unable to provide predictions concerning the charged fermion masses, which should be described by an *ad hoc* set of parameters. To improve this aspect, two suggestions have recently been put forward. If quark and charged-lepton masses cannot be precisely predicted, at least their order of magnitude can be captured by letting the modular weights play the role of Froggatt-Nielsen charges (Criado, Feruglio, and King, 2019; King and King, 2020). Assigning different weights to electroweak singlet fermions, we can achieve a natural relative suppression of charged fermion masses that is similar to what happens in ordinary Abelian symmetries. As a consequence, dimensionless free parameters are not reduced in number, but their values have the same order of magnitude. A second observation is that modular invariance can naturally enforce texture zeros, which are known to increase the predictive power of flavor models. Along these lines, Lu, Liu, and Ding (2019) built several models at level 3. They made use of odd weight modular forms and assigned quarks to both singlet and doublet representations of $\Gamma'_3 \approx T'$, the double covering of Γ_3. In a specific model all 22 fermion mass and mixing observables are reproduced using 17 independent parameters and the best fit value of τ is close to $-1/2 + i\sqrt{3}/2$, a fixed point under the action of ST.

For moderately large values of $\mathrm{Im}(\tau)$, modular forms have a nearly exponential dependence on τ, which at first sight seems ideal for describing the hierarchical mass spectrum that we observe in quarks and charged leptons. This suggests that we might not have fully exploited all the possibilities offered by this approach.

H. Outlook

Modular invariance is an interesting candidate for a realistic flavor symmetry. Compared to the traditional linear realization of discrete symmetries, it allows one to predict not only mixing angles and phases but also neutrino masses. It requires fewer flavons: in minimal realizations no flavon beyond τ is needed. In the most favorable cases, as long as supersymmetry is exact, the superpotential is completely determined by symmetry requirements, to any order in the τ power expansion, up to an overall constant. In the exact supersymmetry limit the superpotential does not receive any perturbative or nonperturbative corrections, a unique feature compared to the models based on linearly realized symmetries. A lesson that we can learn from the proposed models is that a low level N and modular forms of low weights minimize the number of free parameters. Thus far the approach has allowed no prediction to be made for the charged-lepton masses. The charged-lepton sector might require a substantially different description, perhaps in terms of additional moduli (Ferrara, Lust, and Theisen, 1989; de Medeiros Varzielas, King, and Zhou, 2020) or some conventional flavon. The models proposed thus far have relied on a minimal form of the Kähler potential, which, however, is not justified in a bottom-up approach. Modular invariance allows for additional terms in the Kähler potential, and their impact in the parameter region of interest to neutrinos has been shown to be important.

VII. WHAT HAVE WE LEARNED?

The discovery of neutrino oscillations has led to a major advance in our knowledge about the flavor sector. On the one hand, there is still considerable room for improvement in the data. The uncertainty on the absolute neutrino masses is large since only mass-squared differences have been measured. *CP*-odd phases (in particular, the Majorana ones, if present) will not really be known with good precision for a long time. On the other hand, experimental outcomes in recent years have brought neutrino physics into a precision era, with several combinations of mass and mixing parameters known with a precision approaching the percent level. Tracing those parameters back to some fundamental organizing principle is part of an ambitious program, the solution of the flavor puzzle. In this wider context, we cannot avoid considering both leptons and quarks, probably within some kind of unified framework emerging when physics is probed at a high-energy scale. Actually, the need of reconciling the differing features of the quark and lepton sectors might provide important clues to correctly address and solve the puzzle. Quark intergenerational hierarchy is much more pronounced, especially in the up sector. Mixing angles are small, with the third generation feebly coupled to the first two. Conversely, neutrino masses are of the same order of magnitude, with the possible exception of the lightest state, and still compatible with being massless. The lepton mixing pattern is completely different from the quark one, with the smallest mixing angle being similar in size to the Cabibbo angle. Nevertheless, the description of the lepton sector has borrowed many ideas and techniques originally developed in the context of the quark sector.

An appealing approach that has pervaded the entire field for decades is the one based on flavor symmetries, which is supported by the success that symmetry considerations have achieved in the past century in the description of particle interactions. Flavor symmetries of the leptonic sector have been realized in a vast number of ways, as shown by the extensive literature of the field. Perhaps one of the most striking things that captures one's attention is the fact that, despite all past efforts, a baseline model interpreting neutrino masses and mixings in the context of a flavor symmetry is still missing. Many early models have been discarded by gathering increasingly precise data, but the range of remaining possibilities is still large, even taking into account the constraints from the quark sector. This is closely related to the fact that, in any realistic model of lepton masses relying on flavor symmetries and retaining some degree of predictability, the underlying symmetry is cleverly hidden and breaking effects are a decisive factor in constraining the relevant observables.

Actually, one of the few firm points is the fact that there cannot be exact flavor symmetries for either quarks or leptons alone. The observed masses and mixing angles break any initial flavor symmetry, except possibly for the total baryon and lepton numbers. Once it is ruled out that exact flavor symmetries can be allowed by the data, we can wonder whether they can provide at least some reasonable first order approximation to the observed lepton mass and mixing pattern. It turns out that under mild assumptions symmetries compatible with this requirement are not powerful. In the normal mass ordering case, the neutrino mass matrix is completely unconstrained and any neutrino masses and mixings are possible. The flavor symmetry is useless in the neutrino sector, where it leads to anarchy. Therefore, if this hint at normal hierarchy were to be confirmed, we would conclude that symmetry breaking effects would play a leading role in a realistic nontrivial model of lepton masses.

Indeed, the common denominator of most predictive models is the breaking of the flavor symmetry induced by a set of spurions. The prototype of these models makes use of a spontaneously broken Abelian continuous group. While Abelian symmetries have played a pivotal role in the development of the field, they can lead to predictions matching the present experimental accuracy only in the presence of texture zeros, as each entry of the mass matrices is predicted order of magnitude–wise, with intrinsic uncertainties of order 1. Any successful model of neutrino masses and mixings based on flavor symmetries should rely on a sizable departure of the predictions from the symmetric limit, most often of a non-Abelian group.

If they are so well hidden, flavor symmetries can be difficult to identify from the data. Moreover, in model building sizable breaking effects analyzed to the desired level of accuracy typically involve a non-negligible set of parameters, which weakens the aimed-for predictive power of the construction. In the absence of a symmetric limit reasonably close to observation, the entire symmetry approach seems to be undermined. In that case, why do we not abandon it? We believe there are several examples to counter this negative conclusion. Perhaps the most impressive one is provided by the modular symmetry that, being nonlinearly realized, does not allow any limit where the full modular group remains unbroken. The geometrical interpretation of this feature is particularly transparent. The modular transformations can be seen as gauge transformations describing all possible equivalent parametrizations of the same torus in terms of a modular parameter. For the modular group to be unbroken, we would need a torus that does not admit distinct equivalent parametrizations, which is impossible by construction. Thus, in modular invariant flavor models there is no notion of a symmetric limit, and this does not prevent predictability and precision, at least in principle. In the most favorable cases, the neutrino mass matrix is completely determined by symmetry requirements as a function of the modular parameter up to an overall constant. In the exact supersymmetry limit, the superpotential does not receive any perturbative or nonperturbative corrections, a unique feature relative to the models based on linearly realized symmetries.

The requirement of being far from the symmetric limit does not forbid that, separately, the neutrino and the charged-lepton sectors can be approximately invariant under independent symmetries, arising as subgroups of the full symmetry group. This occurs when spurions with different breaking properties are accidentally sequestered. Since the most general symmetry leaving the neutrino mass matrix invariant and its eigenvalues unconstrained is the Klein symmetry, the more economical realizations of such sequestering adopts discrete flavor groups. Owing to unavoidable corrections, exact sequestering can hardly occur and should rather be viewed as an ideal limit useful for identifying approximate mixing patterns. Even considering the smallest discrete groups allowing three-dimensional irreducible representations, semirealistic mixing patterns such as the tri-bimaximal one can easily be obtained. Tri-bimaximal mixing is ruled out by the data and, to identify realistic mixing patterns in the framework of exact sequestering, we should move to larger discrete groups. Otherwise, working with small discrete groups, we can relax sequestering by allowing sizable corrections or by reducing the residual symmetries.

A weak point of this approach is that sequestering requires a specific vacuum alignment that in turn is often realized at the price of a complicated scalar sector and additional *ad hoc* symmetries. Additional ingredients are needed to constrain masses and Majorana phases. The phases can be dealt with by exploiting flavor symmetries incorporating CP. The hierarchical nature of charged-lepton masses can be accounted for with traditional suppression mechanisms, but order-of-magnitude uncertainties cannot be evaded. Modular invariant models have limitations too. The models proposed to date rely on a minimal form of the Kähler potential. Modular invariance alone allows for more general Kähler potentials, which introduces more parameters reducing the predictability of the approach. Such freedom and the related impact on predictability are common to all supersymmetric models independently from the specific flavor group, but they are particularly relevant in the modular case, where the superpotential can be almost uniquely determined and where realistic values of the modular parameter are nonperturbative.

Even considering these limitations, flavor symmetries remain one of the few tools we have to address the flavor puzzle with the desired level of predictability and precision. In spite of the large number of relevant contributions to the field that we have tried to highlight in this review, we believe there are still many directions to examine. We do not know whether this approach will eventually succeed, but we are certainly encouraged by the present results to proceed and further explore new territory.

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