APS Medal for Exceptional Achievement in Research: Topology and other tools in condensed matter physics*

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This article is an extended version of a talk given at the March 2019 meeting of the American Physical Society, in connection with the award to the author of the 2019 APS Medal for Exceptional Achievement in Research. The article will summarize some of the author's work in areas linked to topological aspects of classical and quantum physics. Topics include quantum Hall effects, as well as the theory of melting in to dimensions and effects of vortex motion in a thin superconductor or a superfluid film.

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I. INTRODUCTION

This article is based on a talk I gave at the 2019 March Meeting of the American Physical Society, consequent to my receipt of the APS Medal for Exceptional Achievement in Research. As I understood it, the medal was a kind of lifetime achievement award, and I was selected to receive it because of an accumulation of work, on a variety of subjects, over a period of some 50 years, not for any one contribution. However, the citation did particularly mention contributions to the role of topology in both classical and quantum systems. So I decided that what I should do in my talk, and what I will do in this article, is to describe a few of the projects related to topology that I have worked on, which the committee may have had in mind when they decided to give me this award.

Topology has played a key role in our understanding of many physical systems, both in the classical regime and in the regime where quantum mechanics plays a dominant role. My own work, on some occasions, has made direct use of topological concepts, while in other examples, topological interpretations only became evident afterward. In all cases, topological concepts were most useful in combination with other theoretical methods and physical ideas and methods. As a result, I like to refer to topology as one tool, to be used with others in condensed matter physics. (My attitude here may reflect the fact that many aspects of my own work, not discussed in this article, have had no obvious connection with topology at all.)

Perhaps the earliest appearance of topological concepts in condensed matter physics concerned the role of dislocations in crystalline solids, and the way in which the existence of dislocations affects the strength and malleability of a metal under stress. A more recent, but familiar, concept in condensed matter physics is that of a vortex line in the order-parameter field of a superfluid or superconductor. The motion of vortex lines is well known as the mechanism for residual resistance or the gradual relaxation of a supercurrent in a superconducting circuit. Although the superfluid order parameter is obviously a quantum-mechanical concept at its roots, as it has the form of a quantum-mechanical wave function for the condensate of a macroscopic number of identical bosons, its time evolution can nevertheless be described in the language of classical mechanics. More recently, however, topological concepts have been employed to characterize quantum ground states of fermionic systems, which generally do not have a classical counterpart.

In the following sections, I will discuss some of my work on the quantum Hall effects, as well as on effects of vortex motion in a superfluid or superconductor and on the theory of defect-mediated melting of a two-dimensional crystal.

II. VORTICES IN A SUPERFLUID

A superfluid or superconducting state is characterized by an order parameter $\psi(\mathbf{r}, t)$, which is a complex-valued function of space and time. Although ψ has an overall phase that is an arbitrary quantity, the value of $\psi(\mathbf{r}, t)$ may be uniquely defined throughout the medium, if one fixes the electromagnetic gauge and one fixes the phase of ψ at a single point in space and time. It is generally required that $\psi(\mathbf{r}, t)$ should be a continuous differentiable function of $\mathbf{r}$ and t inside the medium.

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At any point where $\psi \neq 0$, we may define a phase $\phi(\mathbf{r}, t)$, such that $\psi(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|e^{i\phi(\mathbf{r},t)}$, but, in general, $\phi(\mathbf{r}, t)$ will be determined only modulo 2π. Within any small simply connected region of space-time that includes no points where $\psi = 0$, one can require ϕ to vary continuously, so derivatives of ϕ with respect to $\mathbf{r}$ or t can therefore be uniquely defined. More generally, if one integrates the gradient of ϕ around a closed contour C such that ψ does not vanish at any point on the contour, the requirement that $\psi(\mathbf{r}, t)$ must return to its original value at the end of the contour means that the accumulated change in ϕ must be a multiple of 2π. If C is a contour in space at a fixed time t, this may be written

$$\oint \nabla\phi \cdot d\mathbf{r} = 2\pi n, \tag{1}$$

where n is an integer. Moreover, if the contour C is deformed continuously in such a manner that it never passes through a point where $\psi = 0$ (where $\nabla\phi$ would be undefined), the integer n cannot change. This follows from the fact that the value of the integral could at most change continuously as the contour is deformed, assuming that $\psi(\mathbf{r}, t)$ can only vary continuously, while an integer such as n can only change discontinuously. The fact that the value of n must be constant under the imposed restrictions is one of the simplest examples of the results of topology.

A consequence of the above reasoning is that if n is nonzero for some contour C, it means that on any surface spanning the contour, there must be at least one point in space, inside the contour, such that $\psi = 0$. Otherwise, the contour could be continuously shrunk to a single point without changing n, which would be a contradiction. The condition $\psi = 0$ is actually two conditions as it requires that both the real and imaginary parts of ψ must vanish, and this will normally occur only on a space of codimension two inside a superfluid. Thus, in two dimensions, we would expect that ψ will vanish at most at a set of isolated points, denoted vortex points. In three dimensions, the locus of zeros of ψ will define a set of vortex lines or loops.

In two dimensions, we can define a charge n_v for a vortex according to the number of times the phase will change by 2π as one moves in a counterclockwise direction around a small circle enclosing the vortex. In three dimensions, we can assign an integer charge n_v to a vortex line according to to the number of times the phase changes by 2π as one moves around a small circle enclosing the vortex. The magnitude of n_v will be the same at all points on the vortex. The sign of n_v depends on the orientation one assigns to the vortex, but if this is chosen in a continuous manner, the sign will also be the same at all points on the vortex. A vortex line cannot end inside a superconductor or superfluid, so it must either form a closed loop embedded in the material or trace an open curve with ends at a surface. The lowest energy vortices will have $n_v = \pm 1$.

A. Persistent current in a ring

Consider a solid ring of superconducting material, and consider a closed contour C embedded in the superconductor, which circles once around the ring. The winding number n can be nonzero for the contour, even if there are no vortices in the superconducting material, because $\psi = 0$ in the hole through the center of the ring. Moreover, if throughout some period of time, $\psi \neq 0$ at all points within the superconductor, the value of n must be the same for any contour around the ring, and this value must be constant in time. (Note: The arguments in this section apply equally well for a superfluid contained in a closed loop of pipe.)

It is a general property of a superconducting ring, in the absence of an external applied magnetic field, that the equilibrium state at any given temperature T will have $n = 0$ and will carry no electrical current. States with $n \neq 0$ will necessarily have a higher free energy, but if n is not too large, there will exist a metastable ground state, given by the thermodynamic state with lowest free energy for the given value of n, which will generally be separated from states of smaller n by a free-energy barrier. For small n, the difference in free energy between the metastable state with winding number n and the ground state with winding number zero will be proportional to n^2, and the state will carry a persistent electrical current I proportional to n.

A persistent current can only decay if there are large fluctuations in $\psi(\mathbf{r}, t)$, such that for every contour around the ring there is a moment in time where $\psi = 0$ at some point on the contour, in such a way as to decrease the magnitude of the overall winding number n. One way for this to happen is for a short bent vortex line, with $n_v = \pm 1$, to first form at some place near the surface of the superconductor, with the two ends of the vortex terminating at the surface. The vortex line can then move across a cross section of the ring, if it expands enough in size so that its two ends can move along the surface of the ring on opposite sides of the cross section. Alternatively, a small vortex loop could nucleate at some point in the interior of the superconductor and expand across a cross section of the ring. In any case, the length of the vortex must become equal to or larger than the thickness of the ring. Since the energy to produce a vortex increases with the length of the vortex, the minimum free-energy cost ΔF for producing a vortex that can move across the ring will increase with the thickness of the ring.

If ΔF is large compared to the temperature T, the probability for finding such a fluctuation at any instant of time will fall off as $e^{-\Delta F/T}$, with temperature measured in units of energy. Thus, if ΔF is sufficiently large, the decay rate will be unmeasurably small. However, if the temperature is sufficiently close to the superconducting critical temperature T_c, and if the ring is sufficiently thin, ΔF may be small enough for the decay rate to be significant. Also, in the presence of a sufficiently strong magnetic field, there may be vortex lines present even in equilibrium, whose positions will typically be pinned by impurities or inhomogeneities. In this case, the decay rate for persistent current will be controlled by the free-energy barrier for a vortex line to become unpinned from local inhomogeneities, which will generally be much smaller than the barrier to create a vortex line in the absence of a magnetic field.

In principle, thermal fluctuations can produce vortices that cross the ring in the direction that increases the current as well as in the direction which decreases it. However, the

free-energy barrier will be smaller for fluctuations which decrease the current than for those which increase it. As a consequence of the principle of detailed balance, it can be shown that for a superfluid of bosons, the ratio between the rates r^+ and r^- for transitions that change the value of n by ± 1, respectively, must be given by

$$r^+/r^- = e^{-2\pi\tilde{I}\hbar/T}, \tag{2}$$

where $\tilde{I}$ is the particle current. (For electrons in a superconductor, where the order parameter describes a condensate of Cooper pairs, one should equate $\tilde{I}$ to $I/2e$, where I is the electric current.) If the current $\tilde{I}$ is large compared to $(T/\hbar)$, the uphill rate will be negligible compared to the downhill rate, but for smaller currents, both processes must be considered. In the limit of small currents, the net decay rate $d\tilde{I}/dt$, which depends on the difference $(r^- - r^+)$, will become proportional to $\tilde{I}$.

The motion of vortex lines can also lead to a nonzero resistance for current in a segment of superconducting wire that carries a current provided by contacts at the two ends. For a superconductor that is locally close to equilibrium, the rate of change of the phase ϕ at any point should be related to the voltage V at that point by the Josephson relation

$$\frac{\partial\phi}{\partial t} = \frac{2eV}{\hbar}. \tag{3}$$

Thus, in a steady state, there will be a voltage difference ΔV between two points on the wire if and only if there is a difference in the value of $\partial\phi/\partial t$ at the two points, which in turn will be given by 2π times the net rate dn/dt at which vortices move across the wire between the two points in question. This gives the relation

$$\Delta V = \frac{\pi\hbar}{e}\frac{dn}{dt}. \tag{4}$$

As for the persistent current, the rate dn/dt will be proportional to I for sufficiently small values of I, meaning that the resistance $R = \Delta V/I$ will approach a constant.

In order to estimate the decay rate of a persistent current, or the resistance of a superconducting wire, the first step is to estimate the free-energy barrier ΔF for producing a vortex and moving it across the sample. The decay rate or the electrical resistance should be at least roughly proportional to $e^{-\Delta F/T}$. For a more accurate calculation, however, an estimate of the preexponential factor is also necessary.

In summary, while topology tells us that the supercurrent in a ring cannot decay if one imposes the condition that the order parameter $\psi(\mathbf{r}, t)$ cannot pass through zero inside the material, this prohibition is never absolute. In the most interesting situations, zeros of the order parameter do occur at a small but finite rate, and the real challenge, not answered by topology alone, is to understand what the value of this rate is in any given situation.

B. Resistance in a thin superconducting wire

My first published work related directly to topological concepts in condensed matter systems was a 1970 paper with Dean McCumber, "Time-scale for resistance fluctuations in a thin superconducting wire" (McCumber and Halperin, 1970). Here we considered the case of a wire sufficiently thin that one could neglect variations in ψ across the thickness of the wire and could treat it as a function of a single spatial variable, x, and the time t. Thus, for a phase slip of 2π to occur, there should be an isolated point in space-time where $\psi(x, t) = 0$. We assumed a time evolution for ψ given by a time-dependent Ginzburg-Landau equation,

$$\left(\frac{\partial}{\partial t} + \frac{2ieV}{\hbar}\right)\psi = -\Gamma\frac{\delta F}{\delta\psi*} + \eta(x, t), \tag{5}$$

where F is the Ginzburg-Landau free-energy functional,

$$F = \int dx(-a|\psi|^2 + b|\psi|^4 + \gamma|d\psi/dx|^2), \tag{6}$$

and η is a Gaussian white noise source, chosen to give the correct distribution, $P[\psi] \propto e^{-F/T}$, in thermal equilibrium. The parameters b, γ, and Γ are assumed to be independent of temperature close to the bulk transition temperature T_C, while a varies $\propto (T_C - T)$. The free-energy barrier ΔF for producing a phase slip was previously known for the Ginzburg-Landau model, but we were able to calculate also the preexponential factor, as a function of the current in the wire, up to the classical critical current, where $\Delta F \to 0$.

Employing these formulas, we obtained a result for the resistance of a thin superconducting wire, which in the limit of small currents could be written as

$$R = \frac{4\sqrt{3}}{\pi^{1/2}}\frac{\hbar}{e^2}\frac{L}{\xi(T)}\frac{T_c - T}{T}\left[\frac{\Delta F}{T}\right]^{1/2} e^{-\Delta F/T}, \tag{7}$$

$$\Delta F = \frac{\sqrt{2}}{3\pi}\sigma H_c^2(T)\xi(T), \tag{8}$$

where L is the length of the wire, σ is the cross-sectional area, $\xi(T) \propto (T_c - T)^{-1/2}$ is the Ginzburg-Landau coherence length, and $H_C(T) \propto (T_C - T)$ is the critical magnetic field for the bulk material.

C. Resistance in a two-dimensional superconductor or a superfluid film

In order to relax a supercurrent in a two-dimensional film, it is necessary to move a vortex point across the film. However, an isolated vortex will have a logarithmically infinite energy in an isolated film, so one would expect to find no free vortices at low temperatures. On the other hand, a pair of vortices of opposite sign separated by a distance s has a finite energy, proportional to $\log s$, so we would expect to find a finite density of bound pairs of various separations at any finite temperature. As noted by Kosterlitz and Thouless, this background of thermally excited pairs will lead to a downward renormalization of the stiffness constant for fluctuations in the

phase of the order parameter with increasing temperature, and a consequent reduction in the logarithmic energy cost of adding an additional pair with a large separation s (Kosterlitz and Thouless, 1973; Kosterlitz, 1974). Above a certain temperature, commonly denoted as T_{KT}, the logarithmic interaction will be reduced to the point where it is no longer strong enough to prevent the formation of free vortices. Using a novel renormalization-group technique, Kosterlitz argued that slightly above T_{KT}, there would be a small density n_f of free vortices, given by

$$n_f \sim e^{-b/(T-T_{\mathrm{KT}})^{1/2}}, \tag{9}$$

where b is a constant (Kosterlitz, 1974).

Since the motion of a bound vortex-antivortex pair cannot cause a phase slip, dissipation above T_{KT} should be proportional to n_f. Below T_{KT}, there should be no dissipation in the limit of infinitesimal current densities, but there can still be dissipation for finite currents. This is because in the presence of a current density j, a vortex pair can become unstable if its separation s is larger than a critical value inversely proportional to j, and even below T_{KT}, the density of pairs with separation larger than any specified value will be nonzero in thermal equilibrium.

In a series of papers with Vinay Ambegaokar, David Nelson, and Eric Siggia, published in 1978–1980, we explored, in some detail, how vortex motion should affect the transport properties of a thin superconductor or superfluid film close to T_{KT} (Ambegaokar *et al.*, 1978, 1980; Halperin and Nelson, 1979). Among other results we found that for a superconducting film below T_{KT}, the voltage drop generated by a small current density j should vary as $|j|^{\alpha}$, where the exponent α has the value 3, just below T_{KT}, and increases with decreasing temperature, approaching infinity for $T \to 0$. We also were able to provide estimates of quantities such as $\alpha(T)$ and the constant b, and the overall prefactor determining the resistivity above T_{KT}, in terms of other measurable parameters. Among the questions we explored in the case of superfluid He films were effects of vortex motion on dissipation in an oscillating film.

III. DEFECT-MEDIATED MELTING IN TWO DIMENSIONS

As was noted by Kosterlitz and Thouless, there are strong analogies between the melting of a crystal in two dimensions and the superfluid transition, which they studied in detail. For example, one may define an order parameter for translational order for a two-dimensional crystal by

$$\psi_{\mathbf{G}}(\mathbf{r}) = e^{i\mathbf{G}\cdot\tilde{\mathbf{r}}}, \tag{10}$$

where $\tilde{\mathbf{r}}$ is the location of the atom closest to $\mathbf{r}$. As in the case of the order parameter of a two-dimensional superfluid, the correlation function for this order parameter is expected to fall off as a power law at large separations $\mathbf{s}$, due to the effects of thermally excited phonons. That is,

$$C_{\mathbf{G}}(\mathbf{s}) \equiv \langle \psi^*_{\mathbf{G}}(\mathbf{r})\psi_{\mathbf{G}}(\mathbf{r}+\mathbf{s})\rangle \sim s^{-\eta_{\mathbf{G}}}, \tag{11}$$

where the exponent $\eta_{\mathbf{G}}$ goes to zero for $T \to 0$ and approaches a finite maximum value at the melting temperature. Furthermore, there can be topological defects in the crystal, known as dislocations, whose presence could destroy this quasi-long-range order, but which have an energy that diverges logarithmically, if they are far from the system boundary and from neutralizing dislocations of opposite sign, similar to vortices in the superfluid.

More precisely, a dislocation is characterized by a discrete Burgers vector $\mathbf{b}$, which is a vector on the Bravais lattice of the perfect crystal, and which measures the number of extra steps one needs to take in order to return to one's original position if one moves around the dislocation in a counterclockwise direction by steps on the lattice. (See Fig. 1.) The energy of an isolated pair of dislocations with opposite Burgers vectors has the form

$$E = Kb^2 \ln|\mathbf{s}/a| + C(\mathbf{b},\mathbf{s}), \tag{12}$$

where $\mathbf{s}$ is the separation between dislocations, K depends on the elastic constants of the crystal, and C remains finite in the limit of large s. Unlike the case for vortices in a superfluid, however, the value of C depends on the direction of $\mathbf{s}$ (relative to the orientation of $\mathbf{b}$).

My attention was directed to the problem of melting in two dimensions by my Harvard colleague David Nelson, in 1978. This led to a series of papers on the topic, in collaboration with Nelson, and later with others. One early observation we made, which was found independently by Peter Young at U. C. Santa Cruz, was that the angle-dependent terms in the interaction would lead to quantitative differences in the behavior of quantities such as the elastic constants of the crystal as it approaches the melting point, compared to the behavior of the corresponding stiffness constant in the superfluid case (Young, 1979).

More significantly, however, we argued that if the renormalization group holds, melting in two dimensions should actually occur in two stages, with separate transition temperatures T_{C1} and T_{C2} (Nelson and Halperin, 1979). For temperatures T in the range $T_{C1} < T < T_{C2}$, one should find a new liquid-crystal phase, which we termed the *hexatic* phase, having short-range translational order, but quasi-long-range orientational order. (We assume, here, that the low-temperature crystal phase has the sixfold orientational symmetry of a simple triangular lattice.) Specifically, we defined an orientational order parameter as

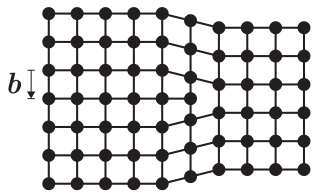

FIG. 1. Dislocation in a two-dimensional crystal. A square lattice is illustrated for ease of visualization. The Burgers vector $\mathbf{b}$, which characterizes the dislocation, is the missing lattice vector when one follows a path around the defect consisting of, say, n steps in the x direction, n steps in the y direction, n steps in the $-x$ direction, and n steps in the $-y$ direction.

$$\Psi(\mathbf{r}) = e^{6i\theta(\mathbf{r})}, \tag{13}$$

where $\theta(\mathbf{r})$ is the orientation, relative to some fixed axis, of the bond between the two atoms nearest to point $\mathbf{r}$. In the hexatic phase, the translational correlation functions $C_{\mathbf{G}}$ fall off exponentially with separation, but the correlation function for Ψ falls off only as a power law:

$$C_\Psi(\mathbf{s}) \equiv \langle \Psi^*(\mathbf{r})\Psi(\mathbf{r}+\mathbf{s})\rangle \sim s^{-\eta_\Psi}. \tag{14}$$

The hexatic phase is made possible because unbound dislocations are much less effective at destroying orientational order than they are at destroying translational order. A small density of dislocations, as one expects to find just above T_{C1}, is enough to cause exponential decay of $C_{\mathbf{G}}$, with a decay length proportional to the typical distance between unbound vortices. However, the stiffness constant for gradients in the orientational order parameter of the hexatic will be large when the distance between free dislocations is large, which leads to a result that the exponent η_Ψ for falloff of C_Ψ will go to zero, when $T \to T_{C1}$ from above.

In order to destroy quasi-long-range orientational order, we need to have a proliferation of defects such as disclinations or grain boundaries, which can cause large angular rotations from one part of the sample to another. (See Fig. 2 for an illustration of a disclination.) In the crystal phase, disclinations have a very large free energy, so the possibility of free disclinations can be completely neglected. In the hexatic phase, however, the energy of an isolated disclination diverges only logarithmically in the size of the system, while the energy of a pair with opposite sign is finite and proportional to the logarithm of the separation s. In the hexatic phase, away from T_{C1}, the presence of a finite density of bound disclination pairs in thermal equilibrium leads to a renormalization of the stiffness constant, completely analogous to the renormalization of the phase-stiffness constant by vortex pairs in a superfluid. Thus, one finds that the maximum allowable value for η_Ψ is $1/4$, and the renormalization-group analysis predicts that T_{C2} is the temperature at which this maximum value is attained. Moreover, the nature of the singularities as one approaches T_{C2} should be identical to corresponding singularities at the Kosterlitz-Thouless superfluid transition. Above T_{C2} we have an isotropic phase, where the orientational correlation function C_Ψ decays exponentially with s, with decay length that diverges strongly for $T \to T_{C2}^+$. We remark that a dislocation in a crystal may be regarded as a tightly bound pair of disclinations of opposite sign.

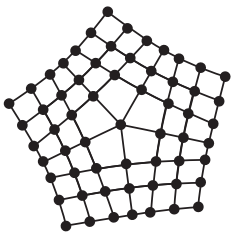

FIG. 2. Disclination in a square lattice. Following a closed path counterclockwise around the defect, the crystal axes are rotated by an angle of $\pi/2$. The disclination may also be characterized as having an extra quadrant of material. A disclination of the opposite sign (not shown), would have a missing quadrant of material and a rotation angle of $-\pi/2$.

Of course, the two-stage melting scenario with a hexatic intermediate phase, predicted by the renormalization-group analysis, could be preempted by a direct first-order transition between the crystal and isotropic liquid phases. Numerical simulations and experiments with colloids suggest that both possibilities can occur, depending on microscopic details, but the hexatic phase does exist, in suitable cases, in a narrow range of density or temperature between the solid and liquid phases (Gasser *et al.*, 2010; Kapfer and Krauth, 2015). [See also Li and Ciamarra (2020), which explores the effects of replacing hard disks by polygonal shapes, and of an attractive van der Waals interaction.] Also, as noted by Robert Birgeneau and David Litster, a hexatic phase, with true long-range orientational order, can occur in three-dimensional smectic liquid crystals (Birgeneau and Litster, 1978; Pindak *et al.*, 1981).

IV. THE QUANTUM HALL EFFECTS

The questions of melting of a crystal or the behavior of a superfluid in two dimensions, referring to the static or dynamical behavior of a system at or near thermal equilibrium at a finite temperature T, are essentially problems of classical statistical physics. There is another set of problems, however, in which the focus is on features of the quantum ground state at $T = 0$, where topology may also play a role. Some of these problems resemble classical problems in that the objects under consideration are required to be a continuous function of space-time in an imaginary-time path integral formulation, and such problems are often closely related to a classical problem in one higher space dimension. Applications of topology to such problems are likely to be focused on the behavior of variables that are functions of position in some region of space-time, and are required to vary continuously throughout the region, subject to certain constraints, similar to the examples discussed above.

In other cases, however, the quantum problem may have no classical analog, and key topological properties refer to the behavior in a more abstract space, such as the momentum variable in the Brillouin zone of a crystal, or the behavior while varying magnetic flux quanta through the holes in a torus. Such is the case in problems concerning the quantum Hall effects. Yet even there, some topological features in real space may be useful for understanding what is going on.

The label "quantum Hall effects" refers to a wide range of peculiar phenomena found in two-dimensional electron systems in strong magnetic fields at low temperatures. The phenomena reflect the existence of a series of peculiar states of matter that are strictly defined only at $T = 0$, but the phenomena can still be observed in laboratory experiments at finite temperature, in many cases with extremely high precision.

A. Early work

The first quantum Hall effect to be studied was the *integer quantized* Hall effect, observed experimentally by von

Klitzing, Dorda, and Pepper (1980). What they found was that in two-dimensional electron systems of sufficiently high quality, at low-enough temperatures, the electrical transport properties could be characterized by a conductivity tensor with

$$\sigma_{xx} = \sigma_{xx} = 0, \qquad \sigma_{yx} = -\sigma_{xy} = \nu e^2/h, \tag{15}$$

where ν was precisely an integer, over a finite interval of magnetic field strengths. The *fractional* quantized Hall effect was discovered by Tsui, Stormer, and Gossard (1982), when they observed additional Hall plateaus, in samples of very high quality, in even stronger magnetic fields and lower temperatures, where the value of ν was a simple rational fraction rather than an integer.

Theoretical predictions that came close to predicting the integer quantized Hall effect had been published already by Ando, Matsumoto, and Uemura (1975). However, the first truly elegant explanation of the exactness of the integer quantized Hall effect was supplied by Robert Laughlin, in a paper published in 1981 (Laughlin, 1981). There was an element of topology in this, as he considered the response of an electron system in a geometry equivalent to an annulus.

My attention was directed to the integer quantized Hall effect, and to Laughlin's explanation, by a phone call that I received from Gloria Lubkin, an editor at Physics Today, in 1981. I became fascinated by Laughlin's explanation, but it raised some vexing questions in my own mind. There were several features that seemed counterintuitive. For example, Laughlin's argument implied the existence of two-dimensional electron states, and even one-dimensional states, that did not get localized by disorder, which was contrary to the folklore at the time.

The integer quantized Hall effect, for a weakly interacting system, occurs when the Fermi level sits in the energy gap between two Landau levels. In the presence of disorder, there may be electron states in the gap, but as long as the states are localized they do not contribute to transport. That states could be easily localized by disorder in the presence of a large magnetic field seemed natural. I realized, however, that Laughlin's argument required that there be at least one extended state below the Fermi level, if the quantized Hall conductance was not zero (Halperin, 1982). More importantly, it required that there be states *at the Fermi level* at the sample boundaries that are extended in the direction parallel to the boundary. These one-dimensional states move in only one direction around the edge, and they could be understood as the quantum analog of skipping orbits that have been discussed previously in the context of three-dimensional metals, where their importance was not as great. Because they move in only one direction, electrons in these one-dimensional states cannot be backscattered or localized by impurities, assuming that the sample is wide enough so that there is negligible overlap between wave functions at opposite edges.

Because tunneling from one edge to another is negligible in a wide sample, the system can reach a state of local equilibrium, where the two edges have a different electrochemical potential. Note that the electrochemical potential includes the electron *chemical* potential as well as the electrostatic potential, and the electrochemical potential difference between the two edges is precisely what is measured by an ideal voltmeter connected to the edges. Laughlin showed by an argument using gauge invariance that the net electric current I transported around the annulus in these circumstances is given by $I = GV$, where at least for weakly interacting electrons, the dimensionless quantity $\nu = Gh/e^2$ must be precisely an integer.

A physical argument for the exactness of a quantized Hall effect can be put in a somewhat different form, which applies equally to the fractional case as to the integer. Consider the split annular geometry illustrated in Fig. 3. The material in the left half of the annulus is an ideal system without disorder. It is supposed to have a clean energy gap separating the many-body ground state from all excited states. Moreover, in the absence of impurities, the Hall conductance should be precisely related to the electron density n by the classical relation $G_H = ne/B$, which means that ν is the Landau-level filling factor. On the right side of the annulus, we suppose there may be localized states at the Fermi energy due to disorder. However, we require that there are no extended states at the Fermi energy crossing between the two edges of the annulus anywhere in the sample, including at the interface line separating the left and right halves.

Now, imagine that we initiate a state with a charge imbalance between the inner and outer edge of the annulus. Since there is no leakage of charge between the edges, each edge will reach a state of local equilibrium, where there may be different voltages on the two edges, but the voltage on a given edge will be precisely the same in the two halves of the annulus. By charge conservation, I must also be the same in the left and right halves of the annulus. Hence the Hall conductance must be exactly the same in the disordered region as in the disorder-free region (Halperin, 1982).

Of course, if the disorder is strong enough, or the electron density is very different in the two regions, the Hall conductance can be different in the two halves. In that case, however, there will inevitably be extended states joining the inner and outer edges of the sample at the junction between the two regions. Then, one cannot set up a steady state with different

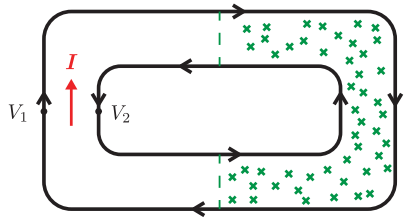

FIG. 3. Split annular geometry filled with a quantized Hall state. The left half of the annulus is occupied by ideal material in a quantized Hall state with filling factor ν. The right half contains a disordered sample. If there are no extended states at the Fermi level in the bulk of the dirty material or at the interface between the two halves, then, at $T = 0$ one can reach a steady state in which there are different voltages, V_1 and V_2, on the inner and outer edges of the annulus, but the voltage on a given edge is constant around the annulus. By conservation of charge, the current I must also be constant around the annulus, so it must be given by $I = \nu e^2(V_1 - V_2)/\hbar$ in both regions.

electrochemical potentials on the inner and outer edges, as was assumed above.

A more sophisticated use of topology for quantized Hall systems came with the work of Thouless and collaborators, who considered the Bloch wave functions for electrons in a magnetic field and a periodic lattice (Thouless *et al.*, 1982). They showed that if one tries to impose the requirement that the wave functions in the occupied bands vary continuously with the Bloch wave vector throughout the Brillouin zone, one may encounter a matching problem at the zone boundary, characterized by an integer topological invariant known as the *first Chern number*. They showed that if the occupied bands are separated from the empty bands by an energy gap, the total Chern number of the occupied bands will coincide with the quantized Hall number ν for the system. In another application, Niu, Thouless, and Wu (1985) showed how the argument could be extended to fractional quantized Hall states, even in the presence of disorder, by considering the Chern number for the many-body wave function on a torus, as one alternately adds and subtracts an integer number of flux quanta through the two holes in the torus. For a fractional quantized Hall state of the form $\nu = p/q$, one must take into account the q-fold degeneracy of the ground state on a torus, so one has to add q flux quanta to come back to the same physical state.

I did not personally make use of the Chern number concept in most of my own work on quantized Hall systems. However, I did use a generalization of the argument of Thouless *et al.* (1982) in 1987, when I analyzed the possible behaviors of a system of noninteracting electrons in a three-dimensional periodic lattice. The key result was that if there is an energy gap separating the filled energy bands from the empty ones, the symmetric part of the conductivity tensor must vanish, and the antisymmetric part must be of the form

$$\sigma_{ij} = \epsilon_{ijk}\mathbf{G}_k e^2/\hbar, \tag{16}$$

where $\mathbf{G}$ is a reciprocal lattice vector of the crystal and ϵ_{ijk} is the unit antisymmetric tensor. This is just the conductivity tensor that would result if there were an integer quantized Hall state in each crystal layer perpendicular to $\mathbf{G}$, and no conduction between the planes. [This work was further developed in Kohmoto, Halperin, and Wu (1992).]

As indicated above, a sufficient condition for the existence of a quantized Hall state with fractional value of ν is that an interacting electron system without impurities would form a stable ground state, with an energy gap, at the corresponding Landau-level filling fraction. The most important questions, then, are why should there be stable states at all in a partially full Landau level, and if there are, what fractions are allowed, and what should be the relative sizes of their energy gaps? The first question was answered by Laughlin, who introduced an explicit wave function for the states at $\nu = 1/3, 1/5, \ldots$, which he argued would be excellent approximations to the true ground state, and which would be separated by an energy gap, which he could estimate, from any excited states (Laughlin, 1983). Extensions of these arguments, using particle-hole symmetry within a Landau level, could explain states at $\nu = 2/3, 4/5, \ldots$, and including the second spin state, at $\nu = 6/5, 4/3, 5/3, 9/5, \ldots$. But quantized Hall states were soon seen at many other odd-denominator fractions, as well.

In 1983, I wrote a paper for the conference proceedings of a meeting of the European Physical Society, published in Helvetica Physica Acta, which proposed various generalizations of Laughlin's wave functions that I argued might be good approximations to the ground states at certain other filling fractions with odd denominator (Halperin, 1983). I also considered generalizations to states with two distinguishable electron species, such as the two spin states in a Landau level that is not completely spin polarized. I argued that at certain filling fractions, such as $\nu = 2/5$, such a state would have a lower Coulomb energy than the ground state of a fully spin-polarized system. Furthermore, I argued that given the very small Zeeman energies for electrons in GaAs, one should consider the possibility of unpolarized quantized Hall states. Indeed, unpolarized states have been seen in GaAs, along with transitions to fully polarized states with increasing magnetic fields at the same filling fraction (Clark *et al.*, 1989; Eisenstein *et al.*, 1989).

In the 1983 paper, I also argued strongly that even-denominator fractional quantized Hall states should be possible, in principle, for a properly designed Hamiltonian, though I could not propose a realistic model where this would be the case. In 1987, a fractional quantized Hall plateau was observed at $\nu = 5/2$ in GaAs (Willett *et al.*, 1987). The precise nature of the ground state in this case is still a matter of debate, but it is clearly not of the type that I considered in my 1983 paper. However, some even-denominator states seen in two-component systems may be described by trial wave functions from that paper (Liu *et al.*, 2019).

Laughlin showed that a fractional quantized Hall state with $\nu = p/q$ will necessarily have quasiparticle excitations with electric charge $\pm e/q$. (In principle, it could also have additional quasiparticles whose charges are submultiples of these.) In 1984, I argued that Laughlin's quasiparticles should also display fractional statistics (Halperin, 1984). The concept of particles with fractional statistics, termed *anyons*, had been introduced earlier, but this was the first example to be found in an actual physical system. [The idea that charged excitations in the $\nu = 1/3$ state obey fractional statistics was independently developed by Arovas, Schrieffer, and Wilczek (1984).]

One definition of fractional statistics is that if a pair of identical quasiparticles is interchanged adiabatically, the many-body wave function will be multiplied by a phase factor other than ± 1, which depends on the direction of interchange and the number of other quasiparticles enclosed by the trajectories. I used a different definition, however, which effectively compared the number of quantum states available to a collection of quasiparticles in a given area with what would have been available if the quasiparticles were fermions, with the same fractional charge, in the lowest Landau level. One could see, using trial wave functions, how, e.g., the $\nu = 2/5$ state could be constructed by adding $e/3$ quasiparticles to the $\nu = 1/3$ state in a Laughlin-type effective wave function, provided one took the fractional statistics into account. I showed that by iterating this construction one could generate arbitrary fractions with odd denominator, and that if one treated the quasiparticles as point anyons interacting via a Coulomb repulsion, one could

estimate the energy gaps and plateau widths at various fractions. It turned out, however, that these estimates were not very useful, because the quasiparticles are actually far from being point charges. Much better estimates were obtained by Jain (1989) using the "composite fermion" approach he developed.

It was shown later that some quantum Hall states could have quasiparticles with non-Abelian statistics (Moore and Read, 1991). [See also Wen (1991).] This means that for a configuration with N quasiparticles, localized at points sufficiently far from each other, there will be an essentially degenerate Hilbert space of ground states, whose dimension grows exponentially with N. If various quasiparticles are moved around each other and/or exchanged, so that the final set of locations is the same as the original one, then the result will be a unitary transformation in the degenerate Hilbert space, with a final state that depends on the order of interchanges. Moreover, the transformation is a topologically robust quantity, determined by the sequence of braiding operations. It is believed that the state at $\nu = 5/2$ in GaAs has non-Abelian quasiparticles with non-Abelian quasiparticles of the Moore-Read type, though the precise form of the ground state is still under debate (Morf, 1998; Banerjee *et al.*, 2018; Willett *et al.*, 2019; Halperin and Jain, 2020).

Non-Abelian statistics may also be realized in other systems, such as hybrid structures of superconductors and semiconductors. There has been great interest in creating such systems and in verifying their properties, since, in principle, such systems, if perfected, could be used for topologically protected quantum computing (Nayak *et al.*, 2008).

In order to completely characterize the topological class of a fractional quantized Hall state, one needs to know more than the charges of the elementary excitations and their fractional or non-Abelian statistics. Another topological quantum number is the shift number, which characterizes a finite mismatch between the particle number and ν times the number of flux quanta in the state on a large sphere or surface of higher genus (Wen and Zee, 1992). The full topological classification for Abelian fractional quantized Hall states is known (Wen, 1995).

B. Later work

Since the early 1990s, much of my work on quantum Hall systems was not so much focused on the integer or fractional quantized Hall effects, but rather on what I like to call the *unquantized quantum Hall effect*. This refers to fractions such as $\nu = 1/2$ and their vicinities, where no plateau is seen in the Hall conductance. Rather, the Hall conductance is found to vary smoothly with magnetic field or electron density, as in a classical Hall conductor, and the longitudinal conductance appears to remain finite, for $T \to 0$, with a value dependent on residual disorder and other details of the sample. Nonetheless, there are anomalies around these filling factors, which are not trivial. The first to be observed was an anomaly in propagation of surface acoustic waves (Willett *et al.*, 1990).

In 1993, Patrick Lee, Nicholas Read, and I published a paper which proposed an explanation based on a unitary transformation of the electrons to a system of composite fermions interacting with a Chern-Simons gauge field (Halperin, Lee, and Read, 1993). In the mean-field approximation, composite fermions in the ground state at $\nu = 1/2$ behave like free fermions, and they fill a Fermi sea, with the Fermi radius determined by the electron density. The anomalous behavior of surface-acoustic-wave propagation is a reflection of the fact that in a clean system, excited composite fermions can travel in straight lines over distances greater than the acoustic wavelength. Slightly away from $\nu = 1/2$, composite fermions move in large effective cyclotron orbits, with a radius inversely proportional to the deviation of the magnetic field from the value at $\nu = 1/2$. As the magnetic field is varied, this leads to several observable oscillatory effects, approximately reflecting commensurability conditions between the effective cyclotron radius and the wavelength of an acoustic wave or of an imposed potential modulation. (Note that these distances are all much larger than the cyclotron radius for a bare electron in the applied magnetic field.)

Although the Halperin-Lee-Read theory has provided explanations for many observed phenomena, there remain some subtle unanswered questions. These are particularly related to the compatibility of the theory with the requirements of particle-hole symmetry, when the bare electron mass is taken to zero, while the electron interaction is held constant, so there is no mixing between Landau levels (Son, 2015, 2018; Wang *et al.*, 2017). Thus, the theory of a half-filled Landau level has been a topic of active research in recent years. The extent to which topological concepts may be helpful in sorting out the remaining issues is not clear.

Other problems in quantum Hall systems that have attracted my attention in recent years have included several related to interferometry experiments, where charged particles are allowed to tunnel between opposite edges of a quantized Hall system at two or more constrictions (Stern and Halperin, 2006; Halperin *et al.*, 2011; Wei *et al.*, 2017). I have also been interested in various phenomena observed in systems with two closely spaced parallel layers, where the Coulomb interaction between electrons in different layers can be comparable to the interaction between neighboring electrons in the same layer (Eisenstein, 2014; Liu *et al.*, 2017, 2019).

V. OUTLOOK

Topological ideas, inspired, in part, by the examples in quantum Hall systems, have played a major role in the science of condensed matter systems in recent times. Applications have included the classification of distinct quantum states of matter at $T = 0$, in three as well as one and two dimensions (Bernevig and Hughes, 2013; Senthil, 2015; Wen, 2017). The first Chern number is only one of the many topological invariants that have been employed in the characterization of quantum materials. As a whole, the field of topological quantum materials is certainly one of the most active areas of theoretical and experimental physics today (Vergniory *et al.*, 2019).

Topological ideas, related to the ones developed for quantum Hall systems, have also been applied to light waves in photonic structures (Lu, Joannopoulos, and Soljačić, 2014), to vibrational modes in constrained systems (Kane and Lubensky, 2014; Stenull, Kane, and Lubensky, 2016), to trapped ultracold atoms (Fetter, 2009; Cooper, Dalibard, and

Spielman, 2019), and to systems subject to a periodic time-dependent perturbation (Nathan and Rudner, 2015; Martin, Refael, and Halperin, 2017).

The area of quantum Hall systems itself continues to be quite exciting. One productive direction of research has been the search for new systems in which to study quantized Hall effects. Over the last decade, fractional quantized Hall states, including even-denominator states, have been observed in a number of new materials, including ZnO, monolayer and bilayer graphene, and Coulomb-coupled graphene double layers (Halperin and Jain, 2020). The quantized anomalous Hall effect, in which an integer quantized Hall conductance can be observed in zero magnetic field, has been observed in thin samples of the three-dimensional topological insulator $(Bi_xSb_{1-x})_2Te_3$, doped with Cr or V (Bestwick *et al.*, 2015; Chang *et al.*, 2015). Here, a combination of ferromagnetism, which produces the necessary broken time-reversal invariance, and spin-orbit coupling leads to a two-dimensional band structure with an energy gap and a nonzero Chern number.

New experimental techniques, and improved accuracy, along with better samples, have allowed more precise tests of our theoretical understanding. Results have confirmed theoretical expectations in many cases, but have also raised questions in others. As one important example, new thermal techniques have confirmed predictions for the quantized thermal conductance for many fractional quantized Hall states, but results at $\nu = 5/2$ have confounded expectations, and so far are lacking in a convincing explanation. See, e.g., Banerjee *et al.* (2018), Ma and Feldman (2019), and Steven *et al.* (2020) and references therein.

Many of these developments are discussed in detail in various chapters of a book that I have edited, together with Jainendra Jain, entitled *Fractional Quantum Hall Effects: New Developments*, published in June, 2020 by World Scientific Press (Halperin and Jain, 2020).

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