# Quantum steering

Roope Uola

*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany and Département de Physique Appliquée, Université de Genève, CH-1211 Genève, Switzerland*

Ana C. S. Costa

*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany and Department of Physics, Federal University of Paraná, 81531-980 Curitiba, Paraná, Brazil*

H. Chau Nguyen and Otfried Gühne

*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany*

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Quantum correlations between two parties are essential for the argument of Einstein, Podolsky, and Rosen in favor of the incompleteness of quantum mechanics. Schrödinger noted that an essential point is the fact that one party can influence the wave function of the other party by performing suitable measurements. He called this phenomenon quantum steering and studied its properties, but only in recent years has this kind of quantum correlation attracted significant interest in quantum information theory. In this paper, the theory of quantum steering is reviewed. First, the basic concepts of steering and local hidden state models are presented and their relation to entanglement and Bell nonlocality is explained. Then various criteria for characterizing steerability and structural results on the phenomenon are described. A detailed discussion is given on the connections between steering and incompatibility of quantum measurements. Finally, applications of steering in quantum information processing and further related topics are reviewed.

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## I. INTRODUCTION

### A. Overview

In 1935, Einstein, Podolsky, and Rosen (EPR) presented their famous argument against the completeness of quantum mechanics (Einstein, Podolsky, and Rosen, 1935). In this argument, a two-particle state is considered, where one party can measure the position or momentum, and the correlations of the state allow one to predict the results of these measurements on the other party if the same measurement is performed there. The EPR argument led to long-lasting discussions, but already directly after its publication Schrödinger noted this phenomenon in the argument: The first party can, by choosing the measurement, steer the state on the other side into an eigenstate of position or momentum. This cannot be used to transmit information, but Schrödinger still considered it to be *magic*.

The early works of Schrödinger (1935, 1936) did not receive much attention; see also Sec. V.M. This changed in 2007, when a formulation in the language of quantum information processing was given and systematic criteria were developed (Wiseman, Jones, and Doherty, 2007). In the modern view, steering denotes the impossibility to describe the conditional states at one party by a local hidden state model. As such, steering denotes a quantum correlation situated between entanglement and Bell nonlocality. In the following years, the theory of steering developed rapidly. It was noted that steering provides a natural formulation for discussing quantum information processing, if for some of the parties the measurement devices are not well characterized. Also, the concept of steering helped scientists to understand and answer open questions in quantum information theory. An important example here is the construction of counterexamples to the Peres conjecture, which states that certain weakly entangled states do not violate any Bell inequality. Finally, steering turned out to be closely related to the concept of joint measurability of generalized measurements in quantum mechanics. More precisely, measurements that are not jointly measurable are exactly the measurements that are useful to reveal the steering phenomenon. This has sparked interest in the question in which sense measurements in quantum mechanics can be considered a resource.

This review aims to give an introduction to the concept and applications of quantum steering. Starting from the basic definitions, we explain steering criteria and structural results on quantum steering. We also discuss in some detail related concepts, such as quantum entanglement and the joint measurability of observables. We focus on the conceptual and theoretical issues and on the finite-dimensional case and mention experiments only briefly. For discussing quantum steering, the tool of semidefinite programming has turned out to be useful. Concerning this, we discuss only the main formulations; concrete examples and algorithms can be found in a different review (Cavalcanti and Skrzypczyk, 2017).

As mentioned, quantum steering is related to several other central concepts in quantum theory, so it may be useful to the reader to mention related relevant literature here. First, a review on the quantitative aspects of the EPR argument was given by Reid *et al.* (2009). The phenomenon of entanglement was extensively discussed by Horodecki *et al.* (2009), and Gühne and Tóth (2009) addressed methods to characterize it. Brunner *et al.* (2014) gave a detailed overview on Bell inequalities and their applications. Finally, Busch *et al.* (2016) developed the theory of quantum measurements in depth.

The structure of this article is as follows: In the remainder of this introduction, we explain the idea of quantum steering and the main definitions. We also provide a short comparison with quantum entanglement and Bell nonlocality, as this is central to the further discussion.

Section II presents different methods for the detection of quantum steering. We discuss in detail how steerability can be inferred if some correlations or the complete quantum state is known. These methods are then used in Sec. III, where we describe key conceptual aspects of steering. This includes the discussion of one-way steering, the superactivation of steering, the steerability of bound entangled states, and the construction of steering maps. In addition, we then present the relation to other types of quantum correlations in detail.

Section IV deals with the connections between steering and the joint measurability of observables. We explain the concept of joint measurability and its various connections to steering. These connections allow one to transfer results from one topic to the other. Section V describes different applications of steering as well as further topics. This includes applications in quantum key distribution and randomness certification, but also topics like multiparticle steering, steering of Gaussian states, the steering ellipsoid, and historical aspects of steering. Finally, Sec. VI presents the conclusion and poses some open questions.

### B. Steering as a formalization of the EPR argument

Let us start by recalling the EPR argument. Originally, EPR used the position and momentum of two particles to explain their line of reasoning (Einstein, Podolsky, and Rosen, 1935), but in the simplest setting the argument can be explained with two spin-1/2 particles or qubits (Bohm, 1951). Consider two particles that are in different locations and that are controlled by Alice and Bob; see Fig. 1. They are in the so-called singlet state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{1}$$

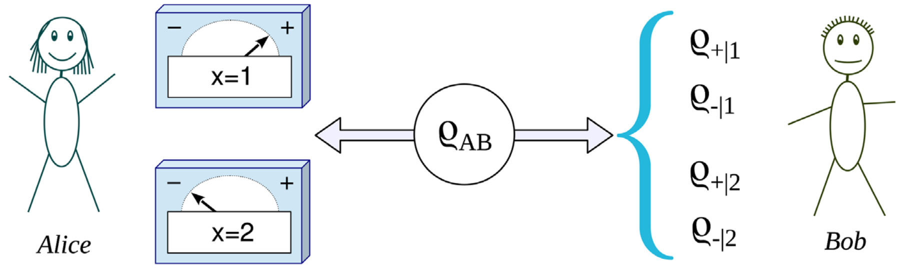


FIG. 1. Schematic description of the steering phenomenon: A state $\varrho_{AB}$ is distributed between two parties. Alice performs a measurement (labeled as $x \in \{1,2\}$) on her particle and obtains the result $a = \pm$. Bob receives the corresponding unnormalized conditional states $\varrho_{a|x}$. If Bob cannot explain this assemblage of states by assuming preexisting states at his location, he has to believe that Alice can influence his state from a distance.

where $|0\rangle = |z^+\rangle$ and $|1\rangle = |z^-\rangle$ denote the two possible spin orientations in the $z$ direction. If Alice measures the spin of her particle in the $z$ direction and obtains the result $+1$ (or $-1$), then, due to the perfect anticorrelations of the singlet state, Bob's state will be in either the state $|1\rangle$ or the state $|0\rangle$. Similarly, if Alice measures the spin in the $x$ direction, Bob's conditional states are given by $|x^+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ if Alice's result is $-1$ and $|x^-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ for the result $+1$.

By choosing her measurement setting, Alice can predict with certainty the values of a $z$ or $x$ measurement on Bob's side. According to EPR, this means that both observables must correspond to "elements of reality," as each of them can be predicted in principle with certainty and without disturbing the system. This raises problems if one assumes that the wave function is a complete description of the physical situation, since the corresponding observables do not commute and the quantum mechanical formalism does not allow one to assign simultaneously definite values to both of them. Consequently, EPR concluded that quantum mechanics is incomplete.

Alice cannot transfer any information to Bob by choosing her measurement directions since Bob's reduced state is independent of this choice. But, as Schrödinger noted, she can determine whether the wave function on his side is in an eigenstate of the Pauli matrix $\sigma_x$ or $\sigma_z$. This *steering* of the wave function is, in Schrödinger's own words, "magic" as it forces Bob to believe that Alice can influence his particle from a distance; see also Sec. V.M for details.

The situation for general quantum states other than the singlet state can be formalized as follows (Wiseman, Jones, and Doherty, 2007): Alice and Bob share a bipartite quantum state $\varrho_{AB}$, and Alice performs different measurements, which do not need to be projective. For each of Alice's measurements setting $x$ and result $a$, Bob remains with an unnormalized conditional state $\varrho_{a|x}$. The set of these states is called the steering assemblage, and the conditional states obey the condition $\sum_a \varrho_{a|x} = \varrho_B$, meaning that the reduced state $\varrho_B = \mathrm{Tr}_A(\varrho_{AB})$ on Bob's side is independent of Alice's choice of measurements.

After characterizing the states $\varrho_{a|x}$, Bob may try to explain their appearance as follows: He assumes that initially his particle was in some hidden state $\sigma_\lambda$ with probability $p(\lambda)$, parametrized by some parameter (or hidden variable) $\lambda$. Then Alice's measurement and result just gave him additional information on the probability of the states. This leads to states of the form (Wiseman, Jones, and Doherty, 2007)

$$\varrho_{a|x} = p(a|x) \int d\lambda p(\lambda|a,x)\sigma_\lambda$$
$$= \int d\lambda p(\lambda) p(a|x,\lambda)\sigma_\lambda. \quad (2)$$

The equivalence between these two expressions is easy to verify if the setting $x$ can be chosen freely and does not depend on the parameter $\lambda$, i.e., $p(x,\lambda) = p(x)p(\lambda)$. The two representations, however, point at different interpretations.

The first representation can be interpreted as if the probability distribution $p(\lambda)$ is updated just to $p(\lambda|a,x)$, depending on the classical information about the result $a$ and setting $x$. Here Bob does not need to believe that Alice has control over his state—her measurements and results just gave him additional information about the distribution of the states $\sigma_\lambda$.

The second representation can be interpreted as a simulation task. Here Alice can simulate the state $\varrho_{a|x}$ by drawing the states $\sigma_\lambda$ according to the distribution $p(\lambda)$ and, at the same time, announcing the result $a$ depending on the known setting $x$ and the parameter $\lambda$. Consequently, Bob does not need to believe that the initial state shared by him with Alice was entangled.

Generally, if a representation as in Eq. (2) exists, Bob does not need to assume any kind of action at a distance to explain the postmeasurement states $\varrho_{a|x}$. Consequently, he does not need to believe that Alice can steer his state by her measurements, and one also says that the state $\varrho_{AB}$ is *unsteerable* or has a local hidden state (LHS) model. If such a model does not exist, Bob is required to believe that Alice can steer the state in his laboratory by some "action at a distance." In this case, the state is said to be *steerable*. Note that steerability is an inherently asymmetric correlation, and there are states where Alice can steer Bob but not the other way round; see also Sec. III.D.

For the wave function in Eq. (1), the corresponding assemblage is formed by the states $|0\rangle\langle 0|/2$, $|1\rangle\langle 1|/2$, $|x^+\rangle\langle x^+|/2$, and $|x^-\rangle\langle x^-|/2$ and one can directly see that no LHS model exists: The four conditional states are, up to normalization, pure and thus cannot be mixtures of other states. Thus, the occurring normalized $\sigma_\lambda$ have to be proportional to the four conditional states. Equation (2) implies that

$|\eta\rangle\langle\eta|/2 = \int d\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda$ for all four $|\eta\rangle\langle\eta|$ and $\sigma_\lambda$ coming from the set $\{|0\rangle\langle 0|, |1\rangle\langle 1|, |x^+\rangle\langle x^+|, |x^-\rangle\langle x^-|\}$. As mixtures are excluded, one must have $p(a|x,\lambda) = 1$ if $\sigma_\lambda$ corresponds to $\varrho_{a|x}$ and therefore $p(\lambda) = 1/2$ for all $\lambda$. But then the probability distribution $p(\lambda)$ cannot be normalized.

For general states and measurements, however, the existence of a LHS model is not straightforward to decide. This leads us to the question of how one can decide for a given state $\varrho_{AB}$ or a given assemblage $\{\varrho_{a|x}\}$ whether it is steerable or not, and this is one of the central questions of this review.

### C. Steering, Bell nonlocality, and entanglement

There is another way to motivate the definition of steering and steerable correlations as in Eq. (2). For that, we briefly explain the notions of local hidden variable (LHV) models and entanglement.

In a general Bell experiment, Alice and Bob perform measurements on their particles, denoted by $A_x$ and $B_y$ and labeled by $x$ and $y$. For the obtained results $a$, $b$, one asks whether their probabilities can be written as

$$p(a,b|x,y) = \int d\lambda p(\lambda)p(a|x,\lambda)p(b|y,\lambda). \tag{3}$$

Such a description is known as a LHV model: The hidden variable $\lambda$ occurs with probability $p(\lambda)$, and Alice and Bob can compute the occurring joint probabilities with local response functions $p(a|x,\lambda)$ and $p(b|y,\lambda)$. For a given finite number of settings $x$, $y$ and outcomes $a$, $b$, the probabilities that can be written as in Eq. (3) form a high-dimensional polytope. The facets of the polytope are described by linear inequalities, the so-called Bell inequalities. Quantum states can result in probabilities that violate the Bell inequalities, but deciding whether a given state violates a Bell inequality or not is not straightforward and is the subject of an entire field of research (Brunner *et al.*, 2014).

Let us now describe the notion of entanglement. In general, a state on a two-particle system is called separable if it can be written as a convex combination of product states,

$$\varrho_{AB} = \sum_k p_k \varrho_k^A \otimes \varrho_k^B. \tag{4}$$

Otherwise, it is called entangled. The separability of a quantum state is not easy to decide, except for systems consisting of two qubits or one qubit and one qutrit, where the method of the partial transposition gives a necessary and sufficient criterion; see also Sec. III.E.

For our discussion, it is important that the measurements on separable states clearly can be explained by a LHV model. A general measurement $M_x$ is given by a positive-operator-valued measure (POVM). This means that one considers a set of effects $E_{a|x}$ that are positive operators, $E_{a|x} \geq 0$, summing up to the identity $\sum_a E_{a|x} = \mathbb{1}$. The probability of the result $a$ in a state $\varrho$ is computed according to $p(a) = \mathrm{Tr}(\varrho E_{a|x})$. Applying this to a separable state, one directly sees that the probabilities of distributed measurements can be written as

$$p(a,b|x,y) = \sum_k p_k \mathrm{Tr}(E_{a|x}\varrho_k^A)\mathrm{Tr}(E_{b|y}\varrho_k^B). \tag{5}$$

This is clearly a LHV model as in Eq. (3), with the extra condition that the response functions $p(a|x,\lambda)$ and $p(b|y,\lambda)$ result from the quantum mechanical description of measurements.

Considering Eqs. (3) and (5), one may ask whether the probabilities can also be described by a hybrid model, where Alice has a general response function, while Bob's function is derived from the quantum mechanical measurement rule. That is, one considers probabilities of the form

$$p(a,b|x,y) = \int d\lambda p(\lambda)p(a|x,\lambda)\mathrm{Tr}(E_{b|y}\sigma_\lambda^B). \tag{6}$$

The point is that such probabilities are exactly the ones that occur in the steering scenario. By linearity, we can rewrite Eq. (6) as

$$p(a,b|x,y) = \mathrm{Tr}(E_{b|y}\varrho_{a|x}), \tag{7}$$

where $\varrho_{a|x} = \int d\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda^B$ are the conditional states, allowing for a LHS model as in Eq. (2).

We can conclude that the steering phenomenon relies on quantum correlations which are between entanglement and violation of a Bell inequality. In fact, any state that violates a Bell inequality can be used for steering, and any steerable state is entangled; see also Fig. 2. These inclusions are strict in the sense that there are entangled states that cannot be used for steering, and there are steerable states that do not violate any Bell inequality. In Sec. III.A, we discuss in detail the relation and the known examples of states in the various subsets.

It is important to note that the indicated hierarchy also represents different levels of trust in the measurement devices in entanglement verification. In general, in quantum information processing tasks, such as cryptography, it makes a difference whether or not one assumes that the measurement devices are well characterized. Completely uncharacterized devices can be seen as a black box, giving just some measurement results without any knowledge about the

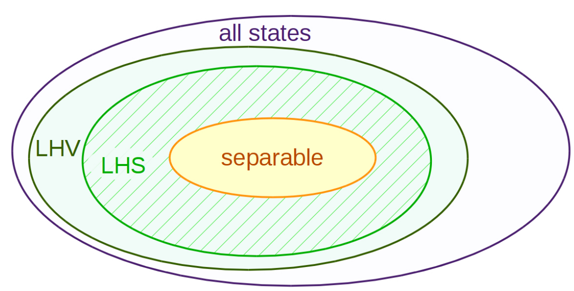


FIG. 2. Inclusion relation between entanglement, steering, and Bell inequality violations. The set of all states is convex. The states which have a LHV model and therefore do not violate any Bell inequality form a convex subset. The states with a LHS model are unsteerable and form a convex subset of the LHV states. Finally, the separable states are a convex subset of the LHS states.

quantum description. There can also be situations where the devices are partly characterized, e.g., if the dimension of the quantum system is known, but not the precise form of the measurement operators.

Entanglement, steering, and Bell nonlocality correspond to different levels of trust in the following sense. The usual schemes of entanglement verification, such as quantum state tomography and entanglement witnesses, require well-characterized measurement devices. The violation of a Bell inequality, however, certifies the presence of entanglement without any assumptions about the measurements or dimension of the system. Steering is between the two scenarios: If a state is steerable, its entanglement can be verified in a one-sided device-independent scenario, where Bob's measurements are characterized, but Alice's are not. In some cases, the assumptions about Bob's system can even be relaxed; see Sec. III.F for an example.

## II. DETECTION OF STEERING

Here we discuss how steering can be verified in different scenarios. Mainly three cases can be distinguished. First, given some expectation values of the form $\langle A_i \otimes B_j \rangle$, one can ask whether these correlations can prove steerability. Second, one can consider the case where Bob's assemblage $\{\varrho_{a|x}\}$ is given and ask whether or not it can be explained by a LHS model. Finally, one can take a complete state $\varrho_{AB}$ and ask whether this state allows one to see the phenomenon of steering if Alice makes appropriate measurements.

### A. Steering detection from correlations

The simplest way to detect steering is to formulate criteria for the correlations between Alice's and Bob's measurement statistics. These can then be directly evaluated in experiments without the need of reconstructing the whole assemblage.

This approach of detecting steerability has a natural connection to the task of entanglement verification (Gühne and Tóth, 2009), and many concepts are similar to the case of entanglement detection. This includes linear criteria that are similar to entanglement witnesses, criteria based on variances or entropic uncertainty relations, and criteria similar to Bell inequalities.

#### 1. Linear and nonlinear steering criteria

Some of the typical ideas for deriving steering criteria are best explained with an example. Consider two qubits and the operator

$$Q = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z. \quad (8)$$

The question is, which values can $\langle Q \rangle$ have for separable states? If one tries to maximize or minimize $\langle Q \rangle$ over separable states, it suffices to consider product states of the form $\varrho_A \otimes \varrho_B$, as these are the extreme points of the separable states. But for product states, the single expectation values factorize and one has (Tóth, 2005)

$$|\langle Q \rangle| = |\langle \sigma_x \rangle_A \langle \sigma_x \rangle_B + \langle \sigma_y \rangle_A \langle \sigma_y \rangle_B + \langle \sigma_z \rangle_A \langle \sigma_z \rangle_B| \le \|\vec{a}\| \|\vec{b}\| \le 1, \quad (9)$$

with $\vec{a} = (\langle \sigma_x \rangle_A, \langle \sigma_y \rangle_A, \langle \sigma_z \rangle_A)$ and $\vec{b}$ defined analogously. Here the Cauchy-Schwarz inequality was used first, and then the fact that, for single-qubit states, $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \le 1$ holds. For the singlet state, however, $\langle Q \rangle = -3$. As a result the operator $\mathcal{W} = \mathbb{1} + Q$ is an entanglement witness, as it has a positive mean value on all separable states but has a negative mean value on some entangled states.

If one wants to estimate $\langle Q \rangle$ for unsteerable states, then in view of Eq. (6) it suffices to consider product distributions again. This time, however, only Bob's results are described by quantum mechanics, so only the norm $\|\vec{b}\| \le 1$ is bounded, while $\|\vec{a}\| = \sqrt{3}$ is possible. Thus $|\langle Q \rangle| \le \sqrt{3}$ is a valid steering inequality that allows for detecting the steerability of the singlet state (Cavalcanti *et al.*, 2009).

A possible modification and generalization is the following: Consider $N$ measurements $A_k$ on Alice's side which can take the two values $\pm 1$ and arbitrary observables $B_k$ on Bob's side. Then, for unsteerable states (Saunders *et al.*, 2010),

$$\sum_{k=1}^{N} |\langle A_k \otimes B_k \rangle| \le \max_{\{a_k\}} \left[ \lambda_{\max} \left( \sum_{k=1}^{N} a_k B_k \right) \right], \quad (10)$$

where $\lambda_{\max}(X)$ denotes the largest eigenvalue of $X$ and $a_k = \pm 1$. To prove this bound, it suffices to consider a product distribution as above; then each $A_k$ can just change the signs of the $B_k$, and the mean value of the resulting sum is bounded by the maximal eigenvalue.

A different kind of generalization uses expectation values on Bob's side that are conditioned on Alice's outcome. If Alice makes a measurement $A_k$ with possible results labeled by $a$, one can denote by $\langle B \rangle|_a$ the mean value of a measurement on Bob's side, conditioned on the outcome $a$. Then one can consider the nonlinear expression

$$T_x^{(k)} = \sum_a p(a|k) (\langle \sigma_x \rangle|_a)^2, \quad (11)$$

and summing this up for three Pauli measurements gives the bound for unsteerable states (Wittmann *et al.*, 2012),

$$T_x^{(1)} + T_y^{(2)} + T_z^{(3)} \le 1. \quad (12)$$

This follows by considering product distributions and $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \le 1$. Note that similar bounds on sums of squared mean values are known for many cases of anticommuting observables or mutually unbiased bases (Tóth and Gühne, 2005; Wu, Yu, and Mølmer, 2009; Wehner and Winter, 2010), so one can directly generalize the previous criteria to broader classes of observables on Bob's side (Evans, Cavalcanti, and Wiseman, 2013). With mutually unbiased bases as a generalization of the Pauli matrices, one can even find steering inequalities with an unbounded violation (Marciniak *et al.*, 2015; Rutkowski *et al.*, 2017).

From these criteria, the following question arises: Which are the best measurement directions for a given state in order to detect steering? For the criterion in Eq. (10), this was studied by Evans and Wiseman (2014). For criteria using Pauli matrices, one can still ask for the best orientation of the coordinate system. McCloskey, Ferraro, and Paternostro (2017) observed that often, but not always, the measurements that correspond to the semiaxes of the so-called steering ellipsoid (i.e., the ellipsoid of the potential conditional states in Bob's Bloch sphere considering all possible measurements for Alice; see Sec. V.B) give strong criteria. For higher-dimensional systems, a systematic study of optimal measurements in restrictive scenarios, i.e., in the context of $N$ measurements of $k$ outcomes, was performed by Bavaresco *et al.* (2017).

So far we have considered criteria that were motivated by concepts in entanglement theory. A different method to design steering inequalities for a given special scenario comes from the theory of semidefinite programs (SDPs). As we see in Sec. II.B, the question of whether a given assemblage $\{\varrho_{a|x}\}$ is steerable can be decided via a SDP. The corresponding dual problem can then be considered as a linear steering inequality. Further details are given in Sec. II.B.2.

The discussed criteria or small variations thereof have been used in several experiments (Saunders *et al.*, 2010; Bennet *et al.*, 2012; Smith *et al.*, 2012; Wittmann *et al.*, 2012; Weston *et al.*, 2018). In the experimental works, it is also important to close loopholes, such as the one arising from inefficient detectors. Theoretical aspects of this issue were discussed in detail by Evans, Cavalcanti, and Wiseman (2013), Vallone (2013), and Jeon and Jeong (2019) and was experimentally studied by Bennet *et al.* (2012), Smith *et al.* (2012), Wittmann *et al.* (2012), and Weston *et al.* (2018).

#### 2. Steering criteria from uncertainty relations

Steering inequalities based on uncertainty relations were proposed long before the formal definition of steerability in the context of the EPR argument (Reid, 1989; Reid *et al.*, 2009). Also, the criterion in Eqs. (11) and (12) can be seen as a criterion in terms of conditional variances.

A systematic approach using entropic uncertainty relations (EURs) was proposed by Walborn *et al.* (2011) for continuous variable systems and tailored to discrete systems by Schneeloch *et al.* (2013). Here we focus on the discrete version. In general, a measurement $M$ results in a probability distribution $\mathcal{P} = (p_1, \ldots, p_n)$ of the outcomes, for which one can consider the Shannon entropy $S(\mathcal{P}) = -\sum_i p_i \log(p_i)$ as the entropy of the measurement $S(M)$. For two projective measurements given by the corresponding Hermitian operators $B_1 = \sum_i \lambda_i |v_i\rangle\langle v_i|$ and $B_2 = \sum_i \mu_i |w_i\rangle\langle w_i|$ on Bob's side, one has the general EUR (Maassen and Uffink, 1988)

$$S(B_1) + S(B_2) \geq -\ln(\Omega_B), \tag{13}$$

where $\Omega_B = \max_{i,j}(|\langle v_i|w_j\rangle|^2)$ is the maximal overlap between the eigenstates. This and similar EURs are central to quantum information theory and quantum cryptography (Coles *et al.*, 2017).

For product measurements $A \otimes B$ on two particles, one can consider the joint distribution and the conditional Shannon entropy $S(B|A) = S(A,B) - S(A)$. Then, for unsteerable states,

$$S(B_1|A_1) + S(B_2|A_2) \geq -\ln(\Omega_B) \tag{14}$$

holds. The intuition behind this criterion is that if Alice can predict from her measurement data Bob's measurement results better than the EUR allows, then there cannot be local quantum states for Bob that reproduce such measurement results.

A generalization of this criterion to other entropies has been developed (Costa, Uola, and Gühne, 2018a, 2018b). The general approach works for any entropy with the following properties: (i) the entropy is (pseudo)additive for independent distributions, (ii) one has a state-independent EUR, and (iii) the corresponding relative entropy is jointly convex. The resulting criteria based on Tsallis entropy are typically stronger than the ones from Shannon entropy. In addition, Kriváchy, Fröwis, and Brunner (2018) obtained tight steering inequalities in terms of the Rényi entropy for scenarios with two measurements per party, and Jia, Wu, and Guo (2017) developed methods of detecting entanglement and steering based on universal uncertainty relations and fine-grained uncertainty relations using majorization.

In the case of continuous variable systems, the entropic criteria proposed by Walborn *et al.* (2011) are connected to one of the first criteria already mentioned. Reid (1989) addressed the following question: To what extent can Alice infer the value of Bob's position $X_B$ or momentum $P_B$ by measuring her own canonical variables? The best estimator of $X_B$ has as uncertainty the minimal variance $\delta^2_{\min}(X_B) = \int dx_A p(x_A)\delta^2(x_B|x_A)$, where $\delta^2(x_B|x_A)$ is the variance of the conditional probability distribution. Then, for quantum states that do not give rise to an EPR argument, the condition

$$\delta^2_{\min}(X_B)\delta^2_{\min}(P_B) \geq \tfrac{1}{4} \tag{15}$$

holds. Later Walborn *et al.* (2011) showed the criterion $S(X_B|X_A) + S(P_B|P_A) \geq \ln(\pi e)$ and demonstrated that this implies Eq. (15). In addition, Walborn *et al.* (2011) reported the experimental observation of states, which can be detected by the entropic criterion, but not by Eq. (15).

Other experiments involving steering criteria from uncertainty relations have been reported in the case of continuous variable systems by Bowen *et al.* (2003), and recently in the case of discrete systems (Wollmann, Uola, and Costa, 2019; Yang *et al.*, 2019).

#### 3. Steering and the CHSH inequality

Given a certain set of measurements, one can ask for the optimal inequalities for detecting quantum correlations. For Bell nonlocality and the case of two measurements with two outcomes each, it is known that the probabilities allow a LHV description if and only if the Clauser-Horne-Shimony-Holt (CHSH) inequality

$$\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle \leq 2 \tag{16}$$

holds (Fine, 1982), where permutations of the measurements and outcomes also have to be taken into account. More precisely, the inequality implies that the probabilities of all outcomes for the measurements $A_i B_j$ can be explained by a LHV model. Note that these probabilities include more information than the full correlations $\langle A_i B_j \rangle$ only, as the marginals $\langle A_i \rangle$ and $\langle B_j \rangle$ are independent of the full correlations. In other words, if the CHSH inequality is fulfilled, there is also no two-setting Bell inequality with marginal terms that is violated.

Similar statements are known from entanglement theory. For instance, one can consider the situation of two qubits, where Alice and Bob perform each the two measurements $\sigma_x$ and $\sigma_z$ only, and not full tomography. For this scenario, all relevant entanglement witnesses have been characterized (Curty, Lewenstein, and Lütkenhaus, 2004).

For quantum steering, one can also consider two measurements with two outcomes per party, where only the measurements of Bob may be characterized. First, one can consider the case in which Bob has a qubit and performs two mutually unbiased measurements (e.g., two Pauli measurements). For this scenario, it was shown by E. G. Cavalcanti *et al.* (2015) that the full correlations $\langle A_i B_j \rangle$ admit a LHS model, if and only if the inequality

$$\sqrt{\langle (A_1 + A_2) B_1 \rangle^2 + \langle (A_1 + A_2) B_2 \rangle^2} + \sqrt{\langle (A_1 - A_2) B_1 \rangle^2 + \langle (A_1 - A_2) B_2 \rangle^2} \leq 2 \quad (17)$$

holds. Note that the resulting inequality and the underlying problem has some similarities to Bell inequalities for orthogonal measurement directions for one or both parties (Uffink and Seevinck, 2008).

For the more general scenario, one has to distinguish carefully whether the LHS model should explain the full correlations $\langle A_i B_j \rangle$ only or in addition the marginal distributions $\langle A_i \rangle$ and $\langle B_j \rangle$.

Concerning full correlations, Girdhar and Cavalcanti (2016) considered the case of uncharacterized projective measurements on Bob's qubit. First, two projective measurements $B_1$ and $B_2$ on a qubit define a plane on the Bloch sphere, and in this plane one can always find a third measurement $B_3$ such that $B_1$ and $B_3$ are mutually unbiased; moreover, the mean values of $B_1$ and $B_2$ can be obtained from the mean values of $B_1$ and $B_3$, and vice versa. Then it was shown that $B_1$ and $B_2$ allow a LHS model if and only if Eq. (17) holds for $B_1$ and $B_3$. In addition, it was shown that if Eq. (17) is violated, then the state violates also the original CHSH inequality and is thus nonlocal, but possibly for a different set of measurements [see also Quan *et al.* (2017) for an independent proof]. Finally, a characterization of POVMs with two outcomes was also given by Girdhar and Cavalcanti (2016).

At the same time, Quan *et al.* (2017) considered the question of whether full correlations and marginals of two dichotomic measurements can be explained via a LHS model. For this case, the equivalence is not true anymore: There are two-qubit states that do not violate any CHSH inequalities; nevertheless, no LHS model can explain the full correlations and marginals of certain $A_1$, $A_2$, $B_1$, $B_2$. One can also find two-qubit states for which steerability from Alice to Bob can be proved by two measurements on each side, but for which steering from Bob to Alice is not possible; see also Sec. III.D. Finally, the interplay between steering and Bell inequality violation for specific families of states was discussed by Costa and Angelo (2016) and Quan *et al.* (2016).

#### 4. Moment matrix approach

Another method that can be used for the characterization of quantum correlations consists of moment matrices or expectation value matrices. In general, one considers a set of operators of the form $M_k = \{A_{i_k} \otimes B_{j_k}\}$ and builds the matrix of expectation values

$$\Gamma_{kl} = \langle M_k^\dagger M_l \rangle. \quad (18)$$

The remaining task is to characterize the possible matrices $\Gamma$ that originate from unsteerable or separable states. Clearly, $\Gamma \geq 0$; i.e., it has no negative eigenvalues.

This approach of moment matrices is a well-known tool in entanglement theory (Shchukin and Vogel, 2005; Häseler, Moroder, and Lütkenhaus, 2008; Moroder, Keyl, and Lütkenhaus, 2008; Miranowicz *et al.*, 2009). There one can argue that, for separable states, the matrix $\Gamma$ inherits a separable structure so that approaches using the partial transposition can be applied. This also allows characterization of entanglement if some of the entries $\Gamma_{kl}$ are not known or if the measurement devices are not trusted (Moroder *et al.*, 2013).

Concerning steering, it follows from Eqs. (5) and (6) that the correlations of unsteerable states can be explained by an underlying separable state, where the measurements of Alice commute (Kogias, Skrzypczyk *et al.*, 2015; Moroder *et al.*, 2016). The commutativity of Alice's measurements together with possible exploitation of the structure of Bob's characterized measurements (e.g., an algebraic structure such as that of the Pauli spin operators) results in constraints on the moment matrix. In the end, for a given set of product operators $\{M_k\}$, one needs to check whether there exist complex parameters for the unknown entries of the moment matrix (such as squares of Alice's measurements) that make the matrix positive. As any moment matrix is positive, proving that such an assignment of parameters is not possible implies that the underlying state is steerable. The main result of Kogias, Skrzypczyk *et al.* (2015) can then be formulated as follows. For any unsteerable correlation experiment

$$\Gamma_R \geq 0 \text{ for some } R, \quad (19)$$

where $\Gamma_R$ is the moment matrix $\Gamma$ for a set of parameters $R$ (fulfilling the requirements inherited from commutativity on Alice's side and possible further structure on Bob's side) as the unknown entries. Kogias, Skrzypczyk *et al.* (2015) further pointed out that checking the existence of such parameters forms a semidefinite program and provided various examples. Note that this approach can still be augmented by using the separable structure of $\Gamma$.

Chen, Budroni *et al.* (2016) used the concept of a moment matrix to characterize steerability in a more refined way. Namely, one can also consider the moment matrices $\Gamma_{a|x}$ for each state in the assemblage $\{\varrho_{a|x}\}$. Using the methods of Moroder *et al.* (2013) then allows one to characterize and quantify steerability in a device-independent way.

#### 5. Steering criteria based on local uncertainty relations

Local uncertainty relations (LURs) are a common tool for entanglement detection, and the underlying idea can be directly generalized to steering detection. For the case of entanglement, the idea is as follows: Consider observables $A_k$ on Alice's side, obeying an uncertainty relation $\sum_k \delta^2(A_k) \geq C_A$, where $\delta^2(X) = \langle X^2\rangle + \langle X\rangle^2$ denotes the variance. An example of such a relation is $\sum_{i=x,y,z} \delta^2(\sigma_i) \geq 2$. For general observables, such bounds can be computed systematically (Huang, 2012; Maccone and Pati, 2014; Schwonnek, Dammeier, and Werner, 2017). Similarly, one can consider observables $B_k$ for Bob, fulfilling $\sum_k \delta^2(B_k) \geq C_B$, and the global observables $M_k = A_k \otimes \mathbb{1} + \mathbb{1} \otimes B_k$. Then, for separable states, the bound $\sum_k \delta^2(M_k) \geq C_A + C_B$ holds (Hofmann and Takeuchi, 2003). This is a very strong entanglement criterion, and its properties have been studied in detail (Gühne *et al.*, 2006; Gittsovich *et al.*, 2008; Zhang *et al.*, 2008).

For steering detection, we can use the same construction. The only difference is that Alice's measurements are not characterized, so no uncertainty relation for them is available (Ji *et al.*, 2015; Zhen *et al.*, 2016). Consequently, unsteerable states obey

$$\sum_k \delta^2(M_k) \geq C_B. \tag{20}$$

The criterion of the LURs can be formulated in terms of covariance matrices (Gühne *et al.*, 2007), and this also works for steering (Ji *et al.*, 2015). For a given quantum state $\varrho$ and observables $\{X_k\}$, the symmetric covariance matrix $\gamma$ is defined by their elements $\gamma_{ij} = (\langle X_i X_j\rangle + \langle X_j X_i\rangle)/2 - \langle X_i\rangle\langle X_j\rangle$. If one considers in a composite system the set of observables $\{X_k\} = \{A_{i_k} \otimes \mathbb{1}, \mathbb{1} \otimes B_{j_k}\}$, then the covariance matrix has a block structure

$$\gamma_{AB} = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}, \tag{21}$$

where $A = \gamma(\varrho_A, \{A_i\})$ and $B = \gamma(\varrho_B, \{B_j\})$ are covariance matrices for the reduced states, and $C$ is the correlation matrix with elements $C_{ij} = \langle A_i \otimes B_j\rangle - \langle A_i\rangle\langle B_j\rangle$.

Given this type of covariance matrices for unsteerable states, it holds that

$$\gamma_{AB} \geq \mathbf{0}_A \oplus \kappa_B, \tag{22}$$

with $\kappa_B = \sum_k p_k \gamma(|b_k\rangle\langle b_k|)$ being a convex combination of covariance matrices of pure states on Bob's system. Here $\mathbf{0}_A$ is an $m \times m$ null matrix, where $m$ is the number of observables on Alice's side. The characterization of the possible $\kappa_B$ has been discussed for typical cases, such as qubit states and local orthogonal observables (Gittsovich *et al.*, 2008).

Finally, it should be noted that the criterion in Eq. (22) is the discrete analog to a criterion for the continuous variable case; see also Sec. V.C.

### B. Steering detection from state assemblages

When full knowledge of the unnormalized conditional ensembles $\varrho_{a|x}$ on Bob's side is available, steerability can be detected more efficiently. As already mentioned, a set of conditional ensembles on Bob's side corresponding to certain measurement settings from Alice is called a steering assemblage. As Alice's choice of measurement cannot be detected on Bob's side, such assemblages are nonsignaling in that $\sum_a \varrho_{a|x} = \sum_a \varrho_{a|x'}$ for different settings $x$, $x'$. Note that, for the bipartite case, any nonsignaling assemblage can be prepared with some shared state and some measurements on Alice's side (Schrödinger, 1936); see also Secs. V.L and V.M. Also, any unsteerable assemblage can be prepared using a separable state and commuting measurements on Alice's side (Kogias, Skrzypczyk *et al.*, 2015; Moroder *et al.*, 2016).

The main point for steering detection is that for *finite* steering assemblages, the question of whether there exists a LHS model described by Eq. (2) can be decided via the so-called SDP technique (Pusey, 2013). The SDP approach also allows one to derive steering inequalities and to quantify the steerability of finite assemblages.

#### 1. Formulation of the semidefinite program

The crucial idea is that, for a finite steering assemblage $\{\varrho_{a|x}\}_{a,x}$, it is sufficient to consider a finite LHS ensemble $\sigma_\lambda$ (Ali *et al.*, 2009; Pusey, 2013). Moreover, the response functions in Eq. (2) can also be chosen to be fixed. The remaining problem is to construct a finite number of positive operators $\sigma_\lambda$. We focus on the conceptual aspects of the SDP formulation; a detailed review on computational aspects was given by Cavalcanti and Skrzypczyk (2017).

Consider a set of $m$ measurement settings on Alice's side, $x \in \{1, 2, \ldots, m\}$. Each has $q$ outcomes $a \in \{1, 2, \ldots, q\}$. Given the shared state $\varrho$, this gives rise to an assemblage $\{\varrho_{a|x}\}_{a,x}$ of $m$ ensembles, each consisting of $q$ conditional states. The space of the hidden variables $\lambda$ can be constructed as follows. The variable $\lambda$ can take $q^m$ values, each of which can be thought of as a string of outcomes ordered according to the measurements $(a_{x=1}, a_{x=2}, \ldots, a_{x=m})$. For such a string $\lambda$, we denote by $\lambda(x)$ the value of the outcome at position $x$. Then $D(a|x,\lambda)$ denotes the deterministic response function defined by $D(a|x,\lambda) = \delta_{a,\lambda(x)}$. This means that $D(a|x,\lambda)$ equals 1 for strings $\lambda$ that predict the outcome $a$ for the measurement $x$, and zero otherwise.

The crucial statement is the following: A finite steering assemblage admits a LHS model described by Eq. (2) if and only if it also admits a LHS model with the constructed set of strings $\lambda$ as the LHV, and the fixed deterministic functions $D(a|x,\lambda)$ as the response functions. The latter means that there exists a set of unnormalized operators $\sigma_\lambda$ satisfying

$$\varrho_{a|x} = \sum_\lambda D(a|x,\lambda)\sigma_\lambda, \quad \text{for all } a, x,$$
$$s.t.\ \sigma_\lambda \geq 0, \quad \text{for all } \lambda. \tag{23}$$

Writing with the explicit definitions of the hidden variable $\lambda$ and the deterministic response function $D(a|x,\lambda)$, the equality in Eq. (23) is simply

$$\varrho_{a|x} = \sum_{\{a_i\}} \delta_{a,a_x} \sigma_{a_1,a_2,\ldots,a_m}. \tag{24}$$

Intuitively, one can think of the hidden states $\sigma_{a_1,a_2,\ldots,a_m}$ as being indexed by $m$ variables. The conditional state $\varrho_{a|x}$ is obtained by summing the function over the values of all variables except for the $x$th one, which is fixed at $a_x = a$.

To give an example, if one considers the case where Alice performs two measurements $x \in \{1,2\}$ with two possible outcomes $a \in \{\pm\}$, the steering assemblage $\{\varrho_{a|x}\}$ is unsteerable if and only if it is possible to find four positive semidefinite operators $\omega_{ij}$, with $i, j = \pm$ such that

$$\begin{aligned} \varrho_{+|1} &= \omega_{++} + \omega_{+-}, & \varrho_{+|2} &= \omega_{++} + \omega_{-+}, \\ \varrho_{-|1} &= \omega_{-+} + \omega_{--}, & \varrho_{-|2} &= \omega_{+-} + \omega_{--}. \end{aligned} \tag{25}$$

In passing from Eq. (2) to Eq. (23), we pass from an arbitrary hidden variable to a finite discrete hidden variable and, at the same time, fix the response functions. One notices that the finiteness of the set of measurements plays a crucial role in this approach.

Given an assemblage $\{\varrho_{a|x}\}_{a,x}$, determining the existence of $\sigma_\lambda$ satisfying Eq. (23) is in fact a well-known problem in convex optimization. More precisely, it is known as a feasibility problem in SDP (Boyd and Vandenberghe, 2004), which can be solved straightforwardly by an appropriate ready-to-use software. Furthermore, it was shown that using the so-called order-monotonic functions, the SDPs can be approximated by simpler ones (Zhu, Hayashi, and Chen, 2016).

#### 2. Steering inequalities from the SDP

The feasibility SDP (23) can be used to construct steering inequalities. First, one can convert such a feasibility problem to an explicit convex maximization,

$$\begin{aligned} &\max\ \mu \\ &w.r.t.\ \mu, \{\sigma_\lambda\} \\ &\quad s.t.\ \varrho_{a|x} = \sum_\lambda D(a|x,\lambda)\sigma_\lambda, \quad \text{for all } a, x, \\ &\qquad \sigma_\lambda \geq \mu \mathbb{1}, \quad \text{for all } \lambda. \end{aligned} \tag{26}$$

If the optimal value of $\mu$ turns out to be negative, then the problem in Eq. (23) is infeasible, indicating that the assemblage is steerable.

To analyze this maximization, there is a powerful tool in convex optimization known as duality theory. In a nutshell, the maximization problem in Eq. (26) is coupled to a so-called dual minimization problem,

$$\begin{aligned} &\min\ \mathrm{Tr}\sum_{a,x} F_{a|x}\varrho_{a|x} \\ &w.r.t.\ \{F_{a|x}\} \\ &\quad s.t.\ \sum_{a,x} F_{a|x} D(a|x,\lambda) \geq 0, \quad \text{for all } \lambda, \\ &\qquad \mathrm{Tr}\sum_{a,x,\lambda} F_{a|x} D(a|x,\lambda) = 1. \end{aligned} \tag{27}$$

The two problems are dual in the sense that the optimal value of the minimization in Eq. (27) is an upper bound for the maximization in Eq. (26). This is known as weak duality. Under weak additional conditions, strong duality also holds: The two optimal values are equal (Boyd and Vandenberghe, 2004).

The duality theory implies that if there exists a collection of observables $\{F_{a|x}\}$ satisfying the constraints in Eq. (27), and if $\mathrm{Tr}\sum_{a,x} F_{a|x}\varrho_{a|x} \leq 0$, then the assemblage is steerable and the dual problem naturally defines a steering inequality. The minimizer of Eq. (27) thus yields optimal steering inequalities for a steerable assemblage. Such steering inequalities find applications in several scenarios; see also Secs. III.E and V.F.

#### 3. Quantification of steerability with SDPs

The SDP approach also allows one to quantify the steerability of an assemblage $\{\varrho_{a|x}\}$. There are several different quantification schemes. We selectively discuss some of those; for an extensive discussion, see the work of Cavalcanti and Skrzypczyk (2017).

An idea for quantifying the steerability of an assemblage is as follows. Let us fix the number of measurements $m$ and the number of outcomes $q$ per measurement at Alice's side. Then the space of all assemblages, i.e., all different $m$ decompositions of Bob's reduced states to $q$-component ensembles, admits a natural convex structure. To be more precise, let $\{\varrho_{a|x}\}$ and $\{\tilde{\varrho}_{a|x}\}$ be two assemblages. Then for $0 \leq p \leq 1$, the set $\{p\varrho_{a|x} + (1-p)\tilde{\varrho}_{a|x}\}$ is also an assemblage. Within the set of all assemblages, the unsteerable assemblages form a subset that is clearly convex. Now, how steerable an assemblage is can be measured by some kind of relative distance to the set of unsteerable assemblages. More precisely, as long as only the linear structure of the state assemblages is concerned, the absolute distance is not meaningful, and one can consider relative ratios of distances only on a line. In practice, one therefore compares the distances between the considered assemblage, the boundary of unsteerable assemblages, and the boundary of all assemblages through a certain line. Different ratios constructed from these distances give rise to different steerability quantifiers.

The first quantification of steerability of an assemblage was proposed by Skrzypczyk, Navascués, and Cavalcanti (2014), known as the *steering weight*. An assemblage $\{\varrho_{a|x}\}$ is first written as a convex combination of an unsteerable assemblage $\{\varrho_{a|x}^{\mathrm{LHS}}\}$ and a general assemblage $\{\gamma_{a|x}\}$,

$$\varrho_{a|x} = p\gamma_{a|x} + (1-p)\varrho_{a|x}^{\mathrm{LHS}} \quad \text{for all } a, x. \tag{28}$$

The steering weight of $\{\varrho_{a|x}\}$, denoted by $\mathrm{SW}(\{\varrho_{a|x}\})$, is the minimal weight $p$ in such a decomposition with respect to all possible choices of the general assemblage $\{\gamma_{a|x}\}$ and the unsteerable assemblage $\{\varrho_{a|x}^{\mathrm{LHS}}\}$. A geometrical illustration of the steering weight is given in Fig. 3 (left panel). As the set of all assemblages and unsteerable assemblages can be characterized via SDPs, the steering weight can also be determined by a SDP. More precisely, $\mathrm{SW}(\{\varrho_{a|x}\})$ is given by

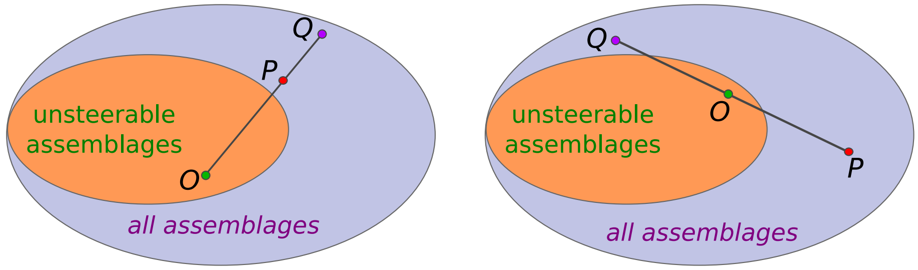


FIG. 3. Geometrical illustrations of (left panel) the steering weight and (right panel) the steering robustness. Here $P$ denotes the assemblage $\{\varrho_{a|x}\}$ under consideration, $Q$ denotes a general realizable assemblage $\{\gamma_{a|x}\}$, and $O$ denotes an unsteerable assemblage $\{\varrho_{a|x}^{\mathrm{LHS}}\}$. The steering weight (left panel) seeks to minimize $p = PO/OQ$ for varying $Q$ and $O$. The steering robustness (right panel) seeks to minimize $t/(1+t) = PO/PQ$ for varying $Q$ and $O$.

$$\begin{aligned} &\min \ 1 - \mathrm{Tr}\sum_\lambda \sigma_\lambda \\ &\text{w.r.t. } \{\sigma_\lambda\} \\ &\quad \text{s.t. } \varrho_{a|x} - \sum_\lambda D(a|x,\lambda)\sigma_\lambda \geq 0, \quad \text{for all } a, x \\ &\qquad \sigma_\lambda \geq 0, \quad \text{for all } \lambda. \end{aligned} \tag{29}$$

A similar quantification of steerability is *steering robustness*, first defined in the context of subchannel discrimination (Piani and Watrous, 2015); see also Sec. V.G. Here the steering robustness $\mathrm{SR}(\{\varrho_{a|x}\})$ is given by the minimal weight on a general assemblage $\{\gamma_{a|x}\}$ considered as noise one needs to mix into the assemblage $\{\varrho_{a|x}\}$ so that it becomes unsteerable. The geometrical illustration is given in Fig. 3 (right panel). Like the steering weight, the steering robustness $\mathrm{SR}(\{\varrho_{a|x}\})$ can be computed via a simple SDP (Piani and Watrous, 2015; Cavalcanti and Skrzypczyk, 2016),

$$\begin{aligned} &\min \ \mathrm{Tr}\sum_\lambda \sigma_\lambda - 1 \\ &\text{w.r.t. } \{\sigma_\lambda\} \\ &\quad \text{s.t. } \sum_\lambda D(a|x,\lambda)\sigma_\lambda - \varrho_{a|x} \geq 0, \quad \text{for all } a, x \\ &\qquad \sigma_\lambda \geq 0, \quad \text{for all } \lambda. \end{aligned} \tag{30}$$

To compare the two quantifiers, we note that whereas the steering weight measures the unsteerable fraction in a given assemblage, the generalized robustness measures the noise tolerance of an assemblage in terms of mixing. It can be shown that whereas robustness relates to the task of subchannel discrimination (Sec. V.G), steering weight relates to the task of subchannel exclusion (Uola, Bullock *et al.*, 2019). It turns out that the steering weight as well as the steering robustness can be related to the maximal violation of all possible steering inequalities with an appropriate normalization (Hsieh, Liang, and Lee, 2016). Moreover, one can make a general comment on the relation between the quantifiers that applies to general resource theories: Any extremal point has maximal weight, but the robustnesses can vary among different extremal points.

Beyond the steering weight and steering robustness, Ku, Chen, Budroni *et al.* (2018) defined a geometric quantifier based on the trace distance between a given assemblage and its corresponding closest assemblage admitting a LHS model. A device-independent quantification of steerability was also proposed by Chen, Budroni *et al.* (2016). This method is based on assemblage moment matrices, a collection of matrices of expectation values, each associated with a conditional quantum state; see Sec. II.A.4. Finally, a different quantifier is given by the critical radius, as explained in Sec. II.C.1.

#### 4. From finite to infinite number of measurements

While the SDP approach was originally designed to construct LHS models when Alice is limited to a finite set of measurements, one can also draw certain conclusions for the case where Alice has an infinite number of measurements (Cavalcanti *et al.*, 2016; Hirsch, Quintino, Vértesi *et al.*, 2016). The idea is as follows. One starts with a finite set of measurements on Alice's side and constructs a LHS model as previously described. In fact, one obtains a bit more. The outcome is a LHS model not only for the original finite set of measurements but for all measurements in its convex hull.

Then one considers the set of all measurements, typically limited to projective ones. One can add certain noise to the set of measurements, e.g., by sending them through a depolarizing channel, and obtains a new set of noisy measurements. For a certain level of noise added, the set of noisy measurements will shrink to fit inside the convex hull of the original finite set of measurements, for which we have a LHS model. One thus has a LHS model for the set of noisy measurements. Alternatively, the noise in the measurements can also be put onto the state instead of the measurement set (Cavalcanti *et al.*, 2016; Hirsch, Quintino, Vértesi *et al.*, 2016). Thus, one can conclude with a LHS model for the set of all measurements, but for a noisier version of the considered state. This construction works similarly for Bell nonlocality (Cavalcanti *et al.*, 2016; Hirsch, Quintino, Vértesi *et al.*, 2016).

While the SDP approach has proven to be useful in algorithmically constructing certain LHS and LHV models (Cavalcanti *et al.*, 2016; Hirsch, Quintino, Vértesi *et al.*, 2016; Cavalcanti and Skrzypczyk, 2017; Fillettaz *et al.*, 2018), it has a significant computational drawback. To reduce the noise needed to add to the state, the original finite set of measurements needs to be sufficiently large. However, the size of the SDP, as one observes, increases exponentially with respect to the number of measurement settings. As illustrated in a systematic study (Fillettaz et al., 2018), this often imposes a significant computational difficulty on the problem of deciding the steerability with high accuracy even for two-qubit states.

### C. Steering detection from full information

When the complete density matrix $\varrho_{AB}$ is exploited, one might expect to have a more complete characterization, i.e., a necessary and sufficient condition for steerability. Like entanglement detection or Bell nonlocality detection, this is a difficult question. There have been exact results only for entanglement detection of low-dimensional or special states. It is thus encouraging that some exact results can also be derived for quantum steering. It was recognized already by Wiseman, Jones, and Doherty (2007) that, for certain highly symmetric states, the problem of determining steerability with projective measurements can be solved completely; see Sec. III.B. Recently, a complete characterization of quantum steering has also been achieved for two-qubit states and projective measurements (Jevtic et al., 2015; Nguyen and Vu, 2016b; Nguyen, Nguyen, and Gühne, 2019).

#### 1. Two-qubit states and projective measurements

From Eq. (2), one sees that, in order to determine the steerability of a given state, one has to consider all possible LHS ensembles $\{p(\lambda), \sigma_\lambda\}$, and for each measurement, one has to solve for the response functions $p(a|x, \lambda)$. The source of difficulty is that the possible choice of the indexing hidden variable $\lambda$ seems to be arbitrary: It can be a discrete variable, a real-valued variable, a multidimensional variable, etc. It is now worth reexamining how the SDP approach discussed in Sec. II.B works: One assumes that Alice can make only a finite number of measurements, which implies the finiteness of a necessary LHS ensemble—a unique choice of the hidden variable is thus singled out. When Alice's set of measurements is not finite, this approach breaks down. Fortunately, one can show (Nguyen et al., 2018; Nguyen, Nguyen, and Gühne, 2019) that for quantum steering there is a canonical choice of the indexing hidden variable, namely, Bob's pure states. This is also true for higher-dimensional systems. In fact, a LHS ensemble can be identified with a probability distribution (or, to be more precise, a probability measure) $\mu$ over Bob's pure states $\mathcal{S}_B$.

The previous discussion implies that the LHS model Eq. (2) can be written as

$$\varrho_{a|x} = \int_{\mathcal{S}_B} d\mu(\sigma)\tilde{p}(a|x,\sigma)\sigma \tag{31}$$

for a certain choice of $\tilde{p}(a|x,\sigma)$ (Nguyen et al., 2018). Note that the response function $\tilde{p}(a|x,\sigma)$ may have no simple relation to $p(a|x,\lambda)$ in Eq. (2) if the association $\lambda$ with $\sigma_\lambda$ is not injective; see the work of Nguyen et al. (2018) for a detailed discussion.

Consider now a system of two qubits. To proceed, let us for now fix a LHS ensemble $\mu$. Given a LHS ensemble $\mu$, one still faces with the problem of solving Eq. (31) for $\tilde{p}(a|x,\sigma)$ for all possible measurements $x$. The next step is to abandon this constructive approach; instead, one only determines a condition for this equation to have a solution. To this end, for a given LHS ensemble $\mu$, one defines (Nguyen and Vu, 2016a, 2016b; Nguyen, Nguyen, and Gühne, 2019)

$$r(\varrho_{AB}, \mu) = \min_C \frac{\int_{\mathcal{S}_B} d\mu(\sigma)|\mathrm{Tr}_B(C\sigma)|}{\sqrt{2}\|\mathrm{Tr}_B[\bar{\varrho}_{AB}(\mathbb{1}_A \otimes C)]\|}, \tag{32}$$

where $\bar{\varrho}_{AB} = \varrho_{AB} - (\mathbb{1}_A \otimes \varrho_B)/2$, the norm is given by $\|X\| = \sqrt{\mathrm{Tr}(X^\dagger X)}$, and the minimization runs over all single-qubit observables $C$ on Bob's space. In fact, the quantity $r(\varrho_{AB}, \mu)$ characterizes the geometry of the set of conditional states Alice can simulate from the LHS ensemble $\mu$, and it is called the principal radius of $\mu$ (Nguyen and Vu, 2016a). It can then be shown (Nguyen and Vu, 2016a; Nguyen, Nguyen, and Gühne, 2019) that Eq. (31) has a solution $\tilde{p}(a|x,\sigma)$ for $x$ running over all possible projective measurements if and only if $r(\varrho_{AB}, \mu) \geq 1$.

So far a fixed choice of LHS ensemble $\mu$ is made. One now defines the critical radius as the maximum of the principal radius (32) over all LHS ensembles,

$$R(\varrho_{AB}) = \max_\mu r(\varrho_{AB}, \mu). \tag{33}$$

Then a two-qubit state is unsteerable if and only if $R(\varrho_{AB}) \geq 1$.

Let $\varrho_{AB}^{(\alpha)} = \alpha\varrho_{AB} + (1-\alpha)(\mathbb{1}_A \otimes \varrho_B)/2$, then $R(\varrho_{AB}^{(\alpha)}) = \alpha^{-1}R(\varrho_{AB})$. This relation also gives an operational meaning to the critical radius. Namely, $1 - R(\varrho_{AB})$ measures the distance from the given state to the surface that separates steerable states from unsteerable states; see Fig. 4. As a consequence, one can in fact equivalently define the critical radius as

$$R(\varrho_{AB}) = \max\{\alpha \geq 0 : \varrho_{AB}^{(\alpha)} \text{ is unsteerable}\}, \tag{34}$$

where unsteerability is considered with respect to projective measurements. We will see that this definition can be naturally generalized to generalized measurements and higher-dimensional systems.

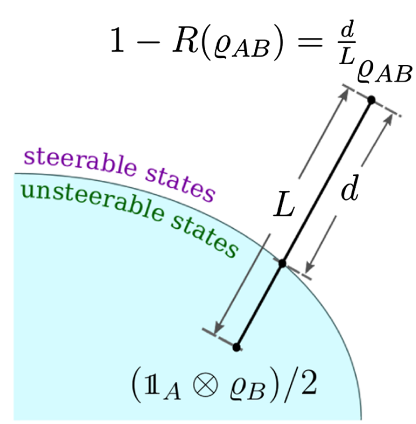


FIG. 4. The operational meaning of the critical radius. $1 - R(\varrho_{AB})$ measures the distance from $\varrho$ to the surface of the unsteerable or steerable state relative to $(\mathbb{1}_A \otimes \varrho_B)/2$. From Nguyen, Nguyen, and Gühne, 2019.

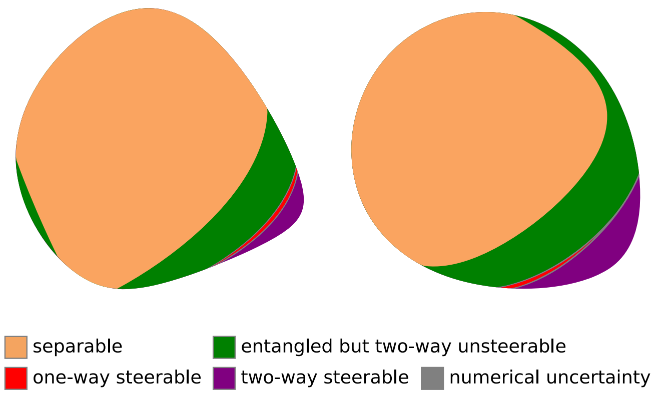


FIG. 5. Two two-dimensional random cross sections of the set of all two-qubit states. From the innermost to the outermost, different areas with different colors denote the set of separable states characterized by the partial transposition (Horodecki, Horodecki, and Horodecki, 1996; Peres, 1996b), entangled states that are unsteerable, one-way steerable states (Alice to Bob, or vice versa), and two-way steerable states (Alice to Bob, and vice versa). The very thin gray lines at the two boundaries of the area corresponding to the one-way steerable states denote those states where the numerical precision used was not sufficient to make an unambiguous decision. From Nguyen, Nguyen, and Gühne, 2019.

The definition of the critical radius by Eq. (33) also allows for its evaluation. Nguyen *et al.* (2018) showed that, by replacing Bob's Bloch sphere by polytopes from inside and outside, one obtains rigorous upper and lower bounds for the critical radius. The computations of the upper and lower bounds are both linear programs, of which the sizes scale cubically with respect to the numbers of vertices of the polytopes. Upon increasing the numbers of vertices, the two bounds quickly converge to the actual value of the critical radius. This approach has been used to access the geometry of the set of unsteerable states via its two-dimensional random cross sections (Nguyen, Nguyen, and Gühne, 2019); see Fig. 5.

Notably, for the so-called Bell-diagonal states, or $T$ states, an explicit formula for the critical radius was obtained,

$$R(\varrho_T) = 2\pi N_T|\det(T)|, \tag{35}$$

where $T$ is the correlation matrix of the $T$ state, $T_{ij} = \mathrm{Tr}(\varrho_T \sigma_i \otimes \sigma_j)$ for $i, j = 1, 2, 3$, and the normalization factor $N_T$ is given by an integration over the Bloch sphere $N_T^{-1} = \int dS(\vec{n})[\vec{n}^T T^{-2}\vec{n}]^{-2}$ (Jevtic *et al.*, 2015; Nguyen and Vu, 2016b). Based on this solution for $T$ states, analytical bounds for the critical radius of a general state can also be derived (Nguyen, Nguyen, and Gühne, 2019).

For further discussions on LHS models for two-qubit states in special cases, see Miller, Colbeck, and Shi (2018), Yu *et al.* (2018a, 2018b), and Zhang and Zhang (2019).

### 2. Steering of higher-dimensional systems and with generalized measurements

Characterizing quantum steering of higher-dimensional systems and with generalized measurements (POVMs) is difficult. Most of the results on quantum steering, in this case, rely on the idea of adding sufficient noise to the state such that a LHS for simpler measurements (e.g., projective measurements) can be turned into a LHS model for POVMs (Hirsch *et al.*, 2013; Quintino *et al.*, 2015; Tischler *et al.*, 2018). More specifically, if a state $\varrho_{AB}$ of dimension $d_A \times d_B$ is unsteerable with respect to two-outcome POVMs, then the state

$$\tilde{\varrho}_{AB} = \frac{1}{d_A}\varrho_{AB} + \frac{d_A - 1}{d_A}\sigma_A \otimes \varrho_B, \tag{36}$$

with an arbitrary choice of state $\sigma_A$ and $\varrho_B = \mathrm{Tr}_A(\varrho_{AB})$, is unsteerable for arbitrary POVMs. One observes that, in Eq. (36), the weight $(d_A - 1)/d_A$ of the separable noise $\sigma_A \otimes \varrho_B$ is close to 1 if the dimension is high. Yet, this technique has played an important role in demonstrating the hierarchy of nonlocality under generalized measurements (see Sec. III.A), superactivation of nonlocality by local filtering (see Sec. III.G), and one-way steering with POVMs (see Sec. III.D).

The critical radius approach explained in the previous subsection gives a promising framework to generally analyze quantum steering with POVMs and higher-dimensional systems. In fact, in any dimension, one can define the critical radius with respect to a certain class of measurements in the same way as in Eq. (34). In particular, considering the set of generalized measurements of $n$ outcomes (POVMs of $n$ outcomes), one can define the critical radius for a bipartite state $\rho_{AB}$ of dimension $d_A \times d_B$ by

$$R_n(\varrho_{AB}) = \max\{\alpha \geq 0 : \varrho_{AB}^{(\alpha)} \text{ is unsteerable}\}, \tag{37}$$

with $\varrho_{AB}^{(\alpha)} = \alpha\varrho_{AB} + (1-\alpha)(\mathbb{1}_A \otimes \varrho_B)/d_A$, $\varrho_B = \mathrm{Tr}_A(\varrho_{AB})$, and unsteerability being considered with respect to POVMs of $n$ outcomes on Alice's side. Defined in this way, $1 - R_n(\varrho_{AB})$ can still be interpreted as measuring the distance from $\varrho_{AB}$ to the surface separating unsteerable and steerable states, here defined with respect to POVMs of $n$ outcomes on Alice's side; see again Fig. 4. However, direct evaluation of the critical radius from the definition Eq. (37) is clearly not possible.

An alternative formula for the critical radius similar to Eqs. (32) and (33) (Nguyen *et al.*, 2018; Nguyen, Nguyen, and Gühne, 2019) can also be found for high-dimensional systems. To this end, for a finite-dimensional bipartite state $\varrho_{AB}$, one can define the principal radius for a given LHS ensemble $\mu$ by

$$r_n^{-1}(\varrho_{AB}, \mu) = \sup_{Z,E} F^{-1}(\varrho_{AB}, \mu, Z, E), \tag{38}$$

with $F^{-1}(\varrho_{AB}, \mu, Z, E)$ defined to be

$$\frac{\sum_{i=1}^n \mathrm{Tr}[\varrho_{AB}(E_i \otimes Z_i)] - (1/d_A)\sum_{i=1}^n \mathrm{Tr}(E_i)\mathrm{Tr}(\varrho_B Z_i)}{\int d\mu(\sigma)\max_i\{\langle Z_i, \sigma\rangle\} - (1/d_A)\sum_{i=1}^n \mathrm{Tr}(E_i)\mathrm{Tr}(\varrho_B Z_i)}, \tag{39}$$

where the supremum is taken over all possible POVMs of $n$ outcomes $E = (E_1, E_2, \ldots, E_n)$ on Alice's side, and all possible $n$ observables $Z = (Z_1, Z_2, \ldots, Z_n)$ on Bob's side. The critical radius as defined by Eq. (37) can be computed as

(Nguyen, Nguyen, and Gühne, 2019; Nguyen and Gühne, 2020)

$$R_n^{-1}(\varrho_{AB}) = \min_{\mu} r_n^{-1}(\varrho_{AB}, \mu). \tag{40}$$

In this way, the problem of computing the critical radius and the princpal radius is in principle an optimization problem. Unfortunately, even in this form, a deterministic algorithm to compute the principal radius and the critical radius with $n \geq 3$ is still unknown, and one has to invoke heuristic techniques in practice (Nguyen *et al.*, 2018; Nguyen, Nguyen, and Gühne, 2019).

One observes that, to study quantum steering, the set of generalized measurements was stratified according to their number of outcomes. This calls for an investigation of the relation between them. Since POVMs of $n$ outcomes form a natural subset of POVMs of $n+1$ outcomes, one has a decreasing chain $R_2(\varrho_{AB}) \geq R_3(\varrho_{AB}) \geq R_4(\varrho_{AB}) \geq \cdots$. As extreme POVMs have at most $d_A^2$ outcomes with $d_A$ being Alice's dimension, this chain turns to equality at $n = d_A^2$. Using the evidence from a heuristic computation for the principal radius, it has been conjectured (Nguyen, Nguyen, and Gühne, 2019) that for two-qubit states ($d_A^2 = 4$) the chain in fact consists of a single number—namely, $R_2(\varrho_{AB}) = R_3(\varrho_{AB}) = R_4(\varrho_{AB})$. In other words, this conjecture implies that measurements of two outcomes (dichotomic measurements) are sufficient to fully demonstrate the quantum steerability of a two-qubit system; measurements of more outcomes are not necessary. Unfortunately, extrapolating this conjecture to higher-dimensional systems fails; it was later shown (Nguyen and Gühne, 2020) that this equality breaks down already for a system of two qutrits; see also Sec. III.B.

#### 3. Full information steering inequality

As discussed, for high-dimensional systems, even when full information about a state is available, a computable necessary and sufficient condition for quantum steerability is not available. In this case, the detection of steerability still relies on steering inequalities. An example of steering inequalities based on full information of the state was given by Zhen *et al.* (2016) in terms of the so-called local orthogonal observables. Embedding in a higher-dimensional space if necessary, we can assume that Alice and Bob have the same local dimension $d$. One can choose a set of $d^2$ orthogonal operators $\{G_k\}$, which serves as a basis for the local observable space; i.e., $\mathrm{Tr}(G_i G_j) = \delta_{ij}$ and $\{G_k\}$ spans the space of operators (Yu and Liu, 2005). The Pauli matrices are a familiar example of such orthogonal operators for a qubit system. By means of the Schmidt decomposition in the operator space, one can choose the orthonormal observables for the local spaces at Alice and Bob, $\{G_k^A\}$ and $\{G_k^B\}$, such that the joint state $\varrho_{AB}$ can be written as

$$\varrho_{AB} = \sum_{k=1}^{d^2} \lambda_k G_k^A \otimes G_k^B, \tag{41}$$

where $\lambda_k \geq 0$. Then, using the local uncertainty relations (see Sec. II.A), Zhen *et al.* (2016) showed that the state $\varrho_{AB}$ is steerable from $A$ to $B$ if

$$\sum_{k=1}^{d^2} \delta^2(g_k G_k^A + G_k^B) < d - 1 \tag{42}$$

for some choice of $g_k$, where $\delta^2(X)$ denotes the variance of operator $X$. By a particular choice of $g_k$, one can easily show that if

$$\sum_k \lambda_k > \sqrt{d}, \tag{43}$$

the state is steerable (Zhen *et al.*, 2016). This elegant inequality resembles the familiar computable cross norm or realignment (CCNR) entanglement criterion (Chen and Wu, 2003; Rudolph, 2005), where $\sum_k \lambda_k > 1$ implies that the state is entangled.

Note that the steering inequalities (42) and (43) are different from the various inequalities discussed in Sec. II.A in the sense that they exploit the full information about the state.

## III. CONCEPTUAL ASPECTS OF STEERING

In this section, we review results on the general properties and structures of quantum steering. We start with a detailed discussion on the connection between steering, entanglement, and Bell nonlocality. We also present in some detail LHS models for different families of states. Then we explain properties like one-way steering, steering of bound entangled states, steering maps, and the superactivation of steering.

### A. Hierarchy of correlations

We explained in Sec. I.C that there is a hierarchy between Bell nonlocality, steering, and entanglement in the sense that one implies the other, but not the other way around. In this section, we first discuss in some detail the known examples of states where the notions differ. Then we explain how the relations between the three concepts can be exploited to characterize one via another. Detailed LHS models are discussed in Sec. III.B.

When discussing the existence of a LHV or LHS model for a given quantum state, one has to distinguish whether the model should explain the results for all projective measurements, or, more generally, for all POVMs. Let us start our discussion with projective measurements. The inequivalence between the notion of entanglement and Bell nonlocality was, in fact, one of the starting points of entanglement theory (Werner, 1989). For that, one may consider the so-called two-qubit Werner state

$$\varrho(p) = p|\psi^-\rangle\langle\psi^-| + (1-p)\tfrac{\mathbb{1}}{4}, \tag{44}$$

where $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state. Using the positivity of partial transpose (PPT) criterion (see Sec. III.E), one can directly verify that this state is entangled if and only if $p > 1/3$. Werner (1989), however, constructed a LHS model for projective measurements for all values $p \leq 1/2$. Moreover, Acín, Gisin, and Toner (2006) and Hirsch *et al.* (2017) showed that a LHV model exists up to $p \leq 1/K_G(3) \approx 0.6829$, where $K_G(3)$ is the Grothendieck constant of the order of 3, so up to this value, no Bell inequality can be violated. These results demonstrate that there are entangled states which do not show Bell nonlocality. Using the fact that the Werner states is steerable for $p > 1/2$ (Wiseman, Jones, and Doherty, 2007), this also proves that steering and Bell nonlocality are inequivalent for projective measurements. States of this type have been prepared experimentally, and their steerability has been demonstrated by Saunders *et al.* (2010).

It remains to discuss the more general case of POVMs. First, Barrett (2002) constructed a LHV model for Werner states with $p \leq 5/12$ that explains all of the measurement probabilities for arbitrary POVMs. In fact, this model can be directly converted into a LHS model (Quintino *et al.*, 2015). Consequently, there are entangled states for which all correlations for POVMs can be explained by a LHV model. In addition, Quintino *et al.* (2015) presented examples of states in a $3 \times 3$ system that are steerable in both directions, but nevertheless a LHV model for all POVMs can be found. This proves the inequivalence of Bell nonlocality and steering for POVMs.

As mentioned, a local model that explains the results of all projective measurements does not necessarily explain all of the correlations for POVMs. It is not clear, however, that POVMs provide an advantage in the detection of steering or Bell nonlocality. As discussed in Sec. II.C.2, there is numerical evidence that two-qubit states that are unsteerable for projective measurements are also unsteerable for POVMs (Nguyen, Nguyen, and Gühne, 2019). Concerning Bell nonlocality, Vértesi and Bene (2010) and Gómez *et al.* (2016) presented Bell inequalities for which the maximal violation requires POVMs, but this does not imply that the states leading to this violation do not also violate some Bell inequality for projective measurements.

Given the similarity in the definitions of Bell nonlocality, quantum steerability, and nonseparability, one may expect that some methods of characterizing the different notions can be related to each other. Specifically, given a state that admits a LHV model as in Eq. (3), one may expect that, by adding suitable separable noise to the state, one can obtain a state that admits a LHS model (2). This was shown to be the case (C. Chen *et al.*, 2018). They showed that if a bipartite qudit-qubit state $\varrho_{AB}$ admits a LHV model, then the state

$$\tilde{\varrho}_{AB} = \mu\varrho_{AB} + (1-\mu)\varrho_A \otimes \frac{\mathbb{1}_B}{2}, \tag{45}$$

with $\mu = 1/\sqrt{3}$, is unsteerable from Alice to Bob. Turning the logic around, if $\tilde{\varrho}_{AB}$ is steerable, then $\varrho_{AB}$ must be Bell nonlocal. A similar statement between steerability and nonseparability has also been obtained (C. Chen *et al.*, 2018; Das, Sasmal, and Roy, 2019). Namely, if a bipartite qubit-qudit state $\varrho_{AB}$ is unsteerable from Alice to Bob, then the state

$$\tilde{\varrho}_{AB} = \mu\varrho_{AB} + (1-\mu)\frac{\mathbb{1}_A}{2} \otimes \varrho_B, \tag{46}$$

with $\mu = 1/\sqrt{3}$, is separable. Or, if the latter is entangled, the former is steerable. Detailed applications of this approach to detecting different nonlocality notions than the others were provided by J.-L. Chen *et al.* (2016), C. Chen *et al.* (2018), and Das, Sasmal, and Roy (2019).

### B. Special states and their local hidden state models

As highlighted in Sec. II.C, the fact that for quantum steering there is a canonical choice for the hidden variable—namely, Bob's pure states—turns out to have far-reaching consequences. The point is that, having simplified the LHS model from Eq. (2) to the form of Eq. (31), the symmetry of the state has stronger implications on the choice of the LHS ensemble (Nguyen *et al.*, 2018). For certain highly symmetric states such as Werner states and isotropic states, the symmetry is then enough to uniquely single out an optimal choice of the LHS ensemble, rendering their exact characterizations of quantum steering with projective measurements possible (Jones, Wiseman, and Doherty, 2007; Wiseman, Jones, and Doherty, 2007). This is in contrast to Bell nonlocality: in Eq. (3), no canonical choice of the LHV is possible. Thus, even for highly symmetric states such as isotropic states and Werner states, no exact characterization of Bell nonlocality is known.

#### 1. Werner states

Suppose that Alice and Bob share the Werner state of dimension $d \times d$ (Werner, 1989), defined by

$$W_d^\eta = \frac{d-1+\eta}{d-1}\frac{\mathbb{1}}{d^2} - \frac{\eta}{d-1}\frac{V}{d}, \tag{47}$$

where $\mathbb{1}$ is the bipartite identity operator and $V$ is the flip operator given by $V|\phi,\psi\rangle = |\psi,\phi\rangle$. Here we follow the parametrization by Wiseman, Jones, and Doherty (2007) so that $W_d^\eta$ is a product state if the mixing parameter $\eta = 0$ and is a state at all only if $\eta \leq 1$. The Werner state is entangled if and only if $\eta > 1/(d+1)$ (Werner, 1989).

In fact, the Werner state was constructed in a way such that it is invariant under the same local unitary transformation at Alice's and Bob's side (Werner, 1989); that is, for any unitary operator $U$ acting in dimension $d$, $W_d^\eta = (U \otimes U)W_d^\eta(U^\dagger \otimes U^\dagger)$. This implies that the optimal LHS ensemble on Bob's Bloch sphere can be chosen to be symmetric under the unitary group $U(d)$, i.e., the Haar measure (Wiseman, Jones, and Doherty, 2007; Nguyen *et al.*, 2018).

Identifying the hidden variable $\lambda$ indexing the LHS with Bob's pure states $|\lambda\rangle$, it remains to construct the response function $p(a|x,\lambda)$ for a projective measurement $\{E_{a|x}\} = \{P_{a|x}\}$ to complete a LHS model. Note that, for a projection outcome $P_{a|x} = |a\rangle\langle a|$ at Alice's side, Bob's conditional state is

$$\varrho_{a|x} = \frac{d-1+\eta}{d(d-1)}\frac{\mathbb{1}}{d} - \frac{\eta}{d(d-1)}|a\rangle\langle a|. \tag{48}$$

The minus sign in front of the last term indicates that the two parties in the Werner state are anticorrelated. To construct the response function, it is natural then to associate a pure state $|\lambda\rangle$ with the outcome that has the least overlap with $P_{a|x}$. The resulting response function is

$$p(a|x,\lambda) = \begin{cases} 1 & \text{if } |\langle\lambda|a\rangle| < |\langle\lambda|a'\rangle|, \quad \forall\ a' \neq a, \\ 0 & \text{otherwise.} \end{cases} \tag{49}$$

With the LHS ensemble and this choice of the response function, it is straightforward (Wiseman, Jones, and Doherty, 2007) to show that the Werner state is unsteerable for

$$\eta \le 1 - \frac{1}{d}. \tag{50}$$

It can also be shown that, for the mixing parameter $\eta > 1 - 1/d$, no construction of response function is possible (Wiseman, Jones, and Doherty, 2007). This threshold together with the threshold for the Werner state to be separable is presented in Fig. 6 (left). Thus, whereas the exact threshold of $\eta$ for which the Werner state is Bell nonlocal is still unknown even in dimension $d = 2$, the threshold for steerability has an analytical expression in all dimensions. Wiseman, Jones, and Doherty (2007) also noted that the previous construction was actually the original construction by Werner (1989) to show that Bell nonlocality and entanglement are distinct notions.

Projective measurements are not, however, the only case where the quantum steerability of the Werner states can be characterized exactly. Specifically, it was shown (Nguyen and Gühne, 2020) that when Alice is limited to making dichotomic measurements, the threshold up to which the Werner state is unsteerable can also be derived in practically closed form,

$$\eta \le (d-1)^2[1 - (1 - 1/d)^{1/(d-1)}], \tag{51}$$

for $d \le 10^5$; see Fig. 6 (left panel). The threshold is also conjectured to hold for all dimensions (Nguyen and Gühne, 2020). For dimension $d \ge 3$, the threshold of Eq. (50) is strictly stronger than that of Eq. (51). These are thus concrete examples illustrating the fact that quantum steering with dichotomic measurements is strictly weaker than that of measurements with more outcomes for higher-dimensional systems, contrasting with the conjecture on their equivalence for two-qubit systems; see Sec. II.C.

### 2. Isotropic states

Another important family of states that allows for exact characterization of quantum steering is that of isotropic states (Wiseman, Jones, and Doherty, 2007). The isotropic state of dimension $d \times d$ at mixing parameter $\eta$, $0 \le \eta \le 1$, is defined by

$$S_d^\eta = (1-\eta)\frac{\mathbb{1}}{d^2} + \eta|\psi_+\rangle\langle\psi_+|, \tag{52}$$

where $|\psi_+\rangle = (1/\sqrt{d})\sum_{i=1}^d |i,i\rangle$. From the definition, one notes that the isotropic state is defined with respect to a particular choice of basis. The isotropic state is entangled if and only if $\eta > 1/(d+1)$ (Horodecki and Horodecki, 1999). Similar to the Werner state, the isotropic state also has a unitary symmetry, namely, $S_d^\eta = (\bar{U} \otimes U)S_q^\eta(\bar{U}^\dagger \otimes U^\dagger)$, for any $d \times d$ unitary matrix $U$, with $\bar{U}$ being its complex conjugate. This again implies that the optimal choice of LHS ensemble is the uniform Haar measure over Bob's Bloch sphere.

For a projection outcome $P_{a|x} = |a\rangle\langle a|$ on her side, with an isotropic state, Alice steers Bob's system to the conditional state

$$\varrho_{a|x} = \frac{1-\eta}{d}\frac{\mathbb{1}}{d} + \frac{\eta}{d}|\bar{a}\rangle\langle\bar{a}|, \tag{53}$$

where $|\bar{a}\rangle$ is the complex conjugate of state $|a\rangle$. In contrast to Eq. (48), the plus sign in front of the last term in Eq. (53) indicates that parties sharing an isotropic state are correlated up to a complex conjugation. This motivates the following choice of response function:

$$p(a|x,\lambda) = \begin{cases} 1 & \text{if } |\langle\lambda|\bar{a}\rangle| > |\langle\lambda|\bar{a}'\rangle|, \quad \forall\ a' \ne a, \\ 0 & \text{otherwise}, \end{cases} \tag{54}$$

where we also again identified the hidden variable $\lambda$ indexing the LHS ensemble with Bob's pure states $|\lambda\rangle$. This construction leads to a LHS model for the isotropic state with

$$\eta \le \frac{H_d - 1}{d-1}, \tag{55}$$

where $H_d = 1 + 1/2 + 1/3 + \cdots + 1/d$. It can again be shown that this threshold is optimal; for $\eta > (H_d - 1)/(d-1)$, no construction for the response function is possible (Wiseman, Jones, and Doherty, 2007). Almeida *et al.* (2007) also obtained this threshold in an attempt to construct a LHV model for the isotropic states before learning of the definition

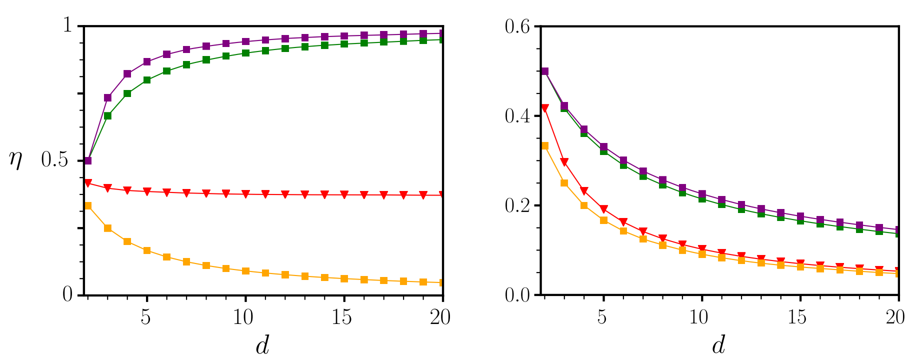


FIG. 6. The exact noise thresholds for (left panel) the Werner states and (right panel) the isotropic states to be steerable with dichotomic measurements (squares, violet, upper) and projective measurements (squares, green, middle), and to be separable (square, orange, lower). The lower bounds for the noise thresholds for them to be unsteerable with all generalized measurements obtained from Eqs. (58) and (60) are also presented (triangles, red). Adapted from Nguyen and Gühne, 2020.

of quantum steering. This threshold is presented in Fig. 6 (right panel) together with that for separability.

Like the Werner state, the quantum steerability of the isotropic states with dichotomic measurements can also be exactly characterized. Nguyen and Gühne (2020) showed that, for $d \leq 10^5$, if

$$\eta \leq 1 - d^{-1/(d-1)}, \tag{56}$$

the isotropic state is unsteerable when Alice's measurements are limited to dichotomic ones; otherwise, it is steerable [see Fig. 6 (right panel)]. The threshold is also conjectured to hold for all dimensions (Nguyen and Gühne, 2020).

### 3. LHS models for generalized measurements

As quantum steering with projective and dichotomic measurements is well understood for the Werner states and the isotropic states, one may hope that certain LHS models with general POVMs for them can also be constructed. This is indeed the case. By an explicit construction, Barrett (2002) demonstrated that sufficiently weakly entangled Werner states do admit a LHV model for all POVMs. Under the light of the formal definition of quantum steering (Wiseman, Jones, and Doherty, 2007), the LHV model turns out to be a LHS model (Quintino *et al.*, 2015). The model was revised recently (Nguyen and Gühne, 2020), and it can be shown that Barrett's original construction works best for the isotropic states; for the Werner states, a better model can be constructed. Further, for two-qubit systems, the construction can also be extended to Bell-diagonal states (Nguyen and Gühne, 2019b).

To construct the model, it is sufficient to consider only POVMs with rank-1 effects, $\{E_{a|x}\} = \{\alpha_{a|x}|a\rangle\langle a|\}$, where $|a\rangle\langle a|$ are rank-1 projections and $0 \leq \alpha_{a|x} \leq 1$ (Barrett, 2002). This is because other POVMs can be postprocessed from these (see also Sec. IV). The optimal choice for the LHS ensemble is again the uniform distribution over Bob's Bloch sphere (Wiseman, Jones, and Doherty, 2007; Nguyen *et al.*, 2018). It is then left to construct the response functions $p(a|x,\lambda)$ for the measurements mentioned.

For the isotropic state, the response function can be given as (Barrett, 2002; Almeida *et al.*, 2007)

$$p(a|x,\lambda) = \alpha_{a|x}|\langle\lambda|\bar{a}\rangle|^2\Theta(|\langle\lambda|\bar{a}\rangle|^2 - 1/d) + \frac{\alpha_{a|x}}{d}\left(1 - \sum_a \alpha_{a|x}|\langle\lambda|\bar{a}\rangle|^2\Theta(|\langle\lambda|\bar{a}\rangle|^2 - 1/d)\right). \tag{57}$$

With this choice of response function, direct computation shows that the isotropic state is unsteerable for arbitrary POVMs on Alice's side (Barrett, 2002; Almeida *et al.*, 2007) if

$$\eta \leq \frac{3d-1}{d+1}(d-1)^{d-1}d^{-d}. \tag{58}$$

As mentioned, this construction was originally suggested as a LHS model for Werner states, and the same threshold for Eq. (58) was found (Barrett, 2002; Quintino *et al.*, 2015). However, it was shown (Nguyen and Gühne, 2020) that for the Werner state, a better choice of the response functions is possible, namely,

$$p(a|x,\lambda) = \frac{\alpha_{a|x}}{d-1}(1 - |\langle\lambda|a\rangle|^2)\Theta(1/d - |\langle\lambda|a\rangle|^2) + \frac{\alpha_{a|x}}{d}\left(1 - \sum_a \frac{\alpha_{a|x}}{d-1}(1 - |\langle\lambda|a\rangle|^2)\Theta(1/d - |\langle\lambda|a\rangle|^2)\right). \tag{59}$$

The Werner state was then shown to be unsteerable for arbitrary POVMs on Alice's side (Nguyen and Gühne, 2020) if

$$\eta \leq \frac{1 + (d-1)^{d+1}d^{-d}}{d+1}. \tag{60}$$

The two bounds, Eqs. (58) and (60), are also presented in Fig. 6. For the Werner states, the bound Eq. (60) is strictly better than the bound given by Eq. (58) for $d \geq 3$. However, both bounds, Eqs. (58) and (60), are strictly within the respective thresholds for the isotropic states and the Werner states to be unsteerable with projective measurements, Eqs. (55) and (50). On the other hand, the constructions of Eqs. (57) and (59) are by no means optimal; in fact, they are not expected to be optimal (Nguyen and Gühne, 2019b). Thus, it is still unclear whether steering with projective measurements is equivalent to steering with generalized measurements even for these highly symmetric states.

The case of two-qubit Werner states ($d = 2$) is slightly better understood. In this case, the bounds of Eqs. (58) and (60) both show that, for $\eta \leq 5/12$, the Werner state is unsteerable for arbitrary POVMs on Alice's side (Barrett, 2002; Quintino *et al.*, 2015). In the range $5/12 \leq p \leq 1/2$, the state is also known to be unsteerable if the POVMs are limited to those with three outcomes (Werner, 2014). For most general POVMs, numerical evidence based on the critical radius approach is available, which indicates that the state is also unsteerable in this range (Nguyen *et al.*, 2018; Nguyen, Nguyen, and Gühne, 2019).

To conclude this section, we refer the interested readers to the work of Augusiak, Demianowicz, and Acín (2014) for further constructions of LHV and LHS models.

## C. Steering and local filtering

For characterizing steerability and other correlations in quantum states, it is relevant to study their behavior under local operations. Given a general quantum state $\varrho_{AB}$, one can consider states of the type

$$\tilde{\varrho}_{AB} = \frac{1}{\mathcal{N}}(T_A \otimes T_B)\varrho_{AB}(T_A^\dagger \otimes T_B^\dagger), \tag{61}$$

where $T_{A/B}$ are some transformation matrices and $\mathcal{N}$ denotes a potential renormalization.

Then one can ask whether the correlations in the state $\varrho_{AB}$ are related to those of the state $\tilde{\varrho}_{AB}$. Clearly, this depends on the properties of the matrices $T_{A/B}$. For them, there are mainly two possible choices: Either one restricts them to be unitary, $T_{A/B} = U_{A/B}$, and therefore one considers local unitary transformations, or one considers general invertible matrices $T_{A/B} = F_{A/B}$, which are the so-called local filtering operations and are more general than local unitaries.

For the case of entanglement, one can directly see from the definition in Eq. (4) that local filtering operations keep the property of a state of being separable or entangled. Filtering operations can, nevertheless, change the amount of entanglement. Any full-rank state can be brought into a normal form under filtering operations, where the reduced states $\tilde{\varrho}_A$ and $\tilde{\varrho}_B$ are maximally mixed (Verstraete, Dehaene, and De Moor, 2003; Leinaas, Myrheim, and Ovrum, 2006). In this form, certain entanglement measures are maximized (Verstraete, Dehaene, and De Moor, 2003), and bringing a state in this normal form can improve many entanglement criteria (Gittsovich *et al.*, 2008). For Bell nonlocality, it can be seen that local unitary transformations keep the property of a state having a LHV model. But local filtering operations are already too general; there are two-qubit states that do not violate the CHSH inequality, but after local filtering, they do (Popescu, 1995; Gisin, 1996).

Steering is a notion between entanglement and nonlocality, so a mixed behavior under local transformations can be expected. In fact, it was noted by Uola, Moroder, and Gühne (2014), Gallego and Aolita (2015), and Quintino *et al.* (2015) that local unitaries on Alice's side and local filtering on Bob's side,

$$\tilde{\varrho}_{AB} = \frac{1}{\mathcal{N}}(U_A \otimes F_B)\varrho_{AB}(U_A^\dagger \otimes F_B^\dagger), \tag{62}$$

do not change the steerability of a state. The critical radius as a steering parameter (see Sec. II.C.1) is also not affected. With these transformations, one can achieve the result that $\tilde{\varrho}_B$ is maximally mixed on its support. If a state is in this form, this can simplify calculations, and therefore it is good starting point to study the steerability of a state (Nguyen, Nguyen, and Gühne, 2019).

### D. One-way steerable states

The asymmetry between the two parties in the definition of quantum steering immediately strikes one with the question as to whether there is a state where Alice can steer Bob, but not the other way around (Wiseman, Jones, and Doherty, 2007). Such one-way steerable states were first constructed for continuous variable systems (Midgley, Ferris, and Olsen, 2010; Olsen, 2013). One-way steerable states for discrete systems were studied later by Bowles *et al.* (2014), Evans and Wiseman (2014), and Skrzypczyk, Navascués, and Cavalcanti (2014). More recently, a simple family of one-way steerable two-qubit states was identified by Bowles *et al.* (2016). This family of states is given by

$$\varrho_{AB}(\alpha,\theta) = \alpha|\psi_\theta\rangle\langle\psi_\theta| + (1-\alpha)\tfrac{\mathbb{1}}{2} \otimes \varrho_B, \tag{63}$$

where $|\psi_\theta\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ and $\varrho_B = \mathrm{Tr}_A|\psi_\theta\rangle\langle\psi_\theta|$, with $0 \le \alpha \le 1$ and $0 < \theta \le \pi/4$. The states can be brought into the two-qubit Werner states with the same mixing probability $\alpha$ by a local filtering on Bob's side and a local unitary on Alice's side. Therefore, it is steerable from Alice to Bob if and only if $\alpha > 1/2$; see Sec. III.C. Using the uniform distribution as an ansatz for the LHS ensemble, Bowles *et al.* showed that the state is unsteerable from Bob to Alice for

$$\cos^2(2\theta) \ge \frac{2\alpha - 1}{(2-\alpha)\alpha^3}.$$

With the complete characterization of steerability for two-qubit states described in Sec. II.C.1, the boundary of the set of unsteerable states from Bob to Alice has been obtained with high accuracy (Nguyen, Nguyen, and Gühne, 2019); see Fig. 7. It is clearly visible from the figure that $\varrho_{AB}(\alpha,\theta)$ is one-way steerable for a large range of parameters.

The one-way steering phenomenon can also be shown to persist when the measurements are extended to POVMs (Quintino *et al.*, 2015). An idea for constructing an example is as follows. One first embeds a state which is unsteerable from Alice to Bob with respect to projective measurements, but steerable for the other direction, into a higher dimension on Alice's side. One then constructs a state that admits a LHS model for all POVMs performed on Alice's side using Eq. (36), with the state $\sigma_A$ chosen to be supported only in the extended dimension on Alice's side. With this choice of $\sigma_A$, it is easy to show that the state is still steerable from Bob to Alice (Quintino *et al.*, 2015). The constructed state is thus also one-way steerable when one considers all POVMs.

The one-way steering phenomenon also attracts attention from the experimental side. Early experiments demonstrating one-way steering were carried out for continuous variable systems and Gaussian measurements (Händchen *et al.*, 2012). The effects of various types of noise on the direction of steering were later analyzed and probed experimentally by Qin *et al.* (2017). Experiments demonstrating one-way steering for discrete systems were performed by Sun *et al.* (2016), Wollmann *et al.* (2016), and Xiao *et al.* (2017). Sun *et al.*, (2016) and Xiao *et al.* (2017) concentrated on demonstrating

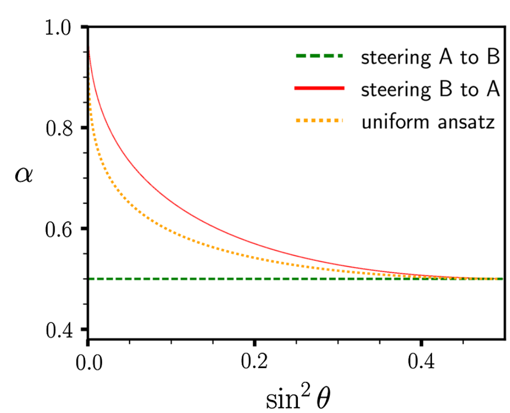


FIG. 7. The border of the one-way steerable area for the family of states given by Eq. (63). The thickness of the border for steering from $B$ to $A$ indicates the uncertain area. The inner bound for the border of steering from $B$ to $A$ with the uniform LHS ensemble as an ansatz is also included (dotted line).

one-way steering when measurements are limited to two and three settings. Wollmann et al. (2016) also demonstrated the persistence of the phenomena for POVMs. Most recently, it was realized (Baker et al., 2018) that existing experiments demonstrating one-way steering committed certain assumptions on the states or the measurements and were therefore inconclusive. A conclusive experiment (Tischler et al., 2018) was then performed shortly thereafter.

### E. Steering with bound entangled states

Now we discuss the steerability of so-called bound entangled states. This provides a relevant example for the fact that the characterization of steering gives new insights into old problems in entanglement theory.

Before presenting the result, we recall some facts about the entanglement criterion of the positivity of the partial transpose (the PPT criterion) and entanglement distillation. Let us start with the PPT criterion (Horodecki, Horodecki, and Horodecki, 1996; Peres, 1996b). Generally, for a two-particle state $\varrho = \sum_{ij,kl} \varrho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l|$, the partial transposition with respect to Bob is defined as

$$\varrho^{T_B} = \sum_{ij,kl} \varrho_{ij,lk} |i\rangle\langle j| \otimes |k\rangle\langle l|. \tag{64}$$

Similarly, one can define a partial transposition with respect to Alice which obeys $\varrho^{T_A} = (\varrho^{T_B})^T$. Note that the partial transposition may change the eigenvalues of a matrix, contrary to the full transposition.

The PPT criterion states that, for separable states, the partial transposition has no negative eigenvalues, $\varrho^{T_B} \geq 0$; such states are also called PPT states. It was further proven that, for systems consisting of two qubits ($2 \times 2$ systems) or one qubit and one qutrit ($2 \times 3$ systems), this criterion is sufficient for separability and all PPT states are separable. For all other dimensions, PPT entangled states exist, these states are, in some sense, weakly entangled, as they cannot be used for certain quantum information tasks.

The main quantum information task where PPT entangled states are useless is the task of entanglement distillation. Entanglement distillation is the process where many copies of some noisy entangled state are distilled to few highly entangled pure states via local operations and classical communication (Horodecki, Horodecki, and Horodecki, 1998). Not all entangled states can be used for distillation, and these undistillable states are called bound entangled. It was shown that PPT entangled states are bound entangled, but there are some states with a nonpositive partial transpose, for which it has been conjectured that they are also bound entangled (Pankowski et al., 2010).

Peres (1999) formulated the conjecture that bound entangled states do not violate any Bell inequality. This conjecture was based on an analogy between a general distillation protocol and Bell inequalities for many observers, but for a long time no proof could be found. In 2013, the conjecture was made that bound entangled states are also useless for steering (Pusey, 2013; Skrzypczyk, Navascués, and Cavalcanti, 2014). This so-called stronger Peres conjecture could potentially open a way to prove the original Peres conjecture, especially as the PPT criterion and the question of steerability are closely related to SDPs.

Moroder et al. (2014), however, showed that some bound entangled states can be used for steering, and an explicit example for two qutrits was given. The idea for finding the counterexample is the following. For a given state of two qutrits and two measurements with three outcomes each, one can determine the steerability of the assemblage $\{\varrho_{a|x}\}$ with a SDP; see also Sec. II.B.1. Considering the dual formulation of the SDP, one finds that the operator

$$W = A_{1|1} \otimes Z_{13} + A_{2|1} \otimes Z_{23} + A_{1|2} \otimes Z_{31} + A_{2|2} \otimes Z_{32} + (A_{3|1} + A_{3|1} - \mathbb{1}) \otimes Z_{33} \tag{65}$$

defines a steering inequality, that is, $\mathrm{Tr}(\varrho W) \geq 0$ for unsteerable states. Here the $A_{a|x}$ are arbitrary measurement operators for Alice, and the set $\{Z_{13}, Z_{23}, Z_{31}, Z_{32}, Z_{33}\}$ consists of five positive operators obeying the four semidefinite constraints $Z_{i3} + Z_{3j} - Z_{33} \geq 0$ for $i, j \in \{1, 2\}$.

Given this steering inequality, one can look for steerable PPT states by an iteration of SDPs: One starts with a random initial steerable state $\varrho$ and fixes Alice's measurements $A_{a|x}$ to be measurements in two mutually unbiased bases. Then, by optimizing the $Z_{ij}$ via a SDP, one can minimize the mean value $\mathrm{Tr}(\varrho W)$ and find the optimal steering inequality $W$. Given this $W$, one can ask for the minimal expectation value of it with respect to PPT states; this is again a SDP. Having found the PPT state with the smallest $\mathrm{Tr}(\varrho W)$, one can optimize over the $Z_{ij}$ again and then iterate. In practice, this procedure converges quickly toward PPT states which are steerable, delivering the desired counterexamples to the stronger Peres conjecture.

Having found the counterexamples, it is a natural question as to whether these states also violate a Bell inequality. Indeed, as has been shown by Vértesi and Brunner (2014), these states are also counterexamples to the original Peres conjecture. Finally, Yu and Oh (2017) presented an analytical approach, giving explicit families of PPT entangled states in any dimension $d \geq 3$ that, for appropriate parameters, violate Bell inequalities or can be used for steering; see Fig. 8.

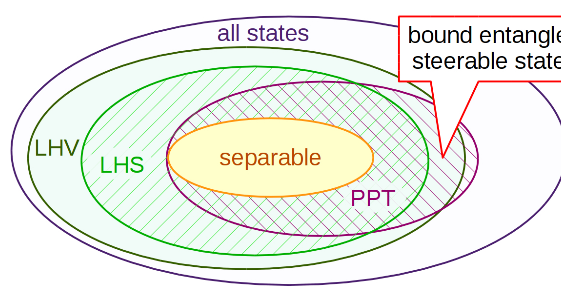


FIG. 8. Inclusion relation between the PPT states and entanglement, steering, and Bell inequality violations. Separable states are PPT, but some entangled states are PPT as well. PPT entangled states are bound entangled, as no pure state entanglement can be distilled from them. There exist, however, PPT states that can be used for steering and also PPT states that violate Bell inequalities. These states are counterexamples to the Peres conjecture.

### F. Steering maps and dimension-bounded steering

In this section, we describe how the steering problem can be viewed as a certain kind of separability problem (Moroder *et al.*, 2016). This allows one to apply the powerful techniques of entanglement theory (Gühne and Tóth, 2009; Horodecki *et al.*, 2009), and to study problems such as the detection of steering, if Bob's system is not well characterized and only its dimension is known.

To formulate the main idea, it suffices to consider the case of two measurements ($x \in \{1,2\}$) with two outcomes ($\pm$) on Alice's side. As discussed in Sec. II.B.1, steerability of the assemblage $\{\varrho_{a|x}\}$ can be decided by a SDP. More precisely, Eq. (25) states that the assemblage is unsteerable if one finds four positive semidefinite operators $\omega_{ij}$ with $i,j=\pm$ such that $\varrho_{+|1}=\omega_{++}+\omega_{+-}$, $\varrho_{+|2}=\omega_{++}+\omega_{-+}$, $\varrho_{-|1}=\omega_{-+}+\omega_{--}$, and $\varrho_{-|2}=\omega_{+-}+\omega_{--}$. These equations are not independent. If one takes $\omega_{++}$ as a free variable, one has the relations $\omega_{+-}=\varrho_{+|1}-\omega_{++}$, $\omega_{-+}=\varrho_{+|2}-\omega_{++}$, and $\omega_{--}=\varrho_B-\varrho_{+|1}-\varrho_{+|2}+\omega_{++}$. Of course, this is a valid solution only if all $\omega_{ij}$ are positive semidefinite.

Then one takes four positive definite operators $Z_{ij}$ with $i,j=\pm$ that obey the relation $Z_{++}=Z_{+-}+Z_{-+}-Z_{--}$, and one considers the bipartite operator

$$\Sigma_{AB}=\sum_{ij}Z_{ij}\otimes\omega_{ij}. \tag{66}$$

This is, after appropriate normalization, a separable state as in Eq. (4). The point is that with the given relations on the $\omega_{ij}$ and $Z_{ij}$, this state can be written as

$$\Sigma_{AB}=Z_{+-}\otimes\varrho_{+|1}+Z_{-+}\otimes\varrho_{+|2}+Z_{--}\otimes(\varrho_B-\varrho_{+|1}-\varrho_{+|2}), \tag{67}$$

as can be verified by direct inspection. Here all of the dependencies on the $\omega_{ij}$ drop out, so $\Sigma_{AB}$ is uniquely determined by the assemblage and the $Z_{ij}$ only. Also, the required normalization follows directly from Eq. (67).

From this, the desired connection to the separability problem follows: Given an unsteerable assemblage and operators $Z_{ij}$ obeying these conditions, the state $\Sigma_{AB}$ in Eq. (67) is separable. Moreover, one can show the opposite direction: If the assemblage is steerable, then there exist a set of operators $Z_{ij}$ such that the state $\Sigma_{AB}$ is entangled. In this case, the entanglement of $\Sigma_{AB}$ can even be detected by a special entanglement witness, namely, the flip operator.

The statement can be generalized to an arbitrary number of measurements and outcomes (Moroder *et al.*, 2016). In fact, it is related to the dual of the original SDP.

This reformulation of the steerability problem can give insights in the detection of steering if the measurements on Bob's side are not fully characterized, but only the dimension of the space that the measurements act on is known. The core idea is the following: For any bipartite state $\varrho_{AB}$ and sets of local orthonormal observables $G_k^A$ and $G_l^B$ [that is, $\mathrm{Tr}(G_i^X G_j^X)=\delta_{ij}$ for $X\in\{A,B\}$], one can build the matrix $\Lambda_{kl}=\mathrm{Tr}(\varrho_{AB}G_k^A\otimes G_l^B)$. Then the CCNR criterion states that if $\varrho_{AB}$ is separable, then the trace norm is bounded by 1, $\|\Lambda\|_1\le 1$ (Chen and Wu, 2003; Rudolph, 2005); see also Sec. II.C.3. This criterion was already used to detect entanglement with uncharacterized devices if the dimension was known: If Alice and Bob make uncharacterized measurements $A_k$ and $B_l$, they can build the expectation value matrix $\Delta_{kl}=\mathrm{Tr}(\varrho_{AB}A_k\otimes B_l)$ and, using the dimension assumption, estimate the trace norm $\|\Lambda\|_1$ (Moroder and Gittsovich, 2012).

A similar approach can be used for steering (Moroder *et al.*, 2016). For a choice of $Z_{ij}$, one considers the state $\Sigma_{AB}$. Then on Alice's side, one takes a set of local orthogonal observables $G_k^A$ and, for Bob's side, uncharacterized measurements $B_l$ and builds an expectation value matrix, which can be used to estimate whether $\Sigma_{AB}$ violates the CCNR criterion. If this is the case, then the original assemblage was steerable. The resulting criteria are strong: For two-qubit Werner states $\varrho(p)=p|\psi^-\rangle\langle\psi^-|+(1-p)\mathbb{1}/4$ and Pauli measurements $\sigma_x$, $\sigma_y$, and $\sigma_z$, one can evaluate from the data the steering inequality in Eq. (12). It detects steerability for $p>1/\sqrt{3}$, which is the same threshold as the steering inequality. So for this case, the approach allows one to draw the same conclusion from the resulting data, but without assuming that the measurements were correct Pauli measurements. The only assumption that is made is that Bob's space is a qubit.

### G. Superactivation of steering

Let us return to the formulation of quantum steering as a simulation task where Alice tries to convince Bob that she can steer his system from a distance as discussed in Sec. I. Note that, in this protocol, Alice has to prepare a large number of pairs of particles in a certain state. One of the particles in each pair is then sent to Bob. Note that it is crucial for Alice to prepare *many copies* of the state so that later on Bob can do tomography to verify the steered states on his side. Alice then declares the set of measurements she can make, or equivalently the assemblage she can steer Bob's system to. To maximize her steering ability, Alice clearly should choose the largest set of measurements. Most often, Alice's measurements are assumed to be projective measurements (or POVMs) on separated particles on her side. This, however, is not yet the maximal set of measurements she can do. As Alice has prepared a large number of bipartite states, she can actually make collective measurements on several particles on her side. We will see that when such collective measurements are considered, the steerability of a state may change. More precisely, for an unsteerable (but entangled) state $\varrho_{AB}$, one asks whether there exists a finite number $n$ such that $\varrho_{AB}^{\otimes n}$ is steerable. In this case, we say that the quantum steerability of $\varrho_{AB}$ can be superactivated.

For nonseparability, a similar question is answered trivially negative for any states, but for Bell nonlocality, it has been extensively investigated since the work of Peres (1996a). For Bell nonlocality, the confirmative answer was first obtained by Palazuelos (2012) and later refined by Cavalcanti *et al.* (2013). They showed that, indeed, for a certain state $\varrho_{AB}$ that admits a LHV model, for sufficiently large $n$, $\varrho_{AB}^{\otimes n}$ can violate a Bell inequality. Note that this is distinct from the notion of superactivation of bound entanglement (Shor, Smolin, and Thapliyal, 2003). While the superactivation of Bell nonlocality that they investigated also implies the ability to superactivate quantum steering, exact characterizations of quantum steering also significantly simplify the understanding of the phenomenon. In fact, Quintino, Brunner, and Huber (2016) and Hsieh, Liang, and Lee (2016) extended the results of Cavalcanti et al. (2013) to show that the steerability of all unsteerable states $\varrho_{AB}$ that satisfy the so-called reduction criterion for entanglement (Horodecki and Horodecki, 1999) can be superactivated. The reduction criterion states that if $\mathbb{1}_A \otimes \varrho_B - \varrho_{AB}$ is not positive, then the state $\varrho_{AB}$ is nonseparable. As satisfying the reduction criterion is a necessary and sufficient condition for a two-qubit or a qubit-qutrit state to be nonseparable (Horodecki and Horodecki, 1999), the steerability of all entangled states of dimension $2 \times 2$ or $2 \times 3$ can be superactivated.

Their idea is based on the exact threshold for quantum steering of the isotropic state in Eq. (55). For convenience, here the isotropic state is reparametrized as

$$S_d^f = f|\phi_+\rangle\langle\phi_+| + (1-f)\frac{\mathbb{1} - |\phi_+\rangle\langle\phi_+|}{d^2-1}, \qquad (68)$$

with the same notation as defined in Eq. (53) and $f = 1 - (1 - 1/d^2)(1 - \eta)$. According to Eq. (55), the isotropic state in Eq. (68) is unsteerable if and only if $f \leq [(1+d)H_d - d]/d^2$. It is known (Horodecki and Horodecki, 1999) that a state that violates the reduction criterion can be brought into an entangled isotropic state with $f > 1/d$ by local filtering on Bob's side and the so-called isotropic twirling operation. As mentioned in Sec. III.C, local filtering on Bob's side does not change the steerability of the state. The isotropic twirling operation consists of averaging the state under certain random local unitary transformations and thus does not increase the steerability. It is sufficient to show that the steerability of the isotropic state $S_d^f$ with $f > 1/d$ can be superactivated. This again can be shown by observing that the isotropic twirling on $(S_d^f)^{\otimes n}$ yields the isotropic state $S_{d^n}^{f^n}$ of dimension $d^n \times d^n$. Thus, $(S_d^f)^{\otimes n}$ is steerable if

$$f > \frac{[(1+d^n)H_{d^n} - d^n]^{1/n}}{d^2}. \qquad (69)$$

At large $n$, the right-hand side asymptotically approaches $1/d$. Therefore, whenever $f > 1/d$, there exists an $n$ such that the inequality is satisfied, or, equivalently, the steerability of $S_d^f$ can be superactivated.

Beyond states that violate the reduction criterion, one may ask whether quantum steerability, or, more generally, Bell nonlocality, can always be superactivated for arbitrary entangled states. This question remains a challenge for future research. If this were the case, the hierarchy of quantum nonlocality would be unified into a single concept (Cavalcanti et al., 2013).

Besides the notion of superactivation of quantum nonlocality via collective measurements on multiple copies of the previously described state, there is also the notion of superactivation of quantum nonlocality via local filtering (on both sides). The phenomenon dates back to the work of Popescu (1995), who showed that the Werner states in dimension $d \geq 5$ that admit LHS models for projective measurements can violate a Bell inequality after appropriate local filtering. Recently, Hirsch et al. (2013) showed that there are states that admit a LHS model for POVMs but become Bell nonlocal after appropriate local filtering. However, Hirsch, Quintino, Bowles et al. (2016) later showed that there are also entangled states whose quantum nonlocality cannot be superactivated by local filtering.

## IV. JOINT MEASURABILITY AND STEERING

In this section, we discuss the problem of joint measurability, to which steering is related in a many-to-one manner (Quintino, Vértesi, and Brunner, 2014; Uola, Moroder, and Gühne, 2014; Uola et al., 2015; Kiukas et al., 2017). Joint measurability is a natural extension of commutativity for general measurements. Operationally, it corresponds to the possibility of deducing the statistics of several measurements from the statistics of a single one. The connection between the concepts of joint measurability and steering unlocks the technical machinery developed within the framework of quantum measurement theory to be used in the context of quantum correlations. It is worth noting that joint measurability was studied extensively for a few decades before steering was formulated in its modern form. We review the connection on three levels: joint measurability on Alice's side (pure states), on Bob's side (mixed states), and on the level of the incompatibility breaking quantum channels (Choi isomorphism). Moreover, we discuss in detail how known results on one field can be mapped to new ones on the other.

### A. Measurement incompatibility

Measurement incompatibility manifests itself in various operationally motivated forms in quantum theory. Maybe the best-known notion is that of noncommutativity. Here by noncommutativity we mean the mutual noncommutativity of the POVM elements of two POVMs; i.e., for POVMs $\{A_a\}_a$ and $\{B_b\}_b$, we ask whether or not $[A_a, B_b] = 0$ for all $a$, $b$. From textbooks on quantum mechanics, we know that the noncommutativity of observables places certain restrictions on the variances of the measured observables. Such restrictions do not, however, give any further operational insight into the involved measurements—they just follow from simple mathematics.

One possible operationally motivated extension of commutativity is that of joint measurability. Namely, one can ask whether two measurements can be performed simultaneously (or jointly), i.e., whether there exists a third measurement whose statistics can be classically processed to match those of the original pair. Further fine-tunings of measurement incompatibility have been presented in the literature, e.g., coexistence, broadcastability, and nondisturbance (Heinosaari and Wolf, 2010; Busch et al., 2016; Heinosaari, 2016), see also Sec. IV.E. Typically, all of the incompatibility related extensions of noncommutativity (on a single system) coincide with noncommutativity for projective measurements, but for the case of POVMs, they form a strict hierarchy (Heinosaari and Wolf, 2010). It is worth mentioning that, in the process matrix formulation of POVMs, even commuting process POVMs can be incompatible (Sedlák et al., 2016).

For investigating steering from the measurement perspective, the notion of joint measurability appears to be most fitting. A set $\{A_{a|x}\}_{a,x}$ of POVMs (i.e., positive operators summing up to the identity for every $x$) is said to be jointly measurable if there exists a POVM $\{G_\lambda\}_\lambda$ together with classical postprocessings $\{p(a|x,\lambda)\}_{a,x,\lambda}$ such that

$$A_{a|x} = \sum_\lambda p(a|x,\lambda)G_\lambda. \tag{70}$$

The POVM $\{G_\lambda\}_\lambda$ is called a joint observable or a joint measurement of the set $\{A_{a|x}\}_{a,x}$.

To give an example of a set of jointly measurable POVMs, one could use a mutually commuting pair of POVMs, in which case a joint measurement is given by a POVM whose elements are products of the original ones. For a more insightful example, we take a pair of noisy Pauli measurements defined as

$$S^\mu_{\pm|x} \coloneqq \tfrac{1}{2}(\mathbb{1} \pm \mu\sigma_x), \tag{71}$$

$$S^\mu_{\pm|z} \coloneqq \tfrac{1}{2}(\mathbb{1} \pm \mu\sigma_z), \tag{72}$$

where $0 < \mu \leq 1$. The question now is how to find candidates for a joint measurement. For this pair, an educated guess [i.e., a candidate with similar symmetry to the pair $(S^\mu_{\pm|x}, S^\mu_{\pm|z})$] gives

$$G^\mu_{i,j} \coloneqq \tfrac{1}{4}(\mathbb{1} + i\mu\sigma_x + j\mu\sigma_z), \tag{73}$$

where $i,j \in \{-,+\}$. One notices straightaway that

$$S^\mu_{\pm|x} = G^\mu_{\pm,+} + G^\mu_{\pm,-}, \tag{74}$$

$$S^\mu_{\pm|z} = G^\mu_{+,\pm} + G^\mu_{-,\pm}. \tag{75}$$

In other words, there exist deterministic postprocessings that give the original measurements. The last thing to check is that $\{G^\mu_{i,j}\}_{i,j}$ forms a POVM. As the normalization follows from the definition, one is left with checking the positivity of the elements, which is equivalent to $\mu \leq 1/\sqrt{2}$. It can be shown that this is indeed the optimal threshold for joint measurability in our example; i.e., beyond this threshold, the POVMs $\{S^\mu_{\pm|x}\}$ and $\{S^\mu_{\pm|z}\}$ do not admit a joint measurement (Busch, 1986).

This example shows that joint measurability is indeed a proper generalization of commutativity. In the literature, many such examples have been discussed in finite and continuous variable quantum systems (Busch et al., 2016). A typical question is as follows: How much noise can be added until measurements become jointly measurable? For small numbers of measurements and outcomes, this can be efficiently checked with SDP (Wolf, Perez-Garcia, and Fernandez, 2009; Uola et al., 2015). For more complicated scenarios, various optimal and semioptimal analytical and numerical techniques have been developed (Kunjwal, Heunen, and Fritz, 2014; Heinosaari, Miyadera, and Ziman, 2016; Uola et al., 2016; Bavaresco et al., 2017; Designolle et al., 2019).

### B. Joint measurability on Alice's side

Comparing the definition of joint measurability with that of unsteerability, one recognizes similarities. Indeed, joint measurability is a question about the existence of suitable postprocessings and a common POVM, whereas unsteerability regards the existence of suitable response functions and a common state ensemble. To make the connection exact, we recall the main result of Quintino, Vértesi, and Brunner (2014) and Uola, Moroder, and Gühne (2014):

A set of measurements $\{A_{a|x}\}_{a,x}$ is not jointly measurable if and only if it can be used to demonstrate steering with some shared state.

To be more precise, using a jointly measurable set of observables on Alice's side, i.e., $A_{a|x} = \sum_\lambda p(a|x,\lambda)G_\lambda$, and a shared state $\varrho_{AB}$ results in a state assemblage

$$\varrho_{a|x} = \sum_\lambda p(a|x,\lambda)\mathrm{tr}_A[(G_\lambda \otimes \mathbb{1})\varrho_{AB}] \tag{76}$$

$$= \sum_\lambda p(a|x,\lambda)\sigma_\lambda, \tag{77}$$

where $\sigma_\lambda = \mathrm{tr}_A[(G_\lambda \otimes \mathbb{1})\varrho_{AB}]$. Hence, the existence of a joint observable for Alice's measurements implies the existence of a local hidden state model. For the other direction, using a full (finite) Schmidt rank state $|\psi\rangle = \sum_i \lambda_i |ii\rangle$, one has

$$\varrho_{a|x} \coloneqq \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})|\psi\rangle\langle\psi|] = CA^T_{a|x}C, \tag{78}$$

where $C = \sum_j \lambda_j |j\rangle\langle j|$ and $X^T$ is the transpose of the operator $X$ in the basis $\{|i\rangle\}_i$. Assuming that the assemblage $\{\varrho_{a|x}\}_{a,x}$ has a local hidden state model, one gets

$$A_{a|x} = \sum_\lambda p(a|x,\lambda)C^{-1}\sigma^T_\lambda C^{-1}, \tag{79}$$

from which it is clear that $\{C^{-1}\sigma^T_\lambda C^{-1}\}_\lambda$ forms the desired joint measurement of $\{A_{a|x}\}_{a,x}$.

To demonstrate a possible use of this result, one can consider a steering scenario where Alice performs measurements on a noisy isotropic state. This noise in the state can be translated to Alice's measurements by writing

$$\mathrm{tr}[(A_{a|x} \otimes \mathbb{1})\varrho^\mu_{AB}] = \mathrm{tr}[(A^\mu_{a|x} \otimes \mathbb{1})\varrho_{AB}], \tag{80}$$

where

$$\varrho^\mu_{AB} = \mu|\psi^+\rangle\langle\psi^+| + \frac{(1-\mu)}{d^2}\mathbb{1}$$

and

$$A^{\mu}_{a|x} = \mu A_{a|x} + \frac{(1-\mu)}{d}\mathrm{tr}[A_{a|x}]\mathbb{1}$$

with $\mu \in [0,1]$. For different sets of measurements on Alice's side, one can either solve the steerability by using known incompatibility results or vice versa. To give an example, consider the known (Wiseman, Jones, and Doherty, 2007) steerability threshold for the noisy isotropic state (with projective measurements)

$$\mu^* = \left(\sum_{n=1}^{d}\frac{1}{n} - 1\right)\Big/(d-1)\cdot$$

Using Eq. (80), one sees that for any $\mu > \mu^*$ there exists a set of projective measurements that remains incompatible with the amount $\mu$ of white noise. On the contrary, any set of projective measurements with an amount $\mu \leq \mu^*$ of white noise results in an unsteerable state assemblage (with the isotropic state), and hence such a set is jointly measurable.

It is worth noting that the connection between the incompatibility of Alice's measurements and the steerability of the resulting assemblage is strongly motivated by a similar work on nonlocality. Wolf, Perez-Garcia, and Fernandez (2009) proved the incompatibility of Alice's measurements to be equivalent to the ability of violating the CHSH inequality (when optimizing over Bob's measurements and the shared state). This connection, however, is known not to be true in general. Bene and Vértesi (2018) and Hirsch, Quintino, and Brunner (2018) presented counterexamples for the nonlocality connection in scenarios with more measurement settings; i.e., there exist sets of measurements that are not jointly measurable but always lead to local correlations. In contrast, joint measurability and steering can both be described in terms of operational contextuality (Tavakoli and Uola, 2019). The CHSH inequality is a criterion for this type of contextuality. It is an open question as to whether there are other contextuality inequalities that fully characterize incompatibility. Tavakoli and Uola (2019) provided numerical evidence that a specific family of contextuality criteria generalizing the CHSH inequality characterizes the incompatibility of sets of binary qubit measurements. Such a characterization, among others, is directly applicable to the steerability of state assemblages by the use of the techniques presented in the following subsection.

### C. Joint measurability on Bob's side

The connection between steering and joint measurements presented in the previous section is based on the use of full Schmidt rank states (on finite-dimensional systems). To loosen the assumption on purity of the state, we recall the main result of Uola et al. (2015):

The question of steerability (of a state assemblage) is a nonnormalized version of the joint measurement problem.

More precisely, by normalizing a state assemblage $\{\varrho_{a|x}\}_{a,x}$, one gets abstract POVMs $\tilde{B}_{a|x} := \varrho_B^{-1/2}\varrho_{a|x}\varrho_B^{-1/2}$, where $\varrho_B = \sum_a \varrho_{a|x}$, and a pseudoinverse is used when necessary. Note that we use a tilde to distinguish between Bob's actual measurements and the normalized state assemblage (which consists of abstract POVMs on a possibly smaller dimensional system than the one Bob's measurements act on).

It is straightforward to show (Uola et al., 2015) that the state assemblage $\{\varrho_{a|x}\}_{a,x}$ is steerable if and only if the abstract POVMs $\{\tilde{B}_{a|x}\}_{a,x}$ are not jointly measurable. Namely, as the normalization keeps the postprocessing functions fixed, the local hidden states map to joint measurements of the normalized assemblage, and joint measurements map to local hidden states.

Such a connection broadens the set of techniques that are translatable between the fields of joint measurability and steering. In general, joint measurability criteria map to steering criteria, and vice versa. To give an example, we take a well-known joint measurability characterization of two qubit POVMs (Busch, 1986). Namely, take two-qubit POVMs of the form

$$A_{\pm|x} := \tfrac{1}{2}(\mathbb{1} \pm \vec{a}_x \cdot \vec{\sigma}), \tag{81}$$

where $x = 1, 2$. This pair is jointly measurable if and only if

$$\|\vec{a}_1 + \vec{a}_2\| + \|\vec{a}_1 - \vec{a}_2\| \leq 2. \tag{82}$$

Note that this criterion is necessary for joint measurability in the more general case, i.e., for pairs of POVMs given as

$$A_{+|x} := \tfrac{1}{2}[(1+\alpha_x)\mathbb{1} \pm \vec{a}_x \cdot \vec{\sigma}], \qquad A_{-|x} = \mathbb{1} - A_{+|x}, \tag{83}$$

where $\alpha_x \in [-1, 1]$ and $\|\vec{a}_x\| \leq 1 + \alpha_x$. Stano, Reitzner, and Heinosaari (2008), Busch and Schmidt (2010), and Yu et al. (2010) gave a necessary and sufficient criterion for joint measurability of such pairs as

$$(1 - F_1^2 - F_2^2)\left(1 - \frac{\alpha_1^2}{F_1^2} - \frac{\alpha_2^2}{F_2^2}\right) \leq (\vec{a}_1 \cdot \vec{a}_2 - \alpha_1\alpha_2)^2, \tag{84}$$

with

$$F_i = \tfrac{1}{2}\left[\sqrt{(1+\alpha_i)^2 - \|\vec{a}_i\|^2} + \sqrt{(1-\alpha_i)^2 - \|\vec{a}_i\|^2}\right],$$

for $i = 1, 2$.

The criteria in Eqs. (82) and (84) are both steering inequalities. The latter characterizes all pairs of binary unsteerable assemblages in the qubit case. Labeling members of such assemblages by $\varrho_{\pm|1} = (1/4)(\mathbb{1} \pm \lambda\sigma_z)$ and $\varrho_{\pm|2} = \beta^{\pm}\mathbb{1} \pm r^{\pm}\sigma_z$, we present a comparison of these criteria in Fig. 9 (Uola et al., 2015).

As another example, we demonstrate how the steering robustness defined in Eq. (30) translates to an incompatibility robustness (Uola et al., 2015). Recall that the steering robustness $\mathrm{SR}(\varrho_{a|x})$ of an assemblage $\{\varrho_{a|x}\}_{a,x}$ can be written as

$$\begin{aligned} &\min\ t \geq 0 \\ &\quad s.t.\ \frac{\varrho_{a|x} + t\gamma_{a|x}}{1+t}\ is \text{ unsteerable for all } a, x. \end{aligned} \tag{85}$$

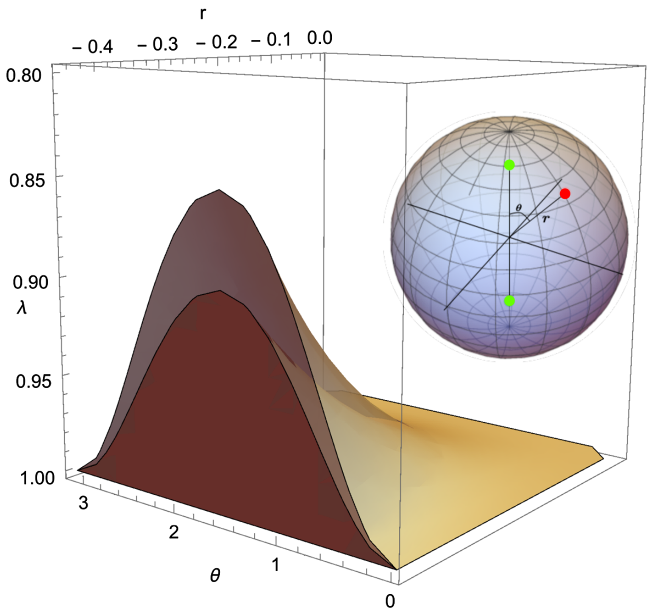


FIG. 9. Regions of the parameters $\lambda$, $r$, $\theta$ allowing for steering, detected by the inequality (82) (inner region) and the inequality (84) (outer region), with $r = \|\vec{r}^{+}\|$ and $\theta$ being the angle between $\vec{r}^{+}$ and the $z$ axis, and $\beta^{+} = 0.45$ (fixed). (Inset) Representation in the Bloch sphere of the reduced states $\varrho_{\pm|1}$ (green points one upon the other) and $\varrho_{+|2}$ (red point on the right). The normalization factor $\beta^{+} = \mathrm{Tr}[\varrho_{+|2}]$ is not represented. From Uola et al., 2015.

Here the optimization is over assemblages $\{\gamma_{a|x}\}_{a,x}$ and positive numbers $t$; see also Fig. 3. Mapping a state assemblage to a set of POVMs, one can define an incompatibility robustness $\mathrm{IR}(\tilde{B}_{a|x})$ for a set $\{\tilde{B}_{a|x}\}_{a,x}$ as (Uola et al., 2015)

$$\min\ t \geq 0$$
$$\text{s.t.}\ \frac{\tilde{B}_{a|x} + tT_{a|x}}{1+t}\ \text{is jointly measurable for all}\ a, x. \quad (86)$$

Here the optimization is performed over POVMs $\{T_{a|x}\}_{a,x}$ and positive numbers $t$. Note that this definition works also for a generic set of POVMs; i.e., the POVMs do not need to originate explicitly from a steering problem. To give the incompatibility robustness $\mathrm{IR}(A_{a|x})$ of a finite set of POVMs $\{A_{a|x}\}_{a,x}$ in the standard SDP form, one writes (Uola et al., 2015)

$$\begin{aligned} &\min\ \mathrm{tr}\left[\frac{\sum_\lambda G'_\lambda}{d}\right] \\ &\text{s.t.}\ \sum_\lambda D(a|x,\lambda)G'_\lambda \geq A_{a|x}, \quad \text{for all}\ a, x, \\ &G'_\lambda \geq 0, \quad \text{for all}\ \lambda, \\ &\sum_\lambda G'_\lambda = \mathbb{1}\ \mathrm{tr}\left[\sum_\lambda G'_\lambda\right]/d, \end{aligned} \quad (87)$$

where $\{D(a|x,\lambda)\}_{a,x,\lambda}$ are deterministic postprocessings.

To finish this section, we stress that even though the normalized state assemblages appear as abstract POVMs that do not have a direct connection to the POVMs actually measured in the correlation experiment, the normalized state assemblages can be interpreted as the POVMs Alice should measure on the canonical purification of $\sum_a \varrho_{a|x}$ in order to prepare $\{\varrho_{a|x}\}_{a,x}$. Hence, we see that the results on steering and joint measurements stated in Sec. IV.B (i.e., nonjointly measurable POVMs allow steering with some shared state) are closely related to the connection presented here.

### D. Incompatibility breaking quantum channels

The extension of the connection between steering and joint measurements to infinite-dimensional systems and measurements with possibly continuous outcome sets was done by Kiukas et al. (2017). The point is to generalize the Choi-Jamiołkowski isomorphism to arbitrary shared states (i.e., not only ones with maximally mixed marginals). The generalization gives a one-to-one connection between states $\varrho$ with a fixed full-rank marginal $\varrho_B$ (on Bob's side) and channels $\Lambda$ from Bob to Alice through

$$\varrho = (\Lambda \otimes \mathbb{1})(|\psi_{\varrho_B}\rangle\langle\psi_{\varrho_B}|), \quad (88)$$

where $|\psi_{\varrho_B}\rangle = \sum_{i=1}^d \sqrt{s_i}|ii\rangle$ is a purification of $\varrho_B = \sum_{i=1}^d s_i|i\rangle\langle i|$. To see a connection to incompatibility, one writes the channel corresponding to a given state (in the Heisenberg picture) as

$$\varrho_B^{1/2}\Lambda^\dagger(A)\varrho_B^{1/2} = \mathrm{tr}_A[(A\otimes\mathbb{1})\varrho]^T, \quad (89)$$

where $^T$ is a transpose in the eigenbasis of $\varrho_B$. Inputting sets of POVMs $\{A_{a|x}\}_{a,x}$ on the left-hand side of Eq. (89) results in transposed state assemblages on the right-hand side. From this, it is clear (at least in the finite-dimensional case in which $\varrho_B$ can be inverted) that the Heisenberg channel $\Lambda^\dagger$ sends Alice's POVMs to normalized state assemblages (i.e., POVMs) on Bob's side. Joint measurability of these POVMs is equivalent to the unsteerability of $\{\varrho_{a|x}\}_{a,x}$. It turns out that this correspondence can be extended to infinite-dimensional systems (Kiukas et al., 2017), resulting in a fully general connection between steering and joint measurements:

A state assemblage $\{\varrho_{a|x}\}_{a,x}$ given by Alice's measurements $\{A_{a|x}\}_{a,x}$ and a state $\varrho$ is steerable if and only if the POVMs $\{\Lambda^\dagger(A_{a|x})\}_{a,x}$ are not jointly measurable.

Note that although the notation here is adapted to the case of discrete POVMs (to avoid technicalities), the connection also works for POVMs with continuous outcome sets.

To demonstrate the power of this result, we list some of its implications (Kiukas et al., 2017). First, the connection is quantitative in the sense that the incompatibility robustness of $\{\Lambda^\dagger(A_{a|x})\}_{a,x}$ coincides with the so-called consistent steering robustness (i.e., a special case of steering robustness; one allows mixing only with assemblages that have the same total state as the original assemblage) of $\{\varrho_{a|x}\}_{a,x}$. Second, for pure states, the corresponding channel $\Lambda^\dagger$ is unitary, hence extending the main results of Quintino, Vértesi, and Brunner (2014) and Uola, Moroder, and Gühne (2014) presented in Sec. IV.B to the infinite-dimensional case. Third, the result characterizes unsteerable states as those whose corresponding Choi-Jamiołkowski channel is incompatibility breaking (i.e., outputs only jointly measurable observables). Finally, seemingly different steering problems (such as certain bipartite states subjected to photon loss and systems with amplitude damping dynamics) can have the same channel $\Lambda$, hence making it possible to solve many steering problems in one attempt.

### E. Further topics on incompatibility

As mentioned at the beginning of this section, measurement incompatibility manifests itself in various ways in quantum theory. In this section, we review briefly two well-known fine-tunings of commutativity and discuss their relation to quantum correlations.

First, in the case of joint measurability, one asks for the existence of a common POVM and a set of postprocessings. One can relax this concept by dropping the assumption on postprocessings. Namely, we say that a set of POVMs $\{A_{a|x}\}_{a,x}$ is *coexistent* if there exists a POVM $\{C_\lambda\}_\lambda$ such that

$$A_{X|x} = \sum_{\lambda\in\tau_{X|x}} C_\lambda, \tag{90}$$

where $\tau_{X|x}$ is a subset of outcomes of $\{C_\lambda\}_\lambda$ for every pair $(X, x)$. Here we used the notation $X$ to emphasize that the definition is required to hold not only for all POVM elements of $\{A_{a|x}\}_{a,x}$ but also for sums of outcomes; e.g., for $X = \{a_1, a_2\}$, one has $A_{X|x} = A_{a_1|x} + A_{a_2|x}$. To give the concept a physical interpretation, Heinosaari, Miyadera, and Ziman (2016) noted that the definition is equivalent to the joint measurability of the set of all binarizations (i.e., coarse grainings to two-valued ones) of the involved measurements. Whereas it is clear that joint measurability implies coexistence (by the use of deterministic postprocessings), the other direction does not hold in general (Reeb, Reitzner, and Wolf, 2013; Pellonpää, 2014).

As coexistence is closely related to joint measurability, one can ask whether anything can be learned from using this concept in the realm of steering. As Uola, Moroder, and Gühne (2014) pointed out, one can reach steering with coexistent measurements on the uncharacterized side (provided that the measurements are not jointly measurable). What has not appeared in the literature so far, but we wish to point out here, is that when using this concept on the characterized side, one can find examples of steerable assemblages that nevertheless form one ensemble. Consider the example of coexistent but not jointly measurable POVMs given by Reeb, Reitzner, and Wolf (2013) by defining a vector

$$|\varphi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$$

and the POVMs

$$A_{i|1} \coloneqq \tfrac{1}{2}(\mathbb{1} - |i\rangle\langle i|), \quad i = 1, 2, 3, \tag{91}$$

$$A_{+|2} \coloneqq \tfrac{1}{2}|\varphi\rangle\langle\varphi|, \qquad A_{-|2} \coloneqq \mathbb{1} - A_{+|2}. \tag{92}$$

To see that these POVMs are coexistent, one can define a POVM $\{C_\lambda\}_\lambda$ through the elements

$$\left(\tfrac{1}{2}|1\rangle\langle 1|, \tfrac{1}{2}|2\rangle\langle 2|, \tfrac{1}{2}|3\rangle\langle 3|, \tfrac{1}{2}|\varphi\rangle\langle\varphi|, \tfrac{1}{2}(\mathbb{1} - |\varphi\rangle\langle\varphi|)\right). \tag{93}$$

For proof that these POVMs are not jointly measurable, see the work of Reeb, Reitzner, and Wolf (2013). Applying the mapping between measurement assemblages and state assemblages (with a full-rank state $\varrho_B$; see also Sec. IV.C) to the previous (or any similar) example, one ends up with a steerable state assemblage that nevertheless fits into a single ensemble.

As another example, we consider the concept of measurement disturbance. A POVM $\{A_a\}_a$ is called *nondisturbing* with respect to a POVM $\{B_b\}_b$ if there exists an instrument, i.e., a collection of completely positive maps summing to a quantum channel, $\{\mathcal{I}_a\}_a$ implementing $\{A_a\}_a$, i.e., $\mathrm{tr}[\mathcal{I}_a(\varrho)] = \mathrm{tr}[A_a\varrho]$ for all states $\varrho$ such that

$$\sum_a \mathrm{tr}[\mathcal{I}_a(\varrho)B_b] = \mathrm{tr}[\varrho B_b] \tag{94}$$

holds for all states $\varrho$ and all outcomes $b$.

Nondisturbance is located between commutativity and joint measurability. Clearly, commutativity implies nondisturbance by the use of the Lüders rule, and nondisturbance implies joint measurability by defining for a nondisturbing scenario $G_{a,b} = \mathcal{I}_a^\dagger(B_b)$, where the dagger refers to the Heisenberg picture. For proof that the implications cannot be reversed in general, and for a more detailed analysis on when the implications are reversible, see the work of Heinosaari and Wolf (2010).

As some disturbing measurements can be jointly measurable, measurement disturbance is necessary but not sufficient for steering. One could, however, ask whether there exist other types of quantum correlations or tasks for which disturbance is necessary and sufficient. It turns out that the question can be answered positively, and one answer is given by violations of typical (i.e., choose between measuring or not measuring) models of macrorealism (Uola, Vitagliano, and Budroni, 2019). More precisely, Uola, Vitagliano, and Budroni (2019) showed that when all classical disturbance (i.e., clumsy measurement implementation) is isolated from a quantum system, the system can violate macrorealism with some initial state if and only if the involved measurements do not fulfill the definition of nondisturbance.

Motivated by the strong connections between quantum measurement theory and quantum correlations presented in this section (see also Secs. V.D and. V.H), it will be a challenge for future research to isolate the measurement resources behind other quantum tasks. Conversely, it will be of interest to see whether other concepts of incompatibility such as broadcastability (Heinosaari, 2016), incompatibility on many copies (Carmeli *et al.*, 2016), and measurement simulability (Oszmaniec *et al.*, 2017) will find counterparts in the realm of quantum correlations. To conclude, we note that, whereas further connections between measurement theory and correlations remain unknown, jointly measurable sets (Carmeli, Heinosaari, and Toigo, 2019; Skrzypczyk, Šupić, and Cavalcanti, 2019), or more generally all convex subsets of measurements (Uola, Kraft *et al.*, 2019), can be characterized through state discrimination tasks.

## V. FURTHER TOPICS AND APPLICATIONS OF STEERING

In this section, we discuss further aspects and applications of steering. We start with multipartite steering, steering of Gaussian states, and temporal steering. Then we discuss applications of steering such as quantum key distribution, randomness certification, and channel discrimination. Finally, we review the resource theory of steering and the phenomenon of postquantum steering.

### A. Multipartite steering

The extension of steering to multipartite systems is an emerging field of research, and different approaches for defining multipartite steering exist. Before explaining them, we point out some peculiarities of the multipartite scenario, if one considers steering across a bipartition.

#### 1. Steering across a bipartition

In order to discuss the different effects that play a role for steering in the multipartite scenario, consider a tripartite state $\varrho_{ABC}$, and investigate steering across a given bipartition, say, $AB|C$ for definiteness. Then there are different scenarios that have to be distinguished, where in all of them Alice and Bob want to steer Charlie.

- Global steering: In the simplest case, Alice and Bob make global measurements on their two particles and steer Charlie. This reduces to a bipartite steering problem for $\varrho_{AB|C}$, and all of the usual methods can be applied.
- Reduced steering: Another simple case arises if Alice (or Bob) tries to steer Charlie, without needing the help of the other. If the action of one of them is not required, this reduces to the bipartite steerability of the reduced state $\varrho_{A|C}$ or $\varrho_{B|C}$. Again, the entire bipartite steering theory can be applied.
- Local steering: An interesting case arises if Alice and Bob try to steer Charlie by local measurements on their respective parties. In this case, one has to consider the assemblage $\varrho_{ab|xy}$ and ask whether its elements can be written as

$$\varrho_{ab|xy} = \int d\lambda p(\lambda) p(a,b|x,y,\lambda)\sigma_\lambda^C. \tag{95}$$

  Here we can distinguish among several cases, depending on the properties of $p(a,b|A,B,\lambda)$. It may be a general probability distribution, it may obey the nonsignaling constraint, or it may factorize,

$$p(a,b|x,y,\lambda) = p(a|x,\lambda)p(b|y,\lambda). \tag{96}$$

  As $p(a,b|x,y,\lambda)$ has the interpretation of a simulation strategy [see Eq. (2)], the latter indicates that Alice and Bob play an independent strategy.

A simple example of the difference between global and local steering can be constructed from the phenomenon of superactivation of steering (Quintino, Brunner, and Huber, 2016); see also Sec. III.G. For certain states, one copy of the state is unsteerable, but many copies of the same state may become steerable. One can consider a state $\varrho_{ABCC'} = \varrho_{AC} \otimes \varrho_{BC'}$, where $\varrho_{BC'}$ is a copy of $\varrho_{AC}$, and $\varrho_{AC}$ is unsteerable, but its steerability can be superactivated [where two copies are already enough (Quintino, Brunner, and Huber, 2016)]. For this state, local measurements give an unsteerable state assemblage as

$$\begin{aligned}\varrho_{ab|xy}^{CC'} &= \mathrm{Tr}_{AB}[(A_{a|x}\otimes B_{b|y}\otimes \mathbb{1}_{CC'})\varrho_{ABCC'}]\\ &= \mathrm{Tr}_{AB}[(A_{a|x}\otimes \mathbb{1}_{C})\varrho_{AC}\otimes (B_{b|y}\otimes \mathbb{1}_{C'})\varrho_{BC'}]\\ &= \int d\lambda d\mu p(\lambda)p(\mu)p(a|x,\lambda)p(b|y,\mu)\sigma_\lambda^C\otimes\tilde{\sigma}_\mu^{C'}\\ &= \int d\nu p(\nu)p(a,b|x,y,\nu)\hat{\sigma}_\nu^{CC'},\end{aligned} \tag{97}$$

and the state is locally unsteerable even with the restriction to factorizing $p(a,b|x,y,\lambda)$. However, because of the superactivation phenomenon, this state is steerable with global measurements.

Another possibility is to consider the bipartition $A|BC$, where Alice wants to steer Bob and Charlie. Here one has to consider the ensemble $\varrho_{a|x}^{BC}$ and ask whether it can be written as $\varrho_{a|x} = \int d\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda^{BC}$. In this case, two possible scenarios emerge, one where the state of Bob and Charlie is a single system, reducing to a bipartite steering problem, and another where the local hidden state of Bob and Charlie factorizes, i.e., $\sigma_\lambda^{BC} = \tilde{\sigma}_\lambda^B\otimes\hat{\sigma}_\lambda^C$.

From this picture, one can see that the extension of steering to multipartite systems leads to different scenarios, making its characterization even more difficult than the ones for entanglement and nonlocality.

#### 2. Different approaches toward multipartite steering

The existing works on multipartite steering can be divided into two different approaches. The first approach sees steering as a one-sided device-independent entanglement verification and translates this to the multipartite scenario. The second approach asks for a multipartite system whether or not steering is possible for a given bipartition.

To discuss the first approach, we need to recall the basic definitions of the different entanglement classes for multipartite systems (Gühne and Tóth, 2009). For a three-partite system $\varrho_{ABC}$, one calls the state fully separable if it can be written as

$$\varrho_{ABC}^{\mathrm{fs}} = \sum_k p_k \varrho_k^A\otimes\varrho_k^B\otimes\varrho_k^C, \tag{98}$$

where the $p_k$ form a probability distribution. If a state is not of this form, it is entangled, but not all particles are necessarily entangled. For instance, a state of the form $\varrho^{\mathrm{bs}}_{A|BC} = \sum_k p_k \varrho^A_k \otimes \varrho^{BC}_k$ may contain entanglement between $B$ and $C$, but it is separable for the bipartition $A|BC$ and therefore called biseparable. More generally, mixtures of biseparable states for the different partitions are also biseparable,

$$\varrho^{\mathrm{bs}}_{ABC} = p_1\varrho^{\mathrm{bs}}_{A|BC} + p_2\varrho^{\mathrm{bs}}_{B|AC} + p_3\varrho^{\mathrm{bs}}_{C|AB}, \tag{99}$$

and states which are not biseparable are genuine multipartite entangled. These definitions can straightforwardly be extended to more than three particles.

One-sided device-independent entanglement detection in the multipartite scenario was first discussed by Cavalcanti *et al.* (2011). There criteria for full separability in the form of Mermin-type inequalities were given, which hold for $k$ trusted sites and $N-k$ untrusted sites. These criteria can be violated in quantum mechanics, and the possible violation increases exponentially with the number of parties. Inequalities for higher-dimensional systems were derived by He, Drummond, and Reid (2011).

In general, if one has a quantum network of $N$ parties where some of the parties perform untrusted measurements, the parties which trust their measurement apparatus can perform quantum state tomography and reconstruct the conditional state after the untrusted parties announce their measurement choices and outcomes. For three parties, there are two one-sided device-independent scenarios: when only one party's device is untrusted, with state assemblage

$$\varrho^{BC}_{a|x} = \mathrm{Tr}_A(A_{a|x} \otimes \mathbb{1}_B \otimes \mathbb{1}_C \varrho_{ABC}), \tag{100}$$

and when two of them are untrusted,

$$\varrho^{C}_{ab|xy} = \mathrm{Tr}_{AB}(A_{a|x} \otimes B_{b|y} \otimes \mathbb{1}_C \varrho_{ABC}). \tag{101}$$

If $\varrho_{ABC}$ is biseparable, this condition imposes constraints on the assemblages. Then, for a given state, to test whether or not the assemblages of the form (100) or (101) obey the conditions, one can use SDPs (D. Cavalcanti *et al.*, 2015). Also, entropic conditions for this scenario have been studied (Costa, Uola, and Gühne, 2018a; Riccardi, Macchiavello, and Maccone, 2018).

The second approach uses steering between the bipartitions to define genuine multipartite steering (He and Reid, 2013). First, for two parties, one can say that they share steering if the first one can steer the other or vice versa. Then, for three parties, one can define genuine multipartite steerability as the impossibility of describing a state with a model where steering is shared between two parties only. This means that the state cannot be described by mixtures of bipartitions as in Eq. (99), where for each partition (e.g., $A|BC$) the two-party state (e.g., $BC$) is allowed to be steerable.

One can then directly see that, for checking this criterion, it is sufficient to consider the bipartitions $AB|C$, $AC|B$, and $BC|A$, where the two-party sites are uncharacterized and the single-party site obeys quantum mechanics. In addition, on the two-party site, only local measurements are allowed, but the results only have to obey the nonsignaling condition. For proving genuine multipartite steering in this sense, several methods are possible. If the state is pure, it suffices to check the steerability for the bipartitions mentioned, as pure state convex combinations into different bipartitions are not possible (He and Reid, 2013). Otherwise, one may derive a linear (or convex) inequality that holds for unsteerable states of all relevant bipartitions. Because of linearity, it also holds for convex combinations, and violation rules out the previously mentioned model. This approach was experimentally used by Armstrong *et al.* (2015) and Li *et al.* (2015).

### B. The steering ellipsoid

Note that the definition of quantum steering, Eq. (2), requires one to consider the ensembles of unnormalized conditional states at Bob's side. However, one can expect important insight to be gained by simply studying the normalized version of these conditional states. Note that in doing so, two things are lost: the steering ensemble to which a conditional state belongs, and the probability with which the conditional state is steered.

For two-qubit states, the normalized conditional states Alice can steer Bob's system to form an ellipsoid inside Bob's Bloch sphere, referred to as the steering ellipsoid (Verstraete, 2002; Shi *et al.*, 2011, 2012; Jevtic *et al.*, 2014). Detailed analysis of their geometry led to the proposal to use them as a tool to represent two-qubit quantum states, in a way similar to the Bloch representation of states of a single qubit. In particular, given the reduced states of both parties, a steering ellipsoid on one side allows recovering of the density operator up to a certain local unitary or antiunitary operation on the other side (Jevtic *et al.*, 2014). Special attention later was given to the volumes of the steering ellipsoids (Jevtic *et al.*, 2014; Milne *et al.*, 2014; Cheng *et al.*, 2016; McCloskey, Ferraro, and Paternostro, 2017; Zhang *et al.*, 2019). In particular, Milne *et al.* (2014, 2015) showed that the volumes of the steering ellipsoids give upper bounds for the entanglement of the state in terms of its concurrence. It is also shown that the volumes of the steering ellipsoids obey certain monogamy relations (Milne *et al.*, 2014, 2015; Cheng *et al.*, 2016), which is discussed in the following.

Consider a system of three qubits $ABC$. Denote the volume of the steering ellipsoids for steering from $A$ to $B$ and $A$ to $C$ by $V_{B|A}$ and $V_{C|A}$, respectively. Milne *et al.*(2014, 2015) showed that, for all pure states of the system of three qubits $ABC$, one has

$$\sqrt{V_{B|A}} + \sqrt{V_{C|A}} \leq \sqrt{4\pi/3}. \tag{102}$$

They also showed that the famous Coffman-Kundu-Wootters monogamous inequality for entanglement (Coffman, Kundu, and Wootters, 2000) can be derived from this inequality. However, Cheng *et al.* (2016) showed that the monogamy relation (102) is violated when the three qubits are in certain mixed states. Instead, Cheng *et al.* showed that a weaker monogamy relation can be derived for all possible states over the three qubits,

$$V_{B|A}{}^{2/3} + V_{C|A}{}^{2/3} \leq (4\pi/3)^{2/3}. \tag{103}$$

Recently, both the monogamy relation (103) and the violation of Eq. (102) were illustrated experimentally (Zhang *et al.*, 2019).

It is in fact the analysis of the geometry of the steering ellipsoids for Bell-diagonal states that leads to the exact characterization of quantum steering for this family of states (Jevtic *et al.*, 2015; Nguyen and Vu, 2016a); see also Sec. II.C. Beyond the Bell-diagonal states, little is known about the extent to which the steering ellipsoids, or their volumes, can characterize the quantum steerability of the state. In particular, a question for future research might be whether the monogamy relations (102) and (103) can induce certain monogamy relations between some measures of steering such as the critical radii defined in Sec. II.C.

### C. Gaussian steering

#### 1. A criterion for steering of Gaussian states

Before discussing Gaussian steering, some preliminary notions are needed. First, Gaussian systems refer to a special class of continuous variable scenarios. Hence, one deals with infinite-dimensional Hilbert spaces $\otimes_{j=1}^{N}\mathcal{L}^2(\mathbb{R})$, where the index $N$ refers to the number of modes. Every Gaussian state is described by a real symmetric matrix, the so-called covariance matrix $V$ satisfying

$$V + i\Omega \geq 0, \tag{104}$$

where $\Omega = \oplus_{j=1}^{N}\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. More precisely, the covariance matrix of a quantum state $\varrho$ is given as $(V)_{ij} = \mathrm{Tr}[\varrho\{R_i - r_i, R_j - r_j\}]$, where $R = (Q_1, P_1, \ldots, Q_n, P_n)^T$ with quadrature operators $Q_i$ and $P_j$ satisfying $[Q_i, P_j] = i\delta_{ij}\mathbb{1}$ and $[Q_i, Q_j] = [P_i, P_j] = 0$, and $r_j = \mathrm{tr}[\varrho R_j]$. Moreover, every real symmetric matrix satisfying Eq. (104) defines a Gaussian state. The use of the word "Gaussian" in this context originates from the fact that the previously described states correspond to the ones whose characteristic function $\hat{\varrho}(x) \coloneqq \mathrm{tr}[W(x)\varrho]$ is Gaussian. Here $W(x) = e^{-ix^T R}$, with $x = (q_1, p_1, \ldots, q_n, p_n)^T$, and

$$\hat{\varrho}(x) = e^{-(1/4)x^T V x - i r^T x}. \tag{105}$$

Second, a Gaussian measurement is a POVM $M_a$ (with values in $a \in \mathbb{R}^d$), whose outcome distribution for any Gaussian state is Gaussian. Such POVMs correspond to triples $(K, L, m)$ satisfying

$$L - iK^T\Omega K \geq 0, \tag{106}$$

where $K$ is an $N \times d$ matrix, $L$ is a $d \times d$ matrix, and $m$ is a displacement vector. The correspondence between the POVM $M_a$ and the triple $(K, L, m)$ is given through the operator-valued characteristic function as

$$\begin{aligned}\hat{M}(p) &\coloneqq \int da\, e^{ip^T a} M_a \\ &= W(Kp)e^{-(1/4)p^T L p - i m^T p}.\end{aligned} \tag{107}$$

With these definitions, we are ready to state the characterization of steerable states in Gaussian systems originally given by Wiseman, Jones, and Doherty (2007).

A bipartite Gaussian state with covariance matrix $V_{AB}$ and displacement $r_{AB}$ is unsteerable with Gaussian measurements if and only if

$$V_{AB} + i(0_A \oplus \Omega_B) \geq 0. \tag{108}$$

Here $0_A$ is a zero matrix on Alice's side, and $\Omega_B$ is the matrix $\oplus_{j=1}^{N}\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ on Bob's side.

In contrast to other steering scenarios, the Gaussian case appears to be special in that the steerability of a state can be characterized through an easy to evaluate inequality. This is, however, not the only special feature for Gaussian steering. Namely, within the Gaussian regime, one can also prove monogamy relations for steering with more than two parties (Reid, 2013; Ji, Kim, and Nha, 2015; Adesso and Simon, 2016; Lami *et al.*, 2016). One should note that the monogamy can break when one is allowed to perform non-Gaussian measurements (Ji *et al.*, 2016).

#### 2. Refining Gaussian steering with EPR-type observables

As a special case of interest in the Gaussian regime, we discuss steering with canonical quadratures. Kiukas *et al.* (2017) showed that steerability of a given state in the Gaussian scenario can be readily detected by a pair of quadrature observables.

To be more concrete, we sketch the construction of the quadratures from Kiukas *et al.* (2017). First, a channel is called Gaussian if it maps Gaussian states to Gaussian states. Gaussian channels between systems of $n$ and $m$ degrees of freedom correspond to triples $(M, N, c)$, with $M$ being a real $2n \times 2m$ matrix, $N$ being a real $2m \times 2m$ matrix, and $c$ being the displacement, that satisfy

$$N - iM^T\Omega M + i\Omega \geq 0. \tag{109}$$

The transformation of Gaussian states on the level of covariance matrices is given as

$$V \mapsto M^T V M + N, \qquad r \mapsto M^T r + c. \tag{110}$$

Given that a bipartite Gaussian state has a corresponding Choi-Jamiołkowski channel with parameters $(M, N, c)$, one first notes that the state is unsteerable with Gaussian measurements if and only if the channel parameters also define a Gaussian measurement (Kiukas *et al.*, 2017). Hence, for a steerable state, there exist two vectors $x$ and $y$ such that $(y^T - ix^T)(N - iM^T\Omega M)(y + ix) < 0$. As the triple $(M, N, c)$ also fulfills Eq. (109), we have $r \coloneqq x^T\Omega y > 0$ and

$$(M\tilde{x})^T\Omega M\tilde{y} > \tfrac{1}{2}(\tilde{x}^T N\tilde{x} + \tilde{y}^T N\tilde{y}), \tag{111}$$

where $\tilde{x} = r^{-1/2}x$ and $\tilde{y} = r^{-1/2}y$. From here, one can construct two canonical quadratures as $Q_{\tilde{x}} = \tilde{x}^T R$ and $P_{\tilde{y}} = \tilde{y}^T R$. These are canonical as, by definition, $\tilde{x}^T \Omega \tilde{y} = 1$. To see that the state is indeed steerable with these measurements, we refer the interested reader to the work of Kiukas *et al.* (2017)). To summarize, we state the following refined characterization of Gaussian steering (Kiukas *et al.*, 2017).

For a bipartite Gaussian state $\varrho_{AB}$ with a covariance matrix $V_{AB}$ and displacement $r_{AB}$, the following are equivalent:

(i) $\varrho_{AB}$ is steerable with Gaussian measurements.

(ii) $\varrho_{AB}$ is steerable with some pair of canonical quadratures.

(iii) $V_{AB} + i(0_A \oplus \Omega_B)$ is not positive semidefinite.

### D. Temporal and channel steering

So far we have concentrated on steering in spatial scenarios, i.e., scenarios where Alice and Bob are spacelike separated. Some efforts to define similar concepts in temporal scenarios, i.e., scenarios where Alice and Bob form a prepare-and-measure-type scenario, and on the level of quantum channels have also been pursued in the literature (Chen *et al.*, 2014, 2017; Piani, 2015; Chen, Lambert *et al.*, 2016). In a temporal scenario (consisting of two measurement times), one can ask whether a state assemblage resulting from measurements at the first time step allows a local hidden state model on the second time step. Of course, in temporal scenarios, signaling is possible, and hence such models are sometimes trivially violated. Despite signaling, temporal steering has found applications in non-Markovianity (Chen, Lambert *et al.*, 2016) and in quantum key distribution (QKD) (Bartkiewicz *et al.*, 2016), and some criteria (Chen *et al.*, 2014) and quantifiers (Bartkiewicz *et al.*, 2016) have been developed. As the criteria and quantifiers strongly resemble those of spatial steering presented in Secs. II.A and II.B, we do not go through them in detail.

In channel steering (Piani, 2015), one is interested in instrument assemblages instead of state assemblages. Namely, given a quantum channel $\Lambda^{C\to B}$ from Charlie to Bob and its extension $\Lambda^{C\to A\otimes B}$, one asks if an assemblage defined as

$$\mathcal{I}_{a|x}(\cdot) \coloneqq \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\Lambda^{C\to A\otimes B}(\cdot)], \tag{112}$$

where $\{A_{a|x}\}_{a,x}$ is a set of POVMs, can be written as

$$\mathcal{I}_{a|x}(\cdot) = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda(\cdot) \tag{113}$$

for some instrument $\{\mathcal{I}_\lambda\}_\lambda$ (i.e., a collection of *CP* maps summing up to a quantum channel) and classical postprocessings $\{p(a|x,\lambda)\}_{a,x,\lambda}$. Whenever this is the case, the instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ is called unsteerable.

The concept of channel steering relates to the coherence of the channel extension. Namely, a channel extension $\Lambda^{C\to A\otimes B}$ is coherent if it cannot be written as

$$\Lambda^{C\to A\otimes B}(\cdot) = \sum_\lambda \mathcal{I}_\lambda(\cdot) \otimes \sigma_\lambda \tag{114}$$

for some instrument $\{\mathcal{I}_\lambda\}_\lambda$ and states $\{\sigma_\lambda\}_\lambda$. Any extension that is of this form is called incoherent. One can show that incoherent extensions always lead to unsteerable instrument assemblages and any unsteerable instrument assemblage can be prepared through some incoherent extension (Piani, 2015). Note that in spatial steering any separable state leads to an unsteerable assemblage, and any unsteerable assemblage can be prepared with a separable state (Moroder *et al.*, 2016).

In the original paper defining channel steering (Piani, 2015), the concept is mainly probed through channel extensions as mentioned. This leads to some connections with state-based correlations. For example, an extension can lead to a steerable instrument assemblage if and only if its Choi state allows Alice to steer Bob, and an extension is incoherent if and only if the Choi state is separable in the cut $A|BC'$, where $C'$ is the extra input system from the isomorphism.

One can investigate the channel protocol by replacing the extension with a (minimal) dilation. For completeness, we note that a minimal dilation of a channel $\Lambda\colon \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{H})$ can be written as $\Lambda(\cdot) = \mathrm{tr}_A[V(\cdot)V^\dagger]$, where $V|\psi\rangle = \sum_{k=1}^n |\varphi_k\rangle \otimes (K_k|\psi\rangle)$ for all $|\psi\rangle \in \mathcal{H}$, $\{K_k\}_{k=1}^n$ forms a linearly independent Kraus decomposition of $\Lambda$, and where $\{|\varphi_k\rangle\}_k$ is an orthonormal basis of the ancillary system. In this case, the correspondence between instruments and POVMs on the dilation is one to one and is given through

$$\mathcal{I}_{a|x}(\cdot) = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})V(\cdot)V^\dagger]. \tag{115}$$

This directly generalizes the connection between joint measurements and spatial steering to the level of channel steering (Uola *et al.*, 2018). Namely, a measurement assemblage $\{A_{a|x}\}_{a,x}$ on the minimal dilation is jointly measurable if and only if the corresponding instrument assemblage is unsteerable. By noticing, furthermore, that channel steering with trivial inputs (i.e., a one-dimensional input system) corresponds to spatial steering, and that in this case a dilation corresponds to a purification of the total state of the assemblage, one recovers the connection between joint measurements and spatial steering.

The dilation technique can also be used to prove that any nonsignaling state assemblage originates from a set of nonsignaling instruments (Uola *et al.*, 2018), hence showing that channel steering captures nontrivial (i.e., nonsignaling) instances of temporal steering (with two time steps), and that in this case a connection between temporal steering and joint measurements follows directly from the one between channel steering and incompatibility. Moreover, using the channel framework, one can translate concepts from spatial to temporal scenarios. One example of this is given by Uola *et al.* (2018) showing that temporal steering and violations of macrorealism respect a similar strict hierarchy as spatial steering and nonlocality. Note that Ku, Chen, Lambert *et al.* (2018) proved the hierarchy independently.

### E. Quantum key distribution

In QKD, two main types of protocols can be distinguished (Scarani *et al.*, 2009). In prepare-and-measure (PM) schemes, such as the BB84 protocol, Alice prepares some quantum states and sends them to Bob, who performs measurements on them. Using classical communication, Alice and Bob can then try to generate a secret key from the measurement data. In entanglement-based (EB) schemes, such as the E91 protocol, an entangled quantum state is distributed to Alice and Bob, and both make measurements on their part of the state. The source of the state might be under the control of an eavesdropper, Eve. Again, the measurement data are then used to generate a secret key.

A central result concerns the role of entanglement for security. Curty, Lewenstein, and Lütkenhaus (2004) proved that entanglement is a necessary precondition for security. For EB schemes, this means that if the measurement data can be explained by a separable state, then no secret key can be distilled. For PM schemes, one can consider an equivalent EB scheme; then the same statement holds. It should be noted, however, that the provable presence of entanglement was not shown to be sufficient for secret key generation. The question as to whether entanglement can be verified depends on the measurement data taken and the assumptions made on the measurements. In a device-independent scheme where no assumptions about the measurements are made, only Bell inequalities can be used to test the presence of entanglement. Still, device-independent QKD can be proved to be secure against certain attacks (Acín *et al.*, 2007).

One can also consider an asymmetric situation where one party trusts its devices and the other one does not. This can be realistic if Alice corresponds to a client of a bank having only a cheap device, while Bob represents the bank itself. Clearly, in such a situation, QKD can work only if the underlying state is steerable. Branciard *et al.* (2012) considered this problem for the BBM92 protocol (Bennett, Brassard, and Mermin, 1992). In this protocol, Alice and Bob share a two-qubit Bell state and measure either $A_1 = B_1 = \sigma_z$ or $A_2 = B_2 = \sigma_x$. The correlations for the measurement $A_1 \otimes B_1$ are used for the key generation, while the correlations in the $A_2 \otimes B_2$ measurement are used to estimate Eve's information.

Branciard *et al.* (2012) studied the security of one-sided device-independent QKD using this protocol against attacks where Eve has no quantum memory; see also Fig. 10. It has been shown that only the detector efficiency of the untrusted party matters and that, for detector efficiencies $\eta \geq 0.659$, a secret key can already be distilled, and a nonzero key rate proves that the underlying states are steerable. The obtainable key rates are higher and the required detector efficiencies lower than in the fully device-independent case.

In addition, results on steering and PM schemes for QKD were obtained by Branciard *et al.* (2012) and Ma and Lütkenhaus (2012). Wang *et al.* (2013) and Zhou *et al.* (2017) conducted analyses of finite key length, and upper bounds on the key rate in one-sided device-independent QKD were obtained by Kaur, Wilde, and Winter (2018). Finally, it should be noted that similar ideas have also been studied and implemented for QKD with continuous variables (Gehring *et al.*, 2015; Walk *et al.*, 2016).

### F. Randomness certification

The task of randomness certification can be defined as follows (Acín, Massar, and Pironio, 2012; Law *et al.*, 2014).

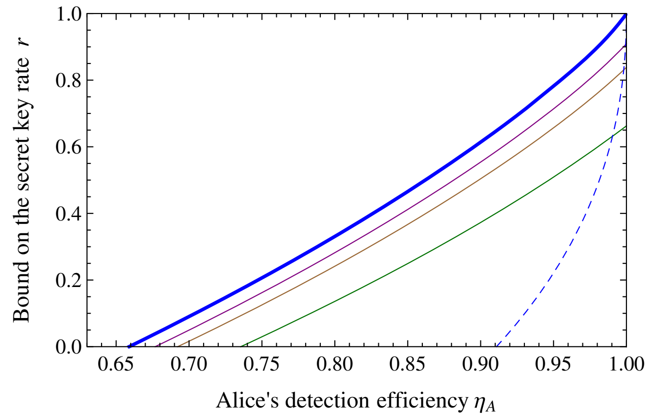


FIG. 10. Key rates for the QKD based on steering. For different visibilities of the initial state ($V \in \{1, 0.99, 0.98, 0.95\}$), lower bounds on the key rate are shown. For perfect visibility (solid blue line), a key can be extracted for detector efficiencies of $\eta \geq 0.659$ for Alice. The dashed line shows a bound (obtained with the same methods) for the fully device-independent scenario for the case of perfect visibility. From Branciard *et al.*, 2012.

On a quantum system $\varrho$, a measurement labeled by $z$ is made, and the result $c$ is obtained. Depending on the situation, the measurement may be a joint measurement on two parties of an entangled state; then the labels for the measurement and result can be written as $z = (x, y)$ and $c = (a, b)$, as in the Bell scenario in Eq. (3). The task is to quantify the extent to which an external adversary Eve can predict the outcome $c$ of the probability distribution $p(c|z)$. Clearly, this depends on the assumptions made about Eve: For instance, one can distinguish the case where the state $\varrho$ is fixed and known to Eve only from the case where Eve indeed provides the state. In the former case, one can furthermore distinguish the knowledge Eve has. She may have only classical information about the state or she may hold a purification of it; see the work of Law *et al.* (2014) for a detailed discussion.

In the simplest case, the state $\varrho = |\psi\rangle\langle\psi|$ is pure and the measurement $z$ is characterized. Then the best strategy for Eve is to guess the $c$ with the maximal probability, and the probability of guessing correctly is given by

$$G(z, \psi) = \max_{c} p(c|z, \psi). \tag{116}$$

If the state $\varrho$ is mixed, then Eve may hold a purification of it and she may know the exact decomposition $\varrho = \sum_k p_k |\phi_k\rangle\langle\phi_k|$ into pure states. Consequently, the maximal guessing probability is

$$G(z, \varrho) = \max_{p_k, \phi_k} \sum_k p_k G(z, \phi_k), \tag{117}$$

where the maximization runs over all decompositions of $\varrho$. If the measurements $z$ are not characterized, one has to optimize over all possible quantum realizations of the classical probability distribution $p(c|z)$. The maximal guessing probability is

$$G(z, p(c|z)) = \max_{\varrho, M_{c|z}} G(z, \varrho), \tag{118}$$

where the maximization runs over all quantum realizations, described by a state $\varrho$ and measurement operators $M_{c|z}$ with $p(c|z) = \mathrm{Tr}(M_{c|z}\varrho)$. Optimizations over this set can be carried out by hierarchies of SDPs (Navascués, Pironio, and Acín, 2007, 2008). In all cases, the number of random bits that one can extract from $p(c|z)$ is given by the minimum entropy, $H_{\min}(G) = -\log_2(G)$.

Initially, the task of randomness certification was studied mainly in the Bell scenario, where $z = (x, y)$ and $c = (a, b)$ describe measurements on an entangled state (Acín, Massar, and Pironio, 2012). Here the devices are not characterized, and, as soon as a Bell inequality is violated, one can prove that the results of a fixed setting cannot be predicted, so the randomness is certified. Law *et al.* (2014) studied randomness certification for the steering scenario: Again, one makes local measurements on an entangled state, but this time the devices on Bob's side are characterized. This leads to additional constraints in the SDP hierarchy (Navascués, Pironio, and Acín, 2007), and, consequently, more randomness can be extracted. For states that are not steerable, the randomness can also be certified; so the violation of a steering inequality is not necessary for randomness certification in the one-sided device-independent scenario.

Passaro *et al.* (2015) studied the task of randomness certification in the distributed one-sided device-independent scenario between two parties. But mainly there, the randomness for a single measurement setting of Alice was considered. It has been shown that this can be directly computed with a SDP, without the need for a convergent hierarchy of SDPs. This is then shown to hold also for the scenario considered by Law *et al.* (2014). Skrzypczyk and Cavalcanti (2018) considered the problem for two $d$-dimensional systems. A steering inequality was derived such that the maximal violation guarantees $\log(d)$ random bits for Alice's outcomes in the one-sided device-independent scenario. Furthermore, any pure entangled state with full Schmidt rank can be used to generate this amount of randomness.

Finally, Curchod *et al.* (2017) showed that if one considers the Bell scenario, then sequential measurements on one party can lead to an unbounded generation of randomness. The extension of this to the steering scenario was discussed by Coyle, Hoban, and Kashefi (2018).

### G. Subchannel discrimination

Piani and Watrous (2015) provided an operational characterization of steerable quantum states. The characterization is in similar spirit as in Piani and Watrous (2009), where the authors show that every entangled state provides an advantage over separable ones in some channel discrimination task. In the case of steering, the related task turns out to be that of subchannel discrimination, i.e., discriminating different branches of a quantum evolution. Namely, take an instrument $\mathcal{I} = \{\mathcal{I}_a\}_a$ (i.e., a collection of completely positive maps summing up to a quantum channel), a POVM $B = \{B_b\}_b$, and an input state $\varrho$, and define the probability of correctly identifying the subchannel (i.e., instrument element) as

$$p_{\mathrm{cor}}(\mathcal{I}, B, \varrho) \coloneqq \sum_a \mathrm{tr}[\mathcal{I}_a(\varrho)B_a]. \tag{119}$$

To find the best strategy for the task, one maximizes over input states and POVMs on the output.

As mentioned previously, entanglement provides an advantage in channel discrimination tasks, i.e., tasks of discriminating between subchannels of the form $\mathcal{I}_a = p(a)\Lambda_a$, where $\{\Lambda_a\}_a$ are quantum channels. To prove a similar result for general subchannels, Piani and Watrous (2015) limited the set of allowed measurements between the system (i.e., the outputs of the instruments) and the ancilla to local measurements supported by forward communication from output to ancilla [i.e. one-way local operations assisted by classical communication (LOCC) measurements]. Such measurements have POVM elements of the form $C_a^{\mathrm{out}\to\mathrm{anc}} = \sum_x A_{a|x} \otimes B_x$, where $\{B_x\}_x$ is a POVM on the output system and $\{A_{a|x}\}_a$ is a POVM on the ancilla for every $x$. The probability of correctly identifying the branch of the evolution with such measurements and a shared state $\varrho_{AB}$ is given as $p_{\mathrm{cor}}(\mathcal{I}, \text{1-LOCC}, \varrho_{AB}) = \sum_{a,x} \mathrm{tr}_{\mathrm{out}}[\mathcal{I}_a^\dagger(B_x)\varrho_{a|x}]$. Note that any unsteerable state can perform, at most, as well as some single system state. One sees this by using a LHS model for the assemblage in the previous equation and by choosing the best performing hidden state as the single system state.

To prove the main result of the paper, Piani and Watrous defined a quantity called steering robustness of a bipartite state $\varrho_{\mathrm{AB}}$ by maximizing the steering robustness of all possible assemblages that the state allows. More formally,

$$R_{\mathrm{steer}}^{A\to B}(\varrho_{AB}) = \sup\{R(A)|\{A_{a|x}\}_{a,x}\}, \tag{120}$$

where $R(A)$ is the steering robustness of the assemblage $\varrho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\varrho_{AB}]$. Clearly, the quantity $R_{\mathrm{steer}}^{A\to B}(\varrho_{AB})$ is zero if and only if the state $\varrho_{AB}$ is unsteerable. The main result now reads as follows:

For any steerable state, there exists a subchannel discrimination task (with forward communication from the output to the ancilla) in which the state performs better than any unsteerable one, i.e.,

$$\sup \frac{p_{\mathrm{cor}}(\mathcal{I}, \text{1-LOCC}, \varrho_{AB})}{p_{\mathrm{cor}}^{NE}(\mathcal{I})} = 1 + R_{\mathrm{steer}}^{A\to B}(\varrho_{AB}). \tag{121}$$

Here the supremum is taken over all instruments and one-way LOCC measurements from the output to the ancilla. The denominator represents the best performance provided by unsteerable states in the corresponding task. Note that this result gives the set of steerable states an operational characterization. Experimental demonstration of this result was presented by Sun *et al.* (2018).

We note that the result on steering robustness can be generalized. One can define a robustness measure for any convex and closed subset of assemblages and can reach a conclusion similar to the previous one using conic programming (Uola, Kraft *et al.*, 2019). Moreover, one can show that a related measure called a convex weight or free fraction has a similar interpretation: Whereas robustness measures discrimination power, the free fraction is a measure of exclusivity (Uola, Bullock *et al.*, 2019).

### H. Device-independent quantification of measurement incompatibility and steering

As steering is closely related to joint measurability and nonlocality, some works (Cavalcanti and Skrzypczyk, 2016; Chen, Budroni *et al.*, 2016) have pursued ways of deriving one-sided device-independent and device-independent bounds on measurement incompatibility. The idea is to show that quantifiers of incompatibility (e.g., incompatibility robustness) are lower bounded by quantifiers of steering (e.g., steering robustness), which in turn are lower bounded by nonlocality quantifiers (e.g., nonlocality robustness). Hence, on top of the one-sided device-independent and fully device-independent lower bounds on incompatibility, one gets also a device-independent lower bound on a quantifier of steering.

We follow the work of Cavalcanti and Skrzypczyk (2016) to make the aforementioned hierarchy more concrete. It is worth mentioning that the hierarchy presented here corresponds to one choice of quantifiers. Analogous results are possible for various fine-tuned quantifiers.

To write down the result, recall the definitions of incompatibility, steering, and nonlocality robustness. Steering robustness is defined in Eq. (30), and analogously to that, one defines the incompatibility robustness $\mathrm{IR}(A_{a|x})$ of a set $\{A_{a|x}\}_{a,x}$ of POVMs as in Eq. (86)

$$\begin{aligned}
&\min\ t\\
&\text{s.t. } \frac{A_{a|x}+tN_{a|x}}{1+t}=\sum_\lambda D(a|x,\lambda)G_\lambda, \quad \text{for all } a,x,\\
&\quad t\geq 0,\\
&\quad N_{a|x}\geq 0, \quad \text{for all } a,x, \qquad \sum_a N_{a|x}=\mathbb{1}, \quad \text{for all } x,\\
&\quad G_\lambda\geq 0, \quad \text{for all } \lambda, \quad \sum_\lambda G_\lambda=\mathbb{1}. \qquad (122)
\end{aligned}$$

Note that here $D(\cdot|x,\lambda)\in\{0,1\}$ is a deterministic assignment for every $x$ and $\lambda$. The interpretation of this robustness is that one mixes the POVMs $\{M_{a|x}\}_{a,x}$ with $\{N_{a|x}\}_{a,x}$ until they become jointly measurable.

Now if Alice's measurements $\{A_{a|x}\}_{a,x}$ in a steering scenario with a state $\varrho_{AB}$ have incompatibility robustness $t$, then replacing Alice's measurements with the jointly measurable POVMs $(A_{a|x}+tN_{a|x})/(1+t)$ shows that $t$ is an upper bound for the steering robustness of $\sigma_{a|x}\coloneqq\mathrm{tr}_A[(A_{a|x}\otimes\mathbb{1})\varrho_{AB}]$. In other words, steering robustness of a given assemblage lower bounds the incompatibility robustness of the measurements on the steering party. This bound, moreover, is one-sided device independent. Notice that with a fine-tuned steering quantifier called consistent steering robustness, the aforementioned inequality is tight for full Schmidt rank states (Cavalcanti and Skrzypczyk, 2016); see also Kiukas *et al.* (2017).

For the device-independent quantification of steering and incompatibility, one can use the nonlocality robustness $\mathrm{NR}[p(a,b|x,y)]$ of a probability table $\{p(a,b|x,y)\}_{a,b,x,y}$, given as (Cavalcanti and Skrzypczyk, 2016)

$$\begin{aligned}
&\min\ r\\
&\text{s.t. } \frac{p(a,b|x,y)+rq(a,b|x,y)}{1+r}\\
&\quad =\sum_{\lambda,\mu}p(\lambda,\mu)D(a|x,\lambda)D(b|y,\mu), \quad \text{for all } a,b,x,y,\\
&\quad r\geq 0, \quad p(\lambda,\mu)\geq 0,\\
&\quad q(a,b|x,y)\in Q, \qquad (123)
\end{aligned}$$

where $Q$ is the set of all possible quantum correlations defined as

$$Q=\{\mathrm{tr}[(A_{a|x}\otimes B_{b|y})\varrho_{AB}]| \times\{A_{a|x}\}_{a,x},\{B_{b|y}\}_{b,y} \text{ POVMs, } \varrho_{AB} \text{ a state}\}. \qquad (124)$$

As in the one-sided device-independent quantification of incompatibility mentioned previously, one sees that, for a given state assemblage $\{\sigma_{a|x}\}_{a,x}$, the nonlocality robustness of any probability table originating from this assemblage, i.e., $p(a,b|x,y)=\mathrm{tr}[\sigma_{a|x}B_{b|y}]$, with $\{B_{b|y}\}_{b,y}$ being POVMs on Bob's side, gives a lower bound for the steering robustness of the assemblage. In other words

$$\mathrm{IR}(A_{a|x})\geq\mathrm{SR}(\sigma_{a|x})\geq\mathrm{NR}[p(a,b|x,y)]. \qquad (125)$$

Hence, using the machinery of Cavalcanti and Skrzypczyk (2016) and Chen, Budroni *et al.* (2016), one finds one-sided device-independent and fully device-independent lower bounds for quantifiers of measurement incompatibility and device-independent lower bounds for quantifiers of steering.

### I. Secret sharing

Secret sharing is a cryptography protocol that allows a dealer (Alice) to send a message to players (Bob and Charlie) in a way such that the message can be decoded only if the players work together, neither of them can decode it by himself. If Alice shared a secret key (see Sec. V.E) with Bob and another with Charlie, she can simply encode the message twice with the two keys to ensure that only Bob and Charlie together can decode the message. Thus, normal quantum key distribution protocols already provide one with protocols for quantum secret sharing. But one can do it more straightforwardly with multipartite entanglement (Hillery, Bužek, and Berthiaume, 1999); see also Żukowski *et al.* (1998) for a related protocol. Take the case where Alice prepares a large number of Greenberger-Horne-Zeilinger (GHZ) states (Hillery, Bužek, and Berthiaume, 1999),

$$|\mathrm{GHZ}\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle). \qquad (126)$$

Alice keeps one particle and sends the other two to Bob and Charlie. Each then measures their particles in random directions, $x$ or $y$. After communicating via a classical public channel, they can identify the triplets that were measured in one of the following directions: $xxx$, $xyy$, $yxy$, or $yyx$; the other triplets are discarded. As one can check, the GHZ state is an eigenstate of the retained measurement operators. In these triplets, Bob and Charlie can use their outcomes to predict Alice's outcomes. However, the outcomes at Bob's or Charlie's side separately are not enough to infer her outcomes. Thus, Alice can use the series of outcomes of her measurements to encode the message in her secret sharing protocol.

The fact that Bob and Charlie have to collaborate to infer the measurement outcomes at Alice's side resembles the distinction between the concepts of local and global steering in the multipartite steering scenarios (He and Reid, 2013; Xiang *et al.*, 2017); see also Sec. V.A. This similarity has been made precise in analyzing the security of secret sharing (He and Reid, 2013; Kogias *et al.*, 2017; Xiang *et al.*, 2017). Specifically, Xiang *et al.* (2017) computed the secrete key-rate bound that guarantees unconditional security of the protocol against eavesdroppers and dishonest players (Kogias *et al.*, 2017) for three-mode Gaussian states and found it to be essentially the quantification of the difference between collective steering and local steering from Bob and Charlie to Alice (Kogias, Lee *et al.*, 2015). To our knowledge, whether this quantitative relation between quantum steering and quantum secret sharing extends beyond Gaussian states is, at the moment, unknown.

### J. Quantum teleportation

Steering shares a close conceptual similarity with state teleportation (Bennett *et al.*, 1993). We follow the work of Cavalcanti, Skrzypczyk, and Šupić (2017) and consider the following abstract teleportation protocol. Alice and Bob share a bipartite quantum state $\varrho^{AB}$. Charlie, perceived as the verifier, draws a pure state $\omega_x$ from a certain set of $|x|$ states indexed by $x$, gives it to Alice and asks her to teleport it to Bob. Without knowing the state $\omega_x$, Alice makes a measurement with POVM elements $\{E_a^{CA}\}_a$ jointly on the received state and her particle that is entangled with Bob's system. Depending on Alice's outcome $a$, Bob's system is then "steered" to a conditional state,

$$\varrho_a^B(\omega_x) = \frac{\mathrm{Tr}_{CA}[(E_a^{CA} \otimes \mathbb{1}^B)(\omega_x \otimes \varrho^{AB})]}{p(a|\omega_x)}, \qquad (127)$$

where the normalization $p(a|\omega_x) = \mathrm{Tr}[(E_a^{CA} \otimes \mathbb{1}^B)(\omega_x \otimes \varrho^{AB})]$ is the probability for Alice to get outcome $a$ in her protocol given that she received state $\omega_x$ from Charlie. Alice communicates her measurement outcome $a$ to Bob, who then makes some appropriate local unitary operation; by the end of the procedure, the state of his system is $U_a \varrho^B_{a|\omega_x} U_a^\dagger$. The design of Alice's measurement and Bob's local unitary operations is such that the final state at Bob's side resembles Charlie's original state $\omega_x$ as much as possible. The quality of the teleportation protocol can be assessed by the so-called average fidelity,

$$\bar{F}_{\mathrm{tel}} = \frac{1}{|x|} \sum_{a,x} p(a|\omega_x) \mathrm{Tr}[\omega_x U_a \varrho_a^B(\omega_x) U_a^\dagger]. \qquad (128)$$

If the teleportation is perfect, the average fidelity is 1 for pure states $\omega_x$.

One can easily observe the similarity of the teleportation protocol to that of quantum steering: Instead of receiving a classical input $x$, Alice receives a quantum state from Charlie $\omega_x$ as an input; see Fig. 11. The idea of receiving quantum inputs from a verifier (Charlie) instead of a classical input was previously considered for entanglement (Buscemi, 2012; Branciard *et al.*, 2013), and later for quantum steering both theoretically and experimentally (Cavalcanti, Hall, and Wiseman, 2013; Kocsis *et al.*, 2015). The benefit of allowing for the quantum inputs is that the verifier can now verify that Alice and Bob share a quantum correlation (entanglement, quantum steering) without trusting their measurement devices or their actual measurements (Hall, 2018); see also Sec. I and Hall and Rivas (2019) for further discussions. Utilizing the similarity, Cavalcanti, Skrzypczyk, and Šupić (2017) showed that all entangled states can demonstrate nonclassical teleportation in a certain sense. One should, however, note that the introduced notion of nonclassical teleportation does not imply high teleportation average fidelity, which has been a standard figure of merit.

There is another line of work which attempts to relate quantum steering with the security of quantum teleportation. In teleporting a state to Bob, Alice does not want an eavesdropper to also obtain some version of the state. Certain security is guaranteed when the average fidelity of teleportation in Eq. (128) is high enough (Pirandola *et al.*, 2015). It is then shown that, for a certain family of bipartite states, to obtain the required fidelity, the state is not only entangled but necessarily two-way steerable (He *et al.*, 2015).

Another way to investigate the security of teleportation is to study its sister protocol known as entanglement swapping. In this protocol, the state given to Alice by Charlie is entangled beforehand with another particle which Charlie keeps. By the end of the teleportation protocol performed by Alice and Bob, the entanglement between Charlie and Alice is transferred to that between Charlie and Bob. In this case, the teleportation

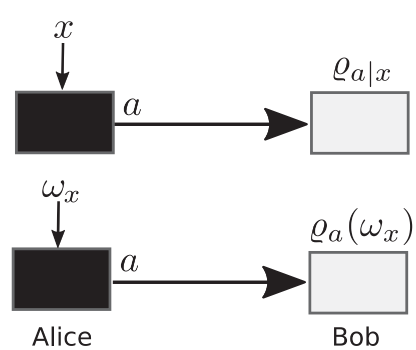


FIG. 11. Quantum steering and quantum teleportation. (Top panel) In quantum steering, Alice receives a classical input $x$, performs a measurement, and communicates the output $a$ to Bob; Bob's system is steered to $\varrho_{a|x}$. (Bottom panel) In teleportation, Alice receives a quantum state $\omega_x$ as input, performs a measurement, and communicates the output to Bob; Bob's system is steered to $\varrho_{a|x}$. Bob further makes a local unitary evolution on his system depending on the outcome he received to obtain the final state.

can be secured by the monogamy of entanglement: If Charlie is sufficiently entangled with Bob, an eavesdropper cannot be entangled with Charlie. Instead of monogamy of entanglement, Reid (2013) then used the monogamy of a certain steering inequality to demonstrate the security of quantum teleportation.

### K. Resource theory of steering

A resource theory is typically seen as consisting of two basic components: free states and free operations. Free states constitute a set that remains unchanged under the actions of free operations. In this sense, one could define a resource theory merely from the free operations. Consequently, any state that is not free has some resource in it, as it cannot be created from the set of free states with free operations. As an example, in the case of entanglement, free states are given by the set of separable states, and free operations are LOCC. Another important aspect of a resource theory are resource measures or monotones. A proper measure should not increase under free operations; i.e., free operations cannot create the resource, should be faithful, i.e., equal to zero only for free states, and should be convex, i.e., randomization should not create a resource either.

In the case of steering, a resource theory has been proposed (Gallego and Aolita, 2015). The free states in this theory are the unsteerable assemblages, and free operations can be any operations on the assemblages that do not map unsteerable assemblages to steerable ones. Gallego and Aolita (2015) showed one-way stochastic LOCC operations to be free operations. In order to introduce one-way stochastic LOCC operations, we adapt the notation of Gallego and Aolita (2015) for state assemblages. Namely, any state ensemble $\{\varrho_a\}_a$ can be embedded in larger space via the correspondence $\{\hat{\varrho}_a\}_a \coloneqq \sum_a |a\rangle\langle a| \otimes \varrho_a$. To make a similar correspondence for state assemblages, one can define a map $\hat{\varrho}_{A|X}(x) \coloneqq \sum_a |a\rangle\langle a| \otimes \varrho_{a|x}$, where $X$ and $A$ label the sets of Alice's inputs and outputs, respectively. Now a one-way stochastic LOCC operation $\mathcal{M}$ is defined on the assemblage $\hat{\varrho}_{A|X}$ as

$$\mathcal{M}(\hat{\varrho}_{A|X}) \coloneqq \sum_\omega (\mathbb{1} \otimes K_\omega)\mathcal{W}_\omega(\hat{\varrho}_{A|X})(\mathbb{1} \otimes K_\omega^\dagger), \tag{129}$$

where $\{K_\omega\}_\omega$ are Kraus operators and $\{\mathcal{W}_\omega\}_\omega$ are wiring maps defined pointwise as

$$\mathcal{W}_\omega(\hat{\varrho}_{A|X})(x_f) \coloneqq \sum_x p(x|x_f,\omega) \sum_{a_f,a} p(a_f|a,x,x_f,\omega) \times (|a_f\rangle\langle a| \otimes \mathbb{1})\hat{\varrho}_{A|X}(x)|a\rangle\langle a_f| \otimes \mathbb{1}. \tag{130}$$

Here $a_f$ and $x_f$ refer to the inputs and outputs of the final assemblage. Physically, such transformations correspond to performing an operation on the characterized party, communicating the information about which operation (i.e., $\omega$) was performed, and the uncharacterized party applying the corresponding classical preprocessings and postprocessings [i.e., $p(\cdot|x_f,\omega)$ and $p(\cdot|a,x,x_f,\omega)$] on their side. Note that the use of one-way stochastic LOCC operations as free operations also has a physical motivation: They can be seen as safe operations in a one-sided device-independent quantum key distribution (Gallego and Aolita, 2015).

A typical resource theory aims at quantifying the resource at hand. For this purpose, one wants to find a mapping (or monotone) $f$ from the set of states of the resource theory to the set of non-negative real numbers that fulfill certain requirements. In the case of steering, the following requirements are considered (Gallego and Aolita, 2015):

- $f(\hat{\varrho}_{A|X}) = 0$ if and only if $\hat{\varrho}_{A|X}$ is unsteerable.
- $f$ is nonincreasing on average under deterministic one-way LOCC.

If in addition the mapping $f$ is convex, it is called a convex steering monotone. Gallego and Aolita (2015) showed the typical steering quantifiers, i.e., steerable weight and robustness of steering (see Sec. II.B), to be convex steering monotones. Furthermore, they introduced a novel convex steering monotone called relative entropy of steering; see also Kaur, Wang, and Wilde (2017) and Kaur and Wilde (2017) for further monotones and alternative definitions of relative entropy of steering.

### L. Postquantum steering

Postquantum steering is the phenomenon that certain assemblages $\{\varrho_{a|x}\}$ may not be realizable by quantum mechanics, although no signaling between the parties is possible. For the case of Bell inequalities, it is known that there are probability distributions which are nonsignaling but cannot come from a quantum state. The most prominent example is the Popescu-Rohrlich (PR) box (Popescu and Rohrlich, 1994), which is a nonsignaling distribution for two parties with two measurements having two outcomes, which leads to a violation of the CHSH inequality with a value $\langle \mathcal{S}_{\mathrm{CHSH}} \rangle = 4$, while in quantum mechanics only values $\langle \mathcal{S}_{\mathrm{CHSH}} \rangle \leq 2\sqrt{2}$ can occur. The analogous question for steering highlights the difference between steering in the bipartite and the multipartite case.

For the bipartite case, one may consider an assemblage $\{\varrho_{a|x}\}$, obeying the no-signaling constraint $\sum_a \varrho_{a|x} = \sum_a \varrho_{a|x'} = \varrho_B$ for all $x$, $x'$. As mentioned in Secs. II.B and V.M, any such assemblage can be realized by quantum mechanics. This means that there is a state $\varrho_{AB}$ and measurements $E_{a|x}$ such that $\varrho_{a|x} = \mathrm{Tr}_A(E_{a|x}\varrho_{AB})$.

This is not the case for the tripartite scenario (Sainz *et al.*, 2015). Here one considers the scenario where Alice and Bob make local measurements in order to steer Charlie's state. Charlie has an assemblage $\{\varrho_{ab|xy}\}$, where $x$ and $a$ ($y$ and $b$) denote the measurement setting and outcome of Alice (Bob). Besides being positive and the normalization constraint $\mathrm{Tr}(\sum_{ab} \varrho_{ab|xy}) = \mathrm{Tr}(\varrho_C) = 1$, this assemblage should fulfill the requirement that neither Alice nor Bob can signal to the other parties, that is,

$$\sum_a \varrho_{ab|xy} = \sum_a \varrho_{ab|x'y}, \quad \text{for all } x, x',$$
$$\sum_b \varrho_{ab|xy} = \sum_b \varrho_{ab|xy'}, \quad \text{for all } y, y'. \tag{131}$$

One can directly check that these conditions also imply that Alice and Bob jointly cannot signal to Charlie by the choice of their measurements.

Contrary to the bipartite case, an assemblage obeying these constraints does not need to have a quantum realization. A simple counterexample can be derived from the PR box mentioned previously: If the conditional states are of the form $\varrho_{ab|xy} = p(ab|xy)|0\rangle\langle 0|_C$, where $p(ab|xy)$ is the probability table of the PR box, the assemblage is clearly nonsignaling but cannot be realized within quantum mechanics. Sainz *et al.* (2015) provided other examples of this behavior. Using iterations of SDPs, they found an example of a qutrit assemblage $\varrho_{ab|xy}$ with the properties that, for any possible measurement $E_{c|z}$ of Charlie, the resulting probability distribution can be explained by a fully local hidden variable model, which is an even stronger requirement than being nonsignaling. Still, the assemblage has no quantum realization, so there is no state $\varrho_{ABC}$ such that $\varrho_{ab|xy} = \mathrm{Tr}_{AB}(E_{a|x} \otimes E_{b|y}\varrho_{ABC})$.

In further works, the theory of postquantum steering was extended. Sainz *et al.* (2018) provided general methods to construct examples of postquantum steering and defined a quantifier of this phenomenon, Hoban and Sainz (2018) established a connection to the theory of quantum channels, and S.-L. Chen *et al.* (2018) used moment matrices to characterize the phenomenon.

## M. Historical aspects of steering

### 1. Discussions between Schrödinger and Einstein

As mentioned in the Introduction, the first observation of the steering phenomenon dates back to Schrödinger's discussions of the EPR argument with Einstein. Schrödinger corresponded with several physicists on this problem; the letters were edited by von Meyenn (2011).

To understand the origin of Schrödinger's idea, it is important to note that Einstein did not like the way the EPR paper was written and how the argument was formulated; for detailed discussions, see Kiefer (2015). Instead, Einstein preferred a somehow simpler version. He published this version much later (Einstein, 1948), but he also explained the basic idea in a letter to Schrödinger on 19 June 1935.

The argument [in the formulation of Einstein (1948)] goes as follows. First, one considers position $X$ and momentum $P$ as noncommuting observables. Then there are, according to Einstein, two possibilities:

(i) One can assume that the position and momentum have definite values before a measurement of them is carried out. Then one has to admit that the wave function $|\psi\rangle$ is not a complete description.

(ii) One can assume that the values of the position or momentum are created during a measurement. This is compatible with the assumption that the wave function $|\psi\rangle$ is a complete description. If $|\psi\rangle$ is a complete description, it follows, according to Einstein, that two different wave functions describe two different physical situations.

These two ways of thinking cannot be distinguished without additional assumptions. Here Einstein introduces a locality principle, stating that if one considers a bipartite system, the real physical situation at one side is independent of what happens on the other side.

In order to conclude the incompleteness of the quantum mechanical description, one can consider a pure entangled wave function, as in the usual EPR argument. Then the conditional wave function $|\phi\rangle_B$ on Bob's side depends on the choice of the measurement on Alice's side. According to the locality principle, however, the physical reality on Bob's side cannot change. One arrives at a contradiction to (ii), and the incompleteness follows. Note that for this argument the perfect correlations between measurements on both sides are not relevant. As Einstein formulated it, "I couldn't care less whether or not $|\phi\rangle_B$ and $|\bar{\phi}\rangle_B$ are eigenstates of some observables."

In the direct reply to this letter (on 13 July 1935), Schrödinger spelled out that the dependence of the conditional state $|\phi\rangle_B$ includes some steering from a distance. Although this phenomenon does not allow signaling between the parties, he considers it to be magic. Recalling discussions with Einstein and colleagues in Berlin during the 1920s, he writes:

"All the others told me that there is no incredible magic in the sense that the system in America gives $X = 6$ if I perform in the European system *nothing* or a *certain* action (you see, we put emphasis on spatial separation), while it gives $X = 5$ if I perform *another* action; but I only repeated myself: It does not have to be *so* bad in order to be silly. I can, by maltreating the European system, steer the American system *deliberately* into a state where either $X$ is sharp, or into a state which is certainly *not* of this class, for example, where $P$ is sharp. This is *also* magic!"

It must be added, of course, that the view of the steering phenomenenon as "nonlocal" is based on a certain interpretation of the wave function, which is not shared by everyone; see Griffiths (2019) for a discussion. In any case, the question remains as to which states this phenomenon can be observed for and which states on Bob's system can be reached by performing measurements on Alice's side. This was also part of the discussion between Schrödinger and other physicists (such as von Laue), and Schrödinger presented his results in two subsequent papers.

### 2. The two papers by Schrödinger

The first paper, entitled "Discussion of probability relations between separated systems" (Schrödinger, 1935), was submitted in August 1935. Schrödinger states that he finds it "rather discomforting" that quantum mechanics allows a system to be steered by performing measurements in a different location. He then presents several results on this phenomenon.

First, he shows that every bipartite pure state can be written as

$$|\psi\rangle = \sum_k s_k |a_k\rangle|b_k\rangle, \tag{132}$$

where the vectors $|a_k\rangle$ and $|b_k\rangle$ form orthogonal sets. This is presently called the Schmidt decomposition. He proves that this is unique if the coefficients $s_k$ are different. He also recognizes that this implies that, for generic states, there is one measurement for Alice (defined by the eigenvectors $|a_k\rangle$) that is perfectly correlated with a measurement on Bob's side (defined by the $|b_k\rangle$).

Then he discusses in more detail the EPR state from the 1935 argument (Einstein, Podolsky, and Rosen, 1935), which is not a generic state, as all of the Schmidt coefficients coincide. He proves that, for any observable $F(X_2, P_2)$ on Bob's side, the value can be predicted by making a suitable measurement $\hat{F}(X_1, P_1)$ on Alice's side. This fact had appeared already in a letter from Schrödinger to von Laue, and it demonstrates the surprising effect that one system seems to know the answers to all possible questions on the other system.

The second paper, entitled "Probability relations between separated systems," was submitted in April 1936 (Schrödinger, 1936). Schrödinger first states that the essence of the previous work was the observation that in quantum mechanics one cannot only determine the wave function at one party by making measurement on the other, but one can also control the state at one side by choosing the measurements on the other side. The question arises, to which extent can the wave function be controlled?

In order to answer this, he first proves a statement on density matrices. A given density matrix may have different decompositions into pure states

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k| = \sum_i q_i |\phi_i\rangle\langle\phi_i|, \qquad (133)$$

and the question arises, which conditions the $|\phi_i\rangle$ and $|\psi_k\rangle$ have to fulfill? Schrödinger proves that two ensembles give the same density matrix if and only if there is a unitary matrix $U$ such that

$$\sqrt{p_k}|\psi_k\rangle = \sum_i U_{ki}\sqrt{q_i}|\phi_i\rangle. \qquad (134)$$

This implies that any $|\phi_i\rangle$ in the range of the space spanned by the $|\psi_k\rangle$ can be an element of a suitable ensemble.

Then Schrödinger applies this to the bipartite state in Eq. (132). Here the reduced state

$$\varrho_B = \sum_k s_k^2 |b_k\rangle\langle b_k| = \sum_i q_i |\beta_i\rangle\langle\beta_i| \qquad (135)$$

has different decompositions. As mentioned, the ensemble $\{s_k^2, |b_k\rangle\}$ can be reached by making the measurement defined by the orthogonal states $|a_k\rangle$ on Alice's side, and the question arises as to whether any other ensemble $\{q_i, |\beta_i\rangle\}$ can be reached. Schrödinger proves that this is the case. Specifically, if the state has a full Schmidt rank and $|b_k\rangle$ span the whole space, *any* state $|\beta_i\rangle$ on Bob's side can be prepared by making a suitable measurement on Alice's side.

Finally, Schrödinger stresses again that he finds the phenomenon of controlling a distant state repugnant and suggests that quantum mechanics may be modified to avoid it. As a potential modification, he suggests that, for a state as in Eq. (132), the phase relations between the $s_k$ may be lost. This means that instead of taking the pure state $\varrho = |\psi\rangle\langle\psi|$, the two-particle system should be described by a diagonal mixed state

$$\varrho = \sum_k s_k^2 |a_k b_k\rangle\langle a_k b_k|, \qquad (136)$$

which he considers to be a possible modification, not contradicting the experimental evidence at the time.

### 3. Impact of these papers

In the following years, Schrödinger's ideas on steering were not further considered in the literature. His mathematical results from the second paper, however, were rederived several times without any reference to him. In the following, we give a short overview; a detailed discussion was given by Kirkpatrick (2006).

A first rediscovery was presented by Jaynes (1957). He derived the first statement in Eq. (134) while studying general properties of density matrices. Based on Jaynes's paper, Hadjisavvas (1981) later presented a simplified proof and an extension to infinite-dimensional systems.

The second mathematical statement [after Eq. (135)] was derived by Gisin in the context of modifications of the Schrödinger dynamics (Gisin, 1989). Here the question arises as to whether the modified dynamics for pure states extends uniquely to mixed states. If it were different for two decompositions such as the ones in Eq. (133), then ensembles will become distinguishable at some point. Given the fact that both ensembles can be prepared by measurements on a distant system, this would lead to a violation of the nonsignaling condition, enforced by special relativity. Finally, both mathematical statements from Schrödinger were also rederived independently by Hughston, Jozsa, and Wootters (1993).

Besides these mathematical results, the notion of steering as a kind of quantum correlation was not discussed for a long time. The situation changed in 1980s. Then Bell inequalities started to attract more attention (Clauser and Shimony, 1978), and the mathematical notions of entangled and separable states were studied (Primas, 1983; Werner, 1984, 1989).

Vujičić and Herbut (1988) were the first to give a clear summary of Schrödinger's ideas and an extension of his results for continuous variable systems. Vujičić and Herbut also argued that steering differs from Bell nonlocality, as it is based on the formalism of quantum mechanics. Independently, Reid (1989) presented quantitative conditions for continuous variable systems to lead to an EPR-type argument. Verstraete (2002) noted in his dissertation the connection of Schrödinger's ideas to quantum teleportation and entanglement transformations, as in both cases, one aims at preparing a quantum state on one side by making measurements on the other. The notion of the steering ellipsoid was also introduced there. Shortly thereafter, steering was recognized to be relevant for foundational questions of quantum mechanics (Clifton, Bub, and Halvorson, 2003; Spekkens, 2007). Finally, Wiseman, Jones, and Doherty (2007) introduced the notion of local hidden state models, laying the foundation for the modern notion of quantum steering.

## VI. CONCLUSION

The notion of quantum steering was motivated by the Einstein-Podolsky-Rosen argument, and it took seven decades until a precise formulation was given. Since then, quantum steering has initiated a new surge of results in quantum information and the foundations of quantum mechanics: Old concepts were put in a new light; long-standing problems gained progress, and some were resolved; connections between areas were established; and novel problems were formulated. In this review, we sketched the dynamic development of the field over the last ten years. Yet future research is facing many challenges. To close the review, we summarize some of the open problems:

- As a complete characterization of quantum steerability has been obtained for two-qubit systems and projective measurements, it is desirable to extend such a characterization to higher-dimensional systems. Although there is an indication that such an extension is possible, much remains to be worked out.
- The question, whether there are states that are unsteerable with projective measurements but are steerable with POVMs, is also relevant. The analogous question in the context of Bell nonlocality has been a long-standing problem without any evidence whether such a state exists. With quantum steering, one now has evidence indicating that there might be no such state for a two-qubit system. Yet, to date, there is no rigorous proof of their nonexistence. In particular, one might still expect such a state to exist in high dimensions.
- We discussed in Sec. III.G the fact that the operational definition of steerability requires multiple copies of the considered state, which implies the possibility of making collective measurements on Alice's side. Thus, apart from being fundamental, the question as to whether all entangled states become steerable upon making collective measurements on Alice's side is also important to the intrinsic consistency of the concept.
- The connection between quantum steering and incompatibility has initiated further questions: Are there other connections between the different notions of incompatibility and different forms of quantum correlations?
- A further open question concerns whether there are other physically motivated properties of state assemblages than that of having a local hidden state model. For instance, one may want to deduce something more than entanglement and incompatibility from such properties. Examples are the preparability from states with a given Schmidt number or properties motivated by incompatibility, such as compatibility on many copies, coexistence, or simulability.
- The study of multipartite steering is in its infancy. A systematic investigation and comparison between different definitions is necessary in the future.
- A closely related research direction is to study the steering of parties who are connected in networks. For a given directed network, one may ask whether there is a quantum state allowing steering along the directed edges.
- For applications, it would be worthwhile to identify tasks in quantum information processing, where the assumptions that can be made are highly asymmetric. Then the methods developed in steering theory may be useful to study the role of correlations therein.
- In experiments, quantum steering is verified by a finite number of measurement settings. There are only a few works on optimizing the measurement settings (when having a fixed number of inputs or outputs), and more research is needed to serve as input for experiments.

This is only a small list of problems, and further challenges remain. Given the current interest in the steering phenomenon, we expect the old observations from Schrödinger to still influence current and future discussions on the foundations of quantum mechanics.

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