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Colloquium: Anomalous metals: Failed superconductors

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The observation of metallic ground states in a variety of two-dimensional electronic systems poses a fundamental challenge for the theory of electron fluids. Here evidence is analyzed for the existence of a regime, called the "anomalous metal regime," in diverse 2D superconducting systems driven through a quantum superconductor to metal transition by tuning physical parameters such as the magnetic field, the gate voltage in the case of systems with a metal-oxide semiconductor field-effect transistor (MOSFET) geometry, or the degree of disorder. The principal phenomenological observation is that in the anomalous metal, as a function of decreasing temperature, the resistivity first drops as if the system were approaching a superconducting ground state, but then saturates at low temperatures to a value that can be orders of magnitude smaller than the Drude value. The anomalous metal also shows a giant positive magnetoresistance. Thus, it behaves as if it were a "failed superconductor." This behavior is observed in a broad range of parameters. It will be moreover exhibited, by theoretical solution of a model of superconducting grains embedded in a metallic matrix, that as a matter of principle such anomalous metallic behavior can occur in the neighborhood of a quantum superconductor to metal transition. However, it will be also argued that the robustness and ubiquitous nature of the observed phenomena are difficult to reconcile with any existing theoretical treatment and speculate about the character of a more fundamental theoretical framework.

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I. INTRODUCTION

A metallic state is defined as a state in which the conductivity $\sigma(T)$ remains finite as $T \to 0$. There is an extraordinarily successful Fermi liquid theory of clean 3D metals with $k_F\ell \gg 1$ and relatively weak interactions. Here k_F and ℓ are the Fermi momentum and the electron mean-free path, respectively. In the Fermi liquid theory there are two types of excitations, fermionic and bosonic: Fermionic excitations (quasiparticles) have a finite density of sates at the Fermi level. Bosonic excitations (e.g., zero sound) roughly can be divided into two groups: Those associated with the charge excitations have a plasmon spectrum. Those associated with spin fluctuations have a sound wave spectrum.[1] In principle electric current can be carried by both fermionic and bosonic excitations. [See, for example, Brazovskii, Matveenko, and Noziéres (1993) and Nayak *et al.* (2001).] However, at low temperatures the contribution of the bosonic excitations to the current is negligible due to their vanishing density of states. Thus the electronic transport properties are controlled by the fermionic excitations (quasiparticles).

The low-temperature conductivity of relatively pure 3D metals is determined by impurity scattering and is given by the Drude formula $\sigma_D = e^2 D\nu$. Here ν is the electron density of states at the Fermi energy, v_F is the Fermi velocity, and $D = v_F\ell/3$ is the diffusion coefficient.

Another well-established paradigm is the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity. It is based on the idea that under some circumstances the electron attraction can dominate the electron repulsion so that at low temperatures electrons form bosonic Cooper pairs which can condense. It is this condensate that carries the supercurrent. As parameters controlling the electronic environment (e.g., band structure, interactions, or external magnetic field) change, the system may exhibit a superconductor to metal transition, which at $T = 0$ is a quantum superconductor to metal transition (QSMT). As we will discuss in detail, it follows from the conventional theory of metals that in zero magnetic field, the QSMT that occurs as the effective interactions between electrons changes from attractive to repulsive *does not have a quantum critical regime*. In other words, as the system approaches the BCS superconducting state from the metallic side, its properties in no way reflect the proximity of another phase. In particular, the conductivity of the system is controlled by the fermionic excitations (quasiparticles) everywhere in the metallic phase.

This picture is supported by a large number of experiments on a variety of systems. However, there exists a variety of experimental systems which exhibit a zero temperature transition from a superconducting state to an “anomalous metallic regime” with $T \to 0$ electronic properties that cannot be understood on the basis of Fermi liquid or Drude theory. Specifically, the $T \to 0$ conductivity in the anomalous metallic regime can be orders of magnitude larger than the Drude conductivity, there is a giant positive magnetoresistance, and as has been observed in at least one case the Hall response is anomalous. Such behavior has been observed for transitions tuned by changing a variety of parameters including the magnetic field, gate voltage, and degree of disorder.

The properties of such anomalous metals is the focus of this article. We will argue that the dramatic signatures in the anomalous metal are due to the fact that it behaves as a “failed superconductor,” a state in which there are significant superconducting correlations but nonetheless the system fails to condense even as $T \to 0$. In other words in the anomalous metal regime current is carried by bosonic quantum fluctuations of the superconducting order parameter. Contrary to popular belief, this anomalous metal appears robust even in two spatial dimensions, $d = 2$. It represents a new paradigm for the electronic properties of a metal that is very different from a Fermi liquid.

A. Background

1. Experimentally observed properties of the anomalous metal

A typical early observation of an anomalous metal was in a study of the onset of superconductivity in ultrathin granular metal films by Jaeger *et al.* (1989) in which the resistance was observed to level off as $T \to 0$ to a value much below the Drude (normal state) value.

The fact that this represents an anomalous metallic phase emerging from a QSMT was first identified in experiments on the magnetic field driven transition in relatively low-resistance ($k_F\ell \gg 1$) amorphous $Mo_{1-x}Ge$ (a-MoGe) films (Ephron *et al.*, 1996; Mason and Kapitulnik, 1999, 2001). There the anomalous metal was observed over a broad range of magnetic fields, exhibiting a low T resistivity that is as much as 3 orders of magnitude smaller than the Drude value. Since then, such a metallic phase proximate to a QSMT has been found in many different systems with different tuning knobs. Next we discuss the main experimental observations in the anomalous metal regime, their robustness, and their significance. Representative results are shown in Sec. II. Generic features seen in a wide variety of material systems and experimental platforms can be summarized as follows:

(i) Most of the evidence for an anomalous metal proximate to a QSMT comes from studies of two-dimensional systems. Nonthermal parameters that have been used to tune from the superconducting to a nonsuperconducting state include microscopic and/or macroscopic disorder, carrier density (typically varied by tuning a gate voltage), screening properties of a nearby ground plane, and a magnetic field (see Sec. II). (Note in Sec. III we show that the theoretical rational for the existence of such a state applies as well in 3D. There have also been numerous experiments on superconducting wires, but since the physics in 1D is quite different than in higher dimensions, we will not survey these results in the present review.)

(ii) The anomalous metallic state is ubiquitously found in metallic films with normal state conductance $\sigma_D^{(2D)}$ that is significantly higher than the quantum of conductance e^2/h. Here the dimensionless conductance per square of the 2D system $\sigma_D^{(2D)}$ is determined either by applying a sufficiently high magnetic field to suppress superconductivity and then extrapolating the measured conductivity to $T \to 0$ or from the value of σ somewhat above the mean field T_c.

(iii) The anomalous metal appears as an intermediate regime; when the systems are tuned further from the

[1]Phonons are also a class of ubiquitous bosonic modes. They are in a sense neutral, although they can make a contribution to charge transport through the mechanism of “phonon drag.”

QSMT, they either exhibit a crossover to a "normal metallic phase," or a further metal-insulator transition (MIT). The range of parameters in which the anomalous metal is observed is often broad (the order of 1).

(iv) While disorder may be present, or even used as a tuning parameter, there is no obvious dependence of the observed phenomena on the detailed morphology of the disorder. The anomalous metallic state has been observed in strongly nonuniform systems, including naturally granular systems and artificially prepared arrays of superconducting "dots" on two-dimensional (2D) semiconductors or metals. It is also observed in what otherwise seem to be homogeneous films, both crystalline and amorphous.

(v) In some cases, typically characterized by strong disorder, a direct superconductor to insulator transition (SIT) is observed. However, we note that much of the published literature that exhibit an anomalous metallic phase has been interpreted in terms of a putative SIT. This sometimes incorrect interpretation was, in turn, motivated by the theoretical belief that metallic phases are forbidden in 2D. The observation that some films exhibit a SIT while others undergo a QSMT can be understood if one posits the existence of a critical disorder strength (Steiner, Breznay, and Kapitulnik, 2008) (corresponding to $k_F\ell \sim 1$) such that there is a SIT in more disordered films and a QSMT (possibly followed by a MIT) in less.

(vi) Measurements of the Hall effect and finite frequency conductivity can also reveal distinguishing characteristics of the anomalous metallic phase.

2. Summary of the theoretical situation

We argue that there is currently no satisfactory theory of anomalous metals that accounts for the full set of key experimental facts, in particular, the robustness of the anomalous metallic state. We view this as one of the major open problems in condensed matter theory. However, there are circumstances in which controlled theory is possible and where the existence of an extended $T = 0$ quantum critical regime beyond a QSMT has been established (Feigel'man and Larkin, 1998; Spivak, Zyuzin, and Hruska, 2001; Spivak, Oreto, and Kivelson, 2008).

Such theoretical considerations are discussed in Sec. III, where we analyze a "model system" of superconducting puddles embedded in a "good" metal.

The finite temperature superconductor-metal transition is driven by classical fluctuations, and it takes place when the intergrain Josephson exchange energy is comparable to the temperature $J_{ij} \sim T$. Neglecting quantum fluctuations of the order parameter, one would infer that such a system is always a superconductor at sufficiently low T. Quantum fluctuations of the phase of an isolated superconducting grain are associated with the charging energy. However, there is no charging energy for a grain embedded in a metal; nonetheless, provided the effects of electron repulsion in the metal are taken into account, it can be shown that there exists a critical concentration of puddles below which long-range phase coherence is destroyed by quantum fluctuations.[2]

In the neighborhood of the resulting QSMT, a substantial fraction of the current is carried by bosonic fluctuations of the superconducting order parameter. Thus, as a point of principle, such a granular system can have an anomalous metallic ground state (i.e., without superconducting long-range phase coherence) with a $T = 0$ conductivity that diverges upon approach to the quantum critical point (QCP). Such a system also exhibits a large positive magnetoresistance.

However, while these considerations address the point of principle, they do not account for the broad range of temperatures and tuning parameters over which anomalous metallic behavior is observed. At issue is the fact that the Ginzburg-Levanyuk parameter (Levanyuk, 1959; Ginzburg, 1961), which typically characterizes the width of the fluctuational regime near a critical point, is very small in most of the relevant experimental systems. This seemingly implies a narrow range of parameters where significant quantum fluctuations exist. In Sec. IV we consider other possible mechanisms for the QSMT that could pertain even to uniform systems.

We also discuss the role of "rare events" on the quantum critical transition in Sec. III.B.8. Systems that exhibit only slight nonuniformities in their electronic structure when far from criticality show amplified effects of small inhomogeneities when tuned close to a QCP. In the case of quantum phase transitions involving a discrete symmetry breaking, such considerations (Fisher, 1992, 1995) can lead to a "quantum Griffiths phase," i.e., a range of parameters of finite measure in the vicinity of the QCP in which the appropriate thermodynamic susceptibility diverges. Spivak, Oreto, and Kivelson (2008) showed that in the case of the QSMT, while there exist circumstances in which the effect of rare regions are highly amplified, they can never be strong enough to produce a true quantum Griffith phase.

3. Is 2D localization relevant in the anomalous metal regime?

Experimentally, most reports of anomalous metals involve 2D samples. Thus, a natural question arises concerning the relevance of 2D localization effects, which have been a key feature of the theory of transport phenomena in the presence of disorder. 2D localization theory is based on the observation that, in the absence of interactions, the first correction to Drude theory in powers of $1/k_F\ell$ diverges logarithmically as $T \to 0$. Renormalization group analysis, assuming one parameter scaling, leads to the inference that $\sigma(T) \to 0$ as $T \to 0$ (Abrahams *et al.*, 1979; Gorkov, Larkin, and Khmelnitskii, 1979). [For a review, see Lee and Ramakrishnan (1985).] This hypothesis has been confirmed by a numerical solution of the Schrödinger equation for a single particle in a disordered medium. [See, for example, Cheraghchi (2006) and Markos (2006).] In other words, in the absence of interactions and spin-orbit scattering, 2D metals do not exist. The question of

[2] A corollary of this analysis is that with attractive interactions only, an electron fluid can only undergo a SIT and will never exhibit an intermediate metallic phase.

2D localization in disordered metals with electron-electron interactions is more complicated, and in spite of extensive theoretical effort there is still no full understanding of the problem.

In order to see the predicted crossover to insulating behavior for systems with $k_F\ell \gg 1$, one would have to measure the conductance at exponentially low temperatures,

$$T < T^\star \sim E_F \exp[-\pi k_F \ell]. \qquad (1)$$

In any case, for this review, we can ignore "localization" effects, including interactional ones (Altshuler, Aronov, and Lee, 1980; Lee and Ramakrishnan, 1985; Finkelshtein, 1987), for several reasons: (i) In most cases the experiments we are interested in are carried out in the range $T \gg T^\star$. (ii) The fact that the low T conductivity is typically orders of magnitude *larger* than $\sigma_D^{(2D)}$ implies that the starting point of the perturbative renormalization group (RG) consideration in Abrahams *et al.* (1979) and Gorkov, Larkin, and Khmelnitskii (1979) is inapplicable in the present circumstances. (iii) Finally, bosonic excitations are not subject to weak localization.

On the other hand, weak localization effects are cut off by the superconducting gap Δ_0. Therefore the superconducting state is robust for $k_F\ell \gg 1$ and $T = 0$ so long as $\Delta_0 > T^\star$.

II. EXPERIMENT

In this section, we discuss multiple examples of experimental systems in which an anomalous metallic phase is found to exist as $T \to 0$ proximate to a superconducting phase. Various "knobs" are used to tune these systems from a superconducting to a nonsuperconducting ground state including gate voltage, film thickness, or an applied magnetic field. It is important to stress that the nature of the anomalous metal is roughly similar in all cases.

The systems we discuss in this section are all in some essential way 2D. One reason for this is that it is relatively easier to tune 2D systems through a QSMT. However, on the basis of the theoretical considerations of Sec. III, there does not appear to be any reason that similar phenomena are excluded in 3D. In relatively pure 3D samples, magnetic field tuned superconductor-metal transitions have been studied for decades. For most 3D superconductors, the transition to the "normal state" can be satisfactorily described by the usual mean-field description of the upper critical field H_{c2}. However, in some circumstances classical melting of the vortex lattice is observed before the mean field H_{c2}, and as $T \to 0$, this may cross over to quantum melting. While the thermal melting transitions are now reasonably well understood (Blatter *et al.*, 1994), the transition to the quantum-dominated regime is not fully understood. We will return to the issue of the QSMT in 3D in Sec. IV.

A. Distinguishing insulators, metals, and superconductors

The defining feature that distinguishes metals from insulators is the value of the conductivity in the limit $T \to 0$; it vanishes in an insulator and approaches a finite limit in a metal. The resistivity vanishes in a superconductor—in some cases below a nonzero critical temperature, but in other cases (as in a 2D superconductor in the presence of a magnetic field) only in the limit $T \to 0$.

Alas, experiments are always confined to nonzero temperatures. We are thus always faced with the task of inferring the character of ground-state phases based on low-temperature measurements. In doing this, it is important to pay attention to both the magnitude and the temperature dependence of the resistivity. Even in conventional, 3D metals there are circumstances in which the resistivity is an increasing or a decreasing function of T at low temperatures, so the sign of $d\rho/dT$ cannot be taken as the defining feature of a metal. Rather, the relevant analysis involves fitting the measured T dependence of ρ to an appropriate functional form, and then using this fit to extrapolate the results to $T \to 0$.

In most cases we will be reviewing, $\rho(T)$ in the anomalous metallic phase is essentially T independent for a range of accessible low temperatures, so the extrapolation to $T = 0$ is obvious. In all cases, it is also important to pay attention to the magnitude of the resistivity: when $\rho(T) \gg h/e^2$ at the lowest temperatures, it is *a priori* reasonable to expect that it will diverge as $T \to 0$, while conversely it would be rather unexpected to encounter a low-temperature regime in which $\rho(T) \ll h/e^2$ in a system that is tending toward an insulating ground state.

B. SIT versus QSMT

As mentioned in the Introduction, early studies of 2D systems were interpreted in the context of a scaling theory (Fisher, 1990) of the SIT. Some of these data appear in the present review, but are now interpreted as showing evidence of a QSMT. To avoid confusion, we begin with a discussion of this "historical" point.

Typically, in studies of a putative SIT, a state was identified as "superconducting" if the resistivity at low T was an increasing function of T and "insulating" if a decreasing function. In some cases the experimental data may be consistent with the assumption that there is a direct transition with no intermediate metallic phase. For example, the early study of Haviland, Liu, and Goldman (1989) (see Fig. 1) shows an evolution of the temperature dependence of the sheet resistance $R(T)$ with increasing thickness of an amorphous Bi film deposited onto Ge. At the separatrix, the resistance is T independent and has a value $\rho \approx (1/4)h/e^2$ corresponding to $k_F\ell \approx 4$.[3]

[3]The value of the resistance on the separatrix was identified by Haviland, Liu, and Goldman (1989) as the Cooper pair quantum of resistance $h/(2e)^2$ in agreement with the prediction of Fisher (1990) which was based on an idea of the localization of Cooper pairs. We note however that the resistivity at the separtrix is T independent up to 10 times or more than the maximal T_c where the Cooper pairs do not exist. Therefore it should be associated with a more conventional Drude theory rather than with quantum critical diffusion of charge $2e$ bosons. Note, in other cases, where a clear separation between the normal state and the regime of superconducting fluctuations is observed, the notion that the critical conductivity is associated with a self-dual point of charge $2e$ bosons, and hence has a value $h/(2e)^2$, has some experimental support; see, for example, Breznay *et al.* (2016).

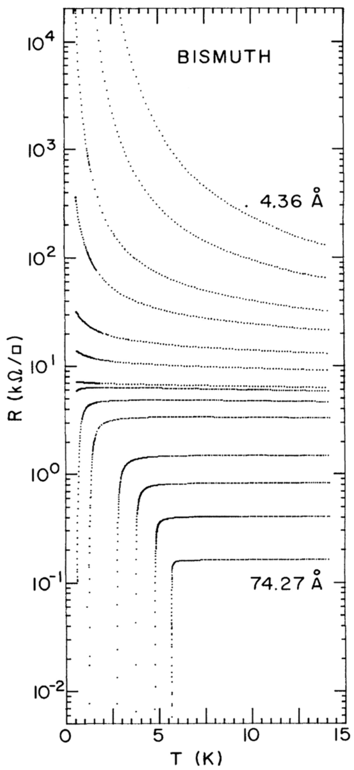

FIG. 1. The iconic figure of a superconductor to insulator transition in amorphous Bi films of varying film thickness. (Thicker films have lower resistance.) From Haviland, Liu, and Goldman, 1989.

Another example of a set of data that was so interpreted is shown in Fig. 2. Here, using a similar technique to gradually increase a film's thickness by depositing at very low temperatures, the evolution of the sheet resistance $R(T)$ with thickness was studied for various metallic elements. Qualitative differences in the T dependences are apparent between the thicker films (with lower normal state sheet resistance) and thinner films, as T decreases below a characteristic scale (presumably associated with the onset of local superconducting pairing), the resistivity of the thicker films drops precipitously while in the thinner films it increases. The existence of an approximately thick independent pairing scale was reasonably taken as indication of a granular morphology of the films. However, importantly from the current perspective, at still lower temperature, the resistance of the near critical films does not vanish at a well-defined finite temperature transition, but rather levels off to a value well below the normal state value.

Similar results are apparent in the data of White, Dynes, and Garno (1986) for thin layers of Sn and Pb on a helium-cooled glass substrate; see Fig. 3.

Data that approximately satisfy scaling relations (Fisher, 1990) expected in the critical regime of a magnetic field driven quantum SIT were obtained by Hebard and Paalanen (1990) (not shown). Here a transverse magnetic field was used to tune a thin film of disordered (mostly amorphous) indium oxide (InO_x) from a state in which the resistance decreases as a function of decreasing temperature to a state where the resistance increases in an activated fashion. The SIT was

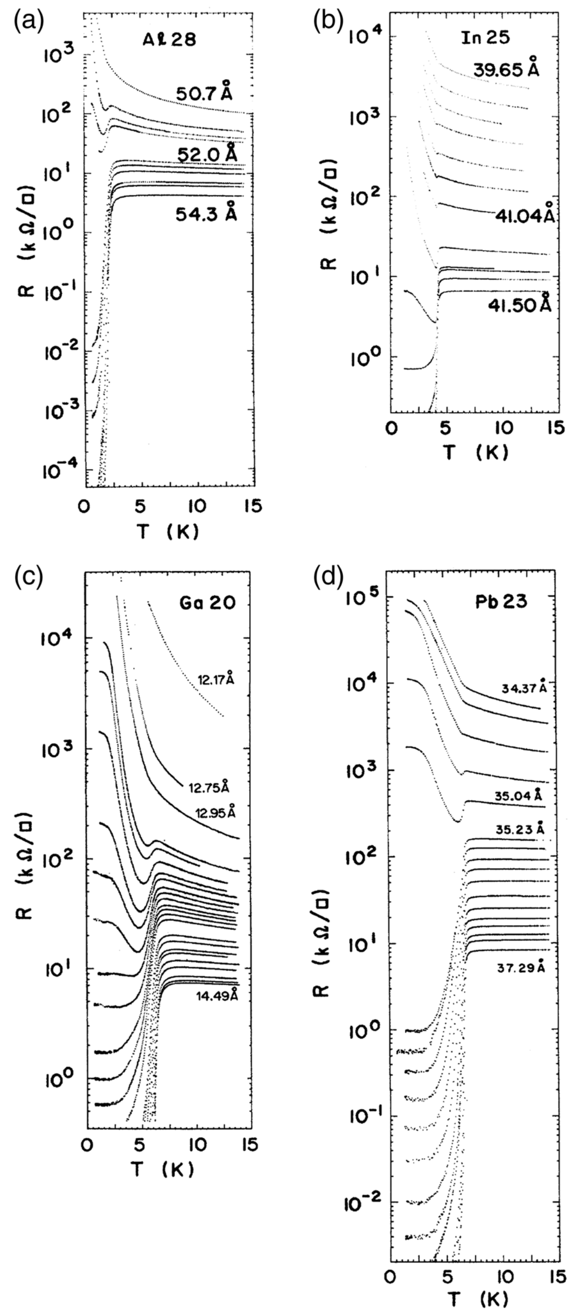

FIG. 2. Resistance (on a logarithmic scale) vs T for a sequence of "granular" films of (a) Al, (b) In, (c) Ga, and (d) Pb where for each subsequent film a small amount of metal is added to the previous film increasing the nominal thickness of the film. From Jaeger *et al.*, 1989.

associated with a thermal "crossing point," corresponding to a critical field at which the sign of the temperature derivative of R goes from positive to negative.

It is an important open question that should be revisited under what circumstances a direct SIT can occur without a possibly narrow intervening metallic phase. In the remainder of this section, we focus exclusively on experiments in which the existence of an anomalous metallic phase is clear. In some cases, these studies involve films with $k_F\ell \gg 1$. As far as we know, whenever $k_F\ell \gg 1$ at the point of the quantum phase transition from the superconducting state, the proximate phase

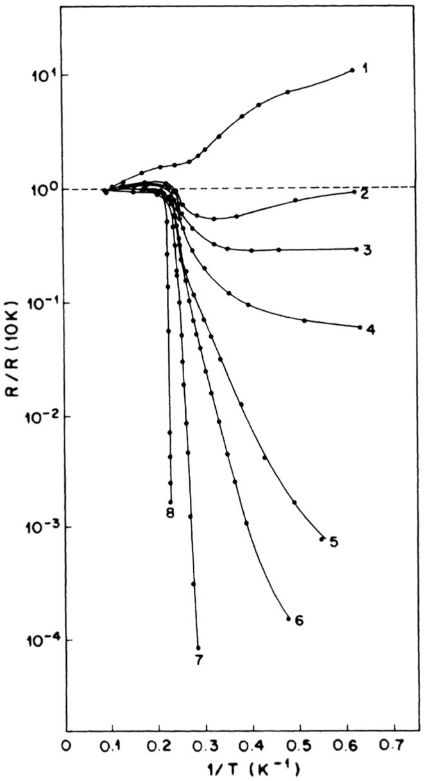

FIG. 3. Resistance (on a logarithmic scale) vs $1/T$ for a sequence of Sn films of varying thicknesses. From White, Dynes, and Garno, 1986.

is always a metal. No similarly categorical statement can be made concerning systems with $k_F\ell \sim 1$; however, as we shall see, many such systems also exhibit clear anomalous metallic phases.[4]

C. Magnetic field driven QSMT

The fact that magnetic fields can be tuned continuously, and that in almost all cases superconductivity can be quenched in accessible field ranges, makes the magnetic field driven transition particularly suitable for experimental study. However, there are possibly special aspects that are associated with field-induced vortices, and with the breaking of time-reversal symmetry that could, in principle, distinguish the field-induced QSMT from other cases. Nonetheless, in Sec. II.D we will show that many aspects of the problem appear to be the same whether or not a magnetic field is present.

Figures 4 and 5 show data from a field driven transition in highly metallic a-MoGe from Ephron *et al.* (1996) and Mason and Kapitulnik (1999, 2001). The "normal state" resistivity of these films $\rho_N = 1/\sigma_D^{(2D)}$ measured at temperatures somewhat above the zero field T_c or at $T = 0$ and large H is small compared to the quantum of resistance, implying that

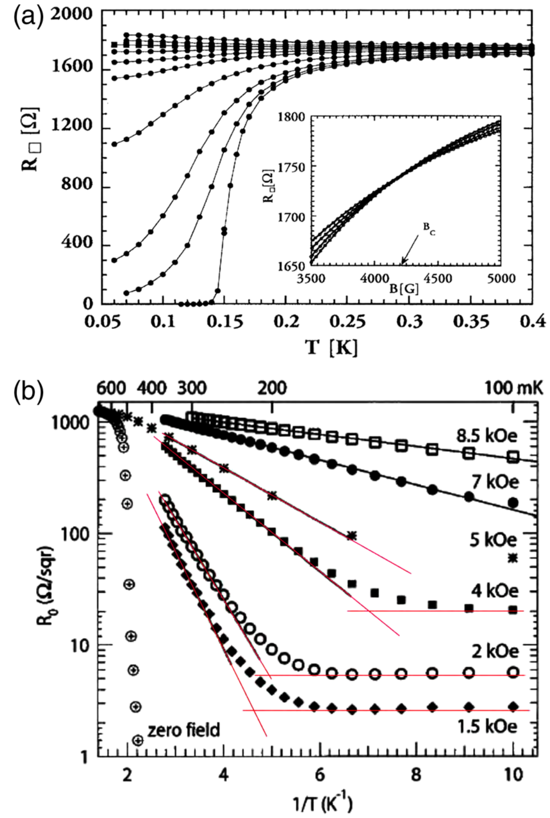

FIG. 4. (a) Resistance vs temperature for an a-MoGe film at a sequence of fixed magnetic fields. (The inset shows the putative crossing point of the isotherms.) From Yazdani and Kapitulnik (1995). (b) Resistivity on a logarithmic scale vs $1/T$ for an a-MoGe film at various values of magnetic field H; the low T saturation is evidence of the existence of an anomalous metallic phase.

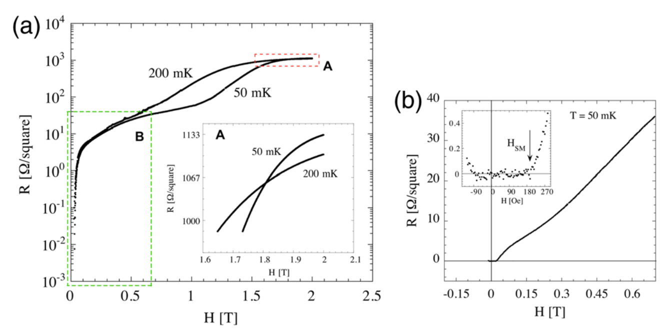

FIG. 5. (a) The magnetic field dependence of the low-temperature resistivity of a highly metallic a-MoGe film, shown on a logarithmic. (Over most of this field range, R is essentially temperature independent below 100 mK.) The inset shows a "crossing point" at an apparent critical field of approximately 1.8 T. (b) An expanded version of the lowest temperature curves shown as the dashed rectangle in (a). The inset shows that, within experimental error, a zero resistance state is found below a QSMT at $H \approx 0.18$ T. From Mason and Kapitulnik, 1999, 2001.

[4]A way to reconcile the differences between systems that exhibit a QSMT versus a SIT was proposed by Steiner, Breznay, and Kapitulnik (2008). Rather than focusing on the value of $k_F\ell$, they proposed that there are two distinct behaviors depending on the value of the critical conductivity σ_c defined as the $T \to 0$ limit of the conductivity at the point at which superconductivity is destroyed. Where $\sigma_c < 4e^2/h$, there is generally a SIT. On the other hand, where $\sigma_c \gg h/4e^2$ there is a QSMT.

$k_F\ell \gg 1$. [Typical values of the Drude conductivity in this case are in the range $\sigma_D^{(2D)} \sim (20\text{–}40) \times e^2/h$.] Moreover, the high field resistance is only weakly H and T dependent, as is expected in a range of fields in which $\omega_c\tau \ll 1$. (Here ω_c is the cyclotron frequency, and $\tau = \ell/v_F$ is the transport lifetime.) At smaller H, there exists a broad range of intermediate fields in which the resistivity first drops dramatically with decreasing temperature and then saturates at a low T "plateau" value that can be as much as 3–4 orders of magnitude smaller than ρ_D. Assuming that a T independent $\sigma(H,T)$ can be extrapolated to $T = 0$, these data imply the existence of a well-defined metallic quantum phase of matter. Moreover, the extent of this phase can be explicitly delimited: On the high field side, it is bounded by the above-mentioned crossing point that was previously associated with a SIT, but which is now associated with either a MIT or possibly a crossover from an anomalous to a more conventional metal. On the low field side, a later study by Mason and Kapitulnik (1999, 2001) identified a critical field that marks the phase transition between a fully superconducting phase (in which, within experimental uncertainty, $\rho \to 0$ as $T \to 0$) at low field and the anomalous metal at higher fields. This situation is sketched in the qualitative phase diagram in Fig. 6.

Similar field driven QSMTs with an anomalous metal regime have been observed in a diverse range of material systems with different morphologies. Next we show data on field-tuned anomalous metal phases for homogeneously disordered superconducting tantalum thin films (Fig. 7), amorphous tantalum-nitride (TaN_x), and indium-oxide (InO_x) films (Fig. 8).

While early field-tuned measurements demonstrated the emergence of a metallic phase in "homogeneously disordered" films, recent results on highly crystalline materials reinforce the idea that the important parameter is the initial high conductance of the films (that is, $k_F\ell \gg 1$), rather than the disorder *per se*. For example, Saito *et al.* (2015) reported transport studies on a single-crystalline flake of ZrNCl, which is ion gated, hence allowing for the tuning of the interface carrier density. In particular, they found that the zero resistance state is destroyed by the application of finite out-of-plane magnetic fields, and a metallic state is stabilized in a wide range of magnetic fields (Fig. 9). In a recent paper, a field driven QSMT was also documented in ion-tuned gated samples of $1T$-$TiSe_2$, and an anomalous metallic phase was observed (Li *et al.*, 2018). It is interesting to note how remarkably similar are the data in both of these systems to the measurements on a-MoGe (Ephron *et al.*, 1996).

D. QSMT at zero magnetic field

Nominally, a disordered superconductor in the presence of a magnetic field forms a glassy state, which implies slow dynamics and even history dependent properties. Indeed, some experiments on field-induced anomalous metals exhibit hysteretic behavior (Mason and Kapitulnik, 1999, 2001). Moreover, one can wonder whether the fact that H breaks time-reversal symmetry is essential for the existence of the anomalous metal. Therefore it is important to study the same phenomena in cases in which a zero field transition can be

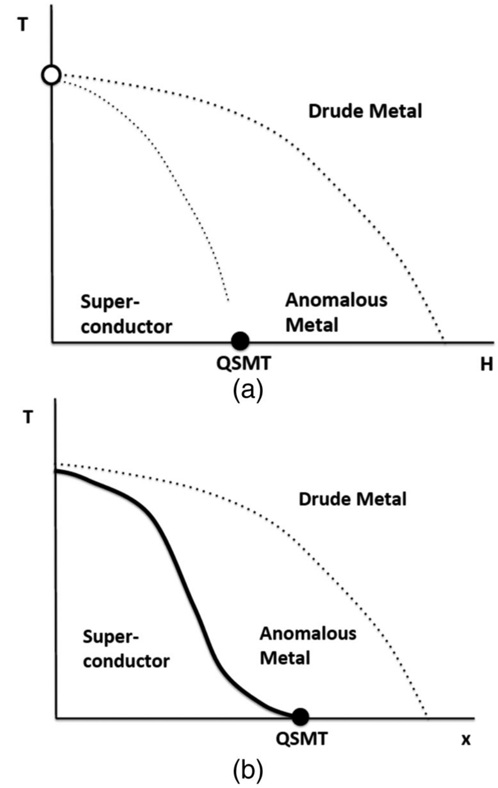

FIG. 6. Schematic phase diagrams for QSMT. (a) The magnetic field driven QSMT. The open circle represents the thermal transition at $H = 0$ and the solid circle the QCP associated with the QSMT. The dashed curves represent possible crossovers. The anomalous metal may be bounded at high H by an insulating phase, in which case there would be a QCP associated with a MIT at the end of the upper crossover line. Alternatively, there could be a quantum crossover to a metallic phase dominated by fermionic excitations. (b) A phase diagram for the gate-tuned QSMT. (More generally, x represents a quantum tuning parameter that does not break time-reversal symmetry.) The solid line represents the superconducting phase boundary and the solid circle the QCP associated with the QSMT. The dashed curve represents a crossover. As in the field-tuned case, the anomalous metal may be bounded at large x by an insulating phase, in which case there would be a QCP associated with a MIT at the end of the crossover line, or there could be a quantum crossover to a metallic phase dominated by fermionic excitations.

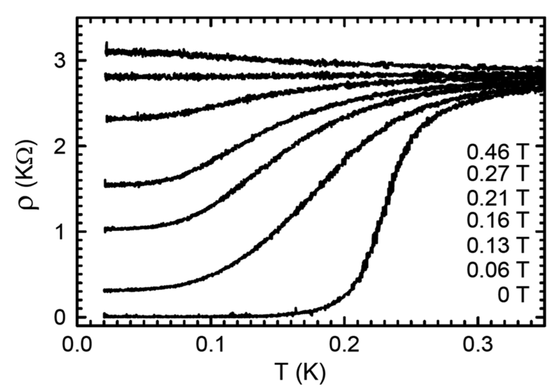

FIG. 7. Resistivity as a function of T for a Ta film for distinct magnetic fields equally spaced between 0 and 5 T. From Qin, Vicente, and Yoon, 2006.

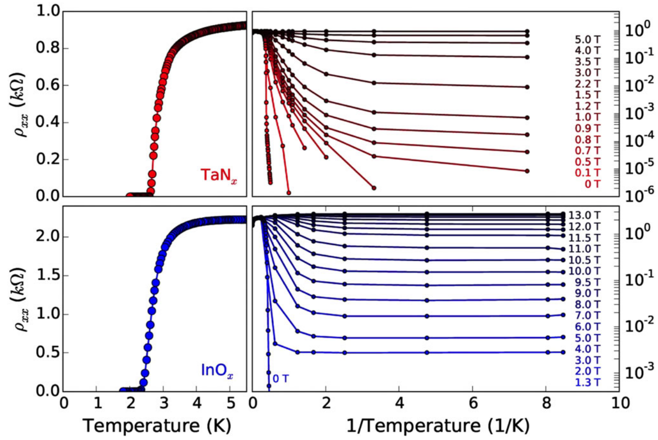

FIG. 8. Resistivity as a function of T for TaN_x and InO_x films. The left-hand panels show the superconducting transition in resistance vs T for $H = 0$. The right-hand panels show the resistance on a logarithmic scale as a function of $1/T$ for various values of the applied magnetic field. From Breznay and Kapitulnik, 2017.

driven by other means. One difference with the field driven case is that, so long as the ground state is superconducting, one expects there to be a finite temperature phase transition as shown in the schematic phase diagram in Fig. 6.

Electrostatic gating is an effective method to introduce doping at the interface of a conducting material. As gating involves introducing a nearby metallic electrode, it also affects the screening of Coulomb interactions and so introduces an additional dissipation channel.

Probably the first study of a gate-controlled QSMT was performed on an array of Al-Al_2O_x-Al Josephson junctions fabricated on a $GaAs/Al_{0.3}Ga_{0.7}As$ heterostructure in which a 2D electron gas (2DEG) was located approximately 100 nm from the surface (Rimberg *et al.*, 1997). In this study, the 2DEG was presumably only coupled capacitively to the Josephson junction array; however, so long as the conductivity of the 2DEG was sufficiently large, screening provided by the 2DEG caused the array to show superconducting behavior despite a large junction resistance. Gating was then used to change the resistance of the 2DEG and hence the dissipation in the electrodynamic environment of the array. As shown in Fig. 10, the temperature dependence of the array is different depending on the resistance of the 2DEG. In all cases, the resistance of the array decreases with decreasing temperature for $T \sim 0.2$ K, presumably reflecting the local superconducting order in the array. However, at lower temperatures, the resistance of the array continues to drop and then to saturate at the lowest temperatures when the resistance of the 2DEG is small. Conversely, the resistance of the array increases strongly with decreasing temperature when the resistance of the 2DEG is large. This behavior is suggestive of the existence of an anomalous metallic state.

There have been a number of other studies of gate-tuned QSMTs. In contrast to the early experiments of Rimberg *et al.* (1997), in these other studies the gate primarily serves to tune an intergrain Josephson coupling: In Fig. 11 we show data from experiments (Han *et al.*, 2014) on artificially prepared samples where a regular array of superconducting Sn disks were placed in a regular lattice on a graphene substrate. The density of electrons in the graphene can be varied by varying the voltage applied by a back gate. There is a proximity effect coupling between the superconducting droplets and the graphene, so the gate voltage (among other things) tunes the effective Josephson coupling between neighboring disks. The distinct colors in the figure represent the resistance as a

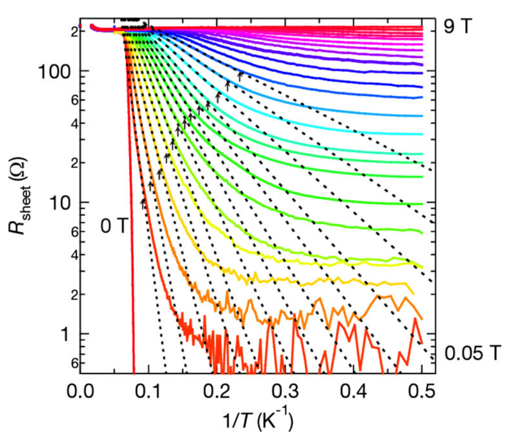

FIG. 9. Arrhenius plot of the sheet resistance of an electric double layer transistor of ZrNCl at gate voltage $V_G = 6.5$ V for different magnetic fields perpendicular to the surface. The black dashed lines demonstrate the activated behavior with activation energy $U(H) \propto \ln(H_0/H)$, similar to Ephron *et al.* (1996). The arrows separate the thermally activated state in the high-temperature limit and the saturated state at lower temperatures. From Saito *et al.*, 2015.

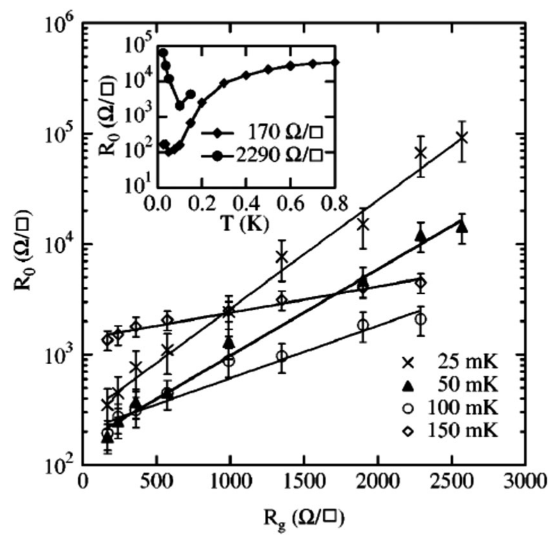

FIG. 10. Resistance of the Josephson array R_0 (on a logarithmic scale) vs resistance R_g of a ground plane (also on a logarithmic scale) to which the array is capacitively coupled. The main figure shows results for a set of increasingly low temperatures. The inset shows the temperature dependence of R_0 for $R_g = 170\ \Omega/\square$ (which exhibits anomalous metallic behavior) and $2290\ \Omega/\square$ which exhibits insulating tendencies, presumably due to quantum fluctuations of the order parameter phase in the Josephson junction array. From Rimberg *et al.*, 1997.

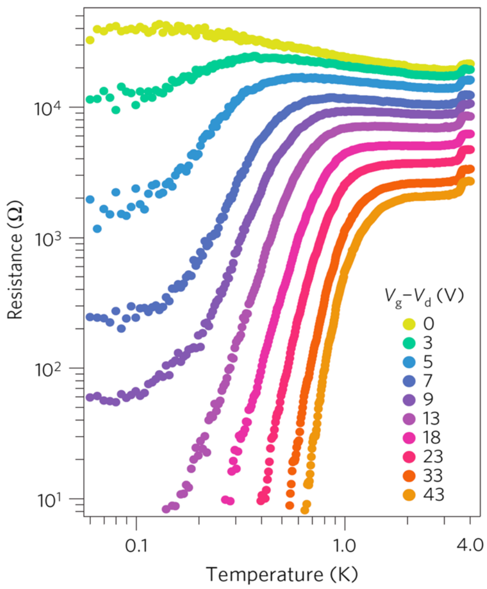

FIG. 11. The resistance vs T on a log-log scale of an ordered array of Sn disks on a graphene substrate; the density of electrons in the graphene is controlled by adjusting the voltage with a back gate. For the largest gate voltages (highest electron densities) there is a clear finite temperature transition to a superconducting state. However, for a broad range of lower gate voltages, we see the familiar several orders of magnitude drop in the resistance that terminates in a temperature independent plateau. From Han *et al.*, 2014.

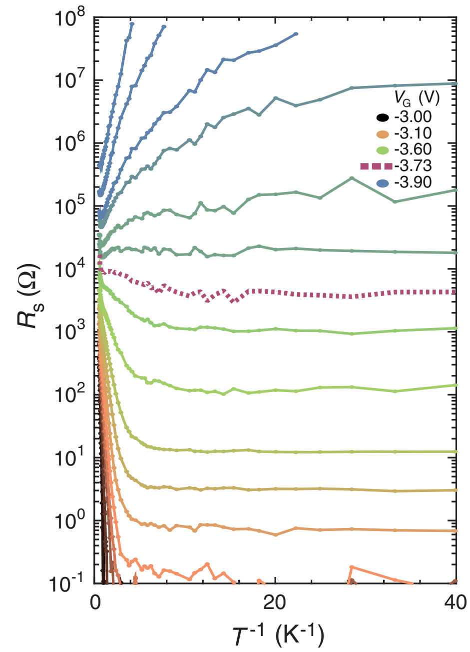

FIG. 12. Log of the sheet resistance R_s as a function of the inverse temperature T^{-1} in a gated InAs heterostructure with epitaxial Al patterned to form a regular array of superconducting islands. Data are shown for a range of gate voltages V_G from -3.0 to -3.9 V. The dashed curve corresponds to $V_G = -3.73$ V; the tendency of the curves with $V_G \geq -3.73$ V to saturate at low T is indicative of the occurrence of a metallic phase. From Bøttcher *et al.*, 2018.

function of T for various values of the gate voltage. The initial drop in the resistance is associated with the onset of superconductivity within the droplets. For large values of the gate voltage (large electron densities in the graphene), the resistance drops sharply at a somewhat lower temperature, extrapolating to zero at a superconducting transition temperature that varies depending on voltage. However, in the anomalous metal regime which appears for somewhat smaller gate voltages, as T decreases, the resistance drops by as much as 3–4 orders of magnitude, but then saturates at a finite plateau value that can be 3–4 orders of magnitude smaller than the Drude value. This behavior is very similar to that seen in the vicinity of the field driven QSMT. In this range of gate voltages, a magnetic field applied at low T produces a giant increase in the resistance which saturates at high fields—a gigantic positive magnetoresistance that recovers the Drude value of the resistance, presumably by suppressing any remnant superconducting coherence.

Figure 12 shows results for the system studied by Bøttcher *et al.* (2018). Here a gated semiconductor heterostructure with epitaxial Al was patterned to form a regular array of superconducting islands connected via a InAs quantum well. Gating the quantum well allowed for variation by many orders of magnitude in resistance, thus unveiling a range of anomalous metal behavior.

All the examples presented so far of a gate-tuned QSMT involved artificially fabricated granular systems, where the gate affects the properties of the intergranular (substrate) electronic structure. However, anomalous metallic states have also been observed in 2D films and interfaces that are considered homogeneous. For example, the same ZrNCl system for which the field-tuned transition is shown in Fig. 9 can also be tuned by tuning an ionic-gate voltage (Saito *et al.*, 2015).

Devices made of exfoliated single crystalline transition metal dichalcogonides, such as MoS_2 (Ye *et al.*, 2012) and WTe_2 (Sajadi *et al.*, 2017) have shown a transition from a superconducting state to an anomalous metallic state upon varying the gate voltage. An example of WTe_2 is shown in Fig. 13 (Sajadi *et al.*, 2017).

An advance in gate-controlled coupling was achieved by Chen *et al.* (2017) utilizing dual electrostatic gates, which, as shown in Fig. 14, were used to manipulate both the mean depth and the asymmetry of the quantum well in a $SrTiO_3$-$LaAlO_3$ heterostructure. Notably, the large (exceeding 20 000) and nonlinear dielectric constant of the $SrTiO_3$ greatly enhances the tunability of this system as compared to conventional gating experiments. On one side, the 2DEG is bounded by the wide gap $LaAlO_3$, where a top gate (V_{TG}) predominantly controlled the density of carriers confined close to the $SrTiO_3/LaAlO_3$ interface. A back gate (V_{BG}) is then used to control the thickness of the conduction layer at the interface, hence the interfacial scattering rate and mobility of the 2DEG.

Turning to "unconventional" superconducting states, Fig. 15 shows the resistivity as a function of T for a liquid ion gated film of the cuprate superconductor $La_2CuO_{4+\delta}$. At small gate

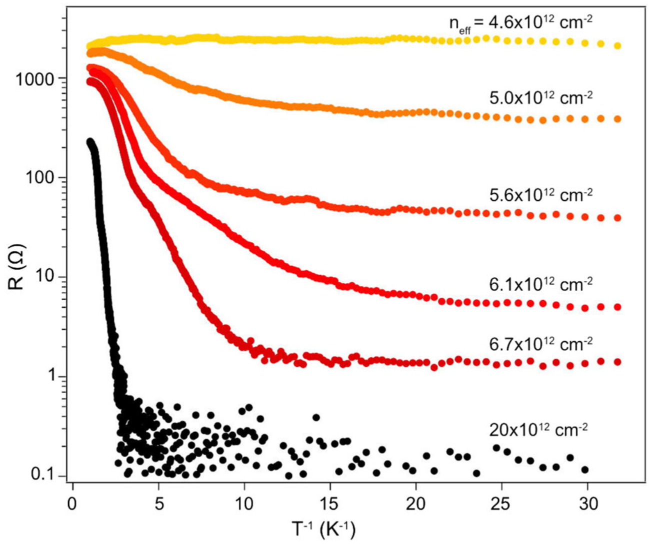

FIG. 13. Four-probe resistance (on a logarithmic scale) vs inverse temperature of a monolayer WTe_2 flake, which is tuned by application of a gate voltage from a superconducting state (seen for the highest electron density $n_{\rm eff} = 20 \times 10^{12}$ cm^{-2}), to an anomalous metallic state (when $n_{\rm eff} = 12$, 8.5, 6.7, 6.1, 5.6, and 5×10^{12} cm^{-2}), and finally to what appears to be a normal metal when $n_{\rm eff} = 4.6 \times 10^{12}$ cm^{-2}. From Sajadi *et al.*, 2017.

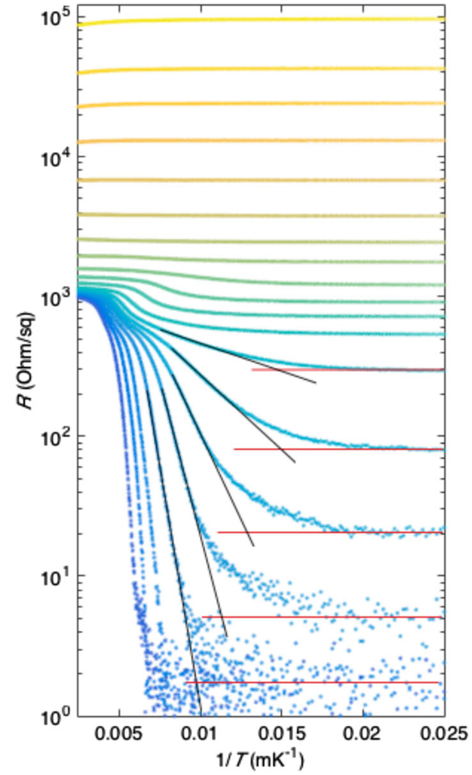

FIG. 14. Resistivity (on a logarithmic scale) vs inverse temperature as a function of the top gate voltage $V_{\rm TG}$ from −0.36 to 1.80 V with fixed bottom gate voltage $V_{\rm BG} = 0$ V in a $SrTiO_3$-$LaAlO_3$ heterostructure. In this voltage interval the electron concentration changes from 0.13 to 1.51 in units of 10^{13} cm^{-2}. The estimation of concentrations is based on the high field limit of the slope (between 13 and 14 T) for the Hall resistivity vs magnetic field curves. The solid lines have been added as guides to the eye. The asymptotic approach of the measured curves to the temperature independent red lines as $T \to 0$ shows the existence of an anomalous metallic phase. From Chen *et al.*, 2017.

voltage, the film exhibits clear insulating behavior, while at large gate voltage it is superconducting below a nonzero superconducting transition temperature. However, as shown in the inset, at intermediate values of the gate voltage, while the resistance drops below a well-defined crossover temperature, it appears to saturate at low T to a small value (sometimes 4 orders of magnitude smaller than ρ_D). Note that with the higher T_c of this system, the temperature where saturation is apparent also increases. For example, at the highest gate voltage in Fig. 15, saturation occurs below ∼10 K. This issue will become important in arguing against a simple heating as explanation of the resistance saturation.

While gate or magnetic field tuning have the advantage that they can be varied continuously, other approaches to the QSMT have been successfully explored as well.

Early studies in which a sequence of presumably granular films is studied for various film thickness have already been presented in Fig. 2. Similarly, a more recent study (Crauste *et al.*, 2009) of presumably homogeneous films of Ni_xSi_{1-x} of various thicknesses have found that as a function of decreasing T, the thinnest films show a strong divergence of the resistivity indicative of approach to an insulating groundstate. The thickest films show a finite temperature transition to a zero resistance state, but films of intermediate thickness show the familiar signatures of an anomalous metal.

Since both the Josephson coupling and the charging energies depend on the size and distance between grains, similar tunability can be achieved by preparing samples with different grain size and periodicity. Indeed, this approach was taken by Eley *et al.* (2012), where an array of Nb dots was deposited on a gold substrate. As seen in Fig. 16, a low-temperature metallic state is clearly revealed at a wide range of distance between grains.

E. Not just the resistivity

Other features of the anomalous metal that illustrate its character as a failed superconductor have been measured in a limited number of cases.

(1) The emergent particle-hole symmetry of the superconducting state suggests that it is natural to expect a reduction of the Hall and thermoelectric responses in the anomalous metal and a tendency for them to vanish upon approach to the QSMT. To explore this issue, simultaneous measurements (Breznay and Kapitulnik, 2017) of ρ_{xx} and ρ_{xy} were performed on InO_x and TaN_x films down to temperatures well below the zero field T_c, and these were used to calculate σ_{xy}. At the lowest temperatures ($T = 120$ mK) ρ_{xy} and σ_{xy} were indeed found to be immeasurably small for a broader range of fields, including both the superconducting range and the anomalous metallic range.

(2) The presence of significant superconducting fluctuations in a system in which long-range superconducting phase coherence has been lost can often be apparent in the finite frequency response $\sigma(\omega)$. Notionally, assuming that some

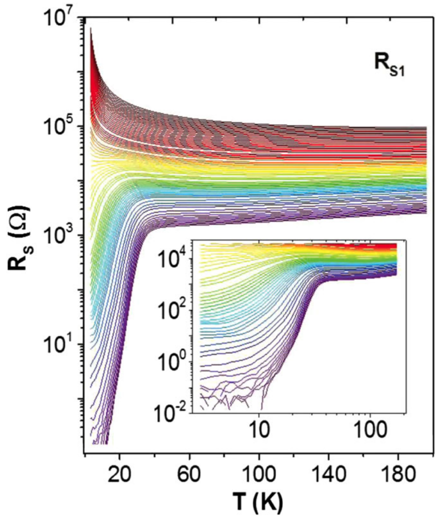

FIG. 15. Sheet resistance (on a logarithmic scale) as a function of temperature for different values of gate voltage V_G measured on a ~5 nm thick $La_2CuO_{4+\delta}$ film. Gate voltages ranged from 1.2 V for the most insulating sample to 3 V for the most superconducting one. The inset shows the less resistive samples on a log-log scale, which expands the low-temperature portion of the curves thereby making clear the saturation of the resistance at low temperatures. From Garcia-Barriocanal *et al.*, 2013.

form of dynamical scaling applies, the finite ω response probes correlations at finite length scales. This strategy has been successfully employed to establish the existence of substantial finite-range superconducting correlations in a-MoGe films (Yazdani *et al.*, 1993; Yazdani, 1994) in a magnetic field, where the high-frequency dynamics was shown to behave as expected upon approach to an ideal classical vortex lattice melting transition. Using a similar approach, finite-range superconducting correlations were also established for the cuprate high-temperature superconductors in a range of temperatures above T_c (Corson *et al.*, 1999; Bilbro *et al.*, 2011) and in the insulating state proximate to an SIT in highly disordered InO_x films (Crane *et al.*, 2007).

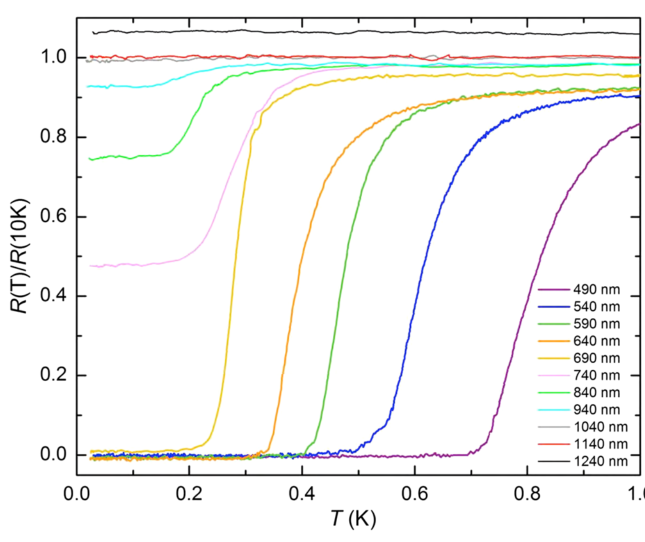

FIG. 16. Normalized resistance as a function of temperature for arrays of widely spaced Nb islands on a Au substrate. For spacings exceeding 700 nm, the Brezinskii-Kosterlitz-Thouless (BKT) transition is interrupted by a low-temperature metallic state. The data for $d \leq 690$ nm and $d \geq 740$ nm come from systems with Nb island heights of 125 and 145 nm, respectively. From Eley *et al.*, 2012.

Recently, the finite ω response of a magnetic field induced anomalous metallic state in weakly disordered InO_x films was measured by Liu *et al.* (2013) and Wang *et al.* (2017). Here broadband microwave measurements were performed in the frequency range from 50 MHz to 8 GHz and the temperature and magnetic field dependences of the complex microwave conductance determined. Strongly non-Drude features are observed in the anomalous metal regime. While intuitively these features support the identification of this regime as a failed superconductor, as far as we know no explicit theoretical account of these observations currently exists. Further study, both theoretical and experimental, of the finite frequency response is clearly warranted.

F. The strange case of granular films

For the most part, the properties of the anomalous metal seen in all the studies so far discussed are similar, independent of system morphology, degree of order, and whether or not a magnetic field is applied. There is, however, another class of systems, which are granular films in some not entirely well-defined sense, which also show evidence of an anomalous metallic phase, but of a very different character. Granular films can be synthesized in various ways (Abeles *et al.*, 1975; Kapitulnik and Deutscher, 1982; Deutcher, Zallen, and Adler, 1983). While local superconductivity can occur within a single grain, global superconducting phase coherence necessarily involves Josephson (pair) tunneling between grains and thus is sensitive to various details of the grain morphology and the nature of the material between grains [see, e.g., Entin-Wohlman, Kapitulnik, and Shapira (1981), Imry and Strongin (1981), and Ioffe and Larkin (1981)]. Here, for completeness, we briefly discuss some such experimental observations.

Pb films are a particular well-studied model system (Imry and Strongin, 1981; Jaeger *et al.*, 1989; Merchant *et al.*, 2001). Figure 17 shows data from Merchant *et al.* (2001) on Pb films. As in the data on other granular materials shown in Fig. 2, there is a clear signature of the onset of local superconductivity within a "grain" at a relatively high T, but then depending on the distance between grains (or more particularly the Pb coverage), the system evolves from a globally insulating to globally superconducting state. Elegantly, in the present case, tunneling studies (not shown here) on the same films show a clean BCS-like superconducting gap opening up at around the same temperature, largely independent of the Pb coverage. Here the resistivity of films with low Pb coverage [Figs. 17(a)–17(d)] shows a clear tendency to diverge in the $T \to 0$ limit and thus can be characterized as insulating. However, there is an intermediate regime of concentrations [Figs. 17(e) and 17(f) and possibly Fig. 17(g)] in which the resistivity decreases strongly with decreasing T, but it does so in a manner such that the T dependence of ρ approximately follows the phenomenological relation

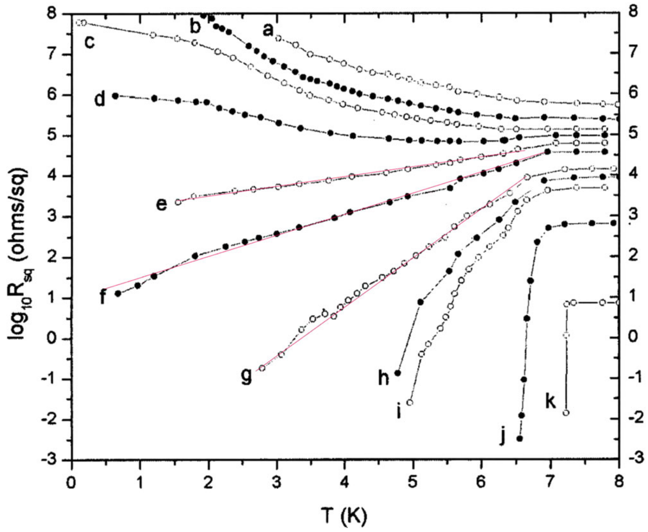

FIG. 17. Resistivity (on a logarithmic scale) as a function of T for a sequence of granular Pb films. Each curve corresponds to data at fixed Pb coverage, and the tuning from one curve to the one immediately below it is accomplished by depositing a small additional quantity of Pb on the previous film. Even in the insulating films with the least Pb coverage (i.e., films a–d), tunneling spectra reveal a well-developed superconducting gap, and this gap is more or less the same even as additional Pb is added, including the "metallic films" (i.e., films e, f, and g) and the clearly superconducting films (i.e., films j and k). From Merchant *et al.*, 2001.

$$\rho(T) \approx \rho_0 \exp[T/T_0], \tag{2}$$

where ρ_0 and T_0 are T independent functions of the concentration of Pb grains. In the case of Fig. 17(g), for example, ρ_0 is roughly 4 orders of magnitude smaller than the normal state resistivity. (See lines fit to the data in Fig. 17.)

Indeed, such behavior has been widely observed in thin superconducting films. For instance, Merchant *et al.* (2001) saw very similar behavior when the coupling between grains of an insulating granular Pb film was gradually increased by the addition of a thin layer of Ag. Behavior of this sort was seen long ago in thin Al films by Masker, Marčelja, and Parks (1969)]. While the fit to Eq. (2) implies the existence of an intermediate metallic state (that is, the resistance extrapolates to a finite value as $T \to 0$), it is difficult to rule out a power-law temperature dependence of the prefactor that could lead to a vanishing resistance at some much lower temperature. More importantly for present purposes, our understanding of the anomalous metallic phase does not include a satisfactory explanation of the $\exp(T/T_0)$ phenomenon. We will therefore (reluctantly) not discuss it further in this article.

G. The issues of heating and nonequilibrium effects

Especially when the saturation of the T dependence of the resistivity occurs at low temperatures, it is essential to establish that this behavior is intrinsic and does not simply represent a point below which nonequilibrium effects interfere with the measurements. While this issue has been discussed by each of the experiments reported, it is still worth briefly enumerating some of the key issues a bit more explicitly.

Heating and other nonequilibrium effects can arise as a consequence of uncontrolled external "noise." Time-varying electric fields couple via mutual capacitances (i.e., electrostatic coupling), and thus inject noise into the system as a current source, while magnetic or inductive interference, which arises from time-varying magnetic fluxes passing through the measurement circuit, can induce fluctuating voltage sources. Some of the systems reported above exhibit saturation at sufficiently elevated temperatures where electron-lattice relaxation rates are high enough that the electron temperature is probably not in question (see Figs. 9 and 15). However, quantum transitions in general, and those involving a metallic state more particularly, are much more sensitive to details of disorder than are classical transitions. In certain models, this can lead to the existence of quantum Griffith phases (Fisher, 1992, 1995). As discussed in Appendix B and Spivak, Oreto, and Kivelson (2008), such Griffith phases cannot occur in the strict asymptotic sense. However, under some circumstances, a broad intermediate region can arise (Del Maestro *et al.*, 2008, 2010; Vojta, Kotabage, and Hoyos, 2009) in which rare events dominate the macroscopic behavior of the system. In such cases, where both the superconducting and the anomalous metal states are extremely fragile, even weak perturbations may destroy these states.

Analog equipment, proper filtering, and electromagnetic shielding are typically used to insure (and demonstrate) that measurements are in the linear response regime [see, e.g., Ephron *et al.* (1996), and Mason and Kapitulnik (1999, 2001)]. On the other hand, it has been shown that in some systems, in a range of parameters in which with proper filtering the resistance can be seen to vanish as $T \to 0$, removing the filters results in saturation of the measured resistivity at low T, giving the spurious appearance of a metallic state (Tamir *et al.*, 2018). This observation highlights the continuing importance of testing the robustness of the various observations. Additional measurements with the same experimental conditions can also be used to test the validity of the basic results. These include a comparison of the "effective electron temperature" (measured independently) to the measured temperature as a function of the applied external parameter [see, e.g., Fig. 4 from Ephron *et al.* (1996)], a comparison of different conditions for the sample that for the same temperature may or may not exhibit saturation [see, e.g., Bøttcher *et al.* (2018), where an anomalous metallic regime is absent when the transition is driven by an in-plane magnetic for the same sample], or measurements in the same cryostat system, where nonsuperconducting samples with similar resistance in the same measurement circuit and on the same type of substrate yield consistent results with no signature of heating (Ephron, 1996; Mason, 2001).

Nonetheless, we stress that these issues are significant and complex. It is therefore important that continuing efforts be made to directly measure the electron temperature in the anomalous metallic regime and to mitigate the effects of any external noise in each system in which such behavior is observed. At the minimum, the fragility of the superconducting state proximate to the QSMT is suggestive of the existence of a highly inhomogeneous electronic structure. This is further discussed in the Theory section next.

III. THEORY

A. The inadequacies of various "obvious" approaches

To begin with, we discuss a variety of theoretical approaches to examine why they are not consistent with the observed phenomena.

1. The inadequacy of classical percolation

One might think to account for the anomalous metallic phase from considerations of classical percolation. Imagine a system that consists of a macroscopic mixture of superconducting regions (with typical radius large compared to the superconducting coherence length ξ) and metallic regions with conductivity σ_D. The conductivity is then given by $\sigma = \sigma_D F(x)$, where x is the volume fraction of a superconductor, and $F(x)$ is a dimensionless function. Above percolation, $x > x_c$, the conductivity is infinite for any x. However, for $x < x_c$, the conductivity is finite. While some aspects of F depend on the details of the ensemble being studied, in general (Stauffer and Aharony, 1994) $F(x) \to 1$ as $x \to 0$ and $F(x)$ diverges as $F(x) \sim (x_c - x)^{-s}$ as x approaches x_c from below, with $s = 4/3$ in 2D and $s \approx 0.73$ in 3D.

The conductivity of an almost percolating superconductor, while finite, can be arbitrarily large. There are several reasons why such an explanation cannot be invoked to account for the observed anomalous metallic phases:

(a) In the framework of the classical percolation to satisfy the condition that $\sigma/\sigma_D \gg 1$, it is necessary that the system be fine-tuned to the very close vicinity of the percolation threshold x_c. In the experiments reviewed earlier, σ/σ_D can be as large as 10^4 which would require $(x - x_c) \sim 10^{-3}$. This is difficult to reconcile with the relatively broad range of parameters and circumstances over which the anomalous metal is observed.

(b) For a classical percolation picture to hold, the distance between superconducting puddles must be larger than $\min[L_T, L_B]$, where $L_T = \sqrt{\hbar D/k_B T}$ is the normal metal coherence length, and $L_B = \sqrt{\Phi_0/2\pi B}$ is the magnetic length. (Φ_0 is the flux quantum.) An inevitable corollary of this picture is that at low enough temperatures, such that L_T grows to be larger than the typical spacing between superconducting regions, global superconducting coherence will be established, leading to a further growth of σ and a superconducting ground state. Manifestly, to describe the quantum superconductor-metal transition at $T = 0$ one has to take into consideration quantum fluctuations of the order parameter.

(c) It can be shown (Stroud and Bergman, 1984) that the effective Hall conductivity in a 2D metal-superconductor mixture is the same as that of the metallic component, independent of x for $x < x_c$. Where this expectation has been tested in InO_x and TaN_x films (Breznay and Kapitulnik, 2017), it has been found that σ_{xy} of the anomalous metal is much smaller than its Drude value.

(d) Finally, there is good reason to doubt that such macroscopic inhomogeneities occur in many of the systems already discussed. Some of these systems consist of ordered arrays of superconducting dots on metallic substrates, and others consist of metallic films whose structural and chemical homogeneity has been scrutinized using various probes. It seems unlikely that there is a hidden inhomogeneity in the structures of these systems on the requisite length scales to justify a percolation analysis.

2. The inadequacy of "conventional" fluctuation superconductivity

The theory of classical superconducting fluctuations upon approach to a transition with a finite T_c is well developed. [For a review, see Larkin and Varlamov (2005)]. In some sense this would seem to provide a prototype for the properties of an anomalous meal. Indeed, the fact that the growing superconducting correlations allow an increasing portion of the current to be carried by collective Cooper pair fluctuations leads to a contribution to the conductivity that diverges as $T \to T_c$. Moreover, since bosonic fluctuational Cooper pairs have a size which diverges as $T \to T_c$, they are not subject to the single-particle interference effects that lead to the weak localization.

There are problems with using this approach to explain properties of the anomalous metal regime: The width of the regime in which fluctuational effects are significant $\delta T \sim T_c \mathcal{G} \ll T_c$ is controlled by the Ginzburg-Levanyuk parameter (Levanyuk, 1959; Ginzburg, 1961) $\mathcal{G} \equiv 1/N_\xi \ll 1$. Here

$$N_\xi = \nu \Delta \xi^D \tag{3}$$

can be interpreted to be the number of electrons per coherence volume that are paired upon entering the superconducting state, ν is the metallic density of states at the Fermi energy, and Δ and ξ are, respectively, the typical gap magnitude and the superconducting coherence length in the superconducting ground state. In many conventional superconductors $\mathcal{G}$ is small. For example, in quasi-2D samples with statistically uniform disorder, $\mathcal{G} \sim e^2/\hbar\sigma_D^{(2D)}$. [See, for example, Larkin and Varlamov (2005).] Note that the celebrated Aslamazov-Larkin (Aslamazov and Larkin, 1968) and Maki-Thompson (Maki, 1968; Thompson, 1970) corrections to the Drude conductivity are calculated in the temperature interval $(T - T_c)/T_c \gg \mathcal{G}$, where they are small. Moreover, these fluctuation corrections exhibit strong temperature dependence as $T \to T_c$, while the measured conductivity in the anomalous metal regime is temperature independent at the lowest temperatures.

3. The inadequacy of local bosonic theories

A theoretical treatment of the transition to a superconducting state can always be treated in terms of an effective action $S^{\text{eff}}[\Delta]$ that is a functional of a charge $2e$ complex scalar field Δ. Formally, S^{eff} can be obtained from a microscopic electronic Hamiltonian by introducing Δ as a Hubbard-Stratonovich field and then integrating out the fermionic electronic degrees of freedom. However, physically there is an important distinction between cases in which S^{eff} is a local functional, when it can be expressed in terms of an integral over $\Delta(\mathbf{r}, t)$ and its derivatives, or a nonlocal functional. In the former case, the low-energy long-wavelength degrees of freedom can be thought of as "purely bosonic." In the latter case, the nonlocality reflects the existence of gapless, delocalized fermionic degrees of freedom that need to be taken into account in one way or another; under these circumstances, the procedure of integrating out the fermionic modes is a formal trick that can be misleading.

The conventional Landau-Ginzburg-Wilson treatment of classical finite temperature phase transitions is an example of a purely bosonic theory.[5] $S^{\rm eff}$ is "local" in an interval of temperatures near T_c, and on spatial scales larger than the coherence length of the normal metal $L_T = \sqrt{\hbar D/k_B T}$ evaluated at $T = T_c$. In other words $S^{\rm eff}$ can be expanded in terms of $\Delta(\mathbf{r}, t)$ and its time and space derivatives. This follows from the fact that the various fermionic response functions that enter $S^{\rm eff}$ decay exponentially on scales larger than L_T. It thus seems natural that the same considerations can be applied to zero temperature quantum phase transitions. However, in the case of a QSMT, $L_T \to \infty$. Consequently, the various electron response functions exhibit power-law decays at long distances and hence $S^{\rm eff}[\Delta]$ is nonlocal.

In the cases we have discussed in which an anomalous metal phase is observed, the single-particle states are presumably gapless.[6] Thus, no purely bosonic theory is adequate.

Currently there are no generally reliable methods to treat nonlocal actions. This is not to say that it is never reasonable to approach the problem from this perspective. Studies of metallic criticality based on the Herz-Millis (Hertz, 1976; Millis, 1993) theory adopt such an approach. In the context of the QSMT, there is a class of model problems which correspond to a quantum version of the phenomenologically defined resistively shunted Josephson junctions (RSJ) model, for which the theoretical solution is clear as discussed in Appendix A.1.

A purely bosonic description may well be possible in a system consisting of superconducting grains coupled by tunnel junction systems such that below a "mean-field" transition temperature there is a negligible density of low-energy fermionic excitations. In this case, a conventional action describing Josephson-coupled superconducting grains supplemented with a quantum capacitance term describing the quantum dynamics of the phase of the order parameter is appropriate. Typically, a proper treatment of such an action yields a quantum superconductor-insulator transition (Fisher, 1986).

4. How BCS theory implies the absence of quantum critical fluctuations at $H = 0$ in systems without competing interactions

It is natural to associate the anomalous metal with growing ground-state superconducting correlations as a QSMT is approached from the metallic side. In Sec. III.B we will discuss theoretically tractable circumstances in which the requisite quantum fluctuations indeed occur. First, however, we discuss why even the existence of a quantum critical regime in the absence of magnetic field is an issue. Specifically, because the uniform susceptibility of a Fermi liquid diverges (logarithmically) as $T \to 0$, even in the presence of weak disorder, any net attractive interaction generally leads to a superconducting ground state. Conversely, weakly repulsive interactions are "irrelevant" and thus can be treated perturbatively. According to this line of reasoning, the QSMT occurs when the effective interactions vanish.

Since this is an important point of perspective, let us consider the QSMT in the context of the Hamiltonian

$$H = H_0 - \int d\mathbf{r} u(\mathbf{r}) \Psi^\dagger_\sigma(\mathbf{r}) \Psi(\mathbf{r})^\dagger_{-\sigma} \Psi_{-\sigma}(\mathbf{r}) \Psi_\sigma(\mathbf{r}), \quad (4)$$

where $\psi^\dagger_\sigma(\mathbf{r})$ creates an electron with spin polarization σ at position $\mathbf{r}$, and the sign convention is chosen such that $u(\mathbf{r}) > 0$ corresponds to a local attractive interaction between electrons. In generalized BCS mean-field theory, the local gap parameter is determined self-consistently in terms of the anomalous expectation value of the pair-field creation operator according to

$$\langle \Delta(\mathbf{r}) \rangle \equiv -u(\mathbf{r}) \langle \psi_\uparrow(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \rangle, \quad (5)$$

where $\langle \rangle$ represents the quantum mechanical average. Thus, the mean-field superconducting transition temperature T_c (if it exists) is the temperature below which the largest eigenvalue $\lambda(T)$ of the linearized gap equation

$$\lambda(T) \Delta(\mathbf{r}) = -u(\mathbf{r}) \int d\mathbf{r}' K(\mathbf{r}, \mathbf{r}') \Delta(\mathbf{r}') \quad (6)$$

is larger than 1, i.e., $\lambda(T_c) = 1$ and $\lambda(T) > 1$ for $T < T_c$. Here $K(\mathbf{r}, \mathbf{r}')$ is the nonlocal order parameter susceptibility, which for noninteracting electrons (or, more generally, for a Fermi liquid) can be expressed in terms of a convolution of single-particle Matsubara Green's functions,

$$K(\mathbf{r}, \mathbf{r}') = T \sum_\omega G_\omega(\mathbf{r}, \mathbf{r}') G_{-\omega}(\mathbf{r}, \mathbf{r}'), \quad (7)$$

where $\omega = (2n + 1)\pi T$. At finite temperature $K(\mathbf{r}, \mathbf{r}') \sim |\mathbf{r} - \mathbf{r}'|^{-(d)}$ for $|\mathbf{r} - \mathbf{r}'| \ll L_T$ and $K(\mathbf{r}, \mathbf{r}') \sim e^{-|\mathbf{r}-\mathbf{r}'|/L_T}$ for $R \gg L_T$. The essential feature is that K is a decreasing function of $|\mathbf{r} - \mathbf{r}'|$ which falls sufficiently slowly with distance that its integral diverges as $T \to 0$:

$$\int d\mathbf{r}' K(\mathbf{r}, \mathbf{r}') \sim \nu \log[E_F/T]. \quad (8)$$

This is nothing more than a reflection of the Cooper instability of a Fermi liquid. That this relation is true even in the presence of disorder (at least out to distance scales comparable to the localization length, if the electronic states are weakly localized) is the essence of "Anderson's theorem" (Anderson, 1959).

[5]While strictly speaking the notion of statistics does not enter the discussion of classical critical phenomena, order parameters always correspond to an even number of electron creation operators and so are "bosonic."

[6]This conclusion is readily supported on theoretical grounds. Direct experimental evidence exists in various specific cases—for instance, a substantial zero energy density is seen in planar tunneling experiments on $SrTiO_3/LaAlO_3$ heterostructures, similar to those reported in Fig. 14. (Fillis-Tsirakis *et al.*, 2016).

To slightly belabor the point, note that a variational lower bound to $\lambda(T)$ can be obtained by considering a trial state $\Delta(\mathbf{r}) = \sqrt{u(\mathbf{r})/\bar{u}}\Delta$, which yields

$$\lambda(T) \geq \Omega^{-1} \int d\mathbf{r} d\mathbf{r}' \sqrt{u(\mathbf{r})u(\mathbf{r}')} K(\mathbf{r}, \mathbf{r}'), \tag{9}$$

where Ω is the "volume" (area in 2D) of the system and $\bar{u}$ is a suitable average of $u(\mathbf{r})$. This gives a lower bound to the mean field $T_c \geq E_F \exp(-1/\bar{u}\nu)$, which is manifestly nonvanishing for any $\bar{u} > 0$. Further, since for small $\bar{u}$ the associated zero temperature coherence length is exponentially long, the assumption that the pairing amplitude is uniform is self-consistently validated as all finite length-scale inhomogeneities are averaged out. (In the literature, this is sometimes referred to as the "Cooper limit.") Finally, the fact that the usual (thermal) Ginzburg-Levanyuk parameter diverges as $\bar{u} \to 0$ implies that the mean-field estimate of T_c becomes asymptotically exact.

More realistic models, for instance, those involving low-energy attractive and high-energy repulsive interactions, when treated using the usual diagramatic approach give rise to the same conclusion: the QSMT is driven by a change in sign of the effective interaction. Consequently, the effective interaction vanishes identically at the point of the quantum phase transition.

As already mentioned, in 2D even for $k_F\ell \gg 1$, all single-particle states are localized (Abrahams *et al.*, 1979; Gorkov, Larkin, and Khmelnitskii, 1979; Lee and Ramakrishnan, 1985), so the divergence of $\lambda_0(T)$ is cut off below $T \sim T^\star$, defined in Eq. (1). In principle, this could result in a nonvanishing interaction strength at criticality. However, the corresponding critical regime is parametrically narrow and any critical effects would be confined to exponentially low temperatures $T \lesssim T^\star$. Thus, for clean metals, these considerations are of no practical importance. (We will return to the issue of localization in Sec. IV.)

When localization physics can be neglected, it is not so much a question of why the metal is not an insulator as how can one understand the existence of a "failed superconductor" in which strong superconducting correlations develop below a nonzero crossover scale, but the ground state fails to be globally phase coherent. In the next section, we will show how this can arise in the case in which $u(\mathbf{r})$ is attractive in some regions of space and repulsive in others. In Sec. IV we discuss other possible origins of critical fluctuations near a QSMT.

B. Theory of the QSMT in granular systems

In order to construct a theory of the quantum critical regime near a QSMT, it is necessary to identify loopholes in the considerations already outlined that lead to a breakdown of BCS theory. One route is to identify processes that cut off the divergence of the superconducting susceptibility in the metallic state, Eq. (8). Another is to consider the case in which there are competing attractive and repulsive interactions, or where a magnetic field cuts off the divergence of the susceptibility so that the QCP occurs at a point in the phase diagram at which interactions have non-negligible effects.

In this section, following the analysis of Feigel'man and Larkin (1998), Spivak, Zyuzin, and Hruska (2001), and Spivak, Oreto, and Kivelson (2008), we consider the QSMT in a system with a spatially nonuniform $u(\mathbf{r})$. This provides an important theoretical paradigm that explains how in principle at zero temperature the conductivity can diverge upon approach to the point of the quantum phase transition. As far as we know, these are the only solvable *microscopically plausible* models of a QSMT with an observable quantum critical regime.

1. Strategy of solution

The system we analyze consists of far separated superconducting puddles embedded in a normal metal background. We consider the limit in which the distance between puddles is large compared to their size, as this separation of scales permits a controlled theoretical approach to the problem. To begin with, we compute the zero temperature superconducting susceptibility of an isolated puddle χ_j. Consistent with general expectations, this susceptibility is always finite, but it can depend exponentially on characteristic properties of the puddle and so can be very large. Then we compute the Josephson coupling J_{ij} between pairs of puddles i and j. Importantly, J_{ij} reflects the quantum diffusion of Cooper pairs through the normal metal and so falls relatively slowly with the separation between puddles, in sharp contrast to the behavior of the Josephson coupling through an insulating region.

What enters thermodynamic considerations is the dimensionless coupling between puddles

$$X_{ij} = \sqrt{|\chi_i J_{ij} \chi_j J_{ji}|}. \tag{10}$$

Two puddles fluctuate essentially independently of each other if $|X_{ij}| \ll 1$, and they are phase locked to each other if $|X_{ij}| \gg 1$. The quantum transition to a globally phase-coherent state occurs at the point at which an infinite cluster of puddles is coupled together by links with $|X_{ij}| \gtrsim 1$. At slightly larger mean spacing between puddles, large clusters of puddles are still phase locked, which thus implies the existence of significant quantum critical effects.

Note that for this procedure to be valid the sum $\sum_j J_{ij}$ must be convergent. At any nonzero T, J falls exponentially with distance R_{ij} between grains, so convergence is guaranteed. However, at $T = 0$, and for repulsive $u_N > 0$, $J_{ij} \sim 1/|R_{ij}|^{-D}(1 + u_N \ln^2 R_{ij})$. In the special case in which there are no interactions in the normal metal ($u_N = 0$) the sum is logarithmically divergent and the ground state is thus always superconducting. Thus the sum is convergent and the transition exists only because of the repulsion in the normal metal.

Since most experiments on the anomalous metal are on 2D devices, we will consider this case.

2. Model of superconducting puddles in a metal

Let us consider an s-wave superconducting grain that is embedded in a normal metal. For simplicity we consider the following spatial structure of the electron interaction:

$$u(\mathbf{r}) = \begin{cases} u_S > 0 & \text{for} \quad |\mathbf{r}| < R, \\ u_N < 0 & \text{for} \quad |\mathbf{r}| < R. \end{cases} \tag{11}$$

Here u_S and u_R are the interaction constants in the superconductor and in the normal metal, respectively. It follows from general statistical mechanical considerations that the quantum mechanical average of the order parameter of a zero-dimensional system $\langle \Delta \rangle = 0$.

Theoretical investigation of the correlation function of the fluctuations of the order parameter has a long history. Here we briefly summarize the main results. At the mean-field level there exists a critical puddle radius R_c, such that for $R > R_c$ there is a nonzero solution of the mean-field equations [Eq. (5)], while for $R < R_c$ no such solution exists. So long as there is no reflection at the puddle boundary we get $R_c \sim \xi$, where ξ is the superconducting correlation length of a bulk superconductor with the interaction constant u_S. The character of the superconducting quantum fluctuations are quite different depending on whether R is less than or greater than R_c.

3. Large puddles with $R \gg R_c$

For puddles with $R \gg R_c$, there is a $T = 0$ mean-field solution for the order parameter with $\Delta_{\mathrm{MF}}(\mathbf{r}) \approx \Delta_0$ for $|\mathbf{r}| < R$ and $\Delta_{\mathrm{MF}}(\mathbf{r}) = 0$ otherwise. In this case the quantum fluctuations of the modulus of the order parameter can be neglected, the order parameter on an individual superconducting puddle can be parametrized as $\Delta_i \equiv |\Delta_0| e^{i\phi_i}$, and the quantum dynamics of the system can be described in terms of phase variables alone. The corresponding phase fluctuations in the ith puddle can be described by the action introduced by Chakravarty (1982)

$$S_i[\phi_i] = -\frac{G_i^{\mathrm{eff}}}{(2\pi)^2} \int dt dt' \frac{\sin^2\{(1/4)[\phi_i(t) - \phi_i(t')]\}}{(t-t')^2}. \tag{12}$$

Here $G_i^{\mathrm{eff}} \gg 1$ is an effective conductance of the medium measured in units of $e^2/\hbar$. As a result[7]

$$\langle e^{-i(\phi_i(t)-\phi_i(0))} \rangle = \begin{cases} \dfrac{1}{|\tau|^{1/G_i^{\mathrm{eff}}}} & \text{for} \quad \tau \ll \tau^*, \\ \left(\dfrac{\tau^*}{\tau}\right)^2 & \text{for} \quad \tau \gg \tau_i^*, \end{cases} \tag{13}$$

where

$$\tau_i^* \sim \tau_0 \exp(2\pi^2 G_i^{\mathrm{eff}}) \tag{14}$$

and

$$\chi_i \sim \tau_i^*. \tag{15}$$

The definition of G^{eff} in Eq. (12) requires clarification. This expression is derived by considering the dynamical screening of charge fluctuations in the superconducting puddle by the surrounding metal. In the 3D case, $G^{\mathrm{eff}} \sim \hbar \nu D R_i$ is the two terminal conductance in units $e^2/\hbar$, which is obtained if one lead is put inside the superconducting puddle and the other is placed on a boundary at infinity. In the 2D case, the conductance defined in this way vanishes in the thermodynamic limit. A more delicate analysis (Feigel'man, Larkin, and Skvortsov, 2001) shows that

$$G^{\mathrm{eff}} \sim \sqrt{\sigma_D^{(2\mathrm{D})} \hbar / e^2}. \tag{16}$$

In this case, G^{eff} is independent of the puddle size.

The action describing a collection of large puddles embedded in a metal has the form

$$S = \sum_i S_i[\phi_i] + S_{\mathrm{int}}[\{\phi\}], \tag{17}$$

where $S_{\mathrm{int}} = \int dt H_{\mathrm{int}}$ and

$$H_{\mathrm{int}} = -\frac{1}{2} \sum_{i \neq j} \tilde{J}_{ij} \cos(\phi_i - \phi_j) \tag{18}$$

is the Josephson Hamiltonian. In the case where the intergrain distance is larger than the zero temperature superconducting coherence length the intergrain coupling is given by a conventional Cooperon (ladder) diagram (Abrikosov, Gor'kov, and Dzyaloshinski, 1975) which in 2D has the form

$$\tilde{J}_{ij} = \frac{\hbar}{2e} \sigma_{2\mathrm{D}} \frac{D}{R^2} \frac{R^2}{|r_{ij}|^2 [1 + 2u_N \ln(|r_{ij}|/R)]^2}. \tag{19}$$

Note that for $u_N > 0$, the sum over couplings Eq. (19), between remote junctions converges, $\sum_j \tilde{J}_{ij} < \infty$, as required. However, in the limit $u_N = 0$, this sum diverges, consistent with our earlier observation that the existence of both attractive and repulsive interactions is a necessary condition for the existence of a critical regime.

4. Near critical puddles with $|R - R_c| \ll R_c$

The value of R_c can be obtained from the linearized mean-field self-consistency equation

$$F_0(\mathbf{r}) = -u(\mathbf{r}) \int d\mathbf{r}_1 K(\mathbf{r}, \mathbf{r}_1) F_0(\mathbf{r}_1), \tag{20}$$

which has a solution with nonzero F_0 when $R = R_c$. We can normalize F_0 such that $\int |F_0|^2 d\mathbf{r} = 1$, and we chose a phase convention such that $F_0(\mathbf{0})$ is real and positive.

For $R < R_c$ so long as $R_c - R \ll R_c$, the important fluctuations can be parametrized as $\Delta(\mathbf{r}, t) = \alpha(t) F_0(\mathbf{r})$; the fluctuations of the shape of $\Delta(\mathbf{r}, t)$ can be neglected and one needs only to calculate the correlation function of the complex amplitude $\alpha(t)$. This is given by the ladder diagrams

[7]The fact that in both limits the correlation functions (13) and (21) decay at large times as $1/t^2$ is a manifestation of a more general principle: whenever the retarded Green's function decays exponentially with time, the causal Green's function decays inversely proportional to time squared.

[see, for example, Abrikosov, Gor'kov, and Dzyaloshinski (1975)]

$$\langle\alpha_i(\omega)\alpha_i^*(-\omega)\rangle_C = \frac{1}{\nu\tau_0(-i|\omega| + 1/\tau_i^*)}, \qquad (21)$$

which in the t representation gives

$$\langle\alpha_i^*(0)\alpha_i(t)\rangle_C = \begin{cases} \frac{1}{\nu\tau_0}\left(\frac{\tau_i^*}{t}\right)^2 & \text{for} \quad \tau_i^* \ll \tau_0, \\ \frac{1}{\nu\tau_0} i[-i\pi + 2\ln(t/\tau_i^*)] & \text{for} \quad \tau_i^* \gg \tau_0. \end{cases} \qquad (22)$$

Here the subscript C refers to the casual (time ordered) Green's function, $\tau_0 = \min[R_c/v_F; R_c^2/D]$ is the time of electron flight through the grain, and

$$\tau_i^* = \frac{\tau_0 R}{R_c - R_i}. \qquad (23)$$

As a result we get an estimate for the superconducting susceptibility of the grain

$$\chi_i = \int dt\langle\alpha_i^*(0)\alpha_i(t)\rangle_C \sim \frac{\tau_i^*}{\nu\tau_0}. \qquad (24)$$

Alternatively one can get the same results for the correlation function using the nonlocal in time effective action obtained by integrating out the fermionic modes which when expanded up to quadratic terms in $\alpha_i(t)$ is

$$S_i[\alpha_i] = \nu\tau_0 \int \frac{d\omega}{2\pi}\left(-i|\omega| + \frac{1}{\tau_i^*}\right)|\tilde{\alpha}_i(\omega)|^2, \qquad (25)$$

where $\tilde{\alpha}_i$ is the Fourier transform of α_i.

Equations (22), (24), and (25) are valid even for $R_i > R_c$ as long as the amplitude of the fluctuations of the order parameter is larger than its mean-field value squared

$$\langle|\delta\Delta_i|^2\rangle \approx \langle|\Delta_i(0) - \Delta_i(t\sim\tau_i^{(*)})|^2\rangle \sim 1/\nu R_c^2\tau_0 \sim \Delta_0^2/\sigma_{2D} \gg \Delta_{MF}^2 \sim \Delta_0^2[(R_i - R_c)/R_c],$$

and the nonlinear terms in the action can be neglected. Here Δ_0 is the value of the order parameter in a bulk sample with interaction constant u_S.

In the opposite limit, $\langle|\delta\Delta_i|^2\rangle \ll \Delta_{MF}^2$ we have

$$\tau_i^* \sim \exp\left[\frac{\Delta_{MF}^2}{\langle|\delta\Delta_i|^2\rangle}\right]. \qquad (26)$$

To describe a system of superconducting puddles embedded into a metalic host we can use the effective action $S[\{\alpha\}] = \sum_i S_i[\alpha_i] + S_{int}[\{\alpha\}]$, where

$$S_{int} = -\sum_{ij}\int dt[J_{ij}\alpha_i^\star\alpha_j + \text{c.c.}] + b\sum_i\int dt|\alpha_i|^4, \qquad (27)$$

where $b \sim R_c^2\nu/aD^2$, and $a \ll \xi$ is the film thickness. Since the quantum fluctuations of the phase of the order parameter are slow compared to the interpuddle electron propagation, the Josephson coupling energy can be calculated using the conventional Cooperon (ladder) diagram (Abrikosov, Gor'kov, and Dzyaloshinski, 1975) which in 2D gives

$$J_{ij} = \frac{\nu R^2}{|r_{ij}|^2[1 + 2u_N\ln(|r_{ij}|/R)]^2}. \qquad (28)$$

Again the essential feature is the logarithmic correction to the long-distance falloff, which makes the sum over J_{ij} convergent.

5. Effect of magnetic field on J_{ij}

The expressions for J_{ij} and $\tilde{J}_{ij}$ are somewhat more complicated in the presence of a magnetic field and/or at finite temperature, in which cases they acquire additional factors $J_{ij} \to \mathcal{F}_{ij}J_{ij}$ and $\tilde{J}_{ij} \to \mathcal{F}_{ij}\tilde{J}_{ij}$, where

$$\mathcal{F}_{ij} \sim \exp\left(-\frac{r_{ij}}{L_T}\right)\exp\left(-\frac{r_{ij}}{L_B}\right)\exp(i\gamma_{ij}), \qquad (29)$$

where $L_T = \min[v_F/T; \sqrt{D/T}]$ is the coherence length of the normal metal, $L_B = \sqrt{\hbar c/eB}$ is the magnetic length, and $\gamma_{ij} = (hc/e)\int_i^j \mathbf{A}(\mathbf{r})\cdot d\mathbf{r}$ is a gauge-dependent phase factor.[8] In this case, the sum over J_{ij} is convergent, even without the logarithmic correction due to nonzero u_N, so the system can exhibit QSM transition with a critical region even without competing electron interactions.

6. Quantum critical region

To illustrate the implications of the above analysis, let us consider the case in which there is a single characteristic puddle size R, and the SMT is driven by changing the concentration of puddles. The critical puddle concentration is thus determined according to $\chi J(r) \sim 1$, where χ is the single puddle susceptibility and r is the typical distance between neighboring puddles. We are on secure theoretical grounds in all our considerations if the metallic portion of the

[8] If the metal is disordered, then the Josephson couplings J_{ij} exhibit sample specific mesoscopic fluctuations, which in the presence of the magnetic field and at $T = 0$ decays only as a power law of the distance $\langle\langle J_{ij}^2\rangle\rangle \sim 1/|r_{ij}|^6$ (Spivak and Zhou, 1995; Zhou and Spivak, 1998). Here double brackets stand for both the quantum mechanical averaging and the averaging over random scattering potential configurations. However, here we are mainly interested in samples with good conductances where the relative amplitude of these fluctuations is small. Therefore in what follows we neglect mesoscopic effects of this sort.

system is relatively clean, so that $k_F\ell \gg 1$ (i.e., in 2D, when $\sigma_D^{(2D)} \gg e^2/h$).

If in addition $R \gg R_c$, this means that χ is exponentially large, and hence that the Josephson coupling at criticality is exponentially small. Therefore in the anomalous metallic regime proximate to the QCP, the temperature below which the T-independent regime can be reached is correspondingly small. If $|(R-R_c)/R_c| \ll e^2/\hbar\sigma_D^{(2D)}$, and if the magnitude of u_N in the metal between the superconducting droplets is sufficiently large, then the criterion $\chi J \sim 1$ yields $(\tau^{(*)}/\tau_0)^2(R^2 N_D) \sim 1$. Here N_D is the concentration of puddles. In this case, the critical regime is of the order of $\mathcal{O}(1)$ both in concentration and in temperature. However, satisfying this condition seemingly involves a certain amount of fine-tuning of the puddle geometry, which appears to be at odds with the robustness of the observed phenomena. This is a worrisome shortcoming of the model.

7. Magnetic field driven QSM transition

The absence of a quantum critical regime discussed in Sec. III.A.3 ultimately reflects the long-range (power-law) falloff of the Cooperon correlation function in the normal metal at $T = 0$. However, in the presence of a magnetic field, these correlations are cut off at the magnetic length. Therefore, in principle, a quantum critical region may exist. In the case of statistically uniform disorder the solution of the mean-field equations has a form of Abrikosov lattice [see, for example, Abrikosov (1988)]. If the magnetic field is close to H_{c2} at $T = 0$ one can write the mean-field Gizburg-Landau equation for $\Delta(\mathbf{r})$, which solution gives us

$$\tilde{\Delta}^2(H) \sim \Delta_0^2 \frac{H_{c2} - H}{H_{c2}}, \tag{30}$$

where the tilde indicates averaging of the modulus of the order parameter over a period of vortex lattice [see, for example, Abrikosov (1988)].

Beyond the mean-field approximation the classical transition sometimes turns out to be the first order (Herbut and Tešanović, 1995). However, the first order is forbidden in disordered 2D samples (Imry and Wortis, 1979; Aizenman and Wehr, 1989; Hui and Berker, 1989; Goswami, Schwab, and Chakravarty, 2008). Ultimately, in high conductance samples the transition is controlled by the formation of rare droplets which are connected by Josephson junctions (Zhou and Spivak, 1998; Spivak, Oreto, and Kivelson, 2008).

However, for this article it is sufficient to say that in the 2D case the width of the critical region as a function of the magnetic field is controlled by the conductance of the film $\sigma_D^{(2D)}$ [see, for example, Blatter *et al.* (1994)]. Therefore, it is narrow in samples with large conductance. Indeed, the amplitude of quantum fluctuations of the order parameter in metal, averaged over an area of order $L^2_{B_{c2}}$, is given by a standard Cooperon diagram (Abrikosov, Gor'kov, and Dzyaloshinski, 1975)

$$\langle \delta\Delta^2 \rangle \sim \frac{\Delta_0}{\nu L^2_{Bc2}} \sim \frac{e^2 \Delta_0^2}{\hbar\sigma_D^{(2D)}}. \tag{31}$$

Thus we arrive at the conclusion that the interval of magnetic field where quantum fluctuations of the order parameter are important is

$$\frac{|H - H_{c2}|}{H_{c2}} \sim \frac{e^2}{\hbar\sigma_D^{(2D)}} \ll 1. \tag{32}$$

In other words Eqs. (30), (31), and (32) imply that in highly conductive samples not very close to H_{c2}, vortices remain macroscopic objects, and their quantum tunneling probability is negligible.

In Sec. IV we will discuss the significance of this result for interpretation of experiments.

8. Role of disorder and Griffiths phenomenon

We conclude this section with a short remark about some possible consequences for the QSMT of rare events associated with certain types of disorder. Note, however, that the wide variations in the character of the disorder of the experimental platforms that exhibit an anomalous metallic phase already suggest that such effects are unlikely to be of central importance.

In classical phase transitions the role of disorder has been discussed in detail in the past. For example, a degree of randomness in the local T_c's can shift the ordering temperature and in some cases can change the universal critical exponents at a transition, but leaves the character of the phases and the general nature of critical scaling intact. In addition, in the neighborhood of any critical point in a disordered system, there are universal "Griffiths phenomena": Even for statistically uniform disorder, due to mesoscopic variations in the local disorder configuration there always exist rare regions that are effectively on one side of the critical point even though the bulk of the system is on the other. In classical systems (i.e., for finite T transitions), this is largely an academic issue, as these rare regions lead to extremely subtle and difficult to detect effects.

The effects of rare regions can be much more important in the neighborhood of a quantum critical point and especially in a metallic system. In the present context, the essential point is that according to Eq. (14), the susceptibility of an individual grain depends exponentially on its properties (i.e., on G_{eff}). Consequently the existence of dilute rare droplets in which the electronic properties differ from average (e.g., in which G^{eff} is anomalously large) can have an enormously amplified effect on the physics close to the QSMT. This can lead to significant spatial inhomogeneities in the electronic properties and the relevance of a percolation analysis even in otherwise highly homogeneous materials.

The superconducting puddles previously discussed can arise from such considerations. However, as shown by Spivak, Oreto, and Kivelson (2008), this amplification is typically not sufficient to lead to a true quantum Griffith phase (Fisher, 1992, 1995) in which the superconducting susceptibility would diverge in a range of parameters proximate to the quantum critical point. In order for this to happen, the susceptibility of an anomalously large puddle would need to diverge exponentially with its "volume" (its area in 2D). Conversely, in most situations relevant to the present discussion, in any puddle that is large compared to the superconducting coherence length, the relevant value of G^{eff} grows at most in proportion to the surface area (the circumference in 2D) of the puddle.[9]

C. The QSMT of a *d*-wave superconductor

One might expect that the symmetry of the superconducting state could have important consequences for the nature of the quantum critical phenomena associated with its demise. For instance, the sign of the Josephson coupling between two *d*-wave grains can be either positive or negative depending on the orientation and shape of the grains and the character of the intervening metal. Random Josephson couplings can lead to the existence of a superconducting glass phase with all the complexity so implied. Thus, at least at zero magnetic field, one might expect that the nature nonsuperconducting state is different proximate to an *s*-wave or a *d*-wave superconductor.

However, while many aspects of the more general theory remain to be developed, it is possible to argue that in some range of parameters the nature of the anomalous metal phase originating from a *d*-wave superconductor is similar to the *s*-wave case. In particular, the phase diagram of a system of *d*-wave superconducting grains of random shape with $R \gg R_c$ in a metallic matrix has been discussed by Spivak, Oreto, and Kivelson (2008, 2009) and Kivelson and Spivak (2015). While each grain has dominantly *d*-wave pairing, an admixture of an *s*-wave component is implied by the (generically) asymmetric character of each grain—effectively each grain is either a $d + s$ or a $d - s$ wave superconductor. When the grains are close to one another, the Josephson couplings have a sign structure that generically leads to frustration and, presumably, the complex physics associated with an XY-spin glass. However, the longest-range portion of the Josephson couplings connected the subdominant *s*-wave components. This results in a situation analogous to that which arises in the Mattis model, in which the couplings between the *d*-wave components are random in sign, but in a way that is thermodynamically equivalent to a ferromagnet (i.e., no frustration). In effect, while the superconducting order is locally *d* wave, on distance scales of the order of the distance between grains and longer, it is only the *s*-wave component of the order that matters. Thus, the physics of the QSMT transition is identical in the two cases.

IV. CONCLUSIONS

In view of the number and diversity of systems in which very similar phenomena have been seen in experiment, we feel that the case for the existence of an anomalous metallic phase proximate to a QSMT is compelling. This conclusion contradicts the widespread "belief," based on perturbative considerations, that no metallic phase can exist in 2D.

The theory of conventional superconducting grains embedded in a weakly interacting metal establishes the fact that such a metallic phase is a valid theoretical possibility. The anomalous metal regime close to the point of the QSMT is very different from a normal Fermi liquid. It is characterized by values of conductivity that diverge upon approach to the transition. However, it seems to us difficult from this approach to account for the robustness of the anomalous metal to variation of circumstances (e.g., whether the QSMT occurs in disordered films, crystalline flakes, or engineered Josephson junction arrays) and for its extension over a relatively broad range of temperatures and quantum tuning parameters.

On a phenomenological level, fluctuational contributions to the conductivity near classical and quantum phase transitions are similar [see, for example, Aslamazov and Larkin (1968), Maki (1968), Thompson (1970), Blatter *et al.* (1994), Larkin and Varlamov (2005), and Davison *et al.* (2016)]. However, in our opinion the central problem in the area is the microscopic origins of the QSMT.

The theory in which the QSMT is driven entirely by quantum fluctuations of the order parameter of superconducting grains embedded in a metal can qualitatively reproduce the salient experimentally observed features of the anomalous metal regime. Note that such grains could arise directly as a consequence of sample inhomogeneities, or arise as an intrinsic feature of the electronic structure. For instance, the interaction between superconducting (SC) and another form of order generally enhances the effects of even weak structural inhomogeneities. In many unconventional superconductors, the superconducting state occurs in close association with charge-density-wave (CDW) phases. While the coupling between the SC order parameter and quenched disorder is constrained by gauge invariance, disorder necessarily destroys long-range incommensurate CDW order, leading to an inhomogeneous state with Larkin-Ovchinnikov-Lee-Rice-Imry-Ma domains (Imry and Ma, 1975; Larkin and Ovchinnikov, 1979; Lee and Rice, 1979). Consequently, if there is strong coupling between the CDW order and the SC order, this can lead to an intrinsically granular SC state.

However, there are reasons to question whether any granular picture can provide an adequate (semi)quantitative account. The central reason is that the underlying Ginzburg-Levanyuk parameter is typically small. In the case of large grains $R > R_c$ and large $\sigma_D^{(2D)}$, it follows from Eqs. (14), (15), and (19) that the critical concentration of superconducting grains at the point of the QSMT is exponentially small; this situation is realized, for example, in the experiments of Bøttcher *et al.* (2018). Consequently, genuinely quantum critical effects should be observable only at exponentially low temperatures.

The situation may be somewhat better if the grain's radius is close to the critical one $|R - R_c|/R_c \ll \hbar\sigma_D^{(2D)}/e^2$. This, however, requires fine-tuning of the puddle geometry which seems to be at odds with the robustness of the experimentally observed phenomena. Moreover, even here the temperature range is small in proportion to $\hbar\sigma_D^{(2d)}/e^2$. In principle, in the case of a magnetic field driven QSMT, it is also possible to

[9]It was correctly shown by Del Maestro *et al.* (2008, 2010) and Vojta, Kotabage, and Hoyos (2009) that a true quantum Griffith phase occurs in a disordered version of a phenomenological model of an array of resistively shunted Josephson junctions discussed in Appendix A.1; while this may well capture correct intermediate scale physics, we argue in that Appendix that ultimately the quantum Griffith phase is a consequence of an unphysical aspect of the model itself.

consider a theoretical approach based on the quantum melting of the vortex lattice. However, as discussed in Sec. III.B, in addition to providing no insight into the anomalous metal observed in zero magnetic field, we think it is likely that this type of theory will encounter similar quantitative problems in accounting for the breadth of the anomalous metallic regime. For example, the magnetic field driven anomalous metal regime has been observed in samples with $\sigma_D^{(2D)}$ as large as $\sim 40 e^2/\hbar$ in Mason and Kapitulnik (1999, 2001). In the absence of a magnetic field, the gate voltage driven anomalous metal regime has been observed in samples with $\sigma_D^{(2D)} \sim 26 e^2/\hbar$ by Chen *et al.* (2017) and Bøttcher *et al.* (2018).

This leads us to speculate that a satisfactory understanding of the anomalous metal will involve augmenting the conventional Fermi liquid description of metals in ways that go beyond just including the effects of quantum fluctuations of the superconducting order parameter itself. For instance, it is possible that the presence of slow (quantum) dynamical fluctuations associated with a proximate broken symmetry state might give rise to intrinsic effects that are similar to those associated with static granular structures in the model problem we have discussed. [Indeed, an inspiring study of the superconducting instability in the neighborhood of a class of QCPs (Raghu, Torroba, and Wang, 2015) found an associated QSMT that occurs at a finite mean BCS coupling—just the sort of situation that can lead to a large quantum critical regime.]

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APPENDIX A: PHENOMENOLOGICAL MODELS WITH DISSIPATIVE HEAT BATHS

There is a considerable body of theoretical analysis that has been carried out on models in which the quantum fluctuations of local superconducting phases (defined, for instance, on the superconducting nodes of a Josephson junction array) are coupled locally to a phenomenological "heat bath." These models have somewhat similar structure to the problem considered in Sec. III, but are simpler to the extent that the heat bath and superconducting degrees of freedom are treated separately. On the plus side, this means that aspects of the solution of these models, some of which we summarize, can be obtained with a greater level of certainty. On the negative side, there is an unphysical aspect of all these models—which we will hightlight as well—in that the presence of distinct heat baths on each site of the system corresponds to the assumed existence of an infinite number of degrees of freedom per unit volume.

1. QSMT in the quantum RSJ model

The quantum fluctuations of a superconducting order parameter in the presence of a dissipative heat bath is a problem with a long history; it was the subject of intense study as the focus of early work on "macroscopic quantum tunneling," a precursor of the studies that led to the study of superconducting Q bits [for a review, see Leggett *et al.* (1987)]. In the context of the QSMT, this problem was studied in the context of an array of resistively shunted Josephson junctions (Chakravarty *et al.*, 1986, 1988; Fisher, 1986; Wagenblast *et al.*, 1997; Kapitulnik *et al.*, 2001; Tewari, Toner, and Chakravarty, 2005; Stiansen, Sperstad, and Sudbø, 2012). The same model was revisited recently in the context of quantum criticality in a dissipative XY model (Zhu, Chen, and Varma, 2015; Hou and Varma, 2016; Zhu, Hou, and Varma, 2016). This model is simple and explicit and has a number of features that capture aspects of the phenomena characteristic of the QSMT, as discussed by Kapitulnik *et al.* (2001). We will summarize some aspects of the solution here. However, it has some physical shortcomings that we will also discuss.

The classical resistively shunted Josephson (RSJ) junction model gives an extremely useful phenomenological description of a resistively shunted Josephson junction. Here there are two contributions to the current across the junction, a supercurrent $I_{sc} = J \sin[\theta]$ and a normal current $I_{\text{norm}} = V/R$, where θ is the difference in phase across the junction, V is the voltage across the junction, and J and R are, respectively, the Josephson coupling and the resistance across the junction. To obtain dynamical equations for the superconducting phase, one invokes the Josephson relation $V = 2e\dot{\theta}$ between the voltage and the phase. It is also sometimes important to account for the capacitance of the junction, according to $\dot{Q} = C\dot{V}$ where $\dot{Q} = I_{sc} + I_{\text{norm}}$ is the time derivative of the charge on the two sides of the junction (treated as the two sides of a capacitor). Combining these considerations leads to the classical equation of motion for the phase across the junction,

$$C\ddot{\theta} + 2eJ\sin(\theta) + (1/R)\dot{\theta} = 0. \tag{A1}$$

Given the success of this description of the collective properties of Josephson junctions at nonzero T, it was natural to ask about its properties thought of as an example of dissipative quantum mechanics. In order to quantize this problem, a representation of the "heat bath" must be introduced. The key assertion (Caldeira and Leggett, 1981) is that the details of the heat bath do not matter—what matters is that it consists of a large number of degrees of freedom, each weakly coupled to the "macroscopic quantum variable" θ, so that the heat bath can be treated in linear response approximation. Then, since it is going to be treated in linear response in any case, the heat bath can always be represented as a collection of harmonic oscillators, with a spectral distribution designed to yield the frictional term in Eq. (A1). One particular representation that is useful is to couple $\dot{\theta}$ to a $1+1$D acoustic boson—a representation that was introduced originally to model the effect of coupling the Josephson junction to an open transmission line (Chakravarty and Schmid, 1986; Zwerger, Dorsey, and Fisher, 1986). The advantage of this representation is that it maps the problem of the dissipative Josephson junction onto a boundary conformal field theory, for which many exact results exist.

Although the effective action obtained by integrating out the gapless heat-bath degrees of freedom is nonlocal in time, it is still of a simple enough form that it can be analyzed. The result is an effective imaginary time action for a quantum resistively shunted Josephson junction:

$$S^{\mathrm{RSJ}}[\theta] = \int d\tau \left\{ \frac{|\dot{\theta}|^2}{2E_C} - J\cos[\theta] \right\} + \frac{\alpha}{4\pi}\int d\tau d\tau' \left| \frac{\theta(\tau)-\theta(\tau')}{\tau-\tau'} \right|^2, \tag{A2}$$

where the "charging energy" $E_C = 1/(4e^2C)$, while $\alpha \propto 1/R$ is the single coupling that reflects the strength of the coupling to the heat bath.

Equation (A2) describes a $(0+1)$-dimensional system. In general, neither finite size quantum systems nor infinite 1D classical (finite T) systems can exhibit phase transitions. However, 1D classical systems with $1/r^2$ interactions are a notable counterexample to this general expectation, which carries over to the $(0+1)$-dimensional quantum system for the special case in which interactions fall as $1/\tau^2$ in imaginary time. [A heat bath that produces such an interaction (which in Fourier transform is linear in $|\omega|$) is also known as an "Ohmic heat bath" (Caldeira and Leggett, 1981).] In the case of the RSJ model at $T=0$, this system exhibits a phase transition (Chakravarty, 1982; Schmid, 1983; Chakravarty and Rudnick, 1995) as a function of α from an ordered ("superconducting") phase for $\alpha > \alpha_c$, in which $\langle[\theta(\tau)-\theta(0)]^2\rangle$ is bounded as $\tau\to\infty$, to a disordered ("metallic") phase in which $\langle[\theta(\tau)-\theta(0)]^2\rangle \to \infty$ as $\tau\to\infty$. [α_c is probably not universal, but $\alpha_c(J/E_c)\to 1$ as $J/E_c \to 0$.] As the names suggest, in the superconducting phase the junction can support a dissipationless supercurrent across the junction in the $T\to 0$ limit. In contrast, in the metallic phase, the junction resistance is finite; there are additive contributions to the conductivity (parallel resistors) from the shunt resistor and the Cooper pair tunneling (Halperin, Refael, and Demler, 2010).

Thus, the single quantum RSJ junction undergoes a nontrivial QSMT, although in this case the superconducting phase exists only at $T=0$. The same considerations have been extended to higher dimensional arrays of resistively shunted Josephson junctions (Chakravarty *et al.*, 1986, 1988; Fisher, 1986; Wagenblast *et al.*, 1997; Zhu, Chen, and Varma, 2015; Hou and Varma, 2016; Zhu, Hou, and Varma, 2016).[10] Perhaps the most interesting thing about this model is that in the limit of small J/E_c, there is a QSMT that occurs as a function of α; for $\alpha > \alpha_c$, the ground state exhibits long-range phase coherence even in the limit $J/E_C \to 0$, while for $\alpha < \alpha_c$, quantum fluctuations destroy long-range phase coherence. Any phase with long-range phase coherence has a nonvanishing helicity modulous (and hence is superconducting); manifestly, because current can always be carried by normal electrons through the shunt resistors, the phase without long-range phase coherence is metallic, despite the fact that it is referred to as "insulating" in much of the theoretical literature on the subject. In fact, for small enough J/E_C, the parallel contribution to the conductivity can be computed perturbatively in powers of J/E_C, from which it can be seen that the $T\to 0$ conductivity diverges continuously as $\alpha\to\alpha_c$ from below.

Clearly this model has many attractive features in the present context: It has a QSMT. The metallic phase proximate to the transition is anomalous in very similar ways to that observed in experiment. Moreover, the model can be (and has been) solved using well-controlled perturbative RG methods and quantum Monte Carlo methods in various limits. However, there are some very peculiar features of the model: It has an effective dynamical exponent $z\to\infty$. Indeed in the metallic phase it has power-law correlations in time while exhibiting exponential fall of correlations in space, something known (Shtengel *et al.*, 2005) as "sliding in time." Along with this, it has nonuniversal critical exponents, corresponding to a line of fixed points rather than a usual fixed point.

Thus, while we view the solution of this model as extremely illuminating, and extremely useful as a caricature of the QSMT, we now focus on some of the unphysical features that prevent it from being considered an entirely satisfactory model. First, the heat bath that has been integrated out has, by construction, an infinite number of degrees of freedom. When thought of as a description of a macroscopic object, this is a reasonable abstraction, but when we come to think of this as representing a local mesoscopic degree of freedom in an extended array of such junctions, this assumption must ultimately break down. It is this feature of the model that is responsible for the $(0+1)$-dimensional character of the critical phenomena. This is a fundamental criticism of all models which assume coupling to a local heat bath. Secondly, in all microscopic derivations of which we know, superconducting fluctuations in a metallic environment couple to the dissipative degrees of freedom through the sine of the phase, as in Eqs. (12) and (B1), rather than through the phase itself, as in Eq. (A2). This observation applies both in the Josephson-junction context (Chakravarty, 1982; Eckern, Schön, and Ambegaokar, 1984) and in the context of superconducting grains embedded in a metal (Feigel'man and Larkin, 1998; Spivak, Oreto, and Kivelson, 2008). As far as the dynamics of small amplitude phase fluctuations are concerned, the two forms of the effective action are identical, but for large amplitude fluctuations the effects are very different, as shown in Appendix B.

APPENDIX B: JOSEPHSON JUNCTION ARRAYS AND QUANTUM GRIFFITH PHASES

A microscopic derivation of the heat bath associated with the normal electrons in a macroscopic Josephson junction was

[10] There are various ways to imagine generalizing the RSJ model to an array. For instance, one can imagine the heat bath is coupled to the phase difference $\theta_i - \theta_j$, across each junction, directly generalizing the model of the single junction, or one could imagine a circumstance in which each node of the array, i.e., each superconducting grain, is capacitively coupled to a heat bath (Emery and Kivelson, 1995) leading to a dissipative term that depends separately on the phase θ_j of each grain separately.

first obtained by Eckern, Schön, and Ambegaokar (1984). This leads to an effective action of the same form as in Eq. (12), where as previously mentioned the Ohmic heat bath is coupled to $e^{i\theta}$. Again at the phenomenological level, it seems reasonable to extend this model to Josephson junction arrays (JJAs), leading to an effective action

$$S^{\rm JJA} = \sum_j \int d\tau \left\{ \frac{|\dot{\theta}_j|^2}{2E_j} - \sum_{i\neq j} \frac{J_{ij}}{2} \cos(\theta_i - \theta_j) \right\} + \sum_j \frac{\alpha_j}{4\pi} \int d\tau d\tau' \left| \frac{e^{i\theta_j(\tau)} - e^{i\theta_j(\tau')}}{\tau - \tau'} \right|^2, \tag{B1}$$

where θ_j is the superconducting phase on node j and the various couplings are typically taken to be random variables reflecting the degree of disorder. In most places where such a model is considered, the Josephson coupling J_{ij} is assumed to be short ranged, although as discussed, for superconducting grains embedded in a metal, this assumption is unphysical, at least in the absence of a magnetic field. In defining the explicit model, we have taken a heat bath coupled separately to the phase on each grain, but similar considerations apply to the model in which we associate a heat bath with each junction (i, j).

First, to illustrate the difference between the JJA and RSJ models, we consider the solution of this model problem for a single grain (i.e., $J_{ij} = 0$). Again, the action is that of a $(0 + 1\mathrm{D})$ system, but now one that is recognizable; if we think of imaginary time as being a spatial dimension, this problem is a version of the classical 1D $1/r^2$ XY ferromagnet at an effective temperature $T \sim 1/\alpha$. 1D is the lower critical dimension for this problem (or if we consider models with interactions of the form $1/r^a$ in 1D, then $a = 2$ is the critical range); while there is no phase transition in this model at any finite T, the correlation length diverges exponentially at low T. In other words, for the quantum model, the superconducting susceptibility and the corresponding correlation time diverge as

$$\tau_0 \sim \chi_{sc} \sim \exp[Z\pi\alpha] \quad \text{as } \alpha \to \infty, \tag{B2}$$

where $Z \approx 1$. On the one hand, this stands in contrast with the RSJ model in which there is a true superconducting phase ($\chi_{sc} = \infty$) for $\alpha > \alpha_c$. On the other hand, for α large, the exponentially large value of χ_{sc} reflects a similar suppression of quantum fluctuations produced by the coupling to the heat bath.

The existence of a susceptibility that depends exponentially on a local parameter is an essential feature needed to obtain a quantum Griffith phase. Indeed, it has been shown by Del Maestro *et al.* (2008, 2010) and Vojta, Kotabage, and Hoyos (2009) that such a model does indeed support a quantum Griffith phase. Loosely, the line of argumentation goes as follows: Consider the case in which J_{ij} is short ranged and the grains are far separated, so that globally the system is not superconducting. However, there will be rare regions in which there is a group of N grains that are strongly coupled by unusually large Josephson couplings $J_{ij} \gg E_i$ and E_j. In such a cluster of grains, the phases are effectively locked leaving only one low-energy phase variable, an average phase for the cluster θ. This phase, is in turn, coupled to a dissipative heat bath with an effective value of $\alpha = \sum_j \alpha_j \sim N\bar{\alpha}$, where the sum runs over grains in the cluster. This implies that the superconducting susceptibility of the cluster grows exponentially with the size of the cluster $\chi_{sc} \sim \exp[Z\pi\bar{\alpha}N]$. Such a cluster is rare for large N—under general circumstances the probability of finding such a cluster is exponentially small in proportion to N: $P(N) \sim \exp[-\gamma N]$. So long as $\gamma > Z\pi\bar{\alpha}$, such rare grains are entirely negligible. However, when $\gamma < Z\pi\bar{\alpha}$, even though large clusters are rare, they make a divergent contribution to the average superconducting susceptibility. Since presumably $\gamma \to 0$ as one approaches the point of a superconducting transition, the susceptibility necessarily diverges before this transition is reached—the defining feature of a quantum Griffith phase.

Appealing as this result is, it ultimately depends on the same unphysical feature of the heat bath already noted. [See discussion in Spivak, Oreto, and Kivelson (2008) and Millis, Morr, and Schmalian (2002) for further details.] For a large superconducting cluster, there is a well-developed superconducting gap and correspondingly a well-defined superconducting coherence length ξ_0. Gapless degrees of freedom associated with the surrounding metallic state can only penetrate at most a distance ξ_0 into the cluster. Therefore, in the end, the coupling to the heat bath can at most grow in proportion to the perimeter (surface area in 3D) of the cluster. In addition, as already mentioned, this model treats J_{ij} as short ranged, whereas in fact in a metal it falls with a slow power law with distance. Thus, while it is an extremely attractive possibility that quantum Griffith-like phenomena may occur in real materials over an interesting intermediate range of scales, we consider the sharp existence of such a phase to be an artifact of the model.

APPENDIX C: CONCERNING "PURELY BOSONIC" APPROACHES

While we feel that the absence of gapless quasiparticle modes in purely bosonic models already disqualifies them as descriptions of the experiments discussed above, in the interest of completeness, we conclude this section with a discussion of a few of these exotic proposals. In some circumstances, which are usually associated with strong correlation effects that can give rise to localized spins, even the sign of the Josephson coupling is a random variable [see, for example, Bulaevskii, Kuzii, and Sobyanin (1977) and Kivelson and Spivak (1992)]. It was further noted in that randomness in the signs of intergrain Josephson couplings can bring the system into the universality class of a quantum XY spin glass. Moreover, it was hypothesized by Phillips and Dalidovich (2003) that such a quantum superconducting glass has finite conductivity. However, given that the anomalous metal regime has been observed in a broad range of systems, some of which are quite pure and with no other signs of strong correlations, it is difficult to believe that the random sign of the Josephson couplings can be a generic property of systems exhibiting anomalous metallic behavior. Not less important, transport properties of quantum spin and superconducting glasses are almost totally uncharted territory, theoretically.

In particular, it remains to be established whether or not the conductivity of the glass phase is finite.

It was proposed by Das and Doniach (1999) that the Bose-Hubbard model can exhibit a "Bose metal" phase which is formally related to a spin liquid with a spinon Fermi surface. However, given that there is no intrinsic frustration in the model (the ground state can be proven to be nodeless), and the fact that subsequent studies have shown that spin liquids arise only when any conventional ordering tendencies are strongly suppressed (extremely frustrated) it is now clear this proposal is not correct. In principle, extensions of this idea involving uniformly frustrated versions of the same model can give rise to spin liquids with a spinon Fermi surface (Sheng, Motrunich, and Fisher, 2009; Yao, Zhang, and Kivelson, 2009; Barkeshli, Yao, and Kivelson, 2013); however, in the absence of an emergent gauge field, such a spinon Fermi surface is inherently unstable (Barkeshli, Yao, and Kivelson, 2013).

A more sophisticated version of a Bose-metal proposal has been mooted (Raghu, Torroba, and Wang, 2015; Mulligan, 2017) in the context of the magnetic field driven QSMT. (Here the magnetic field implicitly introduces the requisite frustration.) A related proposal, but one including gapless fermions, was made by Galitski *et al.* (2005). In 2D there is a precise correspondence (Zhang, Hansson, and Kivelson, 1989; Lopez and Fradkin, 1991; Halperin, Lee, and Read, 1993; Son, 2015; Mulligan and Raghu, 2016) between charged bosons in a magnetic field, and charged fermions in a shifted average magnetic field and coupled to an emergent dynamical gauge field. It was argued by Mulligan and Raghu (2016) and Mulligan (2017) that this mapping provides a rationale for metallic behavior in the neighborhood of a QSMT. This idea builds upon the earlier notions using similar theoretical technology, which establish an analogy between a field driven SIT and various quantum Hall plateau transitions (Kivelson, Lee, and Zhang, 1992; Kapitulnik *et al.*, 2001; Breznay *et al.*, 2016).

While these proposals are interesting in their own right, there are several additional reservations we have about their *application* to the QSMT: First, they do not offer any handle on the observed similarities between the field driven QSMT and the transition in the absence of a magnetic field. Secondly, these theories treat the field-induced vortices as quantum mechanical point particles; however, in the systems of interest with large normal state conductances G, the vortices are quasimacroscopic. Their quantum tunneling amplitude is controlled by a parameter $\exp(-G) \ll 1$, so that they behave as essentially classical objects on all relevant energy and temperature scales.

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