APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory*

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These are notes on some entanglement properties of quantum field theory, aiming to make accessible a variety of ideas that are known in the literature. The main goal is to explain how to deal with entanglement when—as in quantum field theory—it is a property of the algebra of observables and not just of the states.

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I. INTRODUCTION

Ideas of quantum information theory and entanglement have played an increasingly important role in quantum field theory and string theory in recent years. Unfortunately, it is really not possible in a short space to give references to the many developments in this general area that have occurred in the last decade. Many important developments are cited and summarized in the recent review article (Nishioka, 2018).

The present notes are not an overall introduction to this subject. The goal here is more narrow: to make accessible some of the mathematical ideas that underlie some of these developments and which are present in the existing literature but not always so easy to extract. In the process, we will also make contact with some of the older literature on axiomatic and algebraic quantum field theory.

In Sec. II, we describe the Reeh-Schlieder theorem (Reeh and Schlieder, 1961), which demonstrates that, in quantum field theory, all field variables in any one region of spacetime are entangled with variables in other regions. Actually, the entanglement of spatially adjacent field modes is so strong that entanglement entropy between adjoining spacetime regions in quantum field theory is not just large but ultraviolet divergent. [Early references on this ultraviolet divergence include Bombelli *et al.* (1986), Srednicki (1993), Callan and Wilczek (1994), Holzhey, Larson, and Wilczek (1994), McGuigan (1994), and Susskind and Uglum (1994).] This ultraviolet divergence means that the entanglement is not just a property of the states but of the algebras of observables. Explaining this statement and how to deal with it in the context of local quantum field theory is a primary goal in what follows. (We do not consider the implications of quantum gravity.)

An important tool in dealing with entanglement when it is a property of the algebras and not just the states is provided by Tomita-Takesaki theory, which we introduce in Sec. III. It has been used in a number of recent developments, including an attempt to see behind the horizon of a black hole

(Papadodiamas and Raju, 2013), a proof of the quantum null energy condition (Balakrishnan *et al.*, 2017), and too many others to properly cite here. As an inducement for the reader who is not sure this mathematical tool is worthwhile, we describe in Sec. III a rigorous definition—due to Araki (1975, 1976)—of relative entropy in quantum field theory, with a surprisingly simple proof of its main properties, including its monotonicity when one enlarges the region in which measurements are made.

In Sec. IV we explain what Tomita-Takesaki theory means for a quantum system with a finite-dimensional Hilbert space. This motivates the statement of some of the subtler properties of Tomita-Takesaki theory. It also leads—following Araki's work and later developments by Petz (1986) and Nielsen and Petz (2005)—to a natural proof of monotonicity of quantum relative entropy for a finite-dimensional quantum system. Monotonicity of relative entropy and its close cousin, strong subadditivity of quantum entropy, were first proved by Lieb and Ruskai (1973), using a lemma by Lieb (1973). These results underlie many of the deeper statements in quantum information theory.

In Sec. V we describe a fundamental—and fairly well-known—example of entanglement in quantum field theory. This is the case, first analyzed by Bisognano and Wichmann (1976) and Unruh (1976), of two complementary "wedges" or Rindler regions in Minkowski spacetime. In Unruh's formulation, the question is what is seen by an accelerating observer in Minkowski spacetime. We approach this problem both from a path integral point of view, which is important in black hole physics (Hawking, 1977), and following the rigorous approach of Bisognano and Wichman, which was based on analyticity rather than path integrals.

In Sec. VI we explain, following von Neumann and others (von Neumann, 1938; Powers, 1967; Araki and Woods, 1968), a short direct construction of algebras—such as the algebra of quantum field theory observables in a given spacetime region—with the property that a divergent entanglement entropy is built into the structure of the algebra.

Finally, in Sec. VII, we give some examples of the use of Tomita-Takesaki theory to prove statements in quantum field theory that would be more obvious if one could assume a simple factorization of the Hilbert space between degrees of freedom localized in different spacetime regions. All of these statements have been analyzed in previous rigorous papers, in some cases before the relevance of Tomita-Takesaski theory was understood.

The topics discussed in these notes can be treated rigorously, but the presentation here is certainly not rigorous. More complete treatments of most of the points about quantum field theory can be found in Borchers (2000) and Haag (1992). Quantitative measures of entanglement in quantum field theory such as Bell's inequalities have been discussed by Summers and Werner (1987) and from a different standpoint by Narnhofer and Thirring (2012). See also a recent article by Hollands and Sanders (2017) for another point of view on entanglement measures in quantum field theory and much interesting detail. For a general mathematical background on von Neumann algebras, a convenient reference is the lecture notes of Jones (2015).

II. THE REEH-SCHLIEDER THEOREM

A. Statement

Our starting point will be the Reeh-Schlieder theorem (Reeh and Schlieder, 1961), which back in 1961 came as a "surprise" according to Streater and Wightman (1964).

We consider a quantum field theory in Minkowski spacetime M_D of dimension D with spacetime coordinates $x^\mu = (t, \vec{x})$ and metric

$$ds^2 = \sum_{\mu,\nu=0}^{D-1} \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\vec{x}^2. \tag{2.1}$$

We write Ω for the vacuum state and $\mathcal{H}_0$ for the vacuum sector of Hilbert space, which consists of all states that can be created from the vacuum by local field operators. ($\mathcal{H}_0$ is not necessarily the full Hilbert space $\mathcal{H}$ of the given theory, since there may be "superselection sectors"; see Sec. II.C.) For simplicity of notation, we assume that the algebra of local fields of the theory under discussion is generated by a Hermitian scalar field $\phi(x^\mu)$; otherwise, additional generators are included in what follows. Whether $\phi(x^\mu)$ is an "elementary field" is not relevant. For any smooth function f, we write ϕ_f for the smeared field $\int d^D x f(\vec{x}, t)\phi(\vec{x}, t)$. Then states of the form

$$|\Psi_{\vec{f}}\rangle = \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle \tag{2.2}$$

are sufficient to generate $\mathcal{H}_0$ in the Hilbert space sense. (The purpose of smearing is to make sure that these states have finite norm and thus really are Hilbert space states.) In other words, any state in $\mathcal{H}_0$ can be approximated arbitrarily well by a linear combination of states $\Psi_{\vec{f}}$. This is the definition of the vacuum sector $\mathcal{H}_0$.

An initial value hypersurface (or Cauchy hypersurface) Σ is a complete spacelike hypersurface on which, classically, one could formulate initial data for the theory. For example, Σ could be the hypersurface $t = 0$. In Eq. (2.2), we can require that the functions f_i are supported in any given open neighborhood $\mathcal{U}$ of Σ (for example, in the open set $|t| < \epsilon$ for some $\epsilon > 0$ if Σ is defined by $t = 0$), and it is reasonable to hope that such states will still be enough to generate the Hilbert space $\mathcal{H}_0$. This statement is a quantum version of the fact that, classically, a solution of the field equations is determined by initial data (fields and their time derivatives) on Σ. Quantum mechanically, one may view this statement as part of what we mean by quantum field theory; it is Postulate 8 (a) in Haag and Schroer (1962). But actually, we will prove a stronger statement that is known as the Reeh-Schlieder theorem.

The Reeh-Schlieder theorem states that one can further restrict to an arbitrary small open set $\mathcal{V} \subset \Sigma$, and a corresponding small neighborhood $\mathcal{U}_\mathcal{V}$ of $\mathcal{V}$ in spacetime. Thus, even if we restrict the functions $f_1, \ldots, f_n$ to be supported in $\mathcal{U}_\mathcal{V}$, the states $\Psi_{\vec{f}}$ still suffice to generate $\mathcal{H}_0$.

If this were false, there would be some state $|\chi\rangle$ orthogonal to all states $|\Psi_{\vec{f}}\rangle$ such that the f_i are supported in $\mathcal{U}_\mathcal{V}$:

$$0 = \langle \chi | \Psi_{f_1 f_2, \ldots, f_n} \rangle. \tag{2.3}$$

This is true for all functions $f_1, \ldots, f_n$ if and only if it is true without smearing, in other words if and only if

$$\langle \chi | \phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle = 0, \qquad x_1, \ldots, x_n \in \mathcal{U}_{\mathcal{V}}. \tag{2.4}$$

There is not really much difference between the two statements, since the matrix element of a product of local fields, as in Eq. (2.4), has singularities as a function of the x_i and must be interpreted as a distribution. So a precise interpretation of Eq. (2.4) involves a slightly smeared version, as in Eq. (2.3).

B. Proof

To prove the Reeh-Schlieder theorem, we will show that if, for some χ, the left-hand side of Eq. (2.4) vanishes for all $x_1, \ldots, x_n \in \mathcal{U}_{\mathcal{V}}$, then it actually vanishes for all $x_1, \ldots, x_n$ in Minkowski spacetime M_D. This then implies that χ must vanish, by the definition of the vacuum sector. So only the zero vector is orthogonal to all states created from the vacuum by local operators supported in $\mathcal{U}_{\mathcal{V}}$; in other words, such states are dense in $\mathcal{H}_0$.

First let us show[1] that

$$\varphi(x_1, x_2, \ldots, x_n) = \langle \chi | \phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle \tag{2.5}$$

continues to vanish if x_n is moved outside of $\mathcal{U}_{\mathcal{V}}$, keeping the other variables in $\mathcal{U}_{\mathcal{V}}$. We write t for the timelike vector $(1, 0, \ldots, 0)$ and examine the effect of shifting x_n to $x_n + u\mathsf{t}$ for some real u. In other words, we shift x_n by u in the time direction, leaving its spatial coordinates unchanged. Consider the function

$$\begin{aligned} g(u) &= \langle \chi | \phi(x_1)\phi(x_2)\cdots\phi(x_{n-1})\phi(x_n + u\mathsf{t})|\Omega\rangle \\ &= \langle \chi | \phi(x_1)\phi(x_2)\cdots\exp(iHu)\phi(x_n)\exp(-iHu)|\Omega\rangle, \end{aligned} \tag{2.6}$$

where H is the Hamiltonian. We are given that $g(u) = 0$ for sufficiently small real u (since for small enough u, $x_n + u\mathsf{t} \in \mathcal{U}_{\mathcal{V}}$) and we want to prove that it is identically 0. Because $H|\Omega\rangle = 0$, we can drop the last factor of $\exp(-iHu)$ in Eq. (2.6):

$$g(u) = \langle \chi | \phi(x_1)\phi(x_2)\cdots\exp(iHu)\phi(x_n)|\Omega\rangle. \tag{2.7}$$

Because H is bounded below by 0, the operator $\exp(iHu)$ is holomorphic for u in the upper half plane.[2] Thus the function $g(u)$ is holomorphic in the upper half plane, continuous as one approaches the real axis, and vanishes on a segment $I = [-\epsilon, \epsilon]$ of the real axis.

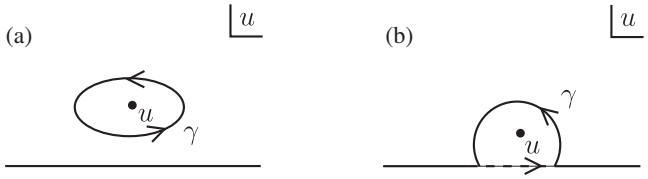

FIG. 1. (a) A function $g(u)$ holomorphic in the upper half u plane can be computed by a Cauchy integral formula: any contour γ in the upper half plane can be used to compute $g(u)$ for u in the interior of γ. (b) If $g(u)$ is continuous on the boundary of the upper half plane, one can take γ to run partly along the boundary. If in addition $g(u) = 0$ along part of the boundary—indicated here by dashed lines—then that part of the contour can be dropped. In this case, the Cauchy integral formula remains holomorphic as u is moved through the gap and into the lower half plane, implying that $g(u)$ is holomorphic on that part of the real axis and is identically zero.

If $g(u)$ were known to be holomorphic along the segment I, its vanishing along I would imply that a Taylor series of $g(u)$ around, say, $u = 0$ must be identically 0 and therefore that $g(u)$ is identically 0. As it is, to begin with, we only have continuity along the real axis and holomorphy in the upper half plane. However, using the fact that $g(u)$ vanishes in a segment of the real axis (and imitating the proof of the Schwarz reflection principle), we can argue as follows. For u in the upper half plane, $g(u)$ can be represented by a Cauchy integral formula

$$g(u) = \frac{1}{2\pi i}\oint_{\gamma} du' \frac{g(u')}{u' - u}. \tag{2.8}$$

Here γ is any contour that wraps counterclockwise once around u [Fig. 1(a)]. For fixed γ, the formula is only valid for u inside the contour, since if we move u across the contour, we meet the pole of the integrand. However, if it is known that $g(u)$ is identically 0 in a segment I of the real axis, we can choose γ to include that segment and then we can drop that part of the integral since $g(u')$ vanishes for $u' \in I$. Once we do this, we are free to move u through the segment I and into the lower half plane [Fig. 1(b)]; in particular, we learn that $g(u)$ is holomorphic along I. As already explained, it follows that $g(u)$ is identically 0.

In this argument, we could replace t by any other timelike vector.[3] Using some other timelike vector instead, we learn that $\langle \chi | \phi(x_1)\phi(x_2)\cdots\phi(x_n')|\Omega\rangle = 0$ if $x_n' - x_n$ is any timelike vector and $x_1, \ldots, x_n \in \mathcal{U}_V$. But now we repeat the process with x_n' replaced by $x_n'' = x_n' + v\mathsf{t}'$ for real v and with some possibly different timelike vector t'. Analyzing the dependence on v in exactly the same way, we learn that $\langle \chi | \phi(x_1)\phi(x_2)\cdots\phi(x_n'')|\Omega\rangle = 0$ for any x_n'' of this form. But since every point in Minkowski spacetime can be reached by starting with $\mathcal{U}_{\mathcal{V}}$ and zigzagging back and forth in different timelike directions, we learn that if, for some $x_1, \ldots, x_{n-1}$,

[1] The following argument is along the lines of that in Streater and Wightman (1964). However, to avoid invoking the multidimensional edge of the wedge theorem, we consider one variable at a time, as suggested by R. Longo.

[2] The rigorous proof of this sort of statement in Streater and Wightman (1964) uses some smearing with respect to x_n to first replace $\phi(x_n)|\Omega\rangle$ with a normalizable vector. So although it is true that the smeared and unsmeared statements (2.3) and (2.4) are equivalent, the smeared version is convenient in the rigorous proof.

[3] In the case of a past-pointing timelike vector, we make the same argument as before using holomorphy in the lower half u plane.

$\varphi(x_1,\ldots,x_{n-1},x_n)$ vanishes for all $x_n\in\mathcal{U}_V$, then it actually vanishes for all x_n, without the restriction $x_n\in\mathcal{U}_V$.

The next step is to remove the restriction $x_{n-1}\in\mathcal{U}_V$. We do this in exactly the same way, now shifting the last two coordinates in a timelike direction. Thus we look now at

$$g(u)=\langle\chi|\phi(x_1)\phi(x_2)\cdots\phi(x_{n-2})\phi(x_{n-1}+u\mathfrak{t})\phi(x_n+u\mathfrak{t})|\Omega\rangle. \tag{2.9}$$

Using again the fact that $H|\Omega\rangle=0$, we have

$$g(u)=\langle\chi|\phi(x_1)\phi(x_2)\cdots\exp(iHu)\phi(x_{n-1})\phi(x_n)|\Omega\rangle. \tag{2.10}$$

Just as before, the function $g(u)$ is holomorphic in the upper half plane and vanishes along a segment of the real axis, so it is identically zero. Repeating this with a second timelike vector, we learn that we can make an arbitrary shift $x_{n-1},x_n\to x_{n-1}+w,x_n+w$ without affecting the vanishing of $\varphi(x_1,\ldots,x_n)$. Since we are also free to shift x_n in an arbitrary fashion, we learn that for $x_1,\ldots,x_{n-2}\in\mathcal{U}_V$, $\varphi(x_1,\ldots,x_n)$ is identically zero, with no restriction on x_{n-1} and x_n.

The rest of the argument is hopefully clear at this point. At the kth step, we make a timelike shift of the last k points, adding $u\mathfrak{t}$ to each of them, and show as before that this does not affect the vanishing of $\varphi(x_1,x_2,\ldots,x_n)$. Repeating this with a shift by $v\mathfrak{t}'$ and combining with the results of previous steps, we learn that vanishing of $\varphi(x_1,x_2,\ldots,x_n)$ is not affected by moving the last k points. At the nth step, we finally learn that $\varphi(x_1,x_2,\ldots,x_n)$ is identically zero for all $x_1,x_2,\ldots,x_n$.

For future reference, a systematic holomorphy statement that can be proved similarly to the above is as follows. The $\mathcal{H}$-valued function

$$F(x_1,x_2,\ldots,x_n)=\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle \tag{2.11}$$

(or the inner product of this function with any other state) is holomorphic if the imaginary part of x_1 and of $x_{i+1}-x_i$, $i=1,\ldots,n-1$ is future timelike. (It is continuous up to the boundary of that domain.) This is proved by writing[4]

$$\begin{aligned}F(x_1,x_2,\ldots,x_n)=&[\exp(-ix_1\cdot P)\phi(0)\exp(ix_1\cdot P)]\\&\times[\exp(-ix_2\cdot P)\phi(0)\exp(ix_2\cdot P)]\cdots\\&\times[\exp(-ix_{n-1}\cdot P)\phi(0)\exp(ix_{n-1}\cdot P)]\\&\times[\exp(-ix_n\cdot P)\phi(0)]|\Omega\rangle.\end{aligned} \tag{2.12}$$

C. Vectors of bounded energy momentum

In proving the Reeh-Schlieder theorem, we used the fact that the energy-momentum operators $P^\mu,\mu=0,\ldots,D-1$ annihilate the vacuum state $|\Omega\rangle$. This implies, in particular, that for any D vector c, $\exp(ic\cdot P)|\Omega\rangle=|\Omega\rangle$. However (Borchers, 1968), in the proof it would be sufficient to know that, for a general D vector c^μ, $\exp(ic\cdot P)|\Omega\rangle$ varies holomorphically with the components $c^0,c^1,\ldots,c^{D-1}$ of c. Then in the above argument, we could not drop the factor $\exp(iu\mathfrak{t}\cdot P)|\Omega\rangle$, but its presence would not affect the discussion of holomorphy.

If a state Ψ has the property that $\exp(ic\cdot P)|\Psi\rangle$ is holomorphic in c, we say that the translation group acts holomorphically on Ψ. This is not true for an arbitrary Ψ, since if c has a future timelike imaginary part, $\exp(ic\cdot P)$ is an unbounded operator and $\exp(ic\cdot P)|\Psi\rangle$ may not make sense in Hilbert space.[5]

A source of many vectors on which the translation group has a holomorphic action is the following. The P^μ are a set of D commuting, self-adjoint operators. This leads to a spectral decomposition of the Hilbert space $\mathcal{H}$ on which the P^μ act. For every closed set S in momentum space, there is a corresponding projection operator Π_S onto the subspace $\mathcal{H}_S$ of Hilbert space consisting of states whose energy momentum is contained in the set S. (We cannot actually diagonalize the P^μ in Hilbert space, since states of definite energy momentum—other than the vacuum—are not normalizable.) If S is compact, then in any Lorentz frame, the energy of a state Ψ that is in the image of Π_S is bounded. This gives, for any c, an upper bound on the norm of $\exp(ic\cdot P)\Psi$ and ensures that the translation group acts holomorphically on Ψ.

If Ψ is any state and S is compact, the projection $\Pi_S\Psi$ to states with energy momentum in S is a state on which the translation group acts holomorphically. Moreover, $\Pi_S\Psi$ is nonzero for sufficiently large S and in fact converges to Ψ as S becomes large. So every state can actually be approximated by states that could be used instead of the vacuum in the Reeh-Schlieder theorem.

As an example of why this is useful, we can consider superselection sectors. In general, the "vacuum sector" $\mathcal{H}_0$, consisting of states that can be created from the vacuum by a product of local operators, is not the full Hilbert space $\mathcal{H}$ of a quantum field theory. In part, this is because there may be conserved charges that are not carried by any local operator. For example, in four spacetime dimensions, a theory with a massless U(1) gauge field has conserved electric and magnetic charges that are not carried by any local operators.[6] Let $\mathcal{H}'$ be the subspace of Hilbert space characterized by particular values of these charges. Such an $\mathcal{H}'$ is called a superselection sector. In a nontrivial superselection sector (not containing the vacuum), there is no state of lowest energy that we could

[4]We work in signature $-+\cdots+$, so $x\cdot P=-tH+\vec{x}\cdot\vec{P}$ where H is the Hamiltonian; this operator is negative semidefinite for $t>|\vec{x}|$, so $|\exp(-ix\cdot P)|\le 1$ for Imx future timelike. This ensures that for such x, the operator $\exp(-ix\cdot P)$ is defined for all states and holomorphically varying.

[5]An unbounded operator on a Hilbert space is defined at most on a dense set of vectors. Suppose, for example, that in some orthonormal basis ψ_n of a Hilbert space $\mathcal{H}$, an operator X acts by $X\psi_n=\lambda_n\psi_n$. For X to be unbounded means that the λ_n are unbounded. In this case, there is a vector $\Psi=\sum_n c_n\psi_n$ with $\sum_n|c_n|^2<\infty$ (so $\Psi\in\mathcal{H}$) but $\sum_n|\lambda_n|^2|c_n|^2=\infty$ (so $X\Psi$ does not make sense as a vector in $\mathcal{H}$).

[6]Below four spacetime dimensions, it may not be possible to fully characterize superselection sectors by conserved charges. An example is given by three-dimensional theories with non-Abelian statistics. [For a treatment of this situation in algebraic quantum field theory, see Fredenhagen, Rehren, and Schroer (1989).] Likewise, soliton sectors in two spacetime dimensions cannot always be fully characterized by conserved charges. However, the following remarks about the Reeh-Schlieder theorem do not depend on whether a given superselection sector can be characterized by conserved charges.

use instead of the vacuum in the Reeh-Schlieder theorem.[7] However, in such a sector, there is no problem to construct states of bounded energy momentum, and for any such state Λ, the analog of the Reeh-Schlieder theorem holds: whatever can be created by local operators acting on Λ can be created by local operators that act on Λ in the small open set $\mathcal{U}_\mathcal{V}$.

What happens to the Reeh-Schlieder theorem if Minkowski spacetime M_D is replaced by another globally hyperbolic spacetime M? In curved spacetime, there is no natural analog of the vacuum state, and there are, of course, also no natural translation generators P^μ. However, it is reasonable to expect that the Reeh-Schlieder theorem should have an analog for any spacetime M that is globally hyperbolic and real analytic. An analog of a vector on which spacetime translations act holomorphically is a vector whose evolution is holomorphic in the following sense. In general, a vector Ψ_Σ defined in quantization on a Cauchy hypersurface $\Sigma \subset M$ can be evolved forward or backward in time to a vector $\Psi_{\Sigma'}$ on any other such hypersurface Σ'. If M is real analytic, it can be "thickened" slightly to a complex analytic manifold $\widehat{M}$, and we can ask whether $\Psi_{\Sigma'}$ evolves holomorphically with Σ' if Σ' is displaced slightly away from M in $\widehat{M}$. If so, we say that Ψ_Σ has holomorphic evolution and a reasonable analog of the Reeh-Schlieder theorem would say that states $\mathsf{a}\Psi_\Sigma$, where a is supported in some given open set, are dense in Hilbert space. For results in this direction, see Strohmaier, Verch, and Wollenberg (2002) and Gérard and Wrochna (2017). There is also a version of the Reeh-Schlieder theorem adapted to anti–de Sitter space and holography (Morrison, 2014), and there are attempts to generalize the theorem to curved spacetime without assuming real analyticity (Sanders, 2009).

D. An important corollary

The Reeh-Schlieder theorem has an important and immediate corollary. Let us assume that the open set $\mathcal{V} \subset \Sigma$ is small enough so that its closure $\bar{\mathcal{V}}$ is not all of Σ. Then the complement of $\bar{\mathcal{V}}$ in Σ is another open set $\mathcal{V}'$, disjoint from $\mathcal{V}$. $\mathcal{V}'$ and $\mathcal{V}$ are spacelike separated, and they are contained in small open sets $\mathcal{U}_\mathcal{V}, \mathcal{U}_{\mathcal{V}'} \subset M_D$ that are also spacelike separated. One also may choose to let $\mathcal{U}_\mathcal{V}$ and $\mathcal{U}_{\mathcal{V}'}$ be as large as possible, while remaining at spacelike separation. The precise choice of $\mathcal{U}_\mathcal{V}$ and $\mathcal{U}_{\mathcal{V}'}$ is not important in this section.

Let a be any operator supported in the spacetime region $\mathcal{U}_\mathcal{V}$, not necessarily constructed from a product of finitely many local operators. Because the regions $\mathcal{U}_\mathcal{V}$, $\mathcal{U}_{\mathcal{V}'}$ are spacelike separated, a commutes with local operators in $\mathcal{U}_{\mathcal{V}'}$;

$$[\phi(x), \mathsf{a}] = 0, \qquad x \in \mathcal{U}_{\mathcal{V}'}. \tag{2.13}$$

Conversely, an operator a' supported in $\mathcal{U}_{\mathcal{V}'}$ satisfies

$$[\phi(x), \mathsf{a}'] = 0, \qquad x \in \mathcal{U}_\mathcal{V}. \tag{2.14}$$

The Reeh-Schlieder theorem applies equally well to $\mathcal{V}$ or to $\mathcal{V}'$, as they are both nonempty open sets in the initial value hypersurface Σ. This has the following consequence. Suppose that an operator a supported in $\mathcal{U}_\mathcal{V}$ annihilates the vacuum state

$$\mathsf{a}|\Omega\rangle = 0. \tag{2.15}$$

Because a commutes with the local operators $\phi(x_i)$, $x_i \in \mathcal{U}_{\mathcal{V}'}$, the vanishing of $\mathsf{a}|\Omega\rangle$ implies that

$$\mathsf{a}\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle = 0, \qquad x_i \in \mathcal{U}_{\mathcal{V}'}. \tag{2.16}$$

But the Reeh-Schlieder theorem tells us that the states $\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle$, $x_i \in \mathcal{U}_{\mathcal{V}'}$ are dense, in the vacuum sector $\mathcal{H}_0$ of Hilbert space. So the vanishing of the left-hand side of Eq. (2.16) for all n and all $x_i \in \mathcal{U}_{\mathcal{V}'}$ implies that the operator a is identically 0, in the vacuum sector.

For an open set $\mathcal{U}$ in spacetime, let us define $\mathcal{A}_\mathcal{U}$ to be the algebra of operators supported in $\mathcal{U}$. We will call this a "local algebra" of the quantum field theory. In Sec. II.F, we will be more specific about what we mean by "all operators." For now we leave this open. In the present discussion, we have considered two open sets, namely $\mathcal{U} = \mathcal{U}_\mathcal{V}$ and $\mathcal{U}' = \mathcal{U}_{\mathcal{V}'}$, which are thickenings of $\mathcal{V}$ and $\mathcal{V}'$, respectively, so there are two algebras to consider, namely $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$.

By way of terminology, a vector Ψ in a Hilbert space $\mathcal{H}_0$ is called a cyclic vector for an algebra such as $\mathcal{A}_\mathcal{U}$ if the states $\mathsf{a}|\Psi\rangle$, $\mathsf{a} \in \mathcal{A}_\mathcal{U}$ are dense in $\mathcal{H}_0$. Ψ is said to be separating for $\mathcal{A}_\mathcal{U}$ if the condition $\mathsf{a}|\Psi\rangle = 0, \mathsf{a} \in \mathcal{A}_\mathcal{U}$ implies that $\mathsf{a} = 0$. The Reeh-Schlieder theorem says that the vacuum vector Ω is cyclic for $\mathcal{A}_\mathcal{U}$ and for $\mathcal{A}_{\mathcal{U}'}$. As we have just explained, a state that is cyclic for one of these algebras is separating for the other, so in fact the vacuum is cyclic and separating for $\mathcal{A}_\mathcal{U}$ and for $\mathcal{A}_{\mathcal{U}'}$.

More generally, the Reeh-Schlieder theorem implies that, in each superselection sector, any vector on which the translation group acts holomorphically is cyclic and separating for $\mathcal{A}_\mathcal{U}$ and for $\mathcal{A}_{\mathcal{U}'}$.

As we have seen, if $\mathcal{U}$ and $\mathcal{U}'$ are a pair of spacelike separated open sets, then many vectors are cyclic and separating for $\mathcal{A}_\mathcal{U}$ and for $\mathcal{A}_{\mathcal{U}'}$, but it is certainly not true that every vector has this property. For a simple counterexample, consider a theory with a complex free fermion ψ. Then for a smearing function f supported in $\mathcal{U}$, $\psi_f = \int d^4x f(x)\psi(x)$ obeys $\psi_f^2 = 0$. It therefore annihilates any vector of the form $\psi_f\chi$. If one defines the local algebras to consist of bosonic operators only [as does Haag (1992)], then one can pick a pair of smearing functions f, g supported in $\mathcal{U}$ and set $\mathcal{O}_{f,g} = \psi_f\psi_g$. Then $\mathcal{O}_{f,g}$ is a bosonic operator supported in $\mathcal{U}$ and obeying $\mathcal{O}_{f,g}^2 = 0$, so $\mathcal{O}_{f,g}$ annihilates any state $\mathcal{O}_{f,g}\chi$. So $\psi_f\chi$ or $\mathcal{O}_{f,g}\chi$ is a state that is not separating for $\mathcal{A}_\mathcal{U}$, or cyclic for $\mathcal{A}_{\mathcal{U}'}$.

The fact that the vacuum is separating for the algebra $\mathcal{A}_\mathcal{U}$ has interesting consequences for the energy density in quantum field theory (Epstein, Glaser, and Jaffe, 1965). Of course, the total energy H is positive semidefinite, and annihilates only the vacuum state. It can be defined as the integral of the energy density T_{00} over an initial value surface $t = 0$. However, in contrast to classical physics, the energy density $T_{00}(x)$ is not positive semidefinite in quantum field

[7] To minimize the energy of, say, a magnetic monopole, we want it to have zero momentum. But such a state is not normalizable.

theory, and the same holds for any smeared operator $T_f = \int_{\mathcal{U}_\mathcal{V}} d^D x f(x) T_{00}(x)$, where f is any real smearing function with support in $\mathcal{U}_\mathcal{V}$. Poincaré invariance and the fact that $H\Omega = 0$ imply that the vacuum has vanishing energy density, $\langle\Omega|T_{00}(x)|\Omega\rangle = 0$. However, the separating property of the vacuum for the algebra $\mathcal{A}_\mathcal{U}$ implies that $T_f|\Omega\rangle \neq 0$. Let χ be some state with $\langle\chi|T_f|\Omega\rangle \neq 0$. Let $\mathcal{W}$ be the two-dimensional subspace of Hilbert space generated by Ω and χ. If we write a vector in $\mathcal{W}$ as a column vector with Ω and χ corresponding to the upper and lower components, then T_f restricted to $\mathcal{W}$ takes the form

$$\begin{pmatrix} 0 & b \\ \bar{b} & c \end{pmatrix}, \tag{2.17}$$

with $b = \langle\chi|T_f|\Omega\rangle \neq 0$. Such a matrix is not positive semi-definite, implying that T_f has a negative expectation value in some state $\widetilde{\chi} \in \mathcal{W} \subset \mathcal{H}$.

E. Discussion

The Reeh-Schlieder theorem may seem paradoxical at first. It implies that by acting on the vacuum with an operator a supported in a small region $\mathcal{U}_\mathcal{V}$, one can create whatever one wants—possibly a complex body such as the Moon—in a faraway, spacelike separated region of spacetime.

To understand this better, let $\mathcal{V}^*$ be a distant region in which we want to create the Moon. Let M be an operator supported in region $\mathcal{U}_{\mathcal{V}^*}$ that to good approximation has expectation value 0 in states that do not contain a moon in region $\mathcal{V}^*$ and 1 in states that do contain one. Thus

$$\langle\Omega|\mathsf{M}|\Omega\rangle \approx 0, \tag{2.18}$$

but according to the Reeh-Schlieder theorem, there is some operator a supported in $\mathcal{U}_\mathcal{V}$ such that the state $\mathsf{a}\Omega$, to very good approximation, contains a moon in region $\mathcal{V}^*$. Thus $\langle\mathsf{a}\Omega|\mathsf{M}|\mathsf{a}\Omega\rangle \approx 1$, so $\langle\Omega|\mathsf{a}^\dagger\mathsf{M}\mathsf{a}|\Omega\rangle \approx 1$. As $\mathsf{a}^\dagger$ is supported in region $\mathcal{U}_\mathcal{V}$ and M is supported in the spacelike separated region $\mathcal{U}_{\mathcal{V}^*}$, these operators commute and thus

$$\langle\Omega|\mathsf{M}\mathsf{a}^\dagger\mathsf{a}|\Omega\rangle \approx 1. \tag{2.19}$$

Is there a conflict between Eqs. (2.18) and (2.19)? If we could choose the operator a to be unitary, we would have $\mathsf{a}^\dagger\mathsf{a} = 1$, and then there would indeed be a conflict. However, the Reeh-Schlieder theorem does not say that there is a *unitary* operator supported in $\mathcal{U}_\mathcal{V}$ that will create the Moon in some distant region; it merely says that there is *some* operator supported in $\mathcal{U}_\mathcal{V}$ that will do this.

If one asks about not mathematical operations in Hilbert space but physical operations that are possible in the real world, then the only physical way that one can modify a quantum state is by perturbing the Hamiltonian by which it evolves, thus bringing about a unitary transformation. If one is able to couple a given quantum field theory to some auxiliary quantum system, then one can implement a unitary transformation on the combined system. It is not possible by such a unitary transformation supported in $\mathcal{U}_\mathcal{V}$ to make any change in observations in a spacelike separated region $\mathcal{V}^*$. That is what we learn from the above computation, which shows that for any operator M supported in $\mathcal{U}_{\mathcal{V}^*}$ and any unitary operator a supported in $\mathcal{V}$, $\langle\mathsf{a}\Omega|\mathsf{M}|\mathsf{a}\Omega\rangle = \langle\Omega|\mathsf{M}|\Omega\rangle$. This computation is unaffected if a acts also on some auxiliary quantum system, as long as a is unitary and commutes with operators in $\mathcal{V}^*$.

While it is not possible for a physical operation in one region to influence a measurement in another region, there can be correlations in the vacuum between operators in the two regions. This happens all the time in quantum field theory, even in free field theory. We are seeing such correlations in Eq. (2.19), which shows that $\langle\Omega|\mathsf{M}\mathsf{a}^\dagger\mathsf{a}|\Omega\rangle \neq \langle\Omega|\mathsf{M}|\Omega\rangle\langle\Omega|\mathsf{a}^\dagger\mathsf{a}|\Omega\rangle$.

The Reeh-Schlieder theorem can be given an intuitive interpretation by considering a finite-dimensional quantum system with a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. For what follows, the most interesting case is that $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same dimension n. We let $\mathcal{A}_1$ be the algebra of $n \times n$ matrices acting on $\mathcal{H}_1$, and $\mathcal{A}_2$ the algebra of $n \times n$ matrices acting on $\mathcal{H}_2$. (In language that we will introduce shortly, these are $*$-algebras and they are each other's commutants.) A generic state Ψ of the composite system is entangled. For any given Ψ, it is possible to choose a basis ψ_k, $k = 1, \ldots, n$ of $\mathcal{H}_1$ and another basis ψ'_k, $k = 1, \ldots, n$ of $\mathcal{H}_2$ such that

$$\Psi = \sum_{k=1}^{n} c_k \psi_k \otimes \psi'_k, \tag{2.20}$$

with some coefficients c_k. It is convenient to write $|k\rangle$ and $|k'\rangle$ for ψ_k and ψ'_k, so that this formula becomes

$$\Psi = \sum_{k=1}^{n} c_k |k\rangle \otimes |k\rangle'. \tag{2.21}$$

The vector Ψ is cyclic and separating for $\mathcal{A}_1$ and for $\mathcal{A}_2$ if and only if the c_k are all nonzero, or equivalently if the reduced density matrices on $\mathcal{H}_1$ and on $\mathcal{H}_2$ are invertible. We will return to this setup in Sec. IV.A.

The Reeh-Schlieder theorem says that, in quantum field theory, if $\mathcal{A}_\mathcal{V}$ and $\mathcal{A}_{\mathcal{V}'}$ are the algebras of operators supported in complementary regions of spacetime, then similarly the vacuum is a cyclic separating vector for this pair of algebras.[8] This might make one suspect that the Hilbert space $\mathcal{H}$ should be factored as $\mathcal{H} = \mathcal{H}_\mathcal{V} \otimes \mathcal{H}_{\mathcal{V}'}$, with the vacuum being a fully entangled vector in the sense that the coefficients analogous to c_k are all nonzero. This is technically not correct. If it were correct, then picking $\psi \in \mathcal{H}_\mathcal{V}$, $\chi \in \mathcal{H}_{\mathcal{V}'}$, we would get a vector $\psi \otimes \chi \in \mathcal{H}$ with no entanglement between observables in $\mathcal{V}$ and those in $\mathcal{V}'$. This is not what happens in quantum field theory. In quantum field theory, the entanglement entropy

[8]This remains so if $\mathcal{V}$ is replaced by a smaller region, and $\mathcal{V}'$ by a correspondingly larger one. That fact would have no natural analog for a finite-dimensional quantum system, and shows in a different way from what is explained in the text the limitations of the analogy between the vacuum of a quantum field theory and a fully entangled state of a finite-dimensional quantum system.

between adjacent regions has a universal ultraviolet divergence, independent of the states considered. The leading ultraviolet divergence is the same in any state as it is in the vacuum, because every state looks like the vacuum at short distances. The universality of this ultraviolet divergence means that it reflects not a property of any particular state but rather the fact that $\mathcal{H}$ cannot be factored as $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{V}'}$.

It is also not correct, technically, to write $\mathcal{H}$ as a direct sum or integral of Hilbert spaces $\mathcal{H}_{\mathcal{V}}^{\zeta}$ and $\mathcal{H}_{\mathcal{V}'}^{\zeta}$, where ζ is some discrete or continuous variable and each $\mathcal{H}_{\mathcal{V}}^{\zeta}$, $\mathcal{H}_{\mathcal{V}'}^{\zeta}$ is supposed to furnish a representation of $\mathcal{A}_{\mathcal{V}}$ or $\mathcal{A}_{\mathcal{V}'}$. If one had $\mathcal{H} = \oplus_{\zeta} \mathcal{H}_{\mathcal{V}}^{\zeta} \otimes \mathcal{H}_{\mathcal{V}'}^{\zeta}$ (where the direct sum over ζ might be a continuous integral), then there would be operators—such as any function of ζ—that commute with both $\mathcal{A}_{\mathcal{V}}$ and $\mathcal{A}_{\mathcal{V}'}$. Bounded functions of the ζ's would be bounded Hilbert space operators, defined on all states. Moreover, because the leading ultraviolet divergence in the entanglement entropy is proportional to the area of the boundary between these two regions, these operators would have to be local along the boundary. There is nothing like that in quantum field theory. What we usually call a local operator $\phi(x)$ has to be smeared just to make a densely defined unbounded operator (let alone a bounded operator, defined on all of Hilbert space), and such a smeared operator does not commute with $\mathcal{A}_{\mathcal{V}}$ and $\mathcal{A}_{\mathcal{V}'}$.

Despite all this, many statements that one could deduce from a naive factorization $\mathcal{H} = \mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{V}'}$ and whose analogs are true for entangled quantum systems of finite dimension are actually true in quantum field theory. Tomita-Takesaki theory, which we introduce in Sec. III, is an important tool in proving such statements.

F. The local algebras

In Sec. II.D, we introduced the notion of associating to an open set $\mathcal{U}$ in spacetime a "local algebra" $\mathcal{A}_{\mathcal{U}}$ consisting of "all operators" supported in $\mathcal{U}$.

But what do we mean by all operators? The operators that we have considered so far are what one might call simple operators, namely, polynomials in smeared local fields. However, there are serious drawbacks to considering only simple operators.[9] For one thing, one wants to be able to claim (Haag and Schroer, 1962) that if $\mathcal{U}$ is an open set in spacetime and $\widehat{\mathcal{U}}$ is a larger open set that is its domain of dependence [Fig. 2(a)] then the algebras $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\widehat{\mathcal{U}}}$ coincide. The logic behind this is that the dynamical time evolution of the theory determines operators in the larger region $\widehat{\mathcal{U}}$ in terms of operators in $\mathcal{U}$. This is true, but operators supported in regions of $\widehat{\mathcal{U}}$ that are to the future or the past of $\mathcal{U}$ are in general exceedingly complex functions of operators in $\mathcal{U}$. Thus we can only get a simple relation $\mathcal{A}_{\mathcal{U}} = \mathcal{A}_{\widehat{\mathcal{U}}}$ if we include in $\mathcal{A}_{\mathcal{U}}$ all operators that can be made from the simple ones.

What sort of operators can we make from simple ones? Some elementary operations come to mind. For example, if f is a real smearing function and $\phi_f = \int d^D x f \phi$, we can consider the operator $\exp(i\phi_f)$, which actually is a *bounded* operator made from ϕ_f. More generally, if F is any bounded function of a complex variable, we can consider $F(\phi_f)$ (now with a possibly complex-valued smearing function f); this again is a bounded operator. Still more generally, if $f_1, \ldots, f_n$ are n smearing functions and F is a bounded function of n complex variables, we can consider $F(\phi_{f_1}, \phi_{f_2}, \ldots, \phi_{f_n})$.

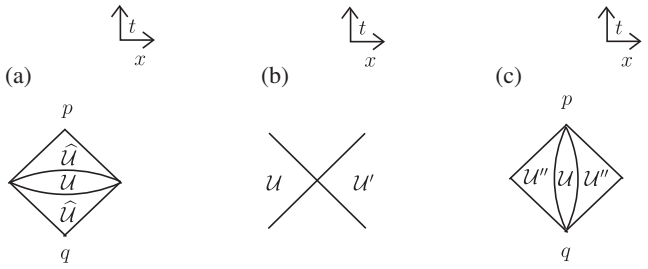

FIG. 2. (a) An open set $\mathcal{U}$ in Minkowski spacetime, and its domain of dependence $\widehat{\mathcal{U}}$ (the union of $\mathcal{U}$ with the regions labeled as $\widehat{\mathcal{U}}$ in the figure), which in this case is a causal diamond and coincides with the causal completion $\mathcal{U}''$ of $\mathcal{U}$. (b) The two open sets $\mathcal{U}$ and $\mathcal{U}'$ are causal complements; each is the largest open set that is spacelike separated from the other. (c) A quite different open set $\mathcal{U}$ whose causal completion $\mathcal{U}''$ (the union of $\mathcal{U}$ and the regions labeled $\mathcal{U}''$) is the same causal diamond as in (a).

The reason to consider bounded operators is that they are defined on all of Hilbert space, so they can be multiplied without any trouble, and naturally form an algebra. Unbounded operators in general cannot be multiplied, as they are defined on different dense subspaces of Hilbert space. If we try to define "all unbounded functions" of the ϕ_f's and hope to make them into an algebra, we will probably have a lot of trouble.

We could go on with elementary constructions. To complete the story, what is really needed is to include limits of the operators we already have. To decide what sort of limits to allow, let us think for a moment about what is involved in *measuring* an operator, such as the weak Hamiltonian that is involved in beta decay. What an experiment gives us is a measurement of finitely many matrix elements of an operator, each with some experimental error. If $\mathsf{a}_1, \mathsf{a}_2, \ldots$ is a sequence of operators all of whose matrix elements $\langle\psi|\mathsf{a}_n|\chi\rangle$ converge for large n to the corresponding matrix elements $\langle\psi|\mathsf{a}|\chi\rangle$ of some operator a, this means that any given experiment will not distinguish a_n from a once n is large enough. In such a situation, it is reasonable physically to say that $\mathsf{a} = \lim_{n\to\infty} \mathsf{a}_n$. What we have just described [following Haag (1992) in this reasoning] is the mathematical notion of a weak limit of a sequence of operators.

It is reasonable to believe that we should define $\mathcal{A}_{\mathcal{U}}$ to be closed under such weak limits.[10] One also expects $\mathcal{A}_{\mathcal{U}}$ to be closed under a more trivial operation. The set of smeared fields in a given region is closed under Hermitian conjugation.

[9]The simple operators also have important advantages, of course; they are the basis of a standard and powerful machinery of renormalization theory, operator product expansions, and so on.

[10]However, a result of von Neumann shows that if we define $\mathcal{A}_{\mathcal{U}}$ to be closed only under a more restricted type of limit called a strong limit, we will actually get the same algebra. A sequence $\mathsf{a}_1, \mathsf{a}_2, \ldots$ of operators has an operator a as its strong limit if for any Hilbert space state χ, $\lim_{n\to\infty} \mathsf{a}_n \chi = \mathsf{a}\chi$.

[If $\phi_f = \int d^D x f(x)\phi(x)$ is a smeared field supported in a given region, then so is $\phi_f^\dagger = \int d^D x \bar{f}(x)\phi(x)$.] Any reasonable set of operations that builds new operators from old ones, starting from a set of operators that is closed under Hermitian conjugation, will give a set of operators that remains closed under Hermitian conjugation. An algebra acting on a Hilbert space and closed under Hermitian conjugation is called a $*$-algebra. Thus any reasonable choice of what we would mean by $\mathcal{A}_\mathcal{U}$ will be a $*$-algebra.

A $*$-algebra of bounded operators on a Hilbert space that is closed under weak limits (and contains the identity operator) is called a von Neumann algebra. Thus we are led in this way to the notion that the local algebra $\mathcal{A}_\mathcal{U}$ of an open set $\mathcal{U}$ should be a von Neumann algebra.

If $\mathcal{A}$ is a $*$-algebra of bounded operators on a Hilbert space $\mathcal{H}$, then its commutant $\mathcal{A}'$, defined as the set of all bounded operators on $\mathcal{H}$ that commute with $\mathcal{A}$, is another $*$-algebra. $\mathcal{A}'$ is always a von Neumann algebra even if $\mathcal{A}$ is not.[11] If $\mathcal{A}$ is a von Neumann algebra, then the relation between $\mathcal{A}$ and $\mathcal{A}'$ is reciprocal: each is the commutant of the other. This is von Neumann's theorem that if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{A}'' = (\mathcal{A}')'$ satisfies $\mathcal{A}'' = \mathcal{A}$.

Operators at spacelike separation commute, so one expects that if $\mathcal{U}$ and $\mathcal{U}'$ are spacelike separated, then[12]

$$[\mathcal{A}_\mathcal{U}, \mathcal{A}_{\mathcal{U}'}] = 0, \tag{2.22}$$

which is an abbreviated way to say that $[\mathsf{a}, \mathsf{a}'] = 0$ if $\mathsf{a} \in \mathcal{A}_\mathcal{U}$, $\mathsf{a}' \in \mathcal{A}_{\mathcal{U}'}$. Thus one expects that $\mathcal{A}_{\mathcal{U}'}$ is always contained in $\mathcal{A}_\mathcal{U}{}'$.

It was proposed by Haag (1963) and by Haag and Schroer (1962) that if $\mathcal{U}$ and $\mathcal{U}'$ are causal complements, meaning that they are maximal open sets under the condition of being spacelike separated, then the corresponding algebras $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$ are commutants, meaning that they are maximal under the condition of commuting with each other. This condition, sometimes called Haag duality, can be written

$$\mathcal{A}_{\mathcal{U}'} = \mathcal{A}_\mathcal{U}{}'. \tag{2.23}$$

This condition is stated in Haag (1992) as part of Tentative Postulate 4.2.1. The rest of the postulate says that if $\mathcal{U}$ is a union of open sets $\mathcal{U}_\alpha$, then $\mathcal{A}_\mathcal{U}$ is the smallest von Neumann algebra containing the $\mathcal{A}_{\mathcal{U}_\alpha}$, and that if $\mathcal{U}, \widetilde{\mathcal{U}}$ are two open sets then $\mathcal{A}_{\mathcal{U}\cap\widetilde{\mathcal{U}}} = \mathcal{A}_\mathcal{U} \cap \mathcal{A}_{\widetilde{\mathcal{U}}}$. Haag duality is known to be true in many circumstances; for example, it was proved by Bisognano and Wichmann (1976) for complementary Rindler regions in Minkowski spacetime (this is explained at the end of Sec. V.B). Haag duality and the rest of Postulate 4.2.1 are apparently valid in an interesting class of quantum field theories and for some open sets in a wider class, but it appears that in some theories and for some classes of open sets, Haag duality and other parts of Tentative Postulate 4.2.1 can fail (Leyland, Roberts, and Testard, 1978; Doplicher and Longo, 1984; Schroer, 2017; Harlow and Ooguri, 2018).

We give an example of the simplification that occurs if two algebras are commutants. If $\mathcal{A}$ and $\mathcal{A}'$ are commutants, then a vector $\Omega \in \mathcal{H}$ is separating for $\mathcal{A}$ if and only if it is cyclic for $\mathcal{A}'$, and vice versa. The "if" part of this statement only depends on $\mathcal{A}$ and $\mathcal{A}'$ commuting and was explained in Sec. II.D. What we gain if $\mathcal{A}$ and $\mathcal{A}'$ are commutants is the "only if" statement. Suppose in fact that a vector Ω is not cyclic for $\mathcal{A}'$. Then the vectors $\mathsf{a}'|\Omega\rangle$, $\mathsf{a}' \in \mathcal{A}'$ generate a Hilbert space $\mathcal{H}'$ that is a proper subspace of $\mathcal{H}$. Let $\Pi : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}'_\perp$. Then Π is bounded and commutes with $\mathcal{A}'$, so if the two algebras are commutants, $\Pi \in \mathcal{A}$. But $\Pi\Omega = 0$ (since $1 \in \mathcal{A}'$, certainly $\Omega = 1 \cdot \Omega$ is of the form $\mathsf{a}'\Omega$, $\mathsf{a}' \in \mathcal{A}'$, and therefore $\Omega \in \mathcal{H}'$). Thus if Ω is not cyclic for $\mathcal{A}'$, then $\Pi \in \mathcal{A}$ annihilates Ω and Ω is not separating for $\mathcal{A}$.

We conclude by describing an analogy between algebras and open sets that is developed in Haag (1992). In the analogy, a $*$-algebra corresponds to an open set in spacetime, a von Neumann algebra corresponds to a causally complete open set, and commutants correspond to causal complements.

Let $\mathcal{A}$ be a $*$-algebra of bounded operators on $\mathcal{H}$ (not necessarily a von Neumann algebra) and $\mathcal{A}'$ its commutant. Then $\mathcal{A}'$ is a von Neumann algebra as explained in footnote 11. In particular, the commutant $\mathcal{A}'' = (\mathcal{A}')'$ of $\mathcal{A}'$ is a von Neumann algebra. Clearly $\mathcal{A} \subset \mathcal{A}''$ ($\mathcal{A}''$ consists of all bounded operators that commute with $\mathcal{A}'$, and the definition of $\mathcal{A}'$ ensures that any element of $\mathcal{A}$ commutes with $\mathcal{A}'$). $\mathcal{A}''$ is called the von Neumann algebra closure of $\mathcal{A}$; it is the smallest von Neumann algebra containing $\mathcal{A}$. If $\mathcal{A}$ was a von Neumann algebra to begin with, then $\mathcal{A} = \mathcal{A}''$. On the other hand $\mathcal{A}'$ is always a von Neumann algebra so one always has $\mathcal{A}' = \mathcal{A}'''$. If $\mathcal{A}$ is a von Neumann algebra, $\mathcal{A}$ and $\mathcal{A}'$ are each other's commutants.

Now consider open sets. If $\mathcal{U}$ is an open set, then as above, its causal complement $\mathcal{U}'$ is the union of all open sets that are spacelike separated from $\mathcal{U}$ (equivalently, it is the largest open set spacelike separated from $\mathcal{U}$). The causal complement $\mathcal{U}'' = (\mathcal{U}')'$ of $\mathcal{U}'$ always contains $\mathcal{U}$, since $\mathcal{U}$ is an open set spacelike separated from $\mathcal{U}'$. One always has $\mathcal{U}''' = \mathcal{U}'$. [Indeed, since $\mathcal{U} \subset \mathcal{U}''$, the condition for a point to be spacelike separated from $\mathcal{U}''$ is stronger than the condition for it to be spacelike separated from $\mathcal{U}$, so $\mathcal{U}''' = (\mathcal{U}'')' \subset \mathcal{U}'$. The opposite inclusion $\mathcal{U}' \subset \mathcal{U}'''$ just says that the open set $\mathcal{U}'$ is contained in $(\mathcal{U}')'' = \mathcal{U}'''$.] $\mathcal{U}$ is said to be causally complete if $\mathcal{U}'' = \mathcal{U}$. The result $\mathcal{U}''' = \mathcal{U}'$ means that $\mathcal{U}'$ is always causally complete. In general, $\mathcal{U}''$ (which also is always causally complete since $\mathcal{U}' = \mathcal{U}'''$ implies $\mathcal{U}'' = \mathcal{U}''''$) is the smallest causally complete set containing $\mathcal{U}$ and is called the causal completion of $\mathcal{U}$. If $\mathcal{U}$ is causally complete, then $\mathcal{U}$ and $\mathcal{U}'$ are each other's causal complements.

[11] The nontrivial point is that $\mathcal{A}'$ is closed under weak limits. If $\mathsf{a}_1', \mathsf{a}_2', \ldots$ is a sequence of bounded operators that commute with $\mathcal{A}$ and has weak limit a', then for any states $\psi, \chi \in \mathcal{H}$ and any $\mathsf{a} \in \mathcal{A}$, one has $\langle\psi|[\mathsf{a}, \mathsf{a}']|\chi\rangle = \lim_{n\to\infty}\langle\psi|[\mathsf{a}, \mathsf{a}_n']|\chi\rangle = 0$; vanishing of $\langle\psi|[\mathsf{a}, \mathsf{a}']|\chi\rangle$ for all ψ, χ means $[\mathsf{a}, \mathsf{a}'] = 0$ and therefore $\mathsf{a}' \in \mathcal{A}'$, showing that $\mathcal{A}'$ is closed under weak limits.

[12] In the presence of fermions, one has anticommutativity as well as commutativity of operators at spacelike separation. In the algebraic approach, one can consider a von Neumann algebra with an automorphism that distinguishes even and odd operators. For one approach, see Guido and Longo (1995).

If Haag duality holds in some theory for all open sets, not necessarily causally complete, then it implies that $\mathcal{A}_{\mathcal{U}} = \mathcal{A}_{\mathcal{U}''}$ for all $\mathcal{U}$, a property stated in Haag (1992), (III.1.10). [Indeed, Haag duality says that $\mathcal{A}_{\mathcal{U}''} = (\mathcal{A}_{\mathcal{U}'})' = (\mathcal{A}_{\mathcal{U}})'' = \mathcal{A}_{\mathcal{U}}$, where in the last step we use the fact that $\mathcal{A}'' = \mathcal{A}$ for any von Neumann algebra $\mathcal{A}$.] The conditions for this to hold do not appear to be known,[13] but it does have a surprisingly wide range of validity. Two illustrative cases are shown in Figs. 2(a) and 2(c). In Fig. 2(a), $\mathcal{U}''$ is a causal diamond, and coincides with the domain of dependence $\widehat{\mathcal{U}}$ of $\mathcal{U}$. Causality would lead us to expect in this example that $\mathcal{A}_{\mathcal{U}} = \mathcal{A}_{\mathcal{U}''}$ and this was indeed an input to the previous discussion. In Fig. 2(c), $\mathcal{U}$ is a thin "timelike tube" (with corners at the top and bottom) whose causal completion $\mathcal{U}''$ is the same causal diamond. In this case, there is no simple reason of causality to expect that $\mathcal{A}_{\mathcal{U}} = \mathcal{A}_{\mathcal{U}''}$, but this can be proved with a more sophisticated use of the ingredients that went into proving the Reeh-Schlieder theorem. The result is sometimes called the Borchers timelike tube theorem (Borchers, 1961, 1996; Araki, 1963; Wightman, 1964).

III. THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEORY

A. Definition and first properties

In some quantum field theory in Minkowski spacetime with Hilbert space $\mathcal{H}$, let $\mathcal{A}_{\mathcal{U}}$ be the algebra of observables in a spacetime region $\mathcal{U}$, and let $\mathcal{A}_{\mathcal{U}}'$ be its commutant. (If Haag duality holds, then $\mathcal{A}_{\mathcal{U}}'$ coincides with $\mathcal{A}_{\mathcal{U}'}$, but we do not need to assume this.) If the context is clear, we sometimes write just $\mathcal{A}$ and $\mathcal{A}'$ for $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\mathcal{U}}'$. Let Ψ be a vector—such as the vacuum vector—that is cyclic and separating for both regions.

The Tomita operator for the state Ψ is an antilinear operator S_Ψ that, roughly speaking, is defined by

$$S_\Psi \mathsf{a}|\Psi\rangle = \mathsf{a}^\dagger|\Psi\rangle \tag{3.1}$$

for all $\mathsf{a} \in \mathcal{A}_{\mathcal{U}}$. To understand this definition, first note that because Ψ is a separating vector for $\mathcal{A}_{\mathcal{U}}$, the state $\mathsf{a}|\Psi\rangle$ is nonzero for all nonzero $\mathsf{a} \in \mathcal{A}_{\mathcal{U}}$. Therefore, we avoid the inconsistency that would arise in this definition if some a would satisfy $\mathsf{a}|\Psi\rangle = 0, \mathsf{a}^\dagger|\Psi\rangle \neq 0$. Second, because the states $\mathsf{a}|\Psi\rangle, \mathsf{a} \in \mathcal{A}_{\mathcal{U}}$ are dense in $\mathcal{H}$, Eq. (3.1) does define the action of S_Ψ on a dense subspace of $\mathcal{H}$.

The definition of Eq. (3.1) will lead to an unbounded operator S_Ψ for the following reason. In the region $\mathcal{U}$, given that it is small enough that its causal complement contains another open set $\mathcal{U}'$, it is not possible to make a mode of definite positive or negative frequency. But by using modes of very short wavelength, we can construct an operator a in region $\mathcal{U}$ that is arbitrarily close to being an annihilation operator (one that lowers the energy) while $\mathsf{a}^\dagger$ is equally close to being a creation operator. So $\mathsf{a}|\Omega\rangle$ can be arbitrarily small while $\mathsf{a}^\dagger|\Omega\rangle$ is not small. Thus S_Ψ is unbounded.

An unbounded operator cannot be defined on all states in Hilbert space (recall footnote 5). But it is important to slightly extend the definition of S_Ω as follows. If a_n, $n = 1, 2, 3, \ldots$ is a sequence of elements of $\mathcal{A}_{\mathcal{U}}$ such that both limits

$$x = \lim_{n\to\infty} \mathsf{a}_n|\Psi\rangle, \qquad y = \lim_{n\to\infty} \mathsf{a}_n^\dagger|\Psi\rangle \tag{3.2}$$

exist, then we define[14]

$$S_\Psi x = y. \tag{3.3}$$

Extending the definition of S_Ψ in this way gives what technically is known as a "closed" operator, meaning that its graph is closed; see Sec. III.F.

The definition (3.1) makes it clear that

$$S_\Psi^2 = 1, \tag{3.4}$$

so in particular S_Ψ is invertible. Another obvious fact is that

$$S_\Psi|\Psi\rangle = |\Psi\rangle. \tag{3.5}$$

We could of course similarly define the modular operator S_Ψ' for the commuting algebra $\mathcal{A}_{\mathcal{U}}'$. In fact, these operators are Hermitian adjoints:

$$S_\Psi' = S_\Psi^\dagger. \tag{3.6}$$

The definition of the adjoint of an antilinear operator W is that for any states Λ, χ,

$$\langle \Lambda|W\chi\rangle = \overline{\langle W^\dagger\Lambda|\chi\rangle} = \langle\chi|W^\dagger\Lambda\rangle. \tag{3.7}$$

A special case of this which we will use shortly is that if W is antiunitary, meaning that it is antilinear and satisfies $W^\dagger W = WW^\dagger = 1$, then

$$\langle W\Lambda|W\chi\rangle = \overline{\langle\Lambda|\chi\rangle} = \langle\chi|\Lambda\rangle. \tag{3.8}$$

To show that $S_\Psi' = S_\Psi^\dagger$, we have to show that for all states Λ, χ, we have $\langle S_\Psi'\Lambda|\chi\rangle = \langle S_\Psi\chi|\Lambda\rangle$. It is enough to check this for a dense set of states, so we can take $\chi = \mathsf{a}\Psi$, $\Lambda = \mathsf{a}'\Psi$, with $\mathsf{a} \in \mathcal{A}_{\mathcal{U}}$, $\mathsf{a}' \in \mathcal{A}_{\mathcal{U}}'$. Using the definitions of S_Ψ and S_Ψ' and of a Hermitian adjoint and the fact that a and a' commute, we get

[13] As a counterexample if $\mathcal{U}$ is not required to be connected, in two-dimensional spacetime, let $\mathcal{U}$ be the union of small balls centered at the two points $(t, x) = (\pm 1, 0)$. Then $\mathcal{U}''$ is again a (slightly rounded) causal diamond. Massless fields are functions only of $x_\pm = x \pm t$. In $\mathcal{U}''$, one can measure modes of massless fields in the whole range $-1 \leq x_\pm \leq 1$, but in $\mathcal{U}$, one only see values of $x_\pm$ near ± 1.

[14] For this definition to make sense, it must be that if $\lim_{n\to\infty}\mathsf{a}_n\Psi = 0$ then also $\lim_{n\to\infty}\mathsf{a}_n^\dagger\Psi = 0$. Suppose that $y = \lim_{n\to\infty}\mathsf{a}_n^\dagger|\Psi\rangle$ exists and is nonzero. As it is separating for $\mathcal{A}_{\mathcal{U}}$, the state Ψ is cyclic for $\mathcal{A}_{\mathcal{U}}'$. So there is $\mathsf{a}' \in \mathcal{A}_{\mathcal{U}}'$ with nonzero $C = \langle\mathsf{a}'\Psi|y\rangle = \lim_{n\to\infty}\langle\mathsf{a}'\Psi|\mathsf{a}_n^\dagger\Psi\rangle$. Then $\bar{C} = \lim_{n\to\infty}\langle\mathsf{a}_n^\dagger\Psi|\mathsf{a}'\Psi\rangle = \lim_{n\to\infty}\langle\mathsf{a}'^\dagger\Psi|\mathsf{a}_n\Psi\rangle$ is also nonzero. This implies that $x = \lim_{n\to\infty}\mathsf{a}_n|\Psi\rangle$ is nonzero. Mathematically, we have proved that the operator S_Ψ is "closeable." The importance will become clear in Sec. III.F.

$$\begin{aligned}\langle S'_\Psi \mathsf{a}'\Psi|\mathsf{a}\Psi\rangle &= \langle \mathsf{a}'^\dagger\Psi|\mathsf{a}\Psi\rangle = \langle\Psi|\mathsf{a}'\mathsf{a}\Psi\rangle \\ &= \langle\Psi|\mathsf{a}\mathsf{a}'\Psi\rangle = \langle \mathsf{a}^\dagger\Psi|\mathsf{a}'\Psi\rangle \\ &= \langle S_\psi \mathsf{a}\Psi|\mathsf{a}'\Psi\rangle \end{aligned} \tag{3.9}$$

as desired.[15]

Since it is invertible, S_Ψ has a *unique* polar decomposition

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2}, \tag{3.10}$$

where J_Ψ is antiunitary and $\Delta_\Psi^{1/2}$ is Hermitian and positive definite. This implies that

$$\Delta_\Psi = S_\Psi^\dagger S_\Psi. \tag{3.11}$$

Δ_Ψ and J_Ψ are called the modular operator and the modular conjugation. Since $S_\Psi\Psi = S_\Psi^\dagger\Psi = \Psi$, we can deduce the important result

$$\Delta_\Psi|\Psi\rangle = |\Psi\rangle. \tag{3.12}$$

From Eq. (3.12), it follows that for any function f,

$$f(\Delta_\Psi)|\Psi\rangle = f(1)|\Psi\rangle. \tag{3.13}$$

In addition, because $S_\Psi^2 = 1$, we have $J_\Psi\Delta_\Psi^{1/2}J_\Psi\Delta_\Psi^{1/2} = 1$ or

$$J_\Psi\Delta_\Psi^{1/2}J_\Psi = \Delta_\Psi^{-1/2}. \tag{3.14}$$

Hence

$$J_\Psi^2(J_\Psi^{-1}\Delta_\Psi^{1/2}J_\Psi) = \Delta_\Psi^{-1/2} = 1\cdot\Delta_\Psi^{-1/2}. \tag{3.15}$$

Since $J_\Psi^{-1}\Delta_\Psi^{1/2}J_\Psi$ is positive, this gives two different polar decompositions of the operator $\Delta_\Psi^{-1/2}$. By the uniqueness of the polar decomposition, we must have

$$J_\Psi^2 = 1. \tag{3.16}$$

Therefore

$$S'_\Psi = S_\Psi^\dagger = \Delta_\Psi^{1/2}J_\Psi = J_\Psi\Delta_\Psi^{-1/2}. \tag{3.17}$$

Comparing this to the polar decomposition $S'_\Psi = J'_\Psi\Delta'_\Psi{}^{1/2}$, we find

$$J'_\Psi = J_\Psi, \qquad \Delta'_\Psi = \Delta_\Psi^{-1}. \tag{3.18}$$

Finally, because $J_\Psi\Delta_\Psi J_\Psi = \Delta_\Psi^{-1}$, we have $J_\Psi f(\Delta_\Psi)J_\Psi = \bar{f}(\Delta_\Psi^{-1})$ for any function f. In particular, taking $f(x) = x^{is}$ for real s, we get

$$J_\Psi\Delta^{is}J_\Psi = \Delta^{is}, \qquad s\in\mathbb{R}. \tag{3.19}$$

The operators that we introduced have a number of other important properties, which we will explain in Sec. IV after exploring these definitions for finite-dimensional quantum systems.

B. The relative modular operator

Now let Φ be a second state. The relative Tomita operator[16] $S_{\Psi|\Phi}$ for the algebra $\mathcal{A}_\mathcal{U}$ is defined by (Araki, 1975)

$$S_{\Psi|\Phi}\mathsf{a}|\Psi\rangle = \mathsf{a}^\dagger|\Phi\rangle. \tag{3.20}$$

In this definition, we usually assume that

$$\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle = 1. \tag{3.21}$$

The definition of $S_{\Psi|\Phi}$ is completed by taking limits as in Eq. (3.2).

As before, for $S_{\Psi|\Phi}$ to make sense as a densely defined operator, the state Ψ must be cyclic and separating for the algebra $\mathcal{A}_\mathcal{U}$. But Φ can be any state at all. If Φ is cyclic separating, then we can define

$$S_{\Phi|\Psi}\mathsf{a}|\Phi\rangle = \mathsf{a}^\dagger|\Psi\rangle. \tag{3.22}$$

In this case $S_{\Phi|\Psi}S_{\Psi|\Phi} = 1$ and in particular $S_{\Psi|\Phi}$ is invertible. A calculation similar to that of Eq. (3.9) shows that $S_{\Psi|\Phi}$ for one algebra $\mathcal{A}_\mathcal{U}$ is the adjoint of $S_{\Psi|\Phi}$ for the commutant $\mathcal{A}_\mathcal{U}'$.

The relative modular operator is defined by

$$\Delta_{\Psi|\Phi} = S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}. \tag{3.23}$$

It is positive semidefinite, and is positive definite if and only if $S_{\Psi|\Phi}$ is invertible. If $\Phi = \Psi$, $S_{\Psi|\Phi}$ reduces to S_Ψ and $\Delta_{\Psi|\Phi}$ reduces to the usual modular operator:

$$\Delta_{\Psi|\Psi} = \Delta_\Psi. \tag{3.24}$$

The polar decomposition of the relative modular operator is

$$S_{\Psi|\Phi} = J_{\Psi|\Phi}\Delta_{\Psi|\Phi}^{1/2}, \tag{3.25}$$

[15]This argument really only shows that $S_\Psi^\dagger$ is an extension of S'_Ψ (meaning that the two operators act in the same way on any vector on which S'_Ψ is defined). For the proof that it is not a proper extension (meaning that $S_\Psi^\dagger$ cannot be defined, consistent with $\langle S_\Psi^\dagger\chi|\Lambda\rangle = \langle S_\Psi\Lambda|\chi\rangle$, on any vector on which S'_Ψ is not defined), see, for example, Theorem 13.1.3 in Jones (2015).

[16]We should warn the reader that what we call $S_{\Psi|\Phi}$ is often denoted $S_{\Phi|\Psi}$ (or $S_{\Phi/\Psi}$, $S_{\Phi,\Psi}$, etc.). The purpose of our convention is to agree with quantum information theory, where it has become standard to define the relative entropy between density matrices ρ, σ as $\mathcal{S}(\rho||\sigma) = \mathrm{Tr}\rho(\log\rho - \log\sigma)$. In the relation to information theory, Ψ and Φ correspond, respectively, to ρ and σ, as we will learn in Sec. IV.A, so we put Ψ before Φ just as ρ is conventionally put before σ in $\mathcal{S}(\rho||\sigma)$. Note that some of the classic papers used the opposite ordering for both $S_{\Psi|\Phi}$ and $\mathcal{S}(\rho||\sigma)$.

where $J_{\Psi|\Phi}$ is the relative modular conjugation. Here we have to be careful. If Φ is not separating, then $S_{\Psi|\Phi}$ has a kernel, which is also a kernel of $\Delta_{\Psi|\Phi}$ and $\Delta_{\Psi|\Phi}^{1/2}$. In such a situation, to make the polar decomposition unique, $J_{\Psi|\Phi}$ is defined to annihilate this kernel. Also, if Φ is not cyclic, then the image of $S_{\Psi|\Phi}$ is not a dense subspace of $\mathcal{H}$. In general, $J_{\Psi|\Phi}$ is an antiunitary map from the orthocomplement of the kernel of $S_{\Psi|\Phi}$ to its image. However, if Φ is cyclic separating, then $J_{\Psi|\Phi}$ is antiunitary.

Now let us discuss what happens if Φ is replaced by $\mathsf{a}'\Phi$, where a' is a *unitary* element of the commuting algebra $\mathcal{A}_{\mathcal{U}}{}'$. For $\mathsf{a}\in\mathcal{A}_{\mathcal{U}}$, we get $S_{\Psi|\mathsf{a}'\Phi}\mathsf{a}\Psi=\mathsf{a}^\dagger\mathsf{a}'\Phi=\mathsf{a}'\mathsf{a}^\dagger\Phi$, since $\mathsf{a}^\dagger$ and a' commute. So $S_{\Psi|\mathsf{a}'\Phi}=\mathsf{a}'S_{\Psi|\Phi}$. With a' unitary, this implies

$$\Delta_{\Psi|\mathsf{a}'\Phi}=\Delta_{\Psi|\Phi}. \tag{3.26}$$

If it is important to specify the region $\mathcal{U}$, we write $\Delta_{\Psi|\Phi;\mathcal{U}}$ for the relative modular operator for the algebra $\mathcal{A}_{\mathcal{U}}$ and the states Ψ, Φ, and similarly for $S_{\Psi|\Phi;\mathcal{U}}$.

The following gives a useful characterization of the relative modular operator:

$$\begin{aligned}\langle\mathsf{a}^\dagger\Psi|\Delta_{\Psi|\Phi}|\mathsf{b}\Psi\rangle&=\langle\mathsf{a}^\dagger\Psi|S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}|\mathsf{b}\Psi\rangle\\&=\langle S_{\Psi|\Phi}\mathsf{b}\Psi|S_{\Psi|\Phi}\mathsf{a}^\dagger\Psi\rangle=\langle\mathsf{b}^\dagger\Phi|\mathsf{a}\Phi\rangle.\end{aligned} \tag{3.27}$$

Remark: For future reference, observe that the definition of $S_{\Psi|\Phi}$ and $\Delta_{\Psi|\Phi}$ does not require that Ψ and Φ are vectors in the same Hilbert space. Let $\mathcal{H}$ and $\mathcal{H}'$ be two different Hilbert spaces with an action of the same algebra $\mathcal{A}_{\mathcal{U}}$. For example, $\mathcal{H}$ and $\mathcal{H}'$ might be different superselection sectors in the same quantum field theory. If Ψ is a cyclic separating vector in $\mathcal{H}$ and Φ is any vector in $\mathcal{H}'$ then Eq. (3.20) makes sense and defines an antilinear operator $S_{\Psi|\Phi}:\mathcal{H}\to\mathcal{H}'$. Its adjoint is an antilinear operator $S_{\Psi|\Phi}^\dagger:\mathcal{H}'\to\mathcal{H}$. The product $S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}$ is a non-negative self-adjoint operator, the modular operator $\Delta_{\Psi|\Phi}:\mathcal{H}\to\mathcal{H}$. When not otherwise noted, we usually assume $\mathcal{H}=\mathcal{H}'$.

C. Relative entropy in quantum field theory

A primary application of the relative modular operator in these notes will be to study the relative entropy. Relative entropy was defined in classical information theory by Kullback and Leibler (1951) and in nonrelativistic quantum mechanics by Umegacki (1962); a definition suitable for quantum field theory was given by Araki (1975, 1976). The relative entropy $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ between two states Ψ and Φ, for measurements in the region $\mathcal{U}$, is

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U})=-\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle. \tag{3.28}$$

(In this section, $\mathcal{U}$ is kept fixed and we write $\Delta_{\Psi|\Phi}$ for $\Delta_{\Psi|\Phi;\mathcal{U}}$.) In general, $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ is a real number or $+\infty$. For example, $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ may be $+\infty$ if $\Delta_{\Psi|\Phi}$ has a zero eigenvalue, which will occur if Φ is not separating for $\mathcal{A}_{\mathcal{U}}$. How this definition is related to what may be more familiar definitions of relative entropy will be explained in Sec. IV. In this section, we simply discuss the properties of the relative entropy.

An important elementary property is that $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ is always non-negative, and vanishes precisely if $\Phi=\mathsf{a}'\Psi$ where a' is a unitary element of the commuting algebra $\mathcal{A}_{\mathcal{U}}{}'$. This condition implies that $\langle\Phi|\mathsf{a}|\Phi\rangle=\langle\Psi|\mathsf{a}|\Psi\rangle$ for all $\mathsf{a}\in\mathcal{A}_{\mathcal{U}}$, so it means that Φ and Ψ cannot be distinguished by a measurement in region $\mathcal{U}$. To see the vanishing if $\Phi=\mathsf{a}'\Psi$, with $\mathsf{a}'\in\mathcal{A}_{\mathcal{U}}{}'$, note that in this case, according to Eqs. (3.24) and (3.26), $\Delta_{\Psi|\Phi}$ is the ordinary modular operator Δ_Ψ. So using Eq. (3.13) with $f(x)=\log x$, we get $\log\Delta_{\Psi|\Phi}|\Psi\rangle=0$ for $\Phi=\mathsf{a}'\Psi$, hence $\mathcal{S}_{\Psi|\Psi}(\mathcal{U})=0$.

To show that $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})>0$ if Φ is not of the form $\mathsf{a}'\Psi$, one uses (Araki, 1976) the inequality for a non-negative real number $\log\lambda\le\lambda-1$. This inequality for numbers implies the operator inequality $\log\Delta_{\Psi|\Phi}\le\Delta_{\Psi|\Phi}-1$, or $-\log\Delta_{\Psi|\Phi}\ge 1-\Delta_{\Psi|\Phi}$. So

$$\begin{aligned}\mathcal{S}_{\Psi|\Phi}(\mathcal{U})\ge\langle\Psi|(1-\Delta_{\Psi|\Phi})|\Psi\rangle&=\langle\Psi|\Psi\rangle-\langle\Psi|S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}|\Psi\rangle\\&=\langle\Psi|\Psi\rangle-\langle\Phi|\Phi\rangle=0,\end{aligned} \tag{3.29}$$

since we assume $\langle\Psi|\Psi\rangle=\langle\Phi|\Phi\rangle=1$.

Because the inequality $\log\lambda\le\lambda-1$ is only saturated at $\lambda=1$, to saturate the inequality (3.29) we need $\Delta_{\Psi|\Phi}$ to equal 1 in acting on Ψ, that is we need $\Delta_{\Psi|\Phi}\Psi=\Psi$. But as we will show, this implies that $\Phi=\mathsf{a}'\Psi$ for some unitary $\mathsf{a}'\in\mathcal{A}_{\mathcal{U}}{}'$. The statement that $\Delta_{\Psi|\Phi}\Psi=\Psi$ implies that for any state χ,

$$\langle\chi|\Delta_{\Psi|\Phi}\Psi\rangle=\langle\chi|\Psi\rangle. \tag{3.30}$$

In particular, this must be so if $\chi=\mathsf{a}\Psi$ for $\mathsf{a}\in\mathcal{A}_{\mathcal{U}}$. We calculate

$$\begin{aligned}\langle\mathsf{a}\Psi|\Delta_{\Psi|\Phi}\Psi\rangle&=\langle\mathsf{a}\Psi|S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}\Psi\rangle=\langle\mathsf{a}\Psi|S_{\Psi|\Phi}^\dagger\Phi\rangle\\&=\langle\Phi|S_{\Psi|\Phi}\mathsf{a}\Psi\rangle=\langle\Phi|\mathsf{a}^\dagger\Phi\rangle=\langle\mathsf{a}\Phi|\Phi\rangle.\end{aligned} \tag{3.31}$$

We used $S_{\Psi|\Phi}\Psi=\Phi$ and the definition of the adjoint of an antilinear operator. The condition (3.30) then is that $\langle\mathsf{a}\Phi|\Phi\rangle=\langle\mathsf{a}\Psi|\Psi\rangle$ for all $\mathsf{a}\in\mathcal{A}_{\mathcal{U}}$. Accordingly, for a, $\mathsf{b}\in\mathcal{A}_{\mathcal{U}}$,

$$\langle\mathsf{a}\Phi|\mathsf{b}\Phi\rangle=\langle\mathsf{b}^\dagger\mathsf{a}\Phi|\Phi\rangle=\langle\mathsf{b}^\dagger\mathsf{a}\Psi|\Psi\rangle=\langle\mathsf{a}\Psi|\mathsf{b}\Psi\rangle. \tag{3.32}$$

Since states of the form $\mathsf{a}\Psi$ or $\mathsf{b}\Psi$ are dense in $\mathcal{H}$, we can define a densely defined linear operator that takes $\mathsf{a}\Psi$ to $\mathsf{a}\Phi$. Equation (3.32) states that this operator is unitary (and so, being bounded, it can be naturally extended to all of $\mathcal{H}$), and as it commutes with $\mathcal{A}_{\mathcal{U}}$, it is given by multiplication by a unitary element $\mathsf{a}'\in\mathcal{A}_{\mathcal{U}}{}'$. Thus $\mathsf{a}\Phi=\mathsf{a}'\mathsf{a}\Psi$ for all a, and in particular $\Phi=\mathsf{a}'\Psi$, as claimed.

Positivity of relative entropy has various applications in quantum field theory, for instance in the interpretation and proof (Casini, 2008; Longo and Xu, 2018) of the Bekenstein bound on the energy, entropy, and size of a quantum system. The more subtle property of monotonicity of relative entropy, to which we come next, also has various applications, for

instance in the proof of a semiclassical generalized second law of thermodynamics that includes black hole entropy (Wall, 2012).

D. Monotonicity of relative entropy

In quantum field theory, in the definition of the algebra of observables and the associated modular operators, we can replace the open set $\mathcal{U}$ by a smaller open set $\widetilde{\mathcal{U}} \subset \mathcal{U}$. Thus, for given Ψ and Φ, we can define Tomita operators $S_{\Psi|\Phi;\mathcal{U}}$ and $S_{\Psi|\Phi;\widetilde{U}}$ and associated modular operators $\Delta_{\Psi|\Phi;\mathcal{U}}$ and $\Delta_{\Psi|\Phi;\widetilde{\mathcal{U}}}$. Then we have the relative entropy $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ for measurements in $\mathcal{U}$,

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle \tag{3.33}$$

and the corresponding relative entropy for measurements in $\widetilde{\mathcal{U}}$,

$$\mathcal{S}_{\Psi|\Phi}(\widetilde{\mathcal{U}}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\widetilde{\mathcal{U}}}|\Psi\rangle. \tag{3.34}$$

Monotonicity of relative entropy says that if $\widetilde{\mathcal{U}}$ is contained in $\mathcal{U}$, then

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) \geq \mathcal{S}_{\Psi|\Phi}(\widetilde{\mathcal{U}}). \tag{3.35}$$

In nonrelativistic quantum mechanics, a version of monotonicity of relative entropy was proved by Lieb and Ruskai (1973), along with strong subadditivity of quantum entropy, to which it is closely related. The proof used a lemma of Lieb (1973). A more general form of monotonicity of relative entropy was proved by Uhlmann (1977). In a form that encompasses the statement (3.35) in quantum field theory, monotonicity of relative entropy was proved by Araki (1975, 1976). Petz (1986), with later elaboration by Nielsen and Petz (2005), formulated a proof for nonrelativistic quantum mechanics that drew partly on Araki's framework. Some of these matters will be explained in Sec. IV, but for now we just concentrate on understanding Eq. (3.35).

The states Ψ and Φ will be kept fixed in the rest of this section, so to lighten the notation we usually just write $S_{\mathcal{U}}$ for $S_{\Psi|\Phi;\mathcal{U}}$ and $\Delta_{\mathcal{U}}$ for $\Delta_{\Psi|\Phi;\mathcal{U}}$, and similarly for $\widetilde{\mathcal{U}}$. The inequality (3.35) is a direct consequence of an operator inequality

$$\Delta_{\widetilde{\mathcal{U}}} \geq \Delta_{\mathcal{U}}. \tag{3.36}$$

A self-adjoint operator P is called positive if $\langle\chi|P|\chi\rangle \geq 0$ for all χ; in that case, one writes $P \geq 0$. If P and Q are bounded self-adjoint operators, one says $P \geq Q$ if $P - Q \geq 0$. (The reason for assuming here that P and Q are bounded is that it ensures that $\langle\chi|P-Q|\chi\rangle = \langle\chi|P|\chi\rangle - \langle\chi|Q|\chi\rangle$ is defined for all χ; we explain shortly how to interpret the statement $P \geq Q$ in general.) If $P, Q \geq 0$, an equivalent statement to $P \geq Q$ is

$$\frac{1}{s+P} \leq \frac{1}{s+Q}, \tag{3.37}$$

for all $s > 0$. (If P and Q are strictly positive, one can take $s = 0$.) To show this, consider the family of operators $R(t) = tP + (1-t)Q$, $t \in \mathbb{R}$. Writing $\dot{R} = dR/dt$, we see that $\dot{R} = P - Q \geq 0$. We have

$$\frac{d}{dt}\frac{1}{s+R(t)} = -\frac{1}{s+R(t)}\dot{R}\frac{1}{s+R(t)}. \tag{3.38}$$

The right-hand side is ≤ 0 since it is of the form $-ABA$ with A self-adjoint and $B \geq 0$. Integrating Eq. (3.38) in t from $t = 0$ to 1, we learn that $1/[s+R(1)] \leq 1/[s+R(0)]$, which is Eq. (3.37). We describe this result by saying that $1/(s+P)$ is a decreasing function of P, or equivalently that $-1/(s+P)$ is an increasing function of P. The opposite inequality that Eq. (3.37) implies $P \geq Q$ is proved in the same way, writing $P = 1/T - s$, with $T = 1/(s+P)$.

So far we assumed that P and Q are bounded. If P and Q are densely defined unbounded operators, but non-negative, then it is reasonable to interpret Eq. (3.37) as the *definition* of what we mean by $P \geq Q$. In general, P and Q are defined on different (dense) subspaces, so it can be hard to interpret the statement that $\langle\chi|P|\chi\rangle \geq \langle\chi|Q|\chi\rangle$ for all χ. But $1/(s+P)$ and $1/(s+Q)$ are bounded, and so defined for all χ. The statement (3.37) just means that

$$\langle\chi|\frac{1}{s+P}|\chi\rangle \leq \langle\chi|\frac{1}{s+Q}|\chi\rangle, \quad \forall\ \chi \in \mathcal{H}. \tag{3.39}$$

This is a much stronger and more useful statement than just saying that $\langle\chi|P|\chi\rangle \geq \langle\chi|Q|\chi\rangle$ for all χ on which both P and Q are defined.

Using

$$\log R = \int_0^\infty ds\left(\frac{1}{s+1} - \frac{1}{s+R}\right), \tag{3.40}$$

we see that since $1/(s+R)$ is a decreasing function of R, $\log R$ is an increasing function of R. Thus $P \geq Q$ or its equivalent $1/(s+P) \leq 1/(s+Q)$ implies

$$\log P \geq \log Q. \tag{3.41}$$

So Eq. (3.36) implies that

$$\log\Delta_{\widetilde{\mathcal{U}}} \geq \log\Delta_{\mathcal{U}}. \tag{3.42}$$

The monotonicity statement (3.35) is simply the expectation value of this operator inequality in the state Ψ.

The proof of the crucial inequality (3.36) is rather short and is explained in Sec. III.F. However, we first explain some background and motivation in Sec. III.E. The goal of Sec. III.E is to ensure that the reader will consider the result obvious before actually getting to the proof.

To conclude this section, we explain another monotonicity statement that will be useful later, and then, to help the reader appreciate the subtlety of such statements, we explain a superficially similar version that is false. For $0 < \alpha < 1$, we have

$$R^\alpha = \frac{\sin\pi\alpha}{\pi}\int_0^\infty ds s^\alpha\left(\frac{1}{s} - \frac{1}{s+R}\right). \tag{3.43}$$

If R depends on a parameter t, and $\dot{R} = dR/dt$, we get

$$\frac{d}{dt}R^\alpha = \frac{\sin\pi\alpha}{\pi}\int_0^\infty ds s^\alpha \frac{1}{s+R}\dot{R}\frac{1}{s+R}. \tag{3.44}$$

This is non-negative if $\dot{R} \geq 0$, so R^α is an increasing function of R in this range of α. If, however, $\alpha > 1$, then R^α is in general not an increasing function of R. For $\alpha > 1$, the representation (3.43) is not valid. But if $1 < \alpha < 2$, we can write $R^\alpha = R \cdot R^\beta$, with $0 < \beta < 1$, and then use Eq. (3.43) for R^β. So in this range of α,

$$R^\alpha = \frac{\sin\pi(\alpha-1)}{\pi}\int_0^\infty ds s^{\alpha-1}\left(\frac{R}{s} - 1 + \frac{s}{s+R}\right), \tag{3.45}$$

and hence

$$\frac{d}{dt}R^\alpha = \frac{\sin\pi(\alpha-1)}{\pi}\int_0^\infty ds s^{\alpha-1}\left(\frac{\dot{R}}{s} - s\frac{1}{s+R}\dot{R}\frac{1}{s+R}\right). \tag{3.46}$$

This is not necessarily non-negative for $\dot{R} \geq 0$, since the last term is negative definite and can dominate. For an example with 2×2 matrices, set $R = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$, $\dot{R} = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, and $\chi = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$. Then

$$\langle\chi|\frac{d}{dt}R^\alpha|\chi\rangle < 0. \tag{3.47}$$

E. Examples

The relation between $S_{\mathcal{U}}$ and $S_{\widetilde{\mathcal{U}}}$ is as follows. They are both defined on a dense set of states by the same formula $S\mathsf{a}\Psi = \mathsf{a}^\dagger\Psi$ [together with limiting cases as described in Eq. (3.2)]. The only difference is that the dense subspace on which $S_{\mathcal{U}}$ is defined is larger than the dense subspace on which $S_{\widetilde{\mathcal{U}}}$ is defined. In the case of $S_{\widetilde{\mathcal{U}}}$, a is an element of the algebra $\mathcal{A}_{\widetilde{\mathcal{U}}}$, while in the case of $S_{\mathcal{U}}$, a is an element of the larger algebra $\mathcal{A}_{\mathcal{U}}$.

Let X and Y be unbounded operators[17] on a Hilbert space $\mathcal{H}$ (either both linear or both antilinear). If X is defined whenever Y is defined and they act in the same way on any vector on which they are both defined, then X is called an extension of Y. In this situation, as we will see, it is always true that $X^\dagger X \leq Y^\dagger Y$, and therefore that $\log X^\dagger X \leq \log Y^\dagger Y$. Applied to the case $X = S_{\mathcal{U}}$, $Y = S_{\widetilde{\mathcal{U}}}$, this is the inequality we want.

The following remarks apply for either $\mathcal{U}$ or $\widetilde{\mathcal{U}}$, so we drop the subscripts from S and Δ. The operator $\Delta = S^\dagger S$ is associated with the Hermitian form $F(\chi,\eta) = \langle S\chi|S\eta\rangle$, which is defined on the dense set of vectors $\chi,\eta \in \mathcal{H}$ in the domain of S. This Hermitian form is positive definite in the sense that $F(\chi,\chi) \geq 0$ with equality only if $\chi = 0$. Formally

$$\langle S^\dagger S\eta|\chi\rangle = \langle S\chi|S\eta\rangle. \tag{3.48}$$

The way we interpret this statement is that if, for some η in the domain of S, the relation $\langle\zeta|\chi\rangle = \langle S\chi|S\eta\rangle$ holds for all χ on which S is defined, then we define

$$S^\dagger S\eta = \zeta. \tag{3.49}$$

In other words, we define $S^\dagger S$ on every vector on which it can be defined so as to make Eq. (3.48) true.

If F and G are two Hermitian forms on $\mathcal{H}$, we say that F is an extension of G if it is defined whenever G is defined and they agree where they are both defined. In our problem, we have two Hermitian forms $W_{\mathcal{U}}(\chi,\eta) = \langle S_{\mathcal{U}}\chi|S_{\mathcal{U}}\eta\rangle$ and $W_{\widetilde{\mathcal{U}}}(\chi,\eta) = \langle S_{\widetilde{\mathcal{U}}}\chi|S_{\widetilde{\mathcal{U}}}\eta\rangle$. $W_{\mathcal{U}}$ is an extension of $W_{\widetilde{\mathcal{U}}}$ because $S_{\mathcal{U}}$ is an extension of $S_{\widetilde{\mathcal{U}}}$. The claim that we motivate here and prove in Sec. III.F is that in this situation, the operators $\Delta_{\mathcal{U}} = S_{\mathcal{U}}^\dagger S_{\mathcal{U}}$ and $\Delta_{\widetilde{\mathcal{U}}} = S_{\widetilde{\mathcal{U}}}^\dagger S_{\widetilde{\mathcal{U}}}$ associated to the two Hermitian forms satisfy $\Delta_{\widetilde{\mathcal{U}}} \geq \Delta_{\mathcal{U}}$. In these statements, it does not matter if S is linear or antilinear or if S maps a Hilbert space $\mathcal{H}$ to itself or to some other Hilbert space $\mathcal{H}'$.

To motivate the claim, we consider a more familiar example. Let M be a compact region in $\mathbb{R}^n$ with boundary N. Let $\mathcal{H}$ be the Hilbert space of square-integrable functions on M, and $\mathcal{H}'$ the Hilbert space of square-integrable 1-forms on M. Roughly speaking, we want to consider the exterior derivative d acting from functions to 1-forms. But we consider two different versions of this operator. We let T_0 be the exterior derivative acting on continuous functions ϕ on M such that $d\phi$ is square integrable and ϕ vanishes along the boundary of M. Such functions are dense in $\mathcal{H}$, so T_0 is a densely defined unbounded operator. We let T_1 be the exterior derivative acting on continuous functions ϕ on M such that $d\phi$ is square integrable but with no restriction on ϕ along the boundary of M. Clearly T_1 is an extension of T_0. The corresponding Hermitian form F_1 is likewise an extension of the Hermitian form F_0:

$$F_0(\phi,\rho) = \langle T_0\phi|T_0\rho\rangle = \int_M d^n x\sum_i \frac{\partial\bar{\phi}}{\partial x_i}\frac{\partial\rho}{\partial x_i}, \tag{3.50}$$

$$F_1(\phi,\rho) = \langle T_1\phi|T_1\rho\rangle = \int_M d^n x\sum_i \frac{\partial\bar{\phi}}{\partial x_i}\frac{\partial\rho}{\partial x_i}. \tag{3.51}$$

The only difference between F_0 and F_1 is that in the definition of F_0, ϕ and ρ are required to vanish along $N = \partial M$, while F_1 is defined without this condition.

Now let us compute the operators $T_0^\dagger T_0$ and $T_1^\dagger T_1$ associated to the quadratic forms F_0 and F_1. Since T_0 and T_1 are both defined by the exterior derivative on some class of functions, it is natural to expect that $T_0^\dagger T_0$ and $T_1^\dagger T_1$ will both equal, in some sense, the Laplacian

$$\Delta = d^\dagger d = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}. \tag{3.52}$$

The identity that we need in order to show that $T^\dagger T\phi = \Delta\phi$ for some function ϕ (where T may be T_0 or T_1) is that

[17] A much more systematic explanation of the requisite facts can be found in Reed and Simon (1972), Chapter VIII, and Simon (2015), Chapter VII.5. The example with the Dirichlet and Neumann Laplacians is analyzed in the latter reference.

$$\int_M d^D x\left(-\sum_{i=1}^{n}\frac{\partial^2\bar{\phi}}{\partial x_i^2}\right)\rho \stackrel{?}{=} \int_M d^n x\sum_{i=1}^{n}\frac{\partial\bar{\phi}}{\partial x_i}\frac{\partial\rho}{\partial x_i} \tag{3.53}$$

for all ρ in the appropriate domain. When we try to prove this identity by integration by parts, we run into a surface term

$$\int_N d\mu(-\partial_\perp\bar{\phi})\rho, \tag{3.54}$$

where $d\mu$ is the Riemann measure of N and $\partial_\perp$ is the inward normal derivative along N.

If we are trying to define $T_0^\dagger T_0$, then ρ and ϕ are constrained to vanish along N. Therefore, the surface term (3.54) vanishes. Accordingly, the identity (3.53) is satisfied for any functions ϕ, ρ in the domain of T_0, that is, any functions (continuous and with square-integrable exterior derivative) that vanish along $N = \partial M$. Thus $T_0^\dagger T_0$ is the Laplacian Δ acting on functions that are constrained to vanish on the boundary. This is usually called the Dirichlet Laplacian, and we denote it as Δ_D.

If we are trying to define $T_1^\dagger T_1$, then there is no constraint on ρ along the boundary, and hence to make the surface term vanish we have to require $\partial_\perp\phi = 0$ along N. The Laplacian acting on such functions is usually called the Neumann Laplacian, and we will denote it as Δ_N.

Thus the inequality $T_0^\dagger T_0 \geq T_1^\dagger T_1$ corresponds in this case to $\Delta_D \geq \Delta_N$. To make it obvious that one should expect such an inequality, we can interpolate between F_0 and F_1 in the following way. For $\lambda \geq 0$, we define the Hermitian form

$$G_\lambda(\phi,\rho) = \int_M d^n x\sum_i \frac{\partial\bar{\phi}}{\partial x_i}\frac{\partial\rho}{\partial x_i} + \lambda\int_N d\mu\bar{\phi}\rho, \tag{3.55}$$

which is defined for continuous functions ϕ, ρ, with square-integrable first derivative, and also square-integrable restriction to N. The associated quadratic form $G_\lambda(\phi,\phi)$ is increasing with λ for generic ϕ and nondecreasing for all ϕ. We therefore expect that the operator associated with this quadratic form, which we call X_λ, will be increasing with λ. X_λ will again be the Laplacian, with some boundary condition, since G_λ coincides with the Hermitian forms considered earlier except for a boundary term.

To identify the boundary condition in X_λ, we observe that in order to have $X_\lambda\phi = \Delta\phi$ for some function ϕ, the identity we need is

$$\langle\Delta\phi|\rho\rangle = G_\lambda(\phi,\rho) = \int_M d^n x\sum_i \frac{\partial\bar{\phi}}{\partial x_i}\frac{\partial\rho}{\partial x_i} + \lambda\int_N \bar{\phi}\rho, \tag{3.56}$$

for all ρ in the domain of G_λ. In trying to prove this identity, we run into a surface term, which now is

$$\int_N d\mu(-\partial_\perp\bar{\phi} + \lambda\bar{\phi})\rho. \tag{3.57}$$

The boundary condition that we need is therefore $-\partial_\perp\phi + \lambda\phi = 0$. The operator X_λ is the Laplacian with this boundary condition.

X_λ coincides with the Neumann Laplacian Δ_N at $\lambda = 0$, and with the Dirichlet Laplacian Δ_D in the limit $\lambda \to \infty$. Since X_λ is increasing with λ, this accounts for the inequality $\Delta_D \geq \Delta_N$.

A more compact way to say some of this is that to go from the Neumann quadratic form to the Dirichlet quadratic form, we impose a constraint on the wave function: it should vanish on the boundary. This naturally increases the energy, so it leads to our inequality.

It is useful, especially with a view to Sec. IV, to consider a somewhat similar situation in finite dimensions. Let X be a positive Hermitian matrix acting on $\mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$. We write

$$X = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}, \tag{3.58}$$

where A and C are blocks of size $n\times n$ and $m\times m$, acting on a column vector $\Psi = \binom{\psi}{\chi}$, with $\psi \in \mathbb{C}^n$, $\chi \in \mathbb{C}^m$. For real $\lambda > 0$, let

$$X_\lambda = \begin{pmatrix} A & B^\dagger \\ B^\dagger & C+\lambda \end{pmatrix}. \tag{3.59}$$

Clearly X_λ is increasing with λ, and in particular, for $s \geq 0$,

$$\frac{1}{s+X} \geq \frac{1}{s+X_\lambda}. \tag{3.60}$$

On the other hand, for very large λ, $1/(s+X_\lambda)$ simplifies, because the upper and lower components decouple:

$$\frac{1}{s+X_\lambda} \sim \begin{pmatrix} 1/(s+A) & \mathcal{O}(1/\lambda) \\ \mathcal{O}(1/\lambda) & 1/\lambda \end{pmatrix}, \qquad \lambda \gg 0. \tag{3.61}$$

The inequality (3.60) means that for any $\Psi \in \mathbb{C}^{n+m}$,

$$\langle\Psi|\frac{1}{s+X}|\Psi\rangle \geq \langle\Psi|\frac{1}{s+X_\lambda}|\Psi\rangle. \tag{3.62}$$

Let us evaluate this for $\Psi = \binom{\psi}{0}$. The right-hand side, for $\lambda\to\infty$, reduces to $\langle\psi|(s+A)^{-1}|\psi\rangle$. If we define an isometric embedding $U:\mathbb{C}^n \to \mathbb{C}^{n+m}$ by $U(\psi) = \binom{\psi}{0}$, then the left-hand side is $\langle\psi|U^\dagger(s+X)^{-1}U|\psi\rangle$. So for $\psi\in\mathbb{C}^n$,

$$\langle\psi|U^\dagger\frac{1}{s+X}U|\psi\rangle \geq \langle\psi|\frac{1}{s+A}|\psi\rangle. \tag{3.63}$$

Integrating over s and using Eq. (3.40), we get

$$\langle\psi|U^\dagger(\log X)U|\psi\rangle \leq \langle\psi|\log A|\psi\rangle. \tag{3.64}$$

Since $A = U^\dagger X U$, this is equivalent to

$$\langle\psi|U^\dagger(\log X)U|\psi\rangle \leq \langle\psi|\log(U^\dagger X U)|\psi\rangle. \tag{3.65}$$

F. The proof

Now we will complete the proof of monotonicity of relative entropy under reducing the size of a region.

Suppose that T is an unbounded, densely defined operator from one Hilbert space $\mathcal{H}$ to a possibly different Hilbert space $\mathcal{H}'$. It is convenient to set $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}'$ and to consider the *graph* Γ of T, which is the set of all vectors $(x, Tx) \in \widehat{\mathcal{H}}$. Γ is obviously a linear subspace of $\widehat{\mathcal{H}}$. The operator T is said to be *closed* if Γ is a closed subspace of $\widehat{\mathcal{H}}$, or equivalently if it is a Hilbert subspace. For Γ to be closed means that if a sequence $(x_n, Tx_n) \in \Gamma$ has a limit $(x, y) \in \widehat{\mathcal{H}}$, then this limit is actually in Γ. In more detail, this amounts to saying that if (x_n, Tx_n) is a sequence of elements of Γ such that both limits

$$x = \lim_{n\to\infty} x_n, \qquad y = \lim_{n\to\infty} Tx_n \tag{3.66}$$

exist, then T is defined on x and $Tx = y$. The reason that in defining the Tomita operator S_Ψ and its relative cousin $S_{\Psi|\Phi}$, we included limit points (3.2) was to ensure that these are closed operators.

If Γ is a closed subspace of a Hilbert space $\widehat{\mathcal{H}}$, then one can define an orthogonal projection $\Pi: \widehat{\mathcal{H}} \to \Gamma$. Π is bounded (with eigenvalues 0,1) and so is defined on all states. Such an orthogonal projection does not exist if Γ is a linear subspace of $\widehat{\mathcal{H}}$ that is not closed.

If Γ is the graph of T, then the orthogonal projector Π onto its graph can be written explicitly as a 2×2 matrix[18] of operators acting on a column vector $\binom{\psi}{\psi}$ with $\psi \in \mathcal{H}$, $\chi \in \mathcal{H}'$:

$$\Pi = \begin{pmatrix} (1+T^\dagger T)^{-1} & (1+T^\dagger T)^{-1}T^\dagger \\ T(1+T^\dagger T)^{-1} & T(1+T^\dagger T)^{-1}T^\dagger \end{pmatrix}. \tag{3.67}$$

It is straightforward to verify that Π is Hermitian and $\Pi^2 = \Pi$, so Π is an orthogonal projection operator. It projects onto the graph of T, since $\Pi\binom{\psi}{\chi} = \binom{\eta}{T\eta}$ with $\eta = (1+T^\dagger T)^{-1}(\psi + T^\dagger\chi)$. Clearly, $\binom{\eta}{T\eta}$ is in the graph of T, and every vector in the graph of T is of this form.

We are finally ready for the proof. Suppose that T_0, T_1 are densely defined operators from $\mathcal{H}$ to $\mathcal{H}'$, with graphs Γ_0 and Γ_1. Let Π_0 and Π_1 be the projectors onto the two graphs. If T_1 is an extension of T_0, then Γ_0 is a subspace of Γ_1. This implies that $\Pi_1 \geq \Pi_0$, so $\langle\Psi|\Pi_1|\Psi\rangle \geq \langle\Psi|\Pi_0|\Psi\rangle$ for any state $\Psi = \binom{\psi}{\chi}$. Specializing to the case $\chi = 0$ and using Eq. (3.67), we get the inequality

$$\langle\psi|\frac{1}{1+T_0^\dagger T_0}|\psi\rangle \leq \langle\psi|\frac{1}{1+T_1^\dagger T_1}|\psi\rangle. \tag{3.68}$$

Repeating this analysis with $T_0/\sqrt{s}$ and $T_1/\sqrt{s}$ instead of T_0 and T_1 for some $s > 0$, we get

$$\langle\psi|\frac{1}{s+T_0^\dagger T_0}|\psi\rangle \leq \langle\psi|\frac{1}{s+T_1^\dagger T_1}|\psi\rangle. \tag{3.69}$$

Thus $T_1^\dagger T_1 \leq T_0^\dagger T_0$ and $\log T_1^\dagger T_1 \leq \log T_0^\dagger T_0$.

Taking $S_{\widetilde{\mathcal{U}}}$ and $S_{\mathcal{U}}$ for T_0 and T_1, this is what we needed to prove Eq. (3.36) and thus the monotonicity of relative entropy. There is perhaps just one more detail to clarify. $S_{\mathcal{U}}$ and $S_{\widetilde{\mathcal{U}}}$ are usually defined as antilinear operators from a Hilbert space $\mathcal{H}$ to itself. However, an antilinear operator from $\mathcal{H}$ to $\mathcal{H}$ is the same as a linear operator from $\mathcal{H}$ to $\overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ is the complex conjugate[19] of the Hilbert space $\mathcal{H}$. So we can regard $S_{\mathcal{U}}$ and $S_{\widetilde{\mathcal{U}}}$ as linear operators $\mathcal{H} \to \mathcal{H}'$, with $\mathcal{H}' = \overline{\mathcal{H}}$, and then the above analysis applies precisely.

We followed Borchers (2000) in this explanation of why $\Delta_{\mathcal{U}}$ increases as the region $\mathcal{U}$ is made smaller. Borchers uses this inequality not to analyze the relative entropy but for another application. The computation involving the projection on the graph is much older (Stone, 1951; Halmos, 1969).

It might be helpful to analyze the graphs Γ_0 and Γ_1 in the example considered in Sec. III.E. In doing this, for simplicity, we work in one dimension, so we take M to be the unit interval [0, 1] on the x axis. The operators T_0 and T_1 reduce to d/dx, acting on functions that are or are not required to vanish at the end points in the case of T_0 or T_1, respectively. The graph Γ_0 consists of pairs $\big(f(x), df(x)/dx\big)$, where f vanishes at the end points, and the graph Γ_1 consists of pairs $\big(g(x), dg(x)/dx\big)$ with no such constraint on g at the end points. We claim that Γ_0 is a proper subspace of Γ_1. To show this, we show that there are pairs $(g, g') \in \Gamma_1$ that are orthogonal to all $(f, f') \in \Gamma_0$. The condition of orthogonality is

$$\int_0^1 dx\bar{f}g + \int_0^1 dx\frac{d\bar{f}}{dx}\frac{dg}{dx} = 0. \tag{3.70}$$

We want to find g such that this is true for all f. The requisite condition is that

$$\left(1 - \frac{d^2}{dx^2}\right)g = 0. \tag{3.71}$$

In verifying that Eq. (3.71) implies Eq. (3.70) for all f, one has to integrate by parts; there is no surface term as f vanishes at the end points. Equation (3.71) has a two-dimensional space of solutions $g(x) = Ae^x + Be^{-x}$, so Γ_0 is of codimension two in Γ_1.

[18] Since Π is bounded, also the operators $(1+T^\dagger T)^{-1}$, $(1+T^\dagger T)^{-1}T^\dagger$, etc., appearing as matrix elements of the following matrix are bounded. In particular these operators are defined on all states. That is actually part of why introducing Π is useful in making a rigorous proof. For example, when we write $\eta = (1+T^\dagger T)^{-1}(\psi + T^\dagger\chi)$, this formula makes sense because, although χ may not be in the domain of $T^\dagger$, it is in the domain of $(1+T^\dagger T)^{-1}T^\dagger$.

[19] The complex conjugate $\bar{\mathcal{H}}$ of a Hilbert space $\mathcal{H}$ is defined as follows. Vectors in $\bar{\mathcal{H}}$ are in 1-1 correspondence with vectors in $\mathcal{H}$. But a complex scalar that acts on $\mathcal{H}$ as multiplication by λ acts on $\bar{\mathcal{H}}$ as multiplication by $\bar{\lambda}$, and inner products in $\bar{\mathcal{H}}$ are complex conjugates of those in $\mathcal{H}$. $\bar{\mathcal{H}}$ satisfies all the axioms of a Hilbert space.

Directly explaining the relation between the unbounded operators T_0 and T_1 is subtle because one has to talk about two dense but nonclosed subspaces of Hilbert space, one of which is larger than the other. Passing to the graphs brings the essential difference into the open, as it now involves a comparison of the Hilbert spaces Γ_0 and Γ_1.

IV. FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

In this section, we will explore the modular operators for finite-dimensional quantum systems and draw some lessons.

A. The modular operators in the finite-dimensional case

In finite dimensions, the interesting case is a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with tensor factors $\mathcal{H}_1$ and $\mathcal{H}_2$. Such a tensor product describes what is called a bipartite quantum system. We let $\mathcal{A}$ be the algebra of linear operators acting on $\mathcal{H}_1$ and $\mathcal{A}'$ the algebra of linear operators acting on $\mathcal{H}_2$. A linear operator $\mathsf{a}:\mathcal{H}_1 \to \mathcal{H}_1$ is taken to act on $\mathcal{H}$ as $\mathsf{a} \otimes 1$, while $\mathsf{a}':\mathcal{H}_2 \to \mathcal{H}_2$ similarly acts on $\mathcal{H}$ as $1 \otimes \mathsf{a}'$. The algebras $\mathcal{A}$ and $\mathcal{A}'$ are each other's commutants, since a linear transformation of $\mathcal{H}$ that commutes with $\mathsf{a} \otimes 1$ for all a is of the form $1 \otimes \mathsf{a}'$, and vice versa. So from Sec. II.F, we know that a vector is cyclic for $\mathcal{A}$ if and only if it is separating for $\mathcal{A}'$, and vice versa.

Any vector $\Psi \in \mathcal{H}$ has an expansion

$$\Psi = \sum_{k=1}^{n} c_k \psi_k \otimes \psi'_k, \tag{4.1}$$

where ψ_k are orthogonal unit vectors in $\mathcal{H}_1$ and ψ'_k are orthogonal unit vectors in $\mathcal{H}_2$. Moreover, we can assume the c_k to be all nonzero (or we could omit some terms from the sum). We have

$$(\mathsf{a} \otimes 1)\Psi = \sum_k c_k \mathsf{a}\psi_k \otimes \psi'_k, \tag{4.2}$$

so $\mathsf{a} \otimes 1$ annihilates Ψ if and only if a annihilates all of the ψ_k. If the ψ_k are a complete basis for $\mathcal{H}_1$, this implies that $\mathsf{a} = 0$; otherwise, there is some nonzero a that annihilates all of the ψ_k. Thus Ψ is separating for the algebra $\mathcal{A}$ if and only if the ψ_k are a basis of $\mathcal{H}_1$; likewise it is separating for $\mathcal{A}'$ if and only if the ψ'_k are a basis for $\mathcal{H}_2$. Since Ψ is cyclic for one algebra if and only if it is separating for the other, it follows that Ψ is cyclic and separating for $\mathcal{A}$ and for $\mathcal{A}'$ precisely if the ψ_k and the ψ'_k are orthonormal bases for their respective spaces. In particular, this is possible precisely if $\mathcal{H}_1$ and $\mathcal{H}_2$ are of equal dimension. Conversely, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are of the same dimension n, then a generic vector $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ has an expansion as in Eq. (4.1) with all c_k nonzero, and thus is cyclic and separating for the two algebras. As a matter of notation, we write $\psi_k = |k\rangle, \psi'_k = |k\rangle'$. We also abbreviate $|j\rangle \otimes |k\rangle'$ as $|j,k\rangle$. Thus

$$\Psi = \sum_{k=1}^{n} c_k |k\rangle|k\rangle' = \sum_{k=1}^{n} c_k |k,k\rangle. \tag{4.3}$$

As a check on some of this, we observe that as $\mathcal{H}_1$ and $\mathcal{H}_2$ have dimension n, $\mathcal{H}$ has dimension n^2. The algebras $\mathcal{A}$ and $\mathcal{A}'$ are algebras of $n \times n$ matrices, so they likewise are of dimension n^2. So the linear map $\mathcal{A} \to \mathcal{H}$ that takes $\mathsf{a} \in \mathcal{A}$ to $(\mathsf{a} \otimes 1)\Psi \in \mathcal{H}$ is surjective if and only if it has trivial kernel. In other words, Ψ is separating for $\mathcal{A}$ if and only if it is cyclic. Both properties are true precisely if the c_k are all nonzero.

We want to find the modular operators in this situation. The definition of $S_\Psi:\mathcal{H} \to \mathcal{H}$ is

$$S_\Psi((\mathsf{a} \otimes 1)\Psi) = (\mathsf{a}^\dagger \otimes 1)\Psi. \tag{4.4}$$

To work out the consequences of this, pick some i and j in the set $\{1, 2, \ldots, n\}$, and let a be the elementary matrix that acts on $\mathcal{H}_1$ by

$$\mathsf{a}|i\rangle = |j\rangle, \qquad \mathsf{a}|k\rangle = 0 \quad \text{if } k \neq i. \tag{4.5}$$

Its adjoint acts by

$$\mathsf{a}^\dagger|j\rangle = |i\rangle, \qquad \mathsf{a}^\dagger|k\rangle = 0 \quad \text{if } k \neq j. \tag{4.6}$$

So

$$(\mathsf{a} \otimes 1)\Psi = c_i|j,i\rangle, \qquad (\mathsf{a}^\dagger \otimes 1)\Psi = c_j|i,j\rangle. \tag{4.7}$$

Thus the definition of S_Ψ implies

$$S_\Psi(c_i|j,i\rangle) = c_j|i,j\rangle. \tag{4.8}$$

Recalling that S_Ψ is supposed to be antilinear, this implies

$$S_\Psi|j,i\rangle = \frac{c_j}{\bar{c}_i}|i,j\rangle. \tag{4.9}$$

That gives a complete description of S_Ψ, since the states $|i,j\rangle$ are a basis of $\mathcal{H}$. The adjoint $S_\Psi^\dagger$ acts by

$$S_\Psi^\dagger|i,j\rangle = \frac{c_j}{\bar{c}_i}|j,i\rangle. \tag{4.10}$$

The modular operator $\Delta_\Psi = S_\Psi^\dagger S_\Psi$ hence acts by

$$\Delta_\Psi|j,i\rangle = \frac{|c_j|^2}{|c_i|^2}|j,i\rangle. \tag{4.11}$$

To get this formula, one must recall that $S_\Psi^\dagger$ is antilinear.

We also want to find the antiunitary operator J_Ψ that appears in the polar decomposition $S_\Psi = J_\Psi \Delta_\Psi^{1/2}$. Since

$$\Delta_\Psi^{1/2}|j,i\rangle = \sqrt{\frac{|c_j|^2}{|c_i|^2}}|j,i\rangle, \tag{4.12}$$

we have

$$J_\Psi|j,i\rangle = \sqrt{\frac{c_j c_i}{\bar{c}_j \bar{c}_i}}|i,j\rangle. \tag{4.13}$$

If Φ is a second state in $\mathcal{H}$, we can work out in a simple way the relative operators $S_{\Psi|\Phi}$ and $\Delta_{\Psi|\Phi}$. In some orthonormal bases ϕ_α of $\mathcal{H}_1$ and ϕ'_α of $\mathcal{H}_2$, $\alpha = 1, \ldots, n$, we have

$$\Phi = \sum_{\alpha=1}^{n} d_\alpha \phi_\alpha \otimes \phi'_\alpha, \tag{4.14}$$

with some coefficients d_α. We write $|\alpha\rangle$ and $|\alpha\rangle'$ for ϕ_α and ϕ'_α, and abbreviate $|\alpha\rangle \otimes |\beta\rangle' = |\alpha,\beta\rangle$, and similarly $|\alpha\rangle \otimes |i\rangle' = |\alpha, i\rangle$, $|i\rangle \otimes |\alpha\rangle' = |i,\alpha\rangle$, etc. The state Φ is cyclic and separating for both algebras if and only if the d_α are all nonzero; we do not assume this. We will determine the operator $S_{\Psi|\Phi}$ directly from the definition

$$S_{\Psi|\Phi}((\mathsf{a}\otimes 1)\Psi) = (\mathsf{a}^\dagger \otimes 1)\Phi, \quad \forall\ \mathsf{a}\in\mathcal{A}. \tag{4.15}$$

For some $i,\alpha \in \{1,2,\ldots,n\}$, suppose that $\mathsf{a}\in\mathcal{A}$ acts by

$$\mathsf{a}|i\rangle = |\alpha\rangle, \qquad \mathsf{a}|j\rangle = 0 \quad \text{for } j\neq i. \tag{4.16}$$

Then

$$\mathsf{a}^\dagger|\alpha\rangle = |i\rangle, \qquad \mathsf{a}^\dagger|\beta\rangle = 0 \quad \text{for } \beta\neq\alpha. \tag{4.17}$$

So

$$(\mathsf{a}\otimes 1)\Psi = c_i|\alpha,i\rangle, \qquad (\mathsf{a}^\dagger\otimes 1)\Phi = d_\alpha|i,\alpha\rangle. \tag{4.18}$$

Accordingly

$$S_{\Psi|\Phi}|\alpha,i\rangle = \frac{d_\alpha}{\bar{c}_i}|i,\alpha\rangle. \tag{4.19}$$

The adjoint is characterized by

$$S^\dagger_{\Psi|\Phi}|i,\alpha\rangle = \frac{d_\alpha}{\bar{c}_i}|\alpha,i\rangle. \tag{4.20}$$

It follows that

$$\Delta_{\Psi|\Phi}|\alpha,i\rangle = \frac{|d_\alpha|^2}{|c_i|^2}|\alpha,i\rangle. \tag{4.21}$$

Some of these formulas can be conveniently described in terms of density matrices. Let us assume that Ψ, Φ are unit vectors:

$$\sum_i |c_i|^2 = \sum_\alpha |d_\alpha|^2 = 1. \tag{4.22}$$

To the state $\Psi\in\mathcal{H}_1\otimes\mathcal{H}_2$, one associates a density matrix $\rho_{12} = |\Psi\rangle\langle\Psi|$. It is a matrix acting on $\mathcal{H}$ by $|\chi\rangle \to |\Psi\rangle\langle\Psi|\chi\rangle$; in other words it is the projection operator onto the subspace generated by $|\Psi\rangle$. In particular, it is positive and has trace 1:

$$\mathrm{Tr}_{12}\rho_{12} = 1. \tag{4.23}$$

Here Tr_{12} represents the trace over $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2$. By taking a partial trace over $\mathcal{H}_2$ or $\mathcal{H}_1$, one defines reduced density matrices $\rho_1 = \mathrm{Tr}_2\rho_{12}$, $\rho_2 = \mathrm{Tr}_1\rho_{12}$. Here ρ_1 and ρ_2 are positive matrices acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. They have trace 1 since for example $\mathrm{Tr}_1\rho_1 = \mathrm{Tr}_1\mathrm{Tr}_2\rho_{12} = \mathrm{Tr}_{12}\rho_{12} = 1$. Likewise, one defines a density matrix $\sigma_{12} = |\Phi\rangle\langle\Phi|$ associated to Φ and reduced density matrices $\sigma_1 = \mathrm{Tr}_2\sigma_{12}$, $\sigma_2 = \mathrm{Tr}_1\sigma_{12}$, all positive and of trace 1.

For the state Ψ defined in Eq. (4.1), the corresponding reduced density matrices are

$$\rho_1 = \sum_i |c_i|^2 |i\rangle\langle i|, \qquad \rho_2 = \sum_i |c_i|^2 |i\rangle'\langle i|'. \tag{4.24}$$

Clearly, ρ_1 and ρ_2 are invertible if and only if the c_i are all nonzero, that is if and only if Ψ is cyclic separating for both algebras. Similarly the reduced density matrices of Φ are

$$\sigma_1 = \sum_\alpha |d_\alpha|^2 |\alpha\rangle\langle\alpha|, \qquad \sigma_2 = \sum_\alpha |d_\alpha|^2 |\alpha\rangle'\langle\alpha|'. \tag{4.25}$$

Comparing these formulas to Eqs. (4.11) and (4.21), the modular operator Δ_Ψ and the relative modular operator $\Delta_{\Psi|\Phi}$ can be conveniently written in terms of the reduced density matrices:

$$\Delta_\Psi = \rho_1\otimes\rho_2^{-1}, \qquad \Delta_{\Psi|\Phi} = \sigma_1\otimes\rho_2^{-1}. \tag{4.26}$$

The density matrix ρ_2 is conjugate to ρ_1 under the exchange $|i\rangle \leftrightarrow |i\rangle'$, and similarly for σ_1 and σ_2.

It can be convenient to pick the phases of the states $|i\rangle'$ relative to $|i\rangle$ to ensure that the c_i are all positive. If we do this, the antiunitary operator J_Ψ becomes a simple flip:

$$J_\Psi|i,j\rangle = |j,i\rangle. \tag{4.27}$$

The existence of a natural antiunitary operator J_Ψ that flips the two bases in this way suggests that it is natural (once a cyclic separating state Ψ is given) to identify $\mathcal{H}_2$ as the dual of $\mathcal{H}_1$, by thinking of an element of $\mathcal{H}_1$ in the basis $|i\rangle$ as a column vector and an element of $\mathcal{H}_2$ in the basis $|i\rangle'$ as a row vector. Then an element of $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2$ is regarded as an $n\times n$ matrix, acting on $\mathcal{H}_1$. The Hilbert space inner product of $\mathcal{H}$ is interpreted in terms of matrices $x, y : \mathcal{H}_1 \to \mathcal{H}_1$ as

$$\langle x|y\rangle = \mathrm{Tr}_{\mathcal{H}_1} x^\dagger y. \tag{4.28}$$

The action of $\mathsf{a}\in\mathcal{A}$ on $\mathcal{H}$ becomes

$$x \to \mathsf{a}x \tag{4.29}$$

and the action of $\mathsf{a}'\in\mathcal{A}'$ on $\mathcal{H}$ becomes

$$x \to x\mathsf{a}'^{\mathrm{tr}}, \tag{4.30}$$

where b^{tr} is the transpose of a matrix b. With states reinterpreted in this way as matrices, the state Ψ becomes

$$\Psi = \rho_1^{1/2}. \tag{4.31}$$

This follows upon comparing Eqs. (4.1) and (4.24), remembering that we now take the c_k to be positive and interpret $\psi_k \otimes \psi_k'$ as a matrix $|k\rangle\langle k|$.

When states are reinterpreted as matrices, Eq. (4.26) for the action of $\Delta_{\Psi|\Phi}$ on a state x becomes $\Delta_{\Psi|\Phi}(x) = \sigma_1 x (\rho_2^{\rm tr})^{-1}$. But once we identify $\mathcal{H}_2$ as the dual of $\mathcal{H}_1$, $\rho_2^{\rm tr} = \rho_1$ so

$$\Delta_{\Psi|\Phi}(x) = \sigma_1 x \rho_1^{-1}. \tag{4.32}$$

For future reference, we note that this implies

$$\Delta^{\alpha}_{\Psi|\Phi}(x) = \sigma_1^{\alpha} x \rho_1^{-\alpha}, \tag{4.33}$$

leading to a formula that will be useful later:

$$\begin{aligned}\langle\Psi|\Delta^{\alpha}_{\Psi|\Phi}|\Psi\rangle &= {\rm Tr}_{\mathcal{H}_1}\rho_1^{1/2}\Delta^{\alpha}_{\Psi|\Phi}(\rho_1^{1/2}) \\ &= {\rm Tr}_{\mathcal{H}_1}\rho_1^{1/2}\sigma_1^{\alpha}\rho_1^{1/2}\rho_1^{-\alpha} = {\rm Tr}_{\mathcal{H}_1}\sigma_1^{\alpha}\rho_1^{1-\alpha}.\end{aligned} \tag{4.34}$$

The identification of $\mathcal{H}_2$ with the dual of $\mathcal{H}_1$ depended on the choice of a cyclic separating vector Ψ, so we do not automatically get an equally simple relation between Φ and its reduced density matrices σ_1 and σ_2. However, if we are only interested in σ_1 and not σ_2, we can act on Φ with a *unitary* element of $\mathcal{A}'$ without changing σ_1. In general, once we identify $\mathcal{H}$ with the space of matrices acting on $\mathcal{H}_1$, Φ corresponds to such a matrix. As such it has a polar decomposition $\Phi = PU$, where P is positive and U is unitary. In general $P = \sigma_1^{1/2}$. Acting with a unitary element of $\mathcal{A}'$ to eliminate U, one reduces to $\Phi = \sigma_1^{1/2}$.

B. The modular automorphism group

All of the properties of the operators S_Ψ, Δ_Ψ, etc., that we deduced in general in Secs. III.A and III.B are of course still true in this finite-dimensional setting.

However, some important additional properties are now more transparent. Most of these involve what is called the modular automorphism group. This is the group of unitary transformations of the form Δ_Ψ^{is}, $s \in \mathbb{R}$. We already know [Eq. (3.19)] that Δ_Ψ^{is} commutes with J_Ψ. In the finite-dimensional setting, we have the explicit formula (4.26) for Δ_Ψ. By virtue of this formula, $\Delta_\Psi^{is} = \rho_1^{is} \otimes \rho_2^{-is}$. So for any $\mathsf{a} \otimes 1 \in \mathcal{A}$,

$$\Delta_\Psi^{is}(\mathsf{a}\otimes 1)\Delta_\Psi^{-is} = \rho_1^{is}\mathsf{a}\rho_1^{-is}\otimes 1. \tag{4.35}$$

The important fact here is that the right-hand side of Eq. (4.35) is of the form $\mathsf{b}\otimes 1$ for some b, so it is in $\mathcal{A}$. In other words, conjugation by the modular group maps $\mathcal{A}$ to itself. It similarly maps $\mathcal{A}'$ to itself. We summarize this as

$$\Delta_\Psi^{is}\mathcal{A}\Delta_\Psi^{-is} = \mathcal{A}, \qquad \Delta_\Psi^{is}\mathcal{A}'\Delta_\Psi^{-is} = \mathcal{A}'. \tag{4.36}$$

On the other hand, conjugation by J_Ψ exchanges the two algebras $\mathcal{A}$ and $\mathcal{A}'$:

$$J_\Psi\mathcal{A}J_\Psi = \mathcal{A}', \qquad J_\Psi\mathcal{A}'J_\Psi = \mathcal{A}. \tag{4.37}$$

For example, if we choose the phases of the states so that J_Ψ flips basis vectors $|i,j\rangle$ as in Eq. (4.27), then $J_\Psi(\mathsf{a}\otimes 1)J_\Psi = 1\otimes \mathsf{a}^*$ (where a^* is the complex conjugate matrix to a) and likewise $J_\psi(1\otimes\mathsf{a})J_\Psi = \mathsf{a}^*\otimes 1$.

The group of unitary transformations $\Delta_{\Psi|\Phi}^{is}$, $s\in\mathbb{R}$, is called the relative modular group. In the finite-dimensional setting, Eq. (4.26) leads to

$$\Delta_{\Psi|\Phi}^{is}(\mathsf{a}\otimes 1)\Delta_{\Psi|\Phi}^{-is} = \sigma_1^{is}\mathsf{a}\sigma_1^{-is}\otimes 1. \tag{4.38}$$

Again, conjugation by the relative modular group maps $\mathcal{A}$ (or $\mathcal{A}'$) to itself. But now we see the additional important property that this conjugation depends only on Φ and not on Ψ. Thus if Ψ and Ψ' are two cyclic separating vectors, we have

$$\Delta_{\Psi|\Phi}^{is}(\mathsf{a}\otimes 1)\Delta_{\Psi|\Phi}^{-is} = \Delta_{\Psi'|\Phi}^{is}(\mathsf{a}\otimes 1)\Delta_{\Psi'|\Phi}^{-is}. \tag{4.39}$$

The properties just stated are regarded as the main theorems of Tomita-Takesaki theory. For general infinite-dimensional von Neumann algebras with cyclic separating vectors, these properties are not so easy to prove. However, there is a relatively simple proof (Longo, 1978) in the case of an infinite-dimensional algebra $\mathcal{A}$ that is a limit of matrix algebras. This is believed to be the case in quantum field theory for the algebra $\mathcal{A}_\mathcal{U}$ associated to an open set $\mathcal{U}$ in spacetime. The statement means roughly that one can think of the degrees of freedom in region $\mathcal{U}$ as an infinite collection of qubits. Taking just n of these qubits, one gets an algebra M_n of $2^n\times 2^n$ matrices that is an approximation of $\mathcal{A}_\mathcal{U}$. Adding qubits, one gets an ascending chain of algebras $M_1\subset M_2\subset\cdots\subset M_n\subset\cdots\subset\mathcal{A}_\mathcal{U}$ with $\mathcal{A}_\mathcal{U}$ as its limit.[20] It is believed that this picture is rigorously valid in quantum field theory. At each finite step in the chain, one defines an approximation[21] $\Delta_\Psi^{(n)}$ to the modular operator (or similarly to J_Ψ or $\Delta_{\Psi|\Phi}$). Each such approximation obeys Eq. (4.35), and the nature of this statement is such that if it is true at each step, it remains true in the limit. Of course the main point of the proof is to show that $\Delta_\Psi^{(n)}$ does in an appropriate sense converge to Δ_Ψ.

Similarly the statements (4.37) and (4.39) have the property that if true in a sequence of approximations, they remain true in any reasonable limit. So one should expect these statements to hold in quantum field theory.

The infinite-dimensional case becomes essentially different from a finite-dimensional matrix algebra when one considers the behavior of Δ_Ψ^{is} (or $\Delta_{\Psi|\Phi}^{is}$) when s is no longer real. For a matrix algebra, there is no problem; $\Delta_\Psi^{iz} = \exp(iz\log\Delta_\Psi)$ is an entire matrix-valued function of z. In quantum field theory, Δ_Ψ is unbounded and the analytic properties of $\Delta_\Psi^{iz}\chi$ for a state χ depend very much on χ. By taking spectral projections, we can find states χ such that $\Delta_\Psi^{iz}\chi$ is entire in z, just as in

[20] We will discuss algebras defined in this way in Sec. VI.

[21] This is done as follows. If $\Psi\in\mathcal{H}$ is a cyclic separating vector, then for each n, $\mathcal{H}_n = M_n\Psi$ is a subspace of $\mathcal{H}$ of dimension 2^{2n}. M_n acts on $\mathcal{H}_n$ with cyclic separating vector Ψ, so one can define the modular operator $\Delta_\Psi^{\langle n\rangle}:\mathcal{H}_n\to\mathcal{H}_n$. One defines $\Delta_\Psi^{(n)}:\mathcal{H}\to\mathcal{H}$ to coincide with $\Delta_\Psi^{\langle n\rangle}$ on $\mathcal{H}_n$ and to equal 1 on the orthocomplement.

Sec. II.C we found vectors on which $\exp(ic\cdot P)$ acts holomorphically. At the opposite extreme, we can also find states χ on which $\Delta_\Psi^{iz}\chi$ can only be defined if z is real.

Frequently, however, we are interested in the action of Δ_Ψ on a vector $\mathsf{a}\Psi$, $\mathsf{a}\in\mathcal{A}$ (or $\mathsf{a}'\Psi$, $\mathsf{a}'\in\mathcal{A}'$). Here we have some simple holomorphy. First of all, $\Delta_\Psi^{1/2}\mathsf{a}\Psi$ has finite norm and so makes sense as a Hilbert space vector:

$$\begin{aligned}|\Delta_\Psi^{1/2}\mathsf{a}\Psi|^2 &= \langle\Delta_\Psi^{1/2}\mathsf{a}\Psi|\Delta_\Psi^{1/2}\mathsf{a}\Psi\rangle = \langle\mathsf{a}\Psi|\Delta_\Psi|\mathsf{a}\Psi\rangle\\ &= \langle\mathsf{a}\Psi|S_\Psi^\dagger S_\Psi|\mathsf{a}\Psi\rangle\\ &= \overline{\langle S\mathsf{a}\Psi|S\mathsf{a}\Psi\rangle} = \overline{\langle\mathsf{a}^\dagger\Psi|\mathsf{a}^\dagger\Psi\rangle} < \infty.\end{aligned}\tag{4.40}$$

On the other hand, for $0\le r\le 1$, the inequality $\lambda^r<\lambda+1$ for a positive real number λ implies $\Delta_\Psi^r<\Delta_\Psi+1$. So

$$\langle\Delta_\Psi^{r/2}\mathsf{a}\Psi|\Delta_\Psi^{r/2}\mathsf{a}\Psi\rangle < \langle\Delta_\Psi^{1/2}\mathsf{a}\Psi|\Delta_\Psi^{1/2}\mathsf{a}\Psi\rangle + \langle\mathsf{a}\Psi|\mathsf{a}\Psi\rangle<\infty,\qquad 0\le r\le 1.\tag{4.41}$$

The unitary operator Δ_Ψ^{is}, $s\in\mathbb{R}$ does not change the norm of a state so $\Delta_\Psi^{is}\Delta_\Psi^{r/2}\mathsf{a}\Psi$ also has finite norm for $s\in\mathbb{R}$, $0\le r\le 1/2$. The upshot of this is that $\Delta_\Psi^{iz}\mathsf{a}\Psi$ is continuous in the strip $0\ge \mathrm{Im}z\ge -1/2$ and holomorphic in the interior of the strip. Replacing $\mathcal{A}$ with $\mathcal{A}'$ has the effect of replacing the modular operator Δ_Ψ with its inverse, as we learned in Sec. III.A, so $\Delta_\Psi^{iz}\mathsf{a}'\Psi$ is continuous in the strip $1/2\ge\mathrm{Im}z\ge 0$ and holomorphic in the interior of the strip.

In Sec. V, we find in a basic quantum field theory example that the holomorphy statements that we just made are the best possible: generically, $\Delta^z\mathsf{a}\Psi$ and $\Delta^z\mathsf{a}'\Psi$ cannot be continued outside the strips that we identified.

Now for $\mathsf{a},\mathsf{b}\in\mathcal{A}$, let us look at the analytic properties of the function

$$F(z)=\langle\Psi|\mathsf{b}\Delta_\Psi^{iz}\mathsf{a}|\Psi\rangle,\tag{4.42}$$

initially defined for real z. If $z=s-ir$, this is

$$\langle\mathsf{b}^\dagger\Psi|\Delta_\Psi^{is}\Delta_\psi^r\mathsf{a}|\Psi\rangle=\langle\Delta_\Psi^{r/2}\mathsf{b}^\dagger\Psi|\Delta_\Psi^{is}|\Delta_\Psi^{r/2}\mathsf{a}\Psi\rangle.\tag{4.43}$$

For $r\le 1$, the states $\Delta_\Psi^{r/2}\mathsf{a}\Psi$ and $\Delta_\Psi^{r/2}\mathsf{b}^\dagger\Psi$ are normalizable, as we have already discussed. So the function $F(z)$ is continuous in the strip $0\ge\mathrm{Im}z\ge -1$ and holomorphic in the interior of the strip. On the upper boundary of the strip, we have

$$F(s)=\langle\Psi|\mathsf{b}\Delta_\Psi^{is}\mathsf{a}|\Psi\rangle.\tag{4.44}$$

Let us determine the boundary values on the lower boundary of the strip. We have

$$\begin{aligned}F(-i+s)&=\langle\Psi|\mathsf{b}\Delta_\Psi^{1+is}\mathsf{a}|\Psi\rangle=\langle\Delta_\Psi^{1/2}\mathsf{b}^\dagger\Psi|\Delta_\Psi^{is}|\Delta_\Psi^{1/2}\mathsf{a}\Psi\rangle\\ &=\langle J_\Psi S_\Psi\mathsf{b}^\dagger\Psi|\Delta_\Psi^{is}|J_\Psi S_\Psi\mathsf{a}\Psi\rangle\\ &=\langle J_\Psi\mathsf{b}\Psi|\Delta_\Psi^{is}|J_\Psi\mathsf{a}^\dagger\Psi\rangle=\langle J_\Psi\mathsf{b}\Psi|J_\Psi\Delta_\Psi^{is}\mathsf{a}^\dagger\Psi\rangle\\ &=\langle\Delta_\Psi^{is}\mathsf{a}^\dagger\Psi|\mathsf{b}\Psi\rangle=\langle\Psi|\mathsf{a}\Delta_\Psi^{-is}\mathsf{b}|\Psi\rangle.\end{aligned}\tag{4.45}$$

We used the fact that J_Ψ is antiunitary and commutes with Δ_Ψ^{is}.

To understand what these statements mean for a finite-dimensional quantum system with $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ and $\mathcal{A}$ acting on the first factor, consider again the density matrix $\rho_{12}=|\Psi\rangle\langle\Psi|$ and the reduced density matrix $\rho_1=\mathrm{Tr}_2\rho_{12}$. The "modular Hamiltonian" H is defined by $\rho_1=\exp(-H)$. In the definition of $F(z)$, $\Delta_\Psi^{iz}\mathsf{a}\Psi$ can be replaced by $\Delta_\Psi^{iz}\mathsf{a}\Delta_\Psi^{-iz}\Psi$ since $\Delta_\Psi\Psi=\Psi$. As in Eq. (4.35), $\Delta_\Psi^{iz}\mathsf{a}\Delta_\Psi^{-iz}\Psi=\rho_1^{iz}\mathsf{a}\rho_1^{-iz}\Psi=e^{-izH}\mathsf{a}e^{izH}\Psi$. Moreover, for any $\mathcal{O}$ that acts on $\mathcal{H}_1$, $\langle\Psi|\mathcal{O}|\Psi\rangle=\mathrm{Tr}_{\mathcal{H}_1}\rho_1\mathcal{O}=\mathrm{Tr}_{\mathcal{H}_1}e^{-H}\mathcal{O}$. Hence

$$F(z)=\mathrm{Tr}_{\mathcal{H}_1}e^{-H}\mathsf{b}e^{-izH}\mathsf{a}e^{izH}.\tag{4.46}$$

From this it is clear that the values for $z=s$ and $z=-i+s$ are

$$\begin{aligned}F(s)&=\mathrm{Tr}_{\mathcal{H}_1}e^{-H}\mathsf{b}e^{-isH}\mathsf{a}e^{isH},\\ F(-i+s)&=\mathrm{Tr}_{\mathcal{H}_2}e^{-H}e^{-isH}\mathsf{a}e^{isH}\mathsf{b}.\end{aligned}\tag{4.47}$$

In the usual physical interpretation, s represents real time, $\mathsf{a}(s)=e^{-isH}\mathsf{a}e^{isH}$ is a Heisenberg operator at time $-s$, and these functions are real time two-point functions in a thermal ensemble with Hamiltonian H (and inverse temperature 1), with different operator orderings. The fact that the different operator orderings can be obtained from each other by analytic continuation is important, for example, in the derivation of a general bound on quantum chaos (Maldacena, Shenker, and Stanford, 2016), and in many other applications.

For a finite-dimensional quantum system, $F(z)$ is an entire function. Let us, however, relax the assumption of finite dimensionality, while still assuming a factorization $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ of the Hilbert space. The definition $\rho_1=e^{-H}$ implies that H is non-negative, but in the infinite-dimensional case, H is inevitably unbounded above, given that $\mathrm{Tr}\rho_1=1$. For the trace in Eq. (4.46) to be well behaved, given that H is unbounded above, both iz and $1-iz$ must have non-negative real part. This leads to the strip $0\ge\mathrm{Im}z\ge -1$, which we identified earlier without assuming the factorization $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$.

Assuming the factorization $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$, one would actually predict further holomorphy of correlation functions. For example, generalizing Eq. (4.46), a three-point function

$$F(z_1,z_2)=\mathrm{Tr}_{\mathcal{H}_1}e^{-H}\mathsf{c}e^{-iz_1H}\mathsf{b}e^{-i(z_2-z_1)H}\mathsf{a}e^{iz_2H}\tag{4.48}$$

should be holomorphic for $\mathrm{Im}z_1$, $\mathrm{Im}(z_2-z_1)$, $-1-\mathrm{Im}z_2<0$. Such statements can actually be proved without assuming a factorization of the Hilbert space. See Sec. III of Araki (1973) and also Appendix A.2.

All statements we have made about holomorphy still apply if Δ_Ψ is replaced by the relative modular operator $\Delta_{\Psi|\Phi}$.

C. Monotonicity of relative entropy in the finite-dimensional case

Using results of Sec. IV.A, we can compare Araki's definition of relative entropy, which we used in discussing quantum field theory, to the standard definition in nonrelativistic quantum mechanics.

We recall that Araki's definition for the relative entropy between two states Ψ, Φ, for measurements in a spacetime region $\mathcal{U}$, is

$$\mathcal{S}_{\Psi|\Phi;\mathcal{U}} = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle. \tag{4.49}$$

Here Ψ is a cyclic separating vector for a pair of commuting algebras $\mathcal{A}_{\mathcal{U}}$, $\mathcal{A}'_{\mathcal{U}}$.

In nonrelativistic quantum mechanics, we do not in general associate algebras with spacetime regions. But we do have the notion of a vector Ψ that is cyclic separating for a commuting pair of algebras $\mathcal{A}$, $\mathcal{A}'$. Given a second vector Φ we have the relative modular operator $\Delta_{\Psi|\Phi}$. Given this, we could imitate in nonrelativistic quantum mechanics Araki's definition, which in terms of the density matrix $\rho_{12} = |\Psi\rangle\langle\Psi|$ is

$$\mathcal{S}_{\Psi|\Phi} = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle = -\mathrm{Tr}_{12}\rho_{12}\log\Delta_{\Psi|\Phi}. \tag{4.50}$$

From Eq. (4.26), $\Delta_{\Psi|\Phi} = \sigma_1\otimes\rho_2^{-1}$, so $\log\Delta_{\Psi|\Phi} = \log\sigma_1\otimes 1 - 1\otimes\log\rho_2$. The relative entropy is then

$$\mathcal{S}_{\Psi|\Phi} = -\mathrm{Tr}_{12}\rho_{12}(\log\sigma_1\otimes 1 - 1\otimes\log\rho_2). \tag{4.51}$$

Here $\mathrm{Tr}\rho_{12}(\log\sigma_1\otimes 1) = \mathrm{Tr}_1\rho_1\log\sigma_1$, as one learns by first taking the trace over $\mathcal{H}_2$. Likewise $\mathrm{Tr}\rho_{12}(1\otimes\log\rho_2) = \mathrm{Tr}_2\rho_2\log\rho_2$. But ρ_1 and ρ_2 are conjugate as explained at the end of Sec. IV.A, so $\mathrm{Tr}_2\rho_2\log\rho_2 = \mathrm{Tr}_1\rho_1\log\rho_1$. Finally then

$$\mathcal{S}_{\Psi|\Phi} = \mathrm{Tr}\rho_1(\log\rho_1 - \log\sigma_1). \tag{4.52}$$

We have arrived at the usual definition of the relative entropy in nonrelativistic quantum mechanics. (Of course, that was Araki's motivation.) The usual approach runs in reverse from what we have said. One starts with a Hilbert space $\mathcal{H}_1$ and two density matrices ρ_1 and σ_1. The relative entropy between them is defined as

$$\mathcal{S}(\rho_1||\sigma_1) = \mathrm{Tr}\rho_1(\log\rho_1 - \log\sigma_1). \tag{4.53}$$

After introducing a second Hilbert space $\mathcal{H}_2$, ρ_1 and σ_1 can be "purified" by deriving them as the reduced density matrices of pure states $\Psi, \Phi\in\mathcal{H}_1\otimes\mathcal{H}_2$. The above formulas make clear that $\mathcal{S}(\rho_1||\sigma_1)$ is the same as $\mathcal{S}_{\Psi|\Phi}$.

Now let us discuss properties of the relative entropy. Using the definition (4.50), the proof of positivity of relative entropy that was described in Sec. III.C carries over immediately to nonrelativistic quantum mechanics.

There is also an analog in nonrelativistic quantum mechanics of the more subtle property of monotonicity of relative entropy. We will recall the statement and then explain how it can be understood in a way similar to what we explained for quantum field theory in Sec. III. In fact, although we explained the idea in Sec. III in the context of quantum field theory, Araki's point of view was general enough to encompass nonrelativistic quantum mechanics. In our explanation, we follow Petz (1986), later elaborated by Nielsen and Petz (2005), who developed an approach based in part on Araki's framework.

To formulate the problem of monotonicity of relative entropy, the first step is to take what we have been calling $\mathcal{H}_1$ to be the Hilbert space of a bipartite system AB. If $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of systems A and B, then the Hilbert space of the combined system AB is $\mathcal{H}_A\otimes\mathcal{H}_B$. In what follows, we call this $\mathcal{H}_{AB}$ rather than $\mathcal{H}_1$. If we are given density matrices ρ_{AB} and σ_{AB} on $\mathcal{H}_{AB}$, then we can define the reduced density matrices $\rho_A = \mathrm{Tr}_B\rho_{AB}$ and $\sigma_A = \mathrm{Tr}_B\sigma_{AB}$ on $\mathcal{H}_A$, and the relative entropies $\mathcal{S}(\rho_{AB}||\sigma_{AB})$ and $\mathcal{S}(\rho_A||\sigma_A)$. Monotonicity of relative entropy is the statement[22]

$$\mathcal{S}(\rho_{AB}||\sigma_{AB}) \geq \mathcal{S}(\rho_A||\sigma_A). \tag{4.54}$$

We want to explain how this inequality can be understood in a way similar to what we said in the quantum field theory case in Sec. III. In proving this inequality, we assume that ρ_{AB} (and therefore ρ_A) is invertible. The general case can be reached from this case by a limit.

In quantum field theory, the starting point was to study two open sets $\mathcal{U}$, $\widetilde{\mathcal{U}}$ with $\widetilde{\mathcal{U}}\subset\mathcal{U}$. We associated to them algebras $\mathcal{A}_{\mathcal{U}}$, $\mathcal{A}_{\widetilde{\mathcal{U}}}$. For the bipartite system AB, we can introduce two algebras that will play a somewhat similar role. These algebras will be simply the algebras of matrices acting on $\mathcal{H}_{AB}$ and $\mathcal{H}_A$, respectively. We write $\mathcal{A}_{AB}$ and $\mathcal{A}_A$ for these algebras.

In the quantum field theory case, the smaller algebra $\mathcal{A}_{\widetilde{\mathcal{U}}}$ is naturally a subalgebra of $\mathcal{A}_{\mathcal{U}}$. The closest analog of this in nonrelativistic quantum mechanics is that there is a natural embedding $\varphi:\mathcal{A}_A\to\mathcal{A}_{AB}$ by $\mathsf{a}\to\varphi(\mathsf{a}) = \mathsf{a}\otimes 1$.

By passing from $\mathcal{H}_{AB}$ to a doubled Hilbert space $\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$, we can "purify" ρ_{AB} and σ_{AB}, in the sense of deriving them as reduced density matrices on $\mathcal{H}_{AB}$ associated to pure states[23] $\Psi_{AB}, \Phi_{AB}\in\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$. Since we assume ρ_{AB} to be invertible, Ψ_{AB} is cyclic separating. Likewise, ρ_A and σ_A are reduced density matrices associated to pure states Ψ_A, Φ_A in a doubled Hilbert space $\mathcal{H}_A\otimes\mathcal{H}'_A$, and Ψ_A is cyclic separating.

In quantum field theory, the two algebras $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\widetilde{\mathcal{U}}}$ naturally act on the same Hilbert space $\mathcal{H}$ with the same cyclic separating vector Ψ. In nonrelativistic quantum mechanics, it is more natural for the smaller algebra $\mathcal{A}_A$ to act on the smaller Hilbert space $\mathcal{H}_A\otimes\mathcal{H}'_A$, while the larger algebra $\mathcal{A}_{AB}$ acts on $\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$. The best we can do in nonrelativistic quantum mechanics to imitate the idea that $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\widetilde{\mathcal{U}}}$ act on the same space is to find a suitable isometric embedding

$$U:\mathcal{H}_A\otimes\mathcal{H}'_A\to\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}. \tag{4.55}$$

The embedding that will enable us to imitate what we had in quantum field theory is

$$U(\mathsf{a}\Psi_A) = (\mathsf{a}\otimes 1)\Psi_{AB}. \tag{4.56}$$

[22]This is the version of monotonicity of relative entropy proved by Lieb and Ruskai (1973). A more general version of Uhlmann (1977) involves an arbitrary quantum channel. It can be reduced to what is stated here by considering the Stinespring dilation of the channel.

[23]The reader may wish to consult Nielsen and Petz (2005), where they make the specific choice $\Psi_{AB} = \rho_{AB}^{1/2}$, $\Psi_A = \rho_A^{1/2}$, etc., as in Eq. (4.31). This leads to short and explicit formulas. The approach below aims to draw out the analogy with the quantum field theory case. See also Narnhofer and Thirring (1985) and Ghosh and Raju (2017) for somewhat similar explanations.

Since Ψ_A is cyclic separating, this formula does define a unique linear transformation $U:\mathcal{H}_A\otimes\mathcal{H}'_A\to\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$, and since Ψ_{AB} is separating, this linear transformation is an embedding. To show that it is an isometry, which means that $\langle\eta|\chi\rangle=\langle U\eta|U\chi\rangle$ for all $\eta,\chi\in\mathcal{H}_A\otimes\mathcal{H}'_A$, we observe that as Ψ_A is cyclic, we can take $\eta=\mathsf{a}\Psi_A$, $\chi=\mathsf{b}\Psi_A$. We need then $\langle\mathsf{a}\Psi_A|\mathsf{b}\Psi_A\rangle=\langle(\mathsf{a}\otimes 1)\Psi_{AB}|(\mathsf{b}\otimes 1)\Psi_{AB}\rangle$. Indeed

$$\begin{aligned}\langle(\mathsf{a}\otimes 1)\Psi_{AB}|(\mathsf{b}\otimes 1)\Psi_{AB}\rangle&=\langle\Psi_{AB}|(\mathsf{a}^\dagger\mathsf{b}\otimes 1)\Psi_{AB}\rangle\\&=\mathrm{Tr}_{\rho_{AB}}\mathsf{a}^\dagger\mathsf{b}\otimes 1\\&=\mathrm{Tr}_{\rho_A}\mathsf{a}^\dagger\mathsf{b}=\langle\Psi_A|\mathsf{a}^\dagger\mathsf{b}|\Psi_A\rangle\\&=\langle\mathsf{a}\Psi_A|\mathsf{b}\Psi_A\rangle.\end{aligned}\tag{4.57}$$

Finally, the isometric embedding that we defined commutes with the action of $\mathcal{A}_A$ in the sense that for any $\chi\in\mathcal{H}_A\otimes\mathcal{H}'_A$, we have $U(\mathsf{a}\chi)=\varphi(\mathsf{a})U(\chi)$. Indeed, if $\chi=\mathsf{b}\Psi_A$, we have

$$\begin{aligned}U(\mathsf{a}\chi)=U(\mathsf{ab}\Psi_A)&=(\mathsf{ab}\otimes 1)\Psi_{AB}\\&=(\mathsf{a}\otimes 1)(\mathsf{b}\otimes 1)\Psi_{AB}=\varphi(\mathsf{a})U(\chi).\end{aligned}\tag{4.58}$$

This shows that, if we identify a with $\varphi(\mathsf{a})$, we can regard $\mathcal{A}_A$ as a subalgebra of $\mathcal{A}_{AB}$ and the action of $\mathcal{A}_A$ on $\mathcal{H}_A\otimes\mathcal{H}'_A$ is unitarily equivalent to its action on a subspace of $\mathcal{H}_{AB}\otimes\mathcal{H}_{AB}{}'$. We are almost ready to imitate the proof of Sec. III, but we still have to compare the relative modular operators.

We have a relative modular operator $\Delta_{\Psi_{AB}|\Phi_{AB}}$ for the algebra $\mathcal{A}_{AB}$ acting on $\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$, and a corresponding relative modular operator $\Delta_{\Psi_A|\Phi_A}$ for the algebra $\mathcal{A}_A$ acting on $\mathcal{H}_A\otimes\mathcal{H}'_A$. To lighten the notation, we write just Δ_{AB} and Δ_A instead of $\Delta_{\Psi_{AB}|\Phi_{AB}}$ and $\Delta_{\Psi_A|\Phi_A}$.

The last fact that we need for the proof of monotonicity of relative entropy is that our isometric embedding $U:\mathcal{H}_A\otimes\mathcal{H}'_A\to\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$ intertwines the relative modular operators, in the sense that

$$U^\dagger\Delta_{AB}U=\Delta_A.\tag{4.59}$$

Here $U^\dagger:\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}\to\mathcal{H}_A\otimes\mathcal{H}'_A$ is the adjoint of $U:\mathcal{H}_A\otimes\mathcal{H}'_A\to\mathcal{H}_{AB}\otimes\mathcal{H}'_{AB}$. It is possible to work out an explicit formula for $U^\dagger$, but we will not need it. To prove Eq. (4.59), it is enough to verify that the left- and right-hand sides have the same matrix elements between arbitrary states $\mathsf{a}^\dagger\Psi$ and $\mathsf{b}\Psi$. This is actually a rather direct consequence of Eq. (3.27). For the matrix element of Δ_A, we have

$$\begin{aligned}\langle\mathsf{a}^\dagger\Psi_A|\Delta_A|\mathsf{b}\Psi_A\rangle&=\langle\mathsf{b}^\dagger\Phi_A|\mathsf{a}\Phi_A\rangle\\&=\langle\Phi_A|\mathsf{ba}|\Phi_A\rangle\\&=\mathrm{Tr}_A\sigma_A\mathsf{ba}.\end{aligned}\tag{4.60}$$

The corresponding matrix element of $U^\dagger\Delta_{AB}U$ is

$$\begin{aligned}\langle\mathsf{a}^\dagger\Psi_A|U^\dagger\Delta_{AB}U|\mathsf{b}\Psi_A\rangle&=\langle U(\mathsf{a}^\dagger\Psi_A)|\Delta_{AB}|U(\mathsf{b}\Psi_A)\rangle\\&=\langle(\mathsf{a}^\dagger\otimes 1)\Psi_{AB}|\Delta_{AB}|(\mathsf{b}\otimes 1)\Psi_{AB}\rangle\\&=\langle(\mathsf{b}^\dagger\otimes 1)\Phi_{AB}|(\mathsf{a}\otimes 1)\Phi_{AB}\rangle\\&=\langle\Phi_{AB}|(\mathsf{ba}\otimes 1)|\Phi_{AB}\rangle\\&=\mathrm{Tr}_{AB}\sigma_{AB}(\mathsf{ba}\otimes 1)=\mathrm{Tr}_A\sigma_A\mathsf{ba}.\end{aligned}\tag{4.61}$$

Equation (3.65) (which was proved for an arbitrary isometric embedding), when combined with Eq. (4.59), gives us an inequality

$$U^\dagger(\log\Delta_{AB})U\le\log\Delta_A.\tag{4.62}$$

Now we are finally ready to compare the relative entropies

$$\begin{aligned}\mathcal{S}(\rho_A||\sigma_A)&=-\langle\Psi_A|\log\Delta_A|\Psi_A\rangle,\\\mathcal{S}(\rho_{AB}||\sigma_{AB}\rangle&=-\langle\Psi_{AB}|\log\Delta_{AB}|\Psi_{AB}\rangle.\end{aligned}\tag{4.63}$$

Using Eq. (4.62), we have

$$\begin{aligned}\mathcal{S}(\rho_A||\sigma_A)&=-\langle\Psi_A|\log\Delta_A|\Psi_A\rangle\\&\le-\langle\Psi_A|U^\dagger(\log\Delta_{AB})U|\Psi_A\rangle\\&=-\langle U\Psi_A|\log\Delta_{AB}|U\Psi_A\rangle\\&=-\langle\Psi_{AB}|\log\Delta_{AB}|\Psi_{AB}\rangle\\&=\mathcal{S}(\rho_{AB}||\sigma_{AB}).\end{aligned}\tag{4.64}$$

This completes the proof.

Was it obvious that this proof would work, or did it depend on checking tricky details? Hopefully, we succeeded in convincing the reader that this explanation—which largely follows (Petz, 1986) and (Nielsen and Petz, 2005)—is the natural analog of what was explained for quantum field theory in Sec. III. Philosophically, it might seem obvious that quantum field theory is not simpler than nonrelativistic quantum mechanics, so that an analogous proof in nonrelativistic quantum mechanics must work somehow.

The only property of the logarithm that we used was that $\log X$ is an increasing function of a positive operator X. Many other functions have the same property; an example, as shown in Sec. III.D, is the function X^α, $0\le\alpha\le 1$. Replacing $-\log\Delta_{AB}$ in Eq. (4.64) with Δ_{AB}^α (and reversing the direction of the inequality because of the sign), we get

$$\langle\Psi_A|\Delta_A^\alpha|\Psi_A\rangle\ge\langle\Psi_{AB}|\Delta_{AB}^\alpha|\Psi_{AB}\rangle.\tag{4.65}$$

Evaluating this with the help of Eq. (4.34), we learn that[24]

$$\mathrm{Tr}_A\sigma_A^\alpha\rho_A^{1-\alpha}\ge\mathrm{Tr}_{AB}\sigma_{AB}^\alpha\rho_{AB}^{1-\alpha},\qquad 0\le\alpha\le 1.\tag{4.66}$$

[24]For recent applications of this inequality, see Bernamonti *et al.* (2018). They consider also the case of $\alpha<0$, which can be analyzed by replacing Eq. (3.43) with $R^\alpha\sim\int_0^\infty dss^\alpha/(s+R)$ (in a certain range of α).

This inequality is saturated at $\alpha = 0$, since $\mathrm{Tr}_A \rho_A = \mathrm{Tr}_{AB} \rho_{AB} = 1$. Expanding around $\alpha = 0$, the leading term in the inequality gives back the monotonicity of relative entropy. Similarly, the only property of the states Ψ_A and Ψ_{AB} that was used was that $U\Psi_A = \Psi_{AB}$. One can derive further inequalities by replacing Ψ_A and Ψ_{AB} by $\mathsf{a}\Psi_A$ and $U(\mathsf{a}\Psi_A) = (\mathsf{a} \otimes 1)\Psi_{AB}$. These inequalities (in a formulation originally in terms of convexity rather than monotonicity) go back to Wigner, Yanase, and Dyson (Wigner and Yanase, 1963) and Lieb (1973), with later work by Araki (1976) and Petz (1986), among others.

We conclude this section by briefly explaining how positivity and monotonicity of relative entropy are related to other important concepts in quantum information theory. The von Neumann entropy $\mathcal{S}(\rho)$ of a density matrix ρ is defined as

$$\mathcal{S}(\rho) = -\mathrm{Tr}\rho \log \rho. \tag{4.67}$$

Consider a bipartite system AB with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, density matrix ρ_{AB} and reduced density matrices $\rho_A = \mathrm{Tr}_B \rho_{AB}, \rho_B = \mathrm{Tr}_A \rho_{AB}$. One sets $\mathcal{S}_{AB} = \mathcal{S}(\rho_{AB})$, $\mathcal{S}_A = \mathcal{S}(\rho_A)$, etc. The mutual information $I(A;B)$ between subsystems A and B is defined as

$$I(A;B) = \mathcal{S}_A + \mathcal{S}_B - \mathcal{S}_{AB}. \tag{4.68}$$

Subadditivity of quantum entropy is the statement that $I(A;B) \geq 0$ for all ρ_{AB}. To prove this, define the product density matrix $\sigma_{AB} = \rho_A \otimes \rho_B$ for system AB. The relative entropy between ρ_{AB} and σ_{AB} is

$$\mathcal{S}(\rho_{AB}||\sigma_{AB}) = \mathrm{Tr}_{AB} \rho_{AB}(\log \rho_{AB} - \log \sigma_{AB}). \tag{4.69}$$

Since $\log \sigma_{AB} = \log \rho_A \otimes 1 + 1 \otimes \log \rho_B$, this is

$$\begin{aligned}\mathcal{S}(\rho_{AB}||\sigma_{AB}) &= \mathrm{Tr}_{AB} \rho_{AB}(\log \rho_{AB} - \log \rho_A \otimes 1 - 1 \otimes \log \rho_B) \\ &= -\mathcal{S}_{AB} + \mathcal{S}_A + \mathcal{S}_B = I(A;B).\end{aligned} \tag{4.70}$$

Thus, subadditivity of quantum entropy follows from positivity of relative entropy. For strong subadditivity of quantum entropy (Lieb and Ruskai, 1973), one considers a tripartite system ABC with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and density matrix ρ_{ABC}. One can define various reduced density matrices, such as $\rho_{AB} = \mathrm{Tr}_C \rho_{ABC}$, with corresponding entropy $\mathcal{S}_{AB}$, and likewise for other subsystems. Strong subadditivity of quantum entropy is the statement that mutual information is monotonic in the sense that

$$I(A;B) \leq I(A;BC). \tag{4.71}$$

Expanding this out using the definition of the mutual information, an equivalent statement is

$$\mathcal{S}_B + \mathcal{S}_{ABC} \leq \mathcal{S}_{AB} + \mathcal{S}_{BC}. \tag{4.72}$$

To deduce strong subadditivity from the monotonicity of relative entropy, we compare the two tripartite density matrices ρ_{ABC} and $\sigma_{ABC} = \rho_A \otimes \rho_{BC}$. As we have just seen, the relative entropy between them is

$$\mathcal{S}(\rho_{ABC}||\sigma_{ABC}) = I(A;BC). \tag{4.73}$$

On the other hand, taking a partial trace over system C, the reduced density matrices for the AB subsystem are ρ_{AB} and $\sigma_{AB} = \rho_A \otimes \rho_B$. The relative entropy between them is

$$\mathcal{S}(\rho_{AB}||\sigma_{AB}) = I(A;B). \tag{4.74}$$

Monotonicity of relative entropy tells us that taking the trace over subsystem C can only make the relative entropy smaller, so

$$\mathcal{S}(\rho_{AB}||\sigma_{AB}) \leq \mathcal{S}(\rho_{ABC}||\sigma_{ABC}). \tag{4.75}$$

Putting the last three statements together, we arrive at strong subadditivity.

V. A FUNDAMENTAL EXAMPLE

A. Overview

A certain simple decomposition of Minkowski spacetime provides an important (and well-known) illustration of some of these ideas.

We factorize D-dimensional Minkowski spacetime M_D as the product of a two-dimensional Lorentz signature spacetime $\mathbb{R}^{1,1}$ with coordinates t, x and a $(D-2)$-dimensional Euclidean space $\mathbb{R}^{D-2}$ with coordinates $\vec{y} = (y_1, \ldots, y_{D-2})$. Thus the metric is

$$ds^2 = -dt^2 + dx^2 + d\vec{y} \cdot d\vec{y}. \tag{5.1}$$

In this spacetime, we let Σ be the initial value surface $t = 0$ (Fig. 3). We let $\mathcal{V}_r$ be the open right half-space in Σ, defined by $x > 0$. The complement of its closure, which we call $\mathcal{V}_\ell$, is the left half-space $x < 0$. The domain of dependence of $\mathcal{V}_r$ is what we call the right wedge $\mathcal{U}_r$, defined by $x > |t|$. And the domain of dependence of $\mathcal{V}_\ell$ is what we call the left wedge $\mathcal{U}_\ell$, defined by $x < -|t|$. These wedgelike regions are also often called Rindler spaces (Rindler, 1966). Finally, we denote as $\mathcal{A}_r$ and $\mathcal{A}_\ell$ the algebras of observables in $\mathcal{U}_r$ and $\mathcal{U}_\ell$, respectively. They commute and we will learn that they are each other's commutants.

Let Ω be the vacuum state of a quantum field theory on M_D. The goal of this section will be to determine the modular operators J_Ψ and Δ_Ψ for observations in region $\mathcal{U}_r$. This problem was first analyzed and solved by Bisognano and Wichmann (1976). Their approach involved the analytic behavior of correlation functions and will be sketched in Sec. V.C. But first, in Sec. V.B, we explain a direct path integral approach. This path integral approach is important in Unruh's thermal interpretation of accelerated motion in Minkowski spacetime (Unruh, 1976), which we explain in Sec. V.D. It is also closely related to analogous path integral derivations of the thermal nature of black hole physics (Hawking, 1975, 1977) and of correlation functions in

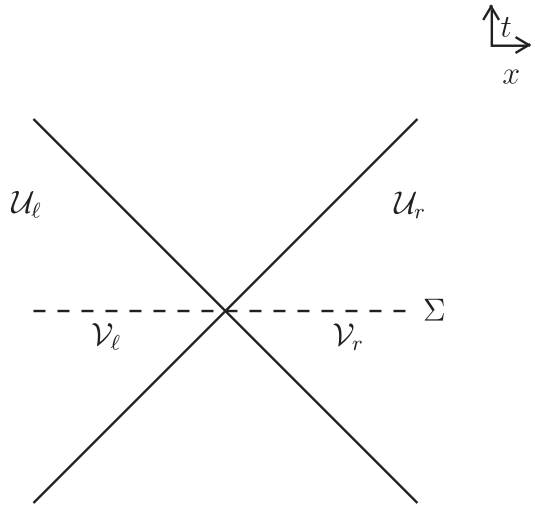

FIG. 3. The right wedge $\mathcal{U}_r$ and the left wedge $\mathcal{U}_\ell$ in Minkowski spacetime. They are the domains of dependence of the right half and left half of the initial value surface $t = 0$, which are labeled as $\mathcal{V}_r$ and $\mathcal{V}_\ell$.

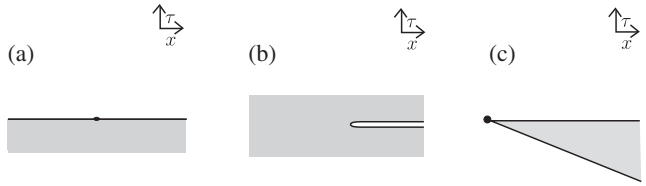

FIG. 4. (a) The path integral on the half-space $\tau < 0$ as a function of boundary values of the fields gives a way to compute the vacuum wave function Ω. (b) To compute the reduced density matrix of the vacuum for the right half of the surface $\tau = 0$ by a Euclidean path integral, we use the path integral on the lower half-space $\tau < 0$ to compute a vacuum bra $\langle\Omega|$, and the path integral on the upper half-space $\tau > 0$ to compute a vacuum ket $|\Omega\rangle$. Then we glue together the left halves of the boundaries of the $\tau < 0$ and $\tau > 0$ half-spaces, identifying the field variables on those boundaries in the bra and the ket. The net effect—a path integral on the upper half-space and the lower half-space together with an integral over field variables on half of the $\tau = 0$ hypersurface—produces a path integral on the space depicted here. It can be obtained from Euclidean space $\mathbb{R}^D$ by making a "cut" along the half-hyperplane $\tau = 0, x \geq 0$. (c) Sketched here is a Euclidean wedge of opening angle θ.

de Sitter spacetime (Figari, Hoegh-Krohn, and Nappi, 1975; Gibbons and Hawking, 1977). As this approach is relatively well known, we will be brief.

The CPT symmetry of quantum field theory will enter in what follows, so we pause to discuss it. CPT acts as -1 on all space and time coordinates. The basic reason that CPT is an unavoidable symmetry of quantum field theory in $3+1$ dimensions is that in Euclidean signature,[25] the transformation that acts as -1 on all four coordinates is in the connected component of the rotation group. (If we factor $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$, then a simultaneous π rotation on each copy of $\mathbb{R}^2$ acts as -1 on all four coordinates.) Therefore, in Euclidean signature this operation is inevitably a symmetry of any rotation-invariant theory. After continuation back to Lorentz signature, this symmetry becomes CPT.

The statement that a transformation of Euclidean space that acts as -1 on all coordinates is in the connected component of the rotation group is true in and only in even spacetime dimension. For odd D, that operation has determinant -1 and is not in the connected component of the rotation group. Accordingly, for odd D, there is no CPT symmetry in general. A better formulation that is uniformly valid in any dimension is to replace parity—a sign change of all spatial coordinates—with a reflection of just one spatial coordinate. We call this operation R. Regardless of the spacetime dimension, a simultaneous sign change of both the time t and one spatial coordinate x is in the identity component of the rotation group in Euclidean signature, as it is a π rotation of the xt plane. Thus, the universal symmetry of quantum field theory in any dimension is CRT rather than CPT. In $3+1$ dimensions, CPT is the product of CRT times a π rotation of two spatial coordinates, so the two are essentially equivalent.

Because CPT or CRT is antiunitary, it reverses the signs of conserved charges. Historically, P and T were defined to be good approximate symmetries of ordinary matter (until the 1950s, they were assumed to be exact symmetries). Since ordinary matter is made of leptons and baryons without antileptons and antibaryons, P and T were defined to commute with baryon number and lepton number. With this choice, the universal discrete symmetry does not coincide with PT or RT and deserves to be called CPT or CRT, to express the fact that it reverses conserved charges.[26]

B. Path integral approach

We continue to Euclidean signature, setting $t = -i\tau$. Euclidean path integrals are an effective way to compute the vacuum state Ω of a quantum field theory. Thus, the path integral on, say, the half-space $\tau \leq 0$, as a function of boundary values on the hyperplane $\tau = 0$, gives a way to compute Ω [Fig. 4(a)].

Suppose it were true that the Hilbert space $\mathcal{H}$ of a quantum field theory has a factorization $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$, where $\mathcal{H}_\ell$ and $\mathcal{H}_r$ are Hilbert spaces of degrees of freedom located at $x < 0$ and $x > 0$ respectively, and thus acted on by the algebras $\mathcal{A}_\ell$ and $\mathcal{A}_r$. In this case, starting with the pure state density matrix $|\Omega\rangle\langle\Omega|$ and taking a partial trace on the degrees of freedom in

[25]The rigorous proof of CPT invariance can be conveniently found in Streater and Wightman (1964). It depends on the holomorphy statement of Eq. (2.11). Holomorphy is built in for free when one starts in Euclidean signature, so if one assumes that a quantum field theory can be obtained by analytic continuation from Euclidean signature, then one can see CPT without a careful discussion of conditions of holomorphy.

[26]Both R and what is usually called CT come from the same operation in Euclidean signature (reflection of one spatial coordinate), continued back to Lorentz signature in different ways. So purely from a relativistic point of view, it would be natural to exchange the names T and CT and refer to the universal discrete symmetry as PT or RT, rather than CPT or CRT. However, this would involve too much conflict with standard terminology.

$\mathcal{H}_\ell$, we could define a reduced density matrix ρ_r on $\mathcal{H}_r$. Technically, it is not quite true that $\mathcal{H}$ has the suggested factorization, but assuming that it does will lead to a correct and illuminating determination of the operators Δ_Ω and J_Ω for the vacuum state.

To formally construct the density matrix ρ_r for the right half-space, we simply reason as follows. Very roughly, think of the vacuum wave function Ω as a function $\Omega(\phi_\ell, \phi_r)$ that depends on field variables ϕ_ℓ in the left half-space and ϕ_r in the right half-space. (We schematically write ϕ_ℓ or ϕ_r for all the field variables at $x < 0$ or $x > 0$.) The density matrix $|\Omega\rangle\langle\Omega|$ is as usual a function $|\Omega(\phi_\ell', \phi_r')\rangle\langle\Omega(\phi_\ell, \phi_r)|$ that depends on two sets of field variables. A partial trace over $\mathcal{H}_\ell$ to get the density matrix ρ_r is carried out by setting $\phi_\ell' = \phi_\ell$ and integrating over ϕ_ℓ:

$$\rho_r(\phi_r', \phi_r) = \int D\phi_\ell |\Omega(\phi_\ell, \phi_r')\rangle\langle\Omega(\phi_\ell, \phi_r)|. \tag{5.2}$$

This has a simple path integral interpretation. The bra $\langle\Omega(\phi_\ell, \phi_r)|$ can be computed, as already noted, by a path integral on the lower half-space $\tau \le 0$, and similarly the ket $|\Omega(\phi_\ell', \phi_r')\rangle$ can be computed by a path integral on the upper half-space. To set $\phi_\ell = \phi_\ell'$, we glue together the portion $x < 0$ of the boundaries of the upper and lower half-spaces. This gluing gives the spacetime $W_{2\pi}$ that is sketched in Fig. 4(b). $W_{2\pi}$ is a copy of Euclidean space except that it has been "cut" along the half-hyperplane $t = 0$, $x > 0$. (The reason for the notation $W_{2\pi}$ will be clear in a moment.) In Eq. (5.2), the path integral over the lower half-space to get $\langle\Omega|$, the path integral over the upper half-space to get $|\Omega\rangle$, and the final integral over ϕ_ℓ to take a partial trace all combine together to make a path integral over $W_{2\pi}$. In this path integral, boundary values ϕ_r and ϕ_r' are specified just below and above the cut.

To identify the modular operator Δ_Ψ, we want to give a Hamiltonian interpretation to the path integral in $W_{2\pi}$. For this, we first consider a path integral on a Euclidean wedge W_θ of opening angle θ [Fig. 4(c)]. This path integral can be viewed as computing an operator. A matrix element of this operator between initial and final states is computed by specifying an initial state at the lower boundary of the wedge and a final state at the upper boundary. The wedge operator is a Euclidean rotation of the τx plane by an angle θ. Thus, the rotation acts by

$$R_\theta \begin{pmatrix} \tau \\ x \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix}. \tag{5.3}$$

To identify in familiar terms the operator that acts in this way in Euclidean signature, let us express the formula in terms of real time $t = -i\tau$:

$$\begin{aligned} R_\theta = \begin{pmatrix} t \\ x \end{pmatrix} &= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\ &= \begin{pmatrix} \cosh(i\theta) & -\sinh(i\theta) \\ -\sinh(i\theta) & \cosh(i\theta) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \end{aligned} \tag{5.4}$$

Looking at the right-hand side, we see a Lorentz boost of the tx plane by an imaginary boost parameter $-i\theta$. The generator of such a Lorentz boost can be written as an integral over the initial value surface $t = 0$:

$$K = \int_{t=0} dx d\vec{y} x T_{00}. \tag{5.5}$$

It has been defined to map the right wedge forward in time, and the left wedge backward in time. Formally we can write

$$K = K_r - K_\ell, \tag{5.6}$$

where K_r and K_ℓ are partial Lorentz boost generators

$$\begin{aligned} K_r &= \int_{t=0, x\ge 0} dx d\vec{y} x T_{00}, \\ K_\ell &= -\int_{t=0, x\ge 0} dx d\vec{y} x T_{00}. \end{aligned} \tag{5.7}$$

The minus sign is included so that K_ℓ boosts the left wedge forward in time, just as K_r does to the right wedge.[27]

The operator K is self-adjoint, and the unitary operator that implements a Lorentz boost by a real boost parameter η is $\exp(-i\eta K)$. Setting $\eta = -i\theta$, we learn that, in real time language, the path integral on the wedge W_θ constructs the operator $\exp(-\theta K_r)$. The path integral on the wedge propagates the degrees of freedom on the right half-space only, so the operator in the exponent is K_r, not K. To get the density matrix ρ_r of the right wedge, we set $\theta = 2\pi$:

$$\rho_r = \exp(-2\pi K_r). \tag{5.8}$$

A precisely similar analysis shows that the density matrix of the left wedge is

$$\rho_\ell = \exp(-2\pi K_\ell). \tag{5.9}$$

We want to combine these results to determine the modular operator Δ_Ω for the vacuum state Ω, for the algebra $\mathcal{A}_r$ of observables in the right wedge. Factoring the Hilbert space as $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$ and using Eq. (4.26) (where we identify $\mathcal{H}_r$ and $\mathcal{H}_\ell$ with $\mathcal{H}_1$ and $\mathcal{H}_2$), the modular operator is

$$\Delta_\Omega = \rho_r \otimes \rho_\ell^{-1} = \exp(-2\pi K_r)\exp(2\pi K_\ell) = \exp(-2\pi K). \tag{5.10}$$

In the last step, we use the fact that formally the operators K_r and K_ℓ commute, since they act respectively on $\mathcal{H}_r$ and $\mathcal{H}_\ell$.

[27] Rather as there is not a rigorous factorization $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$, the operators K_ℓ and K_r are not really well defined as Hilbert space operators, although of course the difference $K = K_r - K_\ell$ is a well-defined Hilbert space operator. K_ℓ and K_r have well-defined matrix elements $\langle\Psi|K_\ell|\chi\rangle$ and $\langle\Psi|K_r|\chi\rangle$ between suitable Hilbert space states χ and Ψ, but if one tries to compute the norm of the state $K_\ell|\chi\rangle$ or $K_r|\chi\rangle$, one will find a universal ultraviolet divergence, near $x = 0$, independent of the choice of χ. This is related to the fact that the factorization $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$ is not really correct.

Now let us consider a state $\mathsf{a}|\Omega\rangle$ obtained by acting on the vacuum with an operator $\mathsf{a}\in\mathcal{A}_r$, supported on the right wedge. For simplicity, we assume that a well-defined operator a can be given by smearing a local operator ϕ in space with no corresponding smearing in time. This is so if the dimension of ϕ, measured in the ultraviolet, is less than $(D-1)/2$. It is not true that the operator product algebra of a quantum field theory is always generated by operators of such relatively low dimension, so in general the following discussion has to be modified to allow a very slight smearing in time, but we omit this.

Under our hypothesis, the state $\mathsf{a}|\Omega\rangle$ can be computed by a path integral on the lower half-space, with an insertion of the operator a on the right half of the boundary [Fig. 5(a)]. Now let us consider the state

$$\begin{aligned}\Delta_\Omega^\alpha \mathsf{a}|\Omega\rangle &= \exp(-2\pi\alpha K)\mathsf{a}|\Omega\rangle\\ &= \exp(2\pi\alpha K_\ell)\exp(-2\pi\alpha K_r)\mathsf{a}|\Omega\rangle.\end{aligned}\tag{5.11}$$

The operator $\exp(-2\pi\alpha K_r)$ is implemented by gluing on a wedge of opening angle $2\pi\alpha$ to the right half of the boundary in Fig. 5(a), while the operator $\exp(2\pi\alpha K_\ell)$ removes such a wedge from the left. If we add one wedge and remove the other, and also rotate the picture so that the boundary is still horizontal, we arrive at Fig. 5(b). There is still a path integral on the lower half-plane, but now the operator a is inserted at an angle $-2\pi\alpha$ relative to where it was before. We can continue in this way until we get to $\alpha=1/2$. This case is depicted in Fig. 5(c). What at $\alpha=0$ was an operator insertion a on the right boundary at $x>0$ has now turned into the insertion of some other operator $\widetilde{\mathsf{a}}$ on the left boundary at $x<0$. As $\widetilde{\mathsf{a}}$ is inserted on the left boundary, it is an element of the algebra $\mathcal{A}'$. Thus for $\mathsf{a}\in\mathcal{A}_r$,

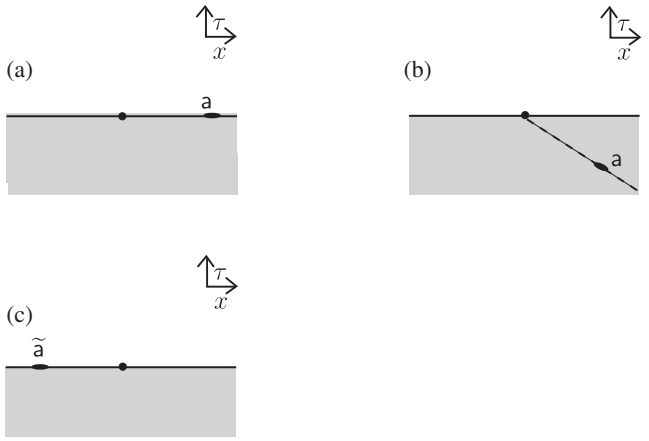

FIG. 5. (a) The state $\mathsf{a}|\Omega\rangle$ can be obtained by a path integral in the lower half-plane, with a inserted on the right half of the boundary. (b) Acting with $\exp(2\pi\alpha K_\ell)\exp(-2\pi\alpha K_r)\mathsf{a}|\Omega\rangle$ adds a wedge of opening angle $2\pi\alpha$ to the right boundary and removes one from the left boundary. If we rotate the picture so that the boundary is again horizontal, it looks like this; the operator a is now inserted on a ray that is at an angle $2\pi\alpha$ from the horizontal. (c) By the time we get to $\alpha=1/2$, a is inserted on the left boundary of the lower half-plane. We cannot extend this process further.

$$\Delta_\Omega^{1/2}\mathsf{a}|\Omega\rangle=\widetilde{\mathsf{a}}|\Omega\rangle,\tag{5.12}$$

for some $\widetilde{\mathsf{a}}\in\mathcal{A}_\ell$. A similar statement holds, of course, with $\mathcal{A}_\ell$ and $\mathcal{A}_r$ exchanged.

We have learned that $\Delta^\alpha\mathsf{a}|\Omega\rangle$ is a well-defined Hilbert space state for $0\le\alpha\le1/2$. But we cannot go farther. The operator Δ^α has removed a wedge of angle $2\pi\alpha$ from the left side of the picture. By the time we reached $\alpha=1/2$, there is no wedge left to remove on that side and we have to stop. On the other hand, there is no problem in acting on any Hilbert space state with the unitary operator Δ^{is}. So a more general conclusion is that, as was claimed in Sec. IV.B, $\Delta_\Omega^{iz}\mathsf{a}|\Omega\rangle$ is holomorphic in the strip $0>\mathrm{Im}z>-1/2$ (and continuous on the boundary of the strip) but not beyond.

Our final goal in this discussion is to determine and exploit the modular conjugation J_Ω. We use the fact that $S_\Omega=J_\Omega\Delta^{1/2}$ is supposed to satisfy

$$S_\Omega\mathsf{a}|\Omega\rangle=\mathsf{a}^\dagger|\Omega\rangle,\qquad\mathsf{a}\in\mathcal{A}_r.\tag{5.13}$$

For simplicity, let us assume that the operator algebra of our theory is generated by a Hermitian scalar field ϕ. To determine what J_Ω must be, it suffices to consider the case that a is equal to either ϕ or $\dot\phi=d\phi/dt$, inserted on the right wedge at the initial value surface $t=0$. Since ϕ and $\dot\phi$ are both Hermitian, we want

$$\begin{aligned}S_\Omega\phi(0,x,\vec y)|\Omega\rangle&=\phi(0,x,\vec y)|\Omega\rangle,\\ S_\Omega\dot\phi(0,x,\vec y)|\Omega\rangle&=\dot\phi(0,x,\vec y)|\Omega\rangle.\end{aligned}\tag{5.14}$$

(One could introduce a smearing function in these statements, but this would not change what follows.) Instead, from Eq. (5.12), we have

$$\begin{aligned}\Delta_\Omega^{1/2}\phi(0,x,\vec y)|\Omega\rangle&=\phi(0,-x,\vec y)|\Omega\rangle,\\ \Delta_\Omega^{1/2}\dot\phi(0,x,\vec y)|\Omega\rangle&=-\dot\phi(0,x,\vec y)|\Omega\rangle.\end{aligned}\tag{5.15}$$

The reason for the minus sign in the second line is that acting with $\Delta_\Omega^{1/2}$ turns a future-pointing time derivative acting on ϕ in Fig. 5(a) into a past-pointing time derivative in Fig. 5(c), so it reverses the sign of $d\phi/dt$. Comparing Eqs. (5.13) and (5.15), we see that we want

$$\begin{aligned}J_\Omega\phi(0,x,\vec y)J_\Omega&=\phi(0,-x,\vec y),\\ J_\Omega\dot\phi(0,x,\vec y)J_\Omega&=-\dot\phi(0,-x,\vec y).\end{aligned}\tag{5.16}$$

In other words, J_Ω is supposed to be an antiunitary operator that maps $x\to-x$, $t\to-t$, $\vec y\to\vec y$.

The antiunitary operator that acts in this way on any Hermitian scalar field (with an analogous action on fields of other types) is the operator CRT that was discussed in Sec. V.A. Thus

$$J_\Omega=\mathsf{CRT}.\tag{5.17}$$

We now pause a moment to explain more explicitly why this operator is traditionally called CRT rather than RT.

Consider a theory with two Hermitian scalar fields ϕ_1 and ϕ_2 rotated by an $SO(2)$ symmetry with generator

$$Q = \int_{t=0} dx d\vec{y}(\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2). \tag{5.18}$$

This charge is odd under J_Ω, since ϕ_1 and ϕ_2 are even while $\dot{\phi}_1$ and $\dot{\phi}_2$ are odd. So J_Ω reverses the sign of Q, and similarly of any other Hermitian conserved charge. Since R and T are traditionally defined to commute with Lorentz-invariant conserved charges while J_Ω reverses their sign, J_Ω corresponds to what is traditionally called CRT rather than RT. CRT is a universal symmetry of relativistic quantum field theory, while there is no universal symmetry corresponding to RT.

In this example, we can explicitly verify the deeper properties of the modular automorphisms Δ_Ω^{is} and J_Ω that were described in Sec. IV.B. Δ_Ω^{is} implements a Lorentz boost with a real boost parameter $2\pi s$, so it is an automorphism of the algebras $\mathcal{A}_\ell$ and $\mathcal{A}_r$ of the two wedges. And $J_\Omega = \mathsf{CRT}$ exchanges the two wedges so it exchanges the two algebras.

In general, in Tomita-Takesaki theory, the modular conjugation J_Ω exchanges an algebra $\mathcal{A}$ with its commutant $\mathcal{A}'$. So in the present context, the fact that J_Ω exchanges $\mathcal{A}_\ell$ and $\mathcal{A}_r$ tells us that these algebras are commutants:

$$\mathcal{A}_\ell' = \mathcal{A}_r, \qquad \mathcal{A}_r' = \mathcal{A}_\ell. \tag{5.19}$$

This is how Bisognano and Wichmann (1976) proved Haag duality for complementary Rindler spaces.

C. The approach of Bisognano and Wichmann

The path integral derivation of the last section is extremely illuminating, and it gives the right result although it is not altogether rigorous. (The flaws all involve an imprecise treatment of the boundary between the two regions at $x = 0$.) Here, following the presentation by Borchers (2000), we briefly sketch the original approach of Bisognano and Wichmann (1976). The main difference is that instead of a Euclidean path integral and a claimed factorization $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$, one uses holomorphy.

Since $J_\Omega = \mathsf{CRT}$ certainly acts as in Eq. (5.16), to determine Δ_Ω and S_Ω, we have to justify the claim that for $\mathsf{a} \in \mathcal{A}_r$,

$$\exp(-2\pi K)\mathsf{a}|\Omega\rangle = \widetilde{\mathsf{a}}|\Omega\rangle, \tag{5.20}$$

where $\widetilde{\mathsf{a}}$ is obtained from a by $t, x, \vec{y} \to -t, -x, \vec{y}$. In checking this, we can take a to be a product of field operators

$$\mathsf{a} = \phi(t_1, x_1, \vec{y}_1)\phi(t_2, x_2, \vec{y}_2)\cdots\phi(t_n, x_n, \vec{y}_n) \tag{5.21}$$

inserted in the right wedge $\mathcal{U}_r$ at points $p_i = (t_i, x_i, \vec{y}_i)$, $i = 1, 2, \ldots, n$. Moreover, we can take the points p_i to be spacelike separated from each other; as the field operators $\phi(t_i, x_i, \vec{y}_i)$ thereby commute, we can order them so that $x_j \geq x_i$ for $j > i$. Even more specifically, we can restrict to

$$x_j - x_i > |t_j - t_i|, \qquad j > i. \tag{5.22}$$

It suffices to consider operators a of this form roughly because states $\mathsf{a}|\Omega\rangle$ with a of this type are dense[28] in $\mathcal{H}$, so in particular they are dense among all states $\mathsf{a}|\Omega\rangle$, $\mathsf{a} \in \mathcal{A}_r$. For a precise statement, see Lemma 3.1.7 in Borchers (2000).

For real s, the Lorentz boost operator $\exp(-2\pi i s K)$ is unitary and its action on a state $\mathsf{a}|\Omega\rangle$ is straightforward to determine. The normal coordinates $\vec{y}$ play no role in what follows so we omit them to simplify the notation. A Lorentz boost $\exp(-2\pi i s K)$ maps $\mathsf{x} = \binom{t}{x}$ to

$$\mathsf{x}'(s) = \begin{pmatrix} t'(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh(2\pi s) & \sinh(2\pi s) \\ \sinh(2\pi s) & \cosh(2\pi s) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \tag{5.23}$$

The corresponding transformation of operators in the Heisenberg picture is

$$\phi(\mathsf{x}(\eta)) = \exp(2\pi i\eta K)\phi(\mathsf{x})\exp(-2\pi i\eta K). \tag{5.24}$$

So for real η, remembering that $K\Omega = 0$,

$$\begin{aligned}\exp(2\pi i\eta K)\phi(\mathsf{x}_1)\phi(\mathsf{x}_2)\cdots\phi(\mathsf{x}_n)|\Omega\rangle \\ = \phi(\mathsf{x}_1'(\eta))\phi(\mathsf{x}_2'(\eta))\cdots\phi(\mathsf{x}_n'(\eta))|\Omega\rangle.\end{aligned} \tag{5.25}$$

We want to analytically continue this formula in η. If it can be continued to $\eta = i/2$, then, since $\mathsf{x}'(i/2) = -\mathsf{x}$, Eq. (5.25) will give the desired result (5.20).

In Sec. II.B, we learned that the $\mathcal{H}$-valued function

$$F(\mathsf{x}_1', \mathsf{x}_2', \ldots, \mathsf{x}_n') = \phi(\mathsf{x}_1')\phi(\mathsf{x}_2')\cdots\phi(\mathsf{x}_n')|\Omega\rangle \tag{5.26}$$

is holomorphic in $\mathsf{x}_1', \ldots, \mathsf{x}_n'$ in a certain domain. To be precise, if $\mathsf{x}_i' = \mathsf{u}_i + i\mathsf{v}_i$ with real u_i, v_i, then $F(\mathsf{x}_1', \mathsf{x}_2', \ldots, \mathsf{x}_n')$ is holomorphic in the domain in which v_1 and $\mathsf{v}_{i+1} - \mathsf{v}_i$ are future timelike.

We claim that if the points $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ are chosen as in Eq. (5.22), then for $1/2 > \mathrm{Im}\eta > 0$, the points $\mathsf{x}_1'(\eta)$, $\mathsf{x}_2'(\eta), \ldots, \mathsf{x}_n'(\eta)$ are in the domain of holomorphy that was just described. Since this statement is manifestly invariant under real Lorentz boosts, it suffices to verify it for imaginary η, say $\eta = ib$, $0 < b < 1/2$. Let x be either x_1 or one of the differences $\mathsf{x}_{i+1} - \mathsf{x}_i$. Our assumptions imply in each case that x is in the right wedge $x > |t|$. We have to show that the imaginary part of $\mathsf{x}'(\eta)$, defined in Eq. (5.23) (with s replaced by $\eta = ib$), is future timelike for the claimed range of b. We compute

$$\begin{pmatrix} t'(\eta) \\ x'(\eta) \end{pmatrix} = \begin{pmatrix} t\cos 2\pi b + ix\sin 2\pi b \\ x\cos 2\pi b + it\sin 2\pi b \end{pmatrix}. \tag{5.27}$$

Since $x > |t|$, the imaginary part is future timelike for $0 < b < 1/2$, which ensures that $\sin 2\pi b > 0$. The $\mathcal{H}$-valued

[28] One can see this by reviewing the proof of the Reeh-Schlieder theorem from Sec. II.B. The proof would go through perfectly well if one begins by assuming only that the functions $\varphi(x_1, x_2, \ldots, x_n) = \langle\chi|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle$ [Eq. (2.5)] vanish under the hypothesis (5.22); one can still prove in the same way that these functions vanish identically for all $x_1, x_2, \ldots, x_n$.

function on the right-hand side of Eq. (5.25) is thus holomorphic for $1/2 > \mathrm{Im}\eta > 0$, and continuous up to the boundary at $\mathrm{Im}\eta = 1/2$. (It cannot be continued holomorphically beyond that.) This is precisely enough to justify setting $\eta = i/2$ in Eq. (5.25), and thus to complete the proof.

D. An accelerating observer

The problem discussed is closely related to Unruh's question (Unruh, 1976) of what is seen by an observer undergoing constant acceleration in Minkowski spacetime, say in the xt plane. The worldline of the observer (Fig. 6) is

$$\begin{pmatrix} t(\tau) \\ x(\tau) \end{pmatrix} = R\begin{pmatrix} \sinh(\tau/R) \\ \cosh(\tau/R) \end{pmatrix}, \tag{5.28}$$

where τ is the observer's proper time; the proper acceleration is $a = 1/R$. As before, we abbreviate $\left(\begin{smallmatrix} t(\tau) \\ x(\tau) \end{smallmatrix}\right)$ as $\mathsf{x}(\tau)$.

We suppose that the observer probes the vacuum Ω of Minkowski spacetime by measuring a local operator $\mathcal{O}$ and its adjoint $\mathcal{O}^\dagger$ along this worldline. For simplicity, we consider only the two-point functions $\mathcal{O}\cdot\mathcal{O}^\dagger$, but we consider both operator orderings. Thus, we suppose that the observer has access to $\langle\Omega|\mathcal{O}(\mathsf{x}(\tau_1))\mathcal{O}^\dagger(\mathsf{x}(\tau_2))|\Omega\rangle$ and $\langle\Omega|\mathcal{O}^\dagger(\mathsf{x}(\tau_2))\mathcal{O}(\mathsf{x}(\tau_1))|\Omega\rangle$. Lorentz invariance implies that these functions depend only on $\tau = \tau_1 - \tau_2$, so there is no essential loss to set $\tau_2 = 0$ and to consider the two functions:

$$\begin{aligned} F(\tau) &= \langle\Omega|\mathcal{O}(\mathsf{x}(\tau))\mathcal{O}^\dagger(\mathsf{x}(0))|\Omega\rangle, \\ G(\tau) &= \langle\Omega|\mathcal{O}^\dagger(\mathsf{x}(0))\mathcal{O}(\mathsf{x}(\tau))|\Omega\rangle. \end{aligned} \tag{5.29}$$

Unruh's basic insight was that these correlation functions have thermal properties. The basic property of real time two-point functions in a thermal ensemble, as already explained in Eqs. (4.46) and (4.47), is that there is a holomorphic function on a strip in the complex plane whose boundary values on the two boundaries of the strip are $F(\tau)$ and $G(\tau)$. In general, the width of the strip is $2\pi\beta$, where β is the inverse temperature; in the derivation of Eqs. (4.46) and (4.47), we took $\beta = 1$ so the width of the strip was 2π. We give two derivations of Unruh's result, first starting in real time and deducing the holomorphic properties of the correlation functions, and second starting in Euclidean signature and analytically continuing back to real time.

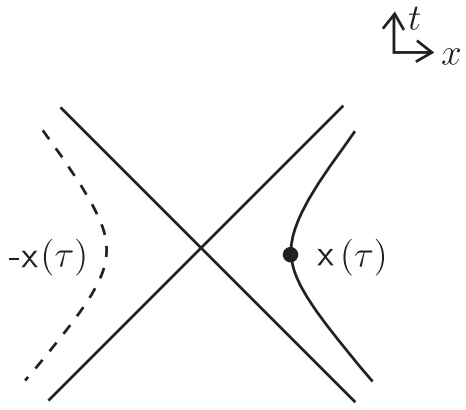

FIG. 6. An accelerating trajectory $\mathsf{x}(\tau)$ in the right quadrant of the xt plane. The point $\tau = 0$ is marked. Shown in dotted lines, on the left, is the mirror trajectory $-\mathsf{x}(\tau)$, which can be obtained from the first by a shift in imaginary time. The two trajectories are spacelike separated.

To understand the analytic properties of the real time correlation functions, we first analytically continue the observer's trajectory. We set $\tau/R = s + i\theta$ with real s, θ and compute that

$$\mathsf{x}(\tau) = R\begin{pmatrix} \sinh s\cos\theta + i\cosh s\sin\theta \\ \cosh s\cos\theta + i\sinh s\sin\theta \end{pmatrix}. \tag{5.30}$$

Thus

$$\mathrm{Im}\mathsf{x}(\tau) = R\sin\theta\begin{pmatrix} \cosh s \\ \sinh s \end{pmatrix}. \tag{5.31}$$

$F(\tau)$ is holomorphic when $\mathrm{Im}\mathsf{x}(\tau)$ is future timelike and $G(\tau)$ is holomorphic when $\mathrm{Im}\mathsf{x}(\tau)$ is past timelike. So $F(\tau)$ is holomorphic in the strip $0 < \theta < \pi$ and continuous on the boundaries of that strip; we describe this more briefly by saying that $F(\tau)$ is holomorphic in the strip $0 \le \theta \le \pi$. Similarly $G(\tau)$ is holomorphic in the strip $\pi \le \theta \le 2\pi$ (or equivalently but less conveniently $-\pi \le \theta \le 0$).

In terms of τ, $F(\tau)$ is holomorphic for $0 \le \mathrm{Im}\tau \le \pi R$. At $\mathrm{Im}\tau = 0$, $F(\tau)$ is simply the original correlation function $\langle\Omega|\mathcal{O}(\mathsf{x}(\tau))\mathcal{O}^\dagger(\mathsf{x}(0))|\Omega\rangle$ on the observer's worldline. On the other boundary of the strip at $\mathrm{Im}\tau = \pi R$, $\mathsf{x}(\tau)$ is again real:

$$\mathsf{x}(\tau + i\pi R) = -\mathsf{x}(\tau) = -R\begin{pmatrix} \sinh(\tau/R) \\ \cosh(\tau/R) \end{pmatrix}. \tag{5.32}$$

So the boundary values at $\tau = R(s + i\pi)$ are

$$F(R(s+i\pi)) = \langle\Omega|\mathcal{O}(-\mathsf{x}(Rs))\mathcal{O}^\dagger(\mathsf{x}(0))|\Omega\rangle. \tag{5.33}$$

Similarly, $G(\tau)$ at $\mathrm{Im}\tau = 2\pi R$ is simply the original correlation function $\langle\Omega|\mathcal{O}^\dagger(\mathsf{x}(0))\mathcal{O}(\mathsf{x}(\tau))|\Omega\rangle$ on the observer's worldline. But at $\mathrm{Im}\tau = \pi R$, we get, similarly to (5.33),

$$G(R(s+i\pi)) = \langle\Omega|\mathcal{O}^\dagger(\mathsf{x}(0))\mathcal{O}(-\mathsf{x}(Rs))|\Omega\rangle. \tag{5.34}$$

Crucially, the operators $\mathcal{O}(-\mathsf{x}(Rs))$ and $\mathcal{O}^\dagger(\mathsf{x}(0))$ commute, since for all real s, $-\mathsf{x}(Rs)$ is spacelike separated from $\mathsf{x}(0)$ (see Fig. 6). So the correlation functions in Eqs. (5.33) and (5.34) are equal.

Thus, we have one function $F(\tau)$ that is holomorphic for $\pi R \ge \mathrm{Im}\tau \ge 0$ and another function $G(\tau)$ that is holomorphic for $2\pi R \ge \mathrm{Im}\tau \ge \pi R$; moreover at $\mathrm{Im}\tau = \pi R$, these two functions are equal. It follows that we can define a single function $H(\tau)$ on the combined strip $2\pi R \ge \mathrm{Im}\tau \ge 0$ by

$$H(\tau) = \begin{cases} F(\tau) & \text{if } \pi R \ge \mathrm{Im}\tau \ge 0, \\ G(\tau) & \text{if } 2\pi R \ge \mathrm{Im}\tau \ge \pi R. \end{cases} \tag{5.35}$$

This function is holomorphic in the combined strip and continuous on its boundaries. (For the proof of holomorphy on the line $\mathrm{Im}\tau = \pi R$ where the two functions were glued together, see Fig. 8 in Appendix A.2.) The boundary values at the top and bottom boundaries of the strip are the two correlation functions that we started with, with the two possible operator orderings.

We have arrived at the usual analytic behavior of a real time two-point correlation function in a thermal ensemble: two-point functions with different operator ordering are opposite boundary values of a single function that is holomorphic in a strip. We found a strip of width $2\pi R$, so the effective temperature is $1/2\pi R$.

A derivation that begins with the Euclidean correlation functions might be more transparent. Let $t_E = it$ be the Euclidean time. A Euclidean version of Eq. (5.28) is

$$t_E = R\sin\theta, \qquad x = R\cos\theta. \tag{5.36}$$

This is the thermal circle that is related to the observations of the accelerated observer. Let $\mathsf{x}_E = \binom{t_E}{x}$. In Euclidean space, one considers the correlation function $\langle \mathcal{O}(\mathsf{x}_E(\theta))\mathcal{O}^\dagger(\mathsf{x}_E(0))\rangle$. *A priori*, a Euclidean correlation function has no operator interpretation. To introduce an operator interpretation, one picks a direction as Euclidean time and introduces a transfer matrix that propagates operators in that direction. Then Euclidean correlation functions acquire an operator interpretation, with the operators being ordered in the direction of increasing Euclidean time. For example, if t_E is chosen as the Euclidean time direction, then a general Euclidean two-point function is interpreted in the transfer matrix formalism as

$$\langle \mathcal{O}(t_E,x)\mathcal{O}^\dagger(t'_E,x')\rangle = \begin{cases} \langle\Omega|\mathcal{O}(t_E,x)\mathcal{O}^\dagger(t'_E,x')|\Omega\rangle & \text{if } t_E \geq t'_E, \\ \langle\Omega|\mathcal{O}^\dagger(t'_E,x')\mathcal{O}(t_E,x)|\Omega\rangle & \text{if } t'_E \geq t_E. \end{cases} \tag{5.37}$$

As before, this is consistent because if $t_E = t'_E$, the operator ordering does not matter. Given this, the operator ordering in the operator interpretation of the Euclidean correlation function $\langle \mathcal{O}(\mathsf{x}_E(\theta))\mathcal{O}^\dagger(\mathsf{x}_E(0))\rangle$ depends on the sign of $t_E = R\sin\theta$, as in the previous derivation. When we analytically continue $\langle \mathcal{O}(\mathsf{x}_E(\theta))\mathcal{O}^\dagger(\mathsf{x}_E(0))\rangle$ from a function of θ to a function of $\tau = R(s + i\theta)$, we get the two operator orderings depending on the sign of $\sin\theta$, as before. This distinction remains in the limit $\theta \to 0^\pm$, where we recover the real time correlation functions with different operator orderings.

VI. ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

A. The problem

Let $\mathcal{U}$ be an open set in Minkowski spacetime. It has a local algebra $\mathcal{A} = \mathcal{A}_\mathcal{U}$ with commutant $\mathcal{A}'$ (which, if Haag duality holds, is $\mathcal{A}_{\mathcal{U}'}$ for some other open set $\mathcal{U}'$). As in Sec. II.F, we understand $\mathcal{A}$ and $\mathcal{A}'$ to be von Neumann algebras of bounded operators (closed under Hermitian conjugation and weak limits, and containing the identity operator). They act on the Hilbert space $\mathcal{H}$ of the theory in question with the vacuum state Ω as a cyclic separating vector.

For a finite-dimensional quantum system, the existence of such a cyclic separating vector would imply a factorization $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, with $\mathcal{A}$ acting on one factor and $\mathcal{A}'$ on the other. Such a factorization cannot exist in quantum field theory, for it would imply the existence of tensor product states $\psi \otimes \chi$ with no entanglement between $\mathcal{U}$ and $\mathcal{U}'$. Instead, in quantum field theory, there is a universal ultraviolet divergence in the entanglement entropy.

The essence of the matter is that in quantum field theory, the divergence in the entanglement entropy is not a property of the states but of the algebras $\mathcal{A}$ and $\mathcal{A}'$. These algebras are not the familiar type I von Neumann algebras which can act irreducibly in a Hilbert space. Instead they are more exotic algebras with the property that the structure of the algebra has the divergence in the entanglement entropy built in. In this section, we explain barely enough about von Neumann algebras to indicate how that comes about.

B. Algebras of type I

A type I von Neumann algebra $\mathcal{A}$ can act *irreducibly* by bounded operators on a Hilbert space $\mathcal{K}$. We will only be interested here in algebras that have trivial centers (consisting only of complex scalars).[29] Under this restriction, $\mathcal{A}$ will actually consist of all bounded operators on $\mathcal{K}$. We also only consider Hilbert spaces of at most countably infinite dimension.

If $\mathcal{K}$ has finite dimension d, then all operators on $\mathcal{K}$ are bounded. We say that the algebra of operators on $\mathcal{K}$ is of type I_d. If $\mathcal{K}$ is infinite dimensional, we call the algebra of bounded operators on $\mathcal{K}$ an algebra of type I_∞. A von Neumann algebra (with trivial center) acting irreducibly on a Hilbert space is always of one of these two types.

A "trace" on a von Neumann algebra is a linear function $\mathsf{a} \to \mathrm{Tr}\mathsf{a}$ that satisfies $\mathrm{Tr}\mathsf{ab} = \mathrm{Tr}\mathsf{ba}$ and $\mathrm{Tr}\mathsf{a}^\dagger\mathsf{a} > 0$ for $\mathsf{a} \neq 0$. Obviously, an algebra of type I_d has a trace. For type I_∞, we can define a trace that has the right properties except that it cannot be defined on the whole algebra as it may diverge; for instance, the trace of the identity operator on an infinite-dimensional Hilbert space is $+\infty$.

In constructing more exotic algebras, we are interested in algebras that can be constructed as limits of matrix algebras. (Such algebras are called hyperfinite.) Such constructions were introduced and developed by von Neumann (1938), Powers (1967), and Araki and Woods (1968).

C. Algebras of type II

The first nontrivial example is the hyperfinite type II_1 factor of Murray and von Neumann. It can be constructed as follows from a countably infinite set of maximally entangled qubit pairs.

[29] A von Neumann algebra with trivial center is called a factor. Factors exhibit the main subtleties of von Neumann algebras, and von Neumann algebras that are not factors are built from factors in a relatively simple way. So it is natural to concentrate on factors here.

Let V be a vector space consisting of 2×2 complex matrices, with Hilbert space structure defined by $\langle v,w\rangle = \mathrm{Tr}v^\dagger w$. Let M_2 and M_2' be two copies of I_2, the algebra of 2×2 complex matrices. We let M_2 and M_2' act on V on the left and right, respectively. Thus $a\in M_2$ acts on $v\in V$ by $v\to av$, and $a'\in M_2'$ acts on v by $v\to va'^{\mathrm{tr}}$ where tr is the transpose. Obviously, M_2 and M_2' are commutants.

We can view V as a tensor product $W\otimes W'$, where W is a space of two-component column vectors acted on by M_2 and W' is a space of two-component row vectors acted on by M_2'. Thus V is a bipartite quantum system. Let I_2 be the 2×2 identity matrix. A normalized maximally entangled vector in V is given by $I_2'=I_2/\sqrt{2}$.

Now consider a countably infinite set of copies of this construction; thus, for $k\geq 1$, let $V^{[k]}$ be a space of 2×2 matrices acted on on the left by $M_2^{[k]}$ and on the right by $M_2'^{[k]}$.

Roughly speaking, we want to consider the infinite tensor product $V^{[1]}\otimes V^{[2]}\otimes\cdots\otimes V^{[k]}\otimes\cdots$. However, taken literally, this infinite tensor product is a vector space of uncountable dimension. To get a Hilbert space of countably infinite dimension, we instead proceed as follows. To start with, we define a space $\mathcal{H}_0$ that consists of tensor products

$$v_1\otimes v_2\otimes\cdots\otimes v_k\otimes\cdots\in V^{[1]}\otimes V^{[2]}\otimes\cdots\otimes V^{[k]}\otimes\cdots \tag{6.1}$$

such that all but finitely many of the v_k are equal to I_2'. This gives a countably infinite-dimensional vector space, but not yet a Hilbert space. To make a Hilbert space, we first define an inner product on $\mathcal{H}_0$. This is done as follows. If $v=v_1\otimes v_2\otimes\cdots$ and $w=w_1\otimes w_2\otimes\cdots$ are elements of $\mathcal{H}_0$, then there is some n such that v_k and w_k both equal I_2' for $k>n$. We truncate v and w at $v_{\langle n\rangle}=v_1\otimes v_2\otimes\cdots\otimes v_n$, $w_{\langle n\rangle}=w_1\otimes w_2\otimes\cdots\otimes w_n$, and define

$$\langle v,w\rangle = \mathrm{Tr}v_{\langle n\rangle}^\dagger w_{\langle n\rangle}. \tag{6.2}$$

This does not depend on where the truncation was made. Having defined a Hermitian inner product on $\mathcal{H}_0$, we complete it to get a Hilbert space $\mathcal{H}$, which is called a restricted tensor product of the $V^{[k]}$. For $v_1\otimes v_2\otimes\cdots\otimes v_n\otimes\cdots$ to be a vector in the restricted tensor product, the v_n must tend rapidly to I_2' for $n\to\infty$.

We do something similar with the algebras. Roughly speaking, we want to define an algebra $\mathcal{A}$ as an infinite tensor product $M_2^{[1]}\otimes M_2^{[2]}\otimes\cdots\otimes M_2^{[n]}\otimes\cdots$. However, a general element $\mathsf{a}=a_1\otimes a_2\otimes\cdots\otimes a_n\otimes\cdots$ cannot act on the restricted tensor product $\mathcal{H}$. (Acting on $v_1\otimes v_2\otimes\cdots\otimes v_n\otimes\cdots$, it would not preserve the condition that the v_n go rapidly to I_2' for $n\to\infty$.) To get around this, we first define an algebra $\mathcal{A}_0$ that consists of elements $\mathsf{a}=a_1\otimes a_2\otimes\cdots\otimes a_n\otimes\cdots$ such that all but finitely many of the a_i are equal to I_2. This algebra acts on $\mathcal{H}$, and it obeys all the conditions of a von Neumann algebra except that it is not closed. To make it closed we add limits. We say that a sequence $\mathsf{a}_{(k)}\in\mathcal{A}_0$ converges if $\lim_{n\to\infty}\mathsf{a}_{(n)}\chi$ exists for all $\chi\in\mathcal{H}$; if so, we define an operator $\mathsf{a}:\mathcal{H}\to\mathcal{H}$ by $\mathsf{a}\chi=\lim_{n\to\infty}\mathsf{a}_{(n)}\chi$, and we define $\mathcal{A}$ to include all such limits. This definition ensures that for $\mathsf{a}\in\mathcal{A}$, $\chi\in\mathcal{H}$, $\mathsf{a}\chi$ is a continuous function of a. Note that the definition of $\mathcal{A}$ depends on a knowledge of the Hilbert space that it is supposed to act on, which entered the question of which sequences $\mathsf{a}_{(n)}$ converge. This will be important in Sec. VI.D.

The commutant of $\mathcal{A}$ is an isomorphic algebra $\mathcal{A}'$ that is defined in just the same way, as a subalgebra of $M_2'^{[1]}\otimes M_2'^{[2]}\otimes\cdots\otimes M_2'^{[n]}\otimes\cdots$.

The vector

$$\Psi = I_2'\otimes I_2'\otimes\cdots\otimes I_2'\otimes\cdots\in\mathcal{H} \tag{6.3}$$

is cyclic separating for $\mathcal{A}$ and for $\mathcal{A}'$. (To show that $\mathsf{a}\Psi\neq 0$ for any nonzero $\mathsf{a}\in\mathcal{A}$, we approximate a by a linear combination of tensor products $a_1\otimes a_2\otimes\cdots\otimes a_n\otimes\cdots$, where in each term $a_n=I_2$ for sufficiently large n, and observe that a nonzero element of this kind certainly does not annihilate Ψ.)

A natural linear function on the algebra $\mathcal{A}$ is defined by $F(\mathsf{a})=\langle\Psi|\mathsf{a}|\Psi\rangle$. Since Ψ is separating for $\mathcal{A}$, any nonzero $\mathsf{a}\in\mathcal{A}$ satisfies $\mathsf{a}\Psi\neq 0$ and hence $F(\mathsf{a}^\dagger\mathsf{a})>0$. We claim that the function F has the defining property of a trace: $F(\mathsf{ab})=F(\mathsf{ba})$. Indeed, if $\mathsf{a}=a_1\otimes a_2\otimes\cdots\otimes a_n\otimes\cdots$, $\mathsf{b}=b_1\otimes b_2\otimes\cdots\otimes b_n\otimes\cdots$ with $a_n,\ b_n=I_2$ for $n>k$, then

$$\begin{aligned}F(\mathsf{ab}) &= \mathrm{Tr}_{M_2^{[1]}\otimes M_2^{[2]}\otimes\cdots\otimes M_2^{[k]}}a_1b_1\otimes a_2b_2\otimes\cdots\otimes a_kb_k\\ &= F(\mathsf{ba}).\end{aligned} \tag{6.4}$$

Since elements a,b of the form just considered are dense in $\mathcal{A}$, the general result $F(\mathsf{ab})=F(\mathsf{ba})$ follows by taking limits, given the way that $\mathcal{A}$ was defined. Since the function $F(\mathsf{a})$ has the properties of a trace, we denote it as $\mathrm{Tr}\mathsf{a}$.

We recall that in the case of a type I_∞ algebra, one can define a trace on a subalgebra but the trace of the identity element is infinite. By contrast, a hyperfinite type II_1 algebra has a trace that is defined on the whole algebra, and which we have normalized so that $\mathrm{Tr}1=1$.

The entanglement entropy in the state Ψ is infinite, since each factor of I_2' represents a perfectly entangled qubit pair shared between $\mathcal{A}$ and $\mathcal{A}'$. Replacing Ψ by another vector in $\mathcal{H}$ will only change the entanglement entropy by a finite or at least less divergent amount, because of the way the restricted tensor product was defined. So the leading divergence in the entanglement entropy is universal, as in quantum field theory.

Another fundamental fact—more or less equivalent to the universal divergence in the entanglement entropy—is that the type II_1 algebra $\mathcal{A}$ has no irreducible representation.

$\mathcal{A}$ acts on the Hilbert space $\mathcal{H}$ that we have constructed, but this action is far from irreducible, as it commutes with the action of $\mathcal{A}'$ on the same space. We can make a smaller representation of $\mathcal{A}$ by projecting $\mathcal{H}$ onto an invariant subspace. Set $J_2=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and consider the following element of $\mathcal{A}'$:

$$\Pi_k' = J_2\otimes J_2\otimes\cdots\otimes J_2\otimes I_2\otimes I_2\otimes\cdots \tag{6.5}$$

with precisely k factors of J_2 and the rest I_2. This is a projection operator with[30] $\mathrm{Tr}\Pi'_k = 2^{-k}$. The subspace $\mathcal{H}\Pi'$ of $\mathcal{H}$ (that is, the set of all elements of $\mathcal{H}$ of the form $\chi\Pi'$ for some $\chi \in \mathcal{H}$) is a representation of $\mathcal{A}$ that, in a sense that was made precise by Murray and von Neumann, is smaller by a factor of 2^k. We can keep going and never get to an irreducible representation. Concretely, Π'_k projects onto vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots \in \mathcal{H}$ such that $v_1, v_2, \ldots, v_k$ are of the form $\left(\begin{smallmatrix} s & 0 \\ t & 0 \end{smallmatrix}\right)$. To get an irreducible representation of $\mathcal{A}$, we must impose such a condition on v_n for all n, but an infinite tensor product of vectors of this type is not in $\mathcal{H}$.

The type II_1 algebra that we considered has some properties in common with local algebras in quantum field theory—they share a universal divergence in the entanglement entropy and the absence of an irreducible representation. But local algebras in quantum field theory do not possess a trace.

D. Algebras of type III

More general algebras can be constructed by proceeding similarly, but with reduced entanglement.

For $0 < \lambda < 1$, define a matrix

$$K_{2,\lambda} = \frac{1}{(1+\lambda)^{1/2}} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{1/2} \end{pmatrix}. \tag{6.6}$$

This matrix describes a pair of qubits with nonzero but also nonmaximal entanglement. (We sometimes include the case $\lambda = 1$; note that $K_{2,1}$ is the matrix I'_2 of Sec. VI.B.)

In the construction of a Hilbert space $\mathcal{H}$ in Sec. VI.C, replace I'_2 everywhere by $K_{2,\lambda}$. Thus, consider the space $\mathcal{H}_0$ spanned by vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots \in V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[n]} \otimes \cdots$ such that all but finitely many of the v_n are equal to $K_{2,\lambda}$. Define $\mathcal{H}_\lambda$ to be the Hilbert space closure of $\mathcal{H}_0$. Similarly, to define an algebra, start with the same $\mathcal{A}_0$ that we used in Sec. VI.C, and take its closure in the space of bounded operators acting on $\mathcal{H}_\lambda$. This gives a von Neumann algebra $\mathcal{A}_\lambda$. $\mathcal{A}_\lambda$ differs from the algebra $\mathcal{A}$ constructed in Sec. VI.C because the Hilbert space $\mathcal{H}_\lambda$ differs from the Hilbert space $\mathcal{H}$ of that section. In other words, the condition for a sequence of operators $\mathsf{a}_n \in \mathcal{A}_0$ to converge depends on which vectors the a_n are supposed to act on, so it depends on the choice of the matrix $K_{2,\lambda}$.

Again, the commutant $\mathcal{A}'_\lambda$ is defined similarly and is isomorphic to $\mathcal{A}_\lambda$. The vector $\Psi = K_{2,\lambda} \otimes K_{2,\lambda} \otimes \cdots \otimes K_{2,\lambda} \otimes \cdots$ is cyclic and separating for $\mathcal{A}_\lambda$ and for $\mathcal{A}'_\lambda$. The corresponding function $F(\mathsf{a}) = \langle\Psi|\mathsf{a}|\Psi\rangle$ does not satisfy $F(\mathsf{ab}) = F(\mathsf{ba})$, and indeed the algebra $\mathcal{A}_\lambda$ does not admit a trace.

The entanglement entropy between $\mathcal{A}_\lambda$ and $\mathcal{A}'_\lambda$ in the state Ψ is divergent, because Ψ describes an infinite collection of qubit pairs each with the same entanglement. As in Sec. VI.C, this divergence is universal; any state in $\mathcal{H}_\lambda$ has the same leading divergence in the entanglement entropy.

As in Sec. VI.C, the action of $\mathcal{A}_\lambda$ on $\mathcal{H}_\lambda$ is far from irreducible; it can be decomposed as finely as one wishes using projection operators in $\mathcal{A}'_\lambda$. In this case, however, although we will not prove it, the invariant subspaces in which $\mathcal{H}_\lambda$ can be decomposed are isomorphic as representations of $\mathcal{A}_\lambda$ to $\mathcal{H}_\lambda$ itself: a hyperfinite von Neumann algebra of type III has only one nontrivial representation, up to isomorphism. All statements in the last three paragraphs also apply to the additional type III algebras that we come to momentarily.

Powers (1967) proved that $\mathcal{A}_\lambda$ and $\mathcal{A}_{\vec{\lambda}}$ for $\lambda \neq \widetilde{\lambda}$ are nonisomorphic. Araki and Woods (1968) considered a generalization of this construction involving a sequence $\lambda_1, \lambda_2, \ldots, 0 < \lambda_i \leq 1$. Now one considers vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots \in V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[n]} \otimes \cdots$ such that $v_n = K_{2,\lambda_n}$ for all but finitely many n. Such vectors make a vector space $\mathcal{H}_{0,\vec{\lambda}}$ whose Hilbert space closure gives a Hilbert space $\mathcal{H}_{\vec{\lambda}}$. To construct an algebra $\mathcal{A}_{\vec{\lambda}}$, we start with the same algebra $\mathcal{A}_0$ as before, and take its closure in the space of bounded operators on $\mathcal{H}_{\vec{\lambda}}$. The commutant $\mathcal{A}'_{\vec{\lambda}}$ is constructed similarly, and

$$\Psi_{\vec{\lambda}} = K_{2,\lambda_1} \otimes K_{2,\lambda_2} \otimes \cdots \otimes K_{2,\lambda_n} \otimes \cdots \tag{6.7}$$

is a cyclic and separating vector for this pair of algebras. (The expectation $\langle\Psi|\mathsf{a}|\Psi\rangle$ is not a trace unless the λ_i are all 1.)

Araki and Woods (1968) showed that if the sequence $\lambda_1, \lambda_2, \ldots$ converges to some λ satisfying $0 < \lambda < 1$, then this construction gives the same type III_λ algebra as before. If the sequence converges to 0, one gets an algebra of type I_∞ if the convergence is fast enough. If it is not fast enough, one gets a new algebra that is defined to be of type III_0.

However, if the sequence $\lambda_1, \lambda_2, \ldots$ does not converge and has at least two limit points in the interval $0 < \lambda < 1$, which are generic in a sense that will be described in Sec. VI.E, then the algebra $\mathcal{A}_{\vec{\lambda}}$ is a new algebra that is defined to be of type III_1.

E. Back to quantum field theory

Local algebras $\mathcal{A}_\mathcal{U}$ in quantum field theory are of[31] type III, since they do not have a trace—even one defined only on part of the algebra. In fact, they are believed to be of type III_1. We will give a somewhat heuristic explanation of this statement, by using the spectrum of the modular operator to distinguish the different algebras.

Because of the way the algebras were constructed from an infinite tensor product of 2×2 matrix algebras, we can understand the modular operator by looking first at the 2×2 case. Let us return to the case of a single product $M_2 \times M'_2$ acting on a Hilbert space V of 2×2 matrices, with the cyclic separating vector $K_{2,\lambda}$. We factorize $V = W \otimes W'$

[30]More generally, for every real x with $0 \leq x \leq 1$, $\mathcal{A}'$ has a projection operator Π'_x with $\mathrm{Tr}\Pi'_x = x$. Projecting on the image of Π'_x (acting on $\mathcal{H}$ on the right) gives a representation of $\mathcal{A}$ whose "dimension" in the sense of Murray and von Neumann is x.

[31]This was first shown for free fields by Araki (1964), before the finer classification of type III algebras was known; see also Longo (1982) and Fredenhagen (1985).

in terms of column and row matrices. The reduced density matrices for the two factors are

$$\rho_1 = \rho_2 = \frac{1}{1+\lambda}\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}. \tag{6.8}$$

According to Sec. IV.A, Δ_Ψ acts on a 2×2 matrix $x \in V$ by $x \to \rho_1 x(\rho_2^{\rm tr})^{-1}$. We see that, in this case, its eigenvalues are $1, \lambda$, and λ^{-1}.

Now let us consider the type III_λ algebra $\mathcal{A}_\lambda$ that was constructed in Sec. VI.D. It has the cyclic separating vector

$$\Psi = K_{2,\lambda} \otimes K_{2,\lambda} \otimes \cdots \otimes K_{2,\lambda} \otimes \cdots \tag{6.9}$$

constructed as an infinite tensor product of copies of $K_{2,\lambda}$. In this case, Δ_Ψ is an infinite tensor product of the answer that we just found in the 2×2 case. The eigenvalues of Δ_Ψ are all integer powers of λ, each occurring infinitely often. The accumulation points of the eigenvalues[32] are the powers of λ and 0 (which is an accumulation point as it is the large n limit of λ^n). More generally, the vector $\Psi_{\vec\lambda} = K_{2,\lambda_1} \otimes K_{2,\lambda_2} \otimes \cdots \otimes K_{2,\lambda_n} \otimes \cdots$ is cyclic separating for $\mathcal{A}_\lambda$ if the λ_k approach λ sufficiently fast. The operator $\Delta_{\Psi_{\vec\lambda}}$ now has a more complicated set of eigenvalues, but 0 and the integer powers of λ are still accumulation points. Still more generally, in the case of a type III_λ algebra, for any cyclic separating vector Ψ, not necessarily of the form $\Psi_{\vec\lambda}$, the integer powers of λ and 0 are accumulation points of the eigenvalues. Roughly this is because any cyclic separating vector can be very well approximated by only changing the original one in Eq. (6.9) in finitely many factors.

For type III_0, the λ_k are approaching 0 and the only unavoidable accumulation points of the eigenvalues of $\Delta_{\Psi_{\vec\lambda}}$ are 0 and 1. These values continue to be accumulation points if $\Psi_{\vec\lambda}$ is replaced by any cyclic separating vector of a type III_0 algebra.

Now let us consider a type III_1 algebra. Suppose that in Eq. (6.7), the λ_k take the two values λ and $\widetilde\lambda$, each infinitely many times. Then the eigenvalues of $\Delta_{\Psi_{\vec\lambda}}$ consist of the numbers $\lambda^n\widetilde\lambda^m$, $n, m \in \mathbb{Z}$, each value occurring infinitely many times. If λ and $\widetilde\lambda$ are generic, then every non-negative real number can be approximated arbitrarily well[33] as $\lambda^n\widetilde\lambda^m$, with integers n, m. So in this case all non-negative real numbers are accumulation points of the eigenvalues. This is the hallmark of a type III_1 algebra: for any cyclic separating vector Ψ, the spectrum of Δ_Ψ (including accumulation points of eigenvalues) comprises the full semi-infinite interval $[0,\infty)$.

[32]Mathematically, the "spectrum" of an unbounded operator is defined to include accumulation points of its eigenvalues, along with the eigenvalues themselves and a possible continuous spectrum. The accumulation points and the possible continuous spectrum are important in the following remarks.

[33]The case that this is not true is that there is some λ' with $\lambda = \lambda'^n$, $\widetilde\lambda = \lambda'^m$, $n, m \in \mathbb{Z}$. Then the spectrum of $\Delta_{\Psi_{\vec\lambda}}$ consists of integer powers of λ', and the algebra is of type $\mathrm{III}_{\lambda'}$.

Now let us return to quantum field theory and consider the case that $\mathcal{U}$ is a wedge region, as analyzed in Sec. V. The modular operator for the vacuum state Ω is $\Delta_\Omega = \exp(-2\pi K)$, where K is the Lorentz boost operator. K has a continuous spectrum consisting of all real numbers, so Δ_Ω has a continuous spectrum consisting of all positive numbers. In particular, all points in $[0,\infty)$ are in that spectrum. Now suppose we replace Ω by some other cyclic separating vector Ψ. At short distances, any state is indistinguishable from the vacuum. So we would expect that acting on excitations of very short wavelength, Δ_Ψ can be approximated by Δ_Ω and therefore has all points in $[0,\infty)$ in its spectrum. See Fredenhagen (1985) and Sec. V.6 of Haag (1992) for more precise statements. Thus the algebra $\mathcal{A}_\mathcal{U}$ is of type III_1.

What about other open sets $\mathcal{U} \subset M$? For an important class of examples, let Σ be an initial value surface, and let $\mathcal{V} \subset \Sigma$ be an open subset whose closure $\bar{\mathcal{V}}$ has a nonempty boundary. Let $\mathcal{U}_\mathcal{V} \subset M$ be the domain of dependence of $\mathcal{V}$. Its closure $\bar{\mathcal{U}}_\mathcal{V}$ has a "corner" along the boundary of $\bar{\mathcal{V}}$. Let $\Delta_\Omega(\mathcal{U}_\mathcal{V})$ be the modular operator of the state Ω for the algebra $\mathcal{A}_{\mathcal{U}_\mathcal{V}}$. For very high energy excitations localized near the corner, $\mathcal{U}_\mathcal{V}$ looks like the wedge region $\mathcal{U}$. So one would expect that for such high energy excitations, $\Delta_\Omega(\mathcal{U}_\mathcal{V})$ looks like the Lorentz boost generators and has all positive real numbers in its spectrum. Again, changing the state will not matter. So again in this case, the algebra $\mathcal{A}_{\mathcal{U}_\mathcal{V}}$ is of type III_1.

According to the Borchers timelike tube theorem, which was already mentioned at the end of Sec. II.F, for many open sets $\mathcal{U}$ that are not of the form $\mathcal{U}_\mathcal{V}$, $\mathcal{A}_\mathcal{U}$ actually coincides with some $\mathcal{A}_{\mathcal{U}_\mathcal{V}}$ where $\mathcal{U} \subset \mathcal{U}_\mathcal{V}$. So then $\mathcal{A}_\mathcal{U}$ is again of type III_1.

VII. FACTORIZED STATES

A. A Question

Let $\mathcal{U}$ and $\mathcal{U}'$ be complementary open sets with local algebras $\mathcal{A}_\mathcal{U}$, $\mathcal{A}_{\mathcal{U}'}$. (We recall that complementary open sets are each other's causal complements and there is no "gap" between them.) If one had a factorization of the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with each algebra acting on one of the two factors, then one could specify independently the physics in $\mathcal{U}$ and in $\mathcal{U}'$. For any $\Psi \in \mathcal{H}_1$, $\chi \in \mathcal{H}_2$, the tensor product state $\Psi \otimes \chi$ would look like Ψ for observations in $\mathcal{U}$ and like χ for observations in $\mathcal{U}'$.

In fact, there is no such factorization and it is not possible to independently specify the state in $\mathcal{U}$ and in $\mathcal{U}'$.

Suppose, however, that there is a "gap" between $\mathcal{U}$ and $\mathcal{U}'$, leaving room for another open set $\mathcal{U}''$ that is spacelike separated from both of them (Fig. 7). Then, given states Ψ, $\chi \in \mathcal{H}$, the question of finding a state looking like Ψ in $\mathcal{U}$ and like χ in $\mathcal{U}'$ is not affected by ultraviolet divergences. But there is still a possible obstruction, which arises if there is some nontrivial operator x (not a multiple of the identity) that is in both $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$. Such an operator is central in both $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$ (since these algebras commute with each other). In Minkowski spacetime, it is reasonable based on what we know from canonical quantization to expect that $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$ have trivial center and trivial intersection, but in general, in more complicated spacetimes, this might fail (Schroer, 2017;

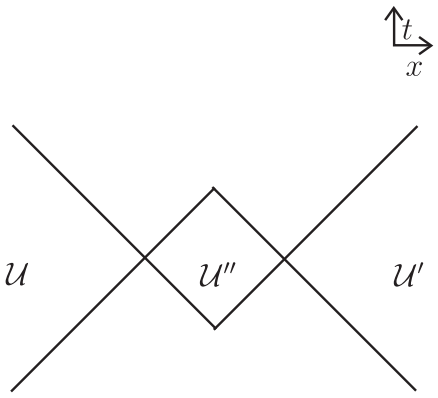

FIG. 7. Two spacelike separated open sets $\mathcal{U}$ and $\mathcal{U}'$ in Minkowski spacetime, with a gap between them.

Harlow and Ooguri, 2018). If there is some $\mathsf{x} \in \mathcal{A}_\mathcal{U} \cap \mathcal{A}_{\mathcal{U}'}$ with $\langle\Psi|\mathsf{x}|\Psi\rangle \neq \langle\chi|\mathsf{x}|\chi\rangle$, then obviously, since x can be measured in either $\mathcal{U}$ or $\mathcal{U}'$, there can be no state that looks like Ψ in $\mathcal{U}$ and like χ in $\mathcal{U}'$.

In proceeding, we assume that there is a gap between $\mathcal{U}$ and $\mathcal{U}'$ and that the intersection of the two algebras is trivial. We impose a further restriction on the boundedness of $\mathcal{U}$ and/or $\mathcal{U}'$ that is discussed later. Given this, it actually is possible,[34] for any Ψ, $\chi \in \mathcal{H}$, to find a state that is indistinguishable from Ψ for measurements in $\mathcal{U}$, and indistinguishable from χ for measurements in $\mathcal{U}'$.

We make use of the gap between $\mathcal{U}$ and $\mathcal{U}'$ in two ways. First, it ensures that the union of the two open sets, $\widehat{\mathcal{U}} = \mathcal{U} \cup \mathcal{U}'$, is "small" enough so that the Reeh-Schlieder theorem applies and the vacuum state Ω is cyclic and separating for the local algebra $\mathcal{A}_{\widehat{\mathcal{U}}}$. (There is another open set $\mathcal{U}''$ that is spacelike separated from $\widehat{\mathcal{U}}$, and this is enough to invoke the theorem.)

Second, we want to use the gap as an ingredient in ensuring that there are no subtleties in building observables in $\widehat{\mathcal{U}}$ from observables in $\mathcal{U}$ and in $\mathcal{U}'$, in the sense that the algebra $\mathcal{A}_{\widehat{\mathcal{U}}}$ is just a tensor product:

$$\mathcal{A}_{\widehat{\mathcal{U}}} = \mathcal{A}_\mathcal{U} \otimes \mathcal{A}_{\mathcal{U}'}. \tag{7.1}$$

However, this point is not straightforward, for several reasons.

First of all, we have to explain what is meant by the tensor product $\mathcal{A}_\mathcal{U} \otimes \mathcal{A}_{\mathcal{U}'}$ of von Neumann algebras. The *algebraic* tensor product $\mathcal{A}_\mathcal{U} \otimes_{\rm alg} \mathcal{A}_{\mathcal{U}'}$ is defined in the familiar way; elements are finite linear combinations $\sum_{i=1}^s \mathsf{a}_i \otimes \mathsf{a}'_i$, with $\mathsf{a}_i \in \mathcal{A}_\mathcal{U}$, $\mathsf{a}'_i \in \mathcal{A}_{\mathcal{U}'}$. Such finite linear combinations are added and multiplied in the familiar way.

However, to get a von Neumann algebra, we have to take a completion of $\mathcal{A}_\mathcal{U} \otimes_{\rm alg} \mathcal{A}_{\mathcal{U}'}$. As usual, what we get when we take a completion depends on what Hilbert space the algebra is acting on. We have seen several examples of this in Sec. VI.

The completion we want is one in which $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$ act completely independently.[35] For this, we introduce a Hilbert space $\widehat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ consisting of two copies of the Hilbert space of our quantum field theory, and we consider the action of $\mathcal{A}_\mathcal{U} \otimes_{\rm alg} \mathcal{A}_{\mathcal{U}'}$ on $\widehat{\mathcal{H}}$ with $\mathcal{A}_\mathcal{U}$ acting on the first factor and $\mathcal{A}_{\mathcal{U}'}$ acting on the second. The von Neumann algebra completion of $\mathcal{A}_\mathcal{U} \otimes_{\rm alg} \mathcal{A}_{\mathcal{U}'}$ acting on $\widehat{\mathcal{H}}$ is the von Neumann algebra tensor product $\mathcal{A}_\mathcal{U} \otimes \mathcal{A}_{\mathcal{U}'}$.

This explains what Eq. (7.1) would mean, but it is not true without some further condition on $\mathcal{U}$ and $\mathcal{U}'$. The gap between them avoids ultraviolet issues that would obstruct the factorization in Eq. (7.1), but there are still infrared issues.

Before explaining this, we consider a simpler question that will actually also be relevant in Sec. VII.C. If a given quantum field theory has more than one vacuum state,[36] does the algebra $\mathcal{A}_\mathcal{U}$ for an open set $\mathcal{U}$ depend on the choice of vacuum? If $\mathcal{U}$ is a *bounded* open set, with compact closure, one expects on physical grounds that the answer will be "no." But in the case of a noncompact region, in general $\mathcal{A}_\mathcal{U}$ does depend on the vacuum.

To understand this, first pick a smooth real smearing function f supported in region $\mathcal{U}$ such that

$$\int_\mathcal{U} d^D x |f|^2 < \infty \tag{7.2}$$

but

$$\int_U d^D x f = \infty. \tag{7.3}$$

Such an f is, of course, not compactly supported. Now pick a local field ϕ and consider the question of whether there exists an operator corresponding to

$$\phi_f = \int_U d^D x f(x)\phi(x). \tag{7.4}$$

A "yes" answer means that there is a dense set of Hilbert space states Ψ such that $|\phi_f\Psi|^2 < \infty$. If so, then bounded functions of ϕ_f such as $\exp(i\phi_f)$ would be included in the algebra $\mathcal{A}_\mathcal{U}$. Actually, since we assume (as part of what we mean by saying that ϕ is a local field) that ϕ_f is a Hilbert space operator if f is compactly supported, the only concern in the noncompact case is a possible infrared divergence in computing $|\phi_f\Psi|^2$. Since any state looks like the vacuum near infinity, such an infrared divergence will not depend on the choice of Ψ and the condition for ϕ_f to be a good operator is just that $|\phi_f\Omega|^2 < \infty$. When we compute $|\phi_f\Omega|^2 = \langle\Omega|\phi_f\phi_f|\Omega\rangle$, we will run into connected and disconnected two-point functions of ϕ. Let us

[34] This question and similar ones are related to what is called the split property in algebraic quantum field theory and have been analyzed with increasing detail by Roos (1970), Buchholz (1974), and Doplicher and Longo (1984).

[35] It is here that we assume that the intersection of the two algebras is trivial. If they have a nontrivial element x in common, it is not possible for them to act independently.

[36] This can happen because of a spontaneously broken symmetry, but there are other possible reasons. For instance, vacuum degeneracy not associated to any symmetry can arise at a first order phase transition, and supersymmetric models often have multiple vacua.

assume for simplicity that our theory has a mass gap. Then the connected correlation function is short range and the condition (7.2) is sufficient to ensure that there is no infrared divergence in the connected part of the correlation function. However, Eq. (7.3) means that the disconnected part of the correlation function will make a divergent contribution to $|\phi_f\Omega|^2$ unless $\langle\Omega|\phi|\Omega\rangle = 0$, that is, unless the disconnected part of the correlation function is 0. The condition that $\langle\Omega|\phi|\Omega\rangle = 0$ certainly depends on the vacuum, and therefore, the question of which ϕ we can use in constructing ϕ_f depends on the vacuum. Thus, for an unbounded open set $\mathcal{U}$, $\mathcal{A}_\mathcal{U}$ depends on the vacuum.

Somewhat similarly, while keeping fixed the vacuum at infinity, one can ask whether $\mathcal{A}_\mathcal{U}$, for noncompact $\mathcal{U}$, depends on the choice of a superselection sector. The general answer to this question is not clear to the author.

Now let us return to the case of $\mathcal{A}_{\widehat{\mathcal{U}}}$ with $\widehat{\mathcal{U}} = \mathcal{U}\cup\mathcal{U}'$. For completely general regions $\mathcal{U}$ and $\mathcal{U}'$, there can be a subtlety analogous to what we encountered in comparing different vacua. For example,[37] suppose that $\mathcal{U}$ and $\mathcal{U}'$ are noncompact and are asymptotically parallel in the sense that there is some fixed vector b such that, at least near infinity, the translation $x \to x + b$ maps $\mathcal{U}$ to $\mathcal{U}'$. Then we can pick local fields ϕ_i and ϕ_i', $i = 1,\dots,s$ and with f as given before, we can attempt to define the operator

$$X_f = \sum_{i=1}^{s}\int_\mathcal{U} d^D x f(x)\phi_i(x)\phi_i'(x+b), \tag{7.5}$$

whose support is in $\widehat{\mathcal{U}} = \mathcal{U}\cup\mathcal{U}'$. Assuming again a mass gap, the condition for X_f to be well defined is that the relevant vacuum expectation value must vanish. In the present case, the operator whose vacuum expectation value must vanish is $X = \sum_i \phi_i(x)\phi_i'(x+b)$. The condition for this to vanish in the vacuum depends on whether $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$ (and hence ϕ_i and ϕ_i') act on the same Hilbert space $\mathcal{H}$ or on the two factors of $\widehat{\mathcal{H}} = \mathcal{H}\otimes\mathcal{H}'$. When the two algebras act on the same copy of $\mathcal{H}$, connected two-point functions contribute in the evaluation of $\langle\Omega|X|\Omega\rangle = \langle\Omega|\sum_i\phi_i(x)\phi_i'(x+b)|\Omega\rangle$. There are no such connected contributions if the two algebras act on two different copies of the Hilbert space. The operators X_f that are well defined are different in the two cases, and thus this gives an example of $\mathcal{U}$ and $\mathcal{U}'$ for which the relation (7.1) that we want is not true.

A sufficient condition that avoids all such questions is to consider bounded open sets only. Indeed, to avoid such issues, and because of a belief that physics is fundamentally local in character, Haag (1992) bases the theory on the $\mathcal{A}_\mathcal{U}$ for bounded open sets $\mathcal{U}$. However, for the specific question under discussion here, we can avoid infrared issues in connected correlation functions if just $\mathcal{U}$ or $\mathcal{U}'$ is bounded. Then the well definedness of an operator such as X_f is the same whether the two algebras act on the same copy or two different copies of $\mathcal{H}$. We make this assumption going forward. For applications discussed in Sec. VII.C that involve just one open set $\mathcal{U}$, we assume that $\mathcal{U}$ is bounded.

Now let us suppose that $\mathcal{U}$ and $\mathcal{U}'$ have been chosen to ensure the factorization (7.1). Since the Reeh-Schlieder theorem applies to $\widehat{\mathcal{U}}$, the algebra $\mathcal{A}_{\widehat{\mathcal{U}}}$ acts on the Hilbert space $\mathcal{H}$ of our quantum field theory with the vacuum vector Ω as a cyclic separating vector. But Eq. (7.1) means by definition that precisely the same algebra can act on $\widehat{\mathcal{H}} = \mathcal{H}\otimes\mathcal{H}'$ with $\mathcal{A}_\mathcal{U}$ acting on the first copy and $\mathcal{A}_{\mathcal{U}'}$ acting on the second. In $\widehat{\mathcal{H}}$, the vector $\Phi = \Omega\otimes\Omega$ is cyclic and separating.

However, whenever the same von Neumann algebra $\mathcal{A}_{\widehat{\mathcal{U}}}$ acts on two different Hilbert spaces $\mathcal{H}$ and $\widehat{\mathcal{H}}$, in each case with a cyclic separating vector, there is always a map between the two Hilbert spaces that maps one action to the other. (It does not generically map one cyclic separating vector to the other.) Applied to our problem, this will enable us to find in $\mathcal{H}$ a state that looks like Ψ for observations in $\mathcal{U}$ and like χ for observations in $\mathcal{U}'$.

We explain the statement about von Neumann algebras in Sec. VII.B. The application to our question, and a few other applications, are discussed in Sec. VII.C.

B. Mapping one representation to another

We assume that the von Neumann algebra $\mathcal{A}$ acts on two Hilbert spaces $\mathcal{H}$ and $\widehat{\mathcal{H}}$ with cyclic separating vectors $\Psi\in\mathcal{H}$ and $\Phi\in\widehat{\mathcal{H}}$. As remarked at the end of Sec. III.B, the relative modular operators $S_{\Psi|\Phi}:\mathcal{H}\to\widehat{\mathcal{H}}$ and $\Delta_{\Psi|\Phi}:\mathcal{H}\to\mathcal{H}$ are defined in this generality.

We will find an isometric or unitary embedding $T:\widehat{\mathcal{H}}\to\mathcal{H}$ that commutes with the action of $\mathcal{A}$. Using the finite-dimensional formulas of Sec. IV.A, one can guess what the map should be. We define a linear map $T:\widehat{\mathcal{H}}\to\mathcal{H}$ by

$$T(\mathsf{a}|\Phi\rangle) = \mathsf{a}\Delta_{\Psi|\Phi}^{1/2}|\Psi\rangle. \tag{7.6}$$

To begin with T is only defined on the dense set of vectors $\mathsf{a}|\Phi\rangle$, $\mathsf{a}\in\mathcal{A}$. But once we show that T is an isometry, this means in particular that it is bounded and it will automatically extend to all of $\widehat{\mathcal{H}}$ as an isometry.

For T to be an isometry means that for all $\mathsf{a},\mathsf{b}\in\mathcal{A}$,

$$\langle\mathsf{b}\Phi|\mathsf{a}\Phi\rangle = \langle\mathsf{b}\Delta_{\Psi|\Phi}^{1/2}\Psi|\mathsf{a}\Delta_{\Psi|\Phi}^{1/2}\Psi\rangle. \tag{7.7}$$

One can show, using formulas of Sec. IV.A, that this statement is true if the Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2$ with each algebra $\mathcal{A}$ and $\mathcal{A}'$ acting on one factor. Very often, statements that are easy to check if one assumes a factorization can be demonstrated in general using Tomita-Takesaski theory. What follows is fairly illustrative of many such arguments.

The right-hand side of Eq. (7.7) is

$$\langle\Psi|\Delta_{\Psi|\Phi}^{1/2}\mathsf{b}^\dagger\mathsf{a}\Delta_{\Psi|\Phi}^{1/2}|\Psi\rangle. \tag{7.8}$$

We want to show that this equals the left-hand side of Eq. (7.7), but first let us consider

[37] This example is discussed in Buchholz (1974) and attributed to Araki.

$$F(s) = \langle\Psi|\Delta_{\Psi|\Phi}^{is}\mathsf{b}^\dagger\mathsf{a}\Delta_{\Psi|\Phi}^{1-is}|\Psi\rangle$$
$$= \langle\Psi|\Delta_{\Psi|\Phi}^{is}\mathsf{b}^\dagger\mathsf{a}\Delta_{\Psi|\Phi}^{-is}S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}|\Psi\rangle \tag{7.9}$$

for real s.

The antiunitarity of $S_{\Psi|\Phi}$ gives

$$F(s) = \langle S_{\Psi|\Phi}\Psi|S_{\Psi|\Phi}\Delta_{\Psi|\Phi}^{is}\mathsf{a}^\dagger\mathsf{b}\Delta_{\Psi|\Phi}^{-is}\Psi\rangle. \tag{7.10}$$

Now we have to remember that conjugation by $\Delta_{\Psi|\Phi}^{is}$ is an automorphism of $\mathcal{A}$, so in particular $\Delta_{\Psi|\Phi}^{is}\mathsf{a}^\dagger\mathsf{b}\Delta_{\Psi|\Phi}^{-is} \in \mathcal{A}$. Moreover, for any $\mathsf{x} \in \mathcal{A}$, $S_{\Psi|\Phi}\mathsf{x}\Psi = \mathsf{x}^\dagger\Phi$. So

$$F(s) = \langle\Phi|\Delta_{\Psi|\Phi}^{is}\mathsf{b}^\dagger\mathsf{a}\Delta_{\Psi|\Phi}^{-is}|\Phi\rangle. \tag{7.11}$$

Now we remember from Sec. IV.B that the automorphism $\mathsf{x} \to \Delta_{\Psi|\Phi}^{is}\mathsf{x}\Delta_{\Psi|\Phi}^{-is}$ of $\mathcal{A}$ depends only on Φ and not on Ψ. So in evaluating this last formula for $F(s)$, we can set $\Psi = \Phi$, whence $\Delta_{\Psi|\Phi}$ reduces to the ordinary modular operator $\Delta_\Phi : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$. Thus

$$F(s) = \langle\Phi|\Delta_\Phi^{is}\mathsf{b}^\dagger\mathsf{a}\Delta_\Phi^{-is}|\Phi\rangle. \tag{7.12}$$

But $\Delta_\Phi|\Phi\rangle = |\Phi\rangle$, so $\Delta_\Phi^{-is}|\Phi\rangle = |\Phi\rangle$. Thus finally for real s

$$F(s) = \langle\Phi|\mathsf{b}^\dagger\mathsf{a}|\Phi\rangle = \langle\mathsf{b}\Phi|\mathsf{a}\Phi\rangle. \tag{7.13}$$

In particular, $F(s)$ is independent of s for real s.

Suppose we know *a priori* that $F(s)$ is holomorphic in the strip $0 > \mathrm{Im}s > -1/2$ and continuous up to the boundary of the strip. Then $F(s)$ has to be constant even if s is not real, so in this case Eq. (7.13) remains valid if we set $s = -i/2$. A look back at the definition (7.9) of $F(s)$ shows that Eq. (7.13) at $s = -i/2$ is what we want. This formula says precisely that Eq. (7.8) equals the left-hand side of Eq. (7.7).

The desired holomorphy goes beyond what was proved in Sec. IV.B and is explained in Appendix A.2.

The result that we found is useful even if the two Hilbert spaces $\mathcal{H}$ and $\widehat{\mathcal{H}}$ are the same. There are many states that are equivalent to Φ for measurements by operators in $\mathcal{A}$; any state $\mathsf{a}'\Phi$, where $\mathsf{a}' \in \mathcal{A}'$ is unitary, has this property. But in that case $\Delta_{\Psi|\mathsf{a}'\Phi} = \Delta_{\Psi|\Phi}$ [Eq. (3.26)] so $\Delta_{\Psi|\mathsf{a}'\Phi}^{1/2}\Psi = \Delta_{\Psi|\Phi}^{1/2}\Psi$. Thus once Ψ is chosen, in every equivalence class of vectors that are equivalent to some Φ for measurements in $\mathcal{A}$, there is a canonical representative $\Delta_{\Psi|\Phi}^{1/2}\Psi$. These representatives make up the canonical cone (Araki, 1974), which has many nice properties.

C. Applications

Our first application of the result of the last section is to a case discussed in Sec. VII.A. Thus, $\mathcal{H}$ is the Hilbert space of a quantum field theory, and $\widehat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ is the tensor product of two copies of $\mathcal{H}$. For open sets $\mathcal{U}, \mathcal{U}'$, at least one of which is bounded, with a gap between them, the same algebra $\mathcal{A}_{\widehat{\mathcal{U}}} = \mathcal{A}_\mathcal{U} \otimes \mathcal{A}_{\mathcal{U}'}$ can act on $\mathcal{H}$ and also on $\widehat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$, within the latter case $\mathcal{A}_\mathcal{U}$ acting on the first factor and $\mathcal{A}_{\mathcal{U}'}$ acting on the second. For cyclic separating vectors, we take $\Psi \in \mathcal{H}$ to be the vacuum vector Ω, and $\Phi \in \widehat{\mathcal{H}}$ to be $\Omega \otimes \Omega$.

The construction of the last section gave an isometric embedding $T : \widehat{\mathcal{H}} \to \mathcal{H}$ that commutes with the action of $\mathcal{A}_{\widehat{\mathcal{U}}}$. Because of the way we chose the action of $\mathcal{A}_\mathcal{U}$ and $\mathcal{A}_{\mathcal{U}'}$ on $\widehat{\mathcal{H}}$, the vector $\Psi \otimes \chi \in \widehat{\mathcal{H}}$ looks like Ψ for measurements in $\mathcal{U}$ and like χ for measurements in $\mathcal{U}'$. So $T(\Psi \otimes \chi)$ is a vector in $\mathcal{H}$ that has the same property.

This sort of reasoning has other applications. For example, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two different superselection sectors in the same quantum field theory. Let $\mathcal{U}$ be a bounded open set; then the same algebra $\mathcal{A}_\mathcal{U}$ acts on both $\mathcal{H}_1$ and $\mathcal{H}_2$. Both $\mathcal{H}_1$ and $\mathcal{H}_2$ contain cyclic separating vectors for $\mathcal{A}_\mathcal{U}$, by the slight extension of the Reeh-Schlieder theorem that was described in Sec. II.C. So we can find an isometric embedding $T : \mathcal{H}_1 \to \mathcal{H}_2$ that commutes with $\mathcal{A}_\mathcal{U}$. If Ψ is a vector in $\mathcal{H}_1$, then $T\Psi$ is a vector in $\mathcal{H}_2$ that cannot be distinguished from Ψ by measurements in the region $\mathcal{U}$. As explained in Haag and Kastler (1964), there is an intuitive reason for this. For example, superselection sectors that are defined by the total magnetic charge cannot be distinguished by measurements in region $\mathcal{U}$, because by such measurements one cannot tell how many magnetic monopoles there are in distant regions.

Similarly, consider a quantum field theory with more than one vacuum state. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces based on these two vacua. For bounded $\mathcal{U}$, the same algebra $\mathcal{A}_\mathcal{U}$ will act in $\mathcal{H}_1$ and in $\mathcal{H}_2$. The same argument as before tells us that measurements in region $\mathcal{U}$ cannot determine which vacuum state we are in. The intuitive reason is that in the Hilbert space built on one vacuum, there can be a state that looks like some other vacuum over a very large region of spacetime.

For a final application, let us consider the following question.[38] Suppose that ρ is a density matrix on $\mathcal{H}$. Is there a pure state $\chi \in \mathcal{H}$ that is indistinguishable from ρ for measurements in region $\mathcal{U}$? If the Hilbert space factored as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathcal{A}_\mathcal{U}$ acting on the first factor, we would answer this question as follows. For measurements in $\mathcal{U}$, we can replace ρ with the reduced density matrix $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}\rho$ on $\mathcal{H}_1$. Then, picking a purification χ of ρ_1 in $\mathcal{H}_1 \otimes \mathcal{H}_2$, χ would be indistinguishable from ρ for measurements in $\mathcal{U}$.

To answer the question without such a factorization, we can use something called the Gelfand-Neimark-Segal (GNS) construction. Consider the function on $\mathcal{A}_\mathcal{U}$ defined by $F(\mathsf{a}) = \mathrm{Tr}_\mathcal{H}\rho\mathsf{a}$; this function is called a faithful normal state on the algebra $\mathcal{A}_\mathcal{U}$. Given this function, the GNS construction produces a Hilbert space $\mathcal{K}$ with action of $\mathcal{A}_\mathcal{U}$ and a cyclic separating vector Ψ such that $F(\mathsf{a}) = \langle\Psi|\mathsf{a}|\Psi\rangle$. The construction is quite simple. To make Ψ cyclic separating, vectors $\mathsf{a}\Psi$ are assumed to satisfy no relations ($\mathsf{a}\Psi \neq \mathsf{b}\Psi$ for $\mathsf{a} \neq \mathsf{b}$) and to comprise a dense subspace $\mathcal{K}_0$ of $\mathcal{K}$. The inner product on $\mathcal{K}_0$ is defined to be $\langle\mathsf{a}\Psi|\mathsf{b}\Psi\rangle = F(\mathsf{a}^\dagger\mathsf{b})$, which in particular ensures that $\langle\Psi|\mathsf{a}|\Psi\rangle = \mathrm{Tr}_\mathcal{H}\rho\mathsf{a}$. All axioms of a Hilbert space are satisfied except completeness. $\mathcal{K}$ is defined as the Hilbert space completion of $\mathcal{K}_0$. Now $\mathcal{A}$ acts on one Hilbert space $\mathcal{H}$ with cyclic separating vector Ω (the vacuum) and on another

[38] See Sec. V.2.2 of Haag (1992), where much more precise results are stated than we explain here.

Hilbert space $\mathcal{K}$ with cyclic separating vector Ψ. So as in Sec. VII.B, we can find an isometric embedding $T:\mathcal{K}\to\mathcal{H}$. Then $T(\Psi)$ is the desired vector in $\mathcal{H}$ that is indistinguishable from ρ for measurements in $\mathcal{U}$.

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APPENDIX A: MORE HOLOMORPHY

1. More on subregions

Here [following Borchers (2000)] we will prove a result relating the modular operators $\Delta_{\Psi;\mathcal{U}}$ and $\Delta_{\Psi;\widetilde{\mathcal{U}}}$ for a pair of open sets $\mathcal{U}$, $\widetilde{\mathcal{U}}$ with $\widetilde{\mathcal{U}}\subset\mathcal{U}$. Ψ is a vector that is cyclic separating for both algebras $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\widetilde{\mathcal{U}}}$; it is kept fixed in the following and will be omitted in the notation. The result we will describe is useful in applications [for example, see Eqs. (6.7) and (6.8) in Balakrishnan *et al.* (2017)].

From Sec. III.F, we know already that $\Delta_{\widetilde{\mathcal{U}}}\geq\Delta_{\mathcal{U}}$, and from Sec. III.E, it follows that

$$\Delta_{\widetilde{\mathcal{U}}}^{\alpha}\geq\Delta_{\mathcal{U}}^{\alpha},\qquad 0\leq\alpha\leq 1. \tag{A1}$$

From this, it follows that for any state χ, and $0\leq\beta\leq 1/2$, we have

$$\begin{aligned}\langle\chi|\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\Delta_{\mathcal{U}}^{2\beta}\Delta_{\widetilde{\mathcal{U}}}^{-\beta}|\chi\rangle&=\langle\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\chi|\Delta_{\mathcal{U}}^{2\beta}|\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\chi\rangle\\&\leq\langle\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\chi|\Delta_{\widetilde{\mathcal{U}}}^{2\beta}|\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\chi\rangle=\langle\chi|\chi\rangle,\end{aligned} \tag{A2}$$

so

$$\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\Delta_{\mathcal{U}}^{2\beta}\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\leq 1,\qquad 0\leq\beta\leq 1/2. \tag{A3}$$

Since $X^{\dagger}X\leq 1$ implies $\|X\|\leq 1$, it follows that

$$\|\Delta_{\mathcal{U}}^{\beta}\Delta_{\widetilde{\mathcal{U}}}^{-\beta}\|\leq 1,\qquad 0\leq\beta\leq 1/2. \tag{A4}$$

An imaginary shift in β does not affect this bound, since the operators $\Delta_{\mathcal{U}}^{is}$, $\Delta_{\widetilde{\mathcal{U}}}^{is}$, $s\in\mathbb{R}$ are unitary. So

$$\|\Delta_{\mathcal{U}}^{-iz}\Delta_{\widetilde{\mathcal{U}}}^{iz}\|\leq 1 \tag{A5}$$

in the strip $1/2\geq\mathrm{Im}z\geq 0$. This bound implies that the operator-valued function $\Delta_{\mathcal{U}}^{-iz}\Delta_{\widetilde{\mathcal{U}}}^{iz}$ is holomorphic in that strip.

2. More on correlation functions

In Sec. VII.B, we needed to know that for $\mathsf{x}=\mathsf{b}^{\dagger}\mathsf{a}\in\mathcal{A}$,

$$\begin{aligned}F(z)&=\langle\Psi|\Delta_{\Psi|\Phi}^{iz}\mathsf{x}\Delta_{\Psi|\Phi}^{1-iz}|\Psi\rangle\\&=\langle\Psi|\Delta_{\Psi|\Phi}^{iz}\mathsf{x}\Delta_{\Psi|\Phi}^{1/2-iz}|\Delta_{\Psi|\Phi}^{1/2}\Psi\rangle\end{aligned} \tag{A6}$$

is holomorphic in the strip $0>\mathrm{Im}z>-1/2$ as well as continuous along the boundaries of the strip. In fact, we will prove that it is holomorphic in a larger strip[39] $0>\mathrm{Im}z>-1$ and again continuous on the boundaries.

As we will see, it helps to consider first the case that the state $\Delta_{\Psi|\Phi}^{1/2}\Psi$ is replaced by $\mathsf{y}\Psi$ for some $\mathsf{y}\in\mathcal{A}$. So we consider the function

$$G(z)=\langle\Psi|\Delta_{\Psi|\Phi}^{iz}\mathsf{x}\Delta_{\Psi|\Phi}^{1/2-iz}|\mathsf{y}\Psi\rangle. \tag{A7}$$

Holomorphy in the strip is now trivial, because the condition $0>\mathrm{Im}z>-1/2$ means that the exponents iz and $1/2-iz$ in Eq. (A7) both have real part between 0 and 1/2, and consequently from Sec. IV.B, we know that both $\Delta_{\Psi|\Phi}^{1/2-iz}|\mathsf{y}\Psi\rangle$ and $\langle\Psi|\Delta_{\Psi|\Phi}^{iz}$ are holomorphic in this strip.

The norm of a state χ is $|\chi|=\sqrt{\langle\chi|\chi\rangle}$, and the norm $\|\mathsf{y}\|$ of a bounded operator y is the least upper bound of $|\mathsf{y}\chi|/|\chi|$ for any state χ. The following proof will depend on getting an upper bound on $|G(z)|$ in the strip by a constant multiple of $|\mathsf{y}\Psi|$. An immediate upper bound is

$$|G(z)|\leq|\Delta_{\Psi|\Phi}^{-i\bar{z}}\Psi|\,||\mathsf{x}||\,|\Delta_{\Psi|\Phi}^{1/2-iz}\mathsf{y}\Psi|. \tag{A8}$$

If $z=s-i\alpha$, with s, $\alpha\in\mathbb{R}$, then the right-hand side of Eq. (A8) only depends on α, since $\Delta_{\Psi|\Phi}^{is}$ is unitary. For $s=0$, the function $G(z)$ is bounded on the compact set $0\leq\alpha\leq 1/2$ [for α in that range it is the inner product of two states that are well defined and bounded in Hilbert space according to Eq. (4.41)], so it is bounded in the whole strip $0\geq\mathrm{Im}z\geq-1/2$. We need to improve this to get a bound by a multiple of $|\mathsf{y}\Psi|$.

Let us look at the function $G(z)$ on the boundaries of the strip. On the lower boundary $z=s-i/2$, $\Delta^{1/2-iz}$ is unitary. Also on that boundary $|\Delta_{\Psi|\Phi}^{-i\bar{z}}\Psi|=|\Delta_{\Psi|\Phi}^{1/2}\Psi|<\infty$. So on the lower boundary, Eq. (A8) bounds $|G(z)|$ by a constant multiple of $|\mathsf{y}\Psi|$. On the upper boundary $z=s$, we write

$$\begin{aligned}|G(z)|&=|\langle\Delta_{\Psi|\Phi}^{1/2+is}\mathsf{x}^{\dagger}\Delta_{\Psi|\Phi}^{-is}\Psi|\mathsf{y}\Psi\rangle|\\&\leq|\Delta_{\Psi|\Phi}^{1/2}\Delta_{\Psi|\Phi}^{is}\mathsf{x}^{\dagger}\Delta_{\Psi|\Phi}^{-is}\Psi|\,|\mathsf{y}\Psi|.\end{aligned} \tag{A9}$$

Reasoning similarly to Eq. (4.40), this implies

$$|G(z)|\leq|\Delta_{\Psi|\Phi}^{is}\mathsf{x}\Delta_{\Psi|\Phi}^{-is}\Phi|\,|\mathsf{y}\Psi|. \tag{A10}$$

Because the operator $\Delta_{\Psi|\Phi}^{is}$ is unitary and $\langle\Phi|\Phi\rangle=1$, we get on the upper boundary

$$|G(z)|\leq||\mathsf{x}||\,|\mathsf{y}\Psi|. \tag{A11}$$

[39]Similarly to Eq. (4.48), one would expect this if one assumes a factorization $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ of the Hilbert space. In this Appendix, we follow Araki's approach to proving such statements without assuming a factorization. See Araki (1973), Sec. III.

So there is a constant C, independent of y and z, such that on the boundaries of the strip

$$|G(z)| \leq C|\mathsf{y}\Psi|. \tag{A12}$$

A holomorphic function, such as $G(z)$, that is bounded and holomorphic in a strip, and obeys a bound $|G(z)| \leq \widehat{C}$ on the boundary of the strip, obeys the same bound in the interior of the strip. This statement is a special case of the Phragmén-Lindelöf principle, and can be proved as follows (we state the argument for our strip $0 > \mathrm{Im}z > -1/2$). For $\epsilon > 0$, the function $G_\epsilon(z) = \exp(-\epsilon z^2)G(z)$ satisfies $|G_\epsilon(z)| \leq \widehat{C}\exp(\epsilon/4)$ on the boundary of the strip. The function $G_\epsilon(z)$ vanishes rapidly for $\mathrm{Re}z \to \pm\infty$, so $|G_\epsilon(z)|$ achieves its maximum somewhere in the interior of the strip or its boundary. By the maximum principle, this maximum is achieved somewhere on the boundary of the strip. Therefore the bound $|G_\epsilon(z)| \leq \widehat{C}\exp(\epsilon/4)$ is satisfied throughout the strip. As this is true for all ϵ, we get $|G(z)| \leq \widehat{C}$ throughout the strip.

Going back to the original definition of $G(z)$ in Eq. (A7), $G(z)$ can be interpreted as a linear functional on the dense subset of $\mathcal{H}$ consisting of states $\mathsf{y}\Psi$, $\mathsf{y} \in \mathcal{A}$. The validity of Eq. (A12) throughout the strip says that this linear functional is bounded. A bounded linear functional on a dense subset of a Hilbert space $\mathcal{H}$ always extends to the whole space, and remains bounded. Moreover a bounded linear functional on a Hilbert space $\mathcal{H}$ is always the inner product with a state in $\mathcal{H}$. So we learn that there is some z-dependent state $\chi(z)$ such that

$$G(z) = \langle\chi(z)|\mathsf{y}\Psi\rangle \tag{A13}$$

for all $\mathsf{y} \in \mathcal{A}$. Moreover $\langle\chi(z)|$ is holomorphic in the strip since $G(z)$ is holomorphic in the strip for all y. The fact that the linear functional in question extends over all of $\mathcal{H}$ means that for any $\Upsilon \in \mathcal{H}$,

$$\langle\chi(z)|\Upsilon\rangle \tag{A14}$$

is well defined and holomorphic in the strip.

The original function $F(z)$ is then

$$F(z) = \langle\chi(z)|\Delta_{\Psi|\Phi}^{1/2}\Psi\rangle. \tag{A15}$$

Here $\Delta_{\Psi|\Phi}^{1/2}\Psi$ is a Hilbert space state [as in Eq. (4.40)], so this is a special case of Eq. (A.14), and therefore is holomorphic in the strip. Moreover the original definition [and bounds such as Eq. (4.41) that were used along the way] make it clear that $F(z)$ has a continuous limit as one approaches the boundaries of the strip.

This is what we needed in Sec. VII.B, but actually the function $F(z)$ is holomorphic in a larger strip. Writing

$$F(z) = \langle\Delta_{\Psi|\Phi}^{1/2}\Psi|\Delta_{\Psi|\Phi}^{-1/2+iz}\mathsf{x}\Delta_{\Psi|\Phi}^{1-iz}|\Psi\rangle, \tag{A16}$$

we make an argument very similar to the previous, but with the role of the bra and the ket exchanged. Thus, we begin by replacing $\Delta_{\Psi|\Phi}^{1/2}\Psi$ with $\mathsf{y}\Psi$ with $\mathsf{y} \in \mathcal{A}$. So we have to study

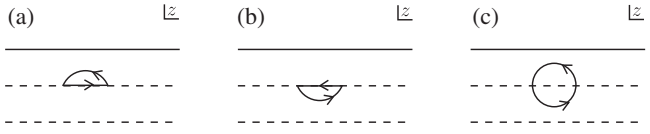

FIG. 8. (a) If a function $F(z)$ is holomorphic in the strip $0 > \mathrm{Im}z > -1/2$ and continuous at the lower boundary of the strip, we can write a Cauchy integral formula with a contour that runs partly on the lower boundary. (b) If $F(z)$ is holomorphic for $-1/2 > \mathrm{Im}z > -1$ and continuous on the upper boundary of that strip, we can write a Cauchy integral formula with a contour that runs partly on the upper boundary. (c) If $F(z)$ satisfies both conditions, we can combine the contours from (a) and (b), choosing them so that the part that runs on the line $\mathrm{Re}z = -1/2$ cancels. The resulting Cauchy integral formula shows that $F(z)$ is holomorphic on that line. The argument sketched in Fig. 1 of Sec. II.B is actually the special case of this in which $F(z)$ vanishes in the lower strip.

$$H(z) = \langle\mathsf{y}\Psi|\Delta_{\Psi|\Phi}^{-1/2+iz}\mathsf{x}\Delta_{\Psi|\Phi}^{1-iz}|\Psi\rangle. \tag{A17}$$

We consider the function $H(z)$ in the strip $-1/2 \geq \mathrm{Im}z \geq -1$. An argument very similar to what we have already seen, reversing the role of the bra and the ket, shows that in this strip $H(z) = \langle\mathsf{y}\Psi|\Upsilon(z)\rangle$, where $\Upsilon(z)$ is holomorphic in the strip. Then $F(z) = \langle\Delta_{\Psi|\Phi}^{1/2}\Psi|\Upsilon(z)\rangle$, and in this representation, holomorphy of $F(z)$ for $-1/2 > \mathrm{Re}z > -1$ is manifest.

We now have a function $F(z)$ that is holomorphic for $0 > \mathrm{Im}z > -1/2$ and for $-1/2 > \mathrm{Im}F(z) > -1$. Moreover, this function is continuous on the line ℓ defined by $\mathrm{Im}z = -1/2$. As sketched in Fig. 8, the Cauchy integral formula can be used to show that $F(z)$ is actually holomorphic on the line ℓ. This fact about holomorphic functions of a single complex variable has a less elementary analog, known as the edge of the wedge theorem, for functions of several complex variables. For some of its applications in quantum field theory, see Streater and Wightman (1964).

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