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B. Nuclear Dynamics, Theoretical*

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* Part A of this report ("Stationary States of Nuclei," by H. A. Bethe and R. F. Bacher) appeared in this journal, April, 1936 (Vol. 8, pp. 82–229). The final part C ("Nuclear Dynamics, Experimental," by M. Stanley Livingston and H. A. Bethe) will appear in the issue of July, 1937.

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IX. Nuclear Processes as Many-Body Problems

§51. Bohr's General Theory (B32[1])

BOHR (B32) was the first to point out that every nuclear process (disintegration, capture of particles, etc.) must be treated as a many-body problem. It is not at all permissible to use a one-body (Hartree) approximation, particularly not in the case of heavy nuclei.

Previously, the one-body approximation had been used extensively for the treatment of nuclear processes (cf. the detailed discussions in §73). The method was taken over from the theory of atomic collisions where it had proved highly successful. Let us consider, as a typical atomic collision, the ionization of an atom by an electron. To treat this process, it is customary to consider the incident as well as the ejected electron as moving in the average field of the residual ion. In this way, "Hartree wave functions" are obtained for the two electrons. Then the interaction between the two electrons is considered as a perturbation, and the matrix element of this perturbation is calculated with the Hartree wave functions for the electrons. The square of the matrix element gives the probability of the collision. This method, known as the "method of the distorted wave functions," gives a good approximation whenever the interaction between the two electrons is small compared to the interaction between one of the electrons and the atom or small compared to the kinetic energy of the incident electron. This is generally true in atomic collisions, and therefore the method described is quite satisfactory for them[2] except for very small energies of the incident electron. It is, in the case of atoms, far superior to the often used Born approximation in which the average interaction between atom and incident electron is neglected.

However, the method of the distorted wave functions is not at all applicable in nuclear physics. As we know (§25[3]), the interaction between one particle and a whole nucleus is only of the same order of magnitude as that between two nuclear particles. Therefore, it has in general no sense to consider the "average potential" acting on the incident particle, in an earlier approximation than the interaction causing the nuclear process itself. Furthermore, the interaction is, in general, large compared to the energy of the incident particle so that the second condition for the applicability of the method of the distorted wave functions is also not fulfilled.

The difference between an atomic and a nuclear collision may be described in a variety of ways. The descriptions are all equivalent but differ in being more or less intuitive and in making use of more classical or more quantum-mechanical concepts.

In an atomic collision, the interaction between the incident particle and the individual electrons of the atom is small, as we have already mentioned. Therefore it is comparatively seldom that a particle in passing through an atom imparts energy to an atomic electron; in other words, inelastic collisions are rare.[4] In most cases, the incident particle will go through the atom without interacting with any particular atomic electron and without losing any energy; it will only be affected by the average field of force of the atom and will be deflected thereby (elastic scattering). The time which the particle spends in the atom is of the order of the atomic dimensions divided by the velocity of the particle. (For electrons, the time is even smaller because the electron is accelerated inside the atom.) Because of this short time spent inside the atom, it is highly improbable that, e.g., radiation is emitted in the collision. Thus atomic collisions are characterized by a very large elastic scattering, a smaller inelastic scattering,[4] and a very small probability of the emission of radiation.

If a particle falls on a *nucleus*, it cannot possibly pass through it without interacting with

[1] A letter and a number, e.g. B32, refer to an original paper. A list of references is given at the end of this article.

[2] It has been applied to the problem of the ionization of the K shell (S17, M9) and to the elastic scattering of electrons by heavy atoms (H25) with considerable success. A wider application has thus far been prevented by mathematical difficulties.

[3] The sections §1 to §50 are contained in part A of this report, Rev. Mod. Phys. **8**, 82 (1936).

[4] This is correct for the collisions in which the incident particle actually passes through the atom (close collisions) and is, in general, deflected by a large angle. In addition, there are numerous inelastic collisions due to particles passing the atom at a distance (B16, W20). These "distant collisions" are due to the long range of the forces between incident particle and atomic electrons (Coulomb forces). Since the nuclear forces are short range forces, there is no analog to the "distant collisions" in nuclear physics.

the individual nuclear particles, for the average distance between them is of the same order as the range of the nuclear forces (§25) due to the peculiar character (short range) of these forces. An incident particle which passes between two nuclear particles must therefore necessarily interact strongly with them; we know that the nuclear forces are very strong (over 10 MV) inside their range of action. The incident particle will therefore lose part of its energy as soon as it strikes the surface of the nucleus, by transferring it to the nuclear particles. As the particle goes on, there will be a further dissipation of its energy among the nuclear particles. As a result, the energy which is initially concentrated in the incident particle, will very soon be distributed among all the particles of the system consisting of the original nucleus and the incident particle (compound nucleus). Each of the particles of the "compound nucleus" will have some energy, but none will have sufficient energy to escape from the rest. Only after a comparatively long time, the energy may "by accident" again be concentrated on one particle so that this particle can escape. Even then, the escaping particle need not be of the same kind as the incident, and even if it is of the same kind, the energy may (and will in general) be divided between the escaping particle and the remaining nucleus in a way different from the initial state of affairs; in other words, the residual nucleus may remain in an excited state. Only if the escaping particle is of the same kind as the incident *and* the internal state of the nucleus is not changed may we speak of an elastic collision. It is obvious that an elastic collision is only a very special case and must therefore be quite a rare event compared to the many kinds of possible inelastic collisions, i.e., such in which either the nucleus is excited (ordinary inelastic collision) and such in which a particle of a different kind is emitted (transmutation). Furthermore it is evident that the time spent by the particle inside the nucleus will in general be very long; it may be several orders of magnitude larger than the time which the particle would need to traverse the nucleus on a straight path without loss of velocity. This provides ample opportunity for the emission of radiation during the collision. Therefore the characteristics of nuclear collisions are: a quite small probability of elastic scattering, a much larger probability of inelastic scattering including transmutations, and a comparatively high probability of the emission of radiation. In some cases radiative processes are even the most probable of all (cf. §58, 61, 62).

The description of the nuclear collisions given in the foregoing shows that the situation during the greater part of the collision may be described as a quasi-stationary state of the "compound nucleus." This compound nucleus will exist during a time which is very large compared with the "characteristic nuclear time" which may be defined as the nuclear radius divided by some average velocity of the nuclear particles, i.e., about[5] $12\cdot10^{-13}/4\cdot10^{+9}=3\cdot10^{-22}$ sec. A theory of nuclear collisions is therefore primarily a theory of the (quasi-stationary) states of the compound nucleus, and of the transitions from these states to states in which one particle or other of the compound nucleus is separated from the rest. The collision may be described by the scheme:

Initial nucleus + incident particle →
compound nucleus → final nucleus
+ outgoing particle.

It is therefore characterized by a double transition, whereas atomic collisions consist in direct transitions from initial to final state.

We have arrived at the idea of the compound nucleus from an almost geometrical argument, *viz.* from the fact that the incident particle cannot find its way through the nucleus in between the nuclear particles but must interact with them. The nucleus is, in Bohr's terminology, a "closed system." An additional particle cannot go through the system but can only be amalgamated with it. In contrast to this, an atom is an "open system" which may be traversed by an external particle without any difficulty, and without the particle's becoming incorporated in the system.

The existence of quasi-stationary states of the compound nucleus, and the nonexistence of an analog in atomic collisions, may also be discussed from an *energetic* rather than geometric point of view. We start from the energy levels of the individual particles (nuclear particles and electrons,

[5] For the value of the nuclear radius, cf. §68. The velocity of $4\cdot10^{+9}$ corresponds to a kinetic energy of about 10 MV, cf. §26.

respectively) in the two cases. It is true that this method of approach, while satisfactory for atoms, does not give a good approximation for nuclei. However, it has the virtue of starting from the extreme opposite to the many-body concept so that the fact that it leads finally to the concept of quasi-stationary states of the compound nucleus is still more convincing.

As we know, the potential energy acting on one nuclear particle (proton or neutron) may, with sufficient accuracy, be represented by a simple "rectangular" hole. In such a hole, the energy levels for the individual particles are almost uniformly distributed from the bottom of the hole to the top. (Strictly speaking, the density of energy levels increases as $E^{\frac{1}{2}}$ where E is the energy above the bottom; cf. §25 and 53.) The depth of the hole is about[6] 18 MV. Since the binding energy of the most loosely bound particle is of the order 8 MV, the potential hole is normally filled with particles up to 8 MV below the top, i.e. to a little more than half its height. There is a great number of empty energy levels between this energy and the top; in fact, the number of empty levels is greater than the number of occupied ones. Let us now consider a state of the nucleus which has, say, 6 MV more energy than the ground state. Such a state may be obtained by exciting *one* of the nuclear particles by 6 MV and leaving the rest unexcited. But equally well, the total excitation energy may be shared between two, three or more particles. It is easily possible that as many as ten particles share in the excitation because the levels of the individual particles lie fairly close together and it is therefore only necessary to supply quite a small amount of energy to each excited particle. (For a quantitative discussion, cf. §53.) Thus it is obvious that there is an enormous variety of ways in which the given total excitation energy (6 MV) may be shared between several particles whereas the configuration in which all the energy is concentrated on one particle represents only a single possibility.

If there were no interaction between the particles, we would thus obtain a very large number of energy levels of the complete nucleus corresponding to the various distributions of the excitation energy among the particles. Besides, we would get a few energy levels corresponding to the excitation of a single particle. In reality, there is interaction between the particles which causes a mixing of the eigenfunctions of the various levels. Therefore, the levels do not fall into different classes corresponding to the excitation of one, two, three . . . particles, but each level is of mixed character. If the nucleus is in any arbitrary state, its excitation energy is part of the time concentrated on one particle, part of the time shared between two, three etc., particles. The exact value of the probability of the concentration of the energy on one particle may be deduced from the eigenfunction of the nuclear state; it is certainly very small because there are so few configurations in which the energy is concentrated and so many in which it is distributed. The number of energy levels, on the other hand, is evidently very great, i.e., the spacing between neighboring energy levels of the nucleus as a whole is extremely small, very much smaller than the spacing between the energy levels of an individual particle would be.

We have chosen the excitation energy of the nucleus lower than the energy necessary to dissociate it into a free particle plus a residual nucleus, in order to avoid dealing with a continuous spectrum for the particle. However, we see now that this restriction is irrelevant because the configuration in which the energy is concentrated on one particle is quite unimportant for the description of a nuclear state. Therefore, the nuclear states which are excited by more than the dissociation energy (about 8 MV) have essentially the same character as those of lower energy; their eigenfunctions are not very different. The only difference is that these more highly excited states may with a certain, quite small, probability disintegrate into a free particle plus a residual nucleus. This probability is given by the coefficient with which the function representing the excitation of a single particle is contained in the eigenfunction of the nuclear state in question. But apart from this comparatively rare event of disintegration, *a nucleus has a series of closely spaced energy levels which have essentially the same*

[6] Cf. (150c) §25. The nuclear radius is about $R=r_0A^{\frac{1}{3}}$ with $r_0=2.05\cdot10^{-13}$ cm (cf. §68) and $N=\frac{1}{2}A$. Therefore the kinetic energy of the most energetic particle $T_0=(9\pi/8)^{\frac{2}{3}}(\hbar^2/2Mr_0^2)=10.3$ MV. The binding energy of this particle is about 8 MV, therefore the depth of the hole $10+8=18$ MV.

character below and above the dissociation energy. The character of these levels will only change decidedly when the excitation energy *per particle* becomes of the order of a few MV, i.e., the excitation energy of the nucleus 100 MV or more.

For atoms, the situation is entirely different. The binding energies of the individual electrons differ by very large amounts: e.g., in the uranium atom the binding energies of successive electrons are approximately 110,000, 20,000, 4000, 800, 200, 40, and 6 volts. The reason for these large differences is, of course, the character of the atomic field, *viz.* a screened Coulomb field which is extremely large near the nucleus and falls off very rapidly at greater distances. The energy region in which there are empty electronic levels, extends over only a few volts. Accordingly, if we excite only outer electrons, we can only obtain excitation energies of a few tens of volts even if we excite many electrons simultaneously. Moreover, since there are only very few electrons in the outer shell as compared to the many particles in a nucleus, there will be comparatively few ways in which a given amount of excitation energy may be shared between the electrons. This makes the configuration in which the whole energy is concentrated on one electron, relatively more probable. The atomic states whose energy is sufficient for a dissociation, will "disintegrate" much more easily into an ion plus an electron than nuclei in corresponding states. (In spite of all this, some states of atoms above the ionization potential, in which two electrons are excited, are known in spectroscopy, especially for the alkaline earths.)

If we excite an inner electron, we obtain, of course, a rather large excitation energy. However, the energy levels of the atom which are obtained in this way, are still restricted to a minute energy region. E.g., if we excite the K electron of uranium and some outer electrons, we obtain a number of energy levels all of which lie between, say, 110,000 and 110,050 volts (above the ground state of the atom). The spectrum of the energy levels of the atom as a whole contains therefore only a few very narrow regions in which there are discrete energy levels, and these regions are separated by immense "empty" spaces in which there is no level at all. The reason is the small region in which there are empty levels for the electrons, a region which is for uranium about 20,000 times smaller than the binding energy of the K electrons, as compared to one-half the binding energy of the most strongly bound particle in nuclei (see above). If we consider the "compound system" formed by the addition of an electron to an atom, i.e., a negative ion, the situation becomes even more extreme because the binding energy of the last electron in a negative ion is considerably less than in an atom. Then the bands in which the compound system possesses energy levels, will extend only over a few volts each. In general, the energy of an incident electron will therefore not fall into one of these regions, and no quasi-stationary state of the compound system will be found in the collision. (It can be shown that the states of the compound system are unimportant even if the energy of the incident electron falls just into one of the energy bands, except for very slow electrons.) Therefore an atomic compound system has no quasi-stationary states of any importance for atomic collisions.

We have seen that the stationary states of the compound nucleus are responsible for quite a different relative magnitude of the various kinds of nuclear collisions (elastic, inelastic, transmutation, radiation) as compared to atomic collisions. But they have another, even more striking consequence: the phenomenon of *resonance.* If the energy of the incident particle is such that the total energy of the system is just equal, or nearly equal, to one of the energy levels of the compound nucleus, the probability of the formation of the compound nucleus will obviously be much greater than if the energy of the particle falls in the region between two resonance levels. Therefore we shall find characteristic fluctuations of the yield of every nuclear process with the energy, from high values at the resonance energies to low values between resonance levels. These resonance phenomena are most pronounced with slow neutrons (Chapter X) but have also been observed in the radiative capture of protons and in transmutations caused by α-particles (Chapter XIII).

The study of the resonance phenomena in nuclear processes is of paramount importance for nuclear physics. First of all, the spacing between neighboring levels of the compound nucleus may be deduced from the resonances. The determination of the spacing as a function of the mass

number and of the excitation energy of the nucleus will enable us to check our theoretical ideas about nuclear structure (cf. §53).

Secondly, the width of the resonance levels is of great interest. Just as in the theory of atoms the width of an excited level is given by the probability of its emitting radiation (W6), so the width of the level of a compound nucleus is given by the total probability of the emission of particles of any kind—neutrons, protons, α-particles, γ-rays etc., by the compound nucleus. Thus the width of the levels enables us to determine the probability of the concentration of energy on any one of the particles in the compound nucleus. This may be supplemented by a study of the relative probability of the emission of various kinds of particles.

The experimental data on nuclear levels are not yet very extensive. The spacing of the energy levels seems to be of the order of 10 volts for nuclei of atomic weight 100 or more and excitation energies just sufficient for a dissociation into a neutron and a residual nucleus (about 8 MV excitation). (These figures are obtained from the slow neutron experiments (§60).) For light nuclei, spacings of a few hundred thousand volts seem to prevail (proton capture (§81), γ-ray spectra (§89), resonances in α-particle disintegrations (§82)). The same order of magnitude has been found for heavy nuclei just above the ground state (from γ-ray evidence and long range α-particles).

A survey of the experimental results for the widths of levels will be given at the end of the next section.

§52. The Dispersion Formula (B51, B15)

The probability of a nuclear process as a function of the energy of the incident particle is given by a formula similar to the ordinary dispersion formula for the scattering of light. This dispersion formula was first derived by Breit and Wigner (B51) for the case of one resonance level of the compound nucleus and then generalized by Bethe and Placzek (B15) for an arbitrary number of resonance levels. For a general derivation, see §55.

Any nuclear process may be described as follows: a particle P falls on a nucleus A which is in a state p. A compound nucleus C is formed which may have a number of energy levels W_r. A particle Q is emitted by the compound nucleus, leaving a residual nucleus B in a state q. The emitted particle Q may or may not be identical with the incident particle P; in the first case, we speak of scattering, in the second case of transmutation. In the special case that Q is a light quantum, we have radiative capture, while we deal with a photoelectric effect, if P is a light quantum and Q a material particle. For the initial and final state energy must be conserved

$$W_{Ap}+W_P+E_P=W_{Bq}+W_Q+E_Q, \quad (254)$$

where W_{Ap} denotes the internal energy of nucleus A in state p, W_P the internal energy and E_P the kinetic energy of the incident particle, etc.[7] For the intermediate state (compound nucleus), of course, conservation of energy is unnecessary.

The probability of the nuclear process

$$A+P\rightarrow C\rightarrow B+Q \quad (254a)$$

may be calculated using the ordinary methods of the Dirac perturbation theory. Since we are dealing with a double process, a second-order perturbation calculation is necessary. The result for the cross section of the process is (B15, and §55)

$$\sigma=4\pi^3\lambda\!\!\!^{-2}\left|\sum_r \frac{H^{APp}{}_{Cr}\,H^{Cr}{}_{BQq}}{W_{Ap}+W_P+E_P-W_{Cr}+\frac{1}{2}i\gamma_r}\right|^2 \quad (255)$$

Here $\lambda\!\!\!^{-}$ is the "wave-length" of the incident particle,

$$\lambda\!\!\!^{-2}=\hbar^2/2ME_P, \quad (256)$$

$$M=M_AM_P/(M_A+M_P). \quad (256a)$$

H is the Hamiltonian of the interaction between the particles concerned, $H^{APp}{}_{Cr}$ its matrix element corresponding to the transition from the initial state (nucleus A in state p+particle P of energy E_P) to the intermediate state (compound nucleus C in state r). To calculate the matrix element, the wave function of the particle P must be normalized per unit energy. γ_r is the total effective width of the level r of the compound nucleus. It is a sum of the partial effective widths due to the emission of various kinds of particles by the compound nucleus:

$$\gamma_r=\sum_{Q'}\gamma^r{}_{Q'}. \quad (257)$$

[7] Throughout this report, we denote by W the energies of nuclear quantum states and by E the kinetic energies of particles.

Among the particles Q' which may be emitted there are the incident particle P, the outgoing particle Q and possibly others. The width $\gamma^r{}_{Q'}$ again may be written in the form

$$\gamma^r{}_{Q'}=\sum_{q'}\gamma^r{}_{Q'q'}, \tag{257a}$$

where $$\gamma^r{}_{Q'q'}=2\pi|H^{Cr}{}_{B'Q'q'}|^2 \tag{257b}$$

is that part of the width of level r which is due to the disintegration of the compound nucleus C into a nucleus B' in state q' and a particle Q' with an energy given by the conservation law (254).

It must be kept in mind that for all quantities occurring in (255) the energy of the corresponding particle (P, Q or Q') must be chosen according to the conservation law (254). Therefore all quantities depend on the energy E_P of the incident particle. This holds also for the effective width γ_r which is therefore in general not the true width of the level r. This true width Γ_r is obtained when the resonance energy

$$E_{Pr}=W_{Cr}-W_{Ap}-W_P \tag{258}$$

is inserted for E_P, so that

$$\Gamma_r=\gamma_r(E_{Pr}). \tag{258a}$$

The cross section (255) shows resonance phenomena. Pronounced maxima occur whenever the energy of the incident particle E_P is near one of the resonance energies (258). In this case, generally one term in (255) will be predominant, namely the one referring to the particular resonance level r in question. Then (255) simplifies to the "one level formula"

$$\sigma^{Pp}{}_{Qq}=\pi\lambda\!\!\!^{-2}\frac{\gamma^r{}_{Pp}\gamma^r{}_{Qq}}{(E_P-E_{Pr})^2+\frac{1}{4}\gamma_r{}^2}, \tag{259}$$

which was first derived by Breit and Wigner. This formula is most useful for applications and sets the resonances in evidence very clearly. The total probability of the production of particles of kind Q summed over all possible levels q in which the nucleus B may be left, is

$$\sigma^{Pp}{}_{Q}=\sum_q\sigma^{Pp}{}_{Qq}=\pi\lambda\!\!\!^{-2}\frac{\gamma^r{}_{Pp}\gamma^r{}_{Q}}{(E_P-E_{Pr})^2+\frac{1}{4}\gamma_r{}^2} \tag{259a}$$

if only one level of C is important.

Special considerations are necessary if some of the nuclear states involved are degenerate. The calculations which have been carried out by Bethe and Placzek replace (255) by the formula

$$\sigma^{Pp}{}_{Qq}=\frac{\pi\lambda\!\!\!^{-2}}{(2i+1)(2s+1)}\sum_{ljl'j'J}(2J+1)\times\left|\sum_r\frac{u^{CrJ}{}_{ApPlj}\,u^{CrJ}{}_{BqQl'j'}}{W_{Ap}+W_P+E_P-W_{CrJ}+\frac{1}{2}i\gamma_{rJ}}\right|^2. \tag{260}$$

Here and in the following we denote by i, J and i' the angular momenta of initial, compound and final nucleus; s and s' are the spins of incident and outgoing particle; ll' and jj' their orbital and total angular momenta. p, r, q mean all quantum numbers of initial, compound and final nucleus other than the angular momenta. The sums over l and l' extend over all values of these quantities from 0 to ∞, the sum over j from $|l-s|$ to $l+s$ and the sum over j' correspondingly; the sums over rJ extend over all states of the compound nucleus. The u's are matrix elements similar to the H's; they are defined so that the partial width of a level is simply equal to the square of u, *viz.*

$$\gamma^{rJ}{}_{Pplj}=(u^{CrJ}{}_{ApPlj})^2, \tag{261}$$

$$\gamma^{rJ}{}_{Pp}=\sum_{lj}\gamma^{rJ}{}_{Pplj}. \tag{261a}$$

Furthermore, the u's are real but may be positive or negative.

If only one level is important, (260) reduces again to the very simple formula

$$\sigma^{Pp}{}_{Qq}=\pi\lambda\!\!\!^{-2}\frac{2J+1}{(2s+1)(2i+1)}\frac{\gamma^{rJ}{}_{Pp}\gamma^{rJ}{}_{Qq}}{(E_P-E_r)^2+\frac{1}{4}\gamma_{rJ}{}^2}, \tag{262}$$

which is the generalization of the Breit-Wigner formula (259) for nonvanishing angular momenta of the nuclei concerned.

The quantities H, u and γ depend on the energy of the particle involved, because the matrix element H (or u) contains the wave function of the incident particle. As we have mentioned, this wave function must be normalized per unit energy. If we assume the particle to be free and to have an orbital momentum l, its normalized wave function has the form (B15, Eq. (15))

$$\psi=\left(\frac{2}{\pi}\frac{dk}{dE}\right)^{\frac{1}{2}}\frac{1}{r}\chi_l(kr)\,Y_{lm}(\vartheta\varphi), \qquad (263)$$

where $k=1/\lambda\!\!\!^{-}$ is the wave number of the particle,

$$E=(\hbar^2/2m)k^2 \qquad (263a)$$

its energy, Y_{lm} a normalized spherical harmonic and χ_l *that* solution of the radial wave equation,

$$\frac{d^2\chi_l}{dr^2}+\left(k^2-\frac{l(l+1)}{r^2}\right)\chi_l=0, \qquad (263b)$$

which behaves asymptotically as $\sin(kr-\frac{1}{2}l\pi)$. At small r,

$$\chi_l=\frac{1}{1\cdot3\cdots(2l-1)(2l+1)}(kr)^{l+1}. \qquad (263c)$$

Now for the matrix element only the behavior of the wave function ψ *inside* the nucleus is important. Therefore, if the wave-length $\lambda\!\!\!^{-}=1/k$ is large compared to nuclear dimensions, we are only interested in the values of ψ for small values of kr. Then, according to (263) (263c), the matrix element will depend on k as

$$(dk/dE)^{\frac{1}{2}}k^{l+1}. \qquad (264a)$$

Since dE/dk is proportional to k (cf. 263a)), the matrix element will be proportional to $k^{l+\frac{1}{2}}$. Thus we may write

$$u^{rJ}{}_{Pp}=b^{rJ}{}_{Pp}\,k^{l+\frac{1}{2}}=b^{rJ}{}_{Pp}\,\lambda\!\!\!^{-(l+\frac{1}{2})} \qquad (264)$$

and, because of (261),

$$\gamma^{rJ}{}_{Pp}=(b^{rJ}{}_{Pp})^2\lambda\!\!\!^{-(2l+1)}, \qquad (264b)$$

where b is a constant. Introducing the true width Γ (cf. (258a)) instead of the effective width γ, and similarly the value of the matrix element at resonance, U, instead of u, we have

$$u^{rJ}{}_{Pp}=U^{rJ}{}_{Pp}(\lambda\!\!\!^{-}_{Pr}/\lambda\!\!\!^{-})^{l+\frac{1}{2}}=U^{rJ}{}_{Pp}(E/E_{Pr})^{\frac{1}{2}l+\frac{1}{4}} \qquad (265)$$

$$\gamma^{rJ}{}_{Pp}=\Gamma^{rJ}{}_{Pp}(\lambda\!\!\!^{-}_{Pr}/\lambda\!\!\!^{-})^{2l+1}=\Gamma^{rJ}{}_{Pp}(E/E_{Pr})^{l+\frac{1}{2}}, \qquad (265a)$$

where $\lambda\!\!\!^{-}_{Pr}$ is the wave-length corresponding to the resonance energy E_{Pr}. It can easily be shown (see B15, p. 454) that the formulae (264) (265) hold also if the particle is not free.

If, as we have assumed, the wave-length of the particle is large compared to nuclear dimensions, then only the partial wave $l=0$ will have appreciable amplitude in the nucleus, as can be seen directly from (263c). Therefore for slow particles (264) (265) become

$$u^{rJ}{}_{Pp}=b^{rJ}{}_{Pp}\lambda\!\!\!^{-\frac{1}{2}}=a^{rJ}{}_{Pp}v^{\frac{1}{2}}=c^{rJ}{}_{Pp}E^{\frac{1}{2}}$$
$$=U^{rJ}{}_{Pp}(v/v_{Pr})^{\frac{1}{2}}, \qquad (266)$$

$$\gamma^{rJ}{}_{Pp}=(c^{rJ}{}_{Pp})^2E^{\frac{1}{2}}=\text{etc.}, \qquad (266a)$$

where v is the velocity of the particle and a, b, c are certain constants whose relation is obvious.

For γ-rays, there is no partial wave $l=0$, so that $l=1$ (dipole radiation) becomes the most important part. The dependence of the matrix element for γ-rays on the wave-length is again given by (264) (cf. B15). However, in contrast e.g., to slow neutrons, the energy of γ-rays is ordinarily very much larger than the spacing between energy levels; therefore the ratio E/E_{Pr} of the actual to the resonance energy will, for γ-rays, be very near unity for all those levels rJ which contribute appreciably to the cross section. Thus we may, for γ-rays, replace the matrix element u by its value at resonance U, and the actual width $\gamma=u^2$ by the true width $\Gamma=U^2$, independent of l.

The same arguments hold for fast particles. This is particularly welcome because the exact behavior of the matrix element as a function of energy cannot be obtained without assuming a special nuclear model, as soon as the wave-length of the particle becomes comparable with or smaller than the nuclear dimensions. But we do not need to know this behavior because we wish to know the matrix elements u only in the neighborhood of the resonance energy, let us say in an energy region of the extension of a few times the spacing between resonance levels. Inasmuch as the spacing of levels is very small compared to the energy of the fast particle, the matrix element u and the effective width γ may be regarded as constant over the energy region considered, and be replaced by their values at resonance, U and Γ. Or, alternatively, formulae (265) (265a) may be used for fast particles: Even though they are not strictly true for this case, their use is permitted over energy regions small compared to the particle energy itself.

Inserting (264) into (260), we have

$$\sigma^{Ppl}{}_{Qql'}=\frac{\pi}{(2s+1)(2i+1)}\lambda^{1-2l}\lambda'^{-(1+2l')}\sum_{jj'J}(2J+1)\times\left|\sum_r\frac{b^{rJ}{}_{ApPlj}b^{rJ}{}_{BqQl'j'}}{E_P-E_{CrJ}+\frac{1}{2}i\gamma_{rJ}}\right|^2 \quad (267)$$

with

$$b^{rJ}{}_{ApPlj}=U^{rJ}{}_{ApPlj}\lambda_{Pr}{}^{l+\frac{1}{2}}, \quad (267a)$$

$$\sigma^{Pp}{}_{Qq}=\sum_{ll'}\sigma^{Ppl}{}_{Qql'}. \quad (267b)$$

λ and λ' are the wave-lengths of incident and outgoing particle; all the other quantities have the same meaning as before. If only one level rJ is important, (267) reduces to

$$\sigma^{Ppl}{}_{Qql'}=\frac{\pi(2J+1)}{(2s+1)(2i+1)}\lambda^{1-2l}\lambda_{Pr}{}^{2l+1}\times\left(\frac{\lambda_{Qr}}{\lambda'}\right)^{2l'+1}\frac{\Gamma^{rJ}{}_{Ppl}\Gamma^{rJ}{}_{Qql'}}{(E_P-E_{Pr})^2+\frac{1}{4}\gamma_{rJ}{}^2}. \quad (268)$$

In this formula, only the factors λ, λ' and E_P itself change with the energy E_P of the incident particle,[8] all the other quantities are constants characterizing the nuclear level.

In order to discuss formulae (267) (268), it is useful to distinguish between fast and slow incident and outgoing particles. A particle is called slow if its wave-length is large compared to nuclear dimensions. This is true for energies up to a few hundred thousand volts for neutrons or protons interacting with heavy nuclei and up to about a million volts if they interact with light nuclei. For α-particles, the limits are one-quarter of these figures. γ-rays can always be regarded as slow or fast at ones discretion, it being simpler to regard them as fast. We have then the following cases:

[8] The effective width γ_{rJ} will in general consist of contributions from fast particles Q and from slow particles which may be emitted by the compound nucleus rJ. The contributions of the fast particles (including γ-rays) will not depend sensitively on the energy of the incident particle, at least not over energy regions of a few thousand or even a few hundred thousand volts. The contributions of the slow emitted particles *will* depend strongly on the energy, according to (265a). But these contributions will in general be small because of the small factor $E^{l+\frac{1}{2}}$. Thus in general, γ_{rJ} will be approximately constant and equal to the true width Γ_{rJ} over fairly large energy regions; and only if the contribution of slow emitted particles happens to be large will there be a noticeable dependence of γ_{rJ} on energy.

A. Incident and outgoing particle fast

In this case the variation of the factors λ and λ' in (267) (268) with energy is irrelevant and may be disregarded (cf. above). The only factor depending on energy is then the resonance factor (last factor in (267) (268)).

B. Incident particle slow, outgoing particle fast

We may put $l=0$ for the incident particle and neglect the dependence on the wave-length λ' of the outgoing particle. Then (267) reduces to

$$\sigma^{Pp}{}_{Qq}=\frac{\pi}{(2s+1)(2i+1)}\lambda\sum_{l'j'J}(2J+1)\times\left|\sum_r\frac{U^{rJ}{}_{ApP0s}\lambda_{Pr}{}^{\frac{1}{2}}U^{rJ}{}_{BqQl'j'}}{E_P-E_{Pr}+\frac{1}{2}i\gamma_{rJ}}\right|^2 \quad (269)$$

and if only one level is important, we obtain from (268)

$$\sigma^{Pp}{}_{Qq}=\frac{\pi(2J+1)}{(2s+1)(2i+1)}\lambda\lambda_{Pr}\frac{\Gamma^{rJ}{}_{Pp}\Gamma^{rJ}{}_{Qq}}{(E_P-E_{Pr})^2+\frac{1}{4}\gamma_{rJ}{}^2}. \quad (270)$$

In both (269) and (270), there appears beside the resonance factor a factor λ which indicates proportionality of the cross section with the reciprocal of the velocity v of the incident particle (again apart from resonance). This "$1/v$-factor" makes the cross section for phenomena produced by slow particles, especially slow neutrons, very large (cf. Chapter X). (About the dependence of γ_{rJ} on energy, cf. footnote 8)

C. Incident particle fast, outgoing particle slow

Following the same considerations as before, (267) reduces to

$$\sigma^{Pp}{}_{Qq}=\frac{\pi}{(2s+1)(2i+1)}\lambda'^{-1}\sum_{ljJ}(2J+1)\times\left|\sum_r\frac{U^{rJ}{}_{ApPlj}U^{rJ}{}_{BqQ0s'}\lambda_{Pr}\lambda_{Qr}{}^{-\frac{1}{2}}}{E_P-E_{Pr}+\frac{1}{2}i\gamma_{rJ}}\right|^2. \quad (271)$$

The cross section is in this case inversely proportional to λ', or directly proportional to the velocity v' of the outgoing particle. Thus slow particles are only rarely produced in nuclear processes.

D. Incident and outgoing particle slow

The cross section contains apart from the resonance factor, a factor $\lambda/\lambda' \propto v'/v$. In the particular case of the elastic scattering of slow particles, $v'=v$ so that the cross section depends on the energy only through the resonance factor while there is no general trend such as the $1/v$ or the v'-law. E.g., if only one level is important, the elastic scattering becomes

$$\sigma^{P_P P_P}=\frac{\pi(2J+1)}{(2s+1)(2i+1)}\lambda_{P_r}{}^2\frac{(\Gamma^{rJ}{}_{P_P})^2}{(E_P-E_{P_r})^2+\frac{1}{4}\gamma_{rJ}{}^2} \tag{272}$$

with the same notation as in (270).

If one or both of the particles involved are slow, there are strict selection rules for the angular momenta. E.g., if the incident particle is slow, its orbital momentum will be zero and therefore its total momentum j equal to its spin s. We have then the selection rules

$$|i-s|\leq J\leq i+s; \qquad J+i\geq s, \tag{273}$$

which means that only a fraction of the levels of the compound nucleus will contribute to the cross section. If the outgoing particle is slow, there will be a similar selection rule determining the angular momentum i' of the final nucleus. γ-rays are to be considered as "slow" particles in this connection, with s being replaced by l, i.e., 1 and 2 for dipole and quadrupole radiation, respectively. For fast material particles no useful selection rule holds.

The application of the dispersion formulae developed in this section to the various nuclear processes will form the main content of this report. From the experimental data, we shall deduce the widths corresponding to the emission of various sorts of particles. The main results known thus far are the following:

The *γ-ray width* seems to be of the order of 0.1 to 1 volt for most of the "slow neutron levels" of medium heavy nuclei, i.e., for levels with an excitation energy around 8 MV. (§61.) For *light* nuclei, the γ-ray width is of the order of 1 to 10 volts for rather large excitation energies ($\sim$15 MV), according to the evidence from the capture of protons (§81).

The *neutron width* is for medium heavy nuclei ($A\sim 100$) and excitation energies around 8 MV, of the order of 10^{-3} volts at a neutron energy of the order of a volt (§60). According to (266a), the neutron width is *cet. par.* proportional to the square root of the neutron energy. It will therefore be about 1 volt for neutrons of 1 MV energy, and near 10 volts if the neutron energy is of the order of the nuclear interaction potentials ($\sim$20 MV). There is not much evidence on the neutron width for light nuclei except that it is very much larger.

The *proton width* is known for some very light nuclei, e.g., Be^8 (§81). It contains one factor representing the transmission of protons through the potential barrier (§68) and one factor giving the proton width without barrier. The latter should be comparable to the neutron width; it is of the order of 50,000 volts for the resonance level of Be^8 at an excitation energy of 17 MV and a proton energy of 400 kv. Data from other light nuclei are similar.

The widths corresponding to the emission of deuterons, α-particles, etc., from the compound nucleus are, apart from the different transmission of the potential barrier, of the same order as for protons and neutrons (Chapter XIII).

§53. The Distribution of Nuclear Energy Levels. (B13, B4, B33, F30, O12)

We have already found in our qualitative discussions in §51 that the energy levels of a heavy nucleus will be very closely spaced and that their spacing will decrease rapidly with increasing energy. In order to find theoretical expressions for the magnitude of the spacing and its dependence on the energy we must, of course, use some model of the nucleus.

Two models suggest themselves which may be considered as opposite extremes:

(a) We may start with *free* individual particles and consider the total energy of the nucleus as equal to the sum of the energies of the individual particles. This amounts to assuming the interaction between the particles to be small; the nucleus would then be comparable to a gas.

(b) We may consider the interaction to be large, more accurately, large compared to the kinetic energy of the particles. Then the nucleus will in first approximation correspond to a drop of liquid, the distance between neighboring

particles remaining almost constant all the time. It seems that assumption (b) will come nearer the truth.

A calculation using assumption (a) was given by Bethe (B13) and also by Oppenheimer (O12), while assumption (b) was proposed by Bohr and Kalckar (B33). An intermediate model, treating the nuclear particles like the electrons in metals, i.e., as half-free particles with correlations between their positions, was treated by Bardeen (B4).

In all cases, we consider the nuclear problem as a statistical one. E.g., if we take assumption (b), we shall first find out the normal modes of vibrations of the particles in a drop of liquid, and then assume that each normal mode has an excitation energy given by the Planck formula. Summing the excitation energies of all normal modes, we obtain the total excitation energy of the nucleus as a function of the "temperature" introduced in the Planck formula. This relation between total excitation energy and temperature should, of course, be considered as a definition of the latter.

Since we are only dealing with excitation energies very small compared to the total binding energy of the nucleus (10 as compared to $\sim$1000 MV), the temperature is always "low." From the relation between energy and temperature we may obtain the specific heat and the entropy of the nucleus using the ordinary thermodynamical relations. The entropy S is then by definition related to the number of states of the nucleus $\rho \sim e^S$. Thus the number of states (per energy interval) can be expressed in terms of the excitation energy of the nucleus.

If we take a different model for the nucleus, e.g., the free particle model, we shall only change the relation between energy and temperature, and therefore between the density of levels and the excitation energy. The principle of the calculation remains unchanged. We shall therefore carry out the calculations first assuming a general relation between energy and temperature, and shall specify the nuclear model only at the end.

Our object is to obtain the average spacing of the nuclear levels

$$D(U)=1/\rho(U), \tag{274}$$

where $\rho(U)dU$ is the number of levels with energy between U and $U+dU$. To obtain the "density of levels" $\rho(U)$, we consider the expression familiar from statistics:

$$\sum_k e^{-E_k/\tau}=e^{-F/\tau}, \tag{275}$$

in which the sum is to be extended over all levels E_k of the nucleus. τ is an arbitrary parameter, and $F(\tau)$ a certain function of τ defined by (275). If the levels are very dense, the expression (275) can be written as an integral

$$\int \rho(E)e^{-E/\tau}dE=e^{-F/\tau}. \tag{275a}$$

It will be possible to find $\tau(U)$ from this equation if we can (1) choose the parameter τ in such a way that only energies near U contribute appreciably to the integral in (275a), (2) determine F as a function of τ and both these quantities as functions of U.

Provided τ *can* be determined so that condition (1) is fulfilled, $\rho(E)e^{-E/\tau}$ will have a sharp maximum near $E=U$. Then we have obviously

$$\frac{\sum E_k e^{-E_k/\tau}}{\sum e^{-E_k/\tau}}=\frac{\int E\rho(E)e^{-E/\tau}dE}{\int \rho(E)e^{-E/\tau}dE}=U. \tag{276}$$

This relation *determines* the parameter τ, as a function of U, if the energy spectrum of the system is known. Conversely, it expresses U as a function of τ. The two functions $U(\tau)$ and $F(\tau)$ are connected by

$$U=\frac{d(F/\tau)}{d(1/\tau)}=F-\tau\frac{dF}{d\tau}, \tag{276a}$$

which follows immediately from (275a) (276). τ, U and F correspond, in ordinary statistical theory, respectively, to kT, to the energy of the system and to the free energy.

As is shown in classical statistics, the condition (1) will be fulfilled by the τ as defined in (276), provided the system contains sufficiently many particles. Then the main contribution to the integral (275a) will come from energy levels E_k near U. Thus we may write instead of (275a)

$$e^{-F/\tau}=\rho(U)e^{-U/\tau}\lambda(U), \tag{277}$$

where $\lambda(U)$ is a quantity of the dimensions of an

energy which is to be calculated by evaluating the integral in (275a). It is a slowly varying function of the excitation energy U as compared to the rapidly varying functions $e^{-F/\tau}$, $e^{-U/\tau}$ and $\rho(U)$. We rewrite (277)

$$\rho(U)=e^{(U-F)/\tau}/\lambda(U). \tag{277a}$$

According to the definition (276), τ is that temperature for which the average energy of the system is U, and F is the corresponding free energy. We introduce the abbreviation

$$(U-F)/\tau=S. \tag{278}$$

S is then the entropy divided by Boltzmann's constant k. Then we have

$$\rho(U)=e^{S(U)}/\lambda(U) \tag{278a}$$

and it remains only to obtain, in specific terms, $S(U)$ and $\lambda(U)$.

For S we have, according to (276a)

$$S=(U-F)/\tau=-dF/d\tau. \tag{278b}$$

Also from (276a) we find

$$\frac{dU}{d\tau}=-\tau\frac{d^2F}{d\tau^2} \tag{278c}$$

so that

$$S=\int\frac{d\tau}{\tau}\frac{dU}{d\tau}=\int\frac{Cd\tau}{\tau}, \tag{279}$$

where C is the specific heat. From (279) the well-known relation

$$\frac{dS}{d\tau}=\frac{1}{\tau}\frac{dU}{d\tau} \tag{279a}$$

follows immediately, and, since τ and U are uniquely related to each other,

$$\frac{dS}{dU}=\frac{1}{\tau}, \tag{279b}$$

expressing the elementary definition of the entropy. S is therefore known as a function of U if the relation between U and τ is known.

To determine λ, we insert (278a) in (275a) and have

$$e^{-F/\tau}=\int e^{S(E)-E/\tau}dE/\lambda(E), \tag{280}$$

where $S(E)$ is the entropy corresponding to the energy E. Since λ varies slowly with the energy, and since the integrand is very small except for $E\approx U$, we may replace $\lambda(E)$ by $\lambda(U)$ and have, using (277) again:

$$\lambda(U)=\int e^{S(E)-S(U)+(U-E)/\tau}dE. \tag{280a}$$

With (279b) the exponent becomes

$$\frac{dS}{dU}(E-U)+\frac{1}{2}\frac{d^2S}{dU^2}(E-U)^2+\frac{U-E}{\tau}=-\frac{1}{2\tau^2}\frac{d\tau}{dU}(E-U)^2. \tag{280b}$$

The integration gives then immediately

$$\lambda(U)=(2\pi)^{\frac{1}{2}}\tau(dU/d\tau)^{\frac{1}{2}}. \tag{280c}$$

With (274) and (278a), the average spacing between levels becomes thus

$$D(U)=1/\rho(U)=\lambda(U)e^{-S(U)}=\tau(2\pi dU/d\tau)^{\frac{1}{2}}e^{-S}. \tag{281}$$

We make now the more special assumption that U depends on the temperature according to a power law

$$U=\alpha\tau^n. \tag{282}$$

This is true for the model (a) proposed above with n equal to 2. For model (b) it is fulfilled for low temperature (low excitation energies) with $n=7/3$, and for high temperature with $n=4$ (see below). With the law (282) we find, using (279),

$$S=\frac{\alpha n}{n-1}\tau^{n-1}=\frac{n}{n-1}(\alpha U^{n-1})^{1/n} \tag{283}$$

and inserting in (281)

$$D(U)=(2\pi n)^{\frac{1}{2}}\left(\frac{U^{n+1}}{\alpha}\right)^{\frac{1}{2}n}\exp\left(-\frac{n}{n-1}\alpha^{1/n}U^{(n-1)/n}\right). \tag{284}$$

The spacing decreases, according to (284), exponentially with increasing energy. The exponent contains $U^{\frac{1}{2}}$ for the Fermi gas model (model a, $n=2$), $U^{4/7}$ for the liquid drop model at low and $U^{3/4}$ at high temperatures. The "liquid" model thus gives a somewhat more rapid decrease of the spacing.

The exact value of α can, of course, only be obtained from a special model. Quite generally α increases with the number of particles A in the nucleus, so that the spacing of nuclear levels *decreases* with increasing A, for constant excitation energy U. The decrease with A is in general more rapid the slower the decrease with increasing energy; e.g., it is more rapid for the free particle model (a) than for the "liquid drop" model (b).

A. Free particle model

In this model, the nucleus is considered as a mixture of two Fermi gases of neutrons and protons. At zero temperature, the particles occupy all the lowest levels while the higher levels are empty. At higher temperatures, there will be some particles in the higher levels and some of the lower levels will be empty.

The statistical treatment given above is not directly applicable to this case because it presupposes that the energy levels of the given system containing, say, N neutrons and Z protons, are known. In the customary treatment of Fermi statistics (and any gas statistics) not only systems of the correct number of particles are considered but also such for which the number of particles is slightly different.

The only way in which the given number of particles is taken into account is by making the statistical probability a maximum for that number of particles. This is achieved by suitably choosing a certain parameter ζ, the Fermi energy. But even with the proper choice of ζ, there remains the fact that the distribution function derived in Fermi statistics gives a finite probability also for the states in which the total number of particles does not exactly have the correct value, so that we must multiply the density $\rho(U)$ obtained in (281) by the probability that the number of particles of each kind has the correct value.

In addition to this, we must remember that, owing to the selection rules, only compound states with one or a few values of the angular momentum are important for a given nuclear process. Therefore, we are interested in the states of the nucleus with a given angular momentum J. This constant of motion should be treated in the same way as N and Z so that we have altogether three constants of motion besides the energy, *viz.* N, Z and J.

For generality, we assume that an arbitrary number m of constants of motion $N_1\, N_2 \cdots N_m$, is given. Then there will be an equal number of parameters $\zeta_1 \cdots \zeta_m$ in the distribution function which must be chosen in such a way that in the average $N_1 N_2 \cdots$ have their correct values. Then we have instead of (275)

$$e^{-\Phi/\tau} = \sum_{k,\, N_1' \ldots N_m'} e^{(-E_k + N_1'\zeta_1 + \ldots + N_m'\zeta_m)/\tau}. \quad (285)$$

The sum extends over all possible values of $N_1' \cdots N_m'$ including their correct values $N_1 \cdots N_m$. (285) may be rewritten in integral form

$$e^{-\Phi/\tau} = \int \rho(E,\, N_1' \cdots N_m') e^{-E/\tau + \alpha_1 N_1' + \ldots + \alpha_m N_m'} \quad (285a)$$

with

$$\alpha_i = \zeta_i/\tau. \quad (285b)$$

Φ must be regarded as a function of the parameters τ, $\alpha_1 \cdots \alpha_m$. We have

$$\left(\frac{\partial(\Phi/\tau)}{\partial(1/\tau)}\right) = U, \quad (286)$$

$$\frac{1}{\tau}\left(\frac{\partial \Phi}{\partial \alpha_i}\right) = -N_i, \quad (286a)$$

where the partial derivative with respect to any parameter α_i is to be taken with the other parameters α_k $(k \neq i)$ and τ being kept constant.

The density of levels may again be written

$$\rho(U,\, N_1 \cdots N_m) = e^S/\lambda \quad (287)$$

with

$$S(U,\, N_1 \cdots N_m) = (U - \Phi)/\tau - \alpha_1 N_1 - \cdots - \alpha_m N_m. \quad (287a)$$

S is a function of the energy U and the "constants of motion" $N_1 \cdots N_m$ only but does not depend explicitly[9] on the parameters τ and α. The derivatives of the entropy with respect to the "constants of motion" are

$$\partial S/\partial U = 1/\tau, \quad (288)$$

$$\partial S/\partial N_i = -\alpha_i. \quad (288a)$$

(288) means that formula (279b) and therefore (283) remains true in our more complicated case provided the number of particles is constant. We assume now again that λ is a slowly varying function of its arguments, and find, similarly to (280a):

$$\lambda(U) = \int e^{S(E) - S(U) + (U-E)/\tau + \alpha_1(N_1' - N_1) + \ldots} \, dE\, dN_1' \cdots dN_m'. \quad (289)$$

With (288, 288a) the exponent becomes

$$\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} \frac{\partial^2 S}{\partial N_i \partial N_j}(N_i' - N_i)(N_j' - N_j) = -\frac{1}{2}\sum_{ij}\frac{\partial \alpha_i}{\partial N_j}(N_i' - N_i)(N_j' - N_j), \quad (289a)$$

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[9] This follows immediately by differentiating (287a) with respect to any of the parameters, keeping the other parameters and U, $N_1 \cdots N_m$ constant. E.g.,

$$\partial S/\partial \alpha_i = -(1/\tau)(\partial \Phi/\partial \alpha_i) - N_i = 0 \quad \text{(cf. (286a))}.$$

S and Φ correspond formally to Hamiltonian and Lagrangian in mechanics.

where we have put

$$E=N_0',\quad U=N_0,\quad 1/\tau=-\alpha_0. \tag{289b}$$

The integration of (289) is then straightforward and gives

$$\lambda(U)=(2\pi)^{\frac{1}{2}(m+1)}/\Delta^{\frac{1}{2}}, \tag{290}$$

where Δ is the Jacobian

$$\Delta=\left\|\frac{\partial\alpha_i}{\partial N_j}\right\|=\left\|-\frac{\partial^2 S}{\partial N_i\partial N_j}\right\|. \tag{290a}$$

If we are treating a mixture of degenerate Fermi gases, then the Fermi energy ζ_i for the ith kind of particles is determined solely by the number N_i of these particles, except for a very small dependence on the excitation energy U which we neglect. Since $\alpha_i=\zeta_i/\tau$, we have (cf. 289b)

$$\frac{\partial\alpha_0}{\partial N_0}=-\frac{\partial(1/\tau)}{\partial U},\quad \frac{\partial\alpha_0}{\partial N_i}=-\frac{\partial(1/\tau)}{\partial N_i},$$

$$\frac{\partial\alpha_i}{\partial N_0}=\zeta_i\frac{\partial(1/\tau)}{\partial U},\quad \frac{\partial\alpha_i}{\partial N_i}=-\frac{1}{\tau}\frac{\partial\zeta_i}{\partial N_i}+\zeta_i\frac{\partial(1/\tau)}{\partial N_i},$$

$$\frac{\partial\alpha_i}{\partial N_k}=\zeta_i\frac{\partial(1/\tau)}{\partial N_k}. \tag{290b}$$

We add ζ_i times the zeroth row to the ith row of the determinant; then there remains in that row only the diagonal term $\tau^{-1}d\zeta_i/dN_i$. Thus the value of the Jacobian is

$$\Delta=-\left(\frac{\partial(1/\tau)}{\partial U}\right)_{N_i}\prod_{i=1}^{m}\frac{1}{\tau}\frac{d\zeta_i}{dN_i} \tag{291}$$

and the spacing between neighboring levels becomes (cf. (287))

$$D(U)=\frac{1}{\rho(U)}=(2\pi)^{\frac{1}{2}(m+1)}\tau^{\frac{1}{2}m+1}\times\left(\frac{dU}{d\tau}\prod_{i=1}^{m}\frac{dN_i}{d\zeta_i}\right)^{\frac{1}{2}}e^{-S(U)}. \tag{292}$$

We now turn to the special features of the free particle model. We assume that the particles move in a spherical potential hole of a certain depth, and that their interaction is small (except inasmuch as it is expressed by the potential hole). Neutrons and protons will obey Fermi statistics, and the number of neutrons having a kinetic energy between ϵ and $\epsilon+d\epsilon$ will be given by the usual formula of Fermi statistics (cf. B13)

$$n(\epsilon)d\epsilon=\tfrac{3}{2}Cf(\epsilon)\epsilon^{\frac{1}{2}}d\epsilon \tag{293}$$

with

$$C=(2^{7/2}/9\pi)(MR^2/\hbar^2)^{\frac{3}{2}}, \tag{293a}$$

$$f(\epsilon)=1/e^{(\epsilon-\zeta_1)/\tau}+1, \tag{293b}$$

R being the nuclear radius and ζ_1 the Fermi energy for neutrons. Integration gives for $\zeta_1\gg\tau$

$$N=\int n(\epsilon)d\epsilon=C\zeta_1^{\frac{3}{2}}(1+(\pi^2/8)(\tau/\zeta_1)^2+\cdots) \tag{294}$$

and for the energy

$$U_1=\tfrac{3}{5}C\zeta_1^{5/2}(1+(5\pi^2/8)(\tau/\zeta_1)^2+\cdots). \tag{295}$$

From (294), we have in sufficient approximation

$$\frac{dN}{d\zeta_1}\approx\frac{3}{2}C\zeta_1^{\frac{1}{2}}=\frac{3}{2}C^{\frac{2}{3}}N^{\frac{1}{3}}=\frac{3}{2}\frac{N}{\zeta_1} \tag{294a}$$

and a corresponding relation for the protons. From (295), we find after a short calculation (cf. B13)

$$U=U_1+U_2-U_0=\tfrac{1}{4}\pi^2C\tau^2(\zeta_1^{\frac{1}{2}}+\zeta_2^{\frac{1}{2}}), \tag{295a}$$

where U_1 is the total energy of the neutrons, U_2 that of the protons, U_0 the zero-point energy which depends only on the numbers of protons and neutrons but not on the temperature τ, and is therefore irrelevant for the determination of the specific heat $dU/d\tau$ and the entropy (cf. 278). Since the numbers of neutrons and protons are not very different, we may put $\zeta_1=\zeta_2=\zeta_0$ where ζ_0 is an average Fermi energy for protons and neutrons. Then

$$U=\tfrac{1}{2}\pi^2C\zeta^{\frac{1}{2}}\tau^2=\tfrac{1}{4}\pi^2(A/\zeta_0)\tau^2, \tag{296}$$

using (294) and putting $N=\frac{1}{2}A$, in analogy with our approximation[10] $\zeta_1=\zeta_2=\zeta_0$. From (294) (296) we find immediately the entropy S, and the quantities $dU/d\tau$, $dN/d\zeta_1$, $dZ/d\zeta_2$ occurring in (292).

We must now introduce a parameter γ analogous to the ζ's which makes a certain value J of the *angular momentum* most probable. It is more convenient to work with the component M of J in a given direction Z because M is the sum of

[10] The justification of these approximations is proved in B13.

the angular momentum components of the individual particles, *viz.*

$$M=\sum_{i=1}^{A} m_i. \tag{297}$$

We write for the probability to find a neutron state of energy and angular momentum m occupied

$$f(\epsilon, m)=1/(e^{(\epsilon-\zeta_1-\gamma m)/\tau}+1). \tag{297a}$$

The expressions (294) (296) for the total number of particles and for the total energy will be changed only by quantities of the order γ^2. The average value of the total angular momentum is

$$M=\int m g(\epsilon, m)\left(\frac{1}{e^{(\epsilon-\zeta_1-\gamma m)/\tau}+1}+\frac{1}{e^{(\epsilon-\zeta_2-\gamma m)/\tau}+1}\right)d\epsilon dm, \tag{297b}$$

where $g(\epsilon, m)$ is the number of states of an individual particle with given ϵ and m.

The number $g(\epsilon, m)$ of states with given ϵ and m of an individual particle in a spherical potential hole of radius R, can be found as follows. Leaving out the spin, we describe each particle state by the three quantum numbers n, l and m where l is the total angular momentum, m its component in the Z direction and n the radial quantum number. According to the WKB method, n is given by

$$n\pi=\int_{\min}^{R}\left(\frac{2M\epsilon}{\hbar^2}-\frac{(l+\frac{1}{2})^2}{r^2}\right)^{\frac{1}{2}}dr, \tag{298}$$

where $r_{\min}$ is the value of r for which the integrand vanishes. Since there is just one state for each integral value nlm, the number of states of given l and m per unit energy is

$$g(\epsilon, l, m)=\frac{dn}{d\epsilon}=\frac{1}{\pi}\frac{M}{\hbar^2}\int\left(\frac{2M\epsilon}{\hbar^2}-\frac{(l+\frac{1}{2})^2}{r^2}\right)^{-\frac{1}{2}}dr$$
$$=\frac{1}{2\pi\epsilon}\left(\frac{2M\epsilon R^2}{\hbar^2}-(l+\tfrac{1}{2})^2\right)^{\frac{1}{2}}. \tag{298a}$$

The total number of states of given m and energy between ϵ and $\epsilon+d\epsilon$ becomes

$$g(\epsilon, m)d\epsilon=d\epsilon\int_{|m|}^{l^0} g(\epsilon, l, m)dl \tag{298b}$$

where l_0 is the value of l for which (298a) vanishes. This expression should be multiplied by 2 for spin. Putting $\zeta_1=\zeta_2=\zeta_0$ and expanding in powers of γ, (297b) becomes then

$$M=-4\gamma\int d\epsilon\frac{d}{d\epsilon}(e^{(\epsilon-\zeta_0)/\tau}+1)^{-1}\int dl\, g(\epsilon, l, m)\int_{-(l+\frac{1}{2})}^{l+\frac{1}{2}} m^2 dm. \tag{299}$$

The elementary evaluation of this integral, using (298a), gives

$$M=c\gamma \tag{299a}$$

with

$$c=\tfrac{2}{5}(MR^2/\hbar^2)A. \tag{299b}$$

The entropy for a given N, Z, U may thus be written (cf. (288a))

$$S(N, Z, U, M)=S(N, Z, U, 0)-(M^2/2c\tau) \tag{300}$$

and the density of states, for given NZU, depends on M as (cf. (287))

$$\rho(N, Z, U, M)=\rho(N, Z, U, 0)\exp(-M^2/2c\tau). \tag{300a}$$

We must now determine the number of states of given J. This is equal to the difference between the number of quantum states with $M=J$, and the number of levels with $M=J+1$ (cf. B13) and therefore (cf. (300a))

$$\rho(N, Z, U, J)=\rho(N, Z, U, 0)\times[(J+\tfrac{1}{2})/c\tau]\exp[-(J+\tfrac{1}{2})^2/2c\tau]. \tag{301}$$

It can be shown (see (299b)) that $c\tau$ is rather large (≈ 1000) so that, for the important values of J, the exponential in (301) may be neglected. Therefore we obtain for the spacing of levels of angular momentum J (cf. (292))

$$D(N, Z, U, J)=D_0/(2J+1), \tag{302}$$

$$D_0=\frac{2c\tau}{\rho(N, Z, U, 0)}=2c(2\pi)^2\tau^{7/2}\times\left(\frac{dU}{d\tau}\frac{dN}{d\zeta_1}\frac{dZ}{d\zeta_2}\frac{dM}{d\gamma}\right)^{\frac{1}{2}}e^{-S(N, Z, U, 0)}. \tag{302a}$$

Inserting (296) (294a) (279) (299a) (299b) we have

$$D_0 = 3\sqrt{2}\pi^3\tau^4(Ac/\zeta_0)^{\frac{1}{2}}e^{-\frac{1}{2}\pi^2 A\tau/\zeta_0}, \quad (302b)$$

$$D = (18/5)^{\frac{1}{2}}(A^2U^2/\zeta_0)e^{-\pi(AU/\zeta_0)^{\frac{1}{2}}}. \quad (302c)^{11}$$

Here the Fermi energy ζ_0 is according to (293a) (294) and with $N=\frac{1}{2}A$:

$$\zeta_0 = (A/2C)^{\frac{1}{2}} = 1.15_5 P, \quad (303)$$

where

$$P = \hbar^2/Mr_0^2, \quad (303a)$$

$$r_0 = RA^{-\frac{1}{3}}. \quad (303b)$$

This radius r_0 is independent of A if the nuclear volume is proportional to the number of particles. From the radii of α-radioactive nuclei (Chapter XI)

$$r_0 = 2.0_5 \cdot 10^{-13} \text{ cm}, \quad (303c)$$

so that

$$P = 10 \text{ MV}, \quad (303d)$$

$$\zeta_0 = 11.5 \text{ MV}, \quad (303e)$$

which is of the same order as the binding energy of neutrons or protons in nuclei

We introduce now the abbreviation

$$x = \pi(AU/\zeta_0)^{\frac{1}{2}} = (AU/1.17)^{\frac{1}{2}} \ (U \text{ in MV}). \quad (304a)$$

Then the spacing becomes (cf. (302c))

$$D_0 = 8 \cdot 10^5 x^4 e^{-x} \text{ volts}. \quad (304)$$

For $U=8$ MV and $A=120$, we have $x=28.7$ and $D_0=0.2$ volt. This result for the spacing is very small indeed[12] and is certainly smaller than the spacing observed in slow neutron experiments (cf. §60) which may be of the order of 10 volts for the atomic weight and excitation energy considered. This shows that our assumptions are inadequate, particularly the assumption of free particles with small interaction.

B. Free particle model with correlations

Bardeen (B4) has pointed out that the free particle model in the form used in A, is not in accord with the assumption of exchange forces between the particles. As Van Vleck has shown (§36 of this report, ref. V 1 of part A), the potential acting on one particle depends strongly on the wave number of that particle, and its absolute value decreases appreciably with increasing wave number, i.e., with increasing kinetic energy of the particle. This means that the total energy of an individual particle,

$$E = E_{\text{kin}} + E_{\text{pot}} \quad (305)$$

depends much more strongly on the wave number k than the kinetic energy itself.

We may put

$$(dE/dk)_{E=\zeta} = \gamma(dE_{\text{kin}}/dk)_{E=\zeta}. \quad (305a)$$

With the range and magnitude of the nuclear forces derived from the binding energy of light nuclei, Bardeen finds

$$\gamma \approx 2. \quad (305b)$$

Now the number of individual particles states per unit energy is proportional to dk/dE, and is therefore reduced by the "correlations" to about one-half of its value. A reduction in the density of individual particle states means a much larger reduction in the density of states of the nucleus as a whole. Quantitatively, we can take account of the change of the density of individual states as follows: The excitation energy U is proportional to the density of individual quantum states times the square of the temperature τ. This follows from (296) if we consider that, with kinetic energy alone, the density of individual states of energy near ζ, is $\frac{3}{2}C\zeta^{\frac{1}{2}}$ (cf. (293)); it also follows from elementary considerations on Fermi statistics.

Thus the constant α in

$$U = \alpha\tau^2 \quad (306)$$

must be multiplied by $1/\gamma$. But, according to (283),

$$S = 2(\alpha U)^{\frac{1}{2}} \quad (306a)$$

Therefore, if U is kept fixed, the entropy "with correlation" is connected to that with completely free particles by

$$S_{\text{corr}} = S_{\text{fr}}\gamma^{-\frac{1}{2}} \quad (306b)$$

and the quantity x in (304) has to be replaced by

$$x' = x\gamma^{-\frac{1}{2}}. \quad (306c)$$

---

[11] This formula differs somewhat from that given by Bethe (B13, Eqs. (49) (50a)). Part of the difference *viz.* a factor $(\log 4)^{\frac{1}{2}}$ is due to the improved treatment of the angular momentum, another part (factor $2^{3/4}$) to a numerical error in Bethe's paper. These errors have also been pointed out by Bardeen.

[12] It seems somewhat surprising that the free particle model gives *too small* a spacing.

TABLE XX. *Spacing of nuclear energy levels according to Bardeen's method. The figures give the spacing D_0 (in volts) of levels of zero angular momentum, at an energy equal to the average neutron dissociation energy Q (cf. Table XXI, row O).*

A	20	50	100	200
Q (MV)	9.5	9.1	8.2	6.9
$D_0(R_0^*=9\cdot10^{-13}$ cm)	$1.1\cdot10^6$	$4.2\cdot10^4$	1000	10
$D_0(R_0^*=12.5\cdot10^{-13}$ cm)	$6\cdot10^4$	300	1.2	$1.2\cdot10^{-3}$

* R_0=nuclear radius for $A_0=230$. The radius for any nucleus is assumed to be $R_0(A/A_0)^{1/3}$.

The resulting spacing of energy levels, as a function of the atomic number, is given in Table XX for various nuclear radii and is compared to the spacing obtained from the free particle model without correlations.

C. Liquid drop model

It was pointed out by Bohr and Kalckar (B33) that a nucleus should be considered as a drop of liquid rather than as a gas because the interaction between the nuclear particles is large and therefore fluctuations of the density very improbable. A liquid drop is capable of two types of vibrations, namely surface waves and volume waves. The surface waves are not connected with any changes in volume; the potential energy is then only due to the surface tension and is comparatively small. Therefore the frequency of the surface waves is rather small, and this type of waves will most easily be excited at "low temperatures" i.e., in all cases relevant for nuclear disintegrations. The volume waves involve a compression of the nuclear liquid which is connected with a large increase in potential energy, a high frequency and therefore small excitation probability at "low" temperatures.

1. The surface waves.—For simplicity, assume first that the surface of the liquid is plane when there are no vibrations. Let the xy plane be the equilibrium surface, and let $\zeta(x, y)$ be the (vertical) displacement at a point x, y of the surface. Then the increase in surface area is for small displacements

$$\tfrac{1}{2}\int dxdy[(\partial\zeta/\partial x)^2+(\partial\zeta/\partial y)^2] \qquad (307)$$

integrated over the surface. If G is the total surface energy of the liquid in *equilibrium* and $S=\int dxdy$ the total surface area, the increase in potential energy is

$$V=(G/2S)\int dxdy[(\partial\zeta/\partial x)^2+(\partial\zeta/\partial y)^2]. \qquad (307a)$$

In order to find the kinetic energy we must know the velocity $\mathbf{u}$ at every point in the liquid. We assume that the motion of the liquid is irrotational; then $\mathbf{u}$ may be written as the gradient of a velocity potential which we call $\partial\psi/\partial t$, *viz.*

$$\mathbf{u}=-\operatorname{grad}\partial\psi/\partial t. \qquad (308)$$

Since the surface waves are not connected with any compression of the liquid, we have

$$\operatorname{div}\mathbf{u}=-\Delta(\partial\psi/\partial t)=0. \qquad (308a)$$

Then the kinetic energy becomes

$$T=\tfrac{1}{2}\rho\int u^2d\tau=\tfrac{1}{2}\rho\int(\operatorname{grad}\partial\psi/\partial t)^2d\tau$$
$$=\tfrac{1}{2}\rho\int\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z\partial t}dxdy, \qquad (308b)$$

where ρ is the density. The last integral is to be extended over the surface $z=0$ of the liquid and the vanishing of the Laplacian has been used.

From (308) it follows that the displacement may be written

$$\boldsymbol{\varrho}=-\operatorname{grad}\psi \qquad (309)$$

and in particular the displacement of the particles on the surface of the liquid

$$\zeta=-(\partial\psi/\partial z)_{z=0}. \qquad (309a)$$

Because of the absence of volume changes we have again

$$\Delta\psi=0. \qquad (309b)$$

We assume now that ψ is a periodic function of the coordinates in the surface plane, x and y, and of t. Then from (309b) it follows that

$$\psi=a\cos(k_xx+k_yy)e^{kz}\cos\omega t, \qquad (309c)$$

where
$$k=(k_x^2+k_y^2)^{\frac{1}{2}}, \qquad (309d)$$

while the relation between ω and k must be calculated. The positive sign of the term kz in the exponent of (309c) makes the displacement and the velocity vanish for large negative z.

The total potential energy (307a) becomes now, with (309a) and (309c)

$$V=(G/2S)a^2\cos^2\omega t\,k^4\int\sin^2(k_xx+k_yy)dxdy$$
$$=\tfrac{1}{4}Ga^2\cos^2\omega t\,k^4, \qquad (310)$$

since the average of $\sin^2$ over the surface is $\frac{1}{2}$. Similarly, the kinetic energy (308b) becomes

$$T=\tfrac{1}{4}\rho Sa^2\omega^2\sin^2\omega t\,k. \qquad (310a)$$

If the sum of potential and kinetic energy is to be

constant, we must have

$$\omega = k^{\frac{3}{2}}(G/\rho S)^{\frac{1}{2}}. \tag{311}$$

From this expression, we can easily find the number of normal modes per unit frequency ω, a number which is needed for calculating the energy of the liquid drop as a function of the temperature. Following the usual method, we obtain for the number of normal modes per $dk_x dk_y$

$$(S/4\pi^2)dk_x dk_y. \tag{312}$$

Therefore the number of vibrations with wave numbers between k and $k+dk$ is

$$(S/4\pi^2)2\pi k dk \tag{312a}$$

and that with frequencies between ω and $\omega+d\omega$, according to (311)

$$p(\omega)d\omega = (S/2\pi)(\rho S/G)^{\frac{1}{3}}\tfrac{2}{3}\omega^{\frac{1}{3}}d\omega. \tag{312b}$$

Essentially the same formula can also be derived for a spherical drop by a more detailed study of its surface vibrations, using an analysis of the deformation in spherical harmonics.

For a spherical nucleus, the surface area is

$$S = 4\pi R^2 \tag{313}$$

and the density $\rho = 3AM/4\pi R^3$. (313a)

The surface energy G can be deduced from empirical data on nuclear binding energies using the method outlined in §30; we find

$$G = \Gamma A^{\frac{2}{3}}, \tag{314}$$

where $\Gamma = 9.6 \text{ MV} \approx 10 \text{ MV}$ (314a)

if the nuclear radius has the value $2.05\cdot 10^{-13}A^{\frac{1}{3}}$ (§68). If we introduce, instead of ω, the quantum energy

$$\epsilon = \hbar\omega \tag{315}$$

(312b) may be written

$$p_S(\epsilon)d\epsilon = 4\cdot 3^{-\frac{1}{3}}(\Gamma P)^{-\frac{2}{3}}A^{\frac{2}{3}}\epsilon^{\frac{1}{3}}d\epsilon \tag{315a}$$

with P as defined in (303a, d). For a heavy nucleus ($A=200$), this gives about 3 normal modes of surface vibrations[12a] with quantum energies below 1 MV. This seems reasonable in connection with the fact that the low excitation levels of radioactive nuclei lie at a few hundred kilovolts.

2. Volume waves.—An ideal liquid will admit only logitudinal volume waves. The number of normal modes is

$$\Omega(2\pi)^{-3}\cdot 4\pi k^2 dk = (2/3\pi)R^3\omega^2 d\omega/u_0^3, \tag{316}$$

where u_0 is the velocity of sound. This quantity is connected to the compressibility of the substance, i.e., to the second derivative of the energy E with respect to the density σ,

$$u_0^2 = \frac{\sigma^2}{MA}\frac{d^2E}{d\sigma^2} = \frac{R^2}{9MA}\frac{d^2E}{dR^2}, \tag{317}$$

where MA is the mass, E the energy, R the radius and $\sigma \propto R^{-3}$ the density of the nucleus ($dE/dR=0$). d^2E/dR^2 may be estimated from the statistical formula for the energy as a function of the radius, (159) (159a):

$$\frac{E}{AT_0} = \frac{3}{10}x^2 - \frac{B}{\pi^{\frac{1}{2}}T_0}\{2x^{-3} - 3x^{-1} + (x^{-1} - 2x^{-3})e^{-x^2} + \pi^{\frac{1}{2}}\Phi(x)\}, \tag{318}$$

where $T_0 = \hbar^2/Ma^2$, $x = \frac{3}{2}\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\frac{a}{r_0}$ (318a)

and Be^{-r^2/a^2} is the interaction between two particles. We use for B, T_0 and x the values derived from the new nuclear radius according to the same method as used in §26, *viz.*

$B = 34$ MV, $T_0 = 1.65$ MV,
$a = 5.0\cdot 10^{-13}$ cm, $x = 3.66$. (318b)

We obtain for the second derivative of E:

$$R^2\frac{d^2E}{dR^2} = x^2\frac{d^2E}{dx^2} = 6AT_0\left[\frac{1}{10}x^2 + \frac{B}{T_0\pi^{\frac{1}{2}}}x^{-3}\{x^2 - 4 + (x^4 + 3x^2 + 4)e^{-x^2}\}\right]. \tag{318c}$$

Inserting the values (318b), this becomes

$$R^2(d^2E/dR^2) = 35A \text{ MV}. \tag{319}$$

Therefore, according to (317)

$$K = Mu_0^2 = (R^2/9A)(d^2E/dR^2) \approx 4 \text{ MV}. \tag{319a}$$

Inserting (319a) into (316), we obtain for the number of longitudinal volume waves with a quantum energy between ϵ and $\epsilon+d\epsilon$

$$p_l(\epsilon)d\epsilon = (2/3\pi)(KP)^{-\frac{3}{2}}A\epsilon^2 d\epsilon. \tag{320}$$

The total number of "longitudinal" modes of vibration is equal to the number of particles, A. The number of modes of surface waves is of the order of the number of particles in the surface, i.e., about $A^{\frac{2}{3}}$. For large A, this is negligible compared to A. Thus longitudinal waves and surface waves together do not give all possible modes of vibration whose number must be $3A$ (number of degrees of freedom). The additional modes must apparently be "transverse" volume waves. If such waves have fairly short wave-length, their frequency will be of the same order as that of longitudinal waves of the same length: For short waves, only the relative displacement of close neighbors

[12a] From a direct analysis of the vibrations of the spherical drop, it follows that the lowest frequency lies actually at about 1.1 MV. Our distribution function (315a) will therefore give a good approximation only for nuclear temperatures of 1 MV or higher. For lighter nuclei, the minimum temperature required increases as $A^{-\frac{1}{3}}$.

matters, but the arrangement and interaction of close neighbors is about the same in a liquid as in a crystal, and in a crystal there is no great difference between the frequencies of waves of the same wave-length and different polarization provided the wave-length is short. Thus, for short waves, we shall get approximately the correct number of modes of vibration per $d\epsilon$ if we multiply (320) by three, i.e. the number of possible polarizations is

$$p_v(\epsilon)d\epsilon=(2/\pi)(KP)^{-3}A\epsilon^2 d\epsilon. \tag{320a}$$

The frequency of long transverse volume waves in a liquid is an unsolved problem. We may only hope that such waves are not very essential for the problem of the specific heat. This may perhaps be justified by the fact that to a certain extent, the long transverse waves will be replaced by surface waves. Moreover, at the low temperatures which are important for nuclear reactions, the influence of the surface waves is much more important than that of the volume waves. In the absence of any correct treatment, we shall therefore assume (320a) to give the correct number of volume waves for low as well as for high quantum energy.

3. Thermal properties of the liquid drop.—The total number of normal modes of quantum energy between ϵ and $\epsilon+d\epsilon$ is (cf. (315a), (320a))

$$p(\epsilon)d\epsilon=(4\cdot 3^{-\frac{1}{2}}(\Gamma P)^{-\frac{3}{2}}A^{\frac{1}{2}}\epsilon^{\frac{1}{2}} +(2/\pi)(KP)^{-3}A\epsilon^2)d\epsilon. \tag{321}$$

The energy at the "temperature" τ is then, according to Planck's formula

$$U=\int_0^\infty \frac{p(\epsilon)d\epsilon}{e^{\epsilon/\tau}-1}=\frac{4}{3^{\frac{1}{2}}}C_{4/3}\frac{A^{\frac{1}{2}}\tau^{7/3}}{(\Gamma P)^{\frac{3}{2}}}+\frac{2}{\pi}C_3\frac{A\tau^4}{(KP)^3}, \tag{322}$$

where
$$C_n=\int_0^\infty \frac{x^n dx}{e^x-1}=n!\zeta(n+1), \tag{322a}$$

$$\zeta(n+1)=1^{-(n+1)}+2^{-(n+1)}+\cdots, \tag{322b}$$

$$C_{4/3}=1.694, \quad C_3=6.50. \tag{322c}$$

We introduce a "critical temperature" τ_0 as that temperature at which the contributions of surface and volume waves to the energy become equal. According to (322),

$$\tau_0=(2\pi C_{4/3}/C_3)^{3/5}3^{-1/5}(K/\Gamma)^{2/5}(KP)^{\frac{1}{2}}A^{-1/5} =1.080(K/\Gamma)^{2/5}(KP)^{\frac{1}{2}}A^{-1/5}. \tag{323}$$

At this temperature, the contributions of surface and volume waves to U each have the value

$$U_0=2^{17/5}\pi^{7/5}3^{-4/5}C_{4/3}{}^{12/5}C_3{}^{-7/5}(K/\Gamma)^{8/5}(KP)^{\frac{1}{2}}A^{1/5} =5.60(K/\Gamma)^{8/5}(KP)^{\frac{1}{2}}A^{1/5}. \tag{324}$$

If we insert the values of P, Γ and K (cf. (303d), (314a), (319a)), we have

$$U_0=8.2A^{1/5}\text{ MV}, \tag{324a}$$

which, for $A=100$, has the value 20.6 MV. In most practical cases, the excitation energy will therefore be considerably less than $2U_0$, and therefore the temperature less than τ_0. This means that surface waves are in general more important than volume waves.

We introduce the abbreviation

$$T=\tau/\tau_0. \tag{325}$$

Then (322) reduces to (cf. (323), (324))

$$U/U_0=T^{7/3}+T^4. \tag{325a}$$

The entropy is

$$S=\int\frac{d\tau}{\tau}\frac{dU}{d\tau}=\frac{U_0}{\tau_0}\left(\frac{7}{4}T^{4/3}+\frac{4}{3}T^4\right) =S_0\left(\frac{21}{16}T^{4/3}+T^4\right) \tag{326}$$

with $S_0=4U_0/3\tau_0=6.93(K/\Gamma)^{6/5}A^{2/5}$. (326a)

With $P=\Gamma=10$ MV and $K=4$ MV, we have

$$S_0=2.31A^{2/5}, \tag{326b}$$

$$\tau_0=4.75A^{-1/5}\text{ MV}. \tag{326c}$$

Finally, the quantity λ in (281) has the value

$$\lambda=(2\pi dU/\partial\tau)^{\frac{1}{2}}\tau=\lambda_0\left(\frac{7}{12}T^{4/3}+T^3\right)^{\frac{1}{2}}T \tag{327}$$

with

$$\lambda_0=(8\pi U_0\tau_0)^{\frac{1}{2}}=12.3(K/\Gamma)(KP)^{\frac{1}{2}}=31\text{ MV}. \tag{327a}$$

The temperature, T, the entropy S/S_0 and the quantity λ/λ_0 are plotted in Fig. 9 as functions of the energy U/U_0. The plots are almost straight lines, especially for the entropy. The values of U_0S_0 and τ_0 for various values of the nuclear mass are given in lines B to D of Table XXI. From these values and from the curves of Fig. 9, the temperature, entropy etc. can be obtained for any excitation energy of the nucleus. The temperatures τ so deduced are listed in lines E to G of Table XXI, for excitation energies of 5, 10 and 20 MV. They are seen to be much smaller than

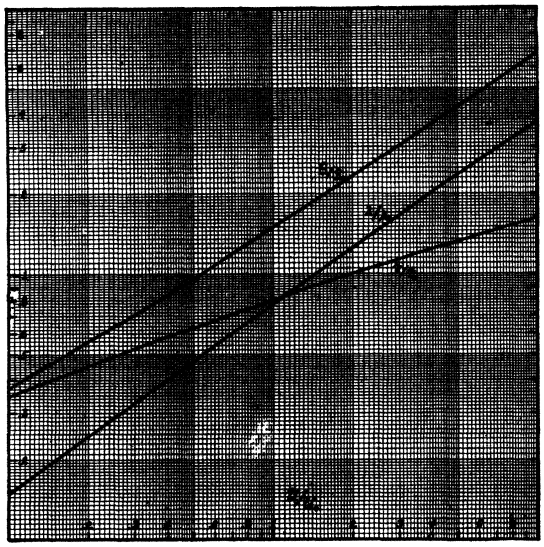

FIG. 9. Nuclear temperature τ, entropy S, and λ (cf. (281)) as functions of the excitation energy U in the liquid drop model. The constants U_0, S_0, τ_0, λ_0 are given in formulae ((324a), (326b), (326c), (327a)).

the excitation energies U, *viz.* about 1 to 2 MV for 10 MV excitation. This is, of course, due to the fact that the excitation energy is, even at low temperatures, distributed among many normal modes of vibration.

From temperature, entropy and λ, the spacing of nuclear levels can be deduced with the help of (281). The result is given in lines H to K of Table XXI, for four different excitation energies (5 to 20 MV).

Table XXI shows clearly the very rapid decrease of the spacing with increasing excitation energy and increasing size of the nucleus which is expected from qualitative considerations (B32). Excitation energies between 15 and 20 MV are easily obtained if fast deuterons (kinetic energy $\sim$5 MV) are used as projectiles, since the large internal energy of the deuteron forms part of the excitation energy of the compound nucleus. With such energies and heavy nuclei, the spacing is seen to be only of the order of a millivolt.

The dependence of the results on the assumptions made in the calculation is shown in lines M and N of Table XXI. In the upper of these lines, we have tabulated the spacing obtained when only surface waves are considered, for $U=10$ MV. The results differ only by a factor 2.5 to 3 from that found with surface and volume waves for the same excitation energy. This is quite welcome because our treatment of volume waves was much less satisfactory than that of the surface waves. This concerns the general procedure, especially the treatment of the transverse volume waves, and also the determination of the constant K which was only possible with the help of the statistical model of nuclei whereas Γ and P could be deduced from empirical data. Therefore the small influence of the volume waves increases our confidence in the results. Of course, a larger influence is to be expected for higher excitation.

In the N line of Table XXI, the spacing as calculated, assuming the old nuclear radius derived by Gamow from the one-body model of the

TABLE XXI. *Spacing of nuclear energy levels in the "liquid drop" model.*

	Atomic Weight A		10	20	50	100	200
B	Characteristic energy U_0 (MV)		13.0	14.9	17.9	20.6	23.7
C	Characteristic temperature τ_0 (MV)		3.00	2.61	2.17	1.89	1.64
D	Characteristic entropy S_0		5.8	7.65	11.05	14.6	19.3
E	Tempera-	for $U=$ 5 MV	1.72	1.43	1.11	0.92	0.75
F	ture τ	10	2.18	1.82	1.41	1.17	0.97
G	(MV)	20	2.78	2.31	1.81	1.49	1.24
H		for $U=$ 5 MV	110,000	36,000	5,900	1,100	150
I	Spacing	$U=10$ MV	13,000	2,700	180	16	0.9
J	$D(U)$ in	$U=15$ MV	2,000	240	9.5	0.35	0.007
K	volts	$U=20$ MV	400	33	0.55	0.015	$1.5\cdot10^{-4}$
L		$U=Q$	16,000	3,300	310	60	15
M	Spacing for	Without volume waves	30,000	9,000	500	40	2.7
N	$U=10$ MV	With $\tau_0=1.48\cdot10^{-13}$	100,000	30,000	4,000	500	60
O	$Q=$neutron dissociation energy (MV)		9.3	9.5	9.1	8.2	6.9

α-particle disintegration, is also given. The corresponding values of the spacing are, as is to be expected, very much (up to 70 times) larger than those derived with the larger radius. A spacing of as much as 500 volts for $A=100$ seems quite irreconcilable with the evidence from slow neutron experiments which may be taken as an argument for the larger nuclear radius independent of the α-particle disintegration (Chapter XI).

The greatest interest of Table XXI lies, of course, in comparisons with experimental data. The most extensive of these are the data obtained from slow neutron experiments which indicate spacings of the order of 10–20 volts for $A=100$ and 200, and 100 to a few hundred volts for $A=50$ (§60, especially Table XXII). The levels of the compound nucleus responsible for neutron capture are those whose excitation energy is about equal to the dissociation energy Q of the compound nucleus into a neutron and a residual nucleus. This dissociation energy Q is estimated in the last line of Table XXI from the semi-empirical formula for mass defects (cf. §30, and B13); the corresponding spacing is given in line L. The values thus obtained for $A=50$, 100 and 200 compare very favorably with the average spacing of neutron levels estimated from experimental data.

Less satisfactory is the agreement with data from proton capture and α-particle disintegrations of very light nuclei. E.g., the levels of P^{31} are well known from the α-particle disintegration of Al^{27}. The spacing is about 0.3 to 0.5 MV for α-particle energies of the order of 5 MV, corresponding to excitation energies of about 12 MV (cf. §82, Table XXXX). Table XXI would indicate a spacing of less than 1000 volts for this case. Similarly, 3 levels below 1 MV are known in the capture of protons by F^{19} (§81), corresponding to an average spacing of about 0.3 MV. The corresponding excitation energy of the compound nucleus, Ne^{20}, is 12 MV so that, from Table XXI, we should expect a spacing of about 1000 volts.

Part of the discrepancy in the latter case may be attributed to the fact that the normal state of Ne^{20} lies particularly low compared to neighboring nuclei. The reasons for this are well known (even number of neutrons as well as protons), but these reasons will only depress the ground state and not the excited states of the nucleus. Therefore the excitation relative to the ground state should not be taken as significant for the "temperature" of the nucleus. Tentatively, we may replace in such a case the ground state by the average of the ground states of two neighboring nuclei, in our case perhaps F^{19} and Ne^{21}. This would reduce the excitation energy by about 4 MV so that a spacing of 10 kv might be expected from Table XXI. But even this spacing is much smaller than the observed one.

It seems therefore that our calculations fail for very light nuclei. This is not very surprising in view of the assumptions made; especially the distinction between surface and volume waves would seem objectionable when almost all the particles in the nucleus are surface particles. Moreover, it might be possible that the nuclear volume increases faster than the number of particles owing to some repulsive interactions; this would make light nuclei smaller and therefore the spacing of their energy levels wider. However, it seems more plausible that the discrepancies are due to a failure of the liquid drop model; the Bardeen model (section B) seems to give much more reasonable values (Table XX) for the spacing of the levels of light nuclei. It is difficult to decide what atomic weight is required for the validity of the liquid drop model; we shall assume it to be valid for $A \gtrsim 50$.

In conclusion, we give here the formulae for the temperature, entropy etc. when only surface waves are important. This assumption is a good approximation as is shown by Table XXI. From (322) we have then

$$U=4\cdot 3^{-\frac{1}{4}}C_{4/3}(\Gamma P)^{-\frac{1}{4}}A^{\frac{1}{4}}\tau^{7/3}$$
$$=4.7(\Gamma P)^{-\frac{1}{4}}A^{\frac{1}{4}}\tau^{7/3}, \tag{328a}$$
$$\tau=0.515(\Gamma P)^{2/7}A^{-2/7}U^{3/7}, \tag{328b}$$
$$S=3.39(A\,U^2/\Gamma P)^{2/7}, \tag{328c}$$
$$\lambda=2.75(\Gamma P/A)^{1/7}U^{5/7}, \tag{328d}$$
$$D=\lambda e^{-S}. \tag{328e}$$

With $\Gamma=P=10$ MV, and U in MV:

$$\tau=1.92A^{-2/7}U^{3/7}\text{ MV}, \tag{329a}$$
$$S=0.91A^{2/7}U^{4/7}, \tag{329b}$$
$$\lambda=5.3A^{-1/7}U^{5/7}\text{ MV}. \tag{329c}$$

§54. The Widths of Nuclear Levels and the Evaporation Model (B33, W7, B15, F30)

There are two ways of calculating the widths of nuclear levels: Either we may start from the assumption that a particle falling on the nucleus will, in a large percentage of all cases, amalgamate with it and form a compound nucleus. This assumption seems plausible because of the large interaction between nuclear particles and is confirmed by slow neutron measurements (cf. Section D); it forms the basis of the "evaporation model" (Sections D, E) which is especially useful for high energy particles. The other, more ambitious method is to try to calculate the matrix elements (widths) from the wave functions (Sections A to C). Although quantitative results have not been obtained from this method, the formalism seems useful especially for the treatment of the widths of low compound states which occur, e.g., in natural radioactivity (§67).

A. Wave function of the incident particle and nuclear potential

The matrix element is defined as

$$H_r^{np} = \int \psi_{Cr}^* H \psi_{An} \psi_{Pp} d\tau, \qquad (330)$$

where ψ_{An}, ψ_{Pp} and ψ_{Cr} are the wave functions of initial nucleus, incident particle and compound nucleus, respectively, the subscripts $n\ p\ r$ specifying the states of the three systems. H is the Hamiltonian of the interaction between incident particle and initial nucleus.

In the calculation of this matrix element, the main uncertainty arises from the function ψ_{Pp} representing the incident particle. All the other factors, *viz.* the Hamiltonian H and the wave functions ψ_{An} and ψ_{Cr} of the initial and compound nucleus, can either be considered as known (Hamiltonian) or can be calculated in principle from the Schrödinger equation. It is true that this calculation would be impracticable in the case of complicated nuclei; but at least the functions ψ_{An} and ψ_{Cr} are uniquely determined by the respective Hamiltonians.

This is not the case for the particle wave function ψ_{Pp}. Here we can make any of three choices:

(a) We can take ψ_{Pp} to be a plane wave,

(b) We can take ψ_{Pp} as the solution of the one-particle Schrödinger equation in an *attractive* potential extending approximately over the volume of the nucleus,

(c) We can take a *repulsive* potential of the same extension.

This ambiguity refers, of course, only to the determination of a "zero-order potential" and of the zero-order wave functions of the particle. It is not an ambiguity in the problem of nuclear disintegration itself but only in our method of solving that problem. If we were able to solve the problem rigorously, we would not need to speak of the particle wave function, and of transitions between compound and dissociated state at all; we would only have to solve the Schrödinger equation for the complete system, and the solution would automatically represent the compound state as long as all particles are close together, and behave asymptotically like the wave function of the initial nucleus, times a spherical wave of the particle P, when this particle is far away from the rest of the system. There would then be no question as to the choice of the phase of the particle wave function because this, among other things, would follow directly from the solution of the wave equation.

As it is, we cannot solve our problem rigorously but must apply a method of successive approximations, the Schrödinger-Dirac perturbation method. Even so, we shall certainly arrive ultimately at the correct result (provided the method converges) from whatever "zero-order wave functions" we start. But it will take a great number of approximations before we arrive at the correct result if we choose an unsuitable zero-order wave function. Therefore we want a particle wave function ψ_{Pp} which makes the perturbation theory converge as rapidly as possible; if possible, we want it such that the second approximation written down in (255) already represents the probability of the process fairly well.

It is evident that the convergence of the successive approximations of the perturbation theory must be very different for the three different choices of the particle potential mentioned above. This follows simply from the fact that, although all zero-order wave functions must ultimately lead to the same result, the value of the matrix element (330) is quite different for different "zero-order" potentials. If we choose an at-

tractive potential, (330) will in the average be much larger than if we start from plane waves, mainly because with an attractive potential there is a chance of having a resonance already in the one-particle wave function ψ_{Pp}, which increases the value of ψ_{Pp} inside the nucleus and hence the matrix element (330). A repulsive potential, on the other hand, will make (330) (in the average) smaller than a zero potential because in a (sufficiently strong) repulsive potential, ψ_{Pp} will decrease exponentially from the surface of the nucleus inwards, and will be smaller than the plane wave function even at the surface itself. The matrix element would decrease indefinitely with increasing repulsive potential except for the fact that the perturbing potential H tends to increase; H is the difference between the actual interaction between particle P and nucleus A, and the "zero-order potential" V used to calculate ψ_{Pp}. Consequently, there will be a certain medium value of the repulsive potential V for which (330) will be a minimum (in the average over the states r).

It seems likely that a repulsive potential which just makes the average matrix elements a minimum will be the best choice for the zero-order potential. Our requirement for a good zero-order wave function is that it shall make the higher approximations of the perturbation theory small compared to the first. This will be the case if the individual matrix elements are small, particularly for high levels r of the compound nucleus. It will be shown below that indeed a repulsive potential will make the matrix elements sufficiently small so that, at least in a particular case, the higher approximations of the perturbation theory are negligible. In contrast to this, we shall find that plane waves make the higher approximations more important than the first, and this will hold *a fortiori* for particle wave functions in an attractive potential.

Our result that we have to choose a repulsive potential to calculate the zero-order wave functions for the incident particle seems very reasonable from the standpoint of the many-body picture of the nucleus. As we have seen in §51, the incident particle will give part of its energy to the nuclear particles as soon as it strikes the surface of the nucleus; it will thus lose energy as it goes inside and its wave function will change to such an extent that we cannot reasonably speak of the free particle inside the nucleus at all. It is therefore reasonable to choose a wave function for the incident particle which "dies out" inside the nucleus.

When it had not yet been realized that the problem of nuclear dynamics is essentially a many-body problem, an attractive potential was used to represent the action of the initial nucleus on the incident particle (B12, A7). This attractive potential was taken as the average interaction energy between nucleus and particle. The use of the average potential seemed very appropriate in the one-body picture when the incident particle was supposed to traverse the nucleus more or less undisturbed. However, in the many-body picture there is no undisturbed traversing of the nucleus, and there appears to be no room for the attractive average potential.

In the theory of radiation, the wave functions of the light quantum which are used in the dispersion formula, are taken simply as plane waves. This appears to be the most straightforward procedure, more logical than to take part of the perturbation already into account when calculating the zero order wave functions. However, the case of radiation is particularly favorable because the interaction between light and matter is very small; in our problem we must apply special tricks to enforce convergence in spite of the large interaction. It is, of course, not clear whether our wave functions are actually suitable to this end, but at any rate they are more likely to be suitable than the other possible functions.

To specify the "zero-order" potential more precisely, we have to define the magnitude of the repulsive potential V_0 and its radius of action. The latter must obviously be taken approximately equal to the nuclear radius R. If the perturbation theory is to have any meaning, the wave functions of the compound states, *plus* those of the dissociated states, should form as nearly a complete set as possible; this obviously requires that the one group of wave functions dies out where the other goes into effect, i.e. that the repulsive potential acts inside a sphere of approximately the nuclear radius.

The height V_0 of the potential barrier must be of the order of nuclear energies (10 MV). A smaller height would be unfavorable because it

would not make the particle wave function ψ_{Pp} small enough inside the nucleus, a larger V_0 would make the perturbing potential H too large, as already mentioned above. If V_0 is chosen of the same order as the actual interaction between nucleus A and particle P, its influence will be moderate.

In order to investigate the influence of the higher approximations of the perturbation theory, we calculate the elastic scattering of slow particles (wave-length $\lambda\!\!\!^{-}$ large compared to the nuclear radius R). The scattering may be divided into three parts:

(a) The "zero-order" scattering contained in the zero-order wave function itself. Since the particle is assumed to be slow, the repulsive potential is practically impenetrable for it so that the nucleus acts like a hard sphere of radius R. The scattering cross section of zero order is therefore

$$\sigma_1 = 4\pi R^2. \tag{331}$$

(b) The terms in the dispersion formula (255) (second-order scattering) which arise from energy levels close to the particle energy. This "resonance scattering" σ_2 may be larger or smaller than the zero-order scattering according to whether the particle energy is near a resonance level or far away. It has no bearing on the question of the convergence of the perturbation calculation.

(c) The terms in the dispersion formula arising from distant energy levels. These terms do not change appreciably with the particle energy and may therefore be regarded as part of the potential scattering. Their ratio to the cross section σ_1 will give us the relative importance of the second as compared to the zero-order approximation which we want to calculate. There is no difficulty in separating "close" and "distant" levels since the contribution of "medium distant" levels can be shown to be very small.

The dispersion formula (255) gives for elastic scattering

$$\sigma_2+\sigma_3 = \pi\lambda\!\!\!^{-2}\left|\sum_r \frac{\gamma^r{}_{Pp}}{E_P - E_{Pr} + \frac{1}{2}i\gamma_r}\right|^2. \tag{332}$$

For given states of initial nucleus and incident particle, the matrix elements $\gamma^r{}_{Pp}$ will be in the average independent of the energy E_r of the compound nucleus, except that for very high compound states $\gamma^r{}_{Pp}$ will fall off (see below). Therefore the contributions to the sum over r in (332) will come mainly from energy regions in which there are many levels, i.e. from high energies. Since for very high energies γ falls off, the main contribution will come from the energy region just before γ starts decreasing.[13] Let us denote the particle energy at which this occurs by E_C. Then we can replace the resonance denominator in (332) by $-E_C$ for all important terms r. The sum over r may be evaluated using the completeness relation:

$$\sum_r \gamma^r{}_{Pp}/(-E_C) = -2\pi\sum_r \left|\int \psi_{Cr}{}^* H\psi_{An}\psi_{Pp}\,d\tau\right|^2/E_C$$

$$= -2\pi\int |H\psi_{An}\psi_{Pp}|^2 d\tau/E_C. \tag{333}$$

The volume element $d\tau$ may be written as $d\tau_A d\tau_P$ where $d\tau_A$ is the volume element in the configuration space of all the particles in the original nucleus A, and $d\tau_P$ the volume element for the incident particle. Integrating over $d\tau_A$ and considering the normalization of ψ_{An}, we have

$$\int |H(r_A, r_P)|^2 |\psi_{An}(r_A)|^2 d\tau_A = U^2(r_P), \tag{333a}$$

where U is an irregularly varying function of the coordinate r_P of the particle, having the magnitude of a nuclear energy. Inserting (333a) in (333) (332), we have

$$\sigma_3 = (4\pi^3\lambda\!\!\!^{-2}/E_C{}^2)\left(\int U^2\psi_P{}^2 d\tau_P\right)^2. \tag{334}$$

The wave function ψ_P is normalized per unit energy, corresponds to the energy E_P and to zero angular momentum of the particle, and satisfies the one-particle Schrödinger equation in a potential which has the value V_0 for $r<R$ and 0 for $r>R$. (The latter assumption is equivalent to assuming the incident particle to be a neutron.) The usual methods give

$$\psi = \begin{cases} \dfrac{1}{\pi(2\hbar v)^{\frac{1}{2}} r}\left(\dfrac{E}{V_0}\right)^{\frac{1}{2}} e^{\kappa(r-R)} & r<R \\ \dfrac{1}{\pi(2\hbar v)^{\frac{1}{2}} r}\sin k(r-R+1/\kappa) & r>R \end{cases} \tag{335}$$

[13] Actually, the condition is that the product of γ and the density of energy levels must have a maximum.

with $k=(2ME)^{\frac{1}{2}}/\hbar$, $\kappa=(2MV_0)^{\frac{1}{2}}/\hbar$. (335a)

Therefore

$$4\pi\int_0^R U^2\psi_P{}^2r^2dr=(U^2)_{\text{Av}}\frac{E}{V_0}\frac{4\pi}{2\kappa}\frac{1}{2\pi^2\hbar v}$$
$$=(U^2)_{\text{Av}}E^{\frac{1}{2}}/2\pi V_0{}^{\frac{3}{2}} \quad (336)$$

and the second-order cross section becomes

$$\sigma_3=(\pi\hbar^2/2M)(U^2)_{\text{Av}}{}^2/V_0{}^3E_C{}^2. \quad (337)$$

The energies U, V_0 and E_C are all of the same order of magnitude. U must be somewhat larger than V_0 because it contains V_0 besides the nuclear potential. We estimate $U\sim 15$ MV, $V_0\sim 10$ MV and (cf. (347a)) $E_C\sim 8$ MV. Therefore the last factor in (337) will be about 0.8 MV^{-1} and σ_3 is about $5\cdot 10^{-25}$ cm^2. It is therefore much smaller than σ_1 which is of the order of 10^{-23} cm^2. This indicates that with our choice of the wave functions, the perturbation calculation actually converges quite rapidly, at least in the case of the elastic scattering.

If we had chosen plane waves, the wave function inside the nucleus would have had the constant value $k/\pi(2\hbar v)^{\frac{1}{2}}$, so that

$$4\pi\int U^2\psi^2r^2dr=(2/3\pi)(U^2)_{\text{Av}}k^2R^3/\hbar v$$
$$=(2M/3\pi\hbar^2)(U^2)_{\text{Av}}R^3/\lambda\!\!\!^{-}, \quad (336a)$$

$$\sigma_3=\frac{16\pi}{9}R^2\frac{(U^2)_{\text{Av}}{}^2}{E_C{}^2(\hbar^2/MR^2)^2}. \quad (337a)$$

With $R=10^{-12}$ cm, $\hbar^2/MR^2$ is about 0.4 MV. For U we take 10 MV instead of 15 MV, because it no longer includes $-V_0$. The critical energy E_C must be chosen higher than before;[14] we take $E_C=17$ MV. Then the last factor in (337a) is about 200, and σ_3 becomes approximately $1000R^2=80\sigma_1$, which shows that the perturbation calculation would diverge with plane waves.

For attractive potentials the perturbation calculation will diverge even more seriously. If $\kappa=(2M|V_0|)^{\frac{1}{2}}/\hbar$ is the wave number inside the nucleus, we have in this case

$$\psi=\frac{k}{\pi(2\hbar v)^{\frac{1}{2}}r(k^2\sin^2\kappa R+\kappa^2\cos^2\kappa R)^{\frac{1}{2}}}\sin\kappa r \quad (335b)$$

for $r<R$. Then

$$4\pi\int U^2\psi_P{}^2r^2dr=\frac{(U^2)_{\text{Av}}R}{\pi\hbar v}\frac{k^2}{k^2\sin^2\kappa R+\kappa^2\cos^2\kappa R}, \quad (336b)$$

$$\sigma_3=\pi R^2\frac{(U^2)_{\text{Av}}{}^2}{E_C{}^2}\frac{1}{(E_C\sin^2\kappa R+V_0\cos^2\kappa R)^2}. \quad (337b)$$

This cross section becomes extremely large in the case of "one-particle resonance," i.e. if

$$\cos\kappa R\approx 0. \quad (337c)$$

In this case there can, of course, be no convergence of the perturbation calculation. We may try to save matters by averaging over all possible phases κR of the particle wave function. Then we find

$$(\sigma_3)_{\text{Av}}=\frac{\pi}{2}R^2\frac{(U^2)_{\text{Av}}{}^2}{E_C{}^2V_0{}^{\frac{1}{2}}E^{\frac{1}{2}}}, \quad (337d)$$

which still becomes extremely large for small energies of the incident particle.

Our considerations show that only with a repulsive zero-order potential can we expect a reasonable convergence of the perturbation expansion.

B. Potential scattering

The two contributions to the potential scattering σ_1 and σ_3 have opposite phase: The phase of the wave scattered from a hard sphere (σ_1) is certainly opposite to the phase of the incident wave (the two waves must just cancel each other at the surface of the sphere). On the other hand, the scattered wave σ_3 due to the dispersion effect of highly excited nuclear states has the same phase as the incident wave, just as light scattered by an atom has when the frequency of the light is smaller than the characteristic frequency of the atom. Thus the total potential scattering cross section is

$$\sigma_{\text{pot}}=(\sqrt{\sigma_1}-\sqrt{\sigma_3})^2. \quad (338)$$

If we estimate σ_3 to be about 5 percent of σ_1 (see above), the total potential scattering will be decreased by 40 percent due to the influence of the high levels of the compound nucleus.

In order to find the interference effects between potential scattering and resonance scattering, we must separate the former into contributions of

[14] The estimate of the critical energy E_C in (344) to (347) is based on a comparison of the experimental width of slow neutron levels with the sum of the neutron widths of all levels as derived from the completeness relation (336). If a different wave function is taken for the neutron (particle P), the completeness relation will give a different result for the sum of all neutron widths. On the other hand, the experimental width of one level is given; therefore a different number of levels below E_C will be deduced. With plane waves, we have according to (336a) instead of (344):

$$N(E_C)=\frac{\Sigma\gamma^r{}_{Pp}}{(\gamma^r{}_{Pp})_{\text{Av}}}=\frac{4}{3}\frac{MR^3}{\hbar^2}\frac{(U^2)_{\text{Av}}}{\lambda\!\!\!^{-}\gamma_{\text{Av}}}=5\cdot 10^9, \quad (344a)$$

taking $U=10$ MV, $\hbar^2/MR^2=0.4$ MV, $R=10^{-12}$ cm, $\lambda\!\!\!^{-}=3.10^{-10}$ cm, $\gamma=2\cdot 10^{-4}$ volts. This number of levels is very much larger than that derived from the repulsive potential wave functions in (344), corresponding to the large amplitude of the plane waves inside the nucleus. With (344a) and the assumptions leading to (347a), we find

$$U_C\approx 25 \text{ MV}, \quad E_C\approx 17 \text{ MV}.$$

various angular momenta J of the whole system. Neglecting σ_3 and assuming the orbital momentum of the incident particle to be zero the contribution of the angular momentum J to the cross section of the potential scattering is

$$\sigma_{1J}=4\pi\frac{2J+1}{(2i+1)(2s+1)}R^2. \tag{339}$$

Considering that only states of the same J interfere, and considering the phase of the potential scattering, the total elastic scattering of slow neutrons becomes

$$\sigma=\frac{\pi}{(2i+1)(2s+1)}\sum_J(2J+1)\times\left|2R+\lambda\sum_r\frac{\gamma^{rJ}{}_{Pp}}{E_P-E_{Pr}+\frac{1}{2}i\gamma_{rJ}}\right|. \tag{340}$$

If only one resonance level is important, this reduces to

$$\sigma=4\pi R^2+\frac{2J+1}{(2i+1)(2s+1)}\frac{\pi\lambda\gamma^{rJ}{}_{Pp}}{(E_P-E_{Pr})^2+\frac{1}{4}\gamma_{rJ}{}^2}\times[4R(E_P-E_{Pr})+\lambda\gamma^{rJ}{}_{Pp}]. \tag{341}$$

Near each resonance level, the cross section has a minimum on the side of low energies and a maximum at an energy slightly higher than the resonance energy E_{Pr}. If we use the fact that $\gamma^{rJ}{}_{Pp}$ varies inversely as the wave-length for slow particles (cf. 266), and if we assume γ_{rJ} to be constant and equal to the true width Γ_{rJ}, we have

$$\sigma=4\pi R^2+\frac{2J+1}{(2i+1)(2s+1)}\frac{\pi\lambda_{Pr}\Gamma^{rJ}{}_{Pp}}{(E_P-E_{Pr})^2+\frac{1}{4}\Gamma_{rJ}{}^2}\times[4R(E_P-E_{Pr})+\lambda_{Pr}\Gamma^{rJ}{}_{Pp}], \tag{341a}$$

$$\sigma/\sigma_1=1+\frac{2J+1}{(2s+1)(2i+1)}\frac{\tau(\tau+4x)}{1+4x^2} \tag{342}$$

with

$$\tau=\lambda_{Pr}\Gamma^{rJ}{}_{Pp}/R\Gamma_{rJ},$$

$$x=(E_P-E_{Pr})/\Gamma_{rJ}. \tag{342a}$$

If the resonance scattering is large compared to the potential scattering ($\tau\gg1$), the minimum cross section occurs at $x=-\frac{1}{2}\tau$ and the minimum at $x=1/2\tau$, the respective values of the cross section being

$$\sigma_{\min}=\left(1-\frac{2J+1}{(2s+1)(2i+1)}\right)\sigma_1, \tag{343}$$

$$\sigma_{\max}=\frac{2J+1}{(2s+1)(2i+1)}\tau^2\sigma_1. \tag{343a}$$

C. Sum rules and dependence of the matrix element on the energy of the compound state

We now want to discuss the average behavior of the matrix element (330) as a function of the energy of the compound nucleus W_{Cr} and the energy of the particle E_P. The wave functions of the various states of the compound nucleus will certainly not be fundamentally different. Consequently, the matrix elements will also not show any particular trend with the energy of the compound state, the state of the particle being kept fixed. This is another point in which nuclear theory is in contrast to atomic theory: Consider, e.g., the matrix elements of the optical transitions from the ground state of an atom to its excited states. (The excited states correspond to the states of the compound nucleus, the ground state to the initial nucleus, the light quantum to the incident particle.) These matrix elements fall off as the inverse third power of the principal quantum number n of the excited state of the atom (cf. B16, p. 442). The reason is simply that in the excited states the valency electron gets farther and farther away from the core of the atom, and its probability of being in the region occupied by the ground state decreases as n^{-3}. On the other hand, a nucleus in an excited state will be just as concentrated as in its ground state, and therefore the wave function of an excited state of the "compound nucleus" will overlap just as much with that of the initial nucleus as the wave function of the ground state of the compound nucleus.

This will, of course, not be true for extremely high excited states of the compound nucleus. The width of such states will be much more accurately given by the evaporation model (Section D) according to which the width of high compound levels decreases as their spacing. This means a decrease in the partial width at high energy and, in addition, it describes the approximate quantitative way in which the decrease takes place.

In order to complete our calculations in A, we want to estimate the critical excitation energy U_C above which the

widths fall off, *not* with the correct dependence of the widths on the energy of the compound state but with the dependence assumed in A, *viz.* constant width up to U_C and width zero at higher energies. Such an estimate may be obtained by combining the experimental results about neutron width with the completeness relation and with some assumptions about the density of nuclear energy levels. From the experiments about neutron capture, we may deduce a neutron width of the order of about $3\cdot10^{-4}$ volts for a neutron energy of about 1 volt and an atomic weight around 100. On the other hand, we have from the completeness relation according to (333) (333a) (336):

$$\sum_r \gamma^r{}_{Pp} = 2\pi\sum_r \left|\int \psi_{Cr}{}^* H \psi_{An}\psi_{Pp} d\tau\right|^2$$

$$= 2\pi\int U^2(r_P)\psi_P{}^2 d\tau_P = U^2{}_{An}E^{\frac{1}{2}}/V_0{}^{\frac{1}{2}}. \quad (344)$$

Since we have assumed $\gamma^r{}_{Pp}$ to be in the average independent of the energy up to the critical excitation energy U_C, the sum on the left of (344) represents just the value of a single matrix element, times the number of states $N(U_C)$ contributing, i.e., the number of states with energy below U_C. We have therefore

$$N(U_C) = (U^2{}_{An}E^{\frac{1}{2}}/V_0{}^{\frac{1}{2}}\gamma_{An}) = (10^{7/2}/3\cdot10^{-4}) = 1\cdot10^7, \quad (345)$$

assuming U and V_0 to be about 10^7 volts. There will thus be about ten million energy levels below the energy U_C.

The critical excitation energy U_C may then be calculated using the values for the spacing of the levels derived from the liquid drop model. Using the general formula (281), we have for the number of levels below U_C

$$N(U_C) = \int_0^{U_C} e^{S(U)}\frac{dU}{\tau}\left(2\pi\frac{dU}{d\tau}\right)^{-\frac{1}{2}}. \quad (346)$$

According to the general thermodynamical relation (279b)

$$dU/\tau = dS. \quad (346a)$$

The last factor in (346) changes slowly compared to the exponential and may therefore, without appreciable error, be replaced by its value for $U = U_C$. Then we have

$$N(U_C) = (2\pi dU/d\tau)_C{}^{-\frac{1}{2}}e^{S(U_C)}. \quad (346b)$$

Using again the general formula for the spacing, this gives

$$N(U_C) = \tau(U_C)/D(U_C). \quad (347)$$

The solution of this equation, for $N = 10^7$ and $A = 100$, is about (cf. Table XXI)

$$U_C = 16 \text{ MV}. \quad (347a)$$

Thus U_C is of the order of a nuclear energy, and $E_C = U_C - Q$ is of the order of the dissociation energy Q itself.

D. The evaporation model, general theory

Frenkel (F30) and Bohr (B33) have proposed to calculate the width of nuclear energy levels, and in particular its dependence on the energy of the compound and final nucleus, by analogy with the process of evaporation. Weisskopf (W7) has given a more detailed treatment of this "evaporation model." The method is especially valuable in predicting the energy distribution of the particles emitted (cf. Section E).

As in the theory of the rate of vaporization, we shall express the probability of emission of a particle by the compound nucleus (vaporization) in terms of the probability of the reverse process (condensation). For this purpose, we shall consider a large box of volume Ω containing nuclear particles (neutrons, protons) in equilibrium with complex nuclei of various kinds. Let $g_{Ak}F(A, W_{Ak}, \mathbf{p}_A)d\mathbf{p}_A$ be the number of nuclei of kind A, whose internal quantum state is W_{Ak} and whose center of gravity moves with the momentum $\mathbf{p}_A$. g_{Ak} is the statistical weight of state k, and $d\mathbf{p}_A$ is the volume element in momentum space. Then, in thermal equilibrium

$$g_{Ak}F(A, W_{Ak}, \mathbf{p}_A)d\mathbf{p}_A = g_{Ak}\frac{\Omega d\mathbf{p}_A}{(2\pi\hbar)^3}e^{-(W_{Ak}+E_A)/kT}, \quad (348)$$

where E_A is the kinetic energy

$$E_A = p_A{}^2/2M_A. \quad (348a)$$

The internal energy W_{Ak} is most conveniently defined as the total energy minus the energy of the free particles constituting nucleus A. The number of nuclei A having an internal energy between W_A and W_A+dW_A and a momentum in the interval $d\mathbf{p}_A$ is

$$F(A, W_A, \mathbf{p}_A)\rho(W_A)dW_A d\mathbf{p}_A, \quad (348b)$$

where $\rho(W_A)$ is the density of energy levels of nucleus A, counting each level as many times as it is degenerate.

We now consider the disintegration of the "compound" nucleus C into two nuclei A and P. The internal states of the latter nuclei are fixed and may be denoted by k and n respectively. The compound nucleus may be in a state r in the interval between W_C and W_C+dW_C. We denote the probability of disintegration of C into A and P by $w_{AP}{}^C$; it will depend on W_C and on the states k and n of the final nuclei A and P. $w_{AP}{}^C$ should be considered as the average of the disintegration probabilities of the compound states in the energy interval dW_C. The probability of the reverse process, i.e., the recombination of A and P to form C, will be denoted by $w_C{}^{AP}$. This recombination will, of course, only be possible when momentum and energy are conserved.

The conservation of momentum requires

$$\mathfrak{p}_A+\mathfrak{p}_P=\mathfrak{p}_C, \tag{349}$$

the conservation of energy

$$W_{Cr}+E_C=W_{Ak}+W_{Pn}+E_A+E_P, \tag{349a}$$

where the E's denote kinetic, the W's internal energies. We introduce, besides $\mathfrak{p}_C$, the momentum of the relative motion of nuclei A and P, *viz.*

$$\mathfrak{p}=(M_A\mathfrak{p}_P-M_P\mathfrak{p}_A)/(M_A+M_P). \tag{349b}$$

Then we have

$$E_A+E_P=E+E_C, \tag{349c}$$

where E is the kinetic energy of the relative motion

$$E=p^2/2M, \tag{349d}$$

M = reduced mass $= M_AM_P/(M_A+M_P)$. The conservation of energy becomes then

$$W_{Cr}=W_{Ak}+W_{Pn}+E. \tag{350}$$

The probabilities $w_{AP}{}^C$ and $w_C{}^{AP}$ are connected by the principle of detailed balancing. The number of disintegration processes $C\rightarrow A+P$ must be equal to the number of recombinations $A+P\rightarrow C$. This gives

$$\rho(W_C)F(C, W_C, \mathfrak{p}_C)d\mathfrak{p}_C dW_C\cdot w_{AP}{}^C = \int_\omega g_{Ak}g_{Pn}F(A, W_{Ak}, \mathfrak{p}_A)F(P, W_{Pn}, \mathfrak{p}_P) \times d\mathfrak{p}_A d\mathfrak{p}_P w_C{}^{AP}. \tag{351}$$

The integral on the right-hand side extends over all *directions* of the relative motion $\mathfrak{p}$, keeping W_C and $\mathfrak{p}_C$ constant. Because of (349) (349b), we may write

$$d\mathfrak{p}_A d\mathfrak{p}_P=d\mathfrak{p}_C d\mathfrak{p} \tag{351a}$$

and divide right and left-hand side of (351) by $d\mathfrak{p}_C$. Understanding $w_C{}^{AP}$ to be an *average* over all directions of $\mathfrak{p}$, we may then carry out the integration over the angle ω and obtain, instead of $d\mathfrak{p}$,

$$4\pi p^2dp=4\pi p^2(dp/dE)dW_C, \tag{351b}$$

using the conservation of energy, (350). (351) divided by dW_C becomes

$$\rho(W_C)F(C, W_C, \mathfrak{p}_C)w_{AP}{}^C = g_{Ak}g_{Pn}F(A, W_{Ak}, \mathfrak{p}_A) \times F(P, W_{Pn}, \mathfrak{p}_P)\cdot 4\pi pM\cdot w_C{}^{AP}. \tag{352}$$

We now insert in (352) the distribution function (348). Because of the conservation of energy (349a), the Boltzmann factors on the left and right-hand side of (352) cancel and we obtain

$$w_{AP}{}^C=w_C{}^{AP}\frac{g_{Ak}g_{Pn}}{\rho(W_C)}\frac{4\pi Mp\Omega}{(2\pi\hbar)^3}. \tag{353}$$

Here we may replace $\hbar w_{AP}{}^C$ by the partial width $\Gamma^{C_r}{}_{AkPn}$ of the compound levels referring to the disintegration into nuclei A and P in the states k and n, respectively (see the definition of Γ in §52). Γ means here the energy value of the width, averaged over all compound states r of energy near W_C. Furthermore, we write

$$\rho(W_C)=1/D'(W_C)=1/D_r', \tag{353a}$$

where D' is the average spacing of the compound levels. Then we have

$$\Gamma^{C_r}{}_{AkPn}/D_r'=\Omega w_C{}^{AP}g_{Ak}g_{Pn}(M^2v/2\pi^2\hbar^2). \tag{354}$$

The probability of recombination $w_C{}^{AP}$ is proportional to the number of collisions between nuclei A and P. Let R be the "collision radius" of the nuclei A and P. If quantum effects can be neglected, i.e., for short de Broglie wave-length $\hbar/Mv$, this radius may be taken as the sum of the actual radii of nuclei A and P. Then we must calculate the probability that a particle P moving with the velocity v hits a stationary sphere A of radius R if the particle P is distributed uniformly over the volume Ω. This probability is, per unit time,

$$w_C{}^{AP'}=4\pi R^2v_n/\Omega, \tag{355}$$

where v_n is the average value of the component of the velocity v normal to the surface of the sphere. As is shown easily, $v_n=\frac{1}{4}v$. We assume now that the average probability for recombination in a collision is ξ. Then the probability of recombination becomes

$$w_C{}^{AP}=\pi R^2v\xi/\Omega \tag{355a}$$

and the partial width

$$\Gamma^{C_r}{}_{AkPn}/D_r'=\frac{g_{Ak}g_{Pn}}{\pi}\cdot\frac{MR^2}{\hbar^2}\cdot\tfrac{1}{2}Mv^2\xi. \tag{355b}$$

Here we introduce

$$P'=\hbar^2/MR^2=PA^{-\frac{2}{3}}, \tag{356a}$$

where P is (cf. (303d)) about 10 MV and A the atomic weight of the compound nucleus. Then we have

$$\Gamma^{C_r}{}_{AkPn}/D_r'=(g_{Ak}g_{Pn}/\pi)(E/P')\xi. \tag{356}$$

The only unknown quantity in (356) is the "sticking probability." Bohr (B33) has suggested that ξ should be almost unity, which would mean that practically every collision between a particle P and a nucleus A would lead to an amalgamation of the two nuclei, and none of the collisions to an elastic reflection. In this case (356) gives a partial

width of the compound levels of the same order as their spacing provided that the kinetic energy E is of order P', i.e., about 1 MV. The total width of the compound levels would be very much larger because in each case a great number of states k are possible for the "final nucleus" A.

The assumption $\xi \approx 1$ is very tempting because of the very large interaction between nuclear particles. Moreover, similar results have been obtained for the probability of condensation of molecules on liquids or solids of the same substance by Polanyi and Wigner (P6a). However, these calculations were carried out assuming equipartition of the energy, i.e., at temperatures high compared to the Debye temperature of the liquid or solid. In the nuclear problem, we are dealing with "low" temperatures. In this case, the transfer of energy from the incident particle (molecule) to the vibrations of the nucleus (crystal lattice) might be impeded. As pointed out by Polanyi and Wigner, this should give rise to smaller condensation probabilities. However, no calculations on this problem have been carried out.

Experimentally, the "sticking probability" ξ seems indeed to be considerably less than unity. From the experiments on the capture of slow neutrons, we can find the partial widths of a fairly large number of compound levels in the low energy region. We may then assume that the width increases as the square root of the neutron energy (§52) as long as the neutron wave-length remains smaller than the nuclear radius, or, in other words, as long as E is smaller than P' (cf. (356a)). Then we find for the neutron width corresponding to the kinetic energy P'

$$\Gamma(P') = \Gamma(E)(P'/E)^{\frac{1}{2}} = \Gamma' P'^{\frac{1}{2}}. \tag{357}$$

$\Gamma' = \Gamma E^{-\frac{1}{2}}$ is given in Table XXVI for various levels of various elements; it is about 0.2–2 millivolts if E is measured in volts. P' is about $\frac{1}{4}$ MV for $A = 100$ (cf. (356a)). Therefore $\Gamma(P') = 0.1$ to 1 volt. On the other hand, the spacing of nuclear levels near the neutron dissociation energy is perhaps 5–20 volts for atomic weights of 100 to 200, both experimentally (§60) and theoretically (§53C). Thus

$$\Gamma(P')/D' \approx 0.005 - 0.2. \tag{357a}$$

Taking 0.03 as an average, and putting $g_{Ak} = g_{Pn} = 1$ (see below), this gives

$$\xi \approx 0.1, \tag{357b}$$

i.e., a "sticking probability" of about ten percent. It seems that in some cases the "sticking probability" of slow neutrons approaches unity while in others it is as low as one percent.

Since ξ is expected to increase with the energy, and is not too far (about 1/10) from its high energy value (unity) even at moderate energy, we may *expect this quantity to change slowly compared to others such as the density of levels.* This seems to be the result of the "evaporation model" which is most valuable in predicting the probability of nuclear disintegrations and the energy distribution of the emitted particles. Quantitatively, we may expect ξ to be proportional to a low power of the temperature, perhaps directly proportional to the total excitation energy; but in the absence of a direct calculation this dependence is merely a guess, and it is also not very important for applications.

A word may be said about the statistical weight factors in (356). Since accidental degeneracies are highly improbable, these weights are given by the spins of the nuclei, i.e.,

$$g_{Ak} = 2i+1, \quad g_{Pn} = 2s+1. \tag{358}$$

We shall now consider particularly the compound levels r with given angular momentum J. The spacing D_r' was defined not taking into account the degeneracy, and is therefore $2J+1$ times smaller than the true spacing D_{rJ}. Therefore

$$\Gamma^{crJ}_{AkPn}/D_{rJ} = \frac{(2i+1)(2s+1)}{2J+1} \cdot \frac{1}{\pi} \cdot \frac{E}{P'} \xi_J, \tag{358a}$$

where ξ_J is the probability that the particles A and P, when colliding, stick together and form a compound nucleus of angular momentum J. If the orbital momentum is zero or small (slow particles), J may have any value between $|i-s|$ and $i+s$. If the probability ξ_J is proportional to the statistical weight, then

$$\xi_J = (2J+1)\xi/(2i+1)(2s+1), \tag{358b}$$

where ξ is the total "sticking probability." Then (358a) reduces to

$$\Gamma^{crJ}_{AkPn}/D_{rJ} = \xi(E/\pi P'). \tag{359}$$

This equation was used for computing ξ in (357b). The computation becomes more complicated when the orbital momentum l of the particle P becomes comparable to the spins s and i (medium fast particles). For very fast particles, it is again easy to compute Γ. Then J can take all values up to

$$l_0 = R/\lambda\!\!\!^{-} = MvR/\hbar \qquad (360a)$$

and approximately

$$\xi_J = \xi(2J+1)/l_0^2 = (2J+1)\xi P'/2E, \qquad (359b)$$

$$\Gamma^{C_rJ}{}_{AkP_n}/D_{rJ} = \xi(2i+1)(2s+1)/2\pi. \qquad (360)$$

The relations given in this section are, of course, only true statistically, i.e., in the average over a sufficiently great number of levels of compound (or final) nucleus. For individual levels, large fluctuations of the probability ξ must be expected. Moreover, all the formulae derived are only valid if the wave-length of the emitted particle is smaller than the nuclear radius R, in other words if the kinetic energy E is greater than the characteristic energy P'. For smaller energies, quantum effects become important; the partial width is then proportional to the square root of the energy (cf. (266a)) and may, according to (359), be written

$$\Gamma^{C_rJ}{}_{AkP_n}/D_{rJ} = (\xi/\pi)(E/P')^{\frac{1}{2}}, \qquad E \ll P',\ J < i+s. \qquad (361)$$

This formula is again only valid in the average over many levels.

E. The energy distribution of the particles produced in nuclear reactions

When the energy levels of the compound nucleus are sufficiently dense (§56), the number of nuclear processes leading to a final nucleus A in state k is simply proportional to the average partial width of the compound levels corresponding to level k of the final nucleus. According to (356), this width is proportional to

$$\Gamma \sim (2i+1)E\xi_k, \qquad (362)$$

where i is the nuclear spin in state k, E the kinetic energy of the particles P emitted when nucleus A is left in state k, and ξ_k the "sticking probability" between particles of energy E and nuclei A in state k. In first approximation, we may consider $E\xi_k$ to be in the average independent of state k; then the probability of formation of state k is proportional to its statistical weight. The same is true of the number of particles P of energy $E = W_C - W_P - W_{Ak}$ emitted, i.e., of the number of particles forming the "group" corresponding to state k.

The assumption made ($E\xi_k$ independent of k) will be best justified for states k having not too different energy and, if possible, otherwise similar properties. A good example is the two lowest states of Li^7 which are supposed to form a "doublet" which, in spectroscopic notation, is to be designated as ${}^2P_{1/2,\ 3/2}$ (F10, R10). These two states are formed in the reaction

$$Li^6 + H^2 = Li^7 + H^1.$$

The intensities of the two proton groups are very nearly in the ratio 2 : 1 which is the ratio of the statistical weights of the two states of Li^7. The ground state of Li^7 has $i = \frac{3}{2}$; accordingly, the more energetic proton group has the higher intensity.

The dependence of ξ on E over large energy intervals is unknown, even statistically. It might be expected that ξ decreases somewhat with decreasing energy of the state k of the final nucleus (i.e., for given compound state, with *increasing* kinetic energy E of the emitted particle) because the transfer of the kinetic energy of the particle to the nuclear vibrations in the condensation process may become increasingly difficult.

In many cases, especially for heavier nuclei, we are not interested in the number of nuclear processes leading to a *definite* final state k but only in the number of processes giving particles of kinetic energy between E and $E+dE$. This probability is proportional to (362), times the number of states of the final nucleus in the energy interval dE, i.e.,

$$w_{AP}{}^C dE \sim \rho(W_A)\xi E dE. \qquad (363)$$

The most important factor here is the density of states

$$\rho(W_A) = \lambda\!\!\!^{-1} e^{S_A}, \qquad (363a)$$

where S is the entropy of nucleus A. We know that the density of states increases with the excitation energy W_A of the final nucleus; therefore the number of particles emitted with a kinetic

energy between E and $E+dE$ will increase rapidly with *decreasing* kinetic energy E of the particle. In other words, *most of the particles emitted in nuclear reactions will be comparatively slow*, leaving the residual nucleus as highly excited as possible.

The quantities ξ and λ in (363), (363a) are known to vary slowly compared to the exponential. Thus, approximately,

$$w_{AP}{}^{C} \sim E\, e^{S(W_{A0}-E)}, \tag{364}$$

where

$$W_{A0} = W_{Cr} - W_P \tag{364a}$$

is the excitation energy of the residual nucleus corresponding to zero kinetic energy of the emitted particle, and $S(W)$ is the entropy of nucleus A corresponding to the energy W. (364) is statistically correct (i.e., correct disregarding fluctuations due to the discrete quantum levels of the final nucleus) no matter what the excitation energy W_{A0}.

We now make an approximation which is only justified if the excitation energy of the residual nucleus, W_{A0}, is sufficiently high. Then we may expand the entropy in (364), *viz.*

$$\begin{aligned} S(W_{A0}-E) &= S(W_{A0}) - E(dS/dW) + \cdots \\ &= S(W_{A0}) - E/\tau + \cdots, \end{aligned} \tag{365}$$

where τ is the "temperature" of nucleus A corresponding to the excitation energy W_{A0}. Since $S(W_{A0})$ is independent of E, we may then write for the distribution function of the emitted particles

$$w_{AP}{}^{C} dE \sim e^{-E/\tau} E\, dE, \tag{366}$$

a distribution strikingly similar to the Maxwell distribution and, indeed, closely connected to it. The distribution was derived by Weisskopf (W7) and also in a simpler but somewhat less rigorous way than used here by Frenkel (F30) and Bohr (B33).

According to (366), the energy distribution of the outgoing particles should have a maximum at $E=\tau$. Since the "temperatures" of nuclei with 5 to 20 MV excitation energy are only of the order of 1 to 4 MV, the temperature τ will always be quite small compared to the available excitation energy W_{A0}. Thus in most nuclear processes, only a small fraction of the available energy will in general be taken up by the outgoing particle, while the larger part remains in the residual nucleus as excitation energy. In some cases, this excitation energy may be large enough for the emission of a second particle: Then we have a so-called three-particle reaction, i.e., a reaction leading to three final products (cf. §85). Examples are $B^{11}+H^1 = Be^{8*}+He^4 = 3\,He^4$; $B^{10}+H^2$ leading to the same products; Be^9+He^4 or $C^{12}+H^1$ or $B^{11}+H^2 \rightarrow Be^{9*}+He^4 = 3\,He^4+n^1$; $N^{14}+H^2 = 4\,He^4$, etc. In the other cases, the excitation energy of the residual nucleus will be given off as γ-radiation (Chapter XIV).

The energy distribution (366) is qualitatively confirmed in many nuclear reactions. A well-known example is the reaction

$$Be^9+He^4 = C^{12}+n^1.$$

With α-particles from radon, the available energy W_{A0} is 13 MV (cf. Table LIII). However, very few neutrons of kinetic energy 13 MV are observed (D21, and §99, Fig. 40) while most of them have energies of about 4 MV which is of the same order as the temperature of a C^{12} nucleus with an excitation energy of 13 MV (cf. Table XX). Similar features are shown by other reactions with large energy evolution, such as

$$B^{10}+H^2 = B^{11}+H^1, \quad N^{14}+H^2 = C^{12}+He^4,$$

and other reactions caused by deuterons, and also reactions caused by α-particles and giving protons such as $P^{31}+He^4 = S^{34}+H^1$ (cf. §99, 101).

According to the distribution formula (366), it is rather improbable that the emitted particles carry away the whole available energy as kinetic energy, leaving the nucleus in its ground state. Therefore it is often difficult to observe the "group" of disintegration particles corresponding to the ground state of the final nucleus (full energy group) and thus to obtain the total energy evolution in the transmutation which is important for the determination of nuclear masses. Quantitatively, the fraction of nuclei left in the ground state should be roughly equal to one over the total number of excitation levels of the residual nucleus with an excitation energy below the total available energy W_{A0}. The *relative* intensity of the full energy group of particles will therefore decrease when the available energy (i.e., also the energy of the incident particle) in-

creases. This is again in agreement with experimental results for disintegrations produced by α-particles: In the reaction $Na^{23}+He^4=Mg^{26}+H^1$, the fastest proton group could be observed by König (K21) with the relatively slow α-particles from polonium (energy 5.3 MV) but not by May and Vadyanathan (M11) with Ra C' alphas (7.7 MV). The danger of missing the full energy group of disintegration particles will be greater for heavier nuclei which have a greater density of levels.

For charged particles, the energy distribution (366) will be modified by the penetration through the potential barrier which favors the emission of high energy particles. In general, the probability of penetration through the potential barrier will increase faster with the particle energy than the Boltzmann factor $e^{-E/\tau}$ in (366) decreases. Therefore the most probable energy of the outgoing particle should be about equal to the height of the potential barrier while the emission of particles of lower and higher energy should be less probable. Fig. 10 gives the approximate theoretical distribution function for protons when the residual nucleus is Hg^{200} and the total available energy W_{A0} is 20 MV.

The influence of the potential barrier will be considered in more detail in Chapter XIII. The total probability of nuclear processes will be discussed in §56. The γ-ray width will be calculated in Chapter XIV.

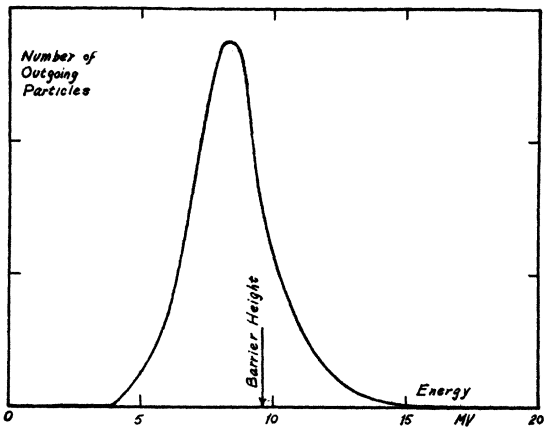

FIG. 10. Energy distribution of protons from a disintegration of Hg^{200}. Total available energy 20 MV, barrier height 9.6 MV.

§55. DERIVATION OF THE DISPERSION FORMULA (B51, B33)

In this section, we shall give the derivation of the dispersion formula (255) for nuclear processes. We do this primarily in order to show the limitations to its validity. The *stationary* method of perturbation theory will be used in our treatment. This seems slightly simpler than the nonstationary method, especially when many levels of the compound nucleus are involved. A proof using the nonstationary (Dirac) method of perturbation theory was given in the original paper of Breit and Wigner.

A. One compound state

1. Notation and fundamental equations.—For simplicity, we treat first the case of only *one* compound state of wave function χ_C, and only two kinds of particles which can be emitted, P and Q. Let χ_A and χ_B be the wave functions of the residual nuclei A and B remaining after emission of the particles P and Q, respectively. χ_A, χ_B, χ_C are supposed to be normalized. If the particles P and Q are complex rather than elementary, there will be two further wave functions χ_P and χ_Q describing the internal motion of the elementary particles inside P and Q. Furthermore, there will be two wave functions ψ_P and ψ_Q describing the motion of the centers of gravity of the particles P and Q, relative to the centers of gravity of the respective residual nuclei. These functions ψ_P and ψ_Q we shall not assume to be normalized. They will be calculated, in the course of our discussion, as to form as well as to normalization. For the moment, we mention only that ψ_P consists of an incident plane wave of given amplitude, plus an outgoing spherical wave (scattering) whose amplitude we wish to calculate. ψ_Q is solely an outgoing wave whose amplitude determines the probability of the disintegration process $A+P\rightarrow B+Q$.

The total Hamiltonian H may be split in the following two ways:

$$H=H_A+H_P+T_P+U_P+V_{AP}, \quad (367)$$

$$H=H_B+H_Q+T_Q+U_Q+V_{BQ}. \quad (367a)$$

Here H_A contains the kinetic energies of and the interactions between the particles contained in nucleus A, while H_P contains the internal energies of the particles inside the incident nucleus P (If P is an elementary particle, $H_P=0$ and $\chi_P=1$).

T_P is the kinetic energy of the motion of the center of gravity of particle P with respect to nucleus A. U_P is an "effective potential energy of the particle P" which depends only on the distance between particle P and nucleus A, and not on the internal coordinates of A and P. At large distances between A and P, this potential U_P will be the Coulomb interaction between nucleus A and particle P, if the latter is charged, while U_P will be zero, if P is neutral (neutron or light quantum). If P is inside the nucleus, a repulsive potential (positive U_P) has to be chosen according to the considerations of §54A.—The last term in (367), V_{AP}, represents the interaction between nucleus A and particle P as far as it is not contained in the effective potential U_P. V_{AP} will depend on the internal coordinates of nuclei A and P as well as on the distance between them.

The internal wave functions χ introduced above satisfy equations such as

$$H_A\chi_A = W_A\chi_A. \tag{368}$$

Similar equations hold for χ_B, χ_P, χ_Q. The wave function of the compound nucleus satisfies

$$H\chi_C = W_C\chi_C \tag{368a}$$

except at the boundary of the compound nucleus: χ_C is supposed to be restricted to a finite region of space, inside of the "nuclear radius," whereas the correct wave function has (see below) continuations extending to infinity.

We write for the wave function of the complete system

$$\Psi = \chi_A\chi_P\psi_P + \chi_B\chi_Q\psi_Q + c\chi_C \tag{369}$$

and try to satisfy the Schrödinger equation

$$H\Psi = W\Psi \tag{369a}$$

as accurately as is possible with this form of Ψ. The energy W is

$$W = W_A + W_P + E_P, \tag{369b}$$

where W_A and W_P are the internal energies of the initial nuclei A and P and E_P the kinetic energy of their relative motion. In order to satisfy (369a) as well as possible we shall fulfill the three equations

$$\int \chi_C^*(H-W)\Psi d\tau_C = 0, \tag{370}$$

$$\int \chi_A^*\chi_P^*(H-W)\Psi d\tau_A d\tau_P = 0, \tag{370a}$$

$$\int \chi_B^*\chi_Q^*(H-W)\Psi d\tau_B d\tau_Q = 0. \tag{370b}$$

The first of these is a scalar equation which will serve to determine the coefficient c of the compound wave function χ_C in (369). The two others are differential equations for the as yet undetermined wave functions $\psi_P\psi_Q$ of incident and outgoing particle. In contrast to (370), the integrations in (370a, b) do not extend over all coordinates but only over the internal ones; the result is therefore a function of the coordinate $\mathbf{r}_{AP}$ of particle P with respect to nucleus A (or of $\mathbf{r}_{BQ}$ in case of 370b).

Using the expressions (367) (367a) for H, the Schrödinger equations (368) (368a) for the χ's and the fact that the χ's are normalized, we find

$$c(W-W_C) = \int \chi_C^*(V_{AP}-L_P)\chi_A\chi_P\psi_P d\tau_C + \int \chi_C^*(V_{BQ}-L_Q)\chi_B\chi_Q\psi_Q d\tau_C, \tag{371}$$

$$L_P\psi_P = c\int \chi_A^*\chi_P^*(V_{AP}-L_P)\chi_C d\tau_A d\tau_P + \int |\chi_A|^2|\chi_P|^2 V_{AP}\psi_P d\tau_A d\tau_P + \int \chi_A^*\chi_P^*(V_{AP}-L_P)\chi_B\chi_Q\psi_Q d\tau_A d\tau_P \tag{372}$$

and a similar equation for ψ_Q. L_P is the operator

$$L_P = E_P - T_P - U_P = E_P + (\hbar^2/2M_P)\Delta_P - U_P \tag{372a}$$

with M_P the reduced mass of P and A, and Δ_P the Laplacian operator with respect to the coordinates of P relative to A.

The second term in (372) gives an additional contribution to the potential scattering. This contribution may cause more irregular variations of the potential scattering from nucleus to nucleus than those predicted by the theory developed in §54. It is obvious how to calculate the effect of this term in first approximation. However, in our considerations we shall disregard the second as well as the last integral in (372): This corresponds to the assumption that most of the nuclear processes $A+P \rightarrow B+Q$ occur by way of formation and decay of the compound state C and only a small correction is supplied by direct transitions. This is probably true in most practical cases.

2. Solution of equation for particle wave functions.—Thus (372) reduces to

$$L_P\psi_P = c\int \chi_A^*\chi_P^*(V_{AP}-L_P)\chi_C d\tau_A d\tau_P. \quad (373)$$

This represents, according to (372a), an inhomogeneous differential equation for ψ_P. Since L_P is a spherically symmetrical operator, (373) may be separated in polar coordinates by putting

$$\psi_P = \sum_{lm}\psi_{Plm}(r_P)\,Y_{lm}(\vartheta_P\varphi_P)/r_P, \quad (373a)$$

where ψ_{Plm} satisfies the differential equation

$$(\hbar^2/2M_P)d^2\psi_{Plm}/dr^2+(E_P-U_P - \hbar^2 l(l+1)/2M_P r^2)\psi_{Plm} = cC_{Plm}(r) \quad (374)$$

with

$$C_{Plm}(r_P) = \int r_P Y_{lm}^*(\vartheta_P\varphi_P)\chi_A^*\chi_P^*(V_{AP}-L_P)\chi_C \times d\tau_A d\tau_P d\omega_P. \quad (374a)$$

$d\omega_P$ is the element of solid angle in the coordinates of particle P. Thus the integration in (374a) goes over all coordinates of all particles in the system except the distance r_P between P and A.

The general solution of the ordinary inhomogeneous differential equation (374) is given by an arbitrary solution of this equation plus any multiple of the regular solution of the homogeneous equation. The right-hand side (inhomogeneity) will be appreciable only as long as $r_P < R$, i.e., as long as particle P is inside or near the nucleus A. For large r_P, the solution of (374) will therefore be identical with a certain solution of the homogeneous equation. But this will in general not be the regular solution, i.e., that solution which remains finite at the origin. Therefore we have for the asymptotic behavior of ψ_{Plm}:

$$\psi_{Plm} \to \alpha_{Plm} f_{Pl}(r_P) + \beta_{Plm} g_{Pl}(r_P), \quad (375)$$

where f and g are the regular and irregular solution of the homogeneous equation

$$(\hbar^2/2M_P)d^2f,\,g/dr^2 + (E_P - U_P - \hbar^2 l(l+1)/2M_P r^2)f,\,g = 0. \quad (375a)$$

At the origin, the regular solution behaves as

$$f \sim r^{l+1} \quad (r \text{ small}), \quad (375b)$$

the irregular one as

$$g \sim r^{-l}. \quad (375c)$$

Asymptotically for large r, the behavior of f is

$$f = \sin(kr - \tfrac{1}{2}l\pi + \delta_l), \quad r\to\infty, \quad (376)$$

where

$$k = (2M_P E_P)^{\frac{1}{2}}/\hbar = M_P v/\hbar \quad (376a)$$

is the wave number, v the velocity of particle P (relative to nucleus A) and δ_l the phase shift of the wave (compared to a free particle wave function, with $U_P = 0$ in (375a)) due to the particle potential U_P. (376) fixes the normalization of f and therefore the magnitude of α and β in (375). For the irregular function g, we may choose any solution of (375a) which is linearly independent of f; the most convenient choice is

$$g = \cos(kr - \tfrac{1}{2}l\pi + \delta_l), \quad r\to\infty, \quad (376b)$$

i.e., the function shifted in phase by $\pi/2$ compared to the regular solution. The two solutions f and g satisfy for *any* r and any potential U_P the relation

$$g\,df/dr - f\,dg/dr = \text{const} = k, \quad (377)$$

which follows easily from (375a) by multiplying the equation for f by g, that for g by f, subtracting and integrating.

Of the constants α and β in (375), α_{Plm} is arbitrary (see above) while β_{Plm} is uniquely determined by the differential equation (374) with the additional condition that ψ must be finite at the origin. A convenient way for calculating β is to use the same procedure which led to (377): Multiply Eq. (375a) for f by ψ, Eq. (374) for ψ by f, subtract and integrate up to a very large distance r. Then we obtain

$$\lim_{r\to\infty}\left(f\frac{d\psi}{dr} - \psi\frac{df}{dr}\right) = \frac{2M_P}{\hbar^2}c\int_0^\infty f_{Pl}C_{Plm}dr. \quad (378)$$

Using the asymptotic expression (375) for ψ and the relation (377) between f and g, we have

$$\beta_{Plm} = -\frac{2M_P}{\hbar^2 k}c\int_0^\infty f_{Pl}C_{Plm}dr. \quad (379)$$

With (374a) and (376a), this may be rewritten

$$\beta_{Plm} = -(2/\hbar v)c\int f_{Pl}(r_P)\,Y_{lm}^*(\vartheta_P\varphi_P)\chi_A^*\chi_P^* \times (V_{AP}-L_P)\chi_C d\tau_A d\tau_P\, d\Omega_P/r_P, \quad (379a)$$

where $d\Omega_P = r_P^2 dr_P d\omega_P$ is the volume element in the space of particle P.[15] It is convenient to introduce in (379a) the function

$$F_{Plm} = f_{Pl}(r_P)\, Y_{lm}(\vartheta_P \varphi_P)/r_P. \quad (379b)$$

According to (372a) (375a), F satisfies the differential equation

$$L_P F_{Plm} = 0. \quad (379c)$$

Since the operator L_P is self-adjoint, its contribution to (379a) obviously vanishes and we find the final result:

$$\beta_{Plm} = -(2/\hbar v) c V^C{}_{APlm}{}^*, \quad (380)$$

where the matrix element $V^C{}_{APlm}$ is defined by

$$V^C{}_{APlm} = \int \chi_C{}^* V_{AP} \chi_A \chi_P F_{Plm} d\tau_C, \quad (380a)$$

$d\tau_C$ being the total volume element

$$d\tau_C = d\tau_A d\tau_P d\Omega_P = d\tau_B d\tau_Q d\Omega_Q. \quad (380b)$$

As already mentioned, the coefficient α in (375) may have any value as far as the differential equation (374) is concerned. In order to fix α, we must consider the required asymptotic behavior of ψ_{Plm}. We know that all ψ_{Qlm}'s (Q=produced particle) and all ψ_{Plm}'s with $m \neq 0$ must behave asymptotically as outgoing waves:

$$\psi_{Plm} \sim \exp(ikr_P), \quad \psi_{Qlm} \sim \exp(ik' r_Q) \quad (381)$$

without any term behaving as e^{-ikr}. The only exceptions are the partial waves $m=0$ for the incident particle which contain a term due to the incident plane wave. If we normalize the incident plane wave per unit current, it may be written in the familiar form (B15)

$$\psi_P{}^{\text{inc}} = v^{-\frac{1}{2}} e^{ikz} = (4\pi/v)^{\frac{1}{2}} (2ikr)^{-1} \sum_l (2l+1)^{\frac{1}{2}} \times Y_{l0}(\vartheta)(e^{i(kr-\frac{1}{2}l\pi)} - e^{-i(kr-\frac{1}{2}l\pi)}). \quad (381a)$$

Therefore we have for particle P and $m=0$:

$$\psi_{Pl0} = \frac{i\pi^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}}{kv^{\frac{1}{2}}} e^{-i(kr-\frac{1}{2}l\pi)} + \gamma e^{ikr}, \quad (382)$$

where γ is an arbitrary coefficient.

We compare the required asymptotic behavior (381) (382) to the known asymptotic behavior of the solutions f and g of the homogeneous equation. (a) In all cases except for particle P and $m=0$, we must have the asymptotic behavior

$$\psi \sim g + if = \text{const} \cdot e^{ikr}. \quad (383)$$

According to (375), this means

$$\alpha_{Plm} = i\beta_{Plm} \quad (383a)$$

or

$$\psi_{Plm} = \beta_{Plm}(g + if) \quad (383b)$$

The number of outgoing particles of the kind P and quantum numbers lm is then

$$\sigma_{Plm} = v_P |\beta_{Plm}|^2. \quad (384)$$

(b) If P is the incident particle and $m=0$, we may put

$$\alpha_{Pl0} = i\beta_{Pl0} + A_l, \quad (385)$$

so that, according to (375) (376) (376b),

$$\psi_{Pl0} = (\beta_{Pl0} - \tfrac{1}{2} i A_l) e^{i(kr - \frac{1}{2}l\pi + \delta_l)} + \tfrac{1}{2} i A_l e^{-i(kr - \frac{1}{2}l\pi + \delta_l)}. \quad (385a)$$

Comparing the coefficient of e^{-ikr} with (382), we find

$$A_l = \frac{2\pi^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}}{kv^{\frac{1}{2}}} e^{i\delta_l}. \quad (386)$$

The outgoing spherical wave is obtained by subtracting the term l of (381a) from (385a):

$$\psi_{Pl0}{}^{\text{out}} = [\beta_{Pl0} e^{i\delta_l} + A_l \sin\delta_l] e^{i(kr - \frac{1}{2}l\pi)}. \quad (386a)$$

Therefore the scattering intensity (number of outgoing particles per second) is

$$\sigma_{Pl0} = |\beta_{Pl0} e^{i\delta_l} + A_l \sin\delta_l|^2 v_P. \quad (387)$$

This formula represents the interference between potential scattering (term $A_l \sin\delta_l$) and scattering by way of the compound state (term β_{Pl0}).

We may now write for any value of r:

$$\psi_{Plm}(r) = \alpha_{Plm} f_{Pl}(r) + \beta_{Plm} \gamma_{Pl}(r), \quad (388)$$

where $\beta_{Plm}\gamma_{Pl}{}^{(r)}$ is a particular solution of the inhomogeneous equation (374) which is chosen in such a way that it goes over asymptotically into βg, *without* any term containing f. It is important to note for the following that γ as well as g and f is *real*, since the differential operators in

[15] In contrast to dr_P which refers to the coordinates of the particles *inside* P.

(375a) and (374) and the asymptotic expressions for f, g and γ are real. (The coefficient β may be complex, if C is complex. However, it follows from general considerations of spatial symmetry, etc. that C must have the same complex phase for all values of r, which is sufficient to make γ real.) We define now, in analogy to (379b),

$$G_{Plm}=\gamma_{Pl}(r_P)\,Y_{lm}(\vartheta_P\varphi_P)/r_P. \qquad (388a)$$

G_{Plm} is regular everywhere. Then we may write (373a):

$$\psi_P=\sum_{lm}\alpha_{Plm}F_{Plm}+\beta_{Plm}G_{Plm}. \qquad (388b)$$

Inserting (380) (383a) (385) this becomes

$$\psi_P=\sum_l A_l F_{Pl0}-2(c/\hbar v)\sum_{lm} V^c{}_{APlm}{}^*(G_{Plm}+iF_{Plm}). \qquad (389)$$

3. Determination of the coefficient of the compound wave function.—(389) expresses the particle wave function ψ_P in terms of calculable functions and of the constants A_l and c. It remains to determine c from Eq. (371) which has not been used yet. We insert (389) and obtain

$$c(W-W_C)=\sum_{lm}(A_l\delta_{m0}-i(2c/\hbar v)\,V^c{}_{APlm}{}^*)$$
$$\times\int\chi_C{}^*(V_{AP}-L_P)\chi_A\chi_P F_{Plm}d\tau_C$$
$$-(2c/\hbar v)\sum_{lm}V^c{}_{APlm}{}^*\int\chi_C{}^*(V_{AP}-L_P)\chi_A\chi_P$$
$$\times G_{Plm}d\tau_C+\text{similar terms with } BQ. \qquad (390)$$

The first integral here is by definition (380a) just $V^c{}_{APlm}$. The second integral is not easy to evaluate, but, as we shall show, it is fortunately unimportant for the theory. This is due to the fact that the integral, if it is complex at all, has the same complex phase as $V^c{}_{APlm}$; the complex character of both quantities arises solely from the dependence of the functions $\chi_C\,\chi_A\,\chi_P\,F$ and G on the angular variables, and this dependence is the same in both cases. Therefore the second sum is real; let us put its value equal to $\frac{1}{2}\hbar v\epsilon_{CP}$. We can then combine the second term on the right of (390), and the corresponding term due to the particles Q, with the term cW_C on the left, by simply putting

$$W_C{}'=W_C-\epsilon_{CP}-\epsilon_{CQ}. \qquad (390a)$$

Then the left-hand side will become $c(W-W_C{}')$. Thus the second term in (390) causes simply a shift of the resonance energy from W_C to $W_C{}'$. Such a shift is known from the theory of dispersion of light (Dirac frequency shift). It is wholly irrelevant for our considerations because we have no way of calculating the correct position W_C of the resonance levels, and therefore no way of telling whether the resonance energy has been shifted by the small amount $\epsilon_{CP}+\epsilon_{CQ}$. We shall, in the following, drop the prime in $W_C{}'$.

Inserting $V^c{}_{APlm}$ for the first integral in (390), and taking the term arising from particle Q into account, we obtain:

$$c=\sum_l A_l V^c{}_{APl0}/(W-W_C+\tfrac{1}{2}i\gamma_C) \qquad (391)$$

with

$$\gamma_C=\frac{4}{\hbar v_P}\sum_{lm}|\,V^c{}_{APlm}|^2+\frac{4}{\hbar v_Q}\sum_{lm}|\,V^c{}_{BQlm}|^2. \qquad (391a)$$

This can immediately be reduced to the form (257b) used in §52. We need only consider the normalization of the radial function f_{Pl} which enters the matrix element V (cf. (379a)). If φ_{Pl} is the corresponding wave function normalized per unit energy, we have (cf. B15, (15))

$$\varphi=(2/\pi)^{\frac{1}{2}}(dk/dE)^{\frac{1}{2}}f=(2/\pi\hbar v)^{\frac{1}{2}}f \qquad (391b)$$

and therefore

$$\gamma_C=2\pi\sum_{P'=P,\,Q}\sum_{lm}|H^c{}_{AP'lm}|^2, \qquad (392)$$

where $H^c{}_{APlm}$ is the same matrix element as $V^c{}_{APlm}$, only taken with wave functions normalized per unit energy, *viz.*:

$$H^c{}_{APlm}=(2/\pi\hbar v)^{\frac{1}{2}}V^c{}_{APlm}. \qquad (392a)$$

Inserting (391) and (386) in (380) (384), we find for the partial cross section corresponding to the production of particle Q with orbital momenta lm:

$$\sigma^P{}_{Qlm}=v_Q|\beta_{Qlm}|^2=\frac{4v_Q}{(\hbar v_Q)^2}\frac{4\pi}{v_P k_P{}^2}$$
$$\times\left|\sum_{l'}(2l'+1)^{\frac{1}{2}}e^{i\delta_{l'}}\frac{V^c{}_{APl'0}V^c{}_{BQlm}{}^*}{W-W_C+\frac{1}{2}i\gamma_C}\right|^2 \qquad (393)$$
$$=4\pi^3\lambda_P{}^2\left|\sum_{l'}(2l'+1)^{\frac{1}{2}}e^{i\delta_{l'}}\frac{H^c{}_{APl'0}H^c{}_{BQlm}{}^*}{W-W_C+\frac{1}{2}i\gamma_C}\right|^2,$$

where $\lambda_P = 1/k_P$ is the wave-length of the incident particle divided by 2π. (393) is similar to (255). The interference terms between the various values of l' for the incident particle disappear when the cross section is summed over m (B15, appendix). If this summation is carried out, as well as the summation over all possible magnetic substates of the compound state C, the final nucleus B and the spin of the outgoing particle Q, and if the average is taken over all orientations of the spins of the initial nucleus A and the incident particle P, the one-level formula (259) is obtained for the total cross section. A much more complicated problem which has not yet been treated generally is the question of the angular distribution of the outgoing particles.

The *scattering* cross section will contain one term analogous to (393), and in addition a term representing the potential scattering and the interference between potential and resonance scattering. According to (386a), this term is for a *given angle* ϑ

$$\sigma^{P}{}_{P}{}^{\text{pot}}(\vartheta) = v_P \Big| \sum_l (A_l \sin\delta_l + \beta_{Pl0} e^{i\delta_l}) Y_{l0}(\vartheta) \Big|^2 - v_P \Big| \sum_l \beta_{Pl0} e^{i\delta_l} Y_{l0} \Big|^2. \quad (393a)$$

Integrated over all angles, this gives

$$\sigma^{P}{}_{P}{}^{\text{pot}} = v_P \sum_l \{ |A_l|^2 \sin^2\delta_l + (A_l \beta_{Pl0}^* e^{-i\delta_l} + A_l^* \beta_{Pl0} e^{i\delta_l}) \sin\delta_l \}$$
$$= 4\pi\lambda_P^2 \left\{ \sum_l (2l+1) \sin^2\delta_l - 2\pi \operatorname{Re} \frac{\sum_{l'} (2l'+1)^{\frac{1}{2}} e^{-i\delta_{l'}} H^C{}_{APl'0}^* \sum_l (2l+1)^{\frac{1}{2}} \sin\delta_l H^C{}_{APl0}}{W - W_C - \frac{1}{2} i\gamma_C} \right\}, \quad (394)$$

where Re denotes the real part. The first term in (394) represents the potential scattering in the strict sense, the second the interference between potential and resonance scattering.

B. Many compound states

We shall now discuss the more general case of many resonance levels which we distinguish by an index r. Moreover, we shall consider any number of different kinds Q of outgoing particles, and also the possibility of various states q of the final nucleus B. Then (369) is replaced by

$$\Psi = \sum_{Qq} \chi_{Bq} \chi_Q \psi_{Qq} + \sum_r c_r \chi_{Cr}, \quad (395)$$

where the sum over Q includes the incident particle P, and the sum over r goes over all states of the compound nucleus. Instead of (371), we obtain one equation for each compound state r:

$$c(W - W_r) = \sum_{Qq} \int \chi_{Cr}^* (V_{BQ} - L_Q) \chi_{Bq} \chi_Q \psi_{Qq} d\tau_C. \quad (396)$$

(373) is replaced by one equation for each possible emitted particle Q and each possible state q of the corresponding residual nucleus B, *viz.*

$$L_Q \psi_{Qq} = \sum_r c_r \int \chi_{Bq}^* \chi_Q^* (V_{BQ} - L_Q) \chi_{Cr} d\tau_B d\tau_Q. \quad (397)$$

The solution of Eqs. (397) is exactly analogous to that of (373). Defining F_{Qqlm} and G_{Qqlm} in exact analogy to (379b) (388a), we have, analogously to (388b),

$$\psi_{Qq} = \sum_{lm} \alpha_{Qqlm} F_{Qqlm} + \beta_{Qqlm} G_{Qqlm}. \quad (397a)$$

The coefficients β are obtained in analogy to (380):

$$\beta_{Qqlm} = -(2/\hbar v_{Qq}) \sum_r c_r V^{Cr}{}_{BQqlm}^*, \quad (398)$$

where v_{Qq} is the velocity of particle Q if nucleus B is left in state q. The coefficients α are determined from the asymptotic behavior of the wave functions, as in (383a) (385):

$$\alpha_{Qqlm} = i\beta_{Qqlm} + A_l \delta_{QP} \delta_{qp} \delta_{m0} \quad (398a)$$

with A_l given by (386). The cross section for the production of particle Q with orbital quantum numbers lm, and with the nucleus B being left in state q, is given by $v_{Qq} |\beta_{Qqlm}|^2$ as in (384): Only for the elastic scattering ($P=Q$, $p=q$, $m=0$) a term has to be added for the potential scattering, as in (387) (394).

The main difference compared to the one level case arises when we want to calculate the coefficients c_r from the equations (396). We insert (397a) in (396) and obtain

$$c_r(W-W_r)=$$

$$\sum_{Q\,q\,l\,m} i\beta_{Q\,q\,l\,m}\int \chi_{C_r}{}^*(V_{BQ}-L_Q)\chi_{B\,q}\chi_Q F_{Q\,q\,l\,m}d\tau_C$$

$$+\sum_{Q\,q\,l\,m}\beta_{Q\,q\,l\,m}\int \chi_{C_r}{}^*(V_{BQ}-L_Q)\chi_{B\,q}\chi_Q G_{Q\,q\,l\,m}d\tau_C$$

$$+\sum_l A_l\int \chi_{C_r}{}^*(V_{AP}-L_P)\chi_{A\,p}\chi_P F_{P\,p\,l\,0}d\tau_C. \qquad (399)$$

Here the second term is again unimportant inasmuch as it only shifts the resonance levels (cf. (390a)).[16] In the first and third terms, L_Q gives no contribution and the integrals are by definition the matrix elements (380a). Thus we have

$$c_r(W-W_r)=i\sum_{Q\,q\,l\,m}\beta_{Q\,q\,l\,m}V^{C_r}{}_{Q\,q\,l\,m}+\sum_l A_l V^{C_r}{}_{P\,p\,l\,0}. \qquad (399a)$$

Here we insert β from (398), A_l from (386) and express V by H (cf. (392a)):

$$c_r(W-W_r)=-i\sum_s C_{rs}c_s+a_r, \qquad (400)$$

where the coefficients C are given in terms of the matrix elements by

$$C_{rs}=\pi\sum_{Q\,q\,l\,m}H^{C_r}{}_{Q\,q\,l\,m}H^{C_s}{}_{Q\,q\,l\,m}{}^* \qquad (400a)$$

and a_r is a constant connected to the amplitude of the incident wave by:

$$a_r=\sqrt{2}\pi\lambda\!\!\!^{-}{}_P\hbar^{\frac{1}{2}}\sum_l(2l+1)^{\frac{1}{2}}H^{C_r}{}_{P\,p\,l\,0}e^{i\delta_l}. \qquad (400b)$$

Whereas (390) was a linear equation for the single unknown c, our result (400) represents a system of linear equations for the (infinitely many) unknowns c_r. Although it is always possible to solve this system rigorously the result will not in general be simple. However, a simple result is *always* obtainable when the spacing between adjacent energy levels W_r is large compared to the widths of the levels which are of the order of the coefficients C_{rs}. Then two cases are possible:

(a) The energy W coincides very nearly with one of the energy levels, let us say, W_n. Then c_n will be very much larger than all the other c_r's. This makes it permissible to neglect the other c_r's in the equation for c_n which gives (cf. (400))

$$c_n=a_n/(W-W_n+\tfrac{1}{2}i\gamma_n) \qquad (401)$$

with

$$\tfrac{1}{2}\gamma_n=C_{nn}. \qquad (401a)$$

The other c_r's are then unimportant for the determination of the amplitudes β of the outgoing wave and may therefore be neglected.[17]

(b) The energy W lies in between two resonance levels, and sufficiently far from each. Then all the c_r's are small, and the sum over s in (400) may be left out entirely. Then

$$c_r=a_r/(W-W_r) \qquad (402)$$

for *all* r's.

The two formulae (401) (402) may be approximated by the same formula:

$$c_r=a_r/(W-W_r+\tfrac{1}{2}i\gamma_r) \quad \text{(all } r\text{)}. \qquad (403)$$

The error in this formula is small in all cases:

1. In case (b) the energy differences $W-W_r$ are all very large compared to the widths of the levels so that the addition of $i\gamma_r$ makes no appreciable difference.

2. In case (a) for $r=n$ formula (403) is identical with the correct formula (401).

3. The terms $r\neq n$ in case (a) are negligible compared to the term n, and likewise the coefficients c_r given by (403) are negligible compared to c_n given by the same formula. (The c_r $(r\neq n)$ are, of course, *not* correctly given by (403), cf. footnote 17.)

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[16] Disregarding the first and third terms on the right-hand side of (399), and inserting β from (398), we have

$$c_r(W-W_r)=\sum_s c_s B_{rs}, \qquad (A)$$

where the B_{rs} are certain real coefficients. This system of equations is identical with that found in a simple perturbation problem. Its solution leads to a new set of eigenvalues W_r', and to a corresponding new set of wave functions $\chi_r'=\sum_s\gamma_{rs}\chi_s$ where the coefficients γ_{rs} follow from (A). The coefficients B_{rs} do not depend sensitively on the energy W of the system. Therefore the corrected resonance energies W_r' and the corrected wave functions χ_{C_r}' will be characteristics of the compound nucleus just as the old resonance energies W_r and wave functions χ_{C_r}. Therefore we may use the set of wave functions χ_{C_r}' just as well as χ_{C_r}, and thus eliminate the second term in (399) entirely.

[17] The value of c_r $(r\neq n)$ may be obtained from (400) by neglecting in the sum all terms except $s=n$. This gives, with (401),

$$c_r=\frac{a_r-iC_{rn}a_n/(W-W_n+iC_{nn})}{W-W_r},$$

which differs from the value (403) by the second term in the numerator. *The resonance for level n influences the amplitudes of all other compound states as well.* However, because of the large denominator $W-W_r$ (as compared to the denominator in (401)) c_r is irrelevant.

Inserting (403) in (398) (384), the dispersion formula (255) for the cross section is obtained. *This formula is therefore proved provided the width of the levels is small compared to their spacing.* For large width, the solution of the system (400) is much more complicated; it will be discussed for some special cases in the next section.

The case of degenerate levels offers no difficulties: Because of the selection rules for the magnetic quantum number, $H^{Cr}{}_{Qqlm}$ and $H^{Cs}{}_{Qqlm}$ cannot simultaneously be different from zero if the magnetic quantum numbers M_r and M_s of the two compound states are different. Therefore C_{rs} is zero if $M_r \neq M_s$. Similarly it can be shown that C_{rs} is only different from zero if r and s have the same *total* angular momentum J (cf. B15, appendix). Thus the system of Eqs. (400) falls into subsets each corresponding to a given J and M. Moreover, it follows from general considerations that the coefficients C_{rs} have the same value for all the magnetic substates M of the given states C_{rs}. Therefore the systems of Eqs. (400) belonging to the same J and different M's will all be equivalent.

§56. Fast Particles: Average Over the Resonances, Wide Levels

If the incident particle is fast, it will in general not be possible to define its energy accurately enough to observe resonance effects. This will be particularly true if the bombarded nucleus is heavy and has therefore closely spaced energy levels. Even the best sources (H6) of nuclear projectiles give particles whose energies fluctuate by about 1 percent, which means 10 kv at 1 MV particle energy. On the other hand (cf. §53) the spacing between neighboring energy levels is less than 10 kv probably for all nuclei containing more than about 50 particles, and it is only a few volts for heavy nuclei ($A=100$ or more, cf. §53). For fast particles and heavy nuclei, it will therefore only be possible to observe the *average* cross section of a nuclear reaction, averaged over a large number of resonance regions. It is the purpose of this section to calculate this average.

A. Width small compared to spacing

We assume first that the width of the levels of the compound states is small compared to their average spacing. Then the general formula (260) is valid and we obtain[18] by averaging over an energy interval ϵ:

$$(\sigma^{Pp}{}_{Qq})_{Av}=\frac{\pi\lambda^2}{(2s+1)(2i+1)}\frac{1}{\epsilon}\int_\epsilon dE\sum_{Jrr'}(2J+1)\times\frac{U^{rJ}{}_{Pp}U^{r'J}{}_{Pp}U^{rJ}{}_{Qq}U^{r'J}{}_{Qq}}{(E-E_{rJ}+\frac{1}{2}i\Gamma_{rJ})(E-E_{r'J}-\frac{1}{2}i\Gamma_{r'J})}. \quad (404)$$

(The u's and γ's have been replaced by the U's and Γ's because each term rJ gives an appreciable contribution only near resonance, cf. §52.) The integration is to be extended over an energy interval ϵ large compared to the spacing of the levels D but small compared to the particle energy E itself. The integration can be carried out immediately and gives

$$(\sigma^{Pp}{}_{Qq})_{Av}=\frac{\pi^2\lambda^2}{(2s+1)(2i+1)}\frac{1}{\epsilon}\sum_J(2J+1)\times\sum_{rr'\,(\text{in }\epsilon)}\frac{U^{rJ}{}_{Pp}U^{rJ}{}_{Qq}U^{r'J}{}_{Pp}U^{r'J}{}_{Qq}(\Gamma_{rJ}+\Gamma_{r'J})}{(E_{rJ}-E_{r'J})^2+\frac{1}{4}(\Gamma_{rJ}+\Gamma_{r'J})^2}. \quad (404a)$$

The sum has to be extended over all levels rr' in the energy interval ϵ. Here the mixed terms $r'\neq r$ can be neglected for two reasons. Firstly, they contain in the denominator the energy differences $E_{rJ}-E_{r'J}$ which are at least of the order of the spacing of the levels, while the terms $r'=r$ contain the width Γ_{rJ} instead; but we have assumed from the beginning that the widths are smaller than the spacing. Secondly, the numerator in the mixed terms will be positive for about as many terms as negative which makes the mixed terms cancel each other. Thus (404a) reduces to (cf. (261))

$$(\sigma^{Pp}{}_{Qq})_{Av}=\frac{2\pi^2\lambda^2}{(2s+1)(2i+1)}\frac{1}{\epsilon}\sum_J(2J+1)\times\sum_{r\,(\text{in }\epsilon)}\frac{\Gamma^{rJ}{}_{Pp}\Gamma^{rJ}{}_{Qq}}{\Gamma_{rJ}}. \quad (404b)$$

[18] We use in this section the kinetic energy $E=E_P$ of the incident particle rather than the energy of the system $W=E+W_A+W_P$, and the resonance energies E_r as defined in (258) rather than the total energy W_{Cr} of the compound nucleus in state r. Moreover, we replace the effective widths γ by the true widths Γ (width at resonance) which is always allowed when dealing with fast particles (cf. §52). The indices $ljl'j'$ are unimportant for our considerations and are therefore left out.

Since the energy interval ϵ is supposed to contain many levels, we may replace the sum over r by the number of levels of angular momentum J times the average value of each term. Now the number of levels is ϵ/D_J, where D_J is the average spacing of levels of angular momentum J in the given energy region, so that we find

$$(\sigma^{Pp}{}_{Qq})_{Av}=\frac{2\pi^2\lambda\!\!\!^{-2}}{(2s+1)(2i+1)}\sum_J(2J+1)\times\left(\frac{\Gamma^J{}_{Pp}\Gamma^J{}_{Qq}}{\Gamma_J D_J}\right)_{Av}, \quad (405)$$

where the index r has been dropped because we are no longer referring to a specific level of the compound nucleus.

B. General formula for high energies

(405) may be simplified if we admit that the average in (405) may be taken by averaging $\Gamma^J{}_{Qq}/\Gamma_J$ and $\Gamma^J{}_{Pp}/D_J$ separately. This amounts to the assumption that the partial widths $\Gamma^J{}_{Pp}$ and $\Gamma_{Qq}{}^J$ have no correlation, i.e., that the levels of the compound nucleus cannot be classified into levels which easily emit a particle P and others which preferably emit Q. This is actually the basic assumption of Bohr's model of nuclear disintegrations and is equivalent to the statement that the nuclear wave functions are very different from "Hartree" wave functions and correspond to "ideal random motion" of the nuclear particles.

The expression $(\Gamma^J{}_{Pp}/D_J)_{Av}$ which now occurs in (405) may be expressed in terms of the "sticking probability" ξ_{PpJ} (cf. (358a)), i.e., the probability that a particle P colliding with nucleus A in state p will stick and form a compound nucleus of angular momentum J. We find (cf. (358a))

$$(\sigma^{Pp}{}_{Qq})_{Av}=2\pi\lambda\!\!\!^{-2}(E/P')\sum_J\xi_{PpJ}(\Gamma^J{}_{Qq}/\Gamma_J)_{Av} \quad (406)$$

or, if we assume further that $\Gamma^J{}_{Qq}/\Gamma_J$ is independent of J, and use the definition of P' in (356a) and $\lambda\!\!\!^{-2}=\hbar^2/2ME$:

$$(\sigma^{Pp}{}_{Qq})_{Av}=\pi R^2\xi_{Pp}(\Gamma_{Qq}/\Gamma)_{Av}, \quad (407)$$

where

$$\xi_{Pp}=\sum_J\xi_{PpJ}$$

is the total sticking probability of particle P.

The very simple result (407) is actually not surprising. It can be expressed as follows: The probability of any nuclear process is equal to the probability of formation of the compound nucleus from the initial particles, times the probability of its disintegration into the particular final particles in question. The probability of formation is

$$\sigma_{Pp}=\pi R^2\xi_{Pp}, \quad (408)$$

i.e., the geometrical cross section of the nucleus times the "sticking probability." In fact, (408) may be considered as the definition of the sticking probability ξ.

The compound nucleus, once formed, must disintegrate in *some* way; therefore (408) must represent the total cross section for all nuclear processes together. The probability of a *particular* mode of disintegration is given by the partial width for that mode divided by the total width, averaged over a sufficiently large number of levels. Expressed in a formula,

$$\sigma^{Pp}{}_{Qq}/\sigma_{Pp}=(\Gamma_{Qq}/\Gamma)_{Av}. \quad (409)$$

This equation expresses the statement that the disintegration of the compound nucleus is independent of the way in which it has been formed, i.e., independent of the nature and the quantum state of the initial particles. This statement may be considered the simplest expression of the ideas of the Bohr theory of nuclear disintegrations. It is equivalent to the absence of correlations between the partial widths Γ_{Pp} and Γ_{Qq} which was used above in the proof of (407).

C. Limitations; angular momentum

The selection rules for the angular momentum may cause "correlations" between the partial widths which invalidate (407). For simplicity, we may suppose that initial nucleus, incident particle, outgoing particle and final nucleus all have zero spin. Then the total angular momentum J of the compound nucleus is equal to the orbital momentum of incident and outgoing particle. These orbital momenta must be smaller than

$$l_{Pp}=R/\lambda\!\!\!^{-}{}_{Pp} \quad \text{and} \quad l_{Qq}=R/\lambda\!\!\!^{-}{}_{Qq}, \quad (410)$$

respectively. Therefore the compound nucleus may be formed in any state with $J<l_{Pp}$, and, according to our assumptions, the probability of a given $J(<l_{Pp})$ is simply proportional to the statistical weight $2J+1$. On the other hand, a com-

pound state J can disintegrate into a particle Q and a residual nucleus in state q only if $J<l_{Qq}$. If $l_{Qq}<l_{Pp}$ the compound states with J between l_{Qq} and l_{Pp} cannot disintegrate in this way. Therefore, if Γ_{Qq} denotes the average of Γ_{QqJ} over all levels for which this quantity is not zero (i.e., with $J<l_{Qq}$), the cross section (407) will be reduced by a factor

$$\eta_{Qq}=(l_{Qq}/l_{Pp})^2=M_QE_{Qq}/M_PE_{Pp} \quad (\text{if } \eta<1). \quad (411)$$

Under these circumstances, the relative probability of two modes of disintegration will *not* be independent of the way in which the compound nucleus is formed. Compare, e.g., the disintegration products Pp and Qq. If $l_{Pp}>l_{Qq}$, the probabilities of emission of Pp as well as Qq are given directly by (407), if the incident particle is Qq. But if the incident particle is Pp, the probability of emission of Qq is reduced by the factor η which is smaller than unity. Thus the selection rule for the angular momentum introduces a certain correlation (in contrast to the fundamental assumption made in B) which favors elastic scattering as compared to other processes, i.e., favors the reemission of a particle of the same kind as the incident one.

This correlation due to the angular momentum can easily be treated in the way indicated in (411). Moreover, the influence of this correlation on the probability of reactions is comparatively small. It will be even smaller if the spins of the nuclei concerned are not zero. However, the fact remains that the existence of a constant of motion such as the angular momentum will necessarily cause a more or less serious breakdown of the assumption of "random motion" of the nuclear particles and of random distribution of the partial widths. According to all our knowledge, there is no other constant of motion besides the energy, the angular momentum and the parity (behavior of wave function with respect to a change of sign of all coordinates). If there were, we should find a more pronounced failure of the "random" assumption.

D. Total width large compared to spacing of levels, but partial width small

As we have shown in §54, the total width of highly excited nuclear states will in general be large compared to their spacing. We shall show that this does not invalidate the formulae derived in A, B provided that (1) the *partial* widths are all (or practically all) small compared to the spacing, and that (2) the partial widths of various nuclear states have no correlation. The second condition is in line with our general assumptions; the first amounts to assuming the sticking probability to be small compared to unity.

Mathematically, assumption (2) means that of the terms in the sum (400a) some will be positive and some negative if $s\neq r$, while they are, of course, all positive for $s=r$. Therefore it is reasonable to expect that the nondiagonal elements C_{rs} are in the average much smaller than the diagonal elements C_{rr}, the ratio being approximately equal to one divided by the square root of the number of terms in the sum (i.e., the number of different modes of disintegration). We shall therefore neglect the "nondiagonal" coefficients C_{rs} entirely. This can be shown to make no appreciable difference in some special cases.

With this assumption, (400) reduces to the form (390) familiar from the one-level case and has the solution

$$c_r=a_r/(E-E_r+\tfrac{1}{2}i\Gamma_r). \quad (412)$$

Then, according to (398) (392a),

$$\sigma^{Pp}{}_{Qqlm}=\frac{2\pi}{\hbar}\sum_{rs}\frac{a_ra_s{}^*H^r{}_{Qqlm}{}^*H^s{}_{Qqlm}}{(E-E_r+\frac{1}{2}i\Gamma_r)(E-E_s-\frac{1}{2}i\Gamma_s)}. \quad (413)$$

The averaging over energy gives, similarly to (404a),

$$(\sigma^{Pp}{}_{Qqlm})_{\text{Av}}=\frac{2\pi^2}{\hbar\epsilon}\sum_{rs\ (\text{in }\epsilon)}\frac{a_ra_s{}^*H^r{}_{Qqlm}{}^*H^s{}_{Qqlm}(\Gamma_r+\Gamma_s)}{(E_r-E_s)^2+\frac{1}{4}(\Gamma_r+\Gamma_s)^2}. \quad (413a)$$

In distinction from (404a), the resonance denominators are *not* much smaller for $r\neq s$ than for $r=s$. But, as we have assumed, H^r and H^s will have opposite signs for about as many pairs rs as for which they have the same sign, while H^rH^{r*} is always positive. Consequently, we expect that the contribution of the terms $r=s$ will be approximately proportional to the number of terms r, i.e., to the number of levels in the energy interval ϵ, which is ϵ/D. On the other hand, the contribution of the terms $r\neq s$ will be approximately proportional to the square root of the number of pairs rs which give an appreciable contribution. Since the contributions are negligible if $E_r-E_s\gg(\Gamma_r)_{\text{Av}}$, there are, for each r, about Γ/D contributing levels s, and therefore altogether $\epsilon\Gamma/D^2$ contributing pairs rs. The contribution of these pairs is, according to the foregoing, proportional to $(\epsilon\Gamma)^{\frac{1}{2}}/D$ as compared to ϵ/D from the diagonal terms. Therefore, if we only average over an energy interval large compared to the average width of the levels, only the diagonal terms will be important and (413a) will reduce to

$$(\sigma^{Pp}{}_{Qqlm})_{\text{Av}}=(2\pi/\hbar D)\,|a_r|^2(\Gamma^r{}_{Qqlm})_{\text{Av}}/(\Gamma_r)_{\text{Av}}, \quad (413b)$$

using the relation (257b) and replacing the sum over r in (413a) by the number of terms times the average of each term. This result is the same as for total width small compared to the spacing.

E. Partial width large compared to the spacing of levels

The case of large partial width can be treated rigorously if only one sort of particles can be emitted by a given compound state. The system of linear equations (400) has the general solution

$$c_1=\frac{\begin{Vmatrix} a_1 & a_2 & a_3 & \cdots \\ -iC_{12} & -iC_{22}+E_2-E & -iC_{32} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{Vmatrix}}{\| -iC_{sr}+\delta_{rs}(E_r-E) \|}=\frac{M_1}{N} \tag{414}$$

and similar expressions for the other c_r. If only one kind of particles P may be emitted and only one value of the angular momentum l is possible ($m=0$), then (400a) reduces to a single term, *viz.*

$$C_{rs}=\pi H_r H_s^*, \tag{414a}$$

where H_r is an abbreviation for $H^{c_r}{}_{Ppl0}$. Similarly, according to (400b):

$$a_r=AH_r, \tag{414b}$$

where A is a constant.

To evaluate the determinant N in the denominator of (414), we take out of the rth row the common factor $-i\pi H_r^*$, and out of the rth column the factor H_r. Then we have

$$N=(-i\pi|H_1|^2)(-i\pi|H_2|^2)\cdots\begin{Vmatrix} 1+i(E_1-E)/\pi|H_1|^2 & 1 & 1 & \cdots \\ 1 & 1+i(E_2-E)/\pi|H_2|^2 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{Vmatrix}. \tag{415}$$

We subtract the first row from each of the other rows which does not change the value of the determinant; then we have

$$N=\prod_r(-i\pi|H_r|^2)\begin{Vmatrix} 1+i(E_1-E)/\pi|H_1|^2 & 1 & 1 & \cdots \\ -i(E_1-E)/\pi|H_1|^2 & i(E_2-E)/\pi|H_2|^2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{Vmatrix}. \tag{415a}$$

The determinant contains now zeros everywhere except in the diagonal, where the general element is $i(E_r-E)/\pi|H_r|^2$, in the first row which contains unity everywhere except for the first element, and in the first column whose elements (except for the first) are all equal to $-i(E_1-E)/\pi|H_1|^2$. The determinant can now be evaluated easily, giving

$$N=\prod_r(-i\pi|H_r|^2)\prod_r[i(E_r-E)/\pi|H_r|^2]\times\left(1+i\sum_s\frac{\pi|H_s|^2}{E_s-E}\right). \tag{415b}$$

With

$$2\pi|H_r|^2=\Gamma_r \tag{416a}$$

this yields

$$N=\prod_r(E_r-E)\left(1+\tfrac{1}{2}i\sum_s\frac{\Gamma_s}{E_s-E}\right). \tag{416}$$

The determinant M in the numerator can be evaluated even more easily. We take out the same factors as before except from the first row, from which we take only the factor A. Then

$$M_1=AH_1\prod_{r=2}^{\infty}(-i\pi|H_r|^2)\times\begin{Vmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1+i(E_2-E)/\pi|H_2|^2 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{Vmatrix}. \tag{417}$$

Subtracting the first row from each of the others, we obtain a diagonal matrix with $i(E_r-E)/\pi|H_r|^2$ as the general diagonal term (except for $r=1$). Therefore

$$M_1=AH_1\prod_{r=2}^{\infty}(E_r-E). \tag{417a}$$

Dividing (417a) by (416), we find

$$c_1=\frac{AH_1}{E_1-E}\left(1+\tfrac{1}{2}i\sum_r\frac{\Gamma_r}{E_r-E}\right)^{-1}. \tag{418}$$

From (418) we obtain immediately the scatter-

ing cross section (exclusive of potential scattering), using (393) (398) (392a) (414b) (400b):

$$\sigma=4\pi\lambda\!\!\!^{-2}(2l+1)\times\left|\sum_r\frac{|H_r|^2}{E_r-E}\left(1+\tfrac{1}{2}i\sum_s\frac{\Gamma_s}{E_s-E}\right)^{-1}\right|^2. \quad (418a)$$

Using (416a) and considering that the parenthesis is independent of r:

$$\sigma=4\pi\lambda\!\!\!^{-2}(2l+1)\frac{\frac{1}{4}(\sum\Gamma_r/E_r-E)^2}{1+\frac{1}{4}(\sum\Gamma_r/E_r-E)^2}. \quad (419)$$

This shows that the cross section is always smaller than $4\pi\lambda\!\!\!^{-2}(2l+1)$ which represents the area of the incident beam corresponding to the angular momentum l. However, if the Γ_r are large, the difference between the cross section and $4\pi\lambda\!\!\!^{-2}(2l+1)$ is, in general, very small. This means that practically every particle of the given angular momentum is scattered if the widths of the levels are large.

There are, however, certain energies at which the scattering will still be small. Between any two energy levels, there is one value of the energy for which $\Sigma\Gamma_r/(E_r-E)$ vanishes. For this sum has the value $-\infty$ if E is just larger than one of the resonance energies E_r, and it is $+\infty$ just below the next level E_{r+1}. In between, it rises gradually and must therefore be zero for some energy about midway between the levels. For this energy, the scattering will therefore vanish. Instead of resonance maxima and practically zero cross section in between, we find now "resonance minima" with practically constant cross section $4\pi\lambda\!\!\!^{-2}$ in between.

We are interested in the cross section averaged over an energy interval large compared to the spacing of the levels. Obviously, only the neighborhood of the minima of the cross section is of interest, because elsewhere the cross section is constant. Near a minimum, we may in first approximation neglect the contribution of all levels but the two neighboring ones, which we denote by 1 and 2. Then we have for the position E_0 of the minimum the relation

$$\Gamma_1/(E_0-E_1)=\Gamma_2/(E_2-E_0). \quad (419a)$$

Near the minimum we find, using (419a),

$$\sum_r\frac{\Gamma_r}{E_r-E}\approx\left(\frac{\Gamma_1}{(E_0-E_1)^2}+\frac{\Gamma_2}{(E_2-E_0)^2}\right)(E-E_0)=\frac{(\Gamma_1+\Gamma_2)^3}{D^2\Gamma_1\Gamma_2}(E-E_0), \quad (419b)$$

where $D=E_2-E_1$ is the spacing of the levels. The average of (419) over the energy interval from E_1 to E_2 is then

$$\sigma_{Av}=4\pi\lambda\!\!\!^{-2}(2l+1)\times\left(1-\frac{1}{D}\int_{E_1}^{E_2}\frac{dE}{1+\frac{1}{4}(\sum\Gamma_r/E_r-E)^2}\right) \quad (419c)$$

$$=4\pi\lambda\!\!\!^{-2}(2l+1)[1-2\pi D\Gamma_1\Gamma_2/(\Gamma_1+\Gamma_2)^3].$$

Replacing Γ_1 and Γ_2 by some average value Γ, we find finally

$$\sigma_{Av}=4\pi\lambda\!\!\!^{-2}(2l+1)(1-\tfrac{1}{4}\pi D/\Gamma). \quad (420)$$

The number of particles *not* scattered is thus proportional to D/Γ.

The theory given in this section applies to the elastic scattering of particles if no inelastic scattering and no transmutations are possible to any appreciable extent, and if furthermore scattered particle and scattering nucleus have zero spin. Then the orbital quantum numbers l and $m=0$ of the scattered particle are identical with the angular momentum quantum numbers J and $M=0$ of the compound nucleus. There will be a set of equations of the form (400) for each J, and each J will give a contribution of the size (420). The total cross section can be obtained by summing over all J up to $J=l_{Pp}$ (cf. (410)) similarly to section A of this §. The total cross section becomes then

$$\sigma_{tot}=4\pi R^2(1-\tfrac{1}{4}\pi D/\Gamma). \quad (420a)$$

The case of large partial width and many kinds of emissible particles seems rather more difficult to treat. However, the total cross section must obviously be again of the order πR^2, and the probability of emission of a given kind of particle will be related to its sticking probability in a way similar to the case of small partial width. This means that the emitted particles should have approximately the Maxwellian distribution discussed in §54D.

X. Neutrons

§57. Slow Neutron Processes. Classification and History. (F13, A7, A10, A11, D23, B12, B32, B51, etc.)

The phenomena produced by slow neutrons have been of the greatest importance for the development of the modern theory of nuclear processes. They supply the most detailed information yet available on the energy levels of heavy nuclei. Therefore in our discussion we shall treat the slow neutron processes first and later proceed to the processes produced by charged particles. In the treatment of the latter we shall make use of the results obtained from neutron experiments.

By "slow neutrons" we understand neutrons of energies up to a few thousand volts, including neutrons of "thermal" energy of the order kT. Slow neutrons are produced by surrounding a source of fast neutrons with paraffin, water or other substances containing hydrogen. The mechanism of the slowing down process and the energy distribution of neutrons in hydrogenic substances will be treated in §59. Sources of fast neutrons will be discussed in §92.

The processes which may occur when a slow neutron interacts with a nucleus are mainly of three types:

A. Elastic scattering

The largest elastic scattering cross section ($12 \cdot 10^{-24}$ cm^2) has been observed in hydrogen (cf. §14); it is used for the production of slow neutrons (§59). The elastic scattering is also the most probable process in most of the other light nuclei (except Li^6, B^{10} and possibly N^{14}, cf. *C* and *D*) and a number of medium heavy nuclei such as Fe, Ni, Cu. The cross sections for these elements vary between about $2 \cdot 10^{-24}$ and $10 \cdot 10^{-24}$ cm^2. For nuclei which capture neutrons strongly (such as Rh, Ag, Cd, Hg) the scattering cross section is known to be small compared to the capture cross section (D23) but it may well be of the same absolute order of magnitude as for the elements mentioned before. No thorough investigation of the scattering of slow neutrons as a function of their energy has yet been made. A more detailed discussion of the scattering will be given in §63.

B. Simple capture of neutron with emission of γ-rays

Processes of the type[19]

$$Z^A + n^1 = Z^{A+1} + \gamma \qquad (421)$$

are known for a great number of nuclei Z^A. In these "simple capture processes," the bombarded nucleus is transmuted into its isotope which is one mass unit heavier. This isotope may be either stable or radioactive. In the latter case, the capture process is most easily detected by the radioactivity produced. This production of artificial radioactivity was discovered by Fermi and his collaborators in 1934 (F13). Since then, 93 radioactive isotopes have been produced by simple capture of neutrons. About the production of stable isotopes by neutron capture, our information is necessarily less complete: We can only infer it from the absorption of the neutrons.

The γ-rays from capture processes have actually been found (A11) and their energy (F25, R3, K10, K11) and intensity (K9, G23) measured. They will be discussed in §90.

Not long after the discovery of the neutron-capture processes, Fermi, Amaldi, d'Agostino, Pontecorvo, Rasetti and Segrè (A7) found that the efficiency of the neutrons in producing radioactive isotopes increased greatly when the neutrons were slowed down by passing through paraffin or water. Cross sections up to $3000 \cdot 10^{-24}$ cm^2 (Cd) were measured for the capture of these slow neutrons, i.e., cross sections more than a hundred times the geometrical cross section of the capturing nuclei.

Attempts were made by several authors (A7, B12, P6) to explain these large cross sections quantum mechanically using a one-body model. The neutron was assumed to move in a certain potential produced by the capturing nucleus. This model gave a capture cross section inversely proportional to the neutron velocity and could thus quite well explain the large increase of the cross section observed when slowing down the neutrons. Moreover, it gave the absolute cross section for slow neutrons reasonably well, and it was also capable of explaining the wide fluctua-

[19] Z^A denotes a nucleus of charge Z and mass number A.

tions of the capture cross section from element to element by differences in the neutron wave functions in the respective potentials.[20]

However, a serious difficulty soon arose in the ratio of scattering to capture cross section. The one-body theory predicted that the two cross sections should, for neutrons of thermal energy, be always of the same order of magnitude so that a large capture cross section would in every case imply large scattering as well. But experiments of Dunning, Pegram, Fink and Mitchell (D23) showed that in the strongly absorbing Cd the scattering cross section was less than 1 percent of the capture cross section.

The final blow to the one-body theory was the selective absorption of neutrons found by Moon and Tillman (M26), Amaldi and Fermi (A7) and Szilard (S29). These authors found that neutrons which made one substance radioactive had very little effect in activating another substance and *vice versa*. This was quite irreconcilable with the prediction of the one-body theory that the cross section should be inversely proportional to the neutron velocity for *any* capturing nucleus.

The phenomenon of selective absorption, while discrediting the one-body theory, led Bohr (B32) and Breit and Wigner (B51) to the correct theory of neutron capture based on the many-body concept of nuclear processes. The selective absorption of given "groups" of neutrons by given nuclei must be interpreted as a resonance effect of the neutrons with a virtual energy level of the "compound nucleus" formed by the temporary addition of neutron and capturing nucleus. Whenever the neutron has an energy coinciding with one of these resonance levels, there will be a large probability of neutron capture. As the resonance levels will be different from nucleus to nucleus, each nucleus will in general have its own characteristic group or groups of neutrons which are easily captured.

The Breit-Wigner theory of neutron capture will be discussed in the next section (§58), its application to the experiments in §§60–62. In these latter sections, we shall discuss the experimental results about the position and spacing of the resonance levels (§60, cf. the theoretical discussions in §53), the width of the levels (§§61, 62, cf. §54) and the absolute magnitude of the cross section (§62).

[20] There was, however, a difficulty about the dependence of the cross section on atomic number (V5).

C. Production of α-particles

The slow neutron is captured and an α-particle emitted, according to the scheme

$$Z^A + n^1 \rightarrow Z^{A+1*} \rightarrow (Z-2)^{A-3} + \mathrm{He}^4. \quad (422)$$

Z^{A+1*} denotes the compound nucleus which is temporarily formed in an excited state (asterisk!). This type of reaction is known to occur with Li^6 and B^{10}. The reactions involved are

$$\mathrm{Li}^6 + n^1 = \mathrm{He}^4 + \mathrm{H}^3, \quad (422a)$$

$$\mathrm{B}^{10} + n^1 = \mathrm{Li}^7 + \mathrm{He}^4. \quad (422b)$$

These reactions were discovered (A8, C12) by observing the heavy particles (H^3 and He^4) produced. The cross sections are very large, about $3000 \cdot 10^{-24}$ and $900 \cdot 10^{-24}$ cm² for the boron and the lithium reaction, respectively, with neutrons of thermal energy. The reactions are therefore used as sensitive methods for the detection of neutrons (of. also §60).

D. Production of protons

This type which may be schematically written

$$Z^A + n^1 = (Z-1)^A + \mathrm{H}^1 \quad (423)$$

has only been observed in a single instance, *viz.*

$$\mathrm{N}^{14} + n^1 = \mathrm{C}^{14} + \mathrm{H}^1. \quad (423a)$$

It is likely that also the process

$$\mathrm{B}^{10} + n^1 = \mathrm{Be}^{10} + \mathrm{H}^1 \quad (423b)$$

occurs; it has not yet been observed, but it is energetically possible. Other processes of this type are probably not possible energetically, which may be seen as follows: The bombarded nucleus Z^A is transmuted into an isobar with a nuclear charge smaller by one unit. Since the nucleus Z^A is known to be stable, $(Z-1)^A$ must have a larger atomic weight than Z^A. On the other hand, the neutron is heavier than the hydrogen atom only by 0.8 MV (cf. Chapter XVIII). Therefore the reaction (423) is only possible energetically if the mass difference $(Z-1)^A - Z^A$ is smaller than 0.8 MV. It is very improbable that this mass difference lies just

between 0 and +0.8 MV, as it has to in order to make the process possible. In the cases (423a) and (423b) this condition happens to be fulfilled, but these are the only cases among the lighter nuclei. For heavier nuclei, a reaction of the type (423) would be extremely improbable even if energetically possible, because the protons produced have very low energy (<0.8 MV) and therefore cannot possibly penetrate the potential barrier of a heavy nucleus with any appreciable probability.

E. Other processes

Other processes produced by slow neutrons seem impossible. This is certainly true for ordinary inelastic collisions because "slow" neutrons do not have sufficient energy to excite even the first excited level of the bombarded nucleus which is usually several 10,000 volts above the ground state (cf. §88). Similarly, it seems energetically impossible that a deuteron could be emitted upon slow neutron bombardment.

Thus we are left, besides the elastic scattering, with only three possible types of slow neutron disintegrations, *viz.* simple capture, α-emission and proton emission. Of these, the γ-emission seems relatively very improbable for light nuclei, simply because the interaction between nuclear matter and the radiation field is so much smaller than between the nuclear particles themselves. On the other hand, for heavy nuclei the emission of charged particles after slow neutron bombardment is very improbable because of the high potential barriers. Thus we have, with a given nucleus, in general only *either* simple capture *or* emission of charged particles. (This is not true for reactions produced by *fast* neutrons, §65). Moreover, only in one case (B^{10}) can α-particles *and* protons be produced from the same initial nucleus. Thus we say that *as a rule slow neutrons can cause only a single type of transmutation*, either simple capture or α-emission or proton emission, *with a given nucleus.*

§58. The Breit-Wigner Theory of Slow Neutron Processes (B51, B15)

We apply our general formula (269) to the special case that the incident particle is a slow neutron. With $s=\frac{1}{2}$, we obtain

$$\sigma^{N0}{}_{Qq}=\frac{\pi\lambda}{2(2i+1)}\sum_{Jl'j'}(2J+1)\times\left|\frac{\lambda_r{}^{\frac{1}{2}}U^{rJ}{}_{N0}U^{rJ}{}_{Qql'j'}}{E-E_r+\frac{1}{2}i\gamma_r}\right|^2. \quad (424)$$

Here N denotes the neutron, the index 0 after N indicates that the initial nucleus is in its ground state, Q is the emitted particle, q denotes the state of the final nucleus, l' and j' the orbital and total angular momentum of the emitted particle. i and J are the angular momenta of initial and compound nucleus, r labels the states of the compound nucleus. E_r is the kinetic energy of a neutron which is in exact resonance with the state r of the compound nucleus (cf. (258)). E is the kinetic energy of the incident neutron, λ and λ_r the neutron wave-lengths (divided by 2π) corresponding to E and E_r. The U's are the matrix elements corresponding to the emission of neutron or particle Q from the state r of the compound nucleus, and γ_r is the effective width of state r. It has been assumed that the emitted particle Q has high energy compared to the neutron, so that $u^{rJ}{}_{Qq}$ does not change appreciably with the neutron energy and may therefore be replaced by $U^{rJ}{}_{Qq}$ (cf. §52, (265)). For the scattering of the slow neutrons themselves, we have according to ((340) (265a))

$$\sigma^{N0}{}_{N0}=\frac{\pi}{2(2i+1)}\sum_{J}(2J+1)\times\left|2R+\sum_r\frac{\lambda_r\Gamma^r{}_N}{E-E_r+\frac{1}{2}i\gamma_r}\right|^2, \quad (425)$$

where $$\Gamma^r{}_N=(U^r{}_{N0})^2 \quad (425a)$$

is the true neutron width of the nuclear level r.

From the formulae (424) (425), the cross section could be calculated as a function of the energy if the matrix elements and the energy levels of the compound nucleus were all known. The mathematical expressions are, however, rather complicated and not easy to interpret. Fortunately, they simplify very much in the cases which are of the greatest practical importance, namely when the cross section (424) is large. This is the case

(a) if one of the resonance denominators is small, i.e., if the energy is near a resonance level.

In this case, the sum over r in (424) reduces to a single term; we obtain a one-level formula similar to (270);

(b) if $\lambda\!\!\!^{-}$ is large, i.e., if the velocity of the incident neutron is very small. In this case, the variation of the resonance factor with energy is negligible compared to the variation of $\lambda\!\!\!^{-}$. The cross section is then proportional to $\lambda\!\!\!^{-}$, i.e., inversely proportional to the neutron velocity v.

A. The resonance case

Formula (424) reduces to the Breit-Wigner one-level formula

$$\sigma^{N0}{}_{Qq}=\frac{\pi}{2}\left(1\pm\frac{1}{2i+1}\right)\frac{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_r\Gamma_N\Gamma_{Qq}}{(E-E_r)^2+\frac{1}{4}\gamma_r{}^2}. \quad (426)$$

The $\pm$ sign stands according to whether the angular momentum J of the resonance level in question is $i+\frac{1}{2}$ or $i-\frac{1}{2}$; for $i=0$, only the $+$ sign is possible and the value of the parenthesis becomes 2.

The width γ in the denominator of (426) contains a term due to the possibility of emitting a neutron and a term due to emission of other particles. The neutron width varies as the square root of the neutron energy while the other term is practically independent of the neutron energy, if all the other particles are fast compared to the neutron. Thus we have

$$\gamma=\Gamma_N(E/E_r)^{\frac{1}{2}}+\sum_{Qq}\Gamma_{Qq}. \quad (427)$$

Because of the small neutron energy E, the neutron width is usually small compared to the other contributions to the width.[21] Then γ becomes independent of the energy and equal to the true width

$$\gamma\approx\Gamma\approx\sum_{Qq}\Gamma_{Qq}. \quad (427a)$$

In general, we are interested in the *total* cross section for the production of particles of the kind Q and of the corresponding final nucleus B, rather than in the *partial* cross section referring to a given state q of the final nucleus B. Therefore we sum (426) over all possible states q and obtain

$$\sigma^N{}_Q=\sum_q\sigma^N{}_{Qq}=\frac{\pi}{2}\left(1\pm\frac{1}{2i+1}\right)\frac{\hbar^2}{2M(EE_r)^{\frac{1}{2}}}\times\frac{\Gamma_N\Gamma_Q}{(E-E_r)^2+\frac{1}{4}\Gamma^2}, \quad (428)$$

where

$$\Gamma_Q=\sum_q\Gamma_{Qq} \quad (428a)$$

is the total "particle Q width" of the nuclear level and Γ_N the total neutron width. Now we know from the general discussion in §57 that ordinarily only *one* kind of particles can be produced with appreciable probability under slow neutron bombardment, *viz.* either γ-rays or α-particles or protons. Furthermore, we know that the neutron width Γ_N is usually small compared to the contribution of the particles Q to the width (cf. (427) and the remarks after (432)). Therefore we may write in all practical cases

$$\Gamma_Q=\Gamma. \quad (429)$$

If we measure the cross section in cm^2 and all energies, including the widths, in volts, (428) may be rewritten thus:

$$\sigma_Q{}^N=(\sigma_0/(1+x^2))(E_r/E)^{\frac{1}{2}} \quad (430)$$

with

$$x=2(E-E_r)/\Gamma, \quad (430a)$$

$$\sigma_0=1.30_5\cdot10^{-18}\left(1\pm\frac{1}{2i+1}\right)\frac{1}{E_r}\frac{\Gamma_N}{\Gamma}. \quad (430b)$$

(430b) shows that *the maximum cross section*, at exact resonance, *gives* a direct measure of the *ratio of the neutron width to the total width*. This is the basis of the experimental determinations of the neutron width.

In (430) the factor E_r/E may be put equal to unity over the whole resonance region provided the width Γ of the resonance level is small compared to the resonance energy E_r, which is true for most of the neutron levels known. In this case, the "shape" of the resonance level reduces, according to (430), exactly to the familiar shape of the optical lines. The cross section is one-half of its maximum value if the energy differs from the resonance energy by one-half of the "width" Γ. The case that E_r and Γ are of the same order will be discussed below.

The *elastic scattering* (425) becomes, if only one

[21] This follows directly from the experimental ratio of scattering to capture probability of slow neutrons, see below.

level is important and the potential scattering is neglected,[22]

$$\sigma^N{}_N=\frac{\pi}{2}\left(1\pm\frac{1}{2i+1}\right)\lambda_r{}^2\frac{\Gamma_N{}^2}{(E-E_r)^2+\frac{1}{4}\Gamma^2}, \quad (431)$$

making again the assumption (427). The ratio of elastic scattering to the capture (or disintegration) cross section (428) is

$$\sigma^N{}_N/\sigma^N{}_Q=(\Gamma_N/\Gamma_Q)(E/E_r)^{\frac{1}{2}}. \quad (432)$$

Experiments by Dunning, Pegram, Fink and D. P. Mitchell (D23) have shown that the scattering of slow neutrons by Cd is very small, probably less than one percent of the capture. Qualitatively the same seems to be true for other strongly absorbing substances such as Ag, Rh etc. (M19, M20). This shows that the *neutron width is generally much smaller than the γ-ray width for medium heavy nuclei.* This justifies our neglecting Γ_N compared to Γ_Q (cf. (427)). The result is furthermore of great importance in order to decide which values to deduce from the capture experiments for the neutron width Γ_N and the emitted-particle width Γ_Q: The capture cross section (428) contains only the product $\Gamma_N\Gamma_Q$ and the sum $\Gamma=\Gamma_N+\Gamma_Q$ of the two widths. From measurements of the capture cross section we can therefore deduce Γ_N and Γ_Q but cannot deduce which is which. The scattering experiments mentioned show that the larger of the two quantities must be identified with Γ_Q, the smaller one with Γ_N.

B. The $1/v$ region

If the energy of the incident neutron is small compared to all the resonance energies[23] E_r, the change of the resonance denominators in (424) with neutron energy is negligible compared to the change of the factor λ. Therefore, *for sufficiently small neutron energy, there will always be a region in which the capture cross section is inversely proportional to the neutron velocity.* Just how large this region is, depends on the position of the first resonance level (of positive or negative energy) and also to some extent on its width (see below).

By virtue of the large factor λ, the cross section will in general be quite large in the $1/v$ region. (Exceptions may occur if there is destructive interference between the contributions of various nuclear levels r for small neutron energies E.) The cross section of a given nucleus for slow neutrons will therefore, besides the resonance maxima, have one maximum at zero energy. While the resonance maxima lie, in general, at different neutron energies for different nuclei, the maximum at low energies is common for all nuclei affected by slow neutrons. In order to separate the characteristic resonance effects from the effect of the very slow neutrons, it is customary to absorb the latter by a suitable absorber. Such an absorber is Cd; it absorbs strongly all neutrons up to about 0.4 volt energy (cf. §61), a thickness of 0.3 mm being sufficient to reduce the intensity of very slow neutrons to less than one percent. The very slow neutrons strongly absorbed in Cd are known in the literature as the "C group" (C = cadmium). The activity produced in most detectors, under conditions specified in the next section, is in the average due in about equal parts to the C neutrons and to the neutrons in the resonance groups.

A closer examination of the validity of the $1/v$ law is possible if we accept the one-level formula (428) as valid in the $1/v$ region. This is in general *not* justified from experiment, as will be shown in §61. Only if the first resonance level lies very near zero[24] and if, in addition, this resonance level is "strong," i.e., has large widths Γ_N and Γ_Q for the neutrons as well as for the produced particles, may we expect the one-level formula to hold. This seems to be the case for Rh and Cd, but, e.g., not for Ag and I. However, the conditions for the validity of the $1/v$ law will not be very different whether the one-level or the many-level formula is to be used.

The one-level formula (428) may be written, after summing over q and putting $\gamma=\Gamma$ (cf. (427a))

$$\sigma^N{}_Q\sim\frac{E^{-\frac{1}{2}}}{(E-E_r)^2+\frac{1}{4}\Gamma^2}. \quad (433)$$

[22] The role of the potential scattering will be treated in §63.

[23] If the resonance energy E_r is *negative*, E must be small compared to $|E_r|$.

[24] "Near" must be understood in comparison with the average spacing of the levels. However, the first level will of course lie at an energy much *higher* than the limit of validity of the $1/v$ law. Cf. remark after (434).

The condition for the $1/v$ law is that the relative variation of $E^{-\frac{1}{2}}$ with energy shall be larger than that of the denominator in (433), *viz.*:

$$|(d/dE)\ (\log E^{-\frac{1}{2}})|$$
$$\gg |(d/dE) \log [(E-E_r)^2+\tfrac{1}{4}\Gamma^2]|. \quad (433a)$$

This yields

$$4E|E_r-E| \ll (E_r-E)^2+\tfrac{1}{4}\Gamma^2. \quad (434)$$

This is certainly fulfilled if E is small compared to the larger of the two quantities[25] E_r and Γ. In words: *The $1/v$ law holds if the neutron energy is small compared to the first resonance energy, or is small compared to the width of this resonance level.*

From this rule, we must expect that the $1/v$ law holds only for a very small energy region in heavy nuclei whose levels are very dense (§53, 60) and narrow (§54, 61), but holds up to rather high energies for light nuclei, corresponding to the large spacing and width of their energy levels. In heavy nuclei, the first resonance level is often found at a neutron energy of about one volt (e.g., Rh, In, Ir, cf. §60) in which case the $1/v$ region will not extend much beyond thermal energies ($kT=1/40$ volt at room temperature). In some cases, such as Cd and Dy, the first level lies even lower (§61) so that even in the thermal region the cross section shows no proportionality with the reciprocal velocity. On the other hand, in light nuclei such as Li, B, N, the average spacing between nuclear levels may be expected to be of the order of hundred thousands of volts, and thus the first level will in general lie at an energy of this same order. But even if, by accident, the resonance level would lie very close to zero in one of these cases, its width would be very large: We know from experiments on the capture of protons by light nuclei that the width of the levels of light nuclei corresponding to the emission of particles is of the order of several ten thousands of volts if the particles are sufficiently fast (more than about 1 MV energy). This latter condition is certainly fulfilled for the slow neutron reactions in Li^6 and B^{10} in which α-particles are produced (cf. (422a) (422b), §64). For these reactions the $1/v$ law will therefore hold up to quite high neutron energies, of a few thousand volts at least. On the strength of these considerations, the absorption coefficient in boron has been used (S29, F33, W5, A7, H8, H9, G17, G18, cf. also §60) for determining the energy of the various resonance groups of neutrons activating various elements.

A very direct *experimental test of the $1/v$ law*, at least in the region of thermal energies, was made by Rasetti, Mitchell, Fink and Pegram (R5) for boron, and by Rasetti, Segrè, Fink, Dunning and Pegram (R4) for cadmium and silver. A steel or aluminum disk of 50 cm diameter was coated with the substance to be investigated and rotated at 6000 r.p.m. so that the linear velocity at the edge was about 140 meters/sec., comparable to the velocity of thermal neutrons (2200 m/sec.). A beam of slow neutrons was sent through the disk near the edge at an angle of about 65° with the normal to the wheel so that the relative velocity of the neutrons with respect to the absorbing nuclei in the wheel differed greatly according to whether the wheel was rotated in the direction or against the direction of the neutron beam. In spite of this change in relative velocity, no change in the absorption of the neutrons is expected when the $1/v$ law holds: For the time the neutrons need for traversing the wheel is independent of the velocity of the wheel, and the $1/v$ law just means that the capture probability per unit time is independent of the velocity. Any deviation from the $1/v$ law must, however, show in a different absorption of the neutrons for the two directions of rotation of the wheel.

The neutron intensity was measured by the number of disintegrations produced in Li with the help of an ionization chamber (cf. §94). No change in the absorption, outside the statistical error, was found for the B and Ag absorber, showing that they obey the $1/v$ law at least in the region of thermal energies. For Cd, however, an increase of 6.3 percent in the absorption coefficient was observed when the Cd disk was moved against the neutron beam. This corresponds under the conditions of the experiment to a cross section almost independent of the velocity (cf. §61).

C. Special cases of the one-level formula

For the applicability of the one-level formula to slow neutrons, it is only necessary that the

[25] This condition is only sufficient if $\Gamma_N \ll \Gamma_Q$. Otherwise, Γ will itself depend on the energy and a more stringent condition holds.

nearest level is very close as compared to the second nearest level, but it is not necessary that the nearest level is a real resonance level of positive energy. It is just as likely that large cross sections for slow neutrons are due to a "negative" level with a negative resonance energy E_r. In this case, of course, the substance in question would only abosorb the "C group" of neutrons and would not show any *characteristic* neutron absorption band, at least not at low energy. The cross section would be given by

$$\sigma^N{}_Q=\frac{\pi}{2}\left(1\pm\frac{1}{2i+1}\right)\frac{\hbar^2}{2M|E_r|^{\frac{1}{2}}E^{\frac{1}{2}}}\times\frac{\Gamma\Gamma_N}{(E+|E_r|)^2+\frac{1}{4}\Gamma^2}, \quad (428b)$$

it would decrease monotonically with increasing energy, first as $E^{-\frac{1}{2}}$, later as $E^{-5/2}$.

Another case in which no characteristic absorption can be observed but only absorption of very slow neutrons, is the case of a resonance level very close to zero energy whose width is comparable to the (positive) resonance energy. Fig. 11 gives the behavior of the cross section as a function of the neutron energy, for various values of the ratio $\epsilon=E_r/\Gamma$. There is a monotonic decrease of the cross section with energy for negative and very small positive ϵ. For $E_r=(5^{\frac{1}{2}}/4)\Gamma=0.56\Gamma$ the curve contains a point of inflection with horizontal tangent at $E=\frac{3}{5}E_r$. For larger ϵ, we find curves containing a minimum and a maximum at the energies

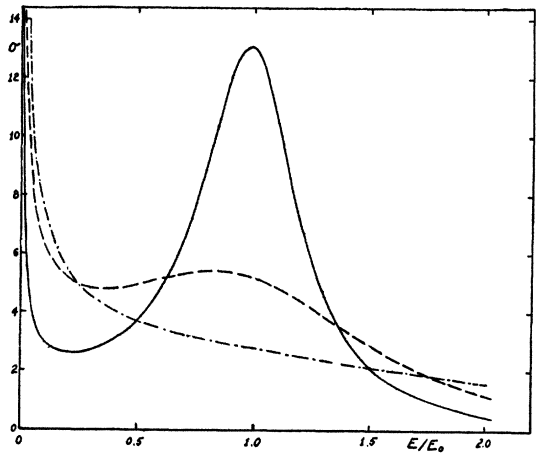

FIG. 11. The cross section for neutron capture as a function of energy for various ratios of width Γ to resonance energy E_r. ——— Width $\Gamma=\frac{1}{2}E_r$ (pronounced resonance), – – – – $\Gamma=\sqrt{2}E_r$ (almost case of Cd), –·–·– $\Gamma=4E_r$ (almost $1/v$ law). The abscissa should be denoted E/E_r rather than E/E_0.

$$E_{\min}{}^{\max}=0.6E_r\pm(0.16E_r{}^2-0.05\Gamma^2)^{\frac{1}{2}}. \quad (435)$$

With increasing ratio E_r/Γ, the two extrema become more pronounced and more separated from each other, until for $E_r\gg\Gamma$ we have (cf. (430), (430a))

$$E_{\min}=\tfrac{1}{5}E_r, \quad E_{\max}=E_r, \quad (435a)$$

$$\sigma_{\min}=(5^{5/2}/64)(\Gamma/E_r)^2\sigma_0=0.87(\Gamma/E_r)^2\sigma_0, \quad (435b)$$

where $\sigma_0=\sigma_{\max}$ is the cross section at exact resonance (430b).

§59. Diffusion of Neutrons (A11, F18, F17)

A. General

Before we can discuss the various methods of determining position, total width and neutron width of resonance levels, we must investigate the behavior of the neutrons in the hydrogen-containing substance (paraffin, water) etc. which is used to slow down the neutrons.

The questions to be answered are mainly the following:

(1) What is the energy distribution of the neutrons in a pure infinitely extended hydrogeneous substance?

(2) What is the distribution of the neutrons in space, and how does it depend on the neutron energy?

(3) How are the distributions affected if an absorbing substance is placed inside the hydrogeneous substance?

There are some further problems which are connected to those mentioned.

The approach to these problems is partly theoretical and partly experimental. We shall start from a few theoretical assumptions which seem reasonably certain, and rely on experimental determinations of the constants involved such as the mean free path of slow neutrons, their lifetime in hydrogeneous substances, etc. Since most experiments were made in paraffin, we speak of "paraffin" as representing hydrogeneous substances in general. We shall follow closely the theoretical treatment of Fermi.

The basic theoretical facts about the scattering of neutrons by protons are the following:

(*a*) In each collision with a proton the neutron will lose energy. As long as the neutron is fast compared to the proton, the probability that the neutron energy lies between E and $E+dE$ after the collision, is

$$w(E)dE=dE/E_0, \tag{436}$$

where E_0 is the neutron energy before the collision. This means that any value of the final energy of the neutron, between 0 and the initial energy E_0, is equally probable.

The proof follows immediately from the energy-momentum considerations given in §14, together with the fact that the scattering of neutrons by protons is practically spherically symmetrical[26] in a system of reference in which the center of gravity of the two particles is at rest (§15). If φ is the deflection of the neutron in the collision, the energy after collision is

$$E=E_0\cos^2\varphi \tag{436a}$$

(cf. (49d)), and the number of neutrons deflected between φ and $\varphi+d\varphi$

$$\text{const}\cdot\sin\varphi\cos\varphi d\varphi \tag{436b}$$

(cf. (72)). Combining (436a, b), we obtain (436).

From (436) it follows, that the arithmetical mean[27] of the energy of the neutron after one collision is $\frac{1}{2}E_0$, while the geometrical mean E_a is given by

$$\log(E_a/E_0)=\int_0^{E_0}\log(E/E_0)dE/E_0=-1 \tag{436c}$$

so that

$$E_a=E_0/e. \tag{437}$$

After n collisions, the geometric mean of the neutron energy will be

$$E_{an}=E_0e^{-n}. \tag{437a}$$

The actual energy distribution of the neutrons after a number of collisions with protons was calculated by Condon and Breit (C31).

(*b*) The mean free path between two collisions decreases very rapidly with decreasing neutron energy. In paraffin, it is about 5 cm for neutrons of 2 MV (theoretical value, cf. (62)) and decreases to about 1 cm (experimental value) for slow neutrons. The mean free path should be constant below, say, 10,000 volts (cf. (62)). For still slower neutrons whose energy is comparable to the vibrational energy of the hydrogen in paraffin ($\sim$0.4 volt) or smaller, the mean free path decreases further (see Sec. C) to about 0.3 cm for thermal neutrons (experimental value) and 0.2 cm for energy zero (theoretical, cf. (451), (464)).

Therefore the first part of the slowing down process will require a great amount of space, but

[26] Recently, additional experimental (H15) and some theoretical material has been advanced which is in contradiction with the spherical distribution of scattered fast neutrons. The theoretical calculations of Morse, Fisk, Schiff and Shockley (M28, M29, F22, F23) were made using a potential between neutron and proton of the form

$$V=-A(e^{-r/a}-\tfrac{1}{2}e^{-2r/a})$$

as suggested by Morse. It was found that such a potential indeed gives larger asymmetries in the scattering than a "rectangular hole" of equivalent radius. The reason is that the Morse potential (and similarly an exponential or a Gauss potential) is finite though small at fairly large distances, and therefore has a greater effect on the "p wave" ($l=1$) which causes the deviation from spherical symmetry. The actual calculations were made assuming a "range of the nuclear forces" which is about 50 percent larger than that expected from the binding energies of light nuclei; this will tend to increase the asymmetry. With this range, the pronounced asymmetry found experimentally by Kurie (K29) and by Harkins, Kamen and Newson (H15) could be duplicated theoretically if a neutron energy of 25 MV was assumed. The average energy of the Be+Rn neutrons used experimentally, is about 5 MV (§99B). Thus, even with the Morse potential which is particularly favorable for an asymmetry, only a few percent of the observed asymmetry would be expected if the correct neutron energy and range of the nuclear forces are used. In any case, an asymmetry in the scattering can certainly only exist for fairly high neutron energies and will be negligible below 1 MV, i.e., in the region in which we are primarily interested. *Added in proof:* The most recent experiments of Dee (private communication of Dr. Cockcroft) and of Kruger and Schoupp with 2.5 MV neutrons from the reaction $H^2+H^2=He^3+n^1$ give a spherical distribution of the recoil protons within the experimental error.

[27] There has been considerable controversy (A7, W15, G19, C31) about which average to use. It seems to us that the geometrical average originally suggested by Fermi (A7) is certainly preferable. This is already borne out by the fact that the arithmetical average of the logarithm of the energy (which is identical with the log of the geometric mean) decreases *linearly* with the number of collisions (cf. 437a) while the arithmetical average of the energy itself decreases exponentially, showing that $(\log E)_{Av}$ is the more suitable quantity. Moreover, the actual values of $\log E$ after n collisions form very nearly a Gaussian distribution around $(\log E)_{Av}$, whereas there is nothing as simple as a Gaussian distribution if we plot the probability against E itself. If any average other than the geometrical is taken, it seems to us that the average efficiency of the neutrons in producing radioactivity etc. should be chosen; assuming this efficiency to be roughly proportional to $1/v\propto E^{-\frac{1}{2}}$, we would have to define the "efficiency average" E^* after one collision by

$$E^{*-\frac{1}{2}}/E_0^{-\frac{1}{2}}=\int_0^{E_0}(E/E_0)^{-\frac{1}{2}}dE/E_0=2;$$

therefore $E^*=\frac{1}{4}E_0$.

when the neutrons have once been decelerated to about 100,000 volts, the remaining energy loss will occur in a relatively very small region of space. Ultimately, after 20 to 30 collisions, the neutron energy will be reduced to thermal energy. There will then, of course, be no further energy loss, but the neutrons will only diffuse in the paraffin. This will continue until they are absorbed, either by the protons in the paraffin themselves, or by some other absorber inserted in the paraffin.

The lifetime of the neutrons in pure paraffin may be calculated from the theoretical probability of the capture of neutrons by protons (cf. §17, (97a)) or from experimental data (see below, subsection E) and is found to be about 10^{-4} sec. This compares to a time of about $1.3 \cdot 10^{-6}$ sec. which is necessary to slow the neutrons down to 1 volt energy. A lifetime of 10^{-4} sec. corresponds to somewhat over 100 collisions (cf. (485)) between neutron and protons after the neutrons have reached thermal equilibrium. Therefore a large fraction of the neutrons present in the paraffin will have thermal energies, and consequently a large part of the radioactivity produced in any detector placed in the paraffin will certainly be due to thermal neutrons.

In spite of the long time during which the neutrons have thermal energies, they will not diffuse over a very large distance in that time, owing to their very small mean free path. Measurements of Fermi give in the average a total "diffusion length" of about $3\frac{1}{2}$ cm, taken from the point at which the neutron first reaches thermal energies. This may be compared with an average path from the *source* to the point of absorption, of over 18 cm. Thus we can say that the neutrons spend most of their time as thermal neutrons, but that the region in which they move as thermal neutrons is small and its position is almost exclusively determined by the path of the neutron while it is fast. The spatial distribution of fast and slow neutrons will show only little difference. This has the practical consequence that it is impossible to shield a region of space against neutrons by just absorbing the slow ones, e.g., by Cd screens: The slow neutrons will be regenerated unless the fast ones are absorbed as well which is only possible by using sufficient amounts of paraffin. About 70 cm of water (50 cm of paraffin) are necessary to reduce the number of neutrons to one percent. (The density of neutrons decreases more rapidly because of the geometrical factor $1/r^2$.)

B. Energy distribution above 1 volt

The energy distribution of neutrons has been derived by Fermi (F18) under the assumptions that the energy distribution of the neutrons after *one* collision with a proton is uniform, from zero up to the initial energy E_0 (cf. (436)). This will be true as long as the neutron energy is large compared to the vibrational energy of the protons in paraffin (cf. Section C). Therefore, the validity of the formula to be derived will be restricted to energies above about one volt. The mean free path $l(E)$ and the probability of absorption (capture) of the neutron per unit time, $1/\tau$, may in the following be arbitrary functions of the energy.

We first derive the energy distribution for the case of *negligible absorption*. This assumption is justified for pure paraffin or water, because the mean lifetime τ of a neutron with respect to capture by a proton, is extremely long compared to the total time required for the slowing down to thermal energies.

Let Q be the total number of neutrons per second emitted by the source, and $N(E)dE$ the number of neutrons of a given energy E present at any time in the whole paraffin. The decrease in the number of neutrons of energy E per second because of elastic collisions is, according to the definition of the mean free path,

$$N(E)vdE/l(E). \tag{438}$$

The number of neutrons thrown into the given energy interval dE by a collision, is

$$QdE/E_0 + \int_E^{E_0} dE' N(E')[v'/l(E')]dE/E'. \tag{438a}$$

The first term represents the neutrons coming into the energy interval dE by the first collision after being emitted from the source, dE/E_0 being the probability that the neutron energy falls into the interval dE after the collision and E_0 being the initial energy of the neutron. The second term gives the neutrons of energy E produced by later collisions: $N(E')v'dE'/l(E')$ gives the total number

of collisions of neutrons in the energy interval E' to $E'+dE'$, and dE/E' gives the probability that the energy of such a neutron is reduced from E' to E.

If the distribution $N(E)$ is to be stationary, (438) must be equal to (438a), i.e.,

$$\frac{N(E)v}{l(E)}=\frac{Q}{E_0}+\int_E^{E_0}\frac{N(E')v'}{l(E')}\frac{dE'}{E'}. \tag{439}$$

To solve this integral equation, we differentiate with respect to E:

$$\frac{d}{dE}\left(\frac{Nv}{l}\right)=-\frac{1}{E}\frac{Nv}{l} \tag{439a}$$

which integrates immediately to

$$Nv/l=\text{const}/E. \tag{439b}$$

Inserting this into (439) we determine the constant and find:

$$N(E)dE=Q\frac{l(E)}{v}\frac{dE}{E}. \tag{440}$$

In the region of slow neutrons (below 10,000 volts), $l(E)$ becomes independent of E, so that

$$N(E)\sim E^{-\frac{1}{2}}. \tag{440a}$$

In the more general case of finite absorption, we have instead of (439):

$$N(E)\left(\frac{v}{l(E)}+\frac{1}{\tau}\right)=\frac{Q}{E_0}+\int_E^{E_0}\frac{N'v'}{l'}\frac{dE'}{E'}, \tag{441}$$

where the second term on the left represents the neutrons of energy E absorbed per unit time. Differentiation gives

$$\frac{d}{dE}\left[N\left(\frac{v}{l}+\frac{1}{\tau}\right)\right]=-\frac{Nv}{lE}$$

$$=-N\left(\frac{v}{l}+\frac{1}{\tau}\right)\frac{v}{(v+l/\tau)E}. \tag{441a}$$

The integral of this is

$$N(E)=\frac{c}{v/l+1/\tau}\exp\left(\int_E^{E_0}\frac{dE}{E}\frac{v}{v+l/\tau}\right). \tag{441b}$$

Inserting this expression for $E=E_0$ into (441) fixes c:

$$c=Q/E_0. \tag{441c}$$

The distribution function (441b, c) simplifies considerably if, below a certain energy limit, l and τ are independent of the energy. This is almost certainly true in pure paraffin in the energy region from 1 to about 10,000 volts. It would also be true (with a smaller τ) if the paraffin (or water) contained a neutron absorber of small atomic weight such as Li and B. It would, however, not be true if an absorber with a large number of resonance absorption bands were contained in the paraffin—i.e., practically any substance of higher atomic weight. If we make the assumption of constant τ and l, the integral in (441b) can be evaluated and we obtain

$$N(E)=\frac{al}{(v+l/\tau)^3}. \tag{442}$$

The constant a can be determined by comparing (442) to (440) because there is a region in which l is practically independent of the energy and at the same time l/τ is negligible. Then we find

$$a=2Q/m, \tag{442a}$$

where m is the neutron mass.

C. Energies below 1 volt. Influence of chemical binding (F17)

1. General remarks.—The ordinary formulae for the scattering of neutrons by protons (§14, 15) are derived assuming that the proton is free. This assumption is justified as long as the neutron energy is very large compared to the "binding energy" of the proton which may be identified with $\hbar$ times the frequency of vibration of the hydrogen in the paraffin molecule. This frequency is about 3000 cm^{-1} for the CH bond, corresponding to about 0.4 volt. For slow neutrons, of energies less than about a volt, the binding of the protons must therefore be considered.

This binding has two effects: The first is that it is no longer possible to freely impart energy to the proton. The vibrations of the paraffin molecules may be divided into two groups. Firstly, vibrations in which a hydrogen atom moves relative to the rest of the molecule without any appreciable motion of other atoms, and secondly, vibrations of whole CH_2 groups with respect to the rest of the molecule. The transfer of energy to the first kind of vibrations will be similar to

that to free protons inasmuch as there is a considerable probability that the neutron loses practically all its energy in one collision. However, such a transfer of energy is only possible as long as the neutron energy is larger than $\hbar$ times the frequency ω (number of vibrations per 2π sec.) of the respective vibrations which is rather high. A neutron of smaller energy can transfer energy only to the vibrations of the second kind; and the effect of the collisions will then be similar to that of collisions with a free[28] CH_2 molecule of mass 14. This fact will reduce greatly the average energy loss of the neutron per collision. Thus the *neutron energy will decrease more slowly once it has reached the "region of the chemical bond."*

The second effect of the proton binding is to change the probability of the collisions. We shall show that the cross section increases so that the *mean free path decreases with decreasing energy.* In the limit of very small energies, the mean free path is theoretically one-quarter of its value for neutrons above 1 volt energy (cf. (451)). As far as the efficiency of the slowing down is concerned, this effect will work in the opposite direction of the first, i.e., it makes the neutron lose energy faster.

Another effect of the binding is a change in the angular distribution of the neutrons after scattering: This distribution will be uniform (per unit solid angle) over the whole sphere at small energies while no neutrons are deflected through more than 90° by free protons (§15).

2. Validity of the Born approximation.—If there is an elastic force (natural frequency ω) on the proton and an interaction V between neutron (coordinate ξ) and proton (coordinate x), the wave equation of the system is

$$(\hbar^2/2m)(\Delta_x+\Delta_\xi)\Psi+(W-\tfrac{1}{2}m\omega^2x^2 - V(|x-\xi|))\Psi=0. \quad (443)$$

This wave equation looks, at first sight, rather unmanageable: On one hand, it is not separable, and on the other hand, it cannot be solved by regarding the interaction V as a small perturbation because V is of the order of 10 MV when neutron and proton are close together, and the Born perturbation method (cf. M32, Chapter VII) is only applicable when the perturbation is small compared to the energy of the particles. However, we can show that V may be replaced by an auxiliary potential U which may be so chosen that it gives the same results as V for the scattering, and at the same time can be treated as a small perturbation.

That this is possible is due to the fact that V is restricted to a very small region of space ($\sim 10^{-13}$ cm) while the oscillator potential $m\omega^2x^2$, the wave functions of the proton in that potential, and the plane wave representing the neutron, change only over very large distances ($\sim 10^{-9}$ cm). Therefore we may solve (443) for small distances $|x-\xi|$ between neutron and proton without paying any attention to the oscillator potential, and join the solution to a solution of (443) which is valid at large distances and in which the nuclear potential is neglected. For this joining on, which may be effected at some medium value s of $|x-\xi|$, it is immaterial how Ψ behaves for small $|x-\xi|<s$. Therefore the solution of (443) at large distances, and consequently the scattering, will remain unaltered if V is replaced by another potential U provided only the asymptotic behavior of the wave function for distances $|\xi-x|$ large compared to the range of the nuclear forces, is the same in the two potentials U and V.

To discuss this asymptotic behavior, it is sufficient to consider the wave equation of the relative motion of neutron and proton, neglecting the oscillator potential:

$$(\hbar^2/m)\Delta_\eta\psi+(E'-V(\eta))\psi=0, \quad (444)$$

where $\mathfrak{n}=\xi-\mathfrak{x}$ and $\frac{1}{2}m$ is the reduced mass. If ψ is expanded in spherical harmonics, the radial wave functions of the partial waves $l\neq 0$ will be the same as for free particles, owing to the short range of the nuclear forces. This means spherical symmetry of the scattering (§15). For $l=0$ we have (cf. §14, Eq. (51))

$$r\psi=\sin K(r+r_0), \quad (445)$$

where $$K=(mE')^{\frac{1}{2}}/\hbar. \quad (445a)$$

(445) corresponds to a scattered wave of amplitude r_0 in any direction.[29] The total scattering cross section σ_0 is

$$\sigma_0=4\pi r_0^2. \quad (445b)$$

(In the notation of §14, $r_0=1/\beta$ if the spins of neutron and proton are opposite, and $r_0=-1/\alpha$ if they are parallel.)

We compare the solution (445) with the solution of the differential equation

$$(\hbar^2/m)\Delta\psi'+(E'-U(\eta))\psi'=0, \quad (446)$$

where the "auxiliary potential" U is defined by

$$\begin{aligned} U&=-U_0 \quad \text{for} \quad \eta<R,\\ U&=0 \quad\ \ \text{for} \quad \eta>R. \end{aligned} \quad (446a)$$

If U is chosen so that the Born method is applicable, the amplitude of the wave scattered in the direction ϑ is

[28] Provided the neutron energy is still large compared to the quantum energies of all the oscillations of the CH_2 groups with respect to each other. For still lower energies of the neutron, the effective mass of the atomic groups with which the neutron collides, would be still larger.

[29] The scattering amplitude is generally (cf. M32, p. 24)

$$f(\vartheta)=\frac{1}{2iK}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\vartheta),$$

where δ_l is the phase shift of the partial wave of angular momentum l (cf. §14). In our case, $\delta_l=0$ for $l\neq 0$ and $\delta_0=Kr_0$ (cf. (51_0), (445)), so that

$$f(\vartheta)=2iKr_0/2iK=r_0.$$

(M32, p. 88, Eq. (5) and p. 87, Eq. (1))

$$f(\vartheta) = -(m/4\pi\hbar^2)\int U(\eta)\exp(iK(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{n})d\tau, \quad (447)$$

where $\mathbf{n}_0$ and $\mathbf{n}$ are unit vectors in the direction of incident and scattered wave.

A. If we want this result to be identical with the scattering from the true nuclear potential, we have to fulfill the following conditions:

(1) $f(\vartheta)$ must be independent of ϑ (cf. above (445)). This means that the extension R of the "auxiliary potential" must be small compared to the wave-length $1/K$, i.e.,

$$KR \ll 1. \quad (447a)$$

(2) The amplitude (447) must be equal to r_0. Since the exponential reduces to unity, owing to (447a), we have with (446a)

$$f(\vartheta) = mU_0R^3/3\hbar^2 = r_0. \quad (448)$$

B. In order that the Born approximation is applicable, U_0 must be small compared to E':

$$U_0 \ll E'. \quad (448a)$$

This condition is reconcilable with (447a) and (448) because combination of (447a) and (448a) gives

$$mU_0R^3/3\hbar^2 \ll mE'/3\hbar^2K^3 = 1/3K, \quad (448b)$$

which is actually fulfilled according to (448) ($r_0 \approx 10^{-12}$ cm; $1/K$ = reciprocal neutron wave-length $\approx 10^{-9}$ cm or more). It is therefore possible to choose U_0 and R so that (447a) and (448a) are fulfilled simultaneously.

3. Calculation of cross section.—The Schrödinger Eq. (443) can now be solved by the Born method: In zero approximation, we take plane waves for the neutron, and oscillator wave functions for the proton. Let ψ_0 and ψ_n be the wave functions of the proton before and after the collision, $\mathbf{k}$ and $\mathbf{k}'$ the initial and final wave vector of the neutron. Then the differential cross section per unit solid angle $d\omega$ is, according to the Born formula

$$\sigma_n(\vartheta)d\omega = d\omega\frac{v'}{v}\left|\frac{2m}{4\pi\hbar^2}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\xi] \times U(|\mathbf{x}-\xi|)\psi_0(\mathbf{x})\psi_n^*(\mathbf{x})d\mathbf{x}d\xi\right|^2. \quad (449)$$

Now the potential U extends only over a region R small compared to the wave-length of the neutron, and small compared to the amplitude of the proton oscillator (cf. (447a)). Therefore in the exponential ξ may be replaced by $\mathbf{x}$. Then the integration over ξ can be carried out, and we obtain, using (448),

$$\sigma d\omega = (v'/v)(2r_0)^2\left|\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}] \times\psi_0(\mathbf{x})\psi_n^*(\mathbf{x})d\mathbf{x}\right|^2 d\omega. \quad (450)$$

In the limiting case of *very small neutron energies*, only elastic collisions are possible so that $v' = v$. Also, k and k' are then very small so that the differential cross section per unit solid angle is $4r_0^2d\omega$, *independent of the angle*. The total cross section is (cf. (445b))

$$\sigma = 16\pi r_0^2 = 4\sigma_0, \quad (451)$$

i.e., *four times the cross section for free protons*. This factor 4 may be directly understood from the Born approximation, once the validity of this approximation has been established. According to (449), the scattering cross section in the Born approximation is proportional to the square of the mass m, for a given potential energy. Now if the protons are tightly bound, the neutron mass has to be inserted for m, while in the case of free protons we must use the reduced mass $m/2$.

For the calculation in the general case (neutron energy comparable to $\hbar\omega$) we assume that the proton is initially in the lowest vibrational state which will practically always be true. The wave functions of a one-dimensional oscillator are:

$$\psi_n = (2\pi)^{-\frac{1}{2}}n!^{-\frac{1}{2}}e^{-\frac{1}{4}\xi^2}H_n(\xi), \quad (452)$$

where

$$\xi = x(\hbar/2m\omega)^{-\frac{1}{2}} \quad (452a)$$

and the Hermitian functions are defined by (Jahnke-Emde, *Tables of Functions*, p. 105)

$$H_n(\xi) = e^{\frac{1}{4}\xi^2}\frac{d^n}{d\alpha^n}(e^{-\frac{1}{4}(\xi-\alpha)^2})_{\alpha=0}. \quad (452a)$$

The normalization of (452) is such that $\int\psi_n^2d\xi = 1$. Our matrix element (450) then is a product of three integrals referring to the x, y, and z coordinates. The x integral is

$$A_x = (2\pi)^{-\frac{1}{2}}n!^{-\frac{1}{2}}\left(\frac{d^n}{d\alpha^n}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(\xi-\alpha)^2+iq_x\xi}d\xi\right)_{\alpha=0} \quad (453)$$

with

$$q_x = (k_x-k_x')(\hbar/2m\omega)^{\frac{1}{2}}. \quad (453a)$$

Elementary integration gives

$$A_x = (iq_x)^n n!^{-\frac{1}{2}}e^{-\frac{1}{4}q_x^2}. \quad (453b)$$

(450) reduces to

$$\sigma d\omega = 4r_0^2(v'/v)d\omega\frac{q_x^{2n_x}q_y^{2n_y}q_z^{2n_z}}{n_x!n_y!n_z!}e^{-q^2} \quad (454)$$

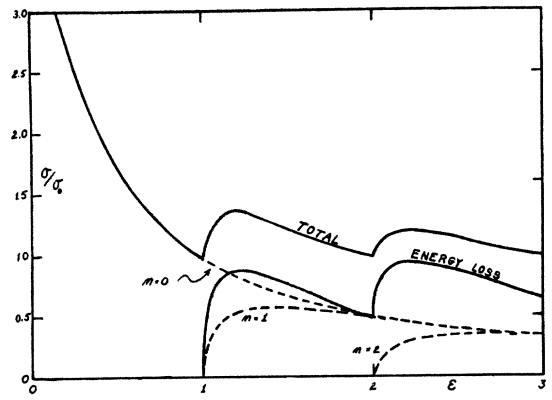

FIG. 12. Influence of the chemical binding of the protons on the slowing down of neutrons in a hydrogenous substance. Isotropic binding of hydrogen atoms assumed. Abscissa: ϵ=ratio of neutron energy to quantum energy of the vibration of the hydrogen in the molecule. Ordinate: Ratio of the cross section to that for free protons. Solid lines: total cross section and energy loss cross section. (The average energy loss per collision is one-half the neutron energy times the ratio of energy loss cross section to total cross section.) Broken lines: contributions of elastic collisions ($n=0$) and of one ($n=1$) and two ($n=2$) quantum excitations to the total cross section.

with

$$q^2=q_x^2+q_y^2+q_z^2=(\hbar/2m\omega)(\mathbf{k}-\mathbf{k}')^2 \\ =(E+E'-2(EE')^{\frac{1}{2}}\cos\vartheta)/\hbar\omega. \quad (454a)$$

ϑ is the angle of deflection of the neutron and $n_xn_yn_z$ the vibrational quantum numbers of the excited state of the proton in the three directions of space. In (454a) it has been tacitly assumed that the vibrational frequencies are the same for the three directions of space, but (454) will be generally true even for anisotropic binding of the proton (cf. below).

4. Isotropically bound protons.—Under the assumption of *isotropic binding*, (454) may be summed over all the substates $n_xn_yn_z$ of the state of total vibrational quantum number

$$n=n_x+n_y+n_z, \quad (455a)$$

the summation giving

$$\sigma_n d\omega=4r_0^2(E'/E)^{\frac{1}{2}}q^{2n}e^{-q^2}2\pi\sin\vartheta d\vartheta/n!. \quad (455)$$

With (454a), we may write

$$dq^2=2(EE')^{\frac{1}{2}}\sin\vartheta d\vartheta/\hbar\omega. \quad (455b)$$

Furthermore, we introduce the abbreviation

$$\epsilon=E/\hbar\omega; \quad \epsilon-n=E'/\hbar\omega. \quad (456)$$

Then
$$\sigma_n(\vartheta)d\omega=\frac{\sigma_0}{\epsilon}\frac{q^{2n}}{n!}e^{-q^2}dq^2. \quad (456a)$$

The total cross section for excitation of the nth vibrational state can be obtained by integration between the limits

$$q_{\min}=\epsilon^{\frac{1}{2}}-(\epsilon-n)^{\frac{1}{2}}, \quad q_{\max}=\epsilon^{\frac{1}{2}}+(\epsilon-n)^{\frac{1}{2}}. \quad (456b)$$

The total cross section for excitation of level n becomes

$$\sigma_n=(\sigma_0/\epsilon)[f_n(\{\epsilon^{\frac{1}{2}}-(\epsilon-n)^{\frac{1}{2}}\}^2) \\ -f_n(\{\epsilon^{\frac{1}{2}}+(\epsilon-n)^{\frac{1}{2}}\}^2)] \quad (457)$$

with

$$f_n(x)=e^{-x}\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right). \quad (457a)$$

For $n=0$ (elastic collisions) σ reduces to

$$\sigma_{n=0}=(\sigma_0/\epsilon)(1-e^{-4\epsilon}), \quad (458)$$

which decreases from $4\sigma_0$ at $\epsilon=0$ to $0.98\sigma_0$ at $\epsilon=1$.

For large energies ϵ and all values of n not very near ϵ, the second term in the square bracket in (457) is negligible while the first is practically unity. Then

$$\sigma_n=\sigma_0/\epsilon, \quad (\epsilon\gg1) \quad (459)$$

independent of n. The maximum value of n is equal to ϵ by definition (456). Therefore the total cross section for large ϵ is σ_0, and any energy loss between zero and the total energy $\epsilon\hbar\omega$ is equally probable, just as we found assuming free protons.

Fig. 12 gives the cross sections for elastic scattering ($n=0$) and excitation of the first and second vibrational state for neutron energies between 0 and $3\hbar\omega$. The total cross section is given, and the "cross section for energy loss"

$$\sigma_l=(2/\epsilon)\sum_n n\sigma_n. \quad (460)$$

This definition of σ_l makes it equal to σ_0 for high energies. For the energies considered, σ_l (in contrast to σ_{total}) remains *smaller* than σ_0, but down to almost $\epsilon=1$ the difference is not large. This means that the efficiency of the paraffin in slowing down the neutrons remains practically the same down to a neutron energy equal to the energy $\hbar\omega$ of the proton vibration.

5. Quantitative considerations. Energy distribution below 1 volt.—The calculations given are not yet directly applicable to our problem because the protons in paraffin are not bound isotropically. The vibration frequency in the direction of the line joining the carbon and hydrogen atom is about $\nu_1=3000$ cm^{-1}, in the directions perpendicular to it, $\nu_2=800$ to 900 cm^{-1}. According to the foregoing, *a noticeable effect on the slowing down of the neutrons will not be found until the neutron energy has decreased to about* $\hbar\omega_2=2\pi\hbar c\nu_2$. This means that the neutron distribution law (440) should hold approximately down to the energy $E=\hbar\omega_2\approx 0.1$ volt, with nearly constant $l(E)$. There will, of course, be a slight deviation from (440) in the direction of a larger number of neutrons for energies below, say, 1 volt, but, according to Fig. 12, this deviation will probably be less than a factor 2 for all energies above 1.25 $\hbar\omega_2\approx 0.13$ volt.

Only for energies smaller than $\hbar\omega_2$, the transfer of energy to the C−H vibration will no longer be possible, and any further slowing down can only occur by transferring energy to the CH_2 groups as a whole. This is equivalent to an increase of the "effective mass" of the hydrogen from $M=1$ to 14 (molecular weight of CH_2). Accordingly, the average energy loss of a neutron per collision is reduced to $2mM/(M+m)^2\approx 1/8$ times the neutron energy. Therefore, if, e.g., we let neutrons pass from a block of hot paraffin to one of cold paraffin, it will take 8 collisions rather than one to "cool" the neutrons to the new temperature (more accurately: to reduce the difference between neutron and paraffin "temperature" to $1/e$ times its initial value).

It might be thought that the distribution function of neutrons at energies below $\hbar\omega_2$ will be affected accordingly, so that the number of neutrons of energy E would be roughly 8 times the number given by (440). This is not the case, simply because energies of this order are no longer in the domain of the validity of any formula similar to (440) but are already in the domain of the Maxwell distribution. The entire energy distribution of the neutrons can be divided into two parts: The high energy region where the neutrons lose energy continuously, and the low energy region where energy gains in a collision are about as frequent as losses. In the first region formula (440) will hold, with possible corrections for the efficiency of the decelerating process. In the second region, we shall have a Maxwell distribution. The total number of neutrons in the Maxwell region will be equal to the number produced per sec. Q, times the mean life τ of a neutron. Therefore the number having an energy between E and $E+dE$ is:

$$N(E)dE=2\pi^{-\frac{1}{2}}Q\tau(kT)^{-\frac{3}{2}}E^{\frac{1}{2}}e^{-E/kT}dE. \quad (461)$$

The limit of validity between distributions (440) and (461) will be given by that energy $\epsilon_0 kT$ for which they give the same number of neutrons. We find

$$\epsilon_0^{-2}e^{\epsilon_0}=\tau(8kT/\pi m)^{\frac{1}{2}}/l(\epsilon_0 kT). \quad (461a)$$

The right-hand side is identical with the number N of collisions before absorption, introduced by Fermi (cf. (484)), except that the mean free path at energy $\epsilon_0 kT$ stands instead of that for energy kT. Assuming the ratio of the mean free paths to be 2.8 (cf. 464), the right-hand side becomes $N/2.8=53$, using Fermi's determination of N. Then (461a) has the solution

$$\epsilon_0=8, \quad (461b)$$

corresponding to about 1600 cm^{-1} at room temperature. *The Maxwell distribution thus extends up to about twice the energy of the weaker hydrogen bond, and joins directly to the distribution (440) with constant l* (of course, all these statements are approximate only).

6. Anisotropic binding of hydrogen.—We now consider the influence of the anisotropic hydrogen bond on the elastic scattering cross section at thermal energies. The differential cross section (455) becomes

$$\sigma_{el}d\omega\sin\chi d\chi=4r_0^2 2\pi\sin\vartheta d\vartheta\tfrac{1}{2}\sin\chi d\chi$$
$$\times e^{-q_x^2-q_y^2-q_z^2}=\sigma_0\sin\vartheta d\vartheta\sin\chi d\chi$$
$$\times e^{-2(1-\cos\vartheta)(\epsilon_1\cos^2\chi+\epsilon_2\sin^2\chi)}, \quad (462)$$

where ϑ is the deflection of the neutron (angle between $\mathbf{k}$ and $\mathbf{k}'$), χ the angle between the vector $\mathbf{k}-\mathbf{k}'$ and the direction of the strong bond, and

$$\epsilon_1=E/\hbar\omega_1, \quad \epsilon_2=E/\hbar\omega_2, \quad (462a)$$

so that

$$\epsilon_1\approx\tfrac{1}{4}\epsilon_2. \quad (462b)$$

Since the energy is supposed to be small com-

pared to both the bonds $\hbar\omega_1$ and $\hbar\omega_2$, we can expand the exponential in (462). Then the integration is straightforward and gives for the total cross section

$$\sigma_{el}=4\sigma_0\Big[1-2\epsilon'+\frac{32}{15}(\epsilon'\epsilon_2+\tfrac{1}{4}\epsilon_1^2)-\frac{64}{35}\Big(\epsilon'\epsilon_2^2+\tfrac{1}{4}\epsilon_1^2\epsilon_2+\frac{5}{24}\epsilon_1^3\Big)+\frac{2048}{1575}(\epsilon'\epsilon_2^3+\cdots)+\cdots\Big], \quad (463)$$

where $$\epsilon'=\tfrac{1}{3}\epsilon_1+\tfrac{2}{3}\epsilon_2. \quad (463a)$$

At room temperature, kT is about 200 cm^{-1}, so that $\epsilon_2\approx\frac{1}{4}$, $\epsilon_1\approx\frac{1}{16}$, $\epsilon'\approx\frac{3}{16}$. Then the elastic cross section becomes

$$\sigma_{el}=2.8\sigma_0. \quad (464)$$

Thus the mean free path of thermal neutrons will be about 2.8 times smaller than that of neutrons above one volt energy.[29a]

7. Conclusions. Mean free path of neutrons at and above thermal energies.—Amaldi and Fermi (A11) have measured the mean free path for thermal neutrons by determining the number of thermal neutrons which were scattered out of a neutron beam by varying thicknesses of paraffin. The number of thermal neutrons was measured by the difference in the radioactivities produced in a piece of rhodium with and without a screen of cadmium. Since Cd is known to absorb the thermal neutrons, such a measurement will indeed yield the number of thermal neutrons fairly accurately. The mean free path found in this way by Amaldi and Fermi was

$$l_{th}=0.3 \text{ cm}. \quad (465)$$

From this together with (464) we would conclude a mean free path for neutrons above one volt of about

$$l_0=0.85 \text{ cm}. \quad (465a)$$

Amaldi and Fermi also measured the mean free path of these faster neutrons directly. A detector for such neutrons, e.g., a sheet of Ag or Rh screened by Cd, was placed on top of a paraffin block. An absorber of the same material was inserted in the paraffin, at various depths x below the surface. The decrease in the activity of the detector δA was measured as a function of the depth x simultaneously with the activity A_0 of the lower surface of the absorber itself. Then $\delta A/A_0$ gives the probability that a neutron with an energy equal to the resonance energy of absorber and detector, is able to travel a distance x in paraffin without losing its property of being in resonance. Since all known resonance levels have a width small compared to the resonance energy, almost every collision will throw a neutron out of resonance. Therefore the measurements referred to will give directly the mean free path l_0 of the resonance neutrons. From both the measurements with Ag and Rh, Amaldi and Fermi deduced

$$l_0=1.1 \text{ cm}. \quad (465b)$$

It seems that the measurement of l_{th} is somewhat more accurate, and we accept therefore

$$l_0=0.9 \text{ cm} \quad (465c)$$

as the true value of the mean free path. Taking the density of paraffin as 0.90, the scattering cross section becomes

$$\sigma_0=\frac{14}{2\cdot 6.05\cdot 10^{23}\cdot 0.90\cdot 0.9}=14\cdot 10^{-24} \text{ cm}^2, \quad (466)$$

where 14 is the molecular weight of CH_2 and 2 the number of hydrogen atoms per CH_2 group. Inserting this into (62), we find for the energy of the virtual ^1S state of the deuteron

$$\epsilon'=105{,}000 \text{ volts}. \quad (467)$$

This figure replaces the figure of 40,000 volts given in (62a) which was erroneous because the effect of the chemical binding of the scattering proton had not been taken into account.

D. Spatial distribution of the neutrons in paraffin

The spatial distribution of the neutrons is a result of their multiple scattering during the process of being slowed down. Fig. 13 gives the distribution as observed by Amaldi and Fermi (A11) for two neutron energies (thermal and Ag resonance[30]). The quantity given is the number

[29a] Fermi (F17) gives 3.3 σ_0, assuming the hydrogen bond to be isotropic with $\nu=3000$ cm^{-1}.

[30] The distribution of the Rh and I resonance neutrons was also given by Amaldi and Fermi (A11, Fig. 7). These distributions are very similar to those of the Ag resonance neutrons.

of neutrons found in a spherical shell of radius r surrounding the source. The distribution curves show an increase at small r because of the increase in the area of the spherical shell, and then a decrease which is for large r practically exponential with a decay constant of 0.106 cm^{-1}. Since we know that the spatial distribution is primarily determined by the diffusion of the neutrons while fast (cf. A), the reciprocal of the decay constant, i.e., 9.4 cm, will be a measure of the mean free path for fast neutrons. The very complicated energy dependence (62) of the cross section for collisions with hydrogen atoms must be taken into account, and it must be considered that the neutrons which have suffered a collision have a smaller mean free path than those which have not. Furthermore, collisions with oxygen atoms are very important for fast neutrons, firstly because of the rapid decrease of the hydrogen cross section with increasing neutron energy, and secondly because a collision with an oxygen atom may cause a deflection of the neutron by a large angle while the deflection in collisions with hydrogen atoms can at most be 90°. The very different effect of oxygen and hydrogen collisions on the energy and the direction of the neutron makes the treatment of their combined effect rather difficult. Moreover, considerable uncertainty is introduced in the calculations because the magnitude of the cross section of oxygen for fast neutrons is known only approximately and its dependence on energy not at all. Finally, the whole problem is further complicated by the inhomogeneity of the neutrons from the source (Rn+Be).

Fermi has given a general formula for the mean square distance $(r^2)_{Av}$ of slow neutrons from the source, taking into account all the effects mentioned (F17). This rather complicated formula was applied by Horvay (H38) to the problem of neutron diffusion in water. Assuming for the cross section of oxygen the reasonable value $2\cdot10^{-24}$ cm^2, Horvay finds that the observed value of $(r^2)_{Av}$ requires an initial average energy of the neutrons of about 3 MV which seems quite reasonable in comparison with the observed distribution of neutrons from a radon-beryllium source (D21).

Bethe (unpublished) has tried to obtain approximately the shape of the distribution curve,

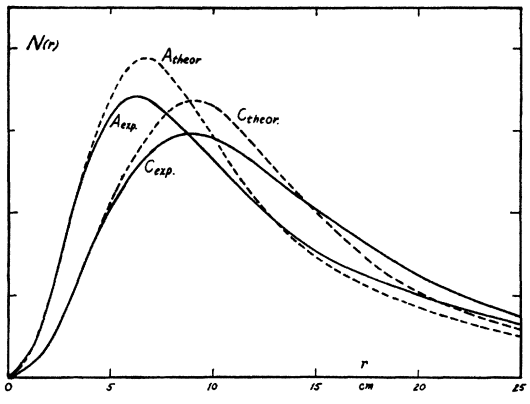

FIG. 13. Distribution of neutrons in a large volume of water, according to Amaldi and Fermi (A 11, Fig. 7). Abscissa: Distance r from source in cm. Ordinate: Number of neutrons of group A and C with distance between r and $r+dr$. Solid curves: observed. Broken curves: calculated with rough assumptions.

using very simplified assumptions about the diffusion process and determining a constant (essentially the mean distance traveled by fast neutrons) from the experimental data. The result of this admittedly rough calculation is also given in Fig. 13.

In this section, we shall limit ourselves to a calculation of the *difference* between the mean square distances of slow neutrons of different energies from the source. This difference can be used for a determination of neutron energies (A11, cf. below) and is comparatively easy to calculate because between 1 and 10,000 volts the mean free path is practically independent of the energy.

Let $\mathbf{r}_1$ be the distance traveled from the source by a neutron of final energy E_1 and $\mathbf{r}_2=\mathbf{r}_1+\mathbf{r}$ that for energy E_2. Then we want to calculate

$$(r_2^2)_{Av}-(r_1^2)_{Av}=(r^2)_{Av}+2(\mathbf{r}_1\cdot\mathbf{r})_{Av}. \qquad (468)$$

The "correlation term" $(\mathbf{r}_1\cdot\mathbf{r})_{Av}$ may be neglected[31] if a large number of collisions is necessary[32] to slow the neutron down from E_1 to E_2. We now denote the number of these collisions by N, and the distance traveled between the $n-1$st and nth collisions by ρ_n. Then

[31] It can be shown that this term is actually compensated by other neglections made in the course of the calculation.

[32] This can always be achieved by choosing E_1 large. E.g., we may choose E_1 equal to 10,000 volts throughout.

$$(r^2)_{Av}=(\sum_{n=1}^{N}\varrho_n)^2{}_{Av}=\sum_n(\rho_n{}^2)_{Av}+2\sum_{n<m}(\rho_m\rho_n\cos\vartheta_{mn})_{Av}, \quad (469)$$

where ϑ_{mn} is the angle between the vectors ϱ_m and ϱ_n. The three quantities in the last term are statistically independent of each other. Denoting the mean free path between two collisions by l, we have[33]

$$(\rho_n{}^2)_{Av}=2l^2, \quad (\rho_n)_{Av}=l, \quad (469a)$$

and therefore

$$(r^2)_{Av}=2Nl^2(1+\sum_{m-n=1}^{\infty}(\cos\vartheta_{mn})_{Av}) \quad (470)$$

$$=2Nl^2\sum_{k=0}^{\infty}(\cos\vartheta_{n,\,n+k})_{Av}. \quad (470a)$$

The sums have been extended to infinity because the terms decrease rapidly with k.

We shall first assume that only collisions with hydrogen atoms occur. Then the average number of collisions necessary is (cf. (437a))

$$N=\log(E_1/E_2). \quad (471)$$

The average cosine of the angle of deflection in a collision is (cf. (436b))

$$(\cos\vartheta_{n,\,n+1})_{Av}=\int_0^{\pi/2}\cos\vartheta\cos\vartheta\sin\vartheta d\vartheta\div\int_0^{\pi/2}\cos\vartheta\sin\vartheta d\vartheta=\tfrac{2}{3}. \quad (472)$$

If φ is the angle between the plane of ϱ_{n+k-1}, ϱ_n and the plane of ϱ_{n+k-1}, ϱ_{n+k}, we have

$$\cos\vartheta_{n,\,n+k}=\cos\vartheta_{n,\,n+k-1}\cos\vartheta_{n+k-1,\,n+k}+\sin\vartheta_{n,\,n+k-1}\sin\vartheta_{n+k-1,\,n+k}\cos\varphi. \quad (472a)$$

The second term vanishes upon averaging over φ. Therefore we have, with (472),

$$(\cos\vartheta_{n,\,n+k})_{Av}=(\cos\vartheta_{n,\,n+k-1})_{Av}(\cos\vartheta_{n+k-1,\,n+k})_{Av}=(\cos\vartheta_{n,\,n+1})_{Av}{}^k=(\tfrac{2}{3})^k. \quad (472b)$$

Therefore the sum over k in (470a) gives 3, and

$$(r^2)_{Av}=6Nl^2=6l^2\log E_1/E_2. \quad (473)$$

This is the formula used by Amaldi and Fermi.

[33] The probability of a given value of ρ_n is $e^{-\rho_n/l}d\rho_n/l$ (definition of the mean free path).

Now we take the collisions with oxygen atoms into account. Let 2α be the ratio of the cross sections of oxygen and hydrogen for slow neutrons (1 to 10,000 volts). Then the number of collisions will now be (one oxygen atom per two hydrogen atoms!)

$$N=(1+\alpha)\log(E_1/E_2), \quad (474)$$

and the mean free path

$$l=l_0/(1+\alpha), \quad (474a)$$

where l_0 is the mean free path with hydrogen only. After a collision with an oxygen atom, the direction of motion of the neutron will be practically random. Therefore the average cosine in (472) must be multiplied by the probability that the collision is with a hydrogen atom, giving

$$(\cos\vartheta_{n,\,n+1})_{Av}=2/3(1+\alpha). \quad (474b)$$

(472b) will remain unchanged except for the last expression. Therefore we obtain instead of (473)

$$(r^2)_{Av}=[6l_0{}^2/(1+3\alpha)]\log(E_1/E_2). \quad (475)$$

l_0 is known from the measurements of Amaldi and Fermi, its value is 0.9 cm in paraffin (cf. (465c)). In water, it should be larger by the ratio of the number of hydrogen atoms per cm³ i.e., by the factor $0.9\cdot18/1.0\cdot14=1.16$, giving $l_0=1.04$ cm. The cross section of oxygen for slow neutrons was measured by Dunning and others (D23 and Table XXVII) and found to be $3.3\cdot10^{-24}$ cm² so that (cf. (466))

$$\alpha=3.3/2\cdot14=0.12. \quad (475a)$$

Inserting in (475), we have

$$(r^2)_{Av}\approx4.8\log(E_1/E_2)\text{ cm}^2. \quad (476)$$

The numerical factor 4.8 differs appreciably from the value used by Amaldi and Fermi (9.7). The difference is due partly to the neglection of the rather large effect of oxygen collisions, partly to a different value used for the mean free path in paraffin (1.1 instead of 0.9 cm). The value given in (476) agrees very much better with the measurements of Amaldi and Fermi (§60) than their old value.

At energies below one volt, (476) will fail. We may divide this low energy region into two parts: The region between 1 volt and thermal energies,

and the Maxwell region. In the first region, the mean free path l, the energy loss, and the angular distribution of the neutrons will be complicated functions of the energy (cf. Section C), so that the increase in r^2 cannot be calculated in any way. In the Maxwell region we have complete diffusion and, according to (487b),

$$(r^2)_{\text{Maxwell}}=6L^2, \tag{477}$$

where L is the "diffusion length" of Maxwell neutrons. L has been measured by Amaldi and Fermi and found to be 2.1 cm in paraffin so that

$$(r^2)_{\text{Maxwell}}=26 \text{ cm}^2 \text{ paraffin}$$
$$=35 \text{ cm}^2 \text{ water.} \tag{477a}$$

On the other hand, Amaldi and Fermi have measured the difference in $(r^2)_{\text{Av}}$ between thermal neutrons and neutrons of about one volt energy (resonance group of Rh). In water, this difference is 50 cm². This would mean that $(r^2)_{\text{Av}}$ increases by $50-35=15$ cm² when the neutrons are slowed down from 1 volt to thermal energies. This figure is, of course, not very accurate.

E. Diffusion of thermal neutrons

When the neutrons have been slowed down to thermal energies, they will diffuse through the paraffin. Assuming that the mean free path is independent of the neutron velocity, and that the scattering of thermal neutrons by protons is isotropic, we have for the diffusion coefficient the well-known formula

$$D=\tfrac{1}{3}l_{\text{th}}v_a, \tag{478}$$

where

$$v_a=(8kT/\pi m)^{\frac{1}{2}} \tag{478a}$$

is the average velocity of thermal neutrons, and l the mean free path of thermal neutrons (cf. (465)).

For a stationary distribution, the diffusion equation takes the form

$$\partial F/\partial t=D\Delta F+q-F/\tau=0. \tag{479}$$

Here F is the number of neutrons per cm³. The first term on the right represents the neutrons entering a given volume element per unit time by diffusion, the last term the number disappearing from it by absorption, τ being the lifetime of a neutron with respect to capture. The second term, q, is the number of thermal neutrons produced in the volume element per second by the neutron source; in a homogeneous block of paraffin, q will be a slowly varying function of the coordinates whose variation we shall neglect. We shall treat the solution of (479) for various cases which are of practical importance:

1. Homogeneous paraffin, no absorber, homogeneous production of neutrons.—We have

$$q=\text{const.}, \ \Delta F=0$$

and therefore

$$F=q\tau. \tag{480}$$

The total number of neutrons throughout the paraffin is

$$\int F dv=Q\tau, \tag{480a}$$

where Q is the total number of neutrons emitted per second by the source.

2. Production of neutrons in a plane $x=0$.—We have $q=q'\delta(x)$ where δ is Dirac's δ function. The solution is

$$F=\tfrac{1}{2}q'(\tau/D)^{\frac{1}{2}}e^{-|x|(D\tau)^{-\frac{1}{2}}}. \tag{481}$$

It satisfies the condition

$$q=0, \text{ i.e., } d^2F/dx^2=F/D\tau \text{ for } x\neq 0. \tag{481a}$$

The current starting from $x=0$ is equal to (cf. (488a))

$$-\lim_{\epsilon=0} 2D\left(\frac{dF}{dx}\right)_{\epsilon}=q'. \tag{481b}$$

(Factor 2 for the two sides of the plane.)

According to (481), the probability that a neutron emitted at $x=0$, reaches the plane x without being absorbed, is proportional to

$$p(x)=e^{-x/L}, \tag{482}$$

where

$$L=(D\tau)^{\frac{1}{2}} \tag{482a}$$

is the "diffusion length."

L has been measured by Amaldi and Fermi in the following way: An *absorber* of thermal neutrons (Cd) is placed at a certain plane $x=0$ in paraffin. Then the neutron distribution will be given by (480) *minus* (481) where q is the number of neutrons produced by the source per cm³ and sec., and q' the number absorbed in the absorber

per cm² and sec. The activity in a detector placed at a distance x from the absorber, is then measured with and without the absorber. The difference turns out to be indeed an exponential function of x; its rate of decay measures the diffusion length L. The result of the measurements is

$$L = 2.1 \text{ cm}. \tag{482b}$$

In the actual experiment, the detector was kept fixed at the surface of a paraffin block and the absorber placed at various depths x. Since the density of neutrons near the surface of a paraffin block is a function of x, it was necessary to measure the number q' of the neutrons absorbed by the absorber itself by measuring its own activity. The decrease in activity of the detector due to the insertion of the absorber, divided by the activity of the absorber, must then be proportional to $e^{-x/L}$.

3. Lifetime of neutron in paraffin.—Using the definition (478) of D and (482a) of L, we have

$$L^2 = \tfrac{1}{3} l_{\text{th}} v_a \tau. \tag{483}$$

We define the "average number of collisions before absorption"

$$N = v_a \tau / l_{\text{th}}. \tag{484}$$

Then we have

$$L^2 = \tfrac{1}{3} l_{\text{th}}^2 N. \tag{484a}$$

With $L = 2.1$, $l_{\text{th}} = 0.30$ (cf. (465)), we find

$$N = 150. \tag{485}$$

Considering the ratio 2.8 of the mean free paths for faster neutrons and thermal neutrons (cf. (464)), we find

$$N_0 = v_a \tau / l_0 = 150/2.8 = 53. \tag{485a}$$

A theoretical calculation of τ on the lines of §17 gives, with the value $|\epsilon'| = 105$ kv (cf. (467)) for the energy of the ^1S state of the deuteron:

$N_0 = 182$ if the ^1S state of the deuteron is stable, (486)

$N_0 = 75$ if the state is virtual.

The experimental value (485a) thus decides uniquely in favor of a *virtual* ^1S state of the deuteron. However, the agreement between the experimental and theoretical value of N_0 is not very good.

4. Point source of neutrons.—We assume now $q = 0$ everywhere except at $r = 0$. We find

$$D\, d^2(rF)/dr^2 = rF/\tau, \tag{487}$$

$$rF = ce^{-r/L}, \tag{487a}$$

and therefore

$$(r^2)_{\text{Av}} = \frac{\int r^2 F r^2 dr}{\int F r^2 dr} = 6L^2 = 2Nl^2, \tag{487b}$$

a result which has been used in (477).

5. Absorption of neutrons by a thick plane absorber.—If there is an absorber at $x = 0$ which absorbs *all* neutrons striking it, the neutron density will, in first approximation, be zero at $x = 0$. If neutrons are produced at a rate q per cm³ per sec., the neutron density at a distance $+x$ from the absorber is (cf. (480), and (481) with q' chosen so as to make $F(0) = 0$)

$$F = q\tau(1 - e^{-x/L}). \tag{488}$$

The neutron current is, according to the definition of the diffusion coefficient

$$S = -D \text{ grad } F. \tag{488a}$$

Therefore the current of neutrons falling on the absorber from the right-hand side

$$S = D(\partial F/\partial x)_0 = qD\tau/L = qL. \tag{489}$$

All the formulae given are valid only inside the paraffin. Near the surface of the paraffin, the density of neutrons of more than Maxwellian energy will be smaller than inside, and will decrease towards the surface. Therefore the rate of production of Maxwell neutrons q will vary fairly rapidly and in a complicated way with the coordinates so that the solutions given are no longer valid. All quantitative measurements of the absolute activity of detectors should therefore be made inside the paraffin rather than at its surface. The effect of an absorber for thermal neutrons, usually Cd, in the interior of the paraffin differs from that of an external surface in that the Cd does not disturb the distribution of the neutrons of more than thermal energy, and leaves therefore the rate of production of thermal neutrons constant all over space although their actual density will not be constant.

F. Angular distribution of the neutrons emerging from paraffin. Activity of detector in interior of paraffin and near absorber. Albedo (F17)

One of the most important practical questions about the distribution of thermal neutrons in paraffin is the problem of the activity produced in various conditions in a detector. The two most important cases are:

(1) A thin detector is placed in the interior of the paraffin, without any absorber near it.

(2) The detector is on one side covered by an absorber for thermal neutrons (Cd).

Case 1.—If F is the density of neutrons at some point in the interior of the paraffin, the number of neutrons crossing unit area of the detector at an angle ϑ with respect to its normal, is per unit time (from both sides together)

$$Fv_a \cos\vartheta \sin\vartheta d\vartheta. \tag{490}$$

The number absorbed per unit time in a thin detector of area s, thickness δ g/cm² and absorption coefficient K cm²/g is accordingly

$$A = s\int_0^{\pi/2} Fv_a \cos\vartheta \sin\vartheta d\vartheta K\delta/\cos\vartheta = Fv_aKw, \tag{490a}$$

where $w=s\delta$ is the weight of the detector. Thus we can *define* the density of neutrons by the activity produced in a detector

$$F = A/v_aKw. \tag{491}$$

Case 2.—The activity produced in a detector placed on the front side of an absorber will be closely connected to the neutron current emerging from the paraffin near the absorber (cf. (489)). The connection will, however, involve the angular distribution of the neutrons emerging, which, near an absorber, will not be uniform as in the interior. Let us assume that the number of neutrons striking unit area of the absorber (and detector) in the direction ϑ per second is given by an expression of the form

$$f(\vartheta)\sin\vartheta d\vartheta = (a\cos\vartheta + b\cos^2\vartheta)\sin\vartheta d\vartheta, \qquad 0<\vartheta<\pi/2 \tag{492}$$

$$f(\vartheta) = 0, \qquad \pi/2<\vartheta<\pi.$$

We shall show below that the distribution actually has this form. If it does, the current is obviously given by

$$S = \int_0^{\pi/2} f(\vartheta)\sin\vartheta d\vartheta = \tfrac{1}{2}a + \tfrac{1}{3}b \tag{492a}$$

and the activity by

$$A = Ks\int f(\vartheta)(\delta/\cos\vartheta)\sin\vartheta d\vartheta = Kw(a+\tfrac{1}{2}b) \tag{492b}$$

so that

$$A = SKw(a+\tfrac{1}{2}b)/(\tfrac{1}{2}a+\tfrac{1}{3}b). \tag{493}$$

We may also define the "effective density" from the activity by means of (491)

$$F_{\text{eff}}(0) = (S/v_a)(a+\tfrac{1}{2}b)/(\tfrac{1}{2}a+\tfrac{1}{3}b). \tag{493a}$$

According to (493a), the neutron density is not exactly zero at the absorber, as we have assumed in (488). On the other hand, it will still be true (cf. (488)) that F is approximately linear in the distance x from the absorber. We may thus write

$$F(x) = \alpha + \beta x \tag{494}$$

and have, according to (488a) (493a) (478)

$$\beta = S/D = 3S/lv_a, \tag{494a}$$

$$\alpha = (S/v_a)(a+\tfrac{1}{2}b)/(\tfrac{1}{2}a+\tfrac{1}{3}b). \tag{494b}$$

We may now use (494) in order to calculate the angular distribution of the emerging neutrons. A neutron which has a collision at the depth x and moves in the direction ϑ, has the probability $\exp(-x/l\cos\vartheta)$ of emerging from the paraffin without further collision. The number of neutron collisions per sec. at a depth between x and $x+dx$ is $dxF(x)v_a/l$, by definition of the mean free path l. Assuming isotropic collision, the fraction $\frac{1}{2}\sin\vartheta d\vartheta$ of the neutrons colliding will have the direction ϑ after collision. Thus the total number of neutrons emerging in the direction ϑ will be

$$f(\vartheta)\sin\vartheta d\vartheta = \tfrac{1}{2}\sin\vartheta d\vartheta\int dxF(x)(v_a/l)e^{-x/l\cos\vartheta}. \tag{495}$$

Inserting (494) and integrating over x, we obtain

$$f(\vartheta) = \tfrac{1}{2}v_a(\alpha\cos\vartheta + \beta l\cos^2\vartheta). \tag{495a}$$

This proves the distribution assumed in (492), with

$$a = \tfrac{1}{2}\alpha v_a, \quad b = \tfrac{1}{2}\beta l v_a. \tag{495b}$$

Inserting this into (494b), we find

$$\alpha=(S/v_a)(\alpha+\tfrac{1}{2}\beta l)/(\tfrac{1}{2}\alpha+\tfrac{1}{3}\beta l). \quad (495c)$$

Comparison with (494a) gives

$$\alpha/\beta l=\tfrac{1}{3}(\alpha+\tfrac{1}{2}\beta l)/(\tfrac{1}{2}\alpha+\tfrac{1}{3}\beta l) \quad (495d)$$

and therefore

$$\beta l=\sqrt{3}\alpha. \quad (496)$$

Inserting this into (492) (495b), we find for the angular distribution of the neutrons

$$f(\vartheta)\sim\cos\vartheta+\sqrt{3}\cos^2\vartheta. \quad (497)$$

This means that relatively more neutrons are emitted in the forward direction than would be expected according to the simple cosine law which is valid in the interior of the paraffin (cf. (490)). This agrees with observations of Fink (F20). The law (497) may be expected to hold not only for thermal neutrons emerging from paraffin near an absorber, but generally for all neutrons leaving a paraffin block at a surface (provided only the mean free path of the neutrons is small compared to the dimensions of the paraffin block).

The density distribution (494) becomes with (496)

$$F=\beta(x+l/\sqrt{3}). \quad (498)$$

This means that the straight line representing the density as a function of the distance x from the absorber, would cut the x-axis at

$$x_0=-l/\sqrt{3}. \quad (498a)$$

Amaldi and Fermi have observed the density distribution (A11, Fig. 8); it is sufficiently nearly linear and may be extrapolated to a cut with the abscissa at

$$x_0=-0.18\text{ cm}. \quad (498b)$$

This would mean $l=0.31$ cm, in very good agreement with the direct determination of the mean free path $l=0.30$ cm (cf. (465)).

As has already been mentioned, these considerations are also valid for neutrons of higher than thermal energy at the surface of paraffin. In calculating the effective mean free path, it must be considered that the directions of motion before and after a collision are correlated. By an argument similar to that in Section D, it may be shown that

$$l=3l_0/(1+3\alpha), \quad (499)$$

where l_0 is the mean free path of the neutrons if only collisions with hydrogen are counted, and α is one-half the ratio of the cross sections of carbon and hydrogen, i.e., about 0.14 (cf. Table XXVII). Thus we should expect for neutrons above thermal energy in paraffin

$$-x_0=\sqrt{3}\cdot 0.9/1.42=1.1\text{ cm}. \quad (499a)$$

The activity of the detector becomes, if we insert (496) (495b) into (493),

$$A_a=\sqrt{3}SKw \quad (500a)$$

and, with (489) (484a),

$$A_a=\sqrt{3}qLKw=ql_{\text{th}}N^{\frac{1}{2}}Kw. \quad (500)$$

This may be compared with an activity in the interior of the paraffin of (cf. (490a), (480), (484))

$$A_i=q\tau v_aKw=ql_{\text{th}}NKw \quad (501)$$

so that

$$A_i/A_a=N^{\frac{1}{2}}. \quad (502)$$

The activity of a thin detector in the interior of the paraffin should thus (cf. (485)) be about 12 times greater than near an absorber. Direct experiments by Amaldi and Fermi gave a ratio of about 11, again in excellent agreement with the value deduced from the ratio of diffusion path to mean free path.

The ratio (502) can be interpreted using the concept of the "albedo." A_a is the activity observed at the surface of a paraffin block. If we have paraffin on both sides of a detector, and if the two blocks of paraffin did not influence each other, the activity would be $2A_a$. In reality, each paraffin block reflects the neutrons coming from the other. If β is the reflection coefficient (albedo), then the number of neutrons once reflected will be $2A_a\beta$, that of the neutrons twice reflected $2A_a\beta^2$ etc., and the total activity

$$A_i=2A_a/(1-\beta) \quad (502a)$$

so that the albedo has the value[33a]

$$\beta=1-2N^{-\frac{1}{2}}\approx 0.83. \quad (503)$$

[33a] *Added in proof:* Halpern, Lueneburg and Clark (H11a) have given a vigorous solution of the albedo problem. They find

$$\beta=1-2.31N^{-\frac{1}{2}} \quad (503a)$$

if the incident neutrons are distributed according to the $\cos\vartheta$ law, and

$$\beta=1-2.48N^{-\frac{1}{2}} \quad (503b)$$

for a $\cos^2\vartheta$ distribution.

§60. Neutron Resonance Energies (G17, G18, G18a, F33, A11, H8, H9, H10)

The most satisfactory method for determining neutron energies is a mechanical device which measures directly the neutron velocity. Such a mechanical velocity selector was constructed by Dunning, Pegram, Fink, Mitchell and Segrè (D25, D26, F19, F20) and used in order to determine the velocity distribution of the neutrons absorbed in Cd. The velocity selector consisted of four Duraluminum disks each of which bore 50 sectors of sheet cadmium, the spacing between the sectors being about the same (3.5°) as the angle subtended by each sector (3.7°). Two disks were mounted on a rapidly rotating shaft (up to about 5000 r.p.m.), a distance $d=54$ cm apart; the other two were fixed within 5 mm of the rotating disks. Each pair of disks, one rotating and the adjacent fixed one, represents a shutter for the neutrons absorbable in Cd, the shutter being opened and closed $50n/60$ times per second where n is the number of revolutions per minute. The two pairs of disks act as a velocity selector: Neutrons of velocity v are strongly absorbed by the shutters if

$$v=(100n/60)d. \qquad (504)$$

Thus, with $d=54$ cm, a speed of about $n=2500$ r.p.m. is necessary to absorb the "Maxwell neutrons" of $v=(2kT/m)^{\frac{1}{2}}=2.2\cdot 10^5$ cm/sec.

The experiments gave in fact a decrease in the number of transmitted neutrons, as detected by the disintegrations produced in boron with the help of an ionization chamber. The decrease was largest when the velocity selector rotated at a speed of about 2500 r.p.m. As a function of the speed of rotation, it followed closely the curve calculated from the assumption of Maxwellian distribution of the neutrons absorbed in cadmium (Fig. 14). This shows that (1) Cd is indeed a strong absorber for neutrons of thermal energies, (2) that boron is a good detector for such neutrons, (3) that thermal neutrons are present in large quantity in the interior of the paraffin and (4) that their distribution is practically Maxwellian, as we have concluded from our theoretical investigation of the diffusion of neutrons in paraffin (§59). A further confirmation of these points is found in experiments which show a shift of the maximum absorption to slower rotation upon

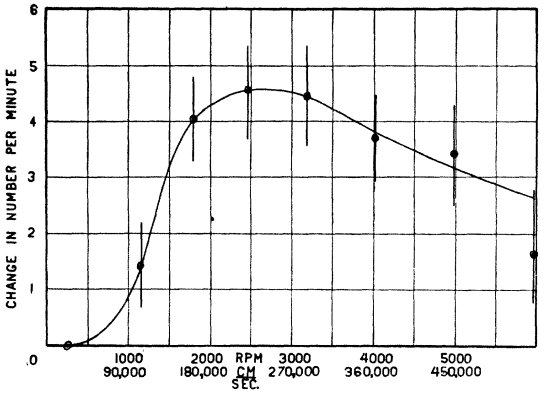

Fig. 14. Velocity selector of neutrons. Distribution of slow neutrons measured with the mechanical velocity selector by Dunning, Pegram, Fink, Mitchell and Segrè. The solid curve represents the Maxwell distribution at room temperature.

cooling of the paraffin used for slowing down the neutrons.

The mechanical velocity selector, while successful with thermal neutrons, does not seem applicable to the resonance groups of neutrons. As we shall see in this section, the energies of the neutrons causing resonance disintegration of nuclei, range from about 1 to several hundred volts, corresponding to velocities from 6 to 100 times thermal velocity. The rotational speed of the velocity selector would have to be increased in the same ratio, i.e., to between 15,000 and 250,000 r.p.m. which appears at present not to be feasible.

Another rather direct method which might in the future become usable is the diffraction of neutrons by a suitable scatterer which would serve as a measurement of the neutron wavelength. Thus far, however, only qualitative indications of neutron diffraction have been obtained (M23).

A. The boron absorption method

According to the foregoing, we are forced to make use of less direct methods for determining the neutron energy. The simplest of these methods is based on the assumption (§58) that the *absorption* coefficient of slow neutrons *in boron* and lithium is *inversely proportional to the neutron velocity*. This assumption is amply justified theoretically by the small density and large width to be expected for the energy levels of light nuclei,

(§58B). In addition, it has been confirmed by the following experiments:[34]

(1) Directly in the region of thermal energies by Rasetti, Mitchell, Fink and Pegram (R4, R5) using a mechanical arrangement which measures directly the deviation from the $1/v$ law (cf. §58B, end);

(2) Indirectly for higher energies by the fact that the absorption coefficients of Li and B change in the same way with energy (G17, H8);

(3) Approximately by the agreement between the observed ratios of the resonance energies of various elements derived by two different methods, *viz.* the absorption in boron and the spatial distribution of the resonance neutrons in paraffin (A11, cf. §60B);

(4) Qualitatively by comparison with experiments showing the order of the resonance levels of various nuclei in the energy scale (H10, cf. §60C).

Of the two light nuclei showing strong absorption of slow neutrons, boron is the more suitable because its absorption coefficient (per g/cm²) is about 10 times as high as that of Li. A high absorption coefficient is desirable in the first place in order to keep the required thickness of the absorber small and thus the geometry manageable. But even more important is the fact that scattering can be expected to be relatively smaller compared to the capture probability if the total absorption coefficient is higher because the scattering cross section will be of the same order for all light nuclei. Now the observed absorption is equal to the absorption by capture plus the absorption by scattering. The capture cross section decreases with increasing energy as $E^{-\frac{1}{2}}$ while the scattering cross section remains constant. Therefore the total absorption coefficient will cease to depend sensitively on the energy when the capture cross section becomes equal to or smaller than the scattering cross section. We estimate the scattering cross section to be about $4\cdot10^{-24}$ cm² (average of other light elements, cf. D23 and Table XXVII). This must be corrected because in the experiments neutrons moving in all directions inside a hemisphere are counted. Therefore we assume the effective scattering cross section to be half the total, i.e., $2\cdot10^{-24}$ cm². Then we find that capture and scattering will become equal at about 7 volts for Li ($\sigma_{\text{capt.}}=34\cdot10^{-24}$ cm² at $E=kT\approx1/40$ volt) and at about 2500 volts for $B(\sigma_{\text{capt.}}=600\cdot10^{-24}$ cm² at $E=kT$). Lithium would therefore be useless for all resonance groups which lie at neutron energies higher than 7 volts (cf. H9). Even for boron, the observed absorption should be corrected for scattering which introduces quite an appreciable correction to the energy especially if the resonance energy lies high. At the same time, since the scattering cross section of boron can only be estimated roughly, a fairly large uncertainty will remain in the energy values given, even after the correction for scattering has been applied.

The principle of the experiment is very simple indeed: the absorption coefficient of boron must be measured, once for thermal neutrons and once for the resonance neutrons of the detector in question. As we know (§58), any neutron detector is made radioactive by thermal neutrons as well as by its particular "resonance neutrons." To separate the two activities, a sheet of Cd is placed between the neutron source and the detector: Cd is known to absorb all the thermal neutrons and to be practically transparent for the neutrons of higher energy ("resonance neutrons") (cf. §61G). Thus the difference of the activity in the detector with and without the Cd absorber will give the activity produced in it by thermal neutrons, while the activity with Cd shield must be attributed to the resonance neutrons. Both the activities, resonance and thermal, must be measured without the boron absorber and with boron, preferably as a function of the thickness of the latter. From these measurements, the absorption coefficients for the two groups of neutrons can be deduced. The energy of the resonance group is then

$$E_{\text{res}}=E_{\text{th}}(K_{\text{th}}/K_{\text{res}})^2. \qquad (505)$$

There is some question as to what to insert for the thermal energy. This problem is, however, well-defined if we remember that both the absorption in boron and the activity produced in the detector are proportional to $1/v$, i.e., to $E^{-\frac{1}{2}}$. Denoting by K_0 the absorption coefficient in

[34] A breakdown of the $1/v$ law for boron was announced by Arsenjewa-Heil, Heil and Westcott (A16) who maintained that all the resonance levels found experimentally lie actually inside the thermal region. This is due to a faulty interpretation of their experimental results, cf. footnote 37.

boron of neutrons of energy kT, and introducing

$$x=(E/kT)^{\frac{1}{2}} \tag{505a}$$

we have for the activity transmitted through a thickness l of boron:

$$I(l)=4\pi^{-\frac{1}{2}}I_0\int x^2e^{-x^2}dxe^{-(K_0/x)l}. \tag{506}$$

The average absorption coefficient K_{th} is defined by

$$I(l)=I_0e^{-K_{\text{th}}l}. \tag{506a}$$

The "effective thermal energy" to be inserted in (505) is then

$$E_{\text{th}}=kT(K_0/K_{\text{th}})^2. \tag{506b}$$

For very small thicknesses of boron, we obtain by a straightforward integration (B15)

$$E_{\text{th}}=(\pi/4)kT. \tag{507}$$

For larger thickness the integral must be evaluated numerically; the result is given in Fig. 15. Here the effective "thermal energy" E_{th} to be inserted in (505) is plotted against the (natural) logarithm of the transmission coefficient I/I_0 which follows directly from experiment. It is seen that E_{th} rises only very slowly with increasing thickness of the boron. For a transmission of 50 percent which is most convenient for experiments, we have E_{th} almost exactly equal kT as was already found by Goldsmith and Rasetti and used in the evaluation of their experiments.[34a]

Another point which requires some attention, is the oblique incidence of the neutrons on the absorber. If the experiments are made at the surface of the paraffin, the angular distribution of the emerging neutrons is approximately given by the law (497), $f(\vartheta)\sim\cos\vartheta+\sqrt{3}\cos^2\vartheta$, for both thermal and resonance neutrons. The absorption will thus not be exactly exponential, but will follow the absorption law

$$I=I_0c(Kl), \tag{508}$$

where the function

$$c(x)=\int_0^1 du(1+\sqrt{3u})e^{-x/u}/(1+\tfrac{1}{2}\sqrt{3}) \tag{508a}$$

has been calculated by Fermi and is given in A11,

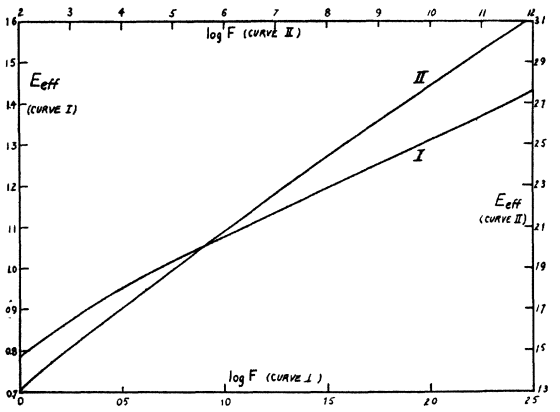

FIG. 15. Absorption of thermal neutrons in boron. The detector is assumed to obey the $1/v$ law in the thermal region. Abscissa: $|\log F|$ where $F=I/I_0$ is the transmission coefficient of the boron absorber. Ordinate: "Effective thermal energy" as a fraction of kT. In determining the energy of resonance neutrons by the boron method, the "effective energy" obtained from this figure should be inserted for the thermal energy.

Fig. 11 and in Fig. 16 of this report. From the given transmission ratio I/I_0, the value of $x=Kl$ can be read from the curve and therefrom the absorption coefficient K obtained. If the absorber thickness for each detector is chosen so that the transmission coefficient is the same for all detectors, the absorption coefficient in boron for the various resonance groups will be inversely proportional to the respective thickness of the boron absorbers used, independently of the angular distribution of the neutrons.

The most extensive measurements have been made by Goldsmith and Rasetti (G17). The results are listed in Table XXII. The values given by Goldsmith and Rasetti have been corrected for scattering, a cross section of $2\cdot10^{-24}$ cm^2 being assumed (cf. above), corresponding to an absorption coefficient of 0.11 cm^2/g due to scattering. This makes all the energy values slightly higher than those given by Goldsmith and Rasetti.

The top row in Table XXII refers to thermal neutrons,[35] measured with a Rh detector as the difference between the activities without and with Cd. The other figures refer to resonance groups. In general, one figure is given for each activity produced. This will represent the energy of the strongest resonance group, or some

[34a] More extensive calculations on the transmission of thermal neutrons by boron were made by Zahn (Z1a) and Laporte (L1b).

[35] According to Fig. 15, the "effective energy" of thermal neutrons corresponding to a boron transmission of 35 percent, is $1.09kT=0.028$ volt.

TABLE XXII. *Neutron resonance energies (from boron method).*

DETECTOR	FILTER	PERIOD	THICKNESS OF *B* ABSORBER g/cm²	TRANSMISSION %	ABS. COEFF. *K* (cm²/g)	*K* CORRECTED FOR SCATTERING	*E* (volt)
Rh[a]		44″	0.02	35.2	28	27.9	0.028
25 Mn	Cd	150′	0.58	43	0.73	0.62	60
29 Cu	Cd	5′	0.58	~50	0.57	0.46	100
33 As	Cd	26 hr.	0.58	54	0.49	0.38	150
35 Br	Cd	18′	0.58	50	0.57	0.46	100
45 Rh	Cd	44″	0.081	50	4.15	4.05	1.3
45 Rh	Cd	3.9′	0.054	~57	4.6	4.5	1.1
47 Ag	Cd	22″	0.108	63	2.8	2.7	3.0
47 Ag	Cd+Ag	22″ [b]	0.108	60	2.1	2.0	5.5
47 Ag	Cd	2.3′	0.58	51.5	0.54	0.43	120
49 In	Cd	13″	0.108	~50	3	2.9	2.6
49 In	Cd	54′	0.081	53	3.8	3.7	1.6
53 I	Cd	25′	0.58	53	0.50	0.39	140
75 Re	Cd	20 hr.	0.23	64	0.83	0.72	40
77 Ir	Cd	19 hr.	0.108	46	3.5	3.4	1.9
79 Au	Cd	2.7 *d*	0.108	53	2.8	2.7	3.0

[a] Thermal neutrons.
[b] "*B* group."

average energy, if there are several resonance groups of about equal intensity.

Only in the case of the short period of Ag (22″), a separation into two resonance groups may easily be carried out, because part of the Ag "resonance neutrons" are very strongly absorbed in Ag (A group), another part (B group) much less strongly. This behavior suggests the presence of two resonance levels, a narrow one (A) and a broader one (B). The neutrons of the A group have an average absorption coefficient of 20 cm²/g in Ag; therefore a layer of 0.25 g/cm² Ag will absorb them almost completely while practically not affecting group B ($K \sim 0.3$ cm²/g in Ag). The activity due to neutrons passing through such a layer of Ag may thus be ascribed to the B group, the rest of the activity to A.

It is seen from Table XXII that there are several resonance levels near 1 volt, *viz.* Rh 44″, Rh 3.9′, In 54′, Ir 19 hr. These resonance levels can be shown to overlap (§61E) ("D group" of Amaldi and Fermi). The same is true for the "A" level of Ag 22″ and the level of Au (A group, energy near 3 volts). With the exception of In 13″ and of the "B" level of Ag 22″ (5 volts), all the remaining levels lie between 40 and 160 volts. The determination of their position is certainly not very accurate, partly because of the uncertainty of the scattering correction. It is plausible to assume (cf. G17) that these higher values actually represent averages over several levels[35a] while the lower values give the true positions of the resonance levels. If the first resonance level lies very low compared with the average spacing between levels, its effect will in general be much larger than that of higher levels, because the activity is cet. par. proportional to $E_r^{-\frac{1}{2}}$ (cf. (550a)). If the first resonance energy is of the order of the spacing of the levels, the second, third ... levels will give contributions of the same order as the first.

From these considerations it seems that the average spacing between the energy levels of medium heavy and heavy nuclei ($Z > 40$) is probably of the order of 10 to 30 volts, near zero kinetic energy of the neutron (cf. §53).

For the lighter nuclei (Mn to Br), the energy values observed are consistently high. It seems reasonable to assume that they represent the true position of the lowest level in spite of being "high," because lighter nuclei are expected to have a larger spacing between the levels (§53) and because no "low" values are observed for this lighter group. This spacing would then appear to be of the order of a few hundred volts, for Z near 30.

Other observations with the boron method have been made by Amaldi and Fermi (A11) and by v. Halban and Preiswerk (H8, H9). Amaldi and Fermi find values of the absorption coefficient in close agreement with Goldsmith and Rasetti, except for the thermal neutrons (38 cm²/g instead of 28 cm²/g). The value of Goldsmith and Rasetti was taken for the computation of Table XXII; it agrees with a value found by the same authors using an approximately parallel beam of neutrons, *viz.* 30 cm²/g, whereas Livingston and Hoffman (L33) find a value (36 cm²/g) near that of Amaldi and Fermi. The values found for the absorption coefficient of "resonance neutrons" in B are

GROUP	Rh 44″	Ag "A"	Ag "B"	I	Br
Goldsmith and Rasetti	4.15	2.8	2.1	0.5	0.57
Amaldi and Fermi	4.7	3	2.3	1.0	—
v. Halban and Preiswerk	6.3	3.4	—	0.7	0.35

The agreement is in general satisfactory. It is likely that for Br the determination of v.

[35a] This view is confirmed by experiments of Ruben and Libby (R17a) who show that the boron absorption coefficient of the neutrons producing activity in iodine decreases by about a factor 7 if the neutrons are first filtered by an iodine absorber.

Halban and Preiswerk is better, since from other evidence (cf. Section C) it seems that the Br level lies higher than the I level.

B. Method of the distribution of resonance neutrons in water

Amaldi and Fermi have also used an alternative method, based on the fact (§59D) that the mean square distance of the neutrons from the source increases with decreasing energy (cf. (476)). The activity of various detectors (Rh, Ag, I), screened by Cd against thermal neutrons, was measured at various points in a large tank of water which contained the neutron source at its center. The distributions are shown in A11, Fig. 7. They show a small but definite shift towards larger distances from the source in the order I, Ag (short period, A+B groups), Rh (short period, marked "D" in figure). The values of $(r^2)_{Av}$ measured are 262.2, 270.6 and 276.6 cm² water, respectively, for I, Ag and Rh. From (476) we have then for the ratio of the energies

$$\text{I} : \text{Ag} : \text{Rh} = 20 : 3.5 : 1. \qquad (509)$$

The agreement with the result of the boron method (160 : 3.5 : 1) is rather satisfactory considering the extreme sensitivity of the energy ratio to small errors in $(r^2)_{Av}$. The corrections suggested in §59D improve the agreement considerably as compared to the values given by Amaldi and Fermi (4.5 : 1.9 : 1).

C. Transformation of one resonance group into another

v. Halban and Preiswerk have pointed out that a neutron resonance group of high energy will be transformed into a group of lower energy by passing through paraffin, and have used this fact for establishing the *order* of various neutron resonance levels in the energy scale. The experimental arrangement is as follows: Neutrons emerging from a large paraffin block pass through seven thin paraffin plates of 3 mm thickness each. At the end of this series of plates, a neutron detector is placed, screened by cadmium. An absorber A can be interposed between any two of the paraffin plates, or at the end of the series right in front of the detector. The activity in the detector is measured as a function of the position of the absorber.

Immediately behind the absorber, the characteristic resonance group of the absorber will be missing in the neutron spectrum while otherwise the spectrum will be unaffected. However, when the neutrons pass through paraffin after traversing the absorber, the "hole" in the neutron distribution will shift to lower energies. Therefore, if the detector is made of the *same* material as the absorber, its activity will be strongly reduced if the absorber is placed right close to it, and will be much less reduced if paraffin is interposed between absorber and detector (v. Halban and Preiswerk, H8, H9). From such measurements, the mean free path of the resonance neutrons in paraffin may be deduced (A11, cf. §59 C7).

If the detector has a resonance level *higher* than that of the absorber, the result will be about the same except that the influence of the absorber is much smaller to start with, because it is only due to the overlapping of the neutron levels of absorber and detector. E.g., v. Halban and Preiswerk found that a certain Ag absorber reduced the activity of their I detector by only 5 percent but that of their Ag detector by 80 percent when there was no paraffin between absorber and detector.

However, when the resonance energy of the detector is *lower* than that of the absorber, the "hole" made by the absorber in the neutron distribution, will gradually cover the resonance region of the detector when the neutrons pass through paraffin. Therefore, in this case, the reduction of the activity of the detector by the absorber will *increase* with increasing thickness of the paraffin interposed between absorber and detector up to a certain maximum, and start decreasing only then. The position of the maximum will be an approximate measure of the ratio of the resonance energies (cf. method B) while its height will be connected to the width of the resonance level of the absorber.

With the method described, v. Halban and Preiswerk were able to determine the relative positions of the resonance levels of various elements. They found that the resonance energies could be arranged as follows:

$$\text{Br}\,(18') > \text{I} > \text{Ag}\,(22'') > \text{Rh}\,(44'') \sim \text{In} \sim \text{Ir}. \qquad (510)$$

This order agrees in general with the order found

by Amaldi and Fermi from the diffusion method B, and with that determined by the boron absorption method A.

The value of the method of transformation of neutron groups into each other lies firstly in the qualitative confirmation of the assumptions made about the boron absorption. Secondly, it is superior for establishing the order of high resonance energies which cannot be measured very well by the boron method (cf. above, scattering correction!). E.g., it shows that the Br level is higher than that of I while the evidence of the boron method is contradictory on this point. Moreover, it will indicate, at least qualitatively, when a nucleus has *several* resonance levels: If one level of nucleus A lies higher, another lower than the resonance level of a given other substance B, then we should expect an increase in the reduction of activity upon interposing paraffin *both* when A is used as absorber and B as detector, *and* when the arrangement is reversed. Applications of the method to the determination of the width of levels will be discussed in the following section.

D. Resonance levels of nuclei which are not activated

The determination of the position of resonance levels is much more difficult if the nucleus in question is not made radioactive by neutron capture. We can, in this case, only measure the absorption of the activity of other detectors in the substance in question. The information which can be obtained about nonactivated neutron absorbers is about the following:

(1) The absorption of thermal neutrons in the substance can be measured. The number of thermal neutrons can be measured either by the activity produced in any detector, or by the disintegration of Li or B. In each case, the difference between the number of disintegrations observed with and without a screen of Cd, gives the number of thermal neutrons. By this method, the absorption coefficients of almost all elements for thermal neutrons has been measured by Dunning, Pegram, Fink and Mitchell (D23) and for some additional elements (rare earths) by Hevesy and Levi (H31).

(2) The average absorption coefficient for faster neutrons can be measured, using a B or Li chamber shielded by Cd. The ratio of this absorption coefficient to that for thermal neutrons R_S is a characteristic of the substance S. It may be compared to the corresponding ratio R_B for boron. If R_S is approximately equal to R_B, the $1/v$-law holds approximately for the substance. Then the substance S will in general not possess any resonance levels at low energies. If $R_S > R_B$, we shall expect a resonance level of S at low energy but above thermal energy. The result $R_S < R_B$ indicates a resonance level in the thermal energy region itself or at small negative energy.

(3) The absorption coefficient for neutrons of certain known energies can be measured, using a detector whose resonance level is known. E.g., we may measure the absorption of one-volt neutrons by using Rh (44″), that of three-volt neutrons by using Ag (22″) as a detector, the detector being in each case screened by Cd. The interpretation is similar to, but more definite than in (2). For a number of absorbers, notably *Cd, Hg, Sm and Gd*, it has been found that the absorption coefficient for D neutrons (Rh resonance group) is almost negligibly small compared to that of the thermal neutrons. The ratio is about 1 : 100 in Cd, Sm and Gd (G17), as compared to 1 : 7 in boron. This shows that the absorbers mentioned must *have very low resonance levels*. Their exact position depends on the width assumed and can therefore be determined only in connection with other experiments measuring the width of the levels (cf. §61G).

It may happen that a given absorber shows very *strong* absorption for the neutrons of a known group. In this case, we would conclude that the absorbing substance has a resonance level coinciding with this group. No such case is known among the nonactivated absorbers, and it is indeed rather unlikely to find the level of an absorber in this way because the energy regions for which we possess specific detectors are very narrow and widely separated.

(4) The approximate position of the resonance levels of nonactivated absorbers may be obtained by the method described in C. A number of detectors must be used whose resonance energies form an increasing series. The absorber to be investigated is placed between the paraffin plates and the reduction in activity measured as

a function of the thickness of paraffin for the various detectors. Then it will be found that for the low-resonance detectors the intensity reduction will increase when a small amount of paraffin is interposed while it will decrease for the high-resonance detectors. The resonance energy of the absorber will lie between that of the last detector which shows the increase and the first one which does not. No experiments of this kind have yet been carried out.

(5) If the absorption of the substance is detectable by the boron or lithium detector, the energy of the resonance level can be estimated by interposing a boron absorber, in addition to the given substance, between neutron source and boron detector. The reduction of the neutron intensity by the substance can then be measured as a function of the thickness of the additional boron absorber, which determines the boron absorption coefficient of the resonance neutrons of the substance.

§61. Total (γ-Ray) Width of Neutron Resonance Levels (B15, A11, G18a, M2a, R4, L32)

According to the general formula (430), the capture cross section for neutrons of a given energy E may be written

$$\sigma_N(E)=\sigma_0/(1+(E-E_r/\tfrac{1}{2}\Gamma)^2) \qquad (511)$$

provided the energy of the neutron is not too different from the resonance energy E_r. σ_0 is the cross section at resonance (cf. 430b), Γ the total "natural width" of the level.

A. Doppler effect

In comparing this formula with experiment, it is in some cases necessary to take account of the *Doppler effect*. The velocities of the capturing nuclei are not negligible compared to the neutron velocities, and the Doppler width introduced in this way is sometimes comparable to, sometimes even larger than, the natural width.

If u_x is the velocity of the capturing nucleus in the direction of motion of the neutron, then $v-u_x$ will be the relative velocity of neutron and nucleus. The relative kinetic energy is then, neglecting terms of the order u^2/v,

$$E'=\tfrac{1}{2}m(v-u_x)^2=E-(2mE)^{\frac{1}{2}}u_x, \qquad (512)$$

where E is the absolute kinetic energy and m the mass of the neutron. Now the probability of finding a nucleus of velocity u_x is given by the Maxwell distribution function

$$w(u_x)du_x=(M/2\pi kT)^{\frac{1}{2}}e^{-Mu_x^2/2kT}du_x, \qquad (512a)$$

where M is the mass of the capturing nucleus. Inserting (512), we find for the probability of a given E' (for fixed E):

$$w(E')dE'=\pi^{-\frac{1}{2}}e^{-(E'-E)^2/\Delta^2}dE'/\Delta, \qquad (513)$$

where

$$\Delta=2(mE_rkT/M)^{\frac{1}{2}} \qquad (513a)$$

is the "Doppler width."[36] E_r has been written instead of E because the neutron energy is supposed to be near the resonance energy E_r.

The energy to be inserted in (511) is, of course, the relative energy E'. The effective cross section for neutrons of energy E is then

$$\sigma(\xi, E)=\int\sigma_N(E')w(E')dE'=\sigma_0\psi(\xi, x) \qquad (514)$$

with

$$x=(E-E_r)/\tfrac{1}{2}\Gamma, \qquad (514a)$$

$$\xi=\Gamma/\Delta, \qquad (514b)$$

$$\psi(\xi, x)=\frac{\xi}{2\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{4}\xi^2(x-y)^2}}{1+y^2}dy, \qquad (515)$$

$$y=(E'-E_r)/\tfrac{1}{2}\Gamma. \qquad (515a)$$

$\psi(\xi, x)$ is in general a complicated function of x. Simple expressions are obtained

(a) for ξ very large (purely natural width)

$$\psi(\xi, x)=1/(1+x^2) \qquad (515b)$$

and (514) (515) reduce to (511);

(b) for ξ very small (pure Doppler width) and $x\ll\xi^{-2}$

$$\psi(\xi, x)=\tfrac{1}{2}\pi^{\frac{1}{2}}\xi e^{-\frac{1}{4}\xi^2x^2}; \qquad (515c)$$

(c) for very large $x\gg\xi^{-2}$ and any ξ

$$\psi(\xi, x)=1/(1+x^2); \qquad (515d)$$

(d) for $x=0$ (exact resonance) and any ξ

$$\psi(\xi, 0)=\tfrac{1}{2}\pi^{\frac{1}{2}}\xi e^{\frac{1}{4}\xi^2}[1-\Phi(\tfrac{1}{2}\xi)], \qquad (516)$$

where Φ is the Gaussian error function

$$\Phi(\tfrac{1}{2}\xi)=2\pi^{-\frac{1}{2}}\int_0^{\frac{1}{2}\xi}e^{-t^2}dt. \qquad (516a)$$

In the special case of large ξ (natural width), (516) reduces to unity, meaning (cf. (514)) that in this case the cross section at exact resonance is equal to σ_0. In the case of small ξ (Doppler width) we have $\psi(\xi, 0)=\frac{1}{2}\pi^{\frac{1}{2}}\xi$, meaning that the cross section at resonance is reduced to

$$\sigma_D(E_r)=\tfrac{1}{2}\pi^{\frac{1}{2}}\sigma_0\Gamma/\Delta. \qquad (516b)$$

Experimentally, the quantities measured most easily are the total activation, and the absorption of this total activity by thin absorbers of the same substance ("absorption coefficient for self-indication"). The total activation is proportional to

$$C=\int n(E)\sigma(\xi, E)dE, \qquad (517)$$

if $n(E)dE$ is the number of neutrons in the energy

[36] Δ is here defined as one-half of the quantity denoted by the same letter in the paper of Bethe and Placzek (B15).

interval dE. Since σ varies rapidly, n slowly with the energy, we have (cf. 514, 514a)

$$C=n(E_r)\sigma_0\tfrac{1}{2}\Gamma\int dx\psi(\xi, x). \qquad (517a)$$

As can easily be seen from the definition (515) of ψ, the integral has the value π independent of ξ so that we obtain

$$C=\tfrac{1}{2}\pi\Gamma n(E_r)\sigma_0. \qquad (518)$$

A thin absorber will absorb a fraction $c\sigma(\eta, E)$ of the activity produced by the neutrons of energy E, where c is a constant depending on the thickness of the absorber in g/cm^2, and $\eta=\Gamma/\Delta'$ where Δ' is the Doppler width for the absorber. This may be different from the Doppler width Δ for the detector because the temperatures may be different. The effective absorption "cross section for self-indication" is then

$$\sigma_s(\xi, \eta)=\frac{\int n(E)\sigma(\xi, E)\sigma(\eta, E)dE}{\int n(E)\sigma(\xi, E)dE}, \qquad (519)$$

The integral in the numerator may be evaluated (B15); then (519) becomes

$$\sigma_s(\xi, \eta)=\tfrac{1}{2}\sigma_0\psi(\zeta, 0) \qquad (520)$$

with

$$\zeta=2\xi\eta(\xi^2+\eta^2)^{-\frac{1}{2}}. \qquad (520a)$$

In the usual case of equal temperature of absorber and emitter this becomes

$$\zeta=\xi\sqrt{2}. \qquad (520b)$$

According to the definition (514b) (513a) of ξ, this means that σ_s is one-half of the resonance cross section at one-half of the temperature of absorber and detector. If the temperatures of absorber and detector are not equal, one-half their arithmetical mean should be inserted, according to (520a).

From the total activity and the effective cross section for self-indication σ_s we can define the "effective width" of the neutron level

$$\Gamma_{\text{eff}}=C/n(E_r)\sigma_s, \qquad (521)$$

so that (cf. 518, 520)

$$\Gamma_{\text{eff}}=\pi\Gamma/\psi(\xi\sqrt{2}, 0). \qquad (521a)$$

For natural width, $\psi(\xi\sqrt{2}, 0)=1$ (cf. (516)) so that

$$\Gamma_{\text{eff}}=\pi\Gamma, \quad (\Gamma\gg\Delta), \qquad (522)$$

while for pure Doppler width $\psi(\xi\sqrt{2}, 0)=(\pi/2)^{\frac{1}{2}}\xi$ and (cf. (514b))

$$\Gamma_{\text{eff}}=(2\pi)^{\frac{1}{2}}\Delta, \quad (\Gamma\ll\Delta). \qquad (523)$$

For intermediate values of $\xi=\Gamma/\Delta$, a power series and a graph for Γ_{eff} have been given by Bethe and Placzek (B15, Fig. 2 and §5).

If the Doppler width is large compared to or of the same order as the natural width, the cross section for self-indication, and therefore also the effective width, will change with temperature. In the extreme case $\xi\ll 1$ (pure Doppler width), σ_s will be inversely proportional to the Doppler width, i.e., proportional to $T^{-\frac{1}{2}}$. This behavior is exactly the same as would be expected for thermal neutrons assuming the validity of the $1/v$ law. A temperature effect of this type has actually been observed for the resonance activity of Ag by Arsenjewa-Heil, Heil and Westcott (A16).[37] If Doppler width and natural width are comparable, the temperature dependence of the cross section σ_s may serve for a determination of the natural width Γ from the known Doppler width (513a).[38]

B. Measurements of the effective width from activity and absorption coefficient

The effective width may be deduced (cf. (521)) from the total activity produced in a given detector by the resonance neutrons and the cross-section for self-indication if the number of neutrons per unit energy $n(E_r)$ is known. Let K be the effective absorption coefficient of the detector in cm^2/g for its resonance neutrons, so that

$$K_r=\sigma_s/M, \qquad (524)$$

where M is the mass of one atom of the detector. Then the total activity of a thin detector of weight w (in grams) is[39]

$$B_r=n(E_r)v_r\Gamma_{\text{eff}}K_rw, \qquad (524a)$$

where $n(E_r)dE$ is the number of neutrons per cm^3 in the energy interval dE about E_r, and therefore $n(E_r)v_rdE$ the number of neutrons in

[37] The effect was incorrectly interpreted by these authors as evidence that the Ag resonance neutrons had actually thermal energies.

[38] This experiment was first suggested and preliminary results reported by Frisch (Copenhagen Conference, June 1936).

[39] The relation between B_r and C (cf. (517), (518), 521)) is

$$B_r=(wv_r/M)C. \qquad (524b)$$

this energy interval striking unit area of the detector per sec. The "activity" is here defined as the number of neutrons captured per sec., or the number of β-rays emitted per sec. by the radioactive substance produced, immediately after an infinite time of irradiation. The detector must be so thin that it absorbs neither the resonance neutrons nor the β-electrons to any appreciable extent; otherwise corrections have to be applied (cf. A11).

The number of neutrons $n(E_r)$ can be expressed by the number emitted from the source with the help of (440):

$$n(E_r)v_r = ql_0/E_r, \tag{525}$$

where q is the number coming from the source per sec. and cm^3 and l_0 the mean free path of neutrons above thermal energy. q is connected to the total number Q coming from the source per sec. by

$$Q = \int q d\Omega, \tag{525a}$$

the integral being extended over the whole volume of the paraffin. We should thus measure the activity of our detector as a function of its position, and integrate over the whole paraffin; then we find

$$\int B_r d\Omega = Ql_0 K_r w \Gamma_{\text{eff}}/E_r. \tag{526}$$

Q may now be obtained from measurements of the activity produced by *thermal* neutrons (in *any* detector). This activity is conveniently measured by determining the activity of the detector when it is covered by Cd from one side, and subtracting therefrom the activity obtained with Cd on both sides. With a Cd cover on one side, the activity of the detector due to thermal neutrons is, according to (500),

$$B_{\text{th}} = \sqrt{3} q' L K_{\text{th}} w, \tag{527}$$

where L is the diffusion length (§59E), K_{th} the absorption coefficient of the detector for thermal neutrons and q' the number of thermal neutrons produced per cm^3 and sec. Owing to the different distribution of thermal and resonance neutrons in the paraffin (cf. §59D, Fig. 13), q' will in general be different from q. However, by integration we obtain

$$\int B_{\text{th}} d\Omega = QL\sqrt{3} K_{\text{th}} w, \tag{527a}$$

where Q has the same meaning as in (526). Combining (526) and (527a), we find[40]

$$\frac{\Gamma_{\text{eff}}}{E_r} = Y \frac{K_{\text{th}}}{K_r} \frac{L\sqrt{3}}{l_0}, \tag{528}$$

where

$$Y = \int B_r d\Omega / \int B_{\text{th}} d\Omega \tag{528a}$$

is the ratio of the activities. According to the measurements of Amaldi and Fermi, $L = 2.1$ cm (cf. (482b)), $l_0 = 0.9$ cm (cf. (465c)) so that

$$\sqrt{3} L/l_0 = 4.0, \tag{528b}$$

$$\Gamma_{\text{eff}}/E_r = 4.0\, Y K_{\text{th}}/K_r. \tag{528c}$$

The actual measurements are complicated by the fact that the activities must be measured all over the paraffin. This can be avoided if only relative values of the "fractional width" Γ_{eff}/E_r are required, i.e., e.g., when Γ_{eff}/E_r has already been determined for one substance. The spatial distribution of various resonance groups is sensibly the same (cf. A11, Fig. 7). Therefore if B_1 and B_2 measure the activities of two detectors at any one place (e.g., near the maximum neutron density), we have according to (524a) (525)

$$\left(\frac{\Gamma_{\text{eff}}}{E_r}\right)_1 \div \left(\frac{\Gamma_{\text{eff}}}{E_r}\right)_2 = \left(\frac{B_r}{K_r w}\right)_1 \div \left(\frac{B_r}{K_r w}\right)_2. \tag{529}$$

A further complication is the finite absorption of neutrons and electrons in the actual target. This correction is somewhat ambiguous since the absorption coefficient for resonance neutrons decreases rather rapidly with increasing thickness of the absorber (cf. below, Section C) and this decrease depends on the ratio of natural to Doppler width (cf. above, Section A). Amaldi and Fermi corrected their results assuming a constant absorption coefficient which means an overcorrection, i.e., an overestimate of the true resonance activity without absorption and therefore an overestimate of the width Γ_{eff} (cf. (528)).

[40] It is, of course, not necessary that the activity of thermal and resonance neutrons is determined in the same detector. This fact was made use of in the determination of the width of the iodine level by Amaldi and Fermi. The activity produced by thermal neutrons in iodine is very small, therefore a Rh detector was used to determine the number of thermal neutrons. The numbers given for thermal neutrons in the I column of Table XXIII refer therefore to Rh, not to I.

TABLE XXIII. *Widths of neutron resonance levels.*

SUBSTANCE	Rh	Ag	I
$Y=\frac{\int B_{\text{res}}d\Omega}{\int B_{\text{th}}d\Omega}$	1.16	0.72	0.045
K_r (cm²/g)	19[40a]	20	0.38
K_{th} (cm²/g)	0.7	0.25	0.7
Γ_{eff}/E_r	0.17	0.036	0.33
E_r(cf. Table XXII)	1.3	3.0	140
Γ_{eff} (volts)	0.22	0.11	50
$(2\pi)^{\frac{1}{2}}\Delta$(cf. (513a))	0.09	0.13	0.9
Γ	0.07	<0.03	15(?)

[40a] This large value for the resonance absorption coefficient of Rh was given recently by Manley, Goldsmith and Schwinger (M2a) on the basis of their own data and those of v. Halban and Preiswerk. It is almost 10 times the value originally given by Amaldi and Fermi (2.0 cm²/g) which causes large changes of all results about the Rh level (cf. B15).

We have recalculated the correction using the absorption curve for resonance neutrons given in Fig. 16, curve *w*. We find that the A group activity in the Ag detector of Amaldi and Fermi (thickness 0.057 g/cm²) is reduced by a factor of 0.45 by absorption (Amaldi and Fermi give 0.35). For Rh, we used the most recent value of the resonance absorption coefficient (M2a), *viz.* 19 cm²/g (A and F give 2.0) and find 0.079 for the absorption factor (A and F give 0.193).

The results of Amaldi and Fermi are given in Table XXIII. For Rh, the effective width obtained is 2.5 times larger than $(2\pi)^{\frac{1}{2}}$ times the Doppler width (cf. (523)), therefore the natural width is certainly much larger than the Doppler width and may be calculated from (522). For Ag, Γ_{eff} as obtained from the activity is exactly equal to $(2\pi)^{\frac{1}{2}}$ times the calculated Doppler width. Therefore the natural width must be very small compared to the Doppler width and cannot be calculated from Γ_{eff}. (Possibly measurements at lower temperatures would allow a determination of the natural width.) For iodine, the situation is the same as for Rh. However, the natural width of 15 volts obtained seems implausibly high (§87). The reason for this is, we believe, that the activity in iodine is due to a large number of resonance levels of various energies, the observed Γ_{eff} being the sum of the widths of all these levels. This assumption is plausible because the "effective" resonance energy of I lies very high (cf. the remarks after Table XXII).[40b]

Thus the measurement in Rh seems to be the only reliable figure for the natural width of nuclear resonance levels which is available at present. According to the considerations in §58, this width must be interpreted as the "γ-ray width," i.e., as giving the probability of γ-ray emission from the excited state of the compound nucleus. The lifetime of this state is $\hbar/\Gamma=1.0\cdot10^{-14}$ sec.

[40b] It is also confirmed by measurements of Ruben and Libby (R17a) (cf. reference 35a).

TABLE XXIV. *Test of the one-level formula.*

SUBSTANCE	Rh	Ag	I
Y	1.16	0.72	~ 3
ξ from (535)	7.5	13	3.3
ξ from Table XXIII	2	≪ 1	40

C. Test of the one-level formula

If we assume that the cross section is, from zero neutron energy up to the first resonance level E_r, determined by the first level alone, we have (cf. (430))

$$\sigma=\sigma_0\left(\frac{E_r}{E}\right)^{\frac{1}{2}}\frac{1}{1+(E_r-E/\frac{1}{2}\Gamma)^2}. \qquad (530)$$

We may apply this formula to obtain the ratio of the activities due to thermal and resonance neutrons. For resonance neutrons we have, according to (518), (524b), (525):

$$B_r=ql_0\tfrac{1}{2}\pi\sigma_0 w\Gamma/ME_r, \qquad (531)$$

independent of the ratio of Doppler width to natural width (cf. (518)). For thermal neutrons, we have simply to insert the average cross section (averaged over the Maxwell distribution) into (527). Assuming that the first resonance energy E_r is large compared to both the level width Γ and the thermal energy $E\approx kT$, we find

$$B_{\text{th}}=\sqrt{3q'L(w/M)}\,\tfrac{1}{4}\sigma_0\Gamma^2E_r^{-\frac{1}{2}}(E^{-\frac{1}{2}})_{\text{Av}}, \qquad (532)$$

where $(E^{-\frac{1}{2}})_{\text{Av}}$ is to be calculated as the average over the thermal neutrons striking the absorber. Since the number of such neutrons in the energy interval dE is proportional to the number of Maxwell neutrons of energy E, $M(E)dE$, times the velocity $v\sim E^{\frac{1}{2}}$, we have

$$(E^{-\frac{1}{2}})_{\text{Av}}=\frac{\int M(E)dE}{\int M(E)E^{\frac{1}{2}}dE}=\frac{1}{(E^{\frac{1}{2}})_a}, \qquad (532a)$$

where $(E^{\frac{1}{2}})_a$ is the average of $E^{\frac{1}{2}}$ over the Maxwell distribution,

$$(E^{\frac{1}{2}})_a=(4/\pi)^{\frac{1}{2}}(kT)^{\frac{1}{2}}. \qquad (532b)$$

Thus (532) becomes

$$B_{\text{th}}=\tfrac{1}{8}(3\pi)^{\frac{1}{2}}q'L(w/M)\sigma_0\Gamma^2E_r^{-\frac{1}{2}}(kT)^{-\frac{1}{2}}. \quad (533)$$

As in Section B, we integrate the activities (531) (533) over all the paraffin and obtain for the ratio

$$Y=\frac{\int B_{\text{res}}d\Omega}{\int B_{\text{th}}d\Omega}=4\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\frac{l_0}{L}\frac{(E_rkT)^{\frac{1}{2}}}{\Gamma}. \quad (534)$$

With the values $l_0=0.9$ cm, $L=2.1$ cm, this gives

$$\frac{\Gamma}{(E_rkT)^{\frac{1}{2}}}=\frac{1.75}{Y}. \quad (534a)$$

In contrast to (528), Γ is the *natural width* of the line. The denominator on the left-hand side is proportional to the Doppler width; therefore we have (cf. (513a), (514b))

$$\xi=\frac{\Gamma}{\Delta}=2\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\left(\frac{M}{m}\right)^{\frac{1}{2}}\frac{l_0}{L}\frac{1}{Y}=\frac{0.88A^{\frac{1}{2}}}{Y}. \quad (535)$$

For the three elements listed in Table XXIII, the results are given in Table XXIV.

The result is fairly satisfactory only for Rh. This substance is just the one for which the one-level formula may be expected to hold best because it has a resonance level at very low energy which in addition is fairly broad and can therefore be expected to give a large contribution to the cross section at thermal energies.

For Ag, there is no agreement at all. From the one-level formula we should conclude a natural width much larger than the Doppler width while actually we found that the natural width is negligible compared to the Doppler width. This means that the cross section for thermal neutrons is much larger (Y much smaller) than would be expected from the one-level formula. This again is understandable because the Ag level is extremely narrow and thus cannot be expected to give a large contribution to the thermal cross section. Probably the "B" level (cf. Table XXII) gives a much larger contribution.

For I, the data on the thermal activity which are available are rather scarce, the value of Y being estimated. The situation is here the opposite of that with Ag, showing that the Γ obtained in Table XXIII is probably grossly overestimated. We suggested already in view of the very high value of Γ obtained in Table XXIII that the "resonance" activity in iodine is presumably due to a number (n, say) of levels. Assuming this, (528) will give n times the average value of Γ_{eff}/E_r. Furthermore, if it is assumed that the interference terms between the contributions of various levels to the thermal cross section average out, B_{th} will be given by a formula of the type (532) with n times the average of $\sigma_0\Gamma^2E_r^{-\frac{1}{2}}$ over all the levels replacing the simple $\sigma_0\Gamma^2E_r^{-\frac{1}{2}}$. Therefore, with this assumption of negligible interference, formula (534) should give the average value of ξ for the resonance levels contributing, while from Table XXIII we should get approximately n times this average value. This would mean that about $40/3.3=12$ levels contribute to the iodine activity which seems plausible. Furthermore, it seems reasonable to assume that the ξ derived from (534) may still be sufficiently in error to make the actual average of ξ of order unity or even smaller, corresponding to natural widths of the individual iodine levels of the same order as for Rh.

D. Widths from the one-level formula

From the considerations of Section C, we may conclude that the use of the one-level formula is justified for substances which have a not too narrow resonance level at low energies. For these substances, the one-level formula may be used to calculate the width from the *absorption* coefficients for resonance and thermal neutrons, in the absence of measurements of the total activity.

For this purpose, we compare the cross section for self-indication (520) with the average cross section for thermal neutrons which we calculate from the one-level formula (530). We have from (520) (520b) (521a)

$$\sigma_s=\tfrac{1}{2}\pi\sigma_0\Gamma/\Gamma_{\text{eff}} \quad (536)$$

and from (530) (532a) (532b)

$$\sigma_{\text{th}}=\tfrac{1}{8}\pi^{\frac{1}{2}}\sigma_0\Gamma^2/E_r^{\frac{1}{2}}(kT)^{\frac{1}{2}}. \quad (537)$$

The ratio of the absorption coefficients is thus

$$\frac{K_r}{K_{\text{th}}}\frac{\sigma_s}{\sigma_{\text{th}}}=4\pi^{\frac{1}{2}}\frac{E_r^{\frac{1}{2}}(kT)^{\frac{1}{2}}}{\Gamma\Gamma_{\text{eff}}}, \quad (537a)$$

a formula which could have been obtained also from (528) and (534). If the Doppler width is small compared to the natural width (which can

TABLE XXV. *Widths from one-level formula.*

SUBSTANCE	Rh 44″	In 54′	Ir	Au	Ag 22″ (Group A)	Ag 22″ (B)
K_r (cm²/g)	19	3.8	1.0	4	20	0.3
K_{th} (cm²/g)	0.7	0.6	1.0	0.25	0.2	0.2
E_r (cf. Table XXII)	1.3	1.6	1.9	3.0	3.0	5.5
Γ (from 538, volts)	0.13	0.34	1.0	0.34	0.14	1.7
$\xi=\Gamma/\Delta$	4	9	30	9	2.7	23
Calculated Y (cf. (535))	2.2	1.0	0.4	1.3	3.4	0.4
Observed Y	1.16	—	—	—	0.72	~0.3

be checked from the result) we have, according to (522):

$$\Gamma=2\pi^{-\frac{1}{2}}E_r^{\frac{3}{4}}(kT)^{\frac{1}{4}}(K_{th}/K_r)^{\frac{1}{2}}. \tag{538}$$

The condition $\Gamma\gg\Delta$ is equivalent to (cf. (513a))

$$\frac{K_r}{K_{th}}\ll\frac{M}{\pi^{\frac{1}{2}}m}\left(\frac{E_r}{kT}\right)^{\frac{1}{2}}. \tag{538a}$$

This is fulfilled for all cases except for the A group of Ag.

The results for various elements are given in Table XXV. The absorption coefficients are due to Amaldi and Fermi.[40c] The absorption of thermal neutrons must be corrected if more than one activity is produced in the same substance, as in Ag and In. In Ag, the ratio of the long period to the short period activity (initial activity after infinite irradiation) is about 1:3 (private communication of Professor Goudsmit) so that $\frac{3}{4}$ of the absorption coefficient is due to the short period. In Indium, the ratio of the activities is $In_{54'}:In_{16''}=3:2$ (cf. A11) so that 60 percent of the absorption of thermal neutrons is due to the long period.[41] These figures have been used in Table XXV. All other substances have only one strong activity. Where possible, the absorption coefficients measured with detector of the same substance as the absorber were taken. kT is taken as 0.026 volt.

As can be seen from the Table XXV, the natural width Γ turns out to be much larger than the Doppler width in all cases except for the A group of Ag. Thus the widths derived ought to be fairly reliable for all cases except the Ag A group. A check is provided by calculating Y from the Γ obtained in Table XXV, and comparing it to the observed ratio Y of resonance to thermal activity wherever the latter is known. The calculated values of Y seem to be of the right order of magnitude (about unity) for all cases except Ag A, the observations giving approximately equal activity from resonance and thermal neutrons for most substances (A11), (G17). A reasonable agreement (as good as for Rh) is also obtained for the B group of Ag, supporting the hypothesis that the B level gives the main contribution to the cross section at thermal energies.

The widths found vary from 0.13 to 1.6 volts which seems reasonable. Broad levels are generally connected with small absorption coefficients for the resonance neutrons. One of the broader levels is that of Ir. The width given for it may, however, not be quite reliable because the absorption coefficient was measured by Amaldi and Fermi with Rh and In as detectors, and the resonance energy of Ir seems to be somewhat higher than for these two substances. This makes it likely that K_r has been underestimated, and therefore Γ overestimated, in Table XXV.

E. Overlapping of levels

In some cases, energy levels of various substances are known to overlap. This is true for Rh, In and Ir ("D group") and for Au and the A level of Ag (A group). In these cases, an estimate of the difference of the resonance energies of the substances in question may be obtained by measuring the mutual absorption coefficients of the resonance radiation and comparing them with the absorption coefficients of each substance for its own radiation. The width must be known for at least one of the substances.

If the resonance energies are E_1 and E_2, the widths Γ_1 and Γ_2, the absorption coefficient of substance 1 for the activity of substance 2 is given by

$$K_{12}=K_1(0)\int\frac{dE}{[1+(E-E_1/\frac{1}{2}\Gamma_1)^2][1+(E-E_2/\frac{1}{2}\Gamma_2)^2]}\Big/\int\frac{dE}{1+(E-E_2/\frac{1}{2}\Gamma_2)^2}, \tag{539a}$$

[40c] Except for Rh, which is due to v. Halban and Preiswerk and to Manley, Goldsmith and Schwinger.

[41] Since we are here only interested in the ratio of the absorption coefficients, the abundance of the isotope responsible for the absorption is immaterial.

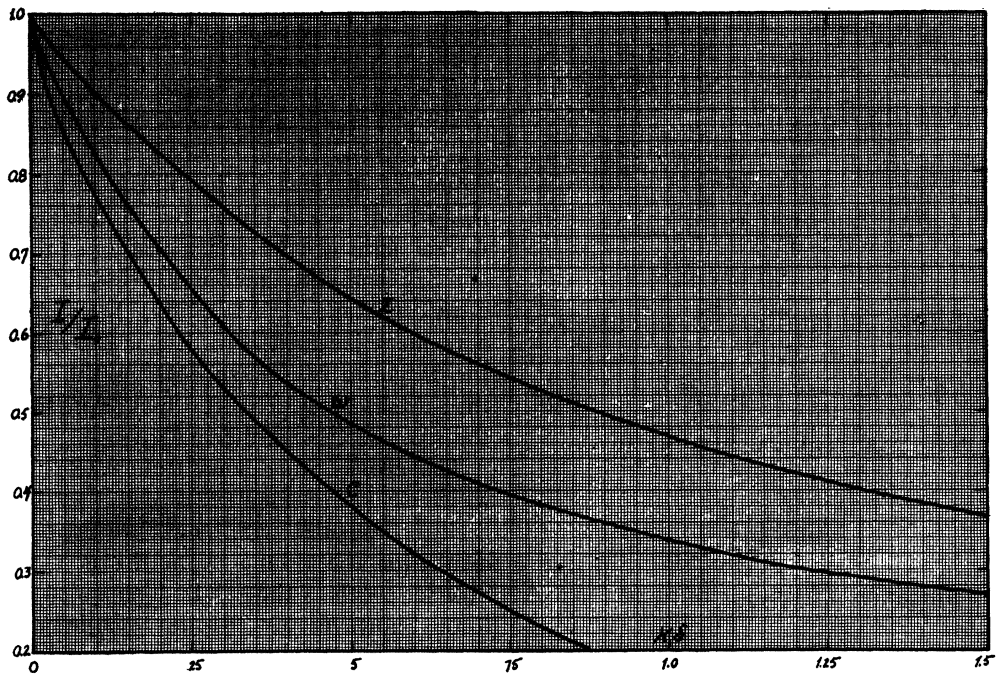

FIG. 16a. Absorption of resonance radiation. Abscissa: $K_r\delta$ where K_r is the average absorption coefficient of the resonance radiation for self-indication and small absorber thickness δ. (The figures on the abscissa should read 0.25, 0.5, 0.75, 1.0 etc.) Ordinate: Transmitted intensity. Curve z corresponds to a collimated incident beam of neutrons, curve w to an angular distribution $\cos\vartheta+\sqrt{3}\cos^2\vartheta$. For comparison, curve c gives the absorption for the same angular distribution with *constant* absorption coefficient, according to Amaldi and Fermi (A11, Fig. 2).

where $K_1(0)$ is the absorption coefficient of substance 1 at its resonance. If the natural width is large compared to the Doppler width, $K_1(0)$ is twice as large as the absorption coefficient for self-indication, K_{11}. Then we obtain by evaluating the integrals

$$\frac{K_{12}}{K_{11}}=\frac{2\Gamma_1(\Gamma_1+\Gamma_2)}{4(E_1-E_2)^2+(\Gamma_1+\Gamma_2)^2}. \tag{539}$$

For $E_1=E_2$ and $\Gamma_1=\Gamma_2$, the right-hand side reduces to unity.

If all four absorption coefficients K_{11} K_{12} K_{21} K_{22} are known, the ratio of the widths can be determined from (539)

$$\Gamma_1/\Gamma_2=K_{12}K_{22}/K_{11}K_{21}. \tag{540}$$

The right-hand side is the product of the two absorption coefficients with the second substance as indicator, divided by the product with the first as indicator. This determination of the ratio of the widths does not make use of the one-level formula except in the immediate neighborhood of the resonance where it is well justified. The difference of the resonance energies becomes, according to (539)

$$|E_1-E_2|=\tfrac{1}{2}(\Gamma_1+\Gamma_2)\times\left[\frac{2K_{11}K_{22}}{K_{12}K_{22}+K_{11}K_{21}}-1\right]^{\frac{1}{2}}. \tag{541}$$

The only case in which all necessary data are available,[41a] is that of Rh (44″) and In (54′). According to Amaldi and Fermi,

$$K_{\mathrm{Rh\,Rh}}=2.0,\quad K_{\mathrm{Rh\,In}}=1.6,\quad K_{\mathrm{In\,Rh}}=3.0,\quad K_{\mathrm{In\,In}}=3.8, \tag{541a}$$

showing that for a given absorber the absorption is noticeably stronger when the detector is of the same material as the absorber. The ratio of the widths becomes according to (540)

$$\Gamma_{\mathrm{In}}/\Gamma_{\mathrm{Rh}}=2.0\cdot 3.0/1.6\cdot 3.8=1.0 \tag{540a}$$

in good agreement with the result found from

[41a] *Note added in proof:* With the large change in $K_{\mathrm{Rh\,Rh}}$ (from 2 to 19) necessary according to Manley, Goldsmith and Schwinger (M2a), the data given in (541a) and the conclusions drawn below seem very unreliable.

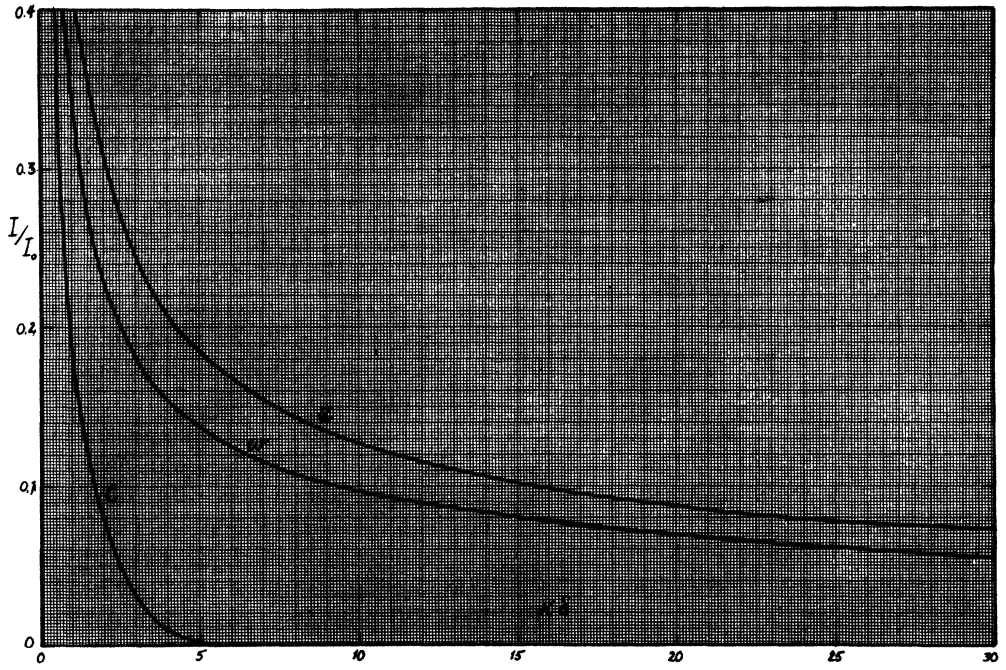

FIG. 16b.

the one-level formula with the use of $K_{\mathrm{Rh\,Rh}}=2.0$, *viz.*

$$\Gamma_{\mathrm{In}}/\Gamma_{\mathrm{Rh}}=0.34/0.40=0.8. \qquad (540b)$$

The difference in the resonance energies is (cf. (531))

$$|E_{\mathrm{In}}-E_{\mathrm{Rh}}|=\tfrac{1}{2}(\Gamma_1+\Gamma_2)\left[\frac{2\cdot 2.0\cdot 3.8}{2.0\cdot 3.0+1.6\cdot 3.8}-1\right]^{\frac{1}{2}}$$
$$=\tfrac{1}{2}(\Gamma_1+\Gamma_2)\cdot 0.50. \qquad (541b)$$

With $\Gamma_{\mathrm{In}}=0.34$ ev (Table XXV) and $\Gamma_{\mathrm{In}}=\Gamma_{\mathrm{Rh}}$, we have 0.17 volt for the energy difference. The sign cannot be determined from these experiments, but it can be found from the absorption coefficients of other substances, measured with a Rh and In detector. According to Amaldi and Fermi, the absorption coefficient of Hg is larger (0.07 cm²/g) with Rh than with In (0.04 cm²/g) as detector, while the ratio is inverted for Ag (0.06 cm²/g with Rh, 0.09 cm²/g with In detector) and Au (0.03 and 0.04). Since Hg has a resonance level at very low energies (cf. end of §60, §61G), these measurements suggest that the In level lies higher, in agreement with the conclusions from the boron absorption measurement (Table XXII).

Measurements of the overlapping of levels would be useful (1) to check the level widths obtained by other methods, (2) to check the positions of the levels obtained from the boron method, (3) to test whether the activity of a given substance is due to a single level or to several—the latter conclusion being suggested in any case where there are serious discrepancies in the tests (1) and (2). The "mutual absorption" method may be one of the most sensitive for indicating more than one level.

F. Absorption of resonance radiation

Thus far, we have only considered the absorption coefficient for very thin absorbers. In the case of thick absorbers, the transmitted intensity is given by

$$I(\delta)=\frac{I_0}{\pi}\int\frac{dx}{1+x^2}e^{-2K_r\delta/(1+x^2)}, \qquad (542)$$

where δ is the thickness of the absorber in g/cm², K_r the absorption coefficient for thin absorbers (=one-half the absorption for exact resonance, cf. (520)) and $x=(E-E_r)/\frac{1}{2}\Gamma$. It has been assumed that the Doppler width is negligible compared to the natural width which is true in every

case investigated thus far, except for the A group of Ag.

The transmission coefficient $z(K_r\delta)=I/I_0$ is given in Fig. 16, a and b, as a function of $y=K_r\delta$. We have

$$z(y)=e^{-y}J_0(iy), \tag{542a}$$

where J_0 is the Bessel function of zero order. For small and large thickness, this gives:

$$z=1-y+\tfrac{3}{2}y^2-(5/2)y^3+\cdots \quad \text{if } y=K_r\delta\ll 1, \tag{542b}$$

$$z=1/(2\pi y)^{\frac{1}{2}} \quad \text{if } y=K_r\delta\gg 1. \tag{542c}$$

According to (542c), the transmitted intensity decreases very slowly with thickness for thick absorbers.

If the incident neutrons are distributed according to the law $\cos\vartheta+\sqrt{3}\cos^2\vartheta$ (cf. (497)), the transmitted intensity is

$$w(y)=\int_0^{\pi/2}\sin\vartheta d\vartheta(1+3^{\frac{1}{2}}\cos\vartheta)\times[1+\tfrac{1}{2}3^{\frac{1}{2}}]^{-1}z(y/\cos\vartheta), \tag{543}$$

where the detector is again assumed to be thin. w is also given in Fig. 16. For small and large argument we have

$$w=1-0.31\,y(\log y+2.3)+0.40\,y^2(\log y+3.75)+\cdots \quad (y\ll 1) \tag{543a}$$

$$w=0.29\,y^{-\frac{1}{2}} \quad (y\gg 1). \tag{543b}$$

Further corrections must be applied for thick detectors.

Investigations of the whole absorption curve of a given detector for its own resonance radiation might show deviations from the laws (542a) to (543b). Such deviations, in particular a still slower decrease of the intensity at large thicknesses than that indicated by (543b), would indicate that the substance possesses more than one resonance level.

G. Very low resonance levels

As already mentioned at the end of §60, several elements have very low resonance levels, near the thermal region, whose energy cannot be measured directly. A method for determining the position and width of these levels is to measure the deviation from the $1/v$ law in two different energy regions.

The possible experiments are:

(1) In the region of very low energies: Determination of the dependence of the capture cross section on temperature.

(2) Near kT: Wheel experiment of Rasetti and collaborators (R4, R6) described in §58B.

(3) Measurement of the "absorption limit" of the substance in question by determining the boron absorption coefficient of the neutrons penetrating through the substance, using boron or lithium indication (L32).

(4) Determination of the absorption coefficient of the substance for D group neutrons (Rh detector) and comparison with the absorption coefficient for thermal neutrons (A11).

The first two methods have the advantage that they make use of the "one-level dispersion formula" only over a very limited energy region (0 to kT). On the other hand, the experiments are not very accurate; in case (2) because of the smallness of the effect, and in case (1) because it is very difficult to cool neutrons efficiently to a temperature below room temperature (F19) so that it is not clear what to insert for the effective temperature of the neutrons. Method (4) depends on the validity of the one-level formula in the substance investigated up to 1 volt (resonance level of Rh); moreover it is experimentally rather difficult to measure the very small absorption of the substance for the D group. However, such measurements have been carried out by Amaldi and Fermi for Cd and Hg; the absorption coefficients for D neutrons are 0.05 and 0.07 cm^2/g, respectively, as compared to 14 and 1 cm^2/g for thermal neutrons. Method (3) is, in principle capable of higher accuracy than (1) and (4), but thus far only preliminary data on Cd are available (F33; Hoffman, Livingston and Bethe, H34, L32).

The experiments are conveniently discussed in terms of the capture *probability* p, i.e., the cross section times the neutron velocity, or the absorption coefficient times $E^{\frac{1}{2}}$. We have

$$p(E)\sim\sigma E^{\frac{1}{2}}\sim[(E-E_r)^2+(\tfrac{1}{2}\Gamma)^2]^{-1}. \tag{544}$$

The measurements (1) (3) and (4) give ratios of the capture probabilities at different energies. Besides the absorption coefficient of Cd at

thermal energy, $p_{th}=15$ cm²/g, we shall use the "absorption limit" of Cd. This absorption limit has been measured (L32) by determining the boron absorption coefficient of the neutrons just able to penetrate $\frac{1}{2}$ mm of Cd. We may estimate that these neutrons have in Cd an absorption coefficient of about

$$p_L \approx 2 \text{ cm}^2/\text{g}.$$

The boron absorption coefficient was found to be 9.5 cm²/g as compared to 36 cm²/g for thermal neutrons. This means that the energy of the absorption limit is

$$E_L = 0.026 \cdot (36/9.5)^2 = 0.37 \text{ volt}. \quad (544a)$$

According to (544), we have

$$\frac{p_{th}}{p_L} = \frac{15}{2}\left(\frac{kT}{E_L}\right)^{\frac{1}{2}} = \frac{(E_L-E_r)^2+\frac{1}{4}\Gamma^2}{(E_r-kT)^2+\frac{1}{4}\Gamma^2}. \quad (544b)$$

Besides, it follows from (544a) that Cd absorbs not only thermal neutrons but also neutrons of considerably higher energy.

Rasetti's wheel experiment gives the logarithmic derivative of the capture probability with respect to the energy, at thermal energy. This derivative can conveniently be expressed by the quantity

$$\epsilon = (dE/d \log p)_{E=kT}, \quad (545)$$

which has the dimension of an energy. If $\Delta p/p$ is the relative change in the absorption coefficient when the direction of rotation of the wheel is reversed, then

$$\epsilon = 2muv_a \sin\vartheta \; p/\Delta p, \quad (545a)$$

where $v_a=(2kT/m)^{\frac{1}{2}}$ is the Maxwell velocity of the neutrons (2200 meter/sec. for room temperature), u the velocity of the edge of the wheel (140 m/sec. in the experiments of Rasetti and collaborators) and ϑ the angle between neutron beam and the normal to the wheel (65° in experiments). The observed value of $\Delta p/p$ was 6.3 percent. Therefore, for Cd,

$$\epsilon = 0.09_5 \text{ volt}. \quad (545b)$$

Theoretically, according to (544) (545)

$$\epsilon = \frac{(E_r-kT)^2+(\frac{1}{2}\Gamma)^2}{2(E_r-kT)}. \quad (546)$$

ϵ is positive (and therefore $E_r>kT$) if the capture probability increases with increasing neutron energy, i.e., if the absorption coefficient is larger when the wheel moves opposite to the neutron beam.

From (544b) and (546) we have

$$E_r = \frac{E_L^2+2\lambda kT-(kT)^2}{2(E_L+\lambda-kT)}, \quad (546a)$$

where

$$\lambda = \epsilon(p_{th}-p_L)/p_L. \quad (546b)$$

We find for Cd

$$E_r = 0.16 \text{ volt}, \quad (547)$$

$$\Gamma = 0.17 \text{ volt}. \quad (547a)$$

This result is, unfortunately, very sensitive to small errors in $\Delta p/p$ (cf. (545a)) and in the "absorption limit" E_L of Cd. E.g., if ϵ is changed to 0.11 volt, the result is $E_r=0.15_7$ and $\Gamma=0.21_6$ volt. Therefore, all that can be said at the moment is that E_r and Γ are of the same order of magnitude and are both about 0.15 volt. Similar conclusions are reached by using the absorption coefficient for D neutrons (B15).

The cross section as a function of the energy is given in Fig. 11 (§58), for $\Gamma/E_r=\sqrt{2}$. The curve shows a very broad region of approximately constant cross section which, for Cd, will extend from about thermal energy (0.026 volt) to about 0.2 volt.

Other elements with very low resonance levels are Sm and Hg. The main evidence comes, in both cases, from the temperature dependence of the cross section. Sm, like Cd, shows hardly any change of the absorption coefficient upon cooling; its resonance level should therefore also lie slightly above thermal energies. For Hg, the increase of the absorption at low temperatures is slightly over normal which suggests a resonance level at *negative* energy. The position of this level may be estimated from the ratio of the absorption coefficients for D neutrons and thermal neutrons which is about 1 : 15 (A11) as compared to 1 : 7 for boron and other substances obeying the $1/v$ law. A value $E_r \approx -0.3$ volt would fit the experimental data.

§62. The Neutron Width and the Absolute Cross Section (B15, A11, D23)

The cross section at exact resonance is, according to (430b),

$$\sigma_0=\frac{1.30_5\cdot 10^{-18}}{E_r}\left(1\pm\frac{1}{2i+1}\right)\frac{\Gamma_N}{\Gamma}, \qquad (548)$$

where Γ_N is the "neutron width" of the resonance level, Γ the total (γ-ray) width, E_r the resonance energy, i the spin of the capturing nucleus. The $\pm$ sign stands according to whether the angular momentum of the compound state (resonance level) is $i+\frac{1}{2}$ or $i-\frac{1}{2}$; since we have no way of knowing this angular momentum, we neglect the term $1/(2i+1)$. σ_0 is measured in cm^2, E_r in volts. The absorption coefficient, in cm^2/g, at exact resonance becomes then

$$K_0=\frac{\sigma_0 La}{A}=7.87\cdot 10^5\frac{a}{A}\frac{\Gamma_N}{\Gamma E_r}, \qquad (548a)$$

where A is the atomic weight of the capturing element, a the abundance of the capturing isotope in the element in question, and L Avogadro's number.

The *observed* resonance absorption coefficient with self-indication is connected to K_0 by [cf. (520) (521a)]

$$K_r=(\pi\Gamma/2\Gamma_{\text{eff}})K_0, \qquad (548b)$$

so that

$$\Gamma_N=\frac{E_r A K_r\Gamma_{\text{eff}}}{1.23\cdot 10^6 a}. \qquad (549)$$

If the total activity is known, Γ_{eff} can be expressed by (528c) so that:

$$\Gamma_N=\frac{E_r^2 A}{1.23\cdot 10^6 a}\frac{3^{\frac{1}{2}}L}{l_0}K_{\text{th}}Y. \qquad (549a)$$

In this formula, only directly observable quantities occur. However, in many cases the resonance absorption coefficient is known when the total activity is unknown. In this case, (549) may be used directly, and Γ_{eff} may be replaced by $\pi\Gamma$ if it is known that the natural width is large compared to the Doppler width (cf. §61A). The results are given in Table XXVI. For Cd, the absorption coefficient at resonance is about 1.7 times that at thermal energies, if the data (547, 547a) are used. The isotope responsible for the large cross section is probably one of odd mass (cf. B13, p. 340). Both the odd isotopes (111, 113) have about the same abundance (10 and 12 percent) so that $A/a\sim 1000$.

The neutron widths observed vary mostly between 0.5 and 2 millivolt, except for iodine for which the data are uncertain because of the superposition of several levels (§60A, 61C). The neutron widths are very fundamental for nuclear theory because they are the only existing data on the particle width for heavy nuclei. They form the basis of the calculation of the nuclear radius from the α-decay (§67) and also the basis for the estimate of the "sticking probability" (§54) of nuclear particles for which they give a value of 0.01 to 1.

According to the theory (§58), the neutron widths should *cet. par.* increase as the velocity of the neutrons so that $\Gamma_N'=\Gamma_N E_r^{-\frac{1}{2}}$ should, in the average, be independent of E_r. In fact, when the very uncertain value for Cd is excluded no trend of $\Gamma_N E_r^{-\frac{1}{2}}$ with the energy can be observed in the data in Table XXVI, all values lying in the very small region between 0.4 and $1.2\cdot 10^{-3}$ volt$^{\frac{1}{2}}$.

Inserting $\Gamma_N' E_r^{\frac{1}{2}}$ for the neutron width Γ_N, we find for the cross section at exact resonance (548)

$$\sigma_{0r}\sim\Gamma_N'/\Gamma E_r^{\frac{1}{2}} \qquad (550)$$

Table XXVI. *Neutron widths.*

Substance	Rh 44″	In 54′	Ir	Au	Ag(A)	Ag(B)	I	Cd
E_r	1.3	1.6	1.9	3.0	3.0	5.5	140	0.16
A/a	100	120	190	200	220	220	130	~1000
K_r (cm^2/g)	19	3.8	1.0	4	20	0.3	0.38	~25
Γ_{eff}* (volts)	0.22*	1.1	3.1	1.1	0.11*	5.3	~1†	0.55
Γ_N (millivolts)	0.44	0.65	0.9	2.1	1.2	1.6	6	1.8
$1000\Gamma_N E_r^{-\frac{1}{2}}$ (volt$^{\frac{1}{2}}$)	0.4	0.5	0.6	1.2	0.7	0.7	0.5	4.5
$1000\Gamma_N/\Gamma$	6	1.8	0.9	6	>30	0.9	?	12

* When Γ_{eff} was obtained from the activity, the value is marked by an asterisk. The other values of Γ_{eff} are from the one-level formula (§61D)
† $(2\pi)^{\frac{1}{2}}\Delta$ (cf. Table XXIII). It is assumed that the width of the iodine levels is primarily determined by Doppler broadening.

and for the total activity (531)

$$B_r \sim \Gamma_N' E_r^{-\frac{1}{2}}. \qquad (550a)$$

Therefore, if we assume Γ_N' and Γ to be in the average independent of the number r of the resonance level, the absorption coefficient for neutrons at exact resonance becomes inversely proportional to the square root of the resonance energy E_r. This "$1/v$ law" holds also for the *average* absorption coefficient, averaged over an energy region large compared to the spacing D of the resonance levels which is (cf. §56)

$$\sigma_a = (\sigma_{0r}\pi\Gamma/2D)_{Av} \sim \Gamma_N'/DE^{\frac{1}{2}}. \qquad (550b)$$

From (550a) it follows that the contribution of a resonance level E_r to the total activity is, under otherwise equal conditions, proportional to $E_r^{-\frac{1}{2}}$, a fact which we used in interpreting Table XXII.

In the last row, we have tabulated the ratio of the neutron width to the γ-ray width. This ratio determines the ratio of resonance scattering to capture (§58, 63). The ratio is seen to be extremely small, of the order of one-thousandth, except for Ag (A group). Only for this latter element may we thus expect to find observable resonance scattering, and even there the predicted magnitude of the scattering is very low ($\sigma_{sc}/\sigma_{capt} > 1/30$).

The values given for the neutron width should be very reliable for Ag (A group) and Rh 44″ (D group) because in these cases the total activity has been determined. The other neutron widths (In, Ir, Au and Ag B) depend on the applicability of the one-level dispersion formula to thermal neutrons (§61D), which, however, is probably justified in these cases. The data for iodine and cadmium are rough estimates.

The cross sections observed at resonance are very large, ranging up to about 10^{-20} cm². They are usually much larger than at thermal energies so that many resonance groups may be recognized by the fact that the activity transmitted by cadmium is more strongly absorbed in the substance of the detector itself than the part of the activity which is absorbed in cadmium. Table XXVII gives the known cross sections for thermal neutrons, resonance neutrons and "fast" neutrons (one to several MV). The measurements of thermal and fast neutrons are mostly due to Dunning, Pegram, Fink and Mitchell (D23) supplemented by data of Amaldi and Fermi (A11), Hevesy and Levi (H31) and of Powers, Fink and Pegram (P14). The resonance data are due to Amaldi and Fermi. The figures refer to the cross section at *exact* resonance which is twice the observable cross section for thin absorbers.

TABLE XXVII. *Neutron cross sections (in 10^{-24} cm²).*

Element	Main Process	Cross Section Therm.	Res.	Fast	Element	Main Process	Cross Section Therm.	Res.	Fast
1 H	E	40*	14*	1.7	47 Ag	22″A†	80	14000	
1 D	E	4.0		1.7		22″B†		200	
3 Li⁶	α	900†		1.8		2.3′†	30		
4 Be	E	5.3		1.9		total	55		
5 B¹⁰	α	3000†		1.6	48 Cd	C	2600	(~80000†)	
6 C	E	4.1		1.3	49 In	54′†	$85/a_1$	$1500/a_1$	
7 N	$E+p$	11.3		1.8		16″†	$55/a_2$		
8 O	E	3.3				total	140		
9 F		2.5			50 Sn		4.0		4.3
11 Na		4.2			51 Sb		8		
12 Mg		3.5			52 Te		8		
13 Al		1.5		2.4	53 I		9.4	150	4.6
14 Si		2.5			56 Ba		140		
15 P		14.7			57 La		80		
16 S		1.4		2.6	58 Ce		~25		
17 Cl		39			59 Pr		25		
19 K		8.2			60 Nd		220		
20 Ca		11.0			62 Sm	C	4300		
21 Sc		25?			63 Eu		2500		
22 Ti		11.9			64 Gd		30000		
23 V		10			65 Tb		<1000		
24 Cr		4.9			66 Dy		700		
25 Mn		14.3			67 Ho		150		
26 Fe	$E+C$	12.0		3.0	68 Er		120		
27 Co		35			69 Tu		500		
28 Ni	$E+C$	15.4			70 Yb		90		
29 Cu		7.5		3.2	71 Lu		~400		
30 Zn		4.7		3.3	73 Ta		27		
32 Ge		~75			74 W		23		5.3
33 As		9			75 Re		90		
34 Se		19			76 Os		27		
35 Br		12			77 Ir	C	280	600	
38 Sr		~9			78 Pt		25		
39 Y		30			79 Au	C	90	2500	
40 Zr		17			80 Hg	C	440		5.8
41 Cb		~14			81 Tl		11		
42 Mo		7			82 Pb		9		5.7
44 Ru		12			83 Bi		8		
45 Rh	44″	125	6000		90 Th		32		
46 Pd		10			92 U		43		

* Cf. §59C. The figure for "resonance" refers to slow neutrons above the thermal region.

† Cross section of the isotope responsible for the absorption. In the case of indium, a_1 and a_2 give the abundance of the two isotopes; it is not known which isotope is responsible for each of the two activities. For Ag, the abundance was taken as 50 percent for each of the two isotopes. For Cd, 12 percent was assumed.

C = capture,
E = elastic scattering,
α, p = α and proton emission.

Remarks on Table XXVII:

The *main process* responsible for the observed absorption *of thermal neutrons* is indicated wherever there was direct experimental evidence about it. Thus hydrogen, deuterium, carbon and oxygen are known to capture neutrons only very slightly, showing that the cross section observed is mainly due to elastic scattering (E). For cadmium, it has been observed that the scattering is extremely small compared to the capture (C) (cf. §63). On the other hand, iron and nickel

are known from scattering experiments to have large elastic scattering besides capture (§63). Mainly capture has also been assumed in all cases where the cross section is known to change rapidly with energy and to show marked resonance effects such as Rh, Ag, In, Sm. Generally, capture may be assumed whenever the cross section is large (more than, say, $10\cdot10^{-24}$ cm^2) for not too light nuclei ($Z>10$, say); but in these cases no remark about the main process has been made in the table. Small cross sections are usually due to scattering. α-disintegration is produced by slow neutrons in Li and B, proton disintegration in nitrogen.—For fast neutrons, scattering, α- and proton disintegration occur.

The data about the rare earths are in general not very reliable because of the difficulties in purification. The cross sections given for some of the rare earths may be spurious due to impurities of Gd, Sm or Eu.

§63. Scattering of Slow Neutrons (D23, M17–20, B5)

A

The elastic cross section for neutrons ($s=\frac{1}{2}$), including potential scattering, is in the case when only one resonance level is important:

$$\sigma^N{}_N=\frac{\pi}{2(2i+1)}\left[(2J+1)\left|2R+\lambda\!\!\!^{-}{}_r\frac{\Gamma_N}{E-E_r+\frac{1}{2}i\Gamma}\right|^2+(4i+1-2J)4R^2\right] \quad (551)$$

$$=4\pi R^2+\frac{\pi}{2}\left(1\pm\frac{1}{2i+1}\right)\lambda\!\!\!^{-}{}_r\Gamma_N\frac{4R(E-E_r)+\lambda\!\!\!^{-}{}_r\Gamma_N}{(E-E_r)^2+\frac{1}{4}\Gamma^2}.$$

Here R is the (effective) nuclear radius, $\lambda\!\!\!^{-}{}_r$ the wave-length at exact resonance, E_r the resonance energy, Γ_N the neutron width and Γ the total width of the resonance level. In the first line, the first term in the square bracket gives the effect of resonance and potential scattering for angular momentum J, the second term is the potential scattering for the other value of the angular momentum possible by selection rules, *viz.* $2i-J$, for which there is no resonance scattering. In the last line of (551), the first term represents the potential scattering, the second the resonance scattering and the interference term between the two.

The behavior of the cross section (551) near the resonance level E_r will be governed by the ratio

$$\rho=\lambda\!\!\!^{-}{}_r\Gamma_N/R\Gamma. \quad (552)$$

This ratio determines the ratio of the cross section at exact resonance to that of the potential scattering alone,

$$\sigma_r/\sigma_p=1+\tfrac{1}{2}(1\pm1/(2i+1))\rho^2. \quad (552a)$$

According to Table XXVI, the ratio Γ_N/Γ in (552) is of the order 10^{-3} for most resonance levels observed, at resonance energies of a few volts, corresponding to $\lambda\!\!\!^{-}{}_r\sim3\cdot10^{-10}$ cm. With $R\sim10^{-12}$ cm, this gives values of ρ smaller than unity. This means that the resonance scattering will, for the resonance levels listed in Table XXVI, not differ markedly from the potential scattering. Only for cases like Ag A (cf. Table XXVI) where Γ_N/Γ is about 30 times larger, may we get an appreciable increase of the scattering cross section at resonance, to 100 or more times the potential scattering.

Introducing the abbreviation ρ, and writing

$$x=(E-E_r)/\tfrac{1}{2}\Gamma, \quad \sigma_p=4\pi R^2, \quad (553a)$$

we have

$$\frac{\sigma^N{}_N}{\sigma_p}=1+\tfrac{1}{2}\left(1\pm\frac{1}{2i+1}\right)\rho\frac{\rho+2x}{1+x^2}. \quad (553)$$

This cross section has a minimum and a maximum near the resonance energy E_r. These extrema occur at

$$x_{\min}{}^{\max}=-\tfrac{1}{2}\rho\pm(\tfrac{1}{4}\rho^2+1)^{\frac{1}{2}} \quad (553b)$$

and have the values

$$\sigma_{\min}{}^{\max}/\sigma_p=1+\tfrac{1}{2}(1\pm1/(2i+1))\rho[\pm(\tfrac{1}{4}\rho^2+1)^{\frac{1}{2}}+\tfrac{1}{2}\rho]. \quad (554)$$

If $\rho\ll1$, the resonance will have no appreciable effect on the scattering. We have a minimum at $E=E_r-\frac{1}{2}\Gamma$ and a maximum at $E_r+\frac{1}{2}\Gamma$, but they will differ only by the relative amount $\rho(1\pm1/(2i+1))$. Essentially we have simply the potential scattering $\sigma_p=4\pi R^2$. Even for $\rho\gg1$, this potential scattering will be the main term if the energy E is far from resonance. In almost all elements, the scattering observed for thermal neutrons will be largely potential scattering.

If ρ is large, the minimum of the cross section occurs at

$$x_{\min}=-\rho; \quad E_{\min}=E_r-\Gamma_N\lambda\!\!\!^{-}{}_r/2R \quad (555a)$$

and has the value

$$\sigma_{\min}=\tfrac{1}{2}\sigma_p(1\mp1/(2i+1)). \quad (555)$$

The maximum occurs practically exactly at E_r, its value is given by (552a).

However, as already mentioned, the resonance scattering will in general be negligible compared to the potential scattering, except in such extraordinary cases as the A level of Ag (22″) where the total width is unusually small and therefore the neutron width a large percentage of the total.

Thus, as a rule, we can consider the scattering cross section as *independent* of the neutron energy.

B

The measurement of the scattering offers some experimental difficulties, mainly because it must be made sure that the observed scattering is actually due to the scatterer investigated. Ordinarily, when scattering is measured, the detector is shielded against the direct neutron beam by absorbing materials. Even then, there will be some activity induced in the detector even without the scatterer in position, because there is always some "stray" scattering material around. It is, in this case, not always sufficient simply to subtract the activity without scatterer from that measured with the scatterer in place, because of the possibility of double scattering from the stray material and the scatterer investigated. All substances containing hydrogen (paraffin, water, wood, etc.) are, of course, particularly dangerous in this respect.

The dependence of the scattering on energy may be measured[42] by using suitable detectors, such as Ag and Rh, and absorbers, such as Cd or the detector material itself. In such experiments, the change of neutron energy in the scattering process must be considered; but this represents a negligible correction unless the scattering cross section depends sensitively on energy in the region investigated, i.e., in all cases except the resonance scattering.

For the resonance scattering, we have to find out whether the change of the neutron energy in the scattering process will throw the neutrons out of the resonance region so that they can no longer be detected by a detector of the same material as the scatterer. For backward scattering by an atom of weight A, the decrease in the netron energy is $4E_r/A$ where E_r is the original neutron energy. The energy loss in the scattering will be immaterial if it is smaller than the effective width of the neutron level:

$$\Gamma_{\text{eff}} \gg 4E_r/A. \tag{556}$$

This is fulfilled for all neutron levels listed in Table XXVI, except for the A level of Ag, for which the two quantities are of the same order. But only in this last case is the investigation of the resonance scattering of interest, because in all other cases the resonance scattering is expected not to differ appreciably from the ordinary potential scattering (cf. above). It is generally true that the resonance scattering is large only if Γ is small (cf. (552)), but that small Γ means, at the same time, that the condition (556) will in general not be fulfilled. Thus, whenever the resonance scattering is large, the neutrons will be thrown off resonance by the scattering process itself, so that the scattered intensity can no longer be measured by a detector of the same material as the scatterer.

[42] In these experiments, even greater care has to be taken to avoid secondary scattering, especially by substances containing hydrogen. Firstly, the scattering of hydrogen is known to depend sensitively on energy (chemical bond, cf. §59C), and secondly, the scattering by hydrogen means a considerable energy loss. Thus neutrons of high energy may be scattered by the primary scatterer, then scattered again by the secondary scatterer and slowed down to the resonance energy of the detector. Then the detector would measure the scattering cross section of the scatterer for a much higher energy than the detector's resonance region.

C

We want to discuss the actual evaluation of scattering experiments for the case of backward scattering, assuming there is no secondary scattering material. For simplicity, we shall treat the problem in one dimension, and shall later discuss the three dimensional problem in some special cases.

Let $f_+(x)$ be the number of neutrons moving to the right (away from the source) at a depth x from the entrance plane of the scatterer, and f_- the number moving to the left. Then of the neutrons moving to the right, a fraction a will be absorbed (captured) per unit path, and a fraction $\frac{1}{2}s$ will be scattered back where a is the absorption and s the scattering coefficient. ($\frac{1}{2}s$ enters because, with isotopic scattering, half of the scattered neutrons will still move forward.) In this way, we find the differential equations

$$\begin{aligned} df_+/dx &= -(a+\tfrac{1}{2}s)f_+ + \tfrac{1}{2}sf_-, \\ -df_-/dx &= -(a+\tfrac{1}{2}s)f_- + \tfrac{1}{2}sf_+. \end{aligned} \tag{557}$$

The solution is

$$\begin{aligned} f_+ &= \alpha s e^{-\lambda x} + \beta s e^{\lambda x}, \\ f_- &= (2a+s-2\lambda)\alpha e^{-\lambda x} + (2a+s+2\lambda)\beta e^{\lambda x} \end{aligned} \tag{557a}$$

with

$$\lambda = (s+a)^{\frac{1}{2}}a^{\frac{1}{2}}. \tag{557b}$$

α and β are two integration constants. If the thickness of the absorber is d, we must have

$$\beta = -\alpha e^{-2\lambda d}(2a+s-2\lambda)/(2a+s+2\lambda). \tag{557c}$$

The intensity of the back scattering (reflection coefficient) is

$$R = \frac{f_-(0)}{f_+(0)} = \frac{s(1-e^{-2\lambda d})}{2a+s+2\lambda-(2a+s-2\lambda)e^{-2\lambda d}}. \tag{558}$$

For small thickness, this reduces to the well-known expression

$$R=\tfrac{1}{2}sd. \tag{558a}$$

For large thicknesses, we have

$$R=\frac{s}{2a+s+2\lambda}=\frac{2a+s}{s}-2\left(\frac{a}{s}\frac{a+s}{s}\right)^{\frac{1}{2}}, \tag{559}$$

a formula first derived by Bayley and Goudsmit (B5). Thick scatterers are, in general, more convenient especially for strong neutron absorbers, of which extremely thin layers would have to be taken in order to make (558a) valid. According to (559), the back scattering from thick scatterers gives directly the ratio of scattering to absorption (capture) cross section $\sigma_s/\sigma_a=s/a$. If $\sigma_s\gg\sigma_a$ (good scatterers), (559) reduces to[43]

$$R=1-2(\sigma_a/\sigma_s)^{\frac{1}{2}} \qquad \sigma_a\ll\sigma_s, \tag{559a}$$

whereas for small scattering cross section

$$R=\sigma_s/4\sigma_a \qquad \sigma_a\gg\sigma_s. \tag{559b}$$

These last two formulae may easily be checked by a three-dimensional calculation. The case of small absorption was treated in §59F for paraffin. For the reflection coefficient (albedo) we found (cf. (503)) exactly the formula (559a) [N in (503) is defined (cf. 484) as σ_s/σ_a]. For strong absorption, the observed back scattering will depend on the angular distribution of the incident neutrons. We may treat the cases

(a) of a collimated beam striking the scatterer perpendicularly

(b) of an angular distribution $f(\vartheta)\sim\cos\vartheta+\frac{3}{2}c\cos^2\vartheta$ of the incident neutrons (c a constant). In case a, the number of neutrons arriving at a depth x, is e^{-ax}. Of these, a fraction $\frac{1}{2}s\sin\vartheta d\vartheta$ is scattered into the direction ϑ. The probability that the scattered neutrons can escape from the scatterer, is $e^{-ax/\cos\vartheta}$. Thus we have

$$R=\tfrac{1}{2}s\int_0^{\pi/2}\sin\vartheta d\vartheta\int_0^{\infty}e^{-ax(1+1/\cos\vartheta)}dx,$$
$$=(s/2a)(1-\log 2)=0.153s/a. \tag{560}$$

In case (b), a somewhat longer calculation gives

$$R=\frac{\frac{2}{3}(1-\log 2)+3c/16}{1+c}\frac{s}{a}=\frac{0.205+0.187c}{1+c}\frac{s}{a}, \tag{560a}$$

so that R is approximately equal to $\sigma_s/5\sigma_a$, in close but not exact agreement with the "one dimensional" formula (559b).

D

Actual *experimental data* on the scattering are very scarce. Measurements have been made by Dunning, Pegram, Fink and D. P. Mitchell (D23) and by A. C. G. Mitchell, Murphy, Langer and Whitaker (M17–20). Dunning and collaborators measured the scattering from cadmium and found that the cross section must be less than 1 percent of the capture cross section. This was of great importance in disproving the one-body theory of neutron phenomena (cf. §57), but is in agreement with the present many-body theory. According to Table XXVI, we expect the resonance scattering at thermal energies to be about 0.45 percent of the capture cross section (with a considerable uncertainty), while the cross section due to potential scattering may be estimated as 6 to $8\cdot10^{-24}$ cm^2 which is 0.2–0.3 percent of the capture cross section.

Mitchell and his collaborators showed that the scattering of thermal neutrons is very small from Ag and Cd but large from C, Mg, Al, S, Fe, Ni, Cu, Zn, Sn, Pb and Bi, with Cr, Mn and Hg giving medium values. This is to be expected from the magnitude of the total absorption cross sections listed in Table XXVII. Assuming a scattering cross section of a few times 10^{-24} in each case, and attributing the rest of the observed absorption to capture, we find that the cross sections of absorption and scattering will be comparable for all the substances for which "large scattering" was found in the experiments but that the scattering should be negligible compared to the capture in Ag and Cd. A quantitative interpretation of the experiments of Mitchell and Murphy seems not possible because the scatterer was placed on top of a paraffin block so that the scattered neutrons could again be scattered by the paraffin, then again by the substance etc., thus increasing greatly the observed scattering coefficient (B5).

§64. Disintegration by Slow Neutrons with Emission of Charged Particles (A8, C12)

Disintegrations by slow neutrons with emission of charged particles have been found in three cases:

$$Li^6+n^1=H^3+He^4, \qquad I$$

$$B^{10}+n^1=Li^7+He^4, \qquad II$$

$$N^{14}+n^1=C^{14}+H^1. \qquad III$$

Probably the process

[43] In this case, measurements of the back scattering will serve to determine the *capture* rather than the scattering cross section, the scattering cross section being practically equal to the total cross section measured in absorption experiments.

$$B^{10}+n^1=Be^{10}+H^1 \qquad IV$$

is also possible. All other processes giving protons are energetically impossible with slow neutrons (§57D), although they are caused with great probability by fast neutrons. Other reactions giving α-particles also seem energetically impossible, on the basis of the atomic masses, for all elements up to phosphorus except for the rare isotopes Ne^{21} and Si^{29} (Table LXI). Even for these, the energy of the emitted α-particle would be so small (1.0 and 1.7 MV, respectively) that it would have a very small probability of escaping through the potential barrier.

The processes are observed through the ionization produced by the charged particle emitted, *viz.* H^3, He^4 and H^1 in the three processes mentioned. They are used as sensitive detectors for slow neutrons, their absorption coefficient following the $1/v$ law (§58B). The absorption in boron is used as an energy gauge for slow neutrons (§60A).

In the reactions I and II above, the particles produced are rather fast, *viz.* 4.6 MV and 3.0 MV, respectively (total kinetic energy of both particles). They will therefore go over the top of the Coulomb potential barrier.[44] Therefore it is reasonable to assume that the "widths" Γ_Q corresponding to the emission of these charged particles are of the same order as for neutrons of the same energy. With this assumption and assuming the width to be proportional to the velocity, we have then for the neutron width

$$\gamma_N(E)=c(E/E_Q)^{\frac{1}{2}}\Gamma_Q, \qquad (561)$$

where E is the neutron energy, E_Q the particle energy and c a constant of order unity. Using the one-level formula (262), we have therefore:

$$\sigma^N{}_Q=\tfrac{1}{2}\pi\lambda^2\left(1\pm\frac{1}{2i+1}\right)\frac{\gamma_N\Gamma_Q}{(E-E_r)^2+\frac{1}{4}\Gamma^2}, \qquad (561a)$$

$$\approx 2\pi c\lambda\lambda'\frac{\Gamma^2}{4(E-E_r)^2+\Gamma^2}, \qquad (561b)$$

where λ' is the wave-length of a neutron of energy E_Q. Γ has been put equal to Γ_Q, and $1/(2i+1)$ has been neglected. With $E_Q=3.0$ MV and a neutron energy of $kT=0.026$ volt, we have $\lambda'=2.6\cdot10^{-13}$ cm and $\lambda=2.8\cdot10^{-9}$ cm. Therefore, neglecting $E=kT$ compared to E_r, we obtain

$$\sigma^N{}_Q=5\cdot10^{-21}c/[1+4E_r^2/\Gamma^2]. \qquad (562)$$

The observed cross section of B^{10} for thermal neutrons is

$$\sigma^N{}_Q=3.0\cdot10^{-21}\text{ cm}^2 \qquad (562a)$$

(cf. Table XXVII). From a comparison of (562) and (562a) we conclude that the resonance level E_r must be of the same order as Γ, which may, by analogy to other levels of light nuclei, be assumed to be of the order of 100,000 volts. For the compound nucleus B^{11} this means an excited level with an excitation energy approximately equal to the difference in mass between $B^{10}+n^1$ and B^{11}, i.e. (cf. Table LXXIII)

$$B^{10}+n^1-B^{11}=11.5\text{ MV}. \qquad (562b)$$

The cross section of Li^6 for thermal neutrons is smaller, *viz.*

$$0.9\cdot10^{-21}\text{ cm}^2. \qquad (562c)$$

Therefore the lithium resonance level should be somewhat farther away from zero neutron energy than the boron level. The corresponding resonance level of the Li^7 compound nucleus lies at an excitation energy around 7 MV.

In the case III above, the protons emitted are very slow, having an energy of about 0.62 MV (including C^{14} recoil, cf. §102B). The potential barrier of C^{14} for protons is probably (599) about 1.7 MV, i.e. much higher than the proton energy. The penetrability of the potential barrier for protons of 0.62 MV is about (cf. 600, Fig. 18)

$$P=e^{-3\cdot3}=1/27. \qquad (563a)$$

The proton width will then be

$$\Gamma_Q=G_QP, \qquad (563)$$

where P is the penetrability of the barrier and G_Q the proton width without barrier. If $E_r\ll\Gamma$ which is certainly true in our case, the cross section is proportional to Γ_Q (cf. (561a)). The observed total cross section is $11\cdot10^{-24}$ cm^2 (Table XXVII), including the elastic scattering which may be $3\cdot10^{-24}$ cm^2. This leaves $8\cdot10^{-24}$ cm^2 for the cross section of process III, which is just one percent of the cross section of Li^6. This is about what would be expected from the penetrability of the proton of 1/30, and from the fact that even

[44] From the general formula (599), the barriers would be 1.4 and 2.2 MV in the two cases.

G_Q must be expected to be smaller than the Γ_Q for the Li^6 reaction because the width without barrier is known to increase with the energy of the particle. Thus the observed cross section of N^{14} for slow neutrons may be interpreted by assuming a resonance level at about the same neutron energy as in Li^6, probably a few hundred kilovolts.

All the numerical values given in this section should be regarded as estimates only; actually at least the cross sections at two different neutron energies (e.g. zero and a few hundred kilovolts) are necessary to determine width and position of the resonance levels.

§65. Fast Neutrons (B32, B33, W7, D2, E1, A7, K7, K9, L19, F13)

A. Classification of processes

Fast neutrons, of energies of several MV, may interact with nuclei in a great variety of ways, *viz.:*

(*a*) Elastic scattering.

(*b*) Inelastic scattering, the initial nucleus being left in an excited state.

(*c*) Simple capture, leading to the formation of an isotope one unit higher in atomic weight.

(*d*) Disintegration with emission of α-particles.

(*e*) Disintegration with emission of protons.

The processes *c*, *d*, *e* may be observed very easily if the nucleus formed in the transmutation is radioactive. Many instances of such transmutations have been found, particularly by Fermi and his collaborators (F13, cf. §102 of this report). If the resultant nucleus is stable, the processes *d*, *e* may be ascertained by observing the particles emitted in the process itself (α-particles or protons). This is usually done in a cloud chamber (cf. §94).

Process *b* (inelastic scattering) is certainly very probable for all nuclei of medium and high atomic weight. This was first shown by Danysz, Rotblat, Wertenstein and Zyw (D2) and then confirmed by Amaldi, d'Agostino, Fermi, Pontecorvo, Rasetti and Segrè (A7) and by Ehrenberg (E1). These authors investigated the change of the radioactivity induced in various substances due to interposing various "scatterers" between a source of fast neutrons and the detector. Scatterers made of C, SiO_2, Ag, Au and Pb produced a marked increase in the radioactivity induced in Ag and Rh indicators but a decrease in the radioactivity in Al and Si. The two latter detectors are made radioactive by processes *d* and *e* mentioned above (α- and proton emission after the capture of the neutron) which, on energetic grounds, are only possible with fast neutrons. On the other hand, the radioactivities in Ag and Rh are produced by radiative neutron capture which is much more probable with slow neutrons (cf. below). Thus the experiments can simply be interpreted as showing that *all the scatterers investigated are very effective in slowing down the neutrons.*

This would be quite impossible if all collisions between the neutron and the scattering nuclei were elastic, because then the maximum possible decrease in energy in one collision would be only $4/A$ times the initial neutron energy, for a scatterer of atomic weight A. Such a decrease would not be noticeable at all for heavy substances such as Ag, Au and Pb. It might be objected that a large number of elastic collisions, even with a heavy substance, would slow down the neutrons: but with the thicknesses of material used (1–3 cm) and the scattering coefficient for fast neutrons known from other measurements (0.2–0.4 cm^{-1}), ordinarily only one collision can take place. This was also proved directly by Ehrenberg, who found a linear increase of the activity of a Ag detector with the thickness of the Ag scatterer interposed.

The slowing down must thus be ascribed to *inelastic scattering* of the neutrons.[45] The nucleus is excited to some excited state while the neutron loses the corresponding amount of kinetic energy. After the neutron has left, the nucleus will lose its excitation energy by emitting one or several γ-rays. These γ-rays from the "noncapture" excitation of nuclei by fast neutrons have actually been found by Lea (L19) and by Kikuchi, Aoki and Husimi (K7, K9) and their intensity has been measured.

No direct evidence is available for the elastic scattering (process *a*), but it may safely be assumed to occur by analogy with the inelastic scattering.

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[45] Inelastic scattering seems much more likely than "radiative scattering," i.e., a process in which an incident fast neutron produces a slower neutron plus a γ-ray in the field of the nucleus which itself is not excited in the process.

B. Theory

The theoretical treatment of fast neutron effects must, of course, again be based on the general theory developed in §52. If the spacing of the nuclear levels is comparable to the neutron energies used, resonance effects may be observed. It was pointed out by Gamow (G5) that some of the resonance levels to be expected could be predicted from the known resonance levels in α-particle reactions. E.g., in the bombardment of Si^{28} by neutrons, the compound nucleus Si^{29} is formed whose resonance levels are known from the reaction $Mg^{25}+He^4 \rightarrow Si^{29*} \rightarrow Al^{28}+H^1$ (observable through the radioactivity of Al^{28}). A great number of similar examples could be given. However, it is not quite certain whether the neutron resonances thus predicted will actually be observable: The α-particles producing the resonances have energies less than or comparable to the potential barrier, and therefore can only penetrate into the nucleus when they have small orbital momentum (§78) while the neutrons have high energy and no potential barrier and therefore can have high orbital momentum. This means that many more resonance levels will interact with neutrons than with α-particles, which may smooth out the resonances in the neutron case.

For heavier nuclei, the nuclear energy levels will become very closely spaced and it will no longer be possible to define the energy of the incident neutron accurately enough to observe any resonances. All we can do in this case is to observe an *average* value of the cross section, averaged over an energy region large compared to the spacing between the levels as discussed in §56.

Applying formula (407) of §56 to our case, and summing over all states q of the final nucleus, we obtain

$$\sigma^{N0}{}_Q = \pi R^2 \xi \Gamma_Q / \Gamma \tag{564}$$

for the total probability of the production of particle Q by a fast neutron. If we further sum over all possible particles Q which may be produced we find

$$\sigma_{N0} = \pi R^2 \xi. \tag{565}$$

This expression does not contain the potential scattering σ_{pot}. Since the mixed terms due to interference of potential scattering and resonance elastic scattering (cf., e.g., the term $4R(E-E_0)$ in (551)) vanish upon integration over a large energy interval, we have for the total cross section

$$\sigma_{N0}' = \pi R^2 \xi + \sigma_{\text{pot}}. \tag{566}$$

This cross section will determine the total "absorption" of fast neutrons as measured by the decrease in the intensity of a collimated beam of fast neutrons when passing through matter. On the other hand, the elastic scattering may be practically eliminated from the measured "absorption" by surrounding, e.g., the source of fast neutrons with a sphere of the material to be investigated and using a detector which responds only to fast neutrons. Now the elastic scattering will be practically equal to the potential scattering.[46] Therefore, if we eliminate experimentally the elastic scattering, we shall measure the cross section (565), which gives immediately the "sticking probability" ξ.

Experimental data

Quantitative data on fast neutrons are not easy to interpret. The reasons are firstly, that the neutrons obtained from most sources are not monochromatic so that it is not known to which neutron energy the results refer. Secondly, the methods for detecting fast neutrons vary enormously in efficiency with changes in the neutron energy: E.g., the number of recoil protons formed by a neutron in a thick layer of a hydrogenous substance is about proportional to the neutron energy (§94). Endoergic reactions such as $Si^{28}+n^1=Al^{28}+H^1$ are only possible above a certain neutron energy, and their yield will depend on the energy even at higher energies. For most of these reactions, the dependence of the yield on the neutron energy is unknown so that no exact interpretation is possible. The experimental data can thus give only qualitative ideas about the probability of various neutron processes.

C. The cross section for inelastic scattering

The cross section for inelastic scattering may be deduced either from measurements of the γ-rays emitted by the nuclei after excitation by

[46] The resonance part of the elastic scattering is certainly for fast neutrons quite small compared to the inelastic scattering, because there is only one possible final state, the ground state, in the elastic resonance scattering but a great number of final states are possible for the inelastic scattering.

fast neutrons (L19, K7, K9) or from measurements of the total absorption of fast neutrons after elimination of the elastic scattering.

The *production of γ-rays* by fast neutrons was first measured by Lea (L19), using neutrons from a Po+Be source which have in the average energies of the order of 5 MV. He deduced from his experiments a cross section of about 2 or $3\cdot10^{-24}$ cm² for Fe and Pb, and much less for C. He could also show that the γ-rays were in all probability due to inelastic scattering rather than capture of the neutrons. This is confirmed by the experiments on the slowing down of neutrons by heavy nuclei (see below). The cross section may be considered an upper limit because the excited nucleus will, in returning to its ground state, often emit more than one quantum (§90).

Kikuchi, Aoki and Husimi (K7, K9) measured the γ-rays produced by the fast neutrons of 2 MV from a D+D source. They investigated a great number of elements all over the periodic table. The cross section was about $1\cdot10^{-24}$ cm² for Cu and Fe, i.e., slightly less than found by Lea with his faster neutrons. For heavier elements, larger cross sections were found,[47] for light elements, smaller ones. No detectable γ-rays (less than 5 percent of those from Cu) were emitted by any element up to oxygen. This is easily explained by assuming that these light nuclei do not possess any excited states below 2 MV, and can therefore not be excited by 2 MV neutrons. This is in agreement with Lea's result that carbon has a small but definite excitation probability with his faster neutrons. The increase of the excitation probability with increasing weight of the nucleus should partly be due to the fact that there will be more and more energy levels below 2 MV, partly simply to the increase in the nuclear radius (cf. (565)), partly perhaps to an increase in the "sticking probability" ξ of the neutron.

Experiments on the *effective absorption of fast neutrons* will give the cross section for inelastic scattering only if all other processes, such as capture, α- and proton disintegration, are rare compared to the inelastic scattering. This condition is probably fulfilled for not too light nuclei (cf. Section D). As a measure of the intensity of fast neutrons, the activity induced in a silicon detector may be used (D2, E1). The activity is due to the process $Si^{28}+n^1=Al^{28}+H^1$ which requires neutrons of at least 3 MV energy (cf. Table LXIII). Ehrenberg found that this activity was reduced by 23 percent by a Ag cylinder of 15 mm. thickness surrounding the source of fast neutrons (Be+Po). This would correspond to a cross section of about $3\cdot10^{-24}$ cm² for Ag, which is compatible with the γ-ray evidence of Lea and Kikuchi.

The *amount of energy lost* by a fast neutron in exciting a nucleus in the Ag scatterer may be estimated experimentally from the increase in the activity induced in a detector, which responds preferably to slower neutrons (e.g. Ag or Rh). According to the experiments (D2, E1) the radioactivity induced in Ag increases about twice as much as that in Si decreases. Thus, if we assume that the decrease in the Si activity measures the number of inelastically scattered neutrons, the scattered neutrons must be captured three times as easily by Ag as the original ones are. Hence we conclude that the energy loss of a fast neutron in an inelastic collision with a nucleus in the Ag scatterer must be fairly large.

This is in agreement with theoretical expectations. The nucleus possesses much more levels at higher excitation energies and each level has, in the average, the same probability of being the final state in the inelastic collision (§54D). Using the theoretical expressions for the level density of the final nucleus (§53), we find that the average kinetic energy of the scattered neutron will be only of the order of the "temperature" of the residual nucleus (evaporation model, §54E). This temperature is only of the order of 1 MV for heavy nuclei and 10 MV excitation energy (§53, Table XXI) so that fast neutrons lose in the average perhaps 90 percent of their kinetic energy in an inelastic scattering process. Weisskopf (W7) has investigated this problem in detail.

In conclusion, it may be pointed out that the observed total cross sections for fast neutrons (cf. Table XXVII) are of the same order as the cross sections for inelastic scattering. This shows that certainly a very large fraction of the "total cross section" observed is due to inelastic scattering. Elastic (potential) scattering can, accordingly, at best be of the same order as the inelastic scattering.

[47] The exceptionally high value found for Cd (more than 4 times the copper value) is probably due to an admixture of slow neutrons in the beam, giving capture γ-rays.

If we take a value of $3\cdot10^{-24}$ for the total cross section for inelastic scattering, and put $R=10^{-12}$ cm (medium sized nuclei), formula (565) gives a "sticking probability" near unity (cf. §54D, E).

D. Capture of fast neutrons

A quantitative estimate of the capture cross section for fast neutrons is even more difficult to obtain than for the inelastic scattering. Since the capture cross section is known to be extremely large for slow neutrons, a small admixture of slow neutrons in a fast neutron beam may produce larger effects than the fast neutrons themselves. Doubts of this kind seem justified in view of the fact that Fermi, Amaldi, d'Agostino, Rasetti and Segrè (F13) found, in their pioneer work with supposedly fast neutrons, large activities in all those elements which were later found to be strongly activated by slow neutrons, and in no other cases of capture reactions. Now it is inconceivable that the capture of fast neutrons should have anything to do with that of slow ones. Large capture of slow neutrons is, as we know, due to strong low resonance levels which have no influence on fast neutrons.

The *activity* produced by fast neutrons should be of the same order of magnitude for *all* heavier nuclei which become radioactive by neutron capture, which must be true of all nuclei of odd charge, for simple stability reasons[47a] (cf. §10). Thus a measure of the capture probability for fast neutrons may be found in the smaller activities observed by Fermi and collaborators with odd elements. In this way, one may estimate cross sections of the order of 10^{-25} cm^2 or less for the capture of fast neutrons.

Such a figure seems compatible with theoretical considerations. If only scattering and capture are possible, the cross-section for capture is

$$\sigma^N{}_\gamma=\pi R^2\xi\Gamma_\gamma/(\Gamma_N+\Gamma_\gamma), \tag{567}$$

where Γ_γ is the total radiation width and Γ_N the total neutron width. The radiation width will presumably not change very much with the neutron energy (cf. §87); its value is, according to the slow neutron experiments, about 0.1 to 1 volt which agrees very well with the theoretical value obtained in §87 ($\sim\frac{1}{2}$ volt). The neutron width Γ_N may be written as

$$\Gamma_N=\sum_{U_q<E}\Gamma_{Nq}, \tag{568}$$

where the sum goes over all levels of the final nucleus (which is, in this case, identical with the initial nucleus) which can be excited by the incident neutron, i.e., whose excitation energy U_q is less than the kinetic energy E of the incident neutron. Each partial width Γ_{Nq} is, at not too high energies, proportional to the velocity of the neutron which is emitted when the nucleus is left in state q, i.e.,

$$\Gamma_{Nq}=\Gamma_{Nq}'(E-U_q)^{\frac{1}{2}}. \tag{568a}$$

The constant Γ_{Nq}' can be estimated from Table XXVI and is about $\frac{1}{2}$ to $1\cdot10^{-3}$ volt$^{\frac{1}{2}}$. Thus Γ_{Nq} is of the same order as Γ_γ if the neutron energy $E-U_q$ is of the order of 1 MV. Therefore, even if no excitation of the nucleus (i.e., no inelastic scattering) is possible, the neutron width will be larger than the radiation width for neutron energies larger than about 1 MV. Then we may write approximately

$$\Gamma_N/\Gamma_\gamma\approx N(E)E^{\frac{1}{2}}, \tag{569}$$

where $N(E)$ is the number of states of the scattering nucleus with an excitation energy less than E, and E is measured in MV.

Since the number of states of a nucleus increases extremely rapidly with increasing excitation energy, the capture cross section (567) becomes negligibly small as soon as the neutron energy is sufficient to excite many levels of the scattering nucleus. Thus really fast neutrons (of several MV energy) should have extremely small capture probability, and only medium fast neutrons (E of the order of a few hundred kilovolts) will be appreciably captured. It should be possible to confirm this point by experiments with neutrons of definite energy such as those from the H^2+H^2 reaction.

We may thus distinguish altogether four energy regions for neutron capture:

(1) Thermal region.

(2) Resonance region ($\frac{1}{2}$ to, perhaps, 1000 or 10,000 volts).

[47a] Only if the radioactive element produced has a very long life can the apparent activity be smaller than usual.

(3) Medium fast neutrons (1 or 10 to about 500 kv).

(4) Fast neutrons ($>\frac{1}{2}$ MV).

In the region of medium fast neutrons, it will scarcely be possible to investigate the individual resonance levels. Only the general trend of the capture cross-section with energy can be found. This is obtained by averaging the dispersion formula over the resonance levels which gives (cf. (405), $s=\frac{1}{2}$, $J=i$)

$$(\sigma^N{}_\gamma)_a=\pi^2\frac{\hbar^2}{2ME}\frac{\Gamma_{N0a}\Gamma_{\gamma a}}{\Gamma_a D}. \tag{570}$$

The subscript a denotes that an average over the resonance levels near the neutron energy E should be taken, D is the average spacing of these levels. Since the region of medium energy is defined by the fact that the radiation width is larger than the neutron width, we have $\Gamma_a=\Gamma_{\gamma a}$. Furthermore, we put

$$\Gamma_{N0a}=\Gamma_{N0}'E^{\frac{1}{2}}. \tag{570a}$$

Then

$$(\sigma^N{}_\gamma)_a=\pi^2\frac{\hbar^2}{2MD}\frac{\Gamma_{N0}'}{E^{\frac{1}{2}}}=2.03\cdot10^{-18}\frac{\Gamma_{N0}'}{DE^{\frac{1}{2}}}\text{ cm}^2, \tag{571}$$

where D and E are to be measured in volts, Γ_{N0}' in volt$^{\frac{1}{2}}$. If we put for medium heavy nuclei (cf. Table XXVI, and after Table XXII)

$$D=10\text{ volts},\ \Gamma_{N0}=10^{-3}\text{ volts}^{\frac{1}{2}}, \tag{571a}$$

we have

$$(\sigma^N{}_\gamma)_a=2\cdot10^{-22}E^{-\frac{1}{2}}\text{ cm}^2. \tag{572}$$

For $E\sim100$ kv, this is of the order of 10^{-24} cm^2.

For *high energies*, the ratio of radiation width to neutron width may be expressed approximately in terms of the entropy, with the help of the formula (359) for the partial neutron width, (347) for the number of levels of the initial nucleus below the excitation energy E, and (729a) for the radiation width. The result is

$$\Gamma_N/\Gamma_\gamma\sim1\cdot10^6EA^{\frac{1}{3}}\xi e^{S(E)-S(E+Q)}. \tag{573}$$

Here E is the neutron energy in MV, ξ the sticking probability, A the atomic weight of the scattering nucleus, $S(E)$ its entropy corresponding to an excitation energy E, and $S(E+Q)$ the entropy of the compound nucleus whose excitation energy is the kinetic energy of the neutron plus the neutron dissociation energy Q. Table XXVIII gives Γ_N/Γ_γ, i.e., the ratio of inelastic scattering to capture, for various nuclei and neutron energies, for $\xi=1$. Table XXVIII shows the extreme smallness of the capture probability for high neutron energy.

E. α-particle and proton emission

Reactions in which an incident fast neutron causes the emission of an α-particle or a proton are known in great number among the lighter nuclei. The cross sections seem to be of the order 10^{-24} cm^2, as should be expected from the inelastic scattering of neutrons (Section C). As long as there is no potential barrier to prevent the charged particles from escaping from the nucleus, the "proton width" and "α-particle width" should be of the same order of magnitude as the fast neutron width.

In many cases, the α-particles and protons produced have been observed in the cloud chamber (for references cf. §102) and their energies measured. As far as these measurements are reliable, they show that the residual nucleus is left in an excited state as often as not.[48] This is to be expected from our general theory, because the

TABLE XXVIII. *Ratio of the probabilities of inelastic scattering and radiative capture for fast neutrons.*

Neutron Energy	$A=20$	50	100	200
2 MV	$1.6\cdot10^4$	$6\cdot10^3$	$4\cdot10^3$	$4\cdot10^3$
5 MV	10^5	$5\cdot10^4$	$4\cdot10^4$	$4\cdot10^4$
10 MV	$3.5\cdot10^5$	$2\cdot10^5$	$1.8\cdot10^5$	$2\cdot10^5$

[48] With a monochromatic group of incident neutrons, the produced particles should fall into several groups according to the state q of the residual nucleus, the energy of the emitted particle being

$$E_Q=E_P+W_A-W_B(q),$$

where W_A is the energy of the initial nucleus in the ground state, $W_B(q)$ that of the final nucleus B in the excited state q. Unfortunately, the neutron sources available do not give monochromatic neutrons, so that the neutron energy itself has to be determined from the resultant momentum of particle Q and recoil nucleus B. Such determinations are extremely uncertain, giving apparent neutron energies of all orders of magnitude, including many which are much higher than the known maximum energy of the neutrons in the beam (B41). Such experiments can therefore easily lead to spurious excitation levels, or even to results entirely irreconcilable with reasonable expectations (K31).

"partial widths" Γ_{Qq} corresponding to the various states q of the residual nucleus, will be of the same order whether q is an excited or the ground state.

For elements heavier than Zn ($Z=30$), the potential barriers become too high to allow the escaping of charged particles from the nucleus to any appreciable extent (B12). Capture of fast neutrons plus subsequent emission of charged particles has not been observed with elements of higher nuclear charge than 30 (F13) except for thorium and uranium (H7, M15). For this element, as for other natural radioactive elements, the large energy set free by spontaneous α-disintegration, plus the kinetic energy of the neutron, are sufficient to bring the produced α-particle over the potential barrier.

XI. α-Radioactivity

§66. Theory of α-Radioactivity According to the One-Body Model (G6, G7, G8, G10, C32, L4, B42, S10, S11)

The theory of the emission of α-particles by radioactive nuclei was the first successful application of quantum theory to nuclear phenomena. As is well known, the theory was given simultaneously by Condon and Gurney (C32) and by Gamow (G7). Subsequently, a great number of authors (G8, G10, G6, L4, B42, S10, S11) have given alternative, and partly more rigorous, mathematical methods for arriving at the same result. All these methods are based on the one-body model, the α-particle being considered as moving in a certain potential created by the nucleus. They must, accordingly, be modified to take into account the principles of the many-body problem (§67).

The starting point of the one-body theory is to assume a suitable potential between an α-particle and a nucleus of charge[49] Z. If the α-particle is far away from the nucleus, the two particles will repel each other according to the Coulomb law, the potential energy being

$$V=zZe^2/r, \tag{574}$$

where $z=2$ is the charge of the α-particle and r the distance between the α-particle and the center of the nucleus. When the α-particle is inside the nucleus, the Coulomb potential (574) breaks down and is to be replaced by a much lower potential energy. This potential energy may be assumed to be constant over the interior of the nucleus, so that

$$V=V_0 \quad (r<R), \tag{574a}$$

where R is the radius of the nucleus. The exact value of V_0 is of no great importance. It is usually positive but in any case smaller than the kinetic energy E of the α-particles which may be emitted from the radioactive nucleus.

The potential as a function of the distance r is shown in Fig. 17. Its maximum occurs at $r=R$ and has the value

$$B=zZe^2/R. \tag{575}$$

B is often called the top of the potential barrier. It is, for the natural α-emitters, much *larger* than the energy E of the emitted α-particles.

If we consider α-particles of a given energy E, we may divide the whole space into three regions, *viz.:*

(1) The interior of the nucleus, $r<R$: Here, the potential energy V_0 is *less* than the energy E of the α-particle.

(2) The region of the potential barrier, i.e., r between R and

$$r_E=zZe^2/E. \tag{575a}$$

In this region the potential energy is *greater* than the energy of the α-particle.

(3) The outside region, $r>r_E$, in which again $V<E$.

According to classical mechanics, the α-particle could only move in regions 1 or 3. Once it was confined in one of these regions, e.g. inside the nucleus, it would be compelled to stay there forever. In wave mechanics, we have the well-known possibility of penetration through the "forbidden region" 2. This enables a particle originally in the nucleus, to "leak out" and to appear, sooner or later, as a free α-particle outside the nucleus (in region 3). Wave mechanics

[49] Z is, of course, the charge of the nucleus which remains after the emission of the α-particle. The radioactive nucleus itself has therefore the charge $Z+2$.

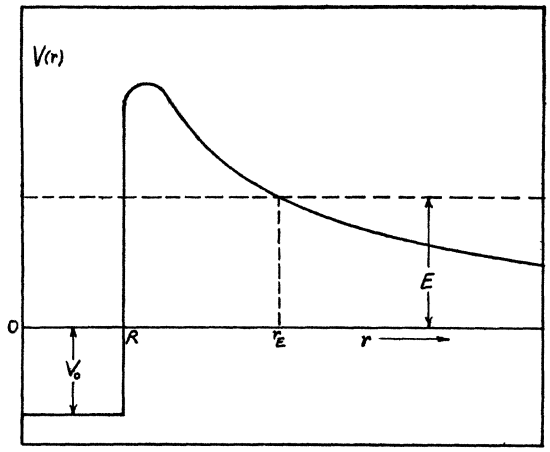

FIG. 17. Potential energy of an α-particle in the field of a nucleus according to Gamow.

allows us to calculate the probability for this to happen in a given time, and thus to compute the lifetime of radioactive nuclei in a perfectly straightforward way.

Hardly any other problem in quantum theory has been treated by so many authors in so many different ways as the radioactive decay. All the proposed methods are, of course, equivalent (insofar as they are correct) but they differ in rigor and complication. We shall give a method which seems about the simplest of the correct ones.

First of all, we shall assume in this section that the wave function of the α-particle is spherically symmetrical, corresponding to an "s state" of the α-particle in the nucleus. Then the wave function of the α-particle can be written

$$\psi = u/r, \tag{576}$$

where u satisfies the equation

$$d^2u/dr^2 + 2M\hbar^{-2}(E-V)u = 0, \tag{576a}$$

V being given by (574, 574a) and E being the experimental α-particle energy. At large distances from the nucleus, V is small (cf. 574) so that the solution of (576a) becomes simply a sine or cosine function, or more generally

$$u = Ae^{ikr} + Be^{-ikr} \tag{577}$$

with

$$k = (2ME)^{\frac{1}{2}}/\hbar \tag{577a}$$

and A and B some constants. The first term represents an outgoing wave, the second a wave converging towards the nucleus, as can be seen by multiplying the time factor $e^{-iEt/\hbar}$ with (577). In the physical problem, an α-particle may leave the nucleus but none comes towards it from outside, and we are therefore interested only in the first term, *viz.*

$$u = Ae^{ikr}. \tag{577b}$$

The constant A determines the number of α-particles leaving the nucleus per unit time:

$$I = v4\pi r^2|\psi|^2 = 4\pi|A|^2 v, \tag{577c}$$

while the *form* of (577b) determines also the form of the wave function at smaller distances from the nucleus.

To obtain the wave functions for smaller values of r, and especially inside the nucleus, it is simplest to use the Wentzel-Kramers-Brillouin (WKB) approximation (W10, K24, B61) in the form due to Kramers (K24). According to this method, the solution of (576a) is approximately

$$u_1(r) = \Phi^{-\frac{1}{2}}(r)\cos\left(\int_{r_E}^{r}\Phi^{\frac{1}{2}}(\rho)d\rho + \tfrac{1}{4}\pi\right) \tag{578}$$

or

$$u_2(r) = \Phi^{-\frac{1}{2}}(r)\cos\left(\int_{r_E}^{r}\Phi^{\frac{1}{2}}(\rho)d\rho - \tfrac{1}{4}\pi\right) \tag{578a}$$

in region (3) ("outside" region) Here

$$\Phi(r) = 2M(E-V)/\hbar^2 \tag{578b}$$

is proportional to the kinetic energy $E-V(r)$ which the α-particle has when at the point r. r_E is the classical distance of closest approach of an α-particle of energy E falling on the nucleus from outside as defined in (575a). The most general solution of the Schrödinger equation "outside" is

$$u = B_1u_1 + B_2u_2. \tag{579}$$

In order to make u have the form (577b) we must choose

$$B_2 = iB_1. \tag{579a}$$

Since for large distances r, the potential energy may be neglected, we have in this case

$$\Phi = 2ME/\hbar^2 = k^2 \tag{580}$$

and, therefore, comparing (577b) and (578):

$$|A| = |B_1|k^{-\frac{1}{2}}. \tag{580a}$$

For r between R and r_E, Φ (cf. 578b) would be negative. The solution u in this region is of an exponential rather than a wave type. The continuations of the functions u_1u_2 (cf. 578, 578a) in the barrier region are:

$$u_1(r) = |\Phi(r)|^{-\frac{1}{2}}\exp\left[+\int_r^{r_E}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right], \tag{581}$$

$$u_2(r) = \tfrac{1}{2}|\Phi(r)|^{-\frac{1}{2}}\exp\left[-\int_r^{r_E}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right], \tag{581a}$$

u_1 decreases from the nucleus outwards, u_2 increases.

Inside the nucleus, we have assumed constant potential energy $V = V_0$ so that the wave function becomes

$$u = c\sin\kappa r, \tag{582}$$

where

$$\kappa = (2M)^{\frac{1}{2}}(E-V_0)^{\frac{1}{2}}/\hbar. \tag{582a}$$

At the boundary of the nucleus ($r = R$), the solution (582) must be identical with the outside solution [(579) to (581a)] as to the value of the wave function and its first derivative. In all practical cases, $u_1(R)$ will be much larger than $u_2(R)$, because u_1 contains an exponential with a large positive exponent, u_2 one with large negative exponent. We may therefore put

$$u(R) = c\sin\kappa R = B_1u_1(R) = B_1|\Phi(R)|^{-\frac{1}{2}}\exp\left[\int_R^{r_E}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right], \tag{583}$$

$$(du/dr)_R = \kappa c\cos\kappa R = -B_1|\Phi(R)|^{\frac{1}{2}}\exp\left[\int_R^{r_E}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right]. \tag{583a}$$

Dividing the lower equation by the upper, we have

$$\kappa \cot \kappa R = -|\Phi(R)|^{\frac{1}{2}}. \tag{583b}$$

Since the Coulomb potential V is very large at the top of the barrier, $\Phi(R)$ is large, and usually much larger than κ. Therefore practically[50]

$$\cot \kappa R = -\infty, \tag{584}$$

$$\kappa R = \pi, \tag{584a}$$

$$\cos \kappa R = -1. \tag{584b}$$

This gives, according to (583a),

$$B_1 = \kappa c |\Phi(R)|^{-\frac{1}{2}} e^{-C} \tag{585}$$

with
$$C = \int_R^{rE} |\Phi(\rho)|^{\frac{1}{2}} d\rho. \tag{585a}$$

The constant c may be obtained by normalizing the internal wave function (582) to unity:

$$4\pi \int u^2 dr = 1, \tag{586}$$

which gives

$$c = (2\pi R)^{-\frac{1}{2}}. \tag{586a}$$

From (577c) (580a) (584a) (585) (586a) we obtain for the number of α-particles emitted per second

$$\lambda' = \frac{2\pi^2 v}{kR^3 |\Phi(R)|^{\frac{1}{2}}} e^{-2C} = \sqrt{2} \frac{\pi^2 \hbar^2 e^{-2C}}{M^{\frac{1}{2}} R^3 (zZe^2 R^{-1} - E)^{\frac{1}{2}}}. \tag{587}$$

The half-life is given by $\tau = (\log 2)/\lambda'$. If we insert[51] for R the value $0.9 \cdot 10^{-12}$ cm derived with this model from experiment in §68, for Z the average nuclear charge of radioactive elements, i.e., about 86, and for E some average energy, let us say, 6 MV, we find

$$\tau = 3.3 \cdot 10^{-21} e^{2C}. \tag{588}$$

The most important factor in this formula is e^{2C}, where C is given by (585a). Inserting Φ from (578b) and V from (574), we have

$$C = (2M)^{\frac{1}{2}} \hbar^{-1} \int_R^{rE} (zZe^2 r^{-1} - E)^{\frac{1}{2}} dr. \tag{588a}$$

The integration is straightforward and yields

$$C = \frac{(2M)^{\frac{1}{2}}}{\hbar} \frac{zZe^2}{E^{\frac{1}{2}}} \left[\arccos \left(\frac{ER}{zZe^2} \right)^{\frac{1}{2}} - \left(\frac{ER}{zZe^2} \right)^{\frac{1}{2}} \left(1 - \frac{ER}{zZe^2} \right)^{\frac{1}{2}} \right] \tag{589}$$

$$= (2zZe^2/\hbar v)[\arccos x^{\frac{1}{2}} - x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}]$$

with (cf. (575)) $x = ER/zZe^2 = E/B.$ (589a)

For a rough approximation, we may neglect E compared to $B = zZe^2/R$, and replace arc cos $x^{\frac{1}{2}}$ by $\frac{1}{2}\pi - x^{\frac{1}{2}}$; then

$$C = \pi zZe^2/\hbar v - (2e/\hbar)(2zZMR)^{\frac{1}{2}}. \tag{589b}$$

The error in C committed when using this approximate formula, is about 1.5 for $E = 6$ MV. Correspondingly, the penetrability e^{-2C} would come out about 20 times too large.

According to (589, 589b) the "Gamow exponent" C is the larger, the higher the nuclear charge Z, the slower the α-particle, and the smaller the nuclear radius R. This is very plausible, since increasing Z increases the height, decreasing v increases the breadth, and decreasing R increases both height and breadth of the potential barrier, and the penetration of the barrier will of course be the more difficult the higher and broader the barrier.

The formula (587) for the lifetime may be written as follows:

$$\tau = \tau_0 / P \tag{590}$$

where P is the transmission coefficient of the potential barrier,

$$P = e^{-2C} \tag{591}$$

with C given in (589), and

$$\tau_0 = 3.3 \cdot 10^{-21} \text{ sec.} \tag{591a}$$

would be the lifetime without potential barrier which is of the same order as the "period of vibration" of the α-particles in the potential of the nucleus.

For the comparison with experiment, cf. §68.

§67. Many-Body Theory of the α-Decay (B14)

According to the many body concept (cf. Chapter IX), the α-particle must not be considered as moving freely in a potential created by the residual nucleus. Only when the α-particle has left the nucleus and is sufficiently far away from it, will it be justified to consider α-particle and residual nucleus as two separate entities. When the α-particle is "inside" the nucleus, its four constituent particles take part in the complicated motion of the compound (initial)

[50] Other solutions, *viz.* $\kappa R = 3\pi$, 5π etc. would correspond to excited α-particle states. Since α-particles obey Bose statistics, there is no exclusion principle for them. In the ground state of the nucleus, all α-particles will therefore be in the lowest level.

[51] Since λ' is not sensitive to the factor multiplying e^{-2C}, we may replace that factor by an approximate value.

nucleus and are in no way distinguishable from the other particles in that nucleus. The α-particle can thus not be considered to "exist" in the radioactive nucleus before its emission but only as formed at the moment of its emission. Accordingly, the probability of α-decay is the product of two factors:

(1) The probability of formation of the α-particle.

(2) The probability of penetration through the potential barrier.

The first named probability will presumably be quite small, making the decay constant of α-radioactive nuclei small even in the absence of a potential barrier. The lifetime of radioactive nuclei would, perhaps, be 10^{-15} sec. (cf. (593a)) if they had no potential barrier. This time, it is true, is much shorter than the actual lifetimes of α-radioactive nuclei, but it is very much longer than the "period of vibration" $\tau_0 \approx 10^{-21}$ sec. (cf. 591a) which would give the lifetime without barrier in the *one*-body model.

Since the lifetime without barrier is much longer in the many-body than in the one-body theory, it is obvious that much lower and narrower barriers are required in order to explain the observed lifetimes of the α-radioactive nuclei. Indeed, the nuclear radii deduced from the many-body concept are about 40 percent larger than those derived from the one-body model (cf. Table XXIX), corresponding to a potential barrier of only seventy percent of the height required in the one-body theory.

In order to derive the nuclear radius from the experimental data, it is, of course, necessary to separate the probability of penetration through the barrier ("2" above) from the probability of formation of the α-particle ("1" above). In order to estimate the latter probability we assume that the *emission of* an *α-particle* by a nucleus would, in the absence of the potential barrier, be just *as probable as that of a neutron of the same energy*. The latter can be obtained from the "neutron widths" measured for *slow* neutrons (§60), if we admit that the neutron width is proportional to the neutron velocity (§54, 56).

According to Table XXVI we have in the average

$$\Gamma_N E_0^{-\frac{1}{2}} \approx 4 \cdot 10^{-4} \text{ volts}^{\frac{1}{2}}, \tag{592}$$

where Γ_N is the neutron width of a resonance level at the neutron energy E_0. Taking $E_0 = 6$ MV which is about the average energy of α-rays, we find

$$\Gamma_N \approx 1 \text{ volt}. \tag{592a}$$

We assume for the α-radioactive nuclei that the "α width in the absence of a potential barrier" would also be

$$G_\alpha = 1 \text{ volt}. \tag{593}$$

The decay constant in the absence of a barrier would then be

$$G_\alpha/\hbar \approx 10^{15} \text{ sec.}^{-1} \tag{593a}$$

and in the presence of the barrier

$$\lambda = \Gamma_\alpha/\hbar = G_\alpha P/\hbar, \tag{594}$$

where P is the penetrability given in (591) Numerically we have;

$$\log_{10} \lambda = \log_{10} P + 15.2 = 15.2 - 0.869C, \tag{594a}$$

where C is given by (589).

§68. Comparison with Experiment (G9, G10)

The relation (588), (589) between the lifetime and the disintegration energy of α-radioactive nuclei was discovered by Geiger and Nuttall (G13) as early as 1911, on a purely empirical basis. With the help of the approximate formula (589b), the relation can be written in the form

$$\log \tau = 2\pi z Ze^2/\hbar v - K, \tag{595}$$

where K is a constant involving, according to (589b), the radius of the radioactive nucleus (or, more accurately, of the nucleus *produced* in the α-disintegration). Assuming the radius to be about the same for all radioactive nuclei, we find *a linear relation between the logarithm of the lifetime and the reciprocal velocity of the α-particle.* Therefore, relatively small changes of the α-particle energy correspond to very large differences in the lifetime: The slowest α-particle, that from thorium, has an energy of 4.3 MV, the fastest (from ThC') one of 8.9 MV, i.e., only a little more than twice as much. The corresponding lifetimes are $2 \cdot 10^{10}$ years and $2 \cdot 10^{-8}$ sec., differing by a factor 10^{25}. This huge variation of the lifetime is correctly represented by (595), with almost the same K throughout, i.e., almost the same nuclear radius.

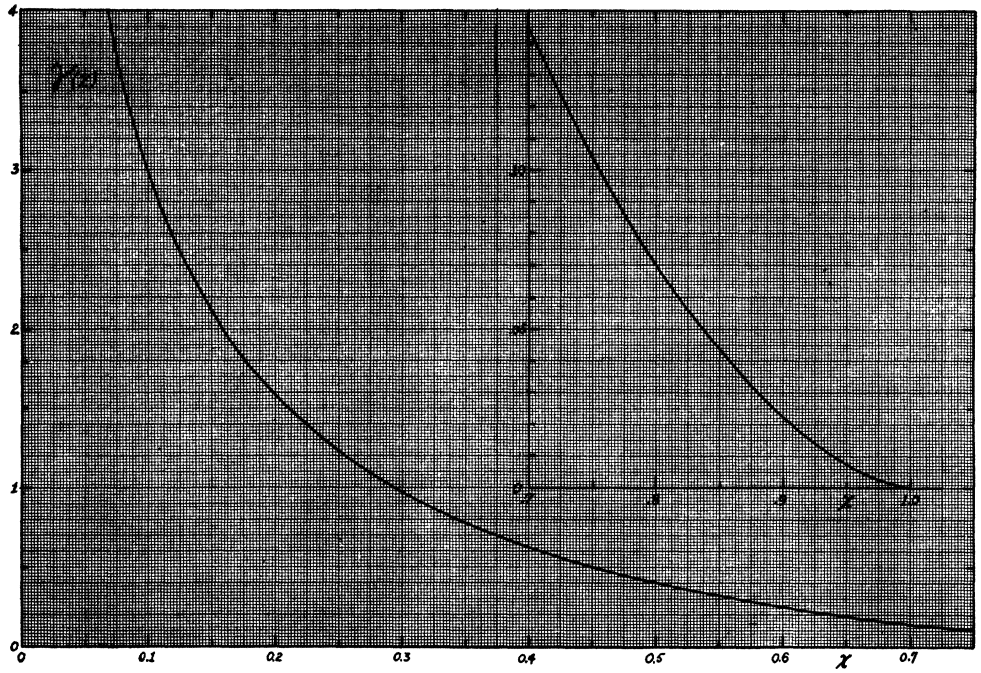

FIG. 18. The function $\gamma(x)$ determining the penetrability of the potential barrier (cf. (600)). x is the ratio of particle energy to height of barrier.

Gamow (G9, G10) has compared all experimental data with the theoretical formula (595). It is most convenient to use the experimental data for the decay constant and the energy of the α-particle, and to compute the nuclear radius from these data, with the help of the theoretical formula. If the formula is correct, the radius R must come out about the same for all radioactive nuclei. This is actually the case for both the one and the many-body theory, as can be seen from Table XXIX. With the one-body method, all radii are between 8.2 and $9.8 \cdot 10^{-13}$ cm, with the many-body model, they vary between 11.3 and $13.2 \cdot 10^{-13}$ cm. Exceptions are, in both cases, the radii for the C and C′ products.

Our values for the nuclear radii from the one-body model are, in the average, about $0.5 \cdot 10^{-13}$ cm larger than Gamow's. The main reason for this is that we have used the exact formula (589) rather than the approximate one (589b), for calculating C.

Much larger are the differences between the radii derived from the one- and the many-body concept. In part A of this report, the radius from the one-body concept was used (cf. §8, 26, 30, etc.). The effect of the introduction of the new, larger radius on calculations concerning the stability of nuclei is considerable. In the present part B the radius derived from the many-body problem has been used throughout.

To judge the accuracy of the nuclear radius obtained from the many-body problem, we investigate the influence of an error in the estimate of G_α (cf. (593)) on the calculated radius. According to (594), a given error in G_α means an equally large error in P, only in the opposite direction. Now, according to (589b), small variations of R and P are connected by the formula

$$\frac{\delta P}{P} = \frac{\delta R}{R} \frac{2e}{\hbar} (2MzZR)^{\frac{1}{2}}. \tag{596}$$

Inserting the numerical values ($R = 12.3 \cdot 10^{-13}$ cm), this gives

$$\delta P/P \approx 50 \delta R/R. \tag{596a}$$

Now our estimate for G_α will probably be correct within a factor 100. This means an

uncertainty of ±10 percent in the nuclear radius. Thus we may write our result for the average radius of the α-radioactive nuclei

$$R=12.3\cdot10^{-13}\text{ cm}\pm10\text{ percent.} \quad (597)$$

From this radius, we may obtain the radii of other nuclei if we assume proportionality between nuclear volume and number of particles in the nucleus:

$$R=r_0A^{\frac{1}{3}}, \quad (598a)$$

where A is the atomic weight. Taking the average of all the radioactive nuclei with the exception of the C products, we find

$$r_0=12.3_5\cdot10^{-13}\cdot219.6^{-\frac{1}{3}}=2.05\cdot10^{-13}\text{ cm.} \quad (598)$$

Accordingly, the height of the potential barriers becomes

$$B=zZe^2/R=0.70zZA^{-\frac{1}{3}}\text{ MV.} \quad (599)$$

In the particular case of light nuclei we may put $A=2Z$ so that

$$B=0.55zZ^{\frac{1}{3}}. \quad (599a)$$

With the values (598) (599) for nuclear radius and height of the potential barrier, the formula for the penetrability of the potential barrier of a given nucleus of charge Z and mass number A by particles of charge z and mass number a takes the simple form (cf. (591a) (589))

$$P=\exp\left[-2g\gamma(E'/B)\right], \quad (600)$$

where

$$g=(2MzZe^2R)^{\frac{1}{2}}/\hbar =0.37_5(ZzAa/A+a)^{\frac{1}{2}}A^{\frac{1}{2}}, \quad (600a)$$

$$\gamma(x)=x^{-\frac{1}{2}}\arccos x^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}, \quad (600b)$$

For M the reduced mass has been inserted. E' is the total kinetic energy (of particle *plus* recoil nucleus) in the system of reference in which the center of gravity of nucleus and particle is at rest. The function $\gamma(x)$ is plotted in Fig. 18 as a function of x. If we neglect in (600a) a compared to A and put then $A=2Z$ we have

$$g=0.42(az)^{\frac{1}{2}}Z^{\frac{1}{3}}. \quad (600c)$$

Values of g for various nuclei are found in Table XXXIII, §70.

In computing Table XXIX, it is of course, necessary to take for v in formula (589) the *relative* velocity of α-particle and recoil nucleus which is $(A+4)/A$ times the velocity of the α-particle itself if A is the atomic weight of the product nucleus. Similarly, the energy E includes the kinetic energy of the recoil nucleus.

Table XXIX includes samarium which is known (L35) to give α-particles of 1.28 cm range, i.e., 2.46 MV energy (the figure in the table includes again the recoil energy). The number of α-particles emitted is about 4 per cm² of Sm_2O_3 per min. (R17). From the range of the α-particles and the stopping power of Sm and O, we may estimate that about $1.5\cdot10^{19}$ Sm atoms per cm² are close enough to the surface so that their α-particles may be detected. Considering the (uniform) angular distribution of the alphas, one in four emitted particles will actually be detected. Assuming that the radioactive isotope of Sm has an abundance of 10 percent, we find for the decay constant $\lambda=4\cdot4/60\cdot1.5\cdot10^{18}=1.7\cdot10^{-19}$ sec.$^{-1}$. The radius turns out to be $9.4\cdot10^{-13}$ cm, i.e., about what would be expected from (598a).

It is of interest to note that no α-particles of less than 4 MV have been observed from nuclei of the radioactive families. Indeed, an energy somewhat below 4 MV would make the penetration of the potential barrier so rare that no radioactivity could be detected. Assuming that one α-particle per hour per cm² constitutes the limit of observability, we may easily calculate the minimum energy of

TABLE XXIX. *α-disintegration.*

Disintegrating Nucleus	Product	Z	Energy E MV	Decay Constant λ in sec.$^{-1}$	Nuclear Radius R in 10^{-13} cm Old	New
Th	MThI	88	4.34	$1.2\cdot10^{-18}$	8.7	11.3
RdTh	ThX	88	5.52	$1.15\cdot10^{-8}$	8.8	12.3
ThX	Thn	86	5.79	$2.20\cdot10^{-6}$	9.0	12.5
Thn	ThA	84	6.40	$1.27\cdot10^{-2}$	9.1	12.3
ThA	ThB	82	6.90	5.0	8.9	12.7
ThC	ThC″	81	6.20	$6.7\cdot10^{-5}$	7.0	10.6
ThC′	ThD	82	8.95	$[4\cdot10^{7}]$*	—	(13.9)
UI	UXI	90	4.15	$5.0\cdot10^{-18}$	9.8	13.2
UII	Io	90	4.76	$(7\cdot10^{-14})$	9.6	13.0
Io	Ra	88	4.67	$2.9\cdot10^{-13}$	9.5	13.1
Ra	Rn	86	4.88	$1.42\cdot10^{-11}$	9.0	12.5
Rn	RaA	84	5.59	$2.10\cdot10^{-6}$	9.0	12.6
RaA	RaB	82	6.11	$3.8\cdot10^{-3}$	9.0	12.5
RaC	RaC″	81	5.61	$2.3\cdot10^{-7}$	7.2	10.9
RaC′	RaD	82	7.83	$7\cdot10^{+4}$	9.4	13.9
RaF	RaG	82	5.40	$5.7\cdot10^{-8}$	8.2	11.5
Pa	Ac	89	5.16	$6.9\cdot10^{-13}$	8.6	11.6
RdAc	AcX	88	6.16	$4.25\cdot10^{-7}$	8.4	11.4
AcX	An	86	5.82	$7.2\cdot10^{-7}$	8.8	12.1
An	AcA	84	6.95	$1.77\cdot10^{-1}$	8.5	12.0
AcA	AcB	82	7.51	$3.5\cdot10^{+2}$	8.8	12.8
AcC	AcC″	81	6.74	$5.3\cdot10^{-3}$	7.3	10.6
AcC′	AcD	82	7.58	$[1\cdot10^{+4}]$*	—	(13.9)
Sm	Nd	60	2.55	$1.7\cdot10^{-19}$	6.7	9.4

* Calculated from radius by means of the theoretical relation. The radius was assumed equal to that of RaC′.

the α-particle necessary to give an observable radioactivity. If we assume a range of the α-particles of 1.5 cm air, and make observations with a pure element, there are about $2 \cdot 10^{19}$ atoms per cm^2 whose α-particles may be detected. Then an activity of one particle per hour per cm^2 is equivalent to a decay constant $\lambda = 1/3600 \cdot 2 \cdot 10^{19}$. According to (594a), this means about $C = 44$. Inserting this into the approximate formula (589b), and assuming the nuclear radius R to be proportional to $Z^{\frac{1}{3}}$, we find for the limit of α-activity

$$E_{\text{min.}} = 3.9Z^2/(44 + 2.41Z^{\frac{1}{3}})^2. \qquad (601)$$

As a function of the nuclear charge, this gives:

Z	10	20	30	40	50	60	70	80	90	
$E_{\text{min.}}$	0.13	0.41	0.77	1.2	1.65	2.15	2.65	3.15	3.7	MV

Thus α-particles of less than 3.7 MV energy would give no observable radioactivity for a nuclear charge of 90, and even for Z as low as 40, an energy of more than 1.2 MV would be required. This fact is important for considerations about the stability of nuclei against α-decay (§8).

Of particular interest among α-emitters is Be^8. This nucleus is known to be formed in some transmutation processes (Chapter XVII) and seems to have a mass just larger than that of two α-particles, the difference being of the order of a few hundred thousand volts. Be^8 should therefore break up spontaneously into two α-particles (This process has not yet been observed.) The lifetime will depend on the "size" of the α-particle, i.e., on the maximum distance R up to which two α-particles exert attractive forces upon each other. For the values $R = 2.5 \cdot 10^{-13}$ and $5 \cdot 10^{-13}$, which are probably too low and too high respectively, and assuming $G_\alpha = 1$ MV for a nucleus as light as Be^8, we find the values for the lifetime given in Table XXX. The lifetime should therefore be exceedingly short, even if the energy of Be^8 is only 50,000 volts higher than that of two α-particles.

The derivation of nuclear radii given here is open to criticism on three points: Firstly, it is questionable whether the "width without barrier" is the same for α-particles as for neutrons; secondly, the width may be larger for the ground state than for a highly excited state; and thirdly the radius obtained is probably the sum of the radii of nucleus and α-particle rather than that of the nucleus alone.

On the first point, we have very little information. It is not known experimentally whether the partial width for the emission of α-particles is in the average larger or smaller than that for the emission of neutrons or protons (leaving out the penetrability of the potential barrier in each case). Some experiments with light nuclei indicate a larger probability for α-emission (e.g., $F^{19} + n^1 \rightarrow N^{16} + He^4$ or $O^{19} + H^1$, cf. §102), others a larger probability for emission of protons (e.g., $Na^{23} + H^2 = Na^{24} + H^1$ or $Ne^{21} + He^4$, cf. §101). The rather scarce evidence on heavy nuclei is compatible with about equal *a priori* probabilities for α and proton emission. Theoretically, it may be argued that the neutron width should be larger because the neutron is an elementary particle, but also that the α-width should be larger because there might be some slight tendency for the preformation of α-particles in nuclei due to their great stability. We are rather inclined to think that, if anything, the neutron width would be larger.

TABLE XXX. *Estimated lifetime of Be^8 as a function of the energy excess E over two α-particles (in seconds).*

E (IN MV)	0.05	0.10	0.20	0.30	0.40
τ for $R = 2.5 \cdot 10^{-13}$ cm	$4 \cdot 10^{-13}$	$3 \cdot 10^{-16}$	$2 \cdot 10^{-18}$	$2 \cdot 10^{-19}$	$5 \cdot 10^{-20}$
τ for $R = 5 \cdot 10^{-13}$ cm	$7 \cdot 10^{-14}$	$4 \cdot 10^{-17}$	$3 \cdot 10^{-19}$	$3 \cdot 10^{-20}$	$1 \cdot 10^{-20}$

As regards the second point, some dependence of the width on the excitation of the nucleus seems plausible. If nuclear states are described in terms of "configurations" of the individual particles (Hartree approximation), the ground state may be a mixture of a smaller number of such configurations (perhaps a hundred) than an excited state (containing perhaps a million configurations). If this is true the partial width of the ground state corresponding to disintegration into a definite state of the residual nucleus, plus an α-particle, might be considerably larger than that of an excited state; this would give some intermediate value for the nuclear radii between the one-body and the many-body result. Perhaps a better estimate of the α-particle width (without barrier) of the ground state will ultimately come from the matrix element of β-disintegrations (Nordheim and Yost, in course of publication).

The third question is: what exactly is meant by "top of potential barrier" and "nuclear radius"? It must be admitted that the α-particle itself has a finite radius, and also that the nuclear forces have a finite range. Therefore it will not be

necessary for a breakdown of the Coulomb law that the center of the α-particle enters the nucleus and not even that the α-particle "touches" the nucleus. The effective radius, measured in the radioactive decay, will be approximately the sum of the radii of the nucleus itself and the α-particle, *plus* a quantity of the order of the range of the nuclear forces. Therefore the radius of the nucleus itself may be appreciably less than the value derived from the radioactive decay constant. Thus, presumably, a smaller radius should be used in calculations of the nuclear properties such as the contribution of the Coulomb energy to the nuclear forces (§9), the semi-empirical formula for nuclear binding energies (§30) and the density of nuclear levels (§53). We have used the larger radius derived from (594a) in all our calculations but mainly because there is at the moment no way of obtaining the radius of the nucleus itself.

For calculating the probability of nuclear disintegrations and especially the transmission coefficients of nuclear potential barriers, it seems to be far preferable to use directly the nuclear radius derived from the many-body picture of radioactive decay. In all transmutation problems, the effective radius will be determined again by the sum of the radii of the initial nucleus and the incident particle, plus a quantity of the order of the nuclear forces. If the particle is an α-particle, the situation will therefore be exactly the same as in the natural α-decay and the nuclear radius may be taken over immediately. For protons and neutrons, the size of the particle itself is zero but the extension of the forces will probably be larger than for the saturated α-particle so that in effect the α-particle radius may again be a good approximation. For deuterons, a larger effective radius may be taken, perhaps $2\cdot 10^{-13}$ may be a fair estimate of the additional term in the effective radius (i.e., of the difference between the radii of α-particle and deuteron).

While these problems about the exact meaning of the nuclear radius remain to be settled, there can be no doubt that in principle the many-body picture must be applied to the natural α-decay just as much as to other nuclear processes. This can most easily be seen by going to the limit of extremely large nuclei (which do not exist in nature), for which all the correction terms mentioned would be negligible.

§69. Complex α-Spectra. Levels of Radioactive Nuclei (R5a, R12, R14, R14a, R22, R23, L23, G6)

In many cases, the α-rays emitted from a radioactive nucleus are not homogeneous but consist of groups differing in energy. These groups have been studied by Rosenblum (R12, R14, R14a) and by Rutherford, Wynn-Williams, Lewis and Bowden (R22, R23, L23) using a magnetic spectrograph. The main results are given in Table XXXI. The elements not quoted, *viz.* UI, UII, Io, Rn, RaA, Po, Pa, AcA, AcC′, Th, Thn, and ThA, emit homogeneous α-rays. The disintegration energy is equal to the kinetic energy of the α-particle times $A/(A-4)$ where A is the atomic weight of the emitting nucleus. The relative numbers of particles in the various groups are given in percent for the spectra denoted as "type I," in particles per million for "type II."

The complex α-spectra fall naturally into two types, I and II. The spectra of type I consist of rather closely spaced lines, the spacing being of the order of 100 kv, of *about equal* intensity, the groups of *lower energy* being usually *less intense.* This type comprises most of the α-spectra. Spectra of type II contain one very strong group and several very *weak groups of* much *higher energy*, the energy differences being of the order of 1 MV and the *ratio of intensities between* 10^4 *and* 10^6. This type is only found for the very short-lived α-emitters ThC′ and RaC′. The interpretation is as follows:

Type I spectra correspond to transitions from the ground state of the initial nucleus to various excited states of the final nucleus. This hypothesis, which was first suggested by Gamow (G6) is confirmed by the internal conversion of the γ-rays emitted by the final nucleus when left in an excited state (§88A). (The energies of the conversion electrons correspond to the electronic energy levels of the atom *produced* in the α-disintegration rather than of the atom emitting the α-ray.) The longest range α-group observed (group 0) must be attributed to a transition to the ground state. The difference in disintegration energy between any group n and group 0 gives the excitation energy of the level in which the final nucleus is left after emission of group n.

The excitation energies derived in this way are listed in the fourth column of Table XXXI. The spacing between adjacent levels is smallest (about 40 kv) for the nucleus produced by α-emission from radioactinium, *viz.* actinium X; in the average over all radioactive elements, the spacing is about 100 kv.

Transitions to excited levels are in general less probable because of the smaller penetrability of the potential barrier for the slow α-particles corresponding to such transitions. From the general formula (589b) we find that the penetrability decreases approximately by a factor

$$P(E)/P(E-\delta E)=\exp\,(170\delta E/E^{\frac{1}{2}}) \quad (602)$$

for a decrease in disintegration energy by δE, where δE and the disintegration energy E itself are measured in MV. For $E=6$ MV, this corresponds to a factor of about 3 for each 100 kv decrease in disintegration energy. This means e.g. that the disintegration probability *without barrier* must be about the same for the groups 1 and 4 in the disintegration of thorium C, since the observed numbers of particles for the two groups are about in the ratio of the expected penetrabilities. On the other hand, groups 7 and 9 of radioactinium have evidently an enormously larger disintegration probability without barrier than group 0 because they contain about as many particles as the latter group although the penetrability is about 30 times smaller for them. The decrease of the penetrability is obviously the explanation for the absence of α-groups of very low disintegration energy, i.e., very high excitation energy of the final nucleus.

The nuclear energy levels obtained from α-groups can be checked by the γ-rays emitted from the product nucleus after the α-disintegration (§88). The most extensive comparison has been made for ThC″, i.e., the nucleus formed by α-decay of ThC. Table XXXII gives the possible combinations between the levels of ThC″ as derived from the α-groups ("calculated" $h\nu$, column 2) and the observed γ-rays (column 3).

TABLE XXXI. *Complex α-spectra.*

Element*	Group No.	Disint. Energy MV	Excit. Energy kv	Relat. Number of Part.
		Type I		
Ra	0	4.879	0	—
	1	4.695	184	—
RaC	0	5.612	0	45
	1	5.550	62	55
RaTh	0	5.517	0	85
	1	5.431	86	15
ThC	0	6.2007	0	27.2
	1	6.1607	40.0	69.8
	2	5.8729	327.8	1.80
	3	5.7283	472.4	0.16
	4	5.7089	491.8	1.10
RaAc†	0	6.159	0	24
	1	6.127	32	2
	2	6.097	62	19
	3	6.075	84	1
	4	6.030	129	3
	5	5.975	184	4
	6	5.921	238	3
	7	5.869	290	22
	8	5.847	312	1
	9	5.822	337	18
	10	5.776	383	3
AcX	0	5.823	0	40
	1	5.709	114	35
	2	5.634	189	18
	3	5.543	280	7
An	0	6.953	0	70
	1	6.683	270	15
	2	6.556	397	11
	3	6.343	610	4
AcC	0	6.739	0	84
	1	6.383	356	16
		Type II		
RaC′	0	7.829	0	10^6
	1	8.437	608	0.43
	2	9.112	1283	0.45
	3	9.242	1412	22
	4	9.493	1664	0.38
	5	9.673	1844	1.35
	6	9.844	2015	0.35
	7	9.968	2139	1.06
	8	10.097	2268	0.36
	9	10.269	2440	1.67
	10	10.342	2513	0.38
	11	10.526	2697	1.12
	12	10.709	2880	0.23
ThC′	0	8.948	0	10^6
	1	9.674	726	34
	2	10.745	1797	190

* Element *emitting* the α-particles.
† Rosenblum, Guillot and Perey (R14a) report 18 instead of 11 α-groups for RaAc and about 10 for AcX. According to the authors, most of these new groups require confirmation as to their exact position and some as to their existence.

TABLE XXXII. *Possible combinations between nuclear levels of ThC″ and observed γ-rays.*

Combination	$h\nu$ (kv) calc.	$h\nu$ (kv) obs.	Designation	Combination	$h\nu$ (kv) calc.	$h\nu$ (kv) obs.	Designation
4→3	19.4	—	—	3→1	432.4	431.7	γ_2
4→2	164.0	—	—	3→0	472.4	470.9	γ_3
4→1	451.8	451.1	γ_1	2→1	287.8	286.9	γ_4
4→0	491.8	—	—	2→0	327.8	326.7	γ_5
3→2	144.6	—	—	1→0	40.0	39.9	γ_6

The agreement is perfect within the experimental accuracy. Of the 10 possible combinations, 6 are "allowed" by optical selection rules. From observations of the internal conversion of the γ-rays it can be shown (§88B) that all the observed γ-rays correspond to quadrupole radiation.

Type II α-spectra are interpreted as being due to various states of the *initial nucleus emitting the α-particle.* Such an emission of α-particles *from* an excited state is only possible if the emission probability is comparable with the probability of the emission of a γ-ray by which the nucleus would go over into a lower quantum state. Since the lifetime of nuclear states against γ-emission is only about 10^{-13} sec. (see below, and §87, 88), α-particle emission from excited states is only possible if the respective α-particles have extremely high energies. Accordingly, the emission of α-rays of longer range than the normal group has only been observed for the nuclei RaC′ and ThC′ for which the normal α-particles are already very energetic.

Neither RaC′ nor ThC′ seems to have any α-spectrum of type I, i.e., any groups of α-particles having energy lower than and intensity comparable to the main group (0). Thus it seems that the final nuclei formed in the disintegrations, *viz.* RaD and ThD, respectively, have no excited levels of importance. The observed long range α-groups should thus give directly the excited levels of the initial nuclei RaC′ and ThC′. The level schemes obtained from the α-groups can again be checked, to a certain extent, by the γ-rays accompanying the α-disintegration. These γ-rays are, in this case, *alternative* to the emission of long range α-particles and not following it.

The number of α-rays in a given long range group is proportional to

$$N_\alpha = F\Gamma_\alpha/(\Gamma_\gamma+\Gamma_\alpha). \qquad (603)$$

Here F is the probability of formation of the excited state from which the group originates, in the preceding disintegration, i.e., in the β-disintegration of RaC and ThC, respectively. Γ_α and Γ_γ are the probabilities of emission of α-rays and γ-rays from the given excited level. The number of γ-rays emitted from the same excited level would be

$$N_\gamma = F\Gamma_\gamma/(\Gamma_\gamma+\Gamma_\alpha). \qquad (603a)$$

By comparing the number of γ-rays and α-rays, we can therefore determine the ratio $\Gamma_\gamma/\Gamma_\alpha$. The results are approximately (cf. §88) $\Gamma_\gamma/\Gamma_\alpha \approx 10^6$, 90, 4000 and 100 for the levels 1 and 3 of RaC′ and the levels 1 and 2 of ThC′, respectively. This means that γ-emission is more probable in all cases which agrees with reasonable theoretical expectations (§88).

The most important application of the observed ratio $\Gamma_\gamma/\Gamma_\alpha$ is to estimate Γ_γ. The probability of emission of α-rays may be estimated by the usual formula $\Gamma_\alpha = G_\alpha e^{-2C}$ where we put the "α-ray width without barrier" G_α again equal to one volt, and calculate the penetrability exponent C from the general formula (589), with the nuclear radius $R = 13.9\cdot 10^{-13}$ cm as observed for the ground state (group 0) of RaC′. Thus we obtain $\hbar/\Gamma_\alpha = 2.7\cdot 10^{-7}$, $9.5\cdot 10^{-9}$, $1.6\cdot 10^{-9}$ and $6\cdot 10^{-11}$ sec., and therefore $\hbar/\Gamma_\gamma = 3$, 800, 4 and $6\cdot 10^{-13}$ sec. for the four levels mentioned. This is the only way available for determining the lifetimes of excited states of radioactive nuclei for γ-radiation. For a discussion of this lifetime, cf. §87, 88C.

Since Γ_γ is always much larger than Γ_α, the number of nuclei *formed* in the excited state is practically equal to the number of emitted γ-quanta. For all the four levels of RaC′ and ThC′ mentioned, the probability of formation is much less than the probability of formation of the ground states of the respective nuclei, *viz.* about 0.4 and 0.0025 for the excited levels no. 1 and 3 of RaC′, and 0.14 and 0.02 for the two excited levels of ThC′. The reason for this small probability of formation is obviously the fact that the probability of β-decay increases with increasing β-energy (§41) and is therefore greater when the β-decay leads to the ground state unless this transition is forbidden.

Complete level schemes using both α- and γ-ray data have been given by Ellis (E2, E3, E4,

E7, E8, E9) for the radioelements ThB, C, C′, C″ and D. From these level schemes, the first evidence for the neutrino hypothesis was derived (§39), since it could be shown that the total energy evolution in the disintegration ThC—C′—D was the same as in the alternative disintegration ThC—C″—D when the maximum energy of the β-rays was considered as the disintegration energy. Ellis (E8, E9) has also given a scheme of levels for RaC′.—In all cases, the relative excitation probabilities of the various levels have been worked out. From these, the average excitation energy can be calculated which may be compared to the total energy evolution in the disintegration as measured calorimetrically. The agreement is fairly good (E2).—Attempts have been made to assign quantum numbers (angular momentum) to the various states, using the results about the character (dipole or quadrupole radiation) of the various γ-rays emitted and the selection rules indicated by the absence of certain γ lines. The results are still questionable.

XII. Scattering of Charged Particles by Nuclei

§70. General Remarks

The scattering of α-particles by nuclei is one of the oldest methods for exploring nuclear fields. More recently, the scattering of protons by protons (§18) gave the most convincing proof for the existence of forces between like particles. The scattering of protons and deuterons by deuterons and α-particles will probably provide an excellent test of our theoretical ideas about the nuclear forces and of the approximate wave functions used for these light nuclei (cf. Chapter III). The scattering by slightly heavier nuclei, from Li to Si or P, has given and will give evidence about nuclear resonance levels. It appears that the scattering of charged particles is very sensitive to small deviations from the Coulomb field and is therefore well suited to investigations of nuclear forces and resonance levels. For this reason, it is regrettable that so little attention has been paid to this field of nuclear physics in more recent years. Of course, it must be admitted that scattering experiments are useful only for the lighter nuclei; for the heavier ones, the Coulomb scattering is too large and masks all other effects even if particles of sufficient energy are available to penetrate through the Coulomb field into the nucleus.

The theoretical treatment of the scattering is, of course, most conveniently carried out in a coordinate system in which the center of mass of scattered particle and scattering nucleus is at rest (C system, §14). If M_1 is the mass of the scattered particle, M_2 that of the nucleus, and ϑ the deflection of the scattered particle in the C system, then the *observed* deflection Θ of the particle in the laboratory coordinate system (R system) is given by

$$\tan\Theta=\frac{M_2\sin\vartheta}{M_1+M_2\cos\vartheta}, \tag{604}$$

provided the scattering nucleus is initially at rest. Conversely, ϑ may be deduced from the observed deflection Θ:

$$\sin(\vartheta-\Theta)=(M_1/M_2)\sin\Theta \tag{604a}$$

or, approximately, for $M_2\gg M_1$:

$$\vartheta=\Theta+(M_1/M_2)\sin\Theta. \tag{604b}$$

The scattering nucleus itself experiences a recoil; the angle between its direction of motion and that of the incident particle is simply

$$\varphi=\tfrac{1}{2}(\pi-\vartheta). \tag{604c}$$

The number of particles scattered through an angle between Θ and $\Theta+d\Theta$ in the ordinary coordinate system, is

$$N(\Theta)\sin\Theta d\Theta=N(\vartheta)\sin\vartheta d\vartheta$$
$$=\frac{N(\vartheta)[M_1\cos\Theta+(M_2^2-M_1^2\sin^2\Theta)^{\frac{1}{2}}]^2}{M_2\qquad(M_2^2-M_1^2\sin^2\Theta)^{\frac{1}{2}}}\sin\Theta d\Theta, \tag{605}$$

where $N(\vartheta)$ is the number scattered per unit solid angle in the C system. For $M_1\ll M_2$ this reduces to:

$$N(\vartheta)[1+2(M_1/M_2)\cos\Theta]\sin\Theta. \tag{605a}$$

The mass to be inserted into the Schrödinger equation is the *reduced mass*

$$M=M_1M_2/(M_1+M_2). \tag{606}$$

The kinetic energy associated with the relative motion of scattered particle and scattering nucleus is

$$E=M_2E_0/(M_1+M_2)=\tfrac{1}{2}Mv^2, \tag{606a}$$

where v is the velocity of the incident particle in the laboratory system and $E_0=\frac{1}{2}M_1v^2$ its kinetic energy measured in the ordinary way. The energy transferred to the scattering nucleus in the collision is

$$E'=4E_0M_1M_2(M_1+M_2)^{-2}\sin^2\tfrac{1}{2}\vartheta. \tag{607}$$

The maximum fraction of the energy of the incident particle which can be transferred is thus

$$E'_{\max}/E_0=4M_1M_2/(M_1+M_2)^2; \tag{607a}$$

the scattered particle goes, in this case, back into the

direction from which it came ($\Theta=\vartheta=180°$) with a velocity equal to $(M_2-M_1)/(M_2+M_1)$ times its original velocity. All these relations follow from elementary application of conservation of energy and momentum.

If only the Coulomb force acts between scatterer and scattered particle, the cross section[52] for scattering through an angle between ϑ and $\vartheta+d\vartheta$, is given by the well-known Rutherford law

$$\sigma(\vartheta)\sin\vartheta d\vartheta = 2\pi(zZe^2/2Mv^2)^2 \times(\sin\tfrac{1}{2}\vartheta)^{-4}\sin\vartheta d\vartheta, \quad (608)$$

where Ze is the charge of the scattering nucleus, ze that of the scattered particle, M the reduced mass (606) and v the velocity of the incident particle.

The Rutherford law (608) has proved correct for the scattering of α-particles and protons by all heavy nuclei. As is well known, the scattering of α-particles by heavy nuclei was the original basis of Rutherford's proposal of the nuclear atom (R26). The accuracy of the Rutherford law for heavy nuclei is so great that the nuclear charge can be determined from the scattering of α-particles (C11).

We are interested in *deviations* from the Rutherford law. Such deviations will occur (1) if the incident particle has an energy great enough to overcome the potential barrier of the nucleus, and (2) if there is resonance, i.e., if the kinetic energy of the incident particle plus the internal energy of the initial nucleus is equal to the energy of one of the states of the compound nucleus.

Experiments on the scattering of particles by nuclei can therefore be used to find resonance levels and also to find the height of the potential barrier. If there is a strong deviation from the Rutherford law which is restricted to a small energy interval, we shall attribute it to resonance. On the other hand, a deviation which sets in rather gradually with increasing energy, and does not disappear upon further increase, is in general interpreted as showing that the particle can go over the top of the potential barrier. (For the exact explanation, cf. §71.)

It must be admitted that the heights of potential barriers obtained from scattering are not very accurate. The reasons for this are (1) that the penetration through the barrier is already fairly probable when the particle energy is still well below the height of the barrier, (2) that the penetration of particles of higher angular momentum keeps increasing even for energies higher than the barrier (§78), (3) that resonances may occur even for energies above the barrier height (§§78, 82).

As far as the evidence goes (§§74, 75), the radii deduced from scattering experiments seem compatible with the assumption that the nuclear volume is proportional to the number of particles in the nucleus. With this assumption, we have found (cf. (598), (598a))

$$R=2.05\cdot10^{-13}A^{\frac{1}{3}}\text{ cm} \quad (608a)$$

and the expressions (599), (599a) for the height of the potential barrier, B. In applying these formulae, it must be kept in mind that only the *relative* kinetic energy is available for overcoming the potential barrier. With the notations used in the beginning of this section, the relative kinetic energy is $M_2/(M_1+M_2)$ times the total kinetic energy of the incident particle, provided the bombarded nucleus is at rest. Therefore, if $a=M_1/M_0$ (M_0=proton mass) is the atomic weight of the incident particle, the kinetic energy must be greater than

$$B=0.70\,zZ(A+a)A^{-4/3}\text{ MV} \quad (609)$$

in order to overcome the potential barrier. Table XXXIII gives the effective heights of the potential barriers of various nuclei for various incident particles according to (609).

The fastest α-particles available are those from ThC′ which have an energy of about 9 MV. Thus

TABLE XXXIII. *Effective heights B of potential barriers in MV. "Characteristic orbital momenta" g(Z).*

Z		2	4	10	20	30	50	70	92
A		4	9	20	40	66	112	174	238
Nucleus		He	Be	Ne	Ca	Zn	Sn	Yb	U
Height of barrier for	Protons	1.1	1.5	2.7	4.2	5.3	7.3	8.8	10.4
	Deuterons	1.3	1.6	2.8	4.3	5.4	7.4	8.9	10.5
	α-particles	3.5	3.9	6.2	9.0	11.0	15.0	18.0	20.9
Critical orbital momentum	Protons	0.6	1.0	1.9	3.1	4.1	5.9	7.5	9.1
	Deuterons	0.8	1.4	2.6	4.3	5.8	8.3	10.6	12.8
	α-particles	1.3	2.5	5.0	8.4	11.5	16.4	21.0	25.5

[52] The cross section is so defined that the *number* of particles scattered by a substance containing N scattering nuclei per cm³, is

$$I=NI_0\,\sigma(\vartheta)\sin\vartheta d\vartheta,$$

where I_0 is the number of incident particles per cm² per sec., and I the number of particles scattered per sec. through an angle between ϑ and $\vartheta+d\vartheta$.

the scattering of α-particles may give evidence about the nuclear field for nuclei up to about $Z=20$ (calcium). Actual experiments have been carried out up to $Z=13$ (Al, cf. §75).

From Table XXXIII we see furthermore that about 10 MV is required for protons and deuterons to go over the top of the potential barrier of uranium. Such energies will probably soon be available from cyclotrons.

An important point for the scattering, especially for the angular distribution of the scattered particles, is the orbital momentum of the incident particle. If the kinetic energy of the particle is large compared to the potential barrier, particles of orbital momentum l will in general approach the nucleus to a minimum distance of $l\lambda\!\!\!^{-}$ (cf., e.g. (54b, c, d)). The particles strongly affected by the nuclear field will therefore be those for which $l\lambda\!\!\!^{-}<R$, i.e., whose orbital momentum is less than

$$l_0=R/\lambda\!\!\!^{-}. \tag{610}$$

At low energies, when the Coulomb field is important, the problem is much more complicated (§72, end). We may, however, formally apply (610) to an energy just sufficient to overcome the potential barrier: Then the wave-length is

$$\lambda\!\!\!^{-}_B=\hbar R^{\frac{1}{2}}/(2Me^2zZ)^{\frac{1}{2}} \tag{610a}$$

and (610) becomes (cf. (600a))

$$l_0(B)=g(Z). \tag{611}$$

§71. General Theory of Scattering of Charged Particles

The general theory of scattering is contained in §55 in which the dispersion formula for nuclear processes was derived. It is only necessary to make a few slight changes in order to represent correctly the asymptotic behavior of the wave functions of a particle in a Coulomb field. This is particularly essential because in the Coulomb scattering the contributions of high orbital momenta is very important, in contrast to the scattering in all other types of fields.

In a pure Coulomb field, the wave function of the scattered particle normalized per unit current is (cf. M32, p. 35, and this article (100))

$$\psi_0=v^{-\frac{1}{2}}e^{ikz+i\alpha\log k(r-z)}+(zZe^2/2Mv^2r\sin^2\tfrac{1}{2}\vartheta)\times v^{-\frac{1}{2}}e^{ikr-i\alpha\log 2kr-i\alpha\log\sin^2\frac{1}{2}\vartheta+i\pi+2i\eta_0}, \tag{612}$$

where M is the reduced mass (606),

$$\alpha=zZe^2/\hbar v, \tag{612a}$$

$$k=Mv/\hbar=(2ME)^{\frac{1}{2}}/\hbar, \tag{612b}$$

$$e^{i\eta_0}=\Gamma(1+i\alpha)/|\Gamma(1+i\alpha)|. \tag{612c}$$

The first term in (612) represents the incident, the second the scattered wave. The square of the absolute value of the second term, multiplied by $r^2d\omega$, gives just the Rutherford scattering cross section for the element of solid angle

$$d\omega=2\pi\sin\vartheta d\vartheta.$$

Alternatively, we may write the Coulomb wave function as a sum over the partial waves of different l, *viz.*:

$$\psi_C=\sum_l A_l f_l(r)\,Y_{l0}(\vartheta)/r. \tag{613}$$

The radial function f_l behaves asymptotically as (cf. M32, p. 39)

$$f_l=\sin(kr-\alpha\log 2kr-\tfrac{1}{2}l\pi+\eta_l), \tag{613a}$$

where

$$e^{i\eta_l}=\frac{\Gamma(l+1+i\alpha)}{|\Gamma(l+1+i\alpha)|}=e^{i\eta_0}\frac{(1+i\alpha)\cdots(l+i\alpha)}{(1+\alpha^2)^{\frac{1}{2}}\cdots(l^2+\alpha^2)^{\frac{1}{2}}} \tag{613b}$$

and

$$A_l=\frac{(4\pi)^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}i^l}{v^{\frac{1}{2}}k}e^{i\eta_l}. \tag{613c}$$

The wave scattered by the nucleus has, according to (373a) (383b), the general form

$$\psi_N=\sum_{lm}\beta_{Plm}(if_l+g_l)\,Y_{lm}(\vartheta\varphi)/r, \tag{614}$$

where f_l is the regular solution (613a) in the Coulomb potential[53] and g_l the solution which is

[53] This means that we identify the particle potential U_P (cf. §55, (367)) with the Coulomb potential. Actually, it would probably be a better approximation to take, inside the nucleus, a constant repulsive potential. We would then have the Coulomb scattering, potential scattering from the repulsive auxiliary potential between nucleus and particle, and resonance scattering. We have not included the nuclear potential scattering in order not to make our formulae too complicated; also, not much useful information can be gained by including this potential scattering.

irregular at $r=0$. We have (cf. (376), (376b)) asymptotically

$$g_l+if_l=e^{i(kr-\alpha \log 2kr-\frac{1}{2}l\pi+\eta_l)}. \quad (614a)$$

The coefficients β can be expressed by the A's (cf. (613c)); we have (cf. 380, 391, 392a)

$$\beta_{Plm}=-\pi\sum_{rl'}\frac{A_{l'}H^{Cr}{}_{Pl'0}H^{Cr}{}_{Plm}{}^*}{E-E_r+\frac{1}{2}i\gamma_r}. \quad (615)$$

A change of the magnetic quantum number of the orbital motion of the particle from 0 to m is, of course, only possible if at the same time the spin of the scattering nucleus or of the scattered particle or both change their direction. We therefore write (615) more fully:

$$\beta_{Plm}{}^{\mu_i'\mu_s'}{}_{\mu_i\mu_s}=$$

$$-\pi\sum_{rl'M}\frac{A_{l'}H^{rM}{}_{Pl'0\mu_i'\mu_s'}H^{rM}{}_{Plm\mu_i\mu_s}{}^*}{E-E_r+\frac{1}{2}i\gamma_r}, \quad (615a)$$

where $\mu_i'\mu_s'$ are the components of the spins of scattering nucleus and scattered particle before the collision, $\mu_i\mu_s$ the same quantities after collision, and M the magnetic quantum number of the compound state. We have the selection rule

$$m+\mu_i+\mu_s=M=\mu_i'+\mu_s'. \quad (615b)$$

We insert (613c, 614a, 615a) into (614) and add the result to (612). Then we obtain for a given pair of spin quantum numbers $\mu_i\mu_s$ of nucleus and particle,

$$\psi^{\mu_i'\mu_s'}{}_{\mu_i\mu_s}=v^{-\frac{1}{2}}e^{ikz+i\alpha \log k(r-z)}\delta_{\mu_i\mu_i'}\delta_{\mu_s\mu_s'}$$

$$-v^{-\frac{1}{2}}e^{ikr-i\alpha \log 2kr+2i\eta_0}\gamma^{-1}f^{\mu_i'\mu_s'}{}_{\mu_i\mu_s} \quad (616a)$$

with

$$f^{\mu_i'\mu_s'}{}_{\mu_i\mu_s}=\frac{zZe^2}{2Mv^2\sin^2\frac{1}{2}\vartheta}e^{-i\alpha \log \sin^2\frac{1}{2}\vartheta}\delta_{\mu_i\mu_i'}\delta_{\mu_s\mu_s'}$$

$$+2\pi^{\frac{3}{2}}\lambda\sum_{lm}Y_{lm}(\vartheta\varphi)\sum_{l'rM}i^{l'-l}(2l'+1)^{\frac{1}{2}}e^{i(\eta_l+\eta_{l'}-2\eta_0)}$$

$$\times H^{CrM}{}_{Pl'0\mu_i'\mu_s'}H^{CrM}{}_{Plm\mu_i\mu_s}{}^*/(E-E_r+\tfrac{1}{2}i\gamma_r). \quad (616)$$

The cross section for elastic scattering through the angle ϑ is then:

$$\sigma(\vartheta)d\omega=\frac{d\omega}{(2i+1)(2s+1)}\sum_{\mu_i'\mu_s'\mu_i\mu_s}|f^{\mu_i'\mu_s'}{}_{\mu_i\mu_s}(\vartheta\varphi)|^2 \quad (616b)$$

In (616b), the average has been taken over all possible directions of the spins of nucleus A and particle P in the initial state and the sum over all directions in the final state. As a result, the cross section (616b) is independent of the azimuth φ.

The expression (616), (616b) for the cross section is rather complicated and no simplified form of it has yet been worked out in the general case. However, in some special cases it is not difficult to reduce the expression to a simpler one.

A. Scattering nucleus and scattered particle having zero spin

In this case, the sum over $\mu_i'\mu_s'\mu_i\mu_s$ in (616b) reduces to a single term $\mu_i'=\mu_s'=\mu_i=\mu_s=0$. The selection rules (615b) require that the compound state r have zero magnetic quantum number M. Moreover, its angular momentum J must be equal to l'. Similarly, $H^{Cr}{}_{Plm}$ can only be different from zero if $l=J=l'$ and $m=0$. Furthermore, we have

$$2\pi|H^{CrJ}{}_{Ppl'0}|^2=\gamma^{rJ}{}_{Pp}, \quad (617)$$

since the given state r of the compound nucleus can only emit particles of angular momentum l' and magnetic quantum number zero, again owing to the selection rules. Inserting $e^{2i(\eta_l-\eta_0)}$ from (613b), and remembering that

$$Y_{l0}=(4\pi)^{-\frac{1}{2}}(2l+1)^{\frac{1}{2}}P_l(\vartheta),$$

we have from (616):

$$\sigma(\vartheta)=\left|\frac{e^2zZ}{2Mv^2\sin^2\frac{1}{2}\vartheta}e^{-i\alpha \log \sin^2\frac{1}{2}\vartheta}\right.$$

$$+\tfrac{1}{2}\lambda\sum_J(2J+1)P_J(\vartheta)$$

$$\left.\times\frac{(1+i\alpha)^2\cdots(J+i\alpha)^2}{(1+\alpha^2)\cdots(J^2+\alpha^2)}\sum_r\frac{\gamma^{rJ}{}_{Pp}}{E-E_r+\frac{1}{2}i\gamma_{rJ}}\right|^2. \quad (618)$$

Of special interest is the case of resonance, in which the sum over r and J reduces to a single term. Replacing the "effective widths" γ by the "true widths" Γ (cf. §52), and introducing

the abbreviations

$$(2J+1)\frac{\lambda\Gamma^{r_J}{}_{Pp}}{\Gamma_{rJ}}\frac{2Mv^2\sin^2\frac{1}{2}\vartheta}{e^2zZ}P_J(\vartheta)=2(2J+1)$$

$$\times(\hbar v/e^2zZ)(\Gamma^r{}_{Pp}/\Gamma_r)\sin^2\tfrac{1}{2}\vartheta P_J(\vartheta)=\rho, \quad (618a)$$

$$e^{i\alpha\log\sin^2\frac{1}{2}\vartheta}\frac{(1+i\alpha)^2\cdots(J+i\alpha)^2}{(1+\alpha^2)\cdots(J^2+\alpha^2)}=e^{i\zeta}, \quad (618b)$$

$$2(E-E_r)/\Gamma_r=x, \quad (618c)$$

$$(e^2zZ/2Mv^2\sin^2\tfrac{1}{2}\vartheta)^2=\sigma_0, \quad (618d)$$

we obtain for the cross section $\sigma(\vartheta)$:

$$\frac{\sigma(\vartheta)}{\sigma_0}=1+\frac{\rho^2+2\rho\sin\zeta+2\rho x\cos\zeta}{1+x^2}, \quad (619)$$

σ_0 is the Rutherford scattering cross section.

The ratio of resonance scattering to Coulomb scattering is primarily determined by ρ. This quantity is (cf. 618a) in general greater for large scattering angles than for small ones. Thus the backward scattering ($\vartheta=\pi$) will show the most pronounced deviations from the Rutherford formula (cf. §74, 75). ρ increases with increasing energy as $v\sim E^{\frac{1}{2}}$ (see, however, remark 1 below). For a given energy, it is larger for light than for heavy particles ($v\sim M^{-\frac{1}{2}}$) and larger for a small charge z of the scattered particle than for a large charge. From this it would follow that protons will show more pronounced resonance scattering than deuterons, and deuterons more than α-particles. However, this factor is probably more than offset by the considerations given below, cf. remark 3. Resonances with levels of high angular momentum J are stronger than for low J, provided they can occur at all (cf. end of §72).

The factor of greatest importance is $\Gamma^r{}_{Pp}/\Gamma_r$. It represents the ratio of the partial width of the resonance level (corresponding to emission of the incident particle P with the scattering nucleus being left in the ground state p) to the total level width. In order that this ratio be large, it is necessary that no other processes but elastic scattering can occur with great probability. The conditions for this are:

(1) The scattered particle must not have too high energy, because otherwise inelastic scattering will occur. Particularly if the energy E_P is high enough so that many levels of the scattering nucleus can be excited, the total width corresponding to all the excited states of the scattering nucleus will in general be much greater than the partial width corresponding to the ground state. This fact will in many cases more than offset the factor v in (618a), and will lead to a *decrease in the strength of resonances with increasing energy.*

(2) The scattering nucleus must not be too heavy because otherwise it will have many excited states of low energy so that again inelastic scattering will be very probable.

(3) The compound nucleus C must not disintegrate with emission of particles other than P with any great probability. This condition will be most easily fulfilled if P is an α-particle. Because of the large mass defect of the α-particle, most nuclear reactions produced by α-particles are endoergic, and unless the kinetic energy of the α-particle is very high, no nuclear reaction can in general occur, especially if the scattering nucleus itself has a very high mass defect. This is the case for most nuclei of zero spin, which are, strictly speaking, the only ones for which our formulae are valid (see above).

Deuterons, and to a lesser extent protons, can almost with every nucleus cause a variety of nuclear processes, owing to their large internal energy. In these cases the total level width Γ_r will in general be much larger than the partial width referring to the incident particle, $\Gamma^r{}_{Pp}$. It is therefore likely that the strongest resonance effects are found in the scattering of α-particles, less strong effects with protons and very weak resonances in the deuteron scattering. This will be true in spite of the factor $1/z$ in (618a) which would tend to make the proton and deuteron resonances stronger than those for α-particles.

To obtain an estimate of the absolute magnitude of ρ, we may consider the α-particles of polonium (energy 5.3 MV). Let us assume that the partial width $\Gamma^r{}_{Pp}$ is equal to the total Γ_r, and that the scattering angle is 180°. Then approximately

$$\rho=7.3(2J+1)/Z. \quad (619a)$$

Thus for elements up to $N(Z=7)$, ρ would be larger than unity even for $J=0$, i.e. for S levels of the compound nucleus. This means, according to (619), that the resonance scattering backward

will, at its maximum, be more than twice the Coulomb scattering whatever J. For elements heavier than nitrogen, the resonance scattering by an S level of the compound nucleus will increase the Coulomb scattering by less than a factor two: With the limited accuracy of the scattering experiments, it will then be difficult to observe resonance scattering of Po α-particles by S resonance-levels. P resonances ($J=1$), on the other hand, will correspond to an increase in the cross section by a factor of about 10 for nitrogen and will therefore be easily observable up to rather high atomic number, probably as long as the nuclear potential barrier is at all penetrable for Po α-particles. This is *a fortiori* true for D resonances.

We shall now discuss the behavior of the scattering cross section (619) near resonance. As in all cases, the resonance scattering is restricted to an energy region of width Γ_r about the resonance energy E_r. An investigation of the scattering cross section as a function of the energy of the incident particle will therefore indicate the width of the resonance level. The cross section has a minimum and a maximum at

$$x_{\min}{}^{\max}=\pm(1+c^2)^{\frac{1}{2}}-c, \tag{620}$$

where

$$c=(\rho/2\cos\zeta)+\tan\zeta. \tag{620a}$$

The maximum and minimum cross section are given by

$$\sigma_{\min}{}^{\max}/\sigma_0=1+\rho[\pm(\tfrac{1}{4}\rho^2+\rho\sin\zeta+1)^{\frac{1}{2}}+\tfrac{1}{2}\rho+\sin\zeta]. \tag{621}$$

If ρ is large, i.e., if the resonance scattering is large compared to the Coulomb scattering, the minimum of the cross section occurs at

$$x_{\min}=-\rho/\cos\zeta, \tag{622}$$

i.e. (cf. 618c),

$$E=E_r-\tfrac{1}{2}\Gamma_r\rho/\cos\zeta. \tag{622a}$$

The maximum occurs at

$$x_{\max}=\cos\zeta/\rho, \tag{622b}$$

which, for large ρ, is practically at the resonance energy. The minimum cross section is, for large ρ:

$$\sigma_{\min}=\sigma_0\sin^2\zeta \tag{623}$$

and the maximum cross section

$$\sigma_{\max}=\sigma_0\rho(\rho+2\sin\zeta)\approx\sigma_0\rho^2. \tag{623a}$$

Inserting the values of σ_0 and ρ from (618a, d), this becomes

$$\sigma_{\max}=(2J+1)^2\lambda\!\!\!^{-2}(\Gamma^r{}_{Pp}/\Gamma_r)^2P_J{}^2(\vartheta). \tag{624}$$

This is the same cross section as without the Coulomb field. It should, however, be kept in mind that the condition $\rho\gg1$ will never be fulfilled for small scattering angles ϑ, so that for small angles the Coulomb scattering is always predominant.

B. Any spin of scattering nucleus and scattered particle, but only zero orbital momentum important

This condition will be fulfilled for slow particles. The sum over l and m in (616) reduces to a single term $l=m=0$, the sum over l' to a single term $l'=0$. The magnetic quantum number of the compound state M is equal to the sum of the magnetic quantum numbers of the spins of scattering nucleus and scattered particle $\mu_i'+\mu_s'=\mu_i+\mu_s$. A fairly simple calculation involving the spacial symmetry properties of the matrix elements H in (616) (similar to the calculation in the appendix of B15) gives for the scattering cross section

$$\frac{\sigma(\vartheta)}{\sigma_0}=1+\frac{2J+1}{(2i+1)(2s+1)}\,\frac{\rho^2+2\rho\sin\zeta+2\rho x\cos\zeta}{1+x^2} \tag{625}$$

similar to (619). Here i and s are the spins of scattering nucleus and scattered particle, σ_0 is the Rutherford scattering cross section (618d), x the distance from resonance divided by half the width of the resonance level (cf. 618c), while ζ and ρ have somewhat simpler forms than in (618a, b) because the orbital momentum of the incident particle is now zero rather than J. We have

$$\zeta=\alpha\log\sin^2\tfrac{1}{2}\vartheta, \tag{625a}$$

$$\rho=(2\hbar v/e^2zZ)(\Gamma^r{}_{Pp}/\Gamma_r)\sin^2\tfrac{1}{2}\vartheta. \tag{625b}$$

The "shape of the resonance line" (625) is the same as for zero spin of nucleus and particle (cf. 619). If we compare the scattering of particles of orbital momentum zero in both cases, we see that the intensity of the resonance scattering is *reduced* by a factor $(2J+1)/(2i+1)(2s+1)$ by the existence of the spins of particle and nucleus.[54] On the other hand, there will be *more* resonances: if the spins s and i are zero, and the orbital momentum $l=0$, only S-states ($J=0$) of the compound nucleus can give rise to resonance scattering, whereas, if s and i are different from zero, any compound state with J between $|s-i|$ and $s+i$ will give resonance scattering.[54]

[54] It may be mentioned that there will be no reduction in the intensity, and no increase in the number of resonances if only *one* of the two spins, s or i, is different from zero. However, this is only true for orbital momentum zero; for arbitrary l, *one* spin different from zero is sufficient to decrease the intensity and increase the number of effective levels.

C. Very high density of levels (high energy)

If we average over an energy interval large compared to the spacing of levels (cf. §56, §65), the interference term between Coulomb scattering and resonance scattering will disappear. Then the scattering cross section will be equal to the Coulomb scattering, plus the average resonance scattering, the latter having the same form as for neutrons. The most probable process will in general be a disintegration or inelastic scattering. The elastic scattering will be primarily Coulomb scattering at small angles and nuclear potential scattering at large angles.

§72. Penetration of the Potential Barrier and Angular Momentum

The scattering is, according to (616, 618a), primarily determined by the matrix elements H, or by the partial width Γ^r_{Pp} of the nuclear resonance level, the latter being proportional to the square of the former. The matrix elements involve the wave function of the incident particle in the nucleus (or at its surface, if the nuclear potential suggested in §54A is accepted). This wave function introduces the well-known penetrability of the potential barrier as a factor into the matrix elements, and therefore into the partial width of the levels and into the scattering cross section.

Accurate tables of the wave functions of a particle in a Coulomb field have been given by Yost, Wheeler and Breit (Y2). They must be used whenever the fundamental assumptions of the theory are sufficiently justified to warrant exact calculations, as, e.g., in the scattering of protons by protons (B53). Similarly, when only one resonance level is of importance, and all the angular momenta concerned are well known, the use of exact wave functions is desirable. Such a case seems to be, e.g., the resonance scattering of protons of 440 kv by Li^7 nuclei (cf. §75).

For estimates, especially when the properties of the nuclei concerned are not well known, it will be sufficient to use the WKB method for calculating wave functions. If we assume the Coulomb potential to hold everywhere (down to $r=0$), the wave function of the particle must *decrease* exponentially as we approach the nucleus. In the notation of §66, we have thus to take the function u_2 (except for the inside part). Therefore, if the wave function behaves asymptotically as (cf. 578a)

$$\psi(r)=v^{-\frac{1}{2}}\cos\left(\int_{r_E}^{r}\Phi^{\frac{1}{2}}(\rho)d\rho-\tfrac{1}{4}\pi\right)$$
$$=v^{-\frac{1}{2}}\sin\left(kr-\alpha\log 2kr+\eta_l\right), \tag{626}$$

its value at the surface of the nucleus ($r=R$) will be

$$\psi(R)=\tfrac{1}{2}(m/\hbar)^{\frac{1}{2}}|\Phi(R)|^{-\frac{1}{4}}$$
$$\times\exp\left(-\int_R^{r_E}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right), \tag{626a}$$

where
$$|\Phi(R)|=2m\hbar^{-2}(zZe^2/R-E), \tag{626b}$$

E is the particle energy, and

$$r_E=zZe^2/E. \tag{626c}$$

The exponent in (626a) is *exactly* the quantity C (cf. (585a)); therefore the square of the exponential is simply the penetrability of the potential barrier.

We may now change our assumptions and suppose that there is a constant repulsive potential V_0 acting on the particle when it is inside the nucleus (cf. §54A). We assume that the transition from the Coulomb potential to this nuclear potential is gradual so that the WKB solution holds throughout. Then we have inside the nucleus

$$\psi(r)=\tfrac{1}{2}(\tfrac{1}{2}m)^{\frac{1}{4}}(V_0-E)^{-\frac{1}{4}}e^{-C}e^{-\kappa(R-r)} \tag{627}$$

with

$$\kappa=(V_0-E)^{\frac{1}{2}}(2m)^{\frac{1}{2}}/\hbar \tag{627a}$$

as in (335). If there were no potential barrier, we should have instead of (627) (cf. 335)

$$\psi_0(r)=(\tfrac{1}{2}m)^{\frac{1}{4}}E^{\frac{1}{4}}V_0^{-\frac{1}{4}}e^{-\kappa(R-r)}. \tag{627b}$$

The ratio is

$$\psi/\psi_0=\tfrac{1}{2}[V_0^2/E(V_0-E)]^{\frac{1}{4}}e^{-C}. \tag{628}$$

The most important factor here is, of course, the penetrability e^{-C}. However, there is, in addition, a factor $\frac{1}{2}[V_0^2/E(V_0-E)]^{\frac{1}{4}}$. This factor arises from the fact that the potential must be assumed to change *suddenly* from zero to V_0 when there is *no potential* barrier, "suddenly" being understood in comparison with the wave-length of the particle outside the nucleus. On the other hand, the potential changes *gradually* when there is a *Coulomb field.* The sudden change of potential causes a reflection of particles at the surface of the nucleus, especially if the particles are very slow ($E\ll V_0$), tending to make the wave function of slow particles inside the nucleus relatively smaller in the case of free particles than in the case of a Coulomb field. The factor $v^{\frac{1}{2}}\sim E^{\frac{1}{4}}$ in the matrix element, and correspondingly $v\sim E^{\frac{1}{2}}$ in the width of the levels, does

not occur in the Coulomb field but only in the case of free particles.

Practically, this point is of little consequence, since very slow charged particles will never be able to penetrate into the nucleus on account of the potential barrier. Therefore we shall, for our purposes, write the "partial width" for charged particles in the form

$$\Gamma^r{}_{Pp}=G^r{}_{Pp}e^{-2C}=G^r{}_{Pp}P, \tag{629}$$

where $G^r{}_{Pp}$ is the width of the level without potential barrier, and P the penetrability of the barrier. A convenient formula for P was given in (600).

The influence of the orbital momentum of the particle on the penetrability may be calculated by the WKB method (cf. §66). The only difference as compared to the case of $l=0$ is that we have instead of (578b)

$$\Phi_l(r)=2m\hbar^{-2}(E-V)-l(l+1)/r^2. \tag{630}$$

Defining C_l in analogy with (585a), *viz.*

$$C_l=\int_R^{r_l}|\Phi_l(\rho)|^{\frac{1}{2}}d\rho, \tag{630a}$$

where r_l is defined as the value of r for which Φ_l vanishes, we obtain

$$\frac{C_l}{g}=\tfrac{1}{2}x^{-\frac{1}{2}}\left(\tfrac{1}{2}\pi+\arcsin\frac{1-2x}{(1+4xy)^{\frac{1}{2}}}\right)-(y+1-x)^{\frac{1}{2}}$$
$$+y^{\frac{1}{2}}\log\frac{1+2y^{\frac{1}{2}}[y^{\frac{1}{2}}+(y+1-x)^{\frac{1}{2}}]}{(1+4xy)^{\frac{1}{2}}}. \tag{631}$$

Here g is the abbreviation introduced in (600a), $x=E/B$ is the ratio of particle energy to barrier height and

$$y=l(l+1)/g^2. \tag{631a}$$

For values of l small compared to the "critical orbital momentum" g, i.e., for $y\ll 1$, and for $x<1$, this may be approximated by

$$C_l=C_0+\tfrac{2}{3}g[(y+1-x)^{\frac{3}{2}}-(1-x)^{\frac{3}{2}}]. \tag{632}$$

For energies not too near the top of the barrier, and below the top, we may further simplify (632) and obtain for the penetrability for the orbital momentum l

$$P_l=e^{-2C_l}=P_0\,e^{-2gy(1-x)^{\frac{1}{2}}}. \tag{632a}$$

This expression falls to $1/e$ times its value for $l=0$ when

$$y_c=\tfrac{1}{2}g^{-1}(1-x)^{-\frac{1}{2}}, \tag{632b}$$

which corresponds to

$$l_c+\tfrac{1}{2}=gy_c^{\frac{1}{2}}=(\tfrac{1}{2}g)^{\frac{1}{2}}(1-x)^{-\frac{1}{4}}. \tag{633}$$

This critical orbital momentum is, for low energy ($x\ll 1$), approximately the square root of one-half the critical momentum g given in Table XXXIII. Thus, e.g., slow α-particles ($E\ll 20$ MV) will be about equally effective in disintegrating uranium when they have orbital momentum 3 as for $l=0$, but much less effective when $l\gg 3$. Therefore the angular distribution of the particles produced in the disintegration of a heavy nucleus by a charged particle will not be spherically symmetrical, even if the wave-length of the incident particle is large compared to the nuclear radius (Teller, unpublished; cf. §78).

With increasing energy, the critical orbital momentum l_c increases slowly. For $x\approx 1$, i.e., for particles which can just go over the top of the barrier, (633) is no longer valid, but (632) reduces to ($C_0=0$)

$$C_l=\tfrac{2}{3}gy^{\frac{3}{2}}. \tag{634}$$

The critical orbital momentum is then

$$l_c(B)=(\tfrac{3}{4})^{\frac{1}{2}}g^{\frac{3}{2}}=0.91g^{\frac{3}{2}}. \tag{634a}$$

For α-particles on uranium, this gives $l_c(B)=8$. Fig. 19 gives l_c as a function of x (see §78).

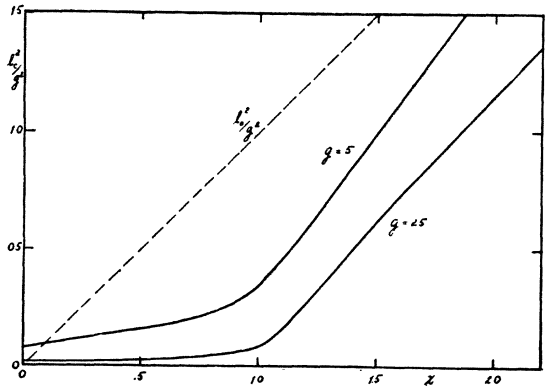

FIG. 19. Critical orbital momentum l_c of charged particles as a function of energy. Abscissa: Ratio of particle energy to barrier height. Ordinate: l_c^2/g^2 where g is the characteristic orbital momentum for the given nucleus. $g=5$ corresponds approximately to deuterons on zinc, $g=25$ to α-particles on uranium. The broken curve corresponds to neutral particles. The curves show clearly the sharp break when the particle energy becomes equal to the barrier height, and the rapid rise at energies above the barrier. They also show that at very low energies the critical angular momentum is larger for charged than for neutral particles.

TABLE XXXIV. *Penetrabilities of potential barriers for various orbital momenta.*

	Beryllium (B=3.9 MV)				Calcium (Z=20, B=9.0 MV)			
		Penetrability P_l				P_l		
x	E(MV)	l=0	l=1	l=2	E	l=0	4	8
1.6	6.2			1	14.4			0.51
1.4	5.5			0.82	12.6		1	0.165
1.2	4.7		1	0.55	10.8		0.95	0.035
1.0	3.9	1	0.74	0.29	9 0	1	0.60	$4\cdot10^{-3}$
0.8	3.1	0.63	0.35	0.105	7.2	0.28	0.039	$1.7\cdot10^{-3}$
0.7	2.7	0.40	0.20	0.051	6.3	0.073	$7.5\cdot10^{-3}$	$2\cdot10^{-5}$
0.6	2.3	0.20	0.092	0.021	5.4	0.011	$7.5\cdot10^{-4}$	$1.3\cdot10^{-6}$
0.5	1.9	0.077	0.028	$6.5\cdot10^{-3}$	4.5	$6.5\cdot10^{-4}$	$3.5\cdot10^{-5}$	$3.5\cdot10^{-8}$
0.4	1.6	0.020	$8.5\cdot10^{-3}$	$1.2\cdot10^{-3}$	3.6	$1.2\cdot10^{-5}$	$7.5\cdot10^{-7}$	$2\cdot10^{-10}$
0.3	1.2	$2.0\cdot10^{-3}$	$8\cdot10^{-4}$	$1.4\cdot10^{-4}$	2.7	$2.5\cdot10^{-8}$	$6\cdot10^{-10}$	$2\cdot10^{-13}$
0.2	0.8	$5\cdot10^{-6}$	$1.4\cdot10^{-5}$	$2\cdot10^{-6}$	1 8	$4\cdot10^{-13}$	$8\cdot10^{-15}$	10^{-18}
0.1	0.4	$6\cdot10^{-9}$	$2\cdot10^{-9}$	$2\cdot10^{-10}$	0.9	10^{-24}	$4\cdot10^{-26}$	10^{-29}

For energies well above the potential barrier, the penetrability is no longer important, and particles of angular momentum up to $l_0=R/\lambda\!\!\!^{-}$ will be effective. As already mentioned at the end of §70, this means that, above the barrier, higher angular momenta become rapidly more important. Therefore there will often be a further increase of the nuclear scattering above the top of the barrier, making the barrier appear higher than it actually is. Indications of this effect can be seen in the data given in §75 where the height of the potential barriers as deduced from scattering is sometimes higher than that following from our general interpolation formula (609).

As illustrations, we give in Table XXXIV the penetrabilities for α-particles of the barriers of two nuclei, a very light one (Be) and a medium heavy one (Ca), as functions of the energy of the particle for various orbital momenta l. The critical orbital momentum g is about 2 for Be and 8 for Ca while l_c (cf.(633)) is, for small energy, about 1 and 2, respectively. It is seen that, for the same ratio of energy to barrier height, the penetrability is much greater for Be than for Ca. Also it is shown by the table that the penetrability is decidedly less for the higher angular momenta than for the lower, in accord with the considerations above.

§73. The One-Body Theory of Scattering (M32)

For light nuclei, it may be a fair approximation to treat the scattering problem as a one-body problem, assuming a suitable potential to act between the scattering nucleus and the scattered particle. This method has been used widely in the past for the analysis and interpretation of experimental results; mainly for this reason, we discuss this method here although we are aware of its limitations.[55]

If we denote by ψ_0 the wave function in the pure Coulomb field (cf. (612)), the wave function in the field of the nucleus may be written (cf. Mott and Massey, p. 24, Eq. (17))

$$\psi=\psi_0+\sum_l[u_l-A_lf_l(r)]Y_{l0}(\vartheta)/r, \tag{635}$$

where A_l is given in (613c), f_l is the radial wave function in the Coulomb field whose asymptotic behavior is indicated in (613a), and u_l is the radial wave function in the actual field, i.e. the Coulomb plus the specifically nuclear field. The asymptotic behavior of u_l will be similar to the Coulomb wave function f_l, except for a phase shift δ_l, *viz.*:

$$u_l=B_l\sin(kr-\alpha\log 2kr-\tfrac{1}{2}l\pi+\eta_l+\delta_l). \tag{635a}$$

δ_l depends on the nuclear potential and can be calculated only if this potential is known. B_l must be determined so that $u_l-A_lf_l$ is purely an outgoing wave. This leads to

$$B_l=A_le^{i\delta_l}. \tag{635b}$$

Inserting the values of A_l, B_l, f_l, u_l and ψ_0 in (635), we find for the scattering cross section per unit solid angle

$$\sigma(\vartheta)=\left|\frac{e^2zZ}{2Mv^2\sin^2\frac{1}{2}\vartheta}e^{-i\alpha\log\sin^2\frac{1}{2}\vartheta+\frac{1}{2}i\lambda\!\!\!^{-}}\right.$$
$$\left.\times\sum_l(2l+1)(e^{2i\delta_l}-1)\frac{(1+i\alpha)^2\cdots(l+i\alpha)^2}{(1+\alpha^2)\cdots(l^2+\alpha^2)}P_l(\cos\vartheta)\right|^2. \tag{636}$$

This formula is similar to that obtained with the many-body model and zero spin of nucleus and particle (618). The main difference is that the sum over the levels r in (618) has been replaced by $i(e^{2i\delta_l}-1)$. The behavior near resonances is also quite similar to that discussed in §71A, only the factor $(\Gamma^r{}_{Pp}/\Gamma_r)^2$ in the maximum cross section (624) is replaced by unity because no processes other than elastic scattering are being considered.

The similarity between (636) and the many-body formula for particles with zero spin is not accidental. If the spins of both scattering nucleus and scattered particle are zero, the scattered wave contains only partial waves of zero magnetic quantum number. Therefore, for large distances from the nucleus, we can certainly write the wave function in the form

$$\psi=\psi_0+\sum_l v_lY_{l0}(\vartheta)/r, \tag{636a}$$

where v_l is an outgoing spherical wave. Furthermore, we know that the orbital momentum l of a particle cannot change in the scattering process; therefore if we analyze the total wave function ψ in spherical harmonics and split the radial function, multiplying each spherical harmonic, into an incident and an outgoing part, the amplitudes of these two parts must have the same absolute value. This leads uniquely to the *form* (636) for the scattering cross section. However, the phases δ_l can in general *not* be found

[55] The method will give incorrect results (1) if the spins of nucleus or particle are different from zero, (2) if disintegrations have a probability comparable to scattering, (3) if inelastic scattering has a probability comparable to the elastic process.

as the phase differences between the solutions of the radial Schrödinger equations with and without the nuclear potential, but they must be regarded as arbitrary constants of much more complicated significance. Thus it will in general be possible to determine "empirical" phases δ_l suitable to represent the observed scattering, provided scattering nucleus and scattered particle have zero spin. But it will usually not be possible to find a suitable nuclear potential which gives the "observed" δ_l's (cf. §74).

§74. Scattering by Very Light Nuclei (up to He^4) (C2, R25, C3, W22, T6, T7, W13)

The scattering of particles by very light nuclei claims particular interest. In some cases it has been possible to draw conclusions about the fundamental forces between elementary nuclear particles from the scattering of very light nuclei (§14, 15, 18, and below). Since the scattering of elementary particles (protons and neutrons) by other elementary particles has already been treated (§18, 14, 15), we shall here be concerned with the scattering of protons, neutrons, deuterons and α-particles by deuterons and α-particles.

In all scattering experiments carried out thus far with protons and deuterons, the effective de Broglie wave-length $\lambda\!\!\!^{-}$ was larger than the presumable range of the nuclear forces. Under these conditions, it is known (§14) that only the particles of angular momentum zero are affected strongly by the nuclear forces. This fact has been made use of in theoretical investigations (T6, T7, M8). When protons of several MV, fast neutrons and deuterons are used for scattering experiments, the situation will be different.

To facilitate estimates of whether angular momenta higher than 0 will play any role in any particular scattering process, we have listed in Table XXXV the effective de Broglie wave-lengths *viz.*

$$\lambda\!\!\!^{-}=4.52\cdot10^{-13}(A+a)/A(aE_0)^{\frac{1}{2}}, \qquad (637)$$

Table XXXV. *Scattering of light nuclei.*

Scattering Nucleus	Wave-lengths $\lambda/2\pi=\lambda\!\!\!^{-}$ (in 10^{-13} cm for $E_0=1$ MV) Scattered Particle			Range of the Forces (in 10^{-13} cm) Scattered Particle		
	H, n	D	He^4	H, n	D	He^4
H, n	9.06	9.60	11.30	2	5*	4.5
D	6.78	6.38	6.78	5*	7*	6*
H^3, He^3	6.04	5.28	5.27	4.5*	6*	5*
He^4	5.65	4.79	4.52	4.5	6*	4

* Estimates (other figures from experiments, see below).

for an energy $E_0=1$ MV. (a and A are the atomic weights of scattered particle and scattering nucleus, E_0 the kinetic energy of the incident particle in MV in the laboratory coordinate system.) In addition, we have tabulated the estimated effective radius of interaction for the particular pair of interacting nuclei (see below, under the discussion of the separate processes). According to the discussion in §14, strong scattering of angular momentum l can be expected only if $\lambda\!\!\!^{-}l$ is amaller than the range of the forces. E.g., for the scattering of protons by α-particles, $\lambda\!\!\!^{-}=5.65E_0^{-\frac{1}{2}}\cdot10^{-13}$ cm while the range of the forces is about $4\cdot10^{-13}$ cm. Therefore for proton energies above 2 MV an influence of the angular momentum $l=1$ is to be expected.

We turn now to the discussion of the various processes.

A. Scattering of neutrons by deuterons

The range of interaction between neutron and deuteron is determined to a large extent by the size of the deuteron. The average distance between proton and neutron in the deuteron is (cf. §12) $4.5\cdot10^{-13}$ cm, therefore the average distance of either proton or neutron from the center of the deuteron is $2.2\cdot10^{-13}$ cm. With a range of the force between two elementary particles of about 2 to $2.5\cdot10^{-13}$ cm (§21), we may estimate that a neutron will begin to be influenced by a deuteron at a distance of about $5\cdot10^{-13}$ cm.

The wave-length characteristic for the process is (Table XXXV) $6.78\cdot10^{-13}E_0^{-\frac{1}{2}}$ where E_0 is the kinetic energy of the neutron in MV. Thus the scattering may be expected to be approximately spherically symmetrical for neutron energies up to about 2 MV. For faster neutrons, the partial waves $l=1$ etc. will be affected so that the scattering will no longer show spherical symmetry. The "spherical symmetry" refers, of course, to the C system (cf. §70); this means (cf. (604)) a distribution of the recoil deuterons according to a law $\sin 2\varphi d\varphi$, φ being the angle between the motions of neutron and recoil deuteron.

If the scattering is spherically symmetrical ($E_0\ll2$ MV), the cross section is determined by the phase δ_0 of the partial wave function $l=0$ (cf. (635a)). The phase shift δ_0 has been calculated by Massey and Mohr (M8) with the assumption that the interaction between deuteron and neutron can be represented by a potential (cf. below). For a potential hole of radius $4.5\cdot10^{-13}$ cm which should about correspond to

reality, they find a pronounced "Ramsauer effect" (Fig. 1 of their paper), i.e., the scattering is low at high energies, rises to a maximum at about 0.8 MV neutron energy and falls off again for smaller energies. The cross section at the maximum is $7 \cdot 10^{-24}$ cm^2, at zero energy only $2.5 \cdot 10^{-24}$ cm^2.

Experimentally, the scattering of neutrons by deuterons has been investigated by Dunning, Pegram, Fink and Mitchell (D23). They found the scattering cross section to be $4 \cdot 10^{-24}$ cm^2 for very slow neutrons (absorbable by Cd, energy probably less than 1 volt, cf. §61G) and $1.7 \cdot 10^{-24}$ for "fast" neutrons, being a mixture of all energies from 1 volt to about 10 MV.

No better agreement between theoretical and experimental value at low energy is to be expected since the influence of the deuteron on the neutron cannot be appropriately represented by a potential. The deuteron should be regarded as a dynamic system, and the proper wave function of all *three* particles, the two neutrons and the proton, be calculated with the actual interaction between them. This proper treatment of the problem as a three-particle problem is of course mathematically very complicated. If the problem is treated as a two-particle one (deuteron+neutron), then at least the exchange of the incident neutron with the one contained in the deuteron and the Majorana type of the forces should be considered. This refinement of the theory should be more necessary for the scattering by the deuteron than by any other nucleus, owing to the small binding energy and the small number of particles in the deuteron.

B. Scattering of protons by deuterons

If the nuclear forces are symmetrical in neutrons and protons (§6, 22), the force between proton and deuteron should be exactly the same as between neutron and deuteron, except for the Coulomb force. The phase shifts δ_l should therefore be identical for the two cases, only the scattering should be calculated from (636) for protons, from (53) for neutrons. A comparison of the scattering of neutrons and protons *of the same energy* by deuterons would therefore provide a very accurate test of the assumption that nuclear forces are symmetrical in neutrons and protons. Of course, it would be necessary to find the angular distributions in both cases. Experiments on the angular distribution and the absolute cross section for the scattering of protons by deuterons have been made by Tuve, Heydenburg and Hafstad (unpublished).

C. Deuteron-deuteron

From the diameter of the deuteron ($4.5 \cdot 10^{-13}$ cm) we estimate that the mutual influence of two deuterons will be appreciable up to distances of about $7 \cdot 10^{-13}$ cm. Accordingly, marked deviations from the Rutherford scattering should set in at very low energies, of the order of a few hundred kv. The partial wave $l=0$ only should be strongly influenced for energies up to about 1 MV. Experiments have been made by Tuve, Heydenburg and Hafstad (unpublished).

D. α-particle—proton

The size (radius) of the α-particle is only about $2 \cdot 10^{-13}$ cm (§21). Therefore the range of the forces between α-particle and protons may be estimated to be about $4 \cdot 10^{-13}$ cm. Consequently, if α-particles are scattered in hydrogen, only the partial wave $l=0$ should be materially affected as long as the α-particle energy is less than $(11.3/4)^2 \approx 8$ MV (cf. Table XXXV).

Very careful experiments on the scattering of α-particles of different velocities in hydrogen have been carried out by Chadwick and Bieler (C2) as early as 1921: The experiments consisted in observing the number and angular distribution of the protons projected by the α-particles from a hydrogen-containing substance such as paraffin. Because of their longer range (§95) the protons are much easier to observe than the scattered α-particles. The angle φ between the directions of proton and incident α is connected to the deflection ϑ of the α-particle in the center-of-mass system by the simple relation $\varphi = \frac{1}{2}(\pi - \vartheta)$ (cf. 604c). The protons emitted in a forward direction correspond thus to maximum deflection ($\vartheta = 180°$) of the α's and therefore their number should show the largest relative deviation from the Rutherford formula (cf. (618a) and text following (619)).

The results of Chadwick and Bieler show that the scattering probability is many times as large as the Rutherford law would indicate. The scattering ratio (ratio of actual to Rutherford scatter-

ing) increases with increasing energy of the α-particles and surpasses 40 for α-particles of the highest energy ($E_0 = 7.5$ MV). For small energies, below about 3 MV, the scattering becomes essentially equal to the Rutherford scattering. This behavior is to be expected. If we assume a range of the nuclear interaction of about $R = 4.5 \cdot 10^{-13}$ cm (see below) the height of the potential barrier is $2e^2/R = 0.6$ MV. But the kinetic energy in the relative coordinate system in only one-fifth of the kinetic energy of the α-particle (cf. 606a) so that the α-particles are able to go over the top of the potential barrier if their energy is more than 3 MV.

The data have been analyzed theoretically by Taylor (T6). He finds that the angular distributions can be explained very satisfactorily on the one-body model by assuming a single phase shift δ_0 for the partial wave of zero angular momentum. This is to be expected for α-particles of less than 8 MV (see above). The "experimental" phases δ_0 can be interpreted theoretically by assuming a potential energy of the rectangular-hole type between proton and α-particle. The depth of the hole which gives best agreement with the experimental data, is 6 MV, the width $4.5 \cdot 10^{-13}$ cm.

In such a potential hole, there should be a stable energy level at -0.8 MV, corresponding to a stable nucleus Li^5 of 0.8 MV binding energy. Such a nucleus is not known and probably does not exist. Its energy must be higher than the energy of He^5 by the amount of the Coulomb repulsion between the proton and the α-particle. But even He^5 probably does not exist (W21c) and is unstable by 0.9 MV against disintegration into an α-particle and a neutron. Thus the potential assumed by Taylor for the scattering, does not correctly represent the interaction between proton and α-particle in the stationary state. However, it can hardly be expected that this interaction can be described by a potential. Moreover, it seems that only a slight change of the potential would be required to make Li^5 unstable, and such a change seems quite possible in view of the crude approximation given by the one-body model.

Experimental results apparently different from those of Chadwick and Bieler have been obtained by Pose and Diebner (P13). They give a scattering probability equal to more than 100 times the Rutherford scattering for α-particles of only 5 MV, for which Chadwick and Bieler found only 10 times the Rutherford scattering. Horsley has pointed out (H37) that such a large scattering ratio could not be explained if only the angular momentum $l = 0$ were affected by the nuclear field, and therefore postulates an influence of the nuclear field on the p-scattering ($l = 1$), contrary to our estimates above. It is true that p-scattering might occur in a limited energy interval due to a p-resonance level. However the discrepancy between Pose-Diebner's and Chadwick-Bieler's results seemed to us far too large to be explained by experimental uncertainties. Indeed, there is obviously an error in the evaluation of the data of Pose and Diebner: They observe all protons emitted in all directions making angles less than 5° with the incident alphas, corresponding to a solid angle $2\pi\ (1 - \cos 5°) = 2.4 \cdot 10^{-3}$. The value of the solid angle given in their paper, and obviously used in their evaluation, is $2.5 \cdot 10^{-3}$. All their results should therefore be divided by 10, which makes them agree with Chadwick and Bieler's values. This change invalidates the conclusions of Horsley about the p-resonance scattering.

E. Neutron—α-particle

The interaction here should be the same as between proton and α-particle, except for the Coulomb force. Therefore the scattering should be spherically symmetrical for neutron energies up to 2 MV. Using Taylor's potential hole for the proton-alpha-interaction, the scattering cross section should be of the order $6 \cdot 10^{-24}$ cm^2 for slow neutrons (below about 1 MV). Experimental investigations of the angular distribution and the scattering cross section are not available, in spite of the fact that recoil helium nuclei have been used as a means of detecting neutrons (B40).

F. α-particle—α-particle

More extensive work has been done on this scattering process, both theoretically and experimentally, than on any of the processes previously mentioned. The theoretical treatment is slightly different from that of the other processes because the symmetry of the wave function in the two α-particles must be considered (M31, T6, T7, and §18 of this article). α-particles obey Bose statistics (§§4, 48) therefore the wave function must be *symmetrical* in the two particles. The particles possess no spin, so that the symmetry requirement applies to the positional wave function directly. This symmetry condition has the immediate consequence that *no odd angular momenta l occur* in an expansion of the wave function in spherical harmonics; for we have shown in §12 that a wave function containing a spherical

harmonic of odd order l, changes sign when the two particles are interchanged, while an even harmonic remains unchanged. This fact affects, of course, the condition for the appearance of phase shifts other than δ_0 in the scattering: The first partial wave besides $l=0$ is $l=2$ rather than $l=1$; but $l=2$ will only be affected by the nuclear field if the range of the forces is more than twice the effective wave-length $\lambda\!\!\!^{-}$. From Table XXXV we find that $\lambda\!\!\!^{-}=4.5\,E_0^{-\frac{1}{2}}\cdot 10^{-13}$ cm for the scattering of α-particles by α-particles. Estimating the range of the forces to be about $4\cdot 10^{-13}$ cm, we would expect that the second spherical harmonic will contribute to the scattering above about 5 MV. This seems to be roughly correct (see below). The higher spherical harmonics $l=4$ etc. should, according to this reasoning, not come in at energies below about 20 MV. However, there seems to be a resonance level for $l=4$ at about 10 or 12 MV α-energy which increases the influence of the partial wave $l=4$.

The scattering is most conveniently expressed in terms of its ratio to the classical scattering in a Coulomb field,

$$\sigma_{\rm cl}\sin\Theta d\Theta=8\pi\left(\frac{4e^2}{M_\alpha v^2}\right)^2\left(\frac{1}{\sin^4\Theta}+\frac{1}{\cos^4\Theta}\right)\times\sin\Theta\cos\Theta d\Theta, \quad (638)$$

where M_α is the mass of the α-particle and $\Theta=\frac{1}{2}\vartheta$ its deflection in the *ordinary* coordinate system. The first term in the bracket, $(\sin\Theta)^{-4}$, gives the number of α-particles deflected through Θ; the second term, $(\cos\Theta)^{-4}$, gives the number of α's deflected through $\frac{1}{2}\pi-\Theta$ each of which is accompanied by a recoil α-particle in the direction Θ (cf. beginning of §14, and (604c)). The ratio of the actual scattering to the Rutherford scattering (638) is (cf. (636); §18; M32, Chapter V, or T6, T7)

$$P=\frac{\left|\dfrac{e^{-i\alpha\log\sin^2\Theta}}{\sin^2\Theta}+\dfrac{e^{-i\alpha\log\cos^2\Theta}}{\cos^2\Theta}+\dfrac{2i}{\alpha}\sum_{l\ \text{even}}(2l+1)(e^{2i\delta_l}-1)\dfrac{(1+i\alpha)^2\cdots(l+i\alpha)^2}{(1+\alpha^2)\cdots(l^2+\alpha^2)}P_l(\cos 2\Theta)\right|^2}{(\sin^{-4}\Theta+\cos^{-4}\Theta)}, \quad (639)$$

where

$$\alpha=4e^2/\hbar v. \quad (639a)$$

The first two terms arise from the symmetrization of the wave function in incident and scattering α-particle. These terms alone give deviations from the classical scattering which were first pointed out by Mott (M31). In particular, for $\Theta=45°$, we have $\sin^2\Theta=\cos^2\Theta=\frac{1}{2}$ which makes the two terms equal and gives $P=2$. Thus at 45° the scattering of α-particles is twice as large as it would be in classical mechanics, even if there is no action of the nuclear force at all. This has actually been proved experimentally for slow α-particles by Chadwick (C3). At angles other than 45°, the deviations from classical scattering change periodically with the velocity and the angle Θ, giving sometimes larger, sometimes smaller scattering than classical. This effect has been checked for various angles and velocities by Blackett and Champion (B27).

The sum over l in (639) represents the influence of the nuclear field. The factor 2 in front of it arises again from the symmetrization of the wave function. The sum extends only over the even values of l, again because of the symmetry requirement. In all other respects, the expression is analogous to (636).

Experiments were carried out by Rutherford and Chadwick (R25), by Chadwick (C3), by Wright (W22) and by Mohr and Pringle (unpublished). A theoretical analysis was made by Taylor (T6, T7), Wright (W22) and recently by Wheeler (to be published in Phys. Rev.).[56] Wheeler, whose analysis is the most complete, represented the observations in terms of the three phase shifts $\delta_0\delta_2\delta_4$ for the three lowest values of l possible. For each energy, the phase shifts are determined provisionally from the scattering at three angles, preferably 19.6°, 27.4° and 35.0° because for these angles either $P_2(2\Theta)$ or $P_4(2\Theta)$ vanishes. In general, 8 different sets of values for $\delta_0\delta_2\delta_4$ satisfy the data for the three angles. The experimental scattering at a fourth angle, e.g. 45°, is then used to decide between these 8 sets, the decision being in general unique.

[56] We are indebted to Professor Wheeler for communication of his results before publication.

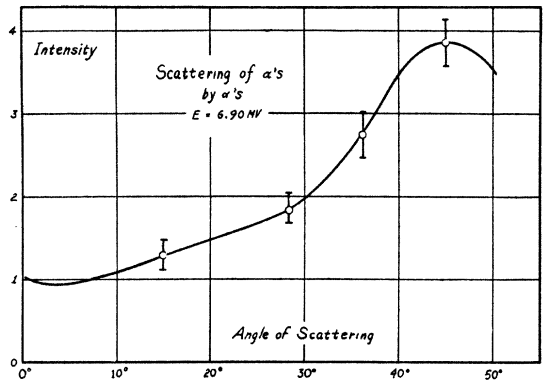

FIG. 20. Scattering of α-particles of 6.9 MV by He nuclei. Solid curve calculated by Wheeler to pass through the four experimental points indicated. (We are indebted to Professor Wheeler for the permission to publish this and the following figure.)

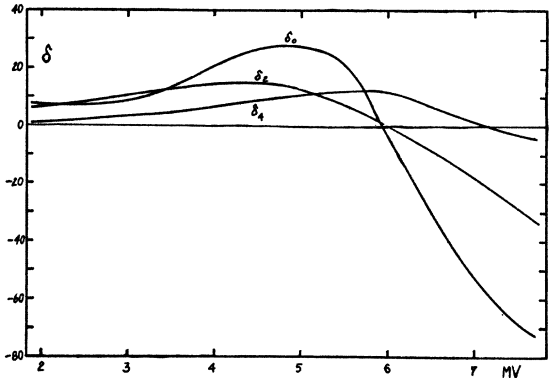

FIG. 21. Phase shifts (in degrees) of the partial waves $l=0$, 2 and 4 in the scattering of α-particles by α-particles according to Wheeler.

The agreement obtained with experiment is very satisfactory, as can be seen from Fig. 20 which shows the scattering of α-particles of 6.9 MV, the solid curve being obtained from (639) with the suitably chosen values of the δ's.

Fig. 21 gives the phase shifts obtained in this way, as functions of the α-particle energy. The δ's are quite small and vary regularly up to about 5.5 MV. Above this energy, a rather sudden change occurs which is probably connected to the level of Be^8 at 3 MV excitation energy which is known from the nuclear reaction $B^{11}+H^1$ (§88). (The energy in the center of gravity system is half the kinetic energy of the incident α-particle!) —Wheeler has also tried to calculate the scattering on purely theoretical grounds, using only the known forces between neutrons and protons. These calculations, being very laborious, are not yet completed.

§75. Other Experimental Results on Scattering (R8, R9, W11)

Most experimental results on scattering have been obtained with α-particles as the scattered particles. The most comprehensive experimental paper is by Riezler (R8). In addition, older experiments by Rutherford and Chadwick (R25) and more recent supplementary experiments by Riezler (R9) must be mentioned.

Wenzel (W11) has analyzed Riezler's results theoretically.[57] His analysis is based on the one-body model (§73) which, as we have mentioned, is justified (to the extent that it is used) if both the scattered particle and the scattering nucleus have spin zero. For the scattered particle (α-particle) this is fulfilled, for the scattering nucleus it is true in the case of C, O and Ne (among the nuclei investigated), but not for Be, B and Al. In fact, Wenzel's analysis gives very satisfactory results for carbon, and much less satisfactory ones for Be and B (the other cases were not analyzed).

If the spins of both nuclei are zero, the orbital momentum l of the particle is identical with the angular momentum J of the compound nucleus. Formula (618) (or (636)) is valid for the scattering, and it is easy to estimate J from the resonance maximum of the scattering (cf. the discussion of carbon, below).

All the results enumerated in the following refer to the scattering of α-particles by the respective nuclei.

Lithium

No experimental data.

Beryllium

Riezler (R8) finds for α-particles of 7.5 MV a scattered intensity exceeding greatly the Rutherford scattering. The ratio (observed scatt./Coulomb scatt.) goes up to 43 for a deflection of 165° (in the center-of-gravity system). Such a large scattering could not be produced if only the wave of orbital momentum zero were affected by

[57] Wenzel's formulae contain an error. In his formula (13) etc. σ_l should be replaced by $2\sigma_l$. The error caused by this is not very great for $l=0$ and 1, but very considerable for $l=2$.

the nuclear field. p-scattering ($l=1$) accounts fairly satisfactorily for the observations if the one-body model is used (Wenzel). In the many-body model a higher orbital momentum is ordinarily required to give the same scattering (§71B). From the size of the Be nucleus and the wave-length of the α-particles it should be expected that just the s-, p- and d-scattering are seriously affected by the nuclear field, which is compatible with our reasoning above. Since the top of the potential barrier lies at about 4 MV for the scattering of α-particles by beryllium, Riezler's α-particles go well over the top. No experiments on the variation of the scattering with the α-particle energy are available.[58]

Boron

Riezler observed the scattering as a function of the angle for 7.5 MV energy, finding a high maximum at 180° as should be expected from (618), (636). He also observed the scattering at 160° as a function of the energy, finding the expected rise with increasing energy. The highest ratio observed is 25 (i.e., observed scattering = 25 times Rutherford scattering) at 167° and 7.5 MV. The increase of the scattering ratio occurs for α-particle energies between 4 and 7 MV. The potential barrier of B for α-particles is expected to be about 5 MV high. Therefore the anomalous scattering is obviously connected with the overcoming of the barrier. The height of the barrier as determined from the experiments turns out in accord with our expectations, thus proving approximate proportionality of nuclear volume and atomic weight (cf. §3, and §25 to §30). From the nuclear radius and the wave-length of the α-particles, we find again that s-, p- and d-particles should be influenced by the nuclear field. In fact, the observed scattering ratio at 7.5 MV is too high to be explained by s scattering alone while p scattering would be sufficient in the one-body model (W11) (cf. the remarks on Be). No resonance maximum of the scattering was observed.

[58] α-particles scattered backwards by Be have an energy of only $(9-4)^2/(9+4)^2=0.15$ times their original energy. If the initial energy is low, the range of the scattered alphas is too short for observation.

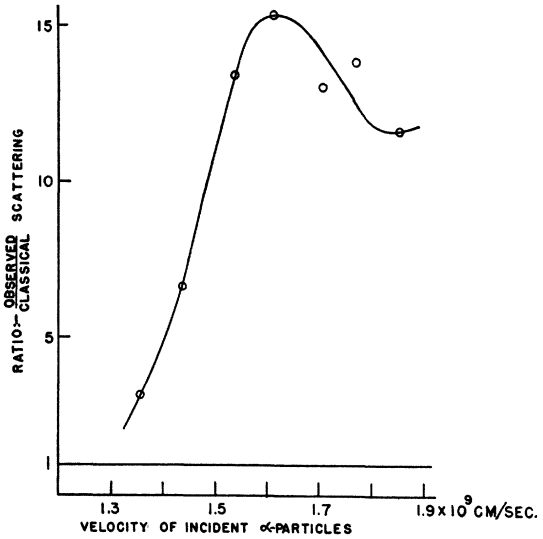

FIG. 22. The scattering of α-particles by carbon, according to experiments of Riezler.

Carbon

In contrast to the scattering by Be and B, the scattering ratio for carbon does not increase continuously with α-particle energy, but shows a marked maximum at about 5 MV (cf. Fig. 22). This is about equal to the height of the potential barrier for C (cf. 609). The observed maximum should be ascribed to resonance, as Wenzel has pointed out. The scattering ratio at the maximum is 15 for 166° deflection. A perturbation of the s scattering alone by the nuclear field could, according to (624), not give a scattering ratio greater than about 3. Therefore the maximum must be ascribed to a resonance level with J different from zero. Indeed, the observed maximum of the scattering ratio agrees closely with the theoretical maximum for p scattering (W11). The width of the resonance peak is of the order of 1 MV, which seems reasonable.

Nitrogen

No observations.

Oxygen

According to the second paper of Riezler (R9), the scattering through about 90° begins to deviate from Rutherford scattering at about 4 MV, *decreases* continuously with the energy, and reaches 0.5 times the Rutherford value at

5.3 MV. The scattering can be explained by assuming a height of the potential barrier of about 6 MV, as compared to about 5.5 from (609).

Neon

Up to energies of 5.3 MV, no deviation from Rutherford scattering was observed by Riezler (R8). This is compatible with the expected height of the barrier (6 MV).

Aluminum

For about 7 MV energy, Riezler found the scattering to be considerably smaller than Rutherford scattering at angles below 158°, and larger for the largest scattering angles. From the magnitude of the deviations, it seems that the energy used (7 MV) is approximately equal to the height of the potential barrier, which agrees with our general formula (609).

Thus the scattering of α-particles by medium light nuclei confirms, at least qualitatively, the proportionality of the nuclear volume with the mass number.

A case of considerable interest, although no experiments are yet available, is the scattering of protons by Li^7. The compound nucleus, Be^8, is known to have a resonance level at 440 kv kinetic energy of the protons. From the general formula (625) it follows that the resonance scattering should increase the Coulomb scattering in the backwards direction by a factor of about 3 (B15, p. 478). In calculating this figure, the assumption has been made that the width of the level is almost entirely due to the disintegration $Be^8 \rightarrow Li^7 + H^1$ which is very plausible because no other process seems possible (cf. §81) except the emission of γ-rays which certainly gives a very small contribution to the width of the level.

XIII. Distintegrations Produced by Charged Particles

§76. Classification of Processes

The general scheme of a nuclear disintegration is, as we know,

$$A + P \rightarrow C \rightarrow B + Q. \qquad (640)$$

The incident particle P falls on the initial nucleus A and combines with it to form the compound nucleus C. The latter splits into an outgoing particle Q and a residual nucleus B. In this chapter, we shall be concerned with processes produced by charged particles P, of which the most important are protons, deuterons and α-particles.

A. The main nuclear process

The processes may first be classified into two types according to the character of the outgoing particle Q. Q may be either a light quantum or a material particle. In the first case, we speak of the simple capture of particle P, in the second case of a disintegration in the strict sense, or a "particle disintegration."

In any nuclear process, charge, mass number, energy and momentum must be conserved. The conservation of energy and momentum will be discussed in detail in §96. It determines the kinetic energy of the outgoing particle in terms of the kinetic energy of the incident particle, and of the difference in internal energy between the initial nuclei $A+P$ and the final nuclei $B+Q$. In the case of simple capture, practically all the momentum will be taken up by the nucleus B, and practically all the energy by the light quantum Q. In the case of particle-disintegrations, energy and momentum will be shared between the two resultant nuclei B and Q (see §96).

A special case of a particle disintegration is scattering (elastic or inelastic). In this case, the outgoing particle is identical with the incident one. Apart from the potential scattering by the Coulomb field, there is no essential difference between scattering and disintegration, and the probabilities of the two processes will, in general, be of the same order. Experimental data on scattering are scarce except for α-particles (§74, 75). Data on inelastic scattering, especially, are almost unavailable and the existence of the process can only be inferred on theoretical grounds and from analogy with neutron experiments (§65C).

Simple capture is ordinarily less probable than particle disintegration, including scattering, because the "radiation width" of nuclear levels is usually smaller than the particle width (§81). Simple capture is therefore probable only when particle disintegrations are impossible or very improbable for some special reason, e.g.,

(1) Because sufficient energy is not available to make up the masses of the possible produced particles, or (2) because the potential barrier is too high for particles which, on energetic grounds, might be emitted, or (3) because only enough energy is available for slow particles to be emitted, so that the particle width is made small by the factor v contained in it (cf. §52B), or (4) because the emission of particles is forbidden by selection rules.

The last case is realized in the simple capture of protons by Li^7 and F^{19} (cf. §81), the first in the capture by C^{12}. The conditions 2 and 3 are fulfilled for the simple capture of slow neutrons by heavy nuclei. The emission of charged particles is then impossible because of the high barriers; the re-emission of neutrons (elastic scattering) is improbable because of the small neutron velocity. In some cases, it is possible to establish the existence of simple capture experimentally, in spite of its small probability. This will be the case when the product nucleus B could not have been produced in any other way and can be detected, e.g., by β-radioactivity.

In any nuclear process, the residual nucleus B may be left either in the ground state or in an excited state. All the excited states which are energetically possible and not improbable because of selection rules (§83) or because of considerations of the penetration through the potential barrier (§78), will in general be formed. To each excited level of nucleus B there corresponds a "group" of outgoing particles Q with a certain definite energy. The group of highest kinetic energy will correspond to the nucleus B being left in the ground state, and the other groups, of decreasing kinetic energy, correspond to excited states of nucleus B of increasing excitation energy. This will be true for simple capture as well as for particle disintegrations. In the case of simple capture, the $h\nu$ of the γ-ray replaces the kinetic energy of the outgoing particle. By measuring the kinetic energies of the various groups of outgoing particles, or the spectrum of the γ-rays from the simple capture process, the excited energy levels of the final nucleus B can be deduced. This has been done in a number of cases in which the outgoing particles are protons, neutrons or α-particles (Chapter XVII, and §109).

The probability is, in general, of the same order of magnitude for an excited final state of the residual nucleus B as for the ground state (cf. §54). Thus, if many excited states are energetically possible, only a small fraction of all disintegration (or simple capture) processes will lead directly to the ground state of the nucleus. Therefore, it may sometimes be difficult to find the group of particles, or the line in the capture γ-ray spectrum, which corresponds to the ground state. On the other hand, the *total* disintegration probability will be greatly enhanced if many final states are possible (§79).

B. Secondary processes due to the residual nucleus

If the residual nucleus is left in an excited state (which, according to the foregoing, is the rule rather than the exception), we may again have two cases. Either the excited state is below the dissociation limit or above it.

1. γ-ray spectrum.—In the first case, the only "particles" which can be emitted by the nucleus B, are light quanta. Therefore the residual nucleus will emit one or more γ-rays until it finally arrives in its ground state. These γ-rays from the residual nucleus must not be confused with the γ-rays emitted in capture processes. The latter are part of the main nuclear process (640) itself, while the former are emitted *after* the main process is finished. γ-rays from the residual nucleus have a discrete spectrum depending only on the levels of nucleus B, while the frequency of the capture γ-rays depends on the kinetic energy of the incident particle so that their spectrum is continuous if the incident particles are not "monochromatic," i.e., if they do not all have the same energy.

The spectrum of the γ-rays from the residual nucleus has been observed in several cases (§89, 99–102). It provides a very important check on the scheme of energy levels of nucleus B derived from the groups of particles emitted in the nuclear process itself. In cases for which these

groups have not been measured, the γ-rays from the residual nucleus may also be used directly to obtain a scheme of energy levels; however, the analysis of the spectrum is often difficult, especially if there are many lines.

It need hardly be mentioned that γ-rays from the residual nucleus may be emitted after capture processes as well as after particle disintegrations. In the former case, it is extremely difficult to separate the two kinds of γ-rays.

2. Three-particle disintegration.—If the residual nucleus B is, in the main nuclear process, left in an excited state above its dissociation energy, it may break up further, emitting material particles according to the scheme

$$B^* \rightarrow D + S. \tag{641}$$

The asterisk denotes an excited state, D is the second residual nucleus, and S the second emitted particle. Experimentally, such a process will appear as a *three-particle disintegration*, according to the scheme

$$A + P \rightarrow C \rightarrow B^* + Q \rightarrow D + S + Q. \tag{641a}$$

Such three-particle disintegrations have been observed in a number of cases (§85, 79, 80, 101D, E, 102D) and it could be shown that the mechanism is actually as indicated by the scheme (641a), at least in the case

$$B^{11} + H^1 \rightarrow C^{12} \rightarrow Be^{8*} + He^4 \rightarrow 3He^4. \tag{641b}$$

It seems that a breaking up into three particles in a single process is very improbable.

The emission of a material particle S by the excited residual nucleus B^*, when energetically possible, is in general more probable than the emission of γ-rays, just as, in the "main" nuclear process, particle disintegrations are more likely than simple capture. The reasons and the exceptions are the same as discussed above, in A.

A particular example of a three-particle disintegration is the case in which one or both of the emitted particles S and Q are of the same kind as the incident particle P, so that we have in effect the reaction

$$A + P \rightarrow D + P + Q. \tag{642}$$

Then it appears from the final result as if particle P had not taken part at all in the reaction, except by giving part of its energy to the other particles. The reaction appears as a noncapture disintegration according to the scheme

$$A \rightarrow D + Q. \tag{642a}$$

However, it should be remembered that actually the mechanism of the reaction is more complicated. The incident particle is first absorbed by the initial nucleus A, to form the compound nucleus C. It is then reemitted, A being left in an excited state. This excited state then disintegrates finally into $D+Q$. Or else, the compound nucleus may first emit particle Q, leaving the residual nucleus B in an excited state, whereupon the reemission of particle P follows as the last stage. The two schemes of "noncapture disintegrations" are thus:

$$A + P \rightarrow C \rightarrow A^* + P, \quad A^* \rightarrow D + Q \tag{642b}$$

or

$$A + P \rightarrow C \rightarrow B^* + Q, \quad B^* \rightarrow D + P. \tag{642c}$$

Both are, in principle, equally probable.

"Noncapture" disintegrations seem to occur in several cases with fast neutrons, the neutron losing part of its energy which is used to knock another neutron out of the nucleus (§85, 102D).

The same statements could be made about nucleus D, as about B. D may again be formed in an excited state and may again emit γ-rays, or, if excited above the dissociation energy, may disintegrate with emission of another particle, etc.

C. β-radioactivity of final nucleus

The nucleus which is finally formed, after all particles, γ-rays, etc. have been emitted, may be β-radioactive. It seems preferable not to include this β-decay in the nuclear reaction at all, not even in the secondary reactions of the type (641), because the time required for the β-decay is of an entirely different order of magnitude than the time of nuclear reactions. Even the slowest nuclear reactions are completed in about 10^{-13} sec., this figure corresponding to a width of the compound nuclear level of about 0.01 volt which is smaller than all widths found experimentally thus far (cf. §61, Table XXVI and §81). The same figure will represent something of an upper limit to the lifetime of the excited levels of the residual nucleus B against γ-emission, except possibly for low metastable levels (§87D). On the other hand, the shortest lifetime ever observed for a β-radioactive nucleus is 1/50 sec., with most

lifetimes ranging from a few seconds to a few years. This is 10^{11}, or more, times longer than the duration of a nuclear reaction.

Thus it seems justified to draw a definite line between the nuclear process itself and the β-emission which may follow it. From this standpoint we have to say that electrons (positive or negative) will never be produced in a nuclear reaction, but can only be given off later, by the nucleus produced in the reaction. The interaction of nuclei with the Fermi field of electrons is so much weaker than all other interactions that we can neglect it entirely in nuclear dynamics. We consider the interaction of the compound nucleus with the "fields" of heavy particles of all kinds, and with the electromagnetic radiation field, but we leave out the extremely small interaction with the β-ray field. For this reason, β-emission was described in part A of this report, which dealt with stationary states of nuclei rather than with dynamic processes.

The chief function of β-radioactivity in the study of nuclear reactions is to provide a convenient indicator of the production of certain product nuclei. In fact, most of the capture processes of neutrons (Chapter X) and many of the reactions produced by charged particles (this chapter) have been discovered through detecting the radioactivity of the product nucleus.

β-emission may also lead to an excited state of the final nucleus which will therefore emit γ-rays. These γ-rays can easily be separated from the γ-rays produced in the original nuclear reaction (capture γ-rays) or emitted by the residual nucleus of the original reaction, because the γ-rays emitted after β-emission appear with a time lag equal to that of the β-emission itself.

§77. General Theory

The general expression for the disintegration cross section is (cf. (260))

$$\sigma^{P_p}{}_{Q_q}=\frac{\pi\lambda^2}{(2s+1)(2i+1)}\sum_J(2J+1)\times\left|\sum_r\frac{u^{rJ}{}_{P_p}\,u^{rJ}{}_{Q_q}}{E-E_r+\frac{1}{2}i\gamma_r}\right|^2. \quad (643)$$

The notations are the same as in §52. The matrix element $u^{rJ}{}_{P_p}$ contains the wave function of the incident particle P, and therefore (cf. §72) the factor $P_{P_p}^{\frac{1}{2}}$ where P_{P_p} is the transmission probability (penetrability) of the potential barrier of the initial nucleus A for the incident particle P. We put

$$u^r{}_{P_p}=w^r{}_{P_p}\,P_{P_p}^{\frac{1}{2}}, \quad (644)$$

where w is the matrix element in the absence of the potential barrier and P the penetrability. Similarly,

$$u^r{}_{Q_q}=w^r{}_{Q_q}\,P_{Q_q}^{\frac{1}{2}}, \quad (644a)$$

where P_{Q_q} is the penetrability for the outgoing particle. It will depend on the kind Q of the outgoing particle as well as on its energy which, for given energy E of the incident particle, is determined by the quantum state q of the residual nucleus. The "matrix elements without barrier" w are connected to the level widths without barrier, G, introduced in (594), by

$$G^r{}_{Q_q}=(w^r{}_{Q_q})^2. \quad (644b)$$

For a given energy E of the incident particle, the penetration factors P are the same for all levels of the compound nucleus (§52). Therefore we may write

$$\sigma^{P_p}{}_{Q_q}=P_{P_p}P_{Q_q}\,s^{P_p}{}_{Q_q}, \quad (645)$$

where

$$s^{P_p}{}_{Q_q}=\frac{\pi\lambda^2}{(2s+1)(2i+1)}\sum_J(2J+1)\times\left|\sum_r\frac{w^r{}_{P_p}\,w^r{}_{Q_q}}{E-E_r+\frac{1}{2}i\gamma_r}\right|^2. \quad (645a)$$

The cross section is thus a product of three factors, *viz.*

(1) The penetrability of the potential barrier for the incident particle P_{P_p}
(2) The penetrability for the outgoing particle P_{Q_q}
(3) The "internal disintegration probability" $s^{P_p}{}_{Q_q}$.

If either the incident or the outgoing particle, or both, have sufficient energy to go over the top of the barrier, the corresponding penetrability factors must be replaced by unity.

The discussion of formula (645, 645a) is conveniently carried out separately for light and for heavy nuclei.

A. Light nuclei

The spacing between the resonance levels is known to be wide, of the order of 1 MV (cf. §81, 82). The resonances will therefore in general be observable if particles of suitable energy are available, and if the width of the resonance levels is small compared to the spacing so that there is actually a marked increase of the scattering cross section at resonance. This latter condition will be fulfilled if the energy of the resonance level is not too high. For very high excitation energy, the ratio of the total width to the spacing is known to increase rapidly (§54). Therefore it is improbable that resonances will be observed with deuterons as the incident particle, since the high internal energy of the deuteron makes the energy of any compound nucleus which is formed by adding a deuteron to an initial nucleus, very high also.

1. In the neighborhood of a fairly narrow resonance level, the penetrabilities P_{Pp} and P_{Qq} may be regarded as constant and be replaced by their values at exact resonance. (645a) reduces to a single term, and (643) to the well-known one level formula

$$\sigma^{Pp}{}_{Qq}=\frac{\pi\lambda\!\!\!^{-2}(2J+1)}{(2s+1)(2i+1)}\,\frac{\Gamma^r{}_{Pp}\Gamma^r{}_{Qq}}{(E-E_r)^2+\frac{1}{4}\Gamma_r{}^2} \tag{646}$$

with

$$\Gamma^r{}_{Pp}=G^r{}_{Pp}P_{Pp},\quad \Gamma^r{}_{Qq}=G^r{}_{Qq}P_{Qq}. \tag{646a}$$

The G's are the partial widths without barrier, the Γ's the widths with barrier, and the P's the penetrabilities of the barriers at resonance. The total width Γ_r may or may not contain other partial widths besides those corresponding to the emission of Pp and Qq. The experimental evidence on resonance disintegration, and the determination of the partial and total widths from the experimental results, will be discussed in §81, 82.

2. For very low energies there will certainly be a region in which the penetrability factor P_{Pp} in (645) changes much more rapidly than the internal disintegration probability s. This region corresponds to the $1/v$ region in neutron disintegrations (§58B). The extension of the "penetrability region" is determined in a way similar to that of the $1/v$ region for neutrons. The penetrability factor will govern the disintegration function as long as the kinetic energy of the incident particle is small compared to the energy or to the width of the first resonance level, whichever is larger.

A narrow resonance level with small partial widths $\Gamma^r{}_{Pp}\Gamma^r{}_{Qq}$ can be disregarded for the purpose of determining the extension of the "penetrability region." Such a level will only interrupt the general trend of the disintegration function for a small energy interval.

In the "penetrability region," the cross section may be written

$$\sigma=\text{const}\cdot P_{Pp}P_{Qq}/E, \tag{647}$$

since the factor $\lambda\!\!\!^{-2}$ in (645a) is proportional to $1/E$. This formula was first suggested by Gamow and is well confirmed for small energies of the incident particle (§78).

For higher energies, the fluctuations of the internal disintegration cross section (645a) with energy will greatly affect the disintegration function. It will then no longer be possible to deduce the penetrability of the barrier directly from the disintegration function, as has been done in the past. Some of the important factors besides the penetrability are discussed in §79.

B. Heavy nuclei

The spacing between the levels of heavy nuclei is known to be of the order of only a few volts, from neutron experiments (§60). No available source gives fast charged particles homogeneous enough in energy to observe such closely spaced levels. Thus we cannot hope to observe resonance levels in heavy nuclei with charged particles. All we can observe is an *average* cross section, averaged over an energy interval large compared to the spacing of levels. This average has been calculated in §56 (405).

The rather intricate influence of the angular momentum is treated in §78B. As will be shown there, we may write

$$\Gamma^J{}_{Pp}=G_{Pp}P_{Pp}\Pi^J{}_{Pp}. \tag{648}$$

Here G_{Pp} is the width without potential barrier which is, in the average, independent of J, and is connected to the "sticking probability" ξ (§§54, 56) approximately by (cf. 360):

$$G_{Pp}=(1/2\pi)(2i+1)(2s+1)D_J\xi_{Pp}. \quad (649)$$

P_{Pp} is the penetrability of the potential barrier for particles of zero orbital momentum, and Π a factor which decreases with increasing J and takes into account the dependence of the penetrability on the angular momentum. A formula similar to (649) holds also for particle Q.

(405) may now be rewritten

$$(\sigma^{Pp}{}_{Qq})_{Av}=\pi\lambda\!\!\!^{-2}\xi_{Pp}P_{Pp}\,l_c^2(\Gamma_{Qq}/\Gamma)_{Av}, \quad (650)$$

where the factor l_c^2 comes from the summation over the angular momenta J (cf. §78) and increases slowly with the energy, while Γ_{Qq} is a suitable average of the $\Gamma^J{}_{Qq}$ over J. Two cases should now be distinguished:

(a) The total level width Γ contains mainly contributions which are not very sensitive to the energy, i.e., contributions from the emission of neutrons or of charged particles of energy high enough to go over the top of the potential barrier. Then it is convenient to replace Γ_{Qq} by $G_{Qq}P_{Qq}$ and to write

$$(\sigma^{Pp}{}_{Qq})_{Av}=s^{Pp}{}_{Qq}P_{Pp}P_{Qq}, \quad (651)$$

where the "internal disintegration probability" s is defined by

$$s^{Pp}{}_{Qq}=\pi\lambda\!\!\!^{-2}l_c{}^2\xi_{Pp}G_{Qq}/\Gamma \quad (652)$$

and changes slowly with the energy as compared to the penetrabilities P_{Pp} and P_{Qq}.

(b) The total width contains mainly contributions from the emission of slow charged particles which must penetrate the potential barrier. Then the influence of the potential barrier on σ is expressed by

$$\sigma\sim P_{Pp}P_{Qq}/P_{Av}, \quad (651a)$$

where P_{Av} is an average of the penetrability over all charged particles which can be emitted.

§78. Penetration Through Potential Barrier

As we mentioned in §77, case A2, there will always be a region at low energies in which the disintegration function is governed by the probability of penetration through the potential barrier, *viz.*

$$\sigma=\text{const}\cdot P_{Pp}P_{Qq}/E. \quad (653)$$

P_{Pp} is the penetrability for the incident, P_{Qq} that for the outgoing particle and E the energy of the incident particle. The penetrability may be written in the form (600):

$$P=e^{-2g\gamma(E/B)}, \quad (600)$$

where g is given in (600a), γ in (600b) and B is the height of the potential barrier (609).

In Table XXXVI (see also Fig. 23), we have compared the "Gamow disintegration function" (653) to the experimental data in a few cases. The agreement is fairly satisfactory for the reactions $Li^6+H^2\rightarrow 2He^4$, $Li^7+H^1\rightarrow 2He^4$ in which only incident particles of energy well below the potential barrier were used. In other cases, the agreement is not so satisfactory especially when the particle energies extend over a wide range. This is to be expected, according to the general theory given in §77, because the internal disin-

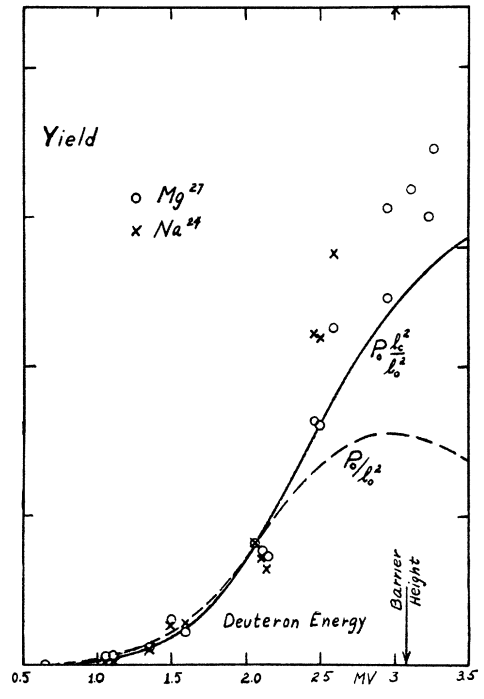

FIG. 23. Yield of the disintegrations of Mg^{26} by deuterons. Solid curve theoretical penetration function, broken curve same without consideration of angular momentum. Circles and crosses observed points for the formation of Mg^{27} and Na^{24} respectively. It is seen that the experimental disintegration functions rise more steeply than the "theoretical" curves. The agreement is slightly better when the angular momentum is taken into account. The remaining discrepancy should be attributed to the increasing number of possible states of the final nucleus (§79).

TABLE XXXVI. *Excitation functions of nuclear reactions. Energies in kv, excitation functions in arbitrary units. All excitation functions are reduced to thin targets.*

	Excit. Funct.			Excitation Function		
Energy kv	Theor.	Exp.	Energy kv	Theor. $l=0$	Theor. all l	Exp.
$Li^6+H^2=2He^4$ Oliphant, Kinsey and Rutherford (O3)			$Be^9+H^2=B^{10}+n^1$ Crane, Lauritsen and Soltan* (C38)			
			320	1.0	0.95	0.3
120	0.33	0.19	400	2.2	2.1	1.5
130	0.50	0.43	440	2.9	2.8	2.5
147	0.92	0.92	480	3.8	3.8	3.7
158	1.26	1.16	520	4.8	4.8	4.8
173	1.52	1.82	560	5.7	5.8	6.6
182	2.13	2.42	600	6.8	7.0	8.8
			640	7.6	7.9	10.7
$Li^7+H^2=2He^4+n^1$ Oliphant, Kinsey and Rutherford (O3)			$F^{19}+H^1=O^{16}+He^4$ Henderson, Lawrence and Livingston† (H22)			
104	0.34	0.10	730	0.74	0.65	0.14
115	0.51	0.29	1050	2.5	2.35	1.1
130	0.94	1.01	1180	3.4	3.3	1.45
151	1.61	1.88	1280	4.0	4.0	4.0
175	3.2	2.8	1420	5.0	5.2	5.7
			1770	7.1	8.3	12.9

* The energies given by the authors have been reduced by 20 percent (see §92C).

† The energies have been deduced from the *ranges* given. They are in the average 8 percent higher than the *energies* given by the authors, which were also deduced from ranges but with the help of an incorrect range-energy relation.

tegration probability s will change appreciably over energy regions of the order of a MV or more.[59]

A. Influence of the angular momentum, theory

Except for the very lightest nuclei, particles with various orbital momenta will be effective for the disintegration. Attention must then be paid to the dependence of the disintegration probability on the orbital momentum l of the incident particle. We shall treat this problem with the following simplifying assumptions which will correspond to reality for sufficiently heavy nuclei:

(1) The density of levels is supposed to be so great that only the average cross section will be measured (cf. §56, §77B).

(2) The contribution of each orbital momentum l is proportional to the corresponding penetrability of the potential barrier P_l (cf. (631) etc.).

Furthermore we assume

(3) The spins of scattering nucleus and scattered particle are zero, so that the orbital momentum l of the incident particle is identical with the angular momentum of the compound state. This restriction can easily be removed.

The averaged cross section is given by (405), *viz.* ($i=s=0$)

$$(\sigma^{P_p Q_q})_{Av}=2\pi^2\lambda^2\sum_J(2J+1)\left(\frac{\Gamma^J{}_{P_p}\Gamma^J{}_{Q_q}}{D_J\Gamma_J}\right)_{Av}. \quad (654)$$

We insert

$$\Gamma^J{}_{P_p}=G_{P_p}P_J, \quad (655)$$

where G_{P_p} is the average partial width without barrier which is supposed to be the same for all values of J, and P_J is the penetrability for particles of orbital momentum J. (655) corresponds to assumption 2 above. For the outgoing particle we assume an energy sufficient to carry it over the top of the barrier, and to give $\Gamma^J{}_{Q_q}$ practically the same value for all values of J for which the penetrability P_J for the incident particle is appreciable. Then (654) may be replaced by

$$\sigma^{P_p}{}_{Q_q}=2\pi^2\lambda^2(G_{P_p}\Gamma_{Q_q}/D\Gamma)_{Av}\sum_J(2J+1)P_J. \quad (655a)$$

The sum over J may, for energies below the top of the barrier, be extended from 0 to ∞ because there is no restriction on J but that imposed by the decrease of the penetrability P_J with increasing J. (655a) may also be expressed in terms of the sticking probability for the incident particle, using (649a):

$$\sigma^{P_p}{}_{Q_q}=\pi\lambda^2\xi_{P_p}(\Gamma_{Q_q}/\Gamma)\sum_J(2J+1)P_J. \quad (656)$$

The penetrability as a function of $l=J$ was calculated at the end of §72. From the formulae given there, we may compute the "effective orbital momentum" l_c which we define by

$$l_c^2=\sum_l(2l+1)P_l/P_0. \quad (657)$$

[59] Henderson, Lawrence and Livingston (H22) to whom the experiments on the disintegration function of the reaction $F^{19}+H^1$ are due, found what seemed to be good agreement with the "Gamow formula" (653). However, they approximated the function γ in the penetrability formula (600) by

$$\gamma'=\tfrac{1}{2}\pi x^{-\frac{1}{2}}-2, \quad (a)$$

which gives for the penetrability the "simplified Gamow formula" (589b), i.e.

$$P=\text{const}\cdot\exp(-2\pi e^2 zZ/\hbar v). \quad (b)$$

The use of this formula is unjustified if the energy of the particle is comparable with the height of the barrier.

The details of the straightforward calculation will be given in a future paper by Bethe and Konopinski. Fig. 19 gives l_c^2/g^2 as a function of x, where x is the ratio of particle energy to barrier height and g the characteristic angular momentum given in Table XXXIII. One curve ($g=5$) corresponds approximately to the disintegration of Zn by deuterons, the other ($g=25$) to the disintegration of uranium by α-particles. For comparison, the figure contains also $l_0^2/g^2=x$ where l_0 is the "critical orbital momentum" of neutral particles $l_0=R/\lambda\!\!\!^{-}$.

The effective orbital momentum l_c becomes equal to l_0 at high energies because then the Coulomb potential may be neglected. At somewhat lower energies but still above the barrier, we have $l_c<l_0$ because the combined action of Coulomb repulsion and centrifugal force produces an "effective potential barrier" for some of the orbital momenta just below l_0. As the energy decreases, this effect becomes more pronounced. For an energy just equal to the potential barrier, (cf. (634a))

$$l_c(B)\approx g^{\frac{1}{2}}, \tag{657a}$$

while $l_0(B)=g$ (by definition, cf. 611). The ratio is here $l_c/l_0\sim g^{-\frac{1}{2}}$ which is about 1/3 for uranium $+\alpha$-particles. For energies below the potential barrier, l_c does not decrease much further while l_0 keeps on decreasing as $E^{\frac{1}{2}}$. At very low energies ($<\frac{1}{2}B$), l_c reaches a *finite limit* (cf. (633))

$$l_c(0)=(\tfrac{1}{2}g)^{\frac{1}{2}}, \tag{658}$$

whereas l_0 goes to zero. This effect, which was already discussed in §72, means that, at low energies, particles of higher angular momentum penetrate the nucleus *relatively* more easily when they are charged than when they are neutral (relatively compared to particles of $l=0$; the absolute penetration probability is of course very much less for charged particles). The Coulomb potential, so to speak, carries the particles smoothly into the nucleus.

With our definition of l_c (cf. (657)), we obtain for the cross section

$$\sigma\sim\lambda\!\!\!^{-2}P_0 l_c^2, \tag{659}$$

if we neglect (cf. 655a) the variation of Γ_{Qq}/Γ and of G_{Pp} with energy. (659) represents the corrected penetration function of the potential barrier which replaces the simple Gamow function $\lambda\!\!\!^{-2}P_0$. The formula, as well as the considerations leading to it, are valid if we assume that the critical orbital momentum l_c' for the outgoing particle is much greater than l_c. If the reverse is true, the critical orbital momentum will be determined by the outgoing rather than by the incident particle. If incident and outgoing particle have about the same l_c, a rule-of-thumb formula for the cross section is

$$\sigma\sim\lambda\!\!\!^{-2}P_0 l_c^2 l_c'^2/(l_c^2+l_c'^2), \tag{659a}$$

where the primed quantities refer to the outgoing, the unprimed ones to the incident particle.

B. Physical consequences of the angular momentum

The behavior of l_c as a function of the energy has consequences for the cross section, the angular distribution of disintegration particles and the occurrence of resonances.

(1) *The cross section* is given by (659).

(a) At very low energies, l_c^2 becomes constant and different from zero. In this region, the cross section depends on the energy as in the Gamow theory, i.e., as $E^{-1}P_0$. However, the magnitude of the cross section is considerably (by a factor l_c^2) larger than would be expected if only particles of zero orbital momentum could enter the nucleus, and also larger than the cross section for neutral particles multiplied by the penetrability P_0. The factor l_c^2 goes, according to (658), up to 13 for α-particles on uranium. The effect of the angular momentum is thus quite appreciable although very small compared to the effect of the penetrability P_0.

(b) At higher energies (about $\frac{1}{3}B$ to $2B$), the cross section is *smaller* than that for neutral particles times the penetrability P_0. The ratio is given by l_c^2/l_0^2 which reaches a minimum of about 0.05 for α-particles on uranium.

(c) The cross section for charged particles *increases considerably for energies higher than the potential barrier*. E.g., for α-particles on uranium, the expression (659) increases by a factor 11 from an energy just equal to the height of the barrier, to twice that energy. Since $P_0=1$ in this region, the increase is entirely due to the increased importance of higher angular mo-

menta. This effect is probably the most important effect of the angular momentum on the cross section; it means that the disintegration function will, in general, keep increasing beyond the top of the potential barrier so that barrier heights deduced from disintegration experiments are apt to be too high.

(2) The angular distribution of the disintegration particles will not be spherically symmetrical, even at zero energy of the incident particle, because even then orbital momenta of the incident particle up to l_c are effective. This was first pointed out by Teller (unpublished). It is remarkable that, in this respect, charged incident particles behave quite differently from neutral ones which always give, at small energy, spherical distribution of the disintegration products.

(3) The rapid increase of the number of effective angular momenta above the top of the potential barrier accounts for the fact that in α-particle disintegrations, resonance maxima can only be observed for α-energies up to approximately the height of the barrier (§82).

Table XXXVI and Fig. 23 give examples of the influence of the angular momentum on the disintegration function. In Table XXXVI we have examples of reactions on which measurements were made only well below the top of the barrier ($B=1.6$ MV for Be^9+H^2 and $B=2.5$ MV for $F^{19}+H^1$). The effect in this energy region is not very great, but the agreement with experiment is slightly improved.

The same is true for the reactions $Mg^{26}+H^2=Mg^{27}+H^1$ and $Mg^{26}+H^2=Na^{24}+He^4$ whose experimental disintegration functions are shown in Fig. 23 (data from Henderson, H23). The reaction giving α-particles has, experimentally, a steeper disintegration function. From the standpoint of the theory presented in this section, the disintegration functions should be approximately the same because in both cases the produced particle can go over the top of the potential barrier. (In §79, the reason for the difference will be explained.) The figure contains the theoretical curves with and without the angular momentum factor l_c^2; the former curve, which rises more steeply, agrees much better with experiment but is still not quite steep enough. As will be explained in the next section, this difference is also understandable.

C. Energies required for given yields

It is often useful to know in advance what energies are required in order to obtain observable results in disintegration experiments. The number of disintegrations per incident particle is approximately

$$p=N\,\delta R\,\pi\lambda\!\!\!^{-2}l_c^2P_P\xi_P\Gamma_Q/\Gamma, \tag{660}$$

where δR is the part of the range of the incident particles in which they are effective and N the number of atoms per cm³ in the disintegrated substance. We want to compute only the order of magnitude and we are especially interested in heavy nuclei because for light ones the observation of disintegrations offers no difficulty. Therefore, we use the following rough approximations: We put $\Gamma_Q=\Gamma$, $\xi_P=1$, $l_c^2=7$, $\lambda\!\!\!^{-}=1.5\cdot10^{-13}$ cm (deuterons of 5 MV). Furthermore we replace $N\delta R$ for the substance by one-third of that quantity for air, corresponding to a stopping power of 3 per atom (cf. §95); we have $N_{\text{air}}=5\cdot10^{19}$ and $\delta R_{\text{air}}\approx5$ cm considering that the particles will only be effective in the first part of their range. With these constants,

$$p\approx10^{-4}P_P. \tag{660a}$$

With a current of incident particles of 10 μa (good average for cyclotrons) the number of disintegrations per sec. is then

$$p\approx10^{10}P_P, \tag{660b}$$

so that, with good detecting apparatus, a penetrability of 10^{-10} should be just sufficient for establishing a process. Table XXXVII gives the energies of protons and α-particles required to give penetrabilities of 10^{-2}, 10^{-5} and 10^{-10}. For deuterons the energies required are about the same as for protons for the larger penetrabilities (10^{-2}) and somewhat higher than for protons for the smaller penetrabilities (10^{-10}).

TABLE XXXVII. *Penetration probabilities for various particles. (The table gives the energy in MV required for given penetration probabilities.)*

Penetration Prob. P	Particle	$Z=3$	7	11	19	29	50	92
10^{-2}	Proton	0.12	0.5	0.8	1.4	2.2	3.7	6.3
	α-particle	0.94	2.2	3.1	4.8	6.7	10.0	16.0
10^{-5}	Proton	0.04	0.15	0.3	0.6	1.1	2.1	4.0
	α-particle	0.34	1.1	1.7	3.0	4.6	7.5	13.0
10^{-10}	Proton	0.02	0.06	0.12	0.3	0.5	1.2	2.5
	α-particle	0.08	0.6	0.9	1.8	2.9	5.2	10.0

§79. The Competition Between the Nuclear Processes

A. General formulae

The cross section for a nuclear process, averaged over an energy region large compared to the spacing between resonance levels, may be written

$$\sigma^{P_p}{}_{Qq} = \pi\lambda\!\!\!^{-2}\xi_{P_p}P_{P_p}l_c^2\Gamma_{Qq}/\Gamma, \qquad (661)$$

where

$$\Gamma_{Qq} = G_{Qq}P_{Qq} \qquad (661a)$$

(cf. (656) to (659)). All the quantities such as Γ and G denote averages over a large number of energy levels in the energy region in question. P_{Qq} is the penetrability for the outgoing particle; it has been assumed that this penetrability does not depend sensitively on the orbital momentum of the particle (otherwise cf. (659a)). P_{P_p} is the penetrability of the incident particle for angular momentum zero, ξ its sticking probability and l_c the critical angular momentum (cf. §78).

(661) may be written as the product of the probability of formation of the compound nucleus and the probability of the particular mode Qq of its disintegration. The probability of formation,

$$\sigma_{P_p} = \pi\xi_{P_p}\lambda\!\!\!^{-2}P_{P_p}l_c^2 \qquad (662)$$

contains the penetration factor $\lambda\!\!\!^{-2}P_{P_p}l_c^2$ which was discussed in detail in the last section, and the "sticking probability" ξ which is expected (§54) to increase slowly with increasing energy. For large energies, $P=1$, $l_c=l_0=R/\lambda\!\!\!^{-}$ (cf. §78) and therefore

$$\sigma_{P_p} = \pi R^2\xi_{P_p}, \quad (E\gg B) \qquad (662a)$$

(cf. (408)). σ_{P_p} is the total cross section of all processes which may be produced by particle P; it depends, at low energies, mainly on the penetrability, at high energies mainly on the sticking probability of particle P.

The fraction of processes which lead to the emission of particle Q and to the state q of the final nucleus, is

$$\Gamma_{Qq}/\Gamma \qquad (663)$$

and the total relative probability for an emission of particle Q by the compound nucleus:

$$\Gamma_Q/\Gamma. \qquad (663a)$$

The probability of emission of a given kind of particles Q is therefore determined by a *competition of this process with other possible processes.*

B. Relative probability of emission of various kinds of particles

From (661a):

$$\Gamma_Q = \sum_q G_{Qq}P_{Qq}. \qquad (664)$$

According to the considerations above and in §54, an individual G_{Qq} will not depend very much on the energy. Therefore the width corresponding to a given kind Q of particles will depend

(1) on the number of possible states q of the final nucleus

(2) on the penetrabilities P_{Qq}.

Both these factors will depend on the energy available in the reaction. Let

$$Q_0 = (M_A + M_P - M_B - M_Q)c^2 \qquad (665)$$

be the energy evolved in the reaction when nucleus B is formed in the ground state; the M's denote the masses of the respective nuclei. Then E_P+Q_0 will be the total energy available. It will, in general, be shared between excitation energy of nucleus B and kinetic energy of particle Q. We may now distinguish between two cases:

(a) The *available energy* E_P+Q_0 is *larger than* the *height of* the *barrier* B_Q for the outgoing particle. Then there will be some states of the final nucleus B for which the penetrabilities P_{Qq} are unity. These states will, then, in general give the largest contribution to the particle width Γ_Q (cf. also Fig. 10, §54). Therefore, if $N(U)$ denotes the number of quantum states of the *final* nucleus B below the excitation energy U, and G_{Qa} the average of G_{Qq} over the various states q of nucleus B, we may write

$$\Gamma_Q = N(E_P+Q_0-B_Q)G_{Qa}. \qquad (666)$$

Thus the particle width increases as the number of available levels of the final nucleus. There will be $N(E_P+Q_0-B_Q)$ different groups of outgoing particles with different kinetic energies. Most of the outgoing particles will have energies between B_Q and $B_Q+\tau_B$ where τ_B is the "temperature" of the residual nucleus B corresponding to the excitation energy $E_P+Q_0-B_Q$ (cf. §54).

(b) The available energy E_P+Q_0 is *smaller than* the *height of* the *barrier* B_Q. Then the particle Q will be compelled to penetrate the barrier, whatever the state q of the final nucleus. In general, the penetrability P_{Qq} will be very much less for highly excited states of B than for the ground state. Therefore the nucleus B will, in this case, in general be formed in a low state. Accordingly, there will be only comparatively few strong groups of outgoing particles. The partial width is of the same order as for the ground state alone, *viz.*

$$\Gamma_Q = G_{Q0}P_{Q0}. \tag{666a}$$

It is obvious that the width Γ_Q in case (a) will ordinarily be much greater than in case (b). Therefore the total width Γ will be determined by those outgoing particles for which the net available energy $E_P+Q_0-B_Q$ is largest. Since E_P is the same for all outgoing particles, the conditions for large contribution to the width are

(1) Large energy evolution Q_0, i.e., the final nucleus and outgoing particle must be as stable as possible.

(2) Low barrier B_Q, i.e., neutrons are preferred compared to protons and deuterons, and these compared to α-particles.

(3) In addition to these factors, of course, the probability of formation G_Q plays a part. This factor is probably not very different for neutrons and protons, but presumably smaller for deuterons and α-particles and very much smaller for γ-rays.

From these conditions we conclude:

(1) The formation of stable nuclei is in general more probable than that of radioactive ones if both can be formed from the same compound nucleus.

(2) The emission of neutrons is, for heavy nuclei, by far the most probable nuclear process whatever the incident particle.

(3) The emission of γ-rays usually gives a negligible contribution to the total width, unless there are special reasons which make the other contributions exceptionally small. Such special reasons seem to exist only in the case of slow neutrons (cf. especially §61, 62) where the γ-ray width is 100 and more times larger than the neutron width. The reason for this is that only *one* final state is possible for the nucleus after neutron emission, *viz.* the ground state of the initial nucleus, but very many states are possible after γ-emission, *viz.* all states of the final nucleus with excitation energies below about 8 MV (binding energy of the neutron).

TABLE XXXVIII. *Average energy evolution for α-particle and proton emission in MV.*

A	50	100	150	200	240
$d\Delta/dA$	−0.9	0	0.9	∼1.9	∼ 2.5
$U_\alpha-U_H$	1.1	3.8	6.5	9.5	11.3
$B_\alpha-B_H$	4.3	6.5	8.0	9.0	10.0

(4) The emission of α-particles will in general have a probability of the same order of magnitude as that of protons in spite of the higher potential barrier for alphas, because this will be compensated by the greater energy evolution. If $\Delta(A)$ is the mass excess (difference between exact mass and mass number) in energy units, as a function of the mass number A, and U_C is the excitation energy of the compound nucleus, the available energy will be for proton emission,

$$U_H = U_C+\Delta(A)-\Delta(H)-\Delta(A-1), \tag{667}$$

for α-emission,

$$U_\alpha = U_C+\Delta(A)-\Delta(\alpha)-\Delta(A-4), \tag{667a}$$

and therefore the difference is:

$$U_\alpha-U_H=\Delta(H)+\Delta(A-1) -\Delta(\alpha)-\Delta(A-4). \tag{667b}$$

If we assume Δ to be a regular function of the atomic number, and insert the values $\Delta(H)=7.5$ MV, $\Delta(\alpha)=3.7$ MV, we have

$$U_\alpha-U_H=3.8\text{ MV}+3d\Delta/dA. \tag{667c}$$

Using the semi-empirical formula for nuclear masses (cf. §30) up to $A=150$, and the average energy of natural α-particles for $A=200$ and 240, we find the values given in Table XXXVIII. In this table, we have also listed the difference in the heights B of the potential barrier for α-particles and protons, according to (609).[60] It can be seen from the table that the difference in energy evolution is, in most cases, almost as great as or even greater than the difference in height of the potential barrier. Apart from the factor G_Q which cannot be estimated easily, the other factors in

[60] The energy evolved in the reaction, U_α and U_H resp., is available in the center-of-gravity system. Therefore it must be compared to the barrier height in the same system

the partial width Γ_Q will be about as favorable for α-particle as for proton emission.

(5) The emission of deuterons is usually energetically unfavorable because of the high internal energy of deuterons.

We therefore come to the conclusion that in most cases the level width Γ will be determined primarily by the neutron emission. If the neutron emission leads to a radioactive nucleus and the nuclear charge is not too high, proton emission may give the main contribution to Γ. For a number of light compound nuclei which are multiples of the α-particle, *viz.* Be^8 C^{12} O^{16} Ne^{20} and, perhaps, Mg^{24} and Si^{28}, the most important mode of disintegration will be the emission of an α-particle. In no known instance is deuteron emission the most probable process. γ-ray emission is the predominant process only for low excited levels of the compound nucleus, *viz.* when no other process is energetically possible or when the available energy for other modes of disintegration is very small.

The ratios of the probabilities of various processes may be of the order ten or a hundred for medium light nuclei. For heavy nuclei, however, these ratios will reach very large values. Consider, e.g., the disintegration of heavy nuclei ($A \approx 200$) by deuterons of about 5 MV. Then the available energy U_B of the residual nucleus will be about 8.5 MV when neutrons are emitted, 9 MV for proton emission and 18 MV for α-emission. Thus protons and α-particles will just be able to go over the top of the barrier when the final nucleus is left in the ground state, and at the best only about 10 levels of the final nucleus will be possible. For neutrons, on the other hand, all the final levels below 8.5 MV can be reached which are perhaps (cf. §53) one million in number. Thus the neutron emission may be about 100,000 times as probable as the emission of a proton or an α-particle.

The probability of the predominant process will have about the same properties as the over-all probability of all possible processes (cf. 661). The probability of all other processes will behave approximately as

$$\sigma^{P_p}{}_Q \approx \sigma_{P_p} \Gamma_Q / \Gamma', \qquad (668)$$

where Γ' is the partial width due to the predominant process. Now the predominant process is usually characterized by a large energy evolution, larger than in the other possible processes. But the partial width Γ_Q increases in general at the greatest rate when the energy evolution is small, whether the increase is due to increased penetrability or to an increase in the number of possible states of the final nucleus. Therefore it is likely that Γ_Q in (668) increases much faster with energy than Γ'. Consequently, *the less probable processes have in general a greater rate of increase with increasing energy.* (An exception is the emission of γ-rays. This process is intrinsically improbable in spite of the fact that the energy evolution is greatest in this case. It therefore becomes even more improbable with increasing energy, cf., e.g., Table XXVIII, §65.)

More quantitatively, we may say that the probability of a less probable process will be approximately proportional to the ratio of the number of possible final levels for this process to the number for the predominant process, if in both cases the produced particle can go over the top of the barrier. If only the particles in the predominant process can do so, the probability of the less probable process will be approximately proportional to the penetrability for the particles produced in this process, divided by the number of final levels possible in the predominant process, the latter factor being presumably less important than the former. Finally, if even the particles of the predominant process have to go through a barrier, the probability of the other process will be proportional to the *ratio* of the transmission coefficients of the barrier for the particles in question and the predominant ones. This is entirely different from the *elementary* theory (one-body theory) in which, of course, a penetrability never appears in the denominator of a cross section formula.

C. Example: The disintegration of Mg by deuterons

An illustration of our considerations is the disintegration of Mg^{26} by H^2 for which the disintegration functions are shown in Fig. 23. The most probable process in this case is no doubt

$$Mg^{26} + H^2 \rightarrow Al^{28} \rightarrow Al^{27} + n^1. \qquad (669)$$

Less probable is

$$Mg^{26} + H^2 \rightarrow Al^{28} \rightarrow Mg^{27} + H^1, \qquad (669a)$$

and still less probable

$$Mg^{26}+H^2\rightarrow Al^{28}\rightarrow Na^{24}+He^4. \quad (669b)$$

The energy evolution in the first case is probably about 5.5 MV (cf. Table LIX), in the second 4.2 MV (Table LVIII), while in the last it is 3.0 MV (Table LVI). The heights of the barriers are 0, 2.8, and 5.35 MV respectively. Therefore we have for the usable energy evolution, i.e. the difference between energy evolution and potential barrier:

$$\begin{aligned} Q-B = {} & 5.5 \text{ MV for neutron emission} \\ & 1.4 \text{ MV for proton emission} \\ & -2.4 \text{ MV for } \alpha\text{-particle emission.} \end{aligned} \quad (669c)$$

In the experiments of Henderson (H22), a deuteron energy up to 3.5 MV was used. Therefore, with neutron emission, all levels of Al^{27} up to an excitation energy of 9.0 MV are possible as final states. The number of these levels will probably amount to about a hundred (cf. Table XXI). With proton emission, Mg^{27} may be left in excited states up to about 4.9 MV excitation energy which may, perhaps, correspond to about 10 possible levels. The nucleus Na^{24} which is left after α-emission, on the other hand, can only have up to 1 MV excitation energy—otherwise the α-particle could not go over the top of the potential barrier. There will hardly be more than one or two excited levels in this region, besides the ground state of Na^{24}. Thus we expect that the neutron emission is much more probable than the proton emission, and the latter in turn more probable than α-emission.

On the other hand, an increase in energy will multiply the number of final levels for Na^{24} at a much faster rate than the number of possible levels for Mg^{27} or Al^{27}. This is due to the fact that in the formula for the density of levels (cf. §53),

$$\log \rho(U)=a+bU^n. \quad (670)$$

(a, b, n constants) the exponent n is smaller than unity ($\frac{1}{2}$ for free particles, 4/7 for the liquid drop model). Therefore the relative increase in the number N of levels below a certain energy U becomes approximately

$$\frac{1}{N}\frac{dN(U)}{dU}\approx\frac{1}{\rho}\frac{d\rho}{dU}=bnU^{n-1}, \quad (670a)$$

which *decreases* with increasing excitation energy U.

These predictions agree with the experimental results of Henderson for the disintegration functions of the reactions (669a, b)which were measured by means of the radioactivity of the produced nuclei. He showed that the disintegration function for the α-reaction (669b) was smaller but increased more steeply (cf. Fig. 23) with energy than that for the proton reaction (669a). The difference in slope cannot be attributed to the penetration of the α-particles through the barrier because the α-particles produced by deuterons of more than 2.5 MV energy are able to go over the top of the barrier. The effect of the angular momentum (§78) can be shown to be small owing to the large mass of the α-particle. Therefore the difference in slope should be attributed to the increase in the number of possible states of the Na^{24} nucleus.

D. Energy distribution of disintegration products

Of great interest is also the distribution of the particles of a given kind among the groups, at a fixed energy of the incident particle. For charged outgoing particles, the intensity in any group is proportional to the penetrability P_{Qq} of the potential barrier for that particular group; for outgoing neutrons, P_{Qq} should be replaced by unity. P_{Qq} (or unity) should be multiplied by $l_c^2(Qq)$ (or $l_0^2(Qq)$) if the critical angular momentum l_c (or l_0) is less for the outgoing than for the incident particle (see (659a) for a more accurate expression). If we assume that the factors l_0^2 and l_c^2 must be taken into account, the number of emitted particles with an energy between E_Q and E_Q+dE_Q is for neutrons

$$n_N(E_Q)dE_Q\sim E_Q\,\rho(E_P+Q_0-E_Q)dE_Q, \quad (671)$$

for charged particles

$$\begin{aligned} n_C(E_Q)dE_Q\sim{} & l_c^2P_Q(E_Q) \\ & \times\rho(E_P+Q_0-B_Q-E_Q)dE_Q. \end{aligned} \quad (672)$$

Here $\rho(U)$ is the density of levels of the final nucleus at an excitation energy U, Q_0 the energy evolution in the reaction, B_Q the height of the potential barrier, $P_Q(E_Q)$ the penetrability of the barrier for particles of energy E_Q.

The density of levels ρ increases with increasing

excitation energy. Therefore neutrons will preferably be emitted with small kinetic energies (cf. §54, 65) of the order of the "temperature" of the residual nucleus B.

For charged particles, the energy distribution (672) is plotted in Fig. 10 (§54). It shows a sharp maximum at a kinetic energy E_Q very near the height B_Q of the potential barrier and falls off rapidly on the low energy side because of the penetration factor, on the high energy side because of the Boltzmann factor. The "width" of the distribution curve is of the order of the "temperature" τ which, for heavy nuclei, is small compared to the barrier height B. The use of the energy distribution of charged particles emitted from heavy nuclei for a determination of the height of the potential barrier seems very promising. Such a determination would probably be more accurate and more direct than that from the excitation function (§78) or the natural α-decay (§68).

E. Secondary processes for very heavy nuclei

As we have shown in B, by far the most probable process for heavy nuclei is the emission of a neutron. According to Section D, the energy E_Q of this neutron is in general of the order of the temperature τ, i.e., about one MV. Therefore the excitation energy of the remaining nucleus is approximately

$$U_B \approx \Delta(P) - \Delta(Q) - (a-1)d\Delta/dA + E_P - E_Q, \quad (673)$$

where a is the mass number and $\Delta(P)$ the mass excess of the incident particle, and the other symbols have the same meaning as above. If the incident particle is a deuteron, we have $\Delta(P) = 13.7$ MV, $\Delta(Q) = 8.3$ MV so that (cf. Table XXXVIII) for $A = 200$:

$$U_B \approx 3.5 + E_P - E_Q. \quad (673a)$$

With deuterons of $E_P = 5$ MV, and a neutron energy of 1 MV, we get

$$U_B \approx 7.5 \text{ MV}. \quad (673b)$$

Now a heavy nucleus is always energetically unstable against α-disintegration, even in its ground state. The energy available for α-decay is approximately, for the ground state and $A = 200$,

$$4d\Delta/dA - 3.7 \approx 4 \text{ MV}. \quad (673c)$$

Therefore a nucleus of 7.5 MV excitation energy may emit α-particles of 11.5 MV. This is more than the energy of the fastest natural α-particles (§69) and, in addition, the nuclear charge is somewhat less than for natural α-emitters. Therefore the emission of α-particles by the residual nucleus will be even more probable than that of the long range α-groups from RaC′ and ThC′. These latter groups have an emission probability somewhat smaller than the probability of γ-emission; consequently, in our case the emission of α-particles will be at least equally probable and perhaps more probable than that of γ-rays. Therefore the primary reaction will, in many cases, be followed by the emission of an α-particle from the residual nucleus. The complete reaction scheme would thus be

$$Z^A + H^2 \rightarrow (Z+1)^{A+2} \rightarrow (Z+1)^{A+1} + n^1 \rightarrow (Z-1)^{A-3} + n^1 + He^4. \quad (674)$$

Some evidence for such a double process was obtained by Cork and Lawrence (C35) in the bombardment of platinum by deuterons. According to our consideration in B, the most probable process should be the capture of the deuteron followed by emission of a neutron. This would lead to the formation of a gold nucleus, and from the most abundant Pt isotopes, $Pt^{194, 195, 196}$, the gold isotopes $Au^{195, 196, 197}$ should be obtained of which the first two should be positron radioactive. Actually, *no gold activity* was observed. This seems very surprising, although not too much emphasis should be placed on negative evidence because the isotopes in question may have very short or very long lives and thus escape detection. Instead of a gold activity, two radioactive isotopes of iridium were found (besides Pt isotopes which are formed by the Oppenheimer-Phillips mechanism, cf. §80). The formation of an Ir isotope would require either a reaction of the type

$$Pt^A + H^2 \rightarrow Ir^{A-2} + He^4 \quad (674a)$$

or a double process of the type (674). The emission of an α-particle in the primary reaction, according to (666), must be (cf. Section B) several thousand times less probable than neutron emission. Thus, if we restrict ourselves to primary processes only, it would be difficult to understand

why radioactive iridium is formed and not radioactive gold. However, the results become immediately understandable when we accept the double process (674). We must only assume that practically all the Au nuclei formed in the primary (dn)-process will disintegrate with α-emission so that as a result Ir remains rather than Au.

According to the scheme (674), the Ir isotopes formed from the most abundant Pt isotopes 194, 195, 196 would be $Ir^{191, 192, 193}$, of which $Ir^{191, 193}$ are the known stable isotopes so that only Ir^{192} can be radioactive. Observed are two active substances, both emitting negative electrons. The second of these must therefore be formed from the somewhat rarer Pt isotope, Pt^{198} (abundance 7.2 percent), and must thus be Ir^{195}. In addition, the formation of Ir^{189} from Pt^{192} might be expected; this iridium isotope should be positron active but should only be formed in very small amounts because the abundance of Pt^{192} is only 0.8 percent.

The emission of an α-particle by the residual nucleus should become even more probable with faster incident particles. It may also occur as a secondary process after the inelastic scattering of very fast neutrons (§65). It is probably restricted to rather heavy nuclei because the energetic instability of the nucleus in the ground state against α-decay is essential. However, very high kinetic energy of the incident deuteron will always produce the effect.

An emission of neutrons by the residual nucleus will occur at still higher deuteron energy. Such a process would be of the type

$$Z^A+H^2=(Z+1)^A+2n, \qquad (674b)$$

i.e., it would lead to an isobar of the original nucleus. If this isobar emits positrons of 1 MV energy, it must be 2 MV heavier than the original nucleus. The two neutrons are 3 MV heavier than the deuteron; in addition, they require about 1 MV kinetic energy each. Thus process (674b) will become probable for deuterons of about 7 MV energy: it will then, by competition, make the process (674) improbable.

F. Validity of assumptions

The considerations given here apply primarily to heavier nuclei and high excitation energies. These two conditions are needed to ensure a sufficient density of levels of the compound nucleus in the energy region in question. The excitation energy of the compound nucleus is

$$U_C=E_P+\Delta(P)+\Delta(A-A_P)-\Delta(A), \qquad (675)$$

where A is the mass number of the compound nucleus and A_P that of the incident particle. If we again assume the mass excess Δ to be a regular function of the mass number—which, of course, is a very crude assumption for all nuclei except the very heavy ones—we have

$$U_C=E_P+\Delta(P)-A_P d\Delta/dA. \qquad (675a)$$

If we take, e.g., nuclei of mass number around 20, for which these considerations probably just begin to become valid, we have $d\Delta/dA\approx -1.3$ MV. Then we find for the most important projectiles:

$$U_C=E_P+13.7+2.6$$
$$=E_P+16 \text{ MV for deuterons} \qquad (675b)$$

$$E_P+\ 8.3+1.3$$
$$=E_P+10 \text{ MV for neutrons} \qquad (675c)$$

$$E_P+\ 7.5+1.3$$
$$=E_P+\ 9 \text{ MV for protons} \qquad (675d)$$

$$E_P+\ 3.9+5.2$$
$$=E_P+\ 9 \text{ MV for } \alpha\text{-particles.} \qquad (675e)$$

Thus the excitation energy of the compound nucleus will always be very high when deuterons are used as projectiles: With 3 MV kinetic energy of the deuterons, the total excitation energy would be about 19 MV (in the average) and the spacing of the energy levels, according to Table XXI only of the order of about 100 volts for atomic weights around 20. Thus we believe that the theory developed here will be applicable to reactions produced by deuterons for atomic weights as low as 20, or even lower. With neutrons, protons and α-particles as projectiles, the excitation energies will be much lower, unless the kinetic energies are extremely high, and the spacing between levels will, for $A=20$, be at least several kilovolts. In fact, in many cases resonance levels have been observed for α-particle disintegrations which have spacings of the order of hundreds of kilovolts. This large spacing is probably partly due to selection rules (§82), partly, perhaps, to incomplete resolution in the

experiments. Quite generally, the properties of the individual levels of the compound nucleus will not average out as completely for neutrons, protons and α-particles as projectiles as for deuterons. However, for atomic weights of 40 to 50 or higher, the "averaged" theory given in this section will probably be true regardless of the incident particle used, if only the energy of the incident particles has an inhomogeneity of several kilovolts or more.

For lighter nuclei the individual properties of the particular nucleus in question will be more important. However, the qualitative results of this section, concerning the relative probability of various processes, the importance of the number of groups of outgoing particles etc. will be valid even for light nuclei, as is shown to a surprising extent by the observed energy distributions of the emitted particles (cf. above, and (366)).

§80. Disintegration by Deuterons (Oppenheimer-Phillips Theory) (O11)

Oppenheimer and Phillips (O11) have suggested that nuclear reactions of the type

$$Z^A + D^2 \rightarrow Z^{A+1} + H^1 \qquad (676)$$

follow a different mechanism from other reactions which makes them more probable and less dependent on energy than would be expected from the ordinary theory. The applicability of the Oppenheimer-Phillips (O-P) theory is restricted to heavy nuclei for which it will, indeed, increase the disintegration probability considerably. For light nuclei (up to about $Z=30$), the difference between the O-P theory and the ordinary Gamow-Condon-Gurney (G-C-G) theory is unobservably small, as will be shown below. The disintegration functions for nuclei such as Na, Al, etc. by deuterons can therefore not be used as evidence for the O-P theory.

The mechanism proposed by Oppenheimer and Phillips may be described as "partial entry" of the deuteron into the nucleus. Since the proton in the deuteron is reemitted in the nuclear process, it is actually not necessary for it to enter the nucleus but it is only necessary for the *neutron* of the deuteron to come inside the nucleus Z^A. This partial entry has the great advantage that the neutron does not need to overcome the potential barrier of the nucleus because the electrostatic repulsion acts only on the proton in the deuteron. The partial entry is possible by virtue of the small dissociation energy of the deuteron (2.2 MV). Due to this fact, the deuteron has a very large radius even in the free state, and the separation between neutron and proton will be increased considerably when the deuteron is subjected to a large electric field near a heavy nucleus which will repel the proton and not act on the neutron.

A. Disproof for light nuclei

The first condition for the applicability of the Oppenheimer-Phillips theory is that it give a greater probability for the entry of a neutron than the ordinary theory of Gamow and Condon-Gurney gives for the entry of the deuteron as a whole. Now the neutron has to overcome the dissociation energy I of the deuteron in order to enter the nucleus alone, whereas the deuteron has to overcome the Coulomb potential barrier B in order to enter as a whole. Therefore the partial entry will be easier than the total entry roughly if

$$B > I. \qquad (676a)$$

Now I is 2.2 MV. The height of the barrier for deuterons is just of the same order of magnitude for nuclear charge around 10. Therefore the Oppenheimer-Phillips theory will give approximately the same result for the entrance proba-

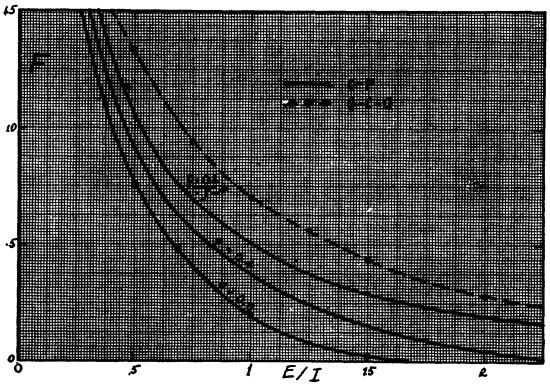

Fig. 24. The penetration function F in the Oppenheimer-Phillips theory. Curves are given for various ratios $\rho = IR/Ze^2$ of the binding energy I of the deuteron to the height of the potential barrier. $\rho=0.6$ corresponds approximately to $Z=17$, 0.4 to $Z=35$, 0.2 to $Z=100$. For $\rho=0.2$ the penetration function following from the Gamow theory is given by the broken line. For $\rho=0.6$, the Gamow points fall on the O-P curve.

bility as the Gamow-Condon-Gurney theory for nuclei with charges around 10.

This qualitative result is confirmed by a quantitative calculation which will be given in more detail in an article to appear shortly in the *Physical Review* by the author (B17). Fig. 24 gives the entrance probability according to the Oppenheimer-Phillips theory as a function of the energy. The entrance probability has been written in the form

$$P=\exp\left[-\frac{2Ze^2}{\hbar}\left(\frac{2M}{I}\right)^{\frac{1}{2}}F(E/I)\right], \quad (677)$$

where M is the mass of the deuteron. F is plotted in Fig. 24 as a function of the ratio of the kinetic energy of the deuteron E to the dissociation energy I of the deuteron, for both the Oppenheimer-Phillips and the Gamow-Condon-Gurney theory. It depends on the ratio of the dissociation energy of the deuteron to the height of the potential barrier, *viz.*

$$\rho=IR/Ze^2. \quad (678)$$

With the expression (609) for the height of the potential barrier, we have

$$\rho=4.0Z^{-\frac{1}{3}}. \quad (678a)$$

The value $\rho=0.6$, which gives the lowest curve in Fig. 24, will therefore correspond to $Z=17$. For this value of ρ, the Gamow-Condon-Gurney theory and the Oppenheimer-Phillips theory differ by an almost unnoticeable amount. The difference in F is 0.04 unit at an energy as low as $\frac{1}{4}I=0.55$ MV; at higher energies, the difference is much less. Now

$$(2Ze^2/\hbar)(2M/I)^{\frac{1}{2}}=0.60Z. \quad (677a)$$

Therefore a difference of 0.04 unit in F means, for $Z=17$, a factor of about 1.5 in the entrance probability (cf. (677)). This means that for $Z=17$ and $E=0.55$ MV, the entry of the neutron alone is 60 percent more probable than that of the deuteron as a whole. The difference is smaller for higher deuteron energy, because higher kinetic energy helps the deuteron to overcome the Coulomb potential barrier while it does not help the dissociation of the deuteron. E.g., for 1.1 MV energy the difference may be about 20 percent, and for deuteron energies above 2.2 MV, it becomes unnoticeable. Such small differences are quite unobservable and are entirely insignificant compared to the changes of the internal disintegration probability (§77) which must be expected over such large energy intervals.

These considerations hold *a fortiori* for elements lighter than $Z=17$, such as Na, Mg, Al, etc. This invalidates the interpretation given for the excitation functions observed in (dp) disintegrations[61] of these elements by Lawrence, McMillan and Thornton (L17) and by Henderson (H23). These authors found that their experimental results did not agree with the excitation function predicted by the simple Gamow-Condon-Gurney (G-C-G) theory (entry of the deuteron as a whole) but agreed well with the Oppenheimer-Phillips (O-P) theory. This is in contrast to our result that the two theories give identical results for the light nuclei in question. The reason for this difference is that Oppenheimer and Phillips calculated the excitation functions only for zero nuclear radius, i.e., for $\rho=0$ (cf. (678)). In this case there is, of course, a very great difference between the probability for partial entry and that for total entry of the deuteron, because the potential barrier for the deuteron would become enormously high if the Coulomb potential continued down to $r=0$. This explains the large difference between the "theoretical" curves for O-P and G-C-G theory obtained in the papers mentioned. Moreover, it happens that the excitation functions for different values of the nuclear radius differ only by an almost constant factor in the O-P theory; at small distances from the center of the nucleus the splitting of the deuteron into neutron plus proton would obviously be more favorable, and the penetration of the neutron into the nucleus depends in a first approximation only on the dissociation energy of the deuteron and not on the Coulomb barrier. On the other hand, the G-C-G excitation function becomes steeper for the smaller nuclear radii. Consequently, we should expect that of the disintegration functions calculated with zero nuclear radius the O-P function will depend on the energy in about the correct

[61] By a (dp) disintegration we understand a process in which the bombarding particle is a deuteron and the emitted one a proton.

way whereas the G-C-G function will not. This is the reason why agreement was found between experiment and the simplified O-P theory.[62]

B. Discussion for heavy nuclei

The difference between O-P and G-C-G theory will be much larger for heavy nuclei. For uranium, ρ (cf. (678a)) will be about 0.2. Then the difference between the values of F for the two theories is about 0.09 for 4.4 MV deuteron energy, which corresponds, according to (677a), to a factor $e^5 = 150$ in the penetration probability, in favor of partial entry. According to this, the O-P theory should give markedly different results from the G-C-G theory for heavy nuclei, and should therefore be applied in this case.

Before it is possible to say anything definite about the applicability of the O-P as against the G-C-G theory to heavy nuclei, it is necessary to investigate both in the light of the Bohr model. The O-P process follows the scheme

$$Z^A + H^2 \rightarrow Z^A + H^1 + n^1 \rightarrow Z^{A+1} + H^1, \quad (679)$$

while the ordinary G-C-G process may be written as

$$Z^A + H^2 \rightarrow (Z+1)^{A+2} \rightarrow Z^{A+1} + H^1. \quad (679a)$$

Thus the O-P process is quite unusual inasmuch as the compound nucleus is identical with the final nucleus. The process is a pure absorption rather than a dispersion phenomenon. It requires, therefore, a treatment different (essentially simpler) from that of the usual processes. The result for the cross section for a given final state q is approximately (cf. B17)

$$(\sigma^{Ap}{}_{Bq})_{\text{O-P}} = 2\pi^2 \lambda\!\!\!^{-2} \kappa \Gamma^q{}_{Np}/E_H. \quad (680)$$

Here $\Gamma^q{}_{Np}$ is the partial width of the compound (=final) state q corresponding to the emission of neutrons and to the state p (=ground state) of the initial nucleus, E_H is the energy of the outgoing proton and κ a constant of the order unity.

[62] As a special argument for the O-P theory of $(d\phi)$ disintegrations, it was pointed out by Henderson (H23) that the reaction $Mg^{26} + H^2 = Na^{24} + He^4$ has a much steeper disintegration function than $Mg^{26} + H^2 = Mg^{27} + H^1$ although the initial particles are the same. This was interpreted as showing that in the first case the whole deuteron must enter the nucleus, in the second only the neutron. The correct explanation is that the number of final levels of the final nucleus increases with the energy of the deuteron more rapidly for the $(d\alpha)$ than for the $(d\phi)$ reaction (§79).

We may express Γ in terms of the penetrability of the barrier and of the sticking probability (cf. 649), *viz.*

$$\Gamma^q{}_{Np} = D_{Bq}\, \xi_N(U_q)\, P_{\text{O-P}}/2\pi, \quad (680a)$$

where D_{Bq} is the spacing of the levels of the final nucleus near the state q, $P_{\text{O-P}}$ the Oppenheimer-Phillips penetrability of the barrier and ξ_N the sticking probability of the neutron which is a function of the excitation energy U_q of nucleus B. Summing over all states q, we have

$$\sum_q \Gamma^q{}_{Np} = (P_{\text{O-P}} \xi_{N\text{Av}}/2\pi) \sum_0^{U'} D_{BQ} = P_{\text{O-P}} \xi_{N\text{Av}} U'/2\pi, \quad (680b)$$

where U' is the maximum possible excitation energy of the final nucleus (see below). Since U' is of the order of E_H, we find finally

$$(\sigma^{Ap}{}_{Bq})_{\text{O-P}} = \pi \lambda\!\!\!^{-2} \kappa\, \xi_{N\text{Av}}(U_q)\, P_{\text{O-P}}. \quad (681)$$

This should be compared with the cross section for total entry of the deuteron, *viz.*

$$(\sigma^{Ap}{}_{Bq})_{\text{G-C-G}} = \pi \lambda\!\!\!^{-2} \xi_D(U_r) P_{\text{G-C-G}} \Gamma_H/\Gamma, \quad (681a)$$

where $P_{\text{G-C-G}}$ is the Gamow-Condon-Gurney penetrability, ξ_D the sticking probability for the deuteron, Γ_H the proton width and Γ the total width of the compound level (compound nucleus $(Z+1)^{A+2}$). The ratio is therefore

$$\frac{\sigma_{\text{O-P}}}{\sigma_{\text{G-C-G}}} = \frac{P_{\text{O-P}}}{P_{\text{G C-G}}} \cdot \frac{\xi_N(U_q)}{\xi_D(U_r)} \cdot \frac{\Gamma}{\Gamma_H} \kappa. \quad (682)$$

Of the factors occurring in (682), the first was found to be about 100 in favor of the Oppenheimer-Phillips theory. As regards the second factor, we have shown in §54 that the sticking probability will presumably increase with increasing excitation energy U. The excitation energy U_r in the case of the ordinary G-C-G processes is (cf. (675b)) extremely high, *viz.* of the order of 20 MV. In the case of the O-P process, we have

$$U_q = \Delta(H^2) - \Delta(H^1) - d\Delta/dA + E_D - E_P, \quad (683)$$

where Δ is the mass excess (cf. (667)), E_D the kinetic energy of the incident deuteron and E_P that of the outgoing proton. It can be shown

(B17) that the proton energy is, in the average, roughly equal to deuteron binding energy plus half the kinetic energy of the incident deuteron, i.e.,

$$E_H = 2.2 \text{ MV} + \tfrac{1}{2}E_D. \tag{683a}$$

With $\Delta(H^2) = 13.7$ MV, $\Delta(H^1) = 7.5$ MV and $d\Delta/dA = 1.9$ MV ($A \approx 200$), we find

$$U_q \approx 2 \text{ MV} + \tfrac{1}{2}E_D. \tag{683b}$$

For $E_D = 5$ MV, this is about two MV less than the excitation energy of a nucleus formed by capture of a slow neutron. Therefore the sticking probability ξ_N will be somewhat less than for slow neutrons, perhaps about 1/100 (cf. 357b). On the other hand ξ_D can be expected to be of the order unity. Therefore the second factor (sticking probability) will approximately cancel the first (penetration probability).

Therefore it will depend on Γ_H/Γ which of the two mechanisms will give the greater cross section. If the proton width Γ_H gives the largest contribution to the total width, the two kinds of processes will be about equally probable. However, for a heavy nucleus by far the largest contribution to the width will come from the emission of neutrons (cf. §79) because the number of possible levels of the residual nucleus is much smaller for proton than for neutron emission. Therefore, $\Gamma_H \ll \Gamma$. This makes the Oppenheimer-Phillips mechanism much more probable for heavy nuclei than the Gamow-Condon-Gurney mechanism.

Our considerations show that the reason for the applicability of the O-P theory is quite different from that naïvely expected. What matters is not so much the facilitated entry of the deuteron as the facilitated escape of the proton. This, again, is not directly affected by the potential barrier but rather indirectly. The fastest protons produced by deuterons of about 5 MV will, in practically all cases, be able to go over the top of the barrier. But the barrier reduces the number of proton groups able to go over the top, and this is the reason which makes the probability of a (dp) reaction small in the ordinary scheme.

At the same time, we see that the probability of emission of a proton according to the Oppenheimer-Phillips mechanism should be about the same as that of a neutron in the ordinary mechanism. This fact may be useful for interpretation of experiments.

A possible experimental test of the O-P theory is afforded by the energy distribution of the emitted protons. According to the ordinary theory, most of the protons should have energies about equal to the height of the potential barrier. According to the O-P theory, the protons do not come as near to the nucleus and therefore do not need to have such high energy in order to escape easily. The most probable energy of the protons will be approximately equal to the deuteron binding energy plus half the kinetic energy of the incident deuterons. Thus, according to the O-P theory, the protons should be slower and have an energy depending on that of the deuterons.

Since the outgoing protons have comparatively small energy, the residual nucleus will be left rather highly excited, the excitation energy being, in the average, about $U_q = 2$ MV$+\frac{1}{2}E_D$ (cf. (683a)). This excitation energy will, for heavy nuclei and very fast deuterons, be sufficient to make the emission of an α-particle by the residual nucleus about as probable as that of a γ-ray (cf. §79E). The available energy for α-emission is, at $A = 200$, about 6 MV$+\frac{1}{2}E_D$ (cf. (673c)). The energy required to make α-emission as probable as γ-emission is about 10 MV (energy of the fastest natural alphas). This means that deuterons of 8 MV will in general produce the double process

$$Z^A + H^2 = (Z-2)^{A-3} + H^1 + He^4 \tag{684}$$

but deuterons of 5 MV will in general not do so to any appreciable extent; they will cause the simple O-P process (676) instead. The process (684) has therefore a higher threshold than the analogous process (674) in which a neutron and an α-particle are emitted. The reason is, of course, that the proton emitted in the O-P process has in general a higher energy than the neutron emitted in a dn process.

A process of the type (684) has actually been observed by Cork and Thornton (C36) who bombarded gold with 7 MV deuterons and observed the formation of a radioactive iridium isotope. Cork and Thornton suggested the reaction

$$Au^{197}+H^2\rightarrow Ir^{194}+He^4+H^1, \quad (684a)$$

which seems to be well justified theoretically by our considerations.

Cork and Thornton observed also the simple O-P reaction

$$Au^{197}+H^2\rightarrow Au^{198}+H^1. \quad (684b)$$

This reaction is presumably caused by the deuterons already slowed down (to 5 MV, perhaps) in the target. It would be of great interest to observe the excitation functions of these two reactions. Up to a certain critical energy E_c, the simple process (684b) should be the most probable and (684a) should be much less probable. At E_c, the relative probabilities will change rather suddenly, owing to the very rapid increase of the α-particle penetrability with increasing energy, so that above E_c the double process (684a) will predominate. Finally, for deuteron energies of, perhaps, 10 MV, the nucleus formed in the primary O-P process will retain enough energy to emit a neutron, so that the complete process will be

$$Z^A+H^2\rightarrow Z^{A+1}+H^1\rightarrow Z^A+H^1+n^1, \quad (684c)$$

which is, of course, unobservable except by the neutron emission itself. For these energies, the double process (684) will therefore cease to be probable, because the emission of α-particles will be less probable (barrier!) than that of neutrons as soon as the latter can occur on energetic grounds.

§81. Resonance Phenomena in the Simple Capture of Protons (H4, H6)

Resonance effects with charged incident particles have been observed for two types of processes, *viz.*

(1) processes produced by α-particles (emission of neutrons or protons).

(2) simple capture of protons.

The simple capture of protons has been studied extensively in the following three cases:

$$Li^7+H^1\rightarrow Be^{8*}\rightarrow Be^8+\gamma, \quad (685)$$

$$C^{12}+H^1\rightarrow N^{13*}\rightarrow N^{13}+\gamma, \quad (685a)$$

$$F^{19}+H^1\rightarrow Ne^{20*}\rightarrow Ne^{20}+\gamma. \quad (685b)$$

In all three cases, the dependence of the yield on the proton energy was studied (H6, H4) and resonance maxima were found. The γ-spectrum has also been investigated (§90, D13). The yield in the reactions (685, 685b) is studied by measuring the γ-rays, whereas in case (685a) the radioactivity of the product nucleus N^{13} is used. Other simple capture phenomena which are known to occur are

$$B^{10}+H^1\rightarrow C^{11}+\gamma, \quad (685c)$$

$$B^{11}+H^1\rightarrow C^{12}+\gamma, \quad (685d)$$

$$O^{16}+H^1=F^{17}+\gamma. \quad (685e)$$

The first and third of these are detected through the radioactivity of C^{11} (C39) and F^{17} (D28), respectively, the second is identified by means of the very energetic γ-rays it produces (C50). The first process has not been investigated for resonance. The second seems to have a resonance maximum at 180 kv proton energy (G13a) with a width of about 15 kv. (The same resonance level appears in the reaction $B^{11}+H^1=Be^8+He^4$, cf. §88 and W21a.) The reaction (685e) does not seem to show resonances (DuBridge, private communication).

The resonances observed by Hafstad, Heydenburg and Tuve in the capture of protons by Li^7 and F^{19} are extremely sharp. The resonance energies and the widths of the levels are given in Table XXXIX. The width in the case of Li is about 11 kv, at a proton energy of 440 kv. For the width of the F levels, only upper limits could be given because the observed width is not larger than the inhomogeneity of the incident proton beam. The data for C are taken from earlier experiments of Hafstad and Tuve in which the voltage definition of the incident particles was

Table XXXIX. *Resonance levels in the simple capture of protons.*

Capturing Nucleus / Compound Nucleus	Li^7 / Be^8	C^{12} / N^{13}	F^{19} / Ne^{20}		
Resonance energy E_0 (kv)	440	420	328	892	942
Width Γ (kv)	11	<40	<4	<12	<15
Angular momentum J (probably)	1	½?	odd (or even)		
Parity (probably)	odd	?	even (or odd)		
Yield at resonance (cm^2)	$\sim 10^{-27}$	—	—	—	—
Integrated yield (10^{-24} volt cm^2)	17	0.9	0.9	20	9
γ-ray width (volts)	4	0.08	0.6*	18*	8*
Reciprocal proton penetrability	3.5	50	6000	13	10.5
Proton width without barrier (kv)	40	<2000	<24000	<160	<160

* For $J=0$. For $J=1$, one-third of these figures should be taken.

less sharp; therefore the width given in the table is probably much larger than the actual width of the resonance level.

The small widths observed make it likely that, in all five cases listed, there is no "probable" process by which the compound nucleus may disintegrate. E.g., the compound state of Be^8 concerned in the capture of protons by Li^7 presumably is not able to disintegrate into two α-particles because such a disintegration would give a very large width of the resonance level. Similarly, the three states of the compound nucleus Ne^{20} probably cannot disintegrate into an O^{16} nucleus in the normal state and an α-particle. *We shall assume that in all cases* listed in Table XXXIX, *the compound nucleus can only disintegrate with the emission of protons or γ-rays.*

There may be two different reasons for such a situation: (a) It may be that the emission of protons and γ-rays are the only processes which are energetically possible. This seems to be the case for the N^{13} state. Disintegrations with emission of neutrons, deuterons and α-particles would lead to the highly unstable or even nonexistent nuclei N^{12}, C^{11} and B^9, respectively; therefore these processes will be energetically impossible.

(b) The emission of other particles may be forbidden by selection rules. It is very easy to see how this may occur for an excited state of Be^8. This nucleus may break up into two α-particles. As is well known, the spin of an α-particle is zero and its internal wave function has even parity.[63] Moreover, two α-particles obey Bose statistics, therefore the wave function describing the relative motion of their centers of gravity must also have even parity and must contain only even orbital momenta (cf. §74F). Thus the complete wave function of a "final state" containing two α-particles must have even parity and even total angular momentum. Therefore a state of the compound nucleus Be^8 which has odd parity and/or odd angular momentum, cannot disintegrate into two α-particles. We therefore ascribe odd J or odd parity or both to the level of Be^8 responsible for the capture of 440 kv protons by Li^7.

In the case of Ne^{20}, we seek a selection rule forbidding the disintegration into $O^{16}+He^4$. These two nuclei have again zero spin and even parity. However, there is no such strict symmetry requirement on the wave function describing the relative motion of the two nuclei with respect to each other as for Be^8. The only requirement is that the wave function should have even parity for even orbital momentum and odd parity for odd l. Thus the disintegration of an excited state of Ne^{20} will be forbidden, if it has even angular momentum combined with odd parity, or *vice versa*. This conclusion is strongly confirmed by the absence of a line corresponding to a transition to the ground state in the γ-ray spectrum (§90).

The disintegration of the compound states of Be^8 and Ne^{20} with emission of neutrons and deuterons is again energetically impossible as can easily be seen from the nuclear masses. There remain therefore only the disintegrations with proton and γ-ray emission. Of these processes, the γ-ray emission certainly does not give a large contribution to the level width, because we know the interaction between matter and radiation to be small. The main contribution to the observed width of the resonance levels must therefore be attributed to the protons, i.e., to the incident particles themselves.[64]

The proton width may be split into a penetrability factor P and a factor G giving the width without barrier. These factors are listed in Table XXXIX. For Be^8, a value of 40 kv is obtained for G, giving

$$G_P' = G_P E_P^{-\frac{1}{2}} = 40{,}000 \cdot 340{,}000^{-\frac{1}{2}} = 70 \text{ volts} \quad (686)$$

for the "width at one volt energy"[65] (cf. (550)). This is to be compared to values of the order of a millivolt for the same quantity deduced from experiments on the capture of slow neutrons by heavy nuclei. The great difference (factor 100,000) is due to the much smaller number of levels in the light nucleus Be^8 as compared to heavy nuclei. The average spacing between the energy levels of Be^8 may be estimated to be of the order of 1 MV, either from theoretical calculations such as those of Wigner and Feenberg (F10), or from the empirical fact that just one resonance level has been

[63] This means that ψ does not change sign upon "inversion," i.e., upon change of the sign of the coordinates of all elementary particles in the α-particle.

[64] For a discussion of other possibilities, cf. H6.

[65] 340 kv is the energy of the proton with respect to the center of gravity; $340 = 440\ (7/8)^2$.

observed in the energy region below 1 MV which was investigated experimentally. This spacing is about 100,000 times larger than the average spacing between neutron resonance levels in heavy nuclei (§60). Therefore the reduced particle width G' seems to be approximately proportional to the spacing of energy levels, corresponding to a sticking probability (§54) independent of the nuclear mass.

For the other resonance levels, only upper limits for the proton width without barrier can be deduced from the experimental data. These upper limits are rather high, and if the proton width is really the only appreciable contribution to the total level width, it is likely that the actual widths of the resonance levels are much smaller than the upper limits given in Table XXXIX.

The γ-ray width Γ_γ may be deduced from the cross section at resonance which, with $s=\frac{1}{2}$ for the proton spin, has the value (cf. (262))

$$\sigma^P{}_\gamma = 2\pi\lambda\!\!\!^{-2}(2J+1/2i+1)\Gamma_\gamma/\Gamma, \qquad (686a)$$

where $\Gamma=\Gamma_P$ is the total width and $\lambda\!\!\!^{-}$ the proton wave-length. The resonance cross section was only determined for the capture of protons by Li^7; it is about 10^{-27} cm^2 in this case. The spin of the compound state is probably $J=1$, because this state has a strong optical transition to the ground state (§90) which is almost certainly a ^1S state. The spin of the capturing nucleus Li^7 is known to be $i=\frac{3}{2}$. The wave-length of the protons is $7.8\cdot10^{-13}$ cm. With these data, we find $\Gamma_\gamma=4$ volts. This seems plausible in comparison with the γ-ray widths of a few tenths of a volt found for heavy nuclei (cf. §61, 90).

The γ-ray width can also be inferred from the integrated (thick-target) cross section (cf. (517))

$$\int \sigma^P{}_\gamma dE = \pi^2\lambda\!\!\!^{-2}(2J+1/2i+1)\Gamma_\gamma. \qquad (687)$$

If the energy loss of the bombarding particle per cm of the bombarded material is written in the form (cf. §95)

$$-dE/dx = N\epsilon \qquad (687a)$$

where N is the number of disintegrable nuclei per cm^3 and ϵ a quantity of the dimension energy times area, then the probability of capture for each incident particle is

$$p^P{}_\gamma = \int \sigma^P{}_\gamma \, dE/\epsilon. \qquad (688)$$

This method may be used to determine the γ-ray widths of the carbon and the fluorine levels.

For carbon, Hafstad and Tuve (H4) found, from the intensity of the produced radioactivity, a capture probability of about 1 in 10^{10}. This value was confirmed by Allison (A5). For protons of 420 kv, we have (§95) about $\epsilon=9\cdot10^{-15}$ volt cm^2. Therefore $\int\sigma dE=9\cdot10^{-25}$ volt cm^2. With $i=0$, $J=\frac{1}{2}$ (most probable because of selection rules), $\lambda\!\!\!^{-}=7.6\cdot10^{-13}$, we find then $\Gamma_\gamma=0.08$ volt. The much smaller value found here as compared to the Li case, is probably due primarily to the smaller energy of the γ-rays (2.3 as compared to 17 MV).

The width of the fluorine levels can only be estimated very roughly by comparing the yields reported by Hafstad and Tuve (H4) for Li and F. They found that the γ-rays from a thick LiOH target produced an ionization of 0.26 divisions/min./μa in an ionization chamber 30 cm from the target, while the γ-rays from CaF_2 which corresponded to the 330 kv resonance capture caused an ionization of 0.21 div./min./μa at a distance of 12 cm. This means a ratio of the ionizations of $0.21\cdot12^2/0.26\cdot30^2=0.13$. We may assume the ionization power of γ-rays to be proportional to the number of Compton electrons produced times their average energy (or range). This gives about equal ionization power for the fluorine and lithium gammas. From the cross section σ_0 of the Li capture at resonance (10^{-27} cm^2) we find then:

Integrated cross section for Li: $\frac{1}{2}\pi\Gamma\sigma_0=\frac{1}{2}\pi\cdot11000\cdot10^{-27}=1.7\cdot10^{-23}$ cm^2 volt.

Energy loss constant (ϵ) for 440 kv protons in air: $1.03\cdot10^{-14}$ cm^2 volt.

Stopping power of LiOH: 2.0 times air per atom of Li, 2.3 times air per atom of Li^7 (considering abundance).

Thus, number of quanta per proton: $1.7\cdot10^{-23}/2.3\cdot1.03\cdot10^{-14}=7\cdot10^{-10}$.

Number of quanta per proton in fluorine: $0.13\cdot7\cdot10^{-10}=9.10^{-11}$.

Energy loss constant for 330 kv protons in CaF_2: $2.3\cdot10^{-14}$ cm^2 volt per F atom.

Integrated cross section for 330 kv level of F: $9\cdot2.3\cdot10^{-25}=2\cdot10^{-24}$ cm^2 volt.

The intensities (integrated cross sections) for the other two levels of F are, according to Hafstad, Heydenburg and Tuve (H4), about 22 and 10 times those of the 330 kv level, respectively. This corresponds to integrated cross sections of 4.5 and $2\cdot10^{-23}$ cm^2 volt. Inserting the nuclear spin $i=\frac{1}{2}$ of the fluorine nucleus, and the proton wave-lengths of 8.4, 5.1 and $4.9\cdot10^{-13}$ cm for the three resonance levels, we obtain the figures listed in Table XXXIX for $J=0$. It is seen that the γ-ray widths of the two higher

levels would appear rather large if $J=0$, especially considering the much lower quantum energy of the fluorine as compared to the lithium radiation.

It may be mentioned that besides the five resonance levels listed in Table XXXIX, some indications of other levels have been obtained for the proton capture by Li^7, C^{12} and F^{19} (H6, H4). There seems to be a very broad level of Be^8 giving rise to capture of protons of 800 kv and more by Li^7, a weak narrow level and a weak broad level of Ne^{20} corresponding to capture of 400–700 kv protons by F^{19}, and a multiplet structure of the level of N^{13}. This shows that, at least for Ne^{20}, the density of nuclear energy levels is already rather large, the spacing being of the order of 100 kv at an excitation energy of about 14 MV. This is, however, a somewhat larger spacing than was calculated theoretically in §53 from the liquid drop model, showing that this model is probably not adequate for such a light nucleus.

Some explanation may be necessary for the fact that no appreciable capture is observed outside the rather narrow resonance regions, while other processes, leading to particle emission, are observable for all energies of the incident particle. Presumably, the latter result is due to the effect of very broad resonance levels whose width is comparable to their distance apart (§84). Now the cross section for γ-ray emission near resonance is inversely proportional to the total width of the level (cf. (686a)) and will therefore be much smaller for a broad level than for a narrow one. On the other hand, it must be admitted that the integrated cross section (687) is independent of the width of the level and depends only on the partial width for γ-emission. Therefore, in thick targets, there should be about as many capture processes due to narrow resonance levels as due to broad ones. The present experimental evidence does not extend over sufficiently many cases to say much about this point; but DuBridge and his collaborators have shown that at least the capture of protons by O^{16} does not arise from narrow resonance levels.

§82. Resonances in α-Particle Disintegrations. Resonance and Barrier Height

A great number of resonance levels have been observed in disintegrations produced by α-particles. A description of the experiments is found in §99. The experimental results are given in Table XXXX. Most reactions have been investigated by several authors, in these cases we have given in the table the results which we consider most reliable; usually, these are the most recent ones.

A. Barrier height and resonances

We have included in Table XXXX the height of the potential barrier for α-particles according to two methods, *viz.* (a) as derived from the interpolation formula (609) and (b) the "experimental" values given by Chadwick and Feather (C10). These authors pointed out that disintegrations should occur at any α-energy higher than the top of the barrier, while for lower α-energies observable intensities can only be obtained in the resonance regions.

This consideration while correct in the old one-body picture of nuclear disintegrations, might at first sight seem somewhat doubtful in the many-body picture. According to this picture, the width of resonance levels is determined by the *most probable* mode of disintegration of the compound nucleus. Now in all cases listed in Table XXXX, the particle *produced* in the reaction

Table XXXX. *Barrier heights and resonance levels from α-particle disintegrations.*

Compound Nucl.	Reaction	Barrier in MV		Resonance Levels		Width MV	Ref.
		Theor.	Exp.	α-energy in MV	Excitation En. of Comp. Nucl. (MV)		
C^{13}	$Be^9+\alpha=C^{12}+n$	3.9	3.5	3.4; 4.8?	12.8; 13.8?	0.3; ~0.3	B7
N^{14}	$B^{10}+\alpha=C^{13}+H$	4.6	3.6	4.2	14.8	0.5	M16
N^{15}	$B^{11}+\alpha=N^{14}+n$	4.3	3.7	3.2?	13.4?	~0.4	C7
F^{18}	$N^{14}+\alpha=O^{17}+H$	5.2	4.1	3.6	8.2	~0.15?	P2
Na^{23}	$F^{19}+\alpha=Ne^{22}+H$	5.7	5.0	3.7; 4.1	14.5; 14.8	0.10; 0.13	C13
Si^{28}	$Mg^{24}+\alpha=Al^{27}+H$	6.8	6.5	5.7; 6.3	13.7; 14.2	0.12; 0.13	C13
P^{31}	$Al^{27}+\alpha=Si^{30}+H$	6.9	6.8	4.0; 4.5;	12.0; 12.4;	~0.10	C10
				4.9; 5.3;	12.8; 13.1;	0.07; 0.13	D19
				5.8; 6.6	13.6; 14.3	~0.12	

(neutron or proton) can go over the top of the barrier even when the energy of the α-particle is low. Therefore the width of the levels will be determined by the proton or neutron emission, especially if the penetrability of the barrier for α-particles is small. From this point of view, there should be little difference in the width of levels below and above the top of the α-particle barrier.

However, the selection rules for the angular momentum may cause a difference in the excitation function below and above the top of the barrier. Below the barrier, we know that only α-particles of small orbital momentum can enter the nucleus (§78) while sufficiently far above the barrier, much higher orbital momenta are possible. Therefore below the barrier there will be resonance only with compound levels of small angular momentum J, while above also compound levels of large J will be important. This will make the spacing between "important" resonance levels much smaller above the barrier than below it, and since the width of the levels will stay about constant, it is to be expected that the resonances will overlap each other at energies above the height of the barrier, and the maxima and minima will thus become unobservable.

This consideration justifies the reasoning of the older theory and makes it possible to deduce approximate barrier heights from the cessation of pronounced resonance phenomena. This has been done by Chadwick and Feather; their results agree surprisingly well with the "theoretical" values obtained from the rule that the nuclear volume is proportional to the number of particles. This can be considered as a fairly good experimental confirmation of this rule.

In accord with our considerations, most of the resonances lie below the calculated top of the barrier. The only exception is a very weak and somewhat doubtful resonance observed by Bernardini for the compound nucleus C^{13} (reaction $Be^9+He^4=C^{12}+n^1$). We must expect that such resonances above the barrier occur occasionally, especially for very light nuclei.—Several resonance levels are found very near the top of the barrier, e.g., the highest levels in the compound nuclei N^{14}, Si^{28} and P^{31}.

B. Spacing of the resonances

In judging the figures in Table XXXX, we must keep in mind that α-particle energies of less than 3–4 MV have not been investigated because for these energies the penetrability of the α-particle would be so small that no observable disintegration effects could be obtained, even at resonance. It is to be hoped that the greater intensity of artificial sources will make the investigation of the region of low α-particle energies possible.

The general impression from the figures in Table XXXX is that the resonance levels become more numerous and therefore more closely spaced for heavier nuclei. This is entirely in accord with out theoretical expectations. The average spacing is apparently slightly more than 1 MV for the lightest, and about $\frac{1}{2}$ MV for the heaviest nuclei listed.

The excitation energies U of the compound nuclei corresponding to resonance are listed in the sixth column of Table XXXX. If M_A, M_α and M_C are the masses of initial nucleus, α-particle and compound nucleus (ground state), respectively, we have

$$U=(M_A+M_\alpha-M_C)c^2+(M_A/M_C)E_\alpha. \quad (689)$$

Most of the excitation energies listed are about 12–15 MV.

C. Width

The width of the resonance levels may be obtained in various ways (cf. C13, C4).

1. If there are few resonances of comparatively large width the simplest procedure is to measure the total yield of the reaction from a thin target, as a function of the α-particle energy. This has been done for the reactions $Be^9+\alpha=C^{12}+n$, $B^{10}+\alpha=C^{13}+H$, $B^{11}+\alpha=N^{14}+n$. The method is apt to give too large widths because of the finite thickness of the target and the straggling of the α-particles. Closely spaced narrow resonances may disappear entirely.

2. With a thick target, the intensity of a given *group* of produced particles (usually protons) may be investigated as a function of the energy of the α-particles entering the target. From the α-particle energy at which the group first appears, and that at which it attains its full intensity, the

width of the level may be inferred. The method is subject to similar criticism as (1). It has been the one most widely used for determining widths (C4).

3. The inhomogeneity in energy of the *protons* of a given group will also give the width of the level, due account being taken of the momentum relations (recoil of the disintegrated nucleus). This method is free from corrections due to the finite thickness of the target and to the straggling of the α-particles but is influenced by the straggling of the protons. Also, the proton energy depends on the direction of emission of the protons which introduces a further inhomogeneity. Thus the widths deduced from this method will again be too large. We have used this method for estimating the widths in the reaction $F^{19}+He^4=Ne^{22}+H^1$.

4. The *maximum* range of the protons of a group may be determined for the α-energy at which the group first appears, and for the α-energy at which it attains full intensity. This method is the most satisfactory in eliminating straggling. Unfortunately, it is probably the most difficult one experimentally. It has been used by Chadwick and Constable to deduce the width of the 4.9 MV level in the reaction $Al^{27}+\alpha=Si^{30}+H$. It was found that the maximum range of the protons increased by about 2 cm (from 28 to 30) when the α-particle range increased from 3.25 (first appearance of group) to 3.55 cm. This corresponds to an increase in proton energy of about 0.18 MV. Considering that the protons were observed *in* the direction of the incident α-particles, we obtain from the momentum relations (§96) that this corresponds to an increase of the α-energy of about 0.17 MV. For the excitation energy only the kinetic energy in the center of gravity system is available; therefore we must reduce the width by a factor 27/31 which gives 0.15 MV. Half this value has been taken in Table XXXX (cf. below).

In accord with our custom, we have given in the table the *width at half-maximum*. Where the shape of the yield-energy curve was not obtained experimentally, it was assumed that the disintegration becomes unobservable when the cross section is about one-fifth of its maximum value. According to the resonance formula, this means that the disintegration will be observable over an energy region equal to *twice* the width Γ of the level. Therefore the values given in the table are smaller than those given, e.g., by Chadwick and Feather, by about a factor of 2.

The width of the resonance levels must probably be attributed mainly to the *produced* particle, i.e., proton or neutron. The partial width due to the α-particle will be smaller because of the potential barrier. In almost all cases, several groups are emitted from each resonance level, corresponding to several states of the final nucleus. E.g., 5 proton groups have been observed for the reaction $B^{10}+\alpha=C^{13}+H$, 4 proton groups for each resonance level of $Al^{27}+\alpha=Si^{30}+H$, and at least three neutron groups for $Be^9+\alpha=C^{12}+n$. The width corresponding to a *given* final state is therefore only about 0.1, 0.06 and 0.02 MV for the compound nuclei C^{13}, N^{14} and P^{31}, respectively. The neutrons and protons emitted have energies of the order of 2–10 MV in each case. Taking an average of 5 MV, we find for the reduced width (width at one volt energy) $\Gamma'=\Gamma E^{-\frac{1}{2}}$ (cf. 685) about 25–50 volts for the lighter nuclei C^{13} and N^{14}, and about 10 volts for the heavier compound nucleus P^{31}. These values fit in very well with the value obtained from the capture of protons by Li^7 (cf. §80) *viz.* 70 volts. They also show the expected tendency to decrease with increasing mass of the nucleus.

D. α-particle width

The α-particle width can be deduced from the integrated disintegration probability if it is admitted

(1) That the α-particle width is smaller than the width for the produced particle, and

(2) That no other process of comparable probability can occur.

Both conditions are probably fulfilled, at least approximately, in the cases listed in Table XXXX.

Chadwick and Constable have given the total yield of protons from α-disintegration of Al and F. For Al, each resonance level gives about $3.5\cdot10^{-7}$ protons per α-particle. The "energy loss constant ϵ" (cf. (687a), §95) is, for α-particles of 5 MV in Al, about $2.3\cdot10^{-14}$ volt cm^2. According to (688), this corresponds to an integrated cross section of $0.8\cdot10^{-20}$ cm^2 volts. Since for α-particles

$s=0$, the formula for the integrated cross section is (cf. (687))

$$\int \sigma dE = 2\pi^2 \lambda_\alpha^2 \Gamma_\alpha (2J+1)/(2i+1). \quad (690)$$

The spin of Al^{27} is $i=\frac{1}{2}$, the wave-length of 5 MV α-particles (with reduced mass!) $1.17\cdot 10^{-13}$ cm. Therefore we find

$$(2J+1)\Gamma_\alpha = 60 \text{ kv}. \quad (690a)$$

Since the total width is only 90 kv, Γ_α must be considerably smaller than this amount. Therefore we must assume that J is not zero. For the given ratio of α-particle energy to barrier height, orbital momenta up to about 3 or 4 will be important. For $J=2$, the α-ray width would come out to be 12 kv. The penetrability of the barrier is 1/14, giving for the α-width without barrier about 170 kv, i.e., somewhat more than the proton width per final level (20 kv).

In a CaF_2 target, the yield is 9.2 and $7.2\cdot 10^{-7}$ protons per α-particle for the two resonance levels of fluorine observed. In pure fluorine, the yield would be 1.77 times as large (C4) since a calcium atom has a stopping power for α-particles about 1.54 times as large as a F atom (§95). This would give in the average for the two levels $14.5\cdot 10^{-7}$ protons per α-particle. The energy loss constant in F is $2.35\cdot 10^{-14}$ volt cm^2 for an α-energy of 3.9 MV (corresponding to the average of the two levels), therefore the integrated cross section is $3.4\cdot 10^{-20}$ volt cm^2. The spin of F^{19} is again $i=\frac{1}{2}$ (Table XVII), the wave-length $1.40\cdot 10^{-13}$ cm. This gives $(2J+1)\Gamma_\alpha = 175$ kv. Since the total widths of the levels are about 100–130 kv, the angular momenta of the levels are obviously again greater than zero.

§83. Selection Rules (G15)

We have repeatedly drawn attention to the importance of selection rules. Selection rules forbidding otherwise probable disintegrations will reduce the width of many nuclear levels and will thus give rise to the very sharp resonance levels which we observe in the simple capture of protons (§81). Selection rules influence greatly the γ-ray spectrum of nuclei (§90). Selection rules combined with the small penetrability of the potential barrier for particles of high orbital momentum will prevent all compound levels of high angular momentum from being effective in disintegrations caused by α-particles as long as the α-particle cannot go over the top of the potential barrier (§82A), thus making the resonances observable for energies below the height of the barrier while they are not for higher energies.

In this section we shall show that selection rules also regulate to a considerable extent the probability of disintegrations which show no marked resonance effects, in particular disintegrations produced by protons and deuterons in light nuclei. This was first pointed out by Goldhaber (G15) who concluded, from such considerations, that the ground states of Li^6 and Li^7 should be a ^3S and a $^2P_{3/2}$ state respectively, and that B^{10} should have a triplet, B^{11} a doublet as its ground state. The prediction for Li^6 which was at the time against evidence from hyperfine structure (G25, and ref S2 of part A), has since then been proved correct by the atomic beams method (M2). Moreover, all the predictions mentioned agree with our present theoretical ideas about nuclear energy levels (F10).

The selection rules may be divided into two classes: Those which should hold generally, for any "coupling scheme" in the nucleus, and those which hold only for Russell-Saunders coupling. The first class concerns only the total angular momentum and the parity of initial nucleus, incident particle, final nucleus and outgoing particle. The second class will give information on the behavior of orbital momentum and spin separately.

The selection rules for total angular momentum and for orbital momentum will involve the orbital momentum of the relative motion of the incident and outgoing particle. The existence of useful selection rules depends therefore on the existence of restrictions on this orbital momentum. Such restrictions will exist (a) for very slow particles for which only the orbital momentum zero gives a high probability of entering the nucleus, and (b) when the two particles (*viz.* incident particle and initial nucleus, or produced particle and residual nucleus) are identical. In case (b) the symmetry requirements for the wave function will forbid certain values of the orbital momentum. The most notable case is that of two

α-particles: Because of the Bose statistics and the zero value of the spin, only even orbital momenta are possible (§73F).

A. Selection rules between initial and final nuclei which hold for any coupling scheme

The vector sum of the total angular momenta of initial nucleus and incident particle must be equal to the vector sum of the total angular momenta of final nucleus and outgoing particle. Let ss' be the spins of incident and outgoing particle, i and i' the (total) intrinsic angular momenta of initial and final nucleus. If, then, both incident and outgoing particle are slow we have

$$\begin{aligned} |i-s| &\leq i'+s' \\ i+s &\geq |i'-s'|. \end{aligned} \tag{691}$$

This selection rule very seldom gives any useful information because it leaves too many possibilities.

The more useful one of the general selection rules is that relating to the parity. Any nuclear energy level has a certain parity, e.g., the ground states of all nuclei up to He^4 have even parity, those from He^4 to O^{16} are probably even and odd for even and odd mass number, respectively. The wave function of the relative motion of two particles has even or odd parity according to whether the orbital momentum is even or odd. The selection rule requires that the parity of the system as a whole, i.e., the product of the intrinsic parity and that of the center-of-gravity motion, remains unchanged in the process. If both incident and outgoing particle are slow, and none of these particles is heavier than He^4, this means that the parities of initial and final nucleus must be the same. According to the above rule, this would forbid all processes in which a nucleus of even mass between 4 and 16 is transmuted into one of odd mass in the same region in the ground state or *vice versa*, i.e., all dp, dn, αp and αn reactions (cf. Chapter XVII). This very restrictive selection rule holds, however, only if both incident and outgoing particle are slow (wave-length larger than nuclear radius), and in no practical case are *both* particles slow at the same time.

However, there is one reaction which is definitely "forbidden" according to this parity rule, if only the *incident* particle is slow. This is the oldest of all reactions produced by artificially accelerated particles, *viz.*

$$Li^7+H^1=2He^4. \tag{692}$$

The ground state of Li^7 has almost certainly odd parity (see above) while the intrinsic parity of all other particles is even. The wave function describing the relative motion of the two α-particles must be even (Bose statistics). Therefore the process can only occur at all if the wave function of the motion of the proton relative to the Li^7 nucleus is odd. This requires at least $l=1$, which makes the process improbable at low proton energy. In fact, its probability is 30 times smaller than that of the "probable" reaction

$$Li^6+H^2=2He^4. \tag{692a}$$

We may, conversely, conclude from the observed relative probabilities of reactions (692), (692a) that the parity of Li^6 is even, that of Li^7 odd in agreement with theoretical views.

As Breit (O15) has pointed out, the energy dependence of the disintegration cross section is not appreciably affected by the orbital momentum for slow charged particles, in contrast to slow neutral particles. The ratio of the penetrabilities of the potential barrier for various orbital momenta is practically independent of the energy of the particle if this is small compared to the barrier height (§72, 78). The reason for this is that the particle energy is negligible compared to the Coulomb potential and the centrifugal force.

B. Selection rules between initial and final nucleus holding only for Russell-Saunders coupling

It is reasonable to assume that in light nuclei the resultant spin σ and the resultant orbital momentum λ of all particles contained in the nucleus are good quantum numbers (F10, I1, R10). This will be true if the magnetic spin-orbit interaction is small compared to the average spacing between levels. It is probable (R10) that the spin-orbit interaction is of the order $\frac{1}{2}$ MV. Therefore Russell-Saunders coupling will break down when the spacing between the energy levels of the *compound nucleus* (not the initial or final nucleus!) becomes of the order $\frac{1}{2}$ MV or less. This will probably occur for atomic weights around 15 or 20, so that the Russell-Saunders

coupling is restricted to the lightest nuclei.

The first selection rule to be added to the two general ones mentioned in A, requires that the resultant spin of the system does not change, i.e.,

$$|s-\sigma| \leq s'+\sigma', \quad |s'-\sigma'| \leq s+\sigma, \quad (693)$$

where the primed quantities refer to the final, the unprimed ones to the initial nuclei and s refers to the particle, σ to the nucleus. It was from this selection rule that Goldhaber derived the spins of Li^6 Li^7 B^{10} and B^{11}. E.g., from the fact that (692a) is a probable reaction, and that the α-particles have zero and the deuteron unit spin, it follows that Li^6 must also have unit spin. The same follows from the probable reaction $B^{10}+H^2 = 3He^4$ for B^{10}. Similarly, the reaction $B^{11}+H^1 = 3He^4$ is probable, therefore B^{11} presumably has spin $\frac{1}{2}$. For Li^7 the situation is a little more complicated. From the reactions $Li^6+H^2=Li^7+H^1$, $Li^7+H^2=2He^4+n^1$ which are both "probable" we can only conclude that the spin of Li^7 is either $\frac{1}{2}$ or $\frac{3}{2}$. On the other hand, the reaction $Li^7+H^1=2He^4$ is improbable (cf. above (692a)). But we had already found a reason for this, *viz.* the change of parity provided the orbital momentum of the incident proton is zero. It may be argued that the reaction is not improbable enough for a reaction violating *two* independent selection rules so that the total spin does not change. This would lead to $\sigma=\frac{1}{2}$ for Li^7, in agreement with present theories.

The selection rule for l is not very useful because there are in general no restrictions on the orbital momentum of the outgoing particle.[66] Where there are restrictions; e.g. in the reactions (692, 692a), this rule gives the same result as the parity rule.

C. Selection rules in resonance disintegrations

As already pointed out in the beginning of this section, more stringent selection rules hold for resonance disintegrations than for nonresonance processes. Angular momentum as well as parity have to be conserved. This means quite generally that either only even or only odd values of the orbital momentum are possible when a compound nucleus in a given state breaks up into two given final nuclei. For the parity is determined for the compound state C as well as for the final nuclei B and Q. Therefore the parity of the motion of the nuclei B and A with respect to each other is also given, and this parity determines whether the orbital momentum is even or odd. Explicitly, we have:

Only even l if all the nuclei C, B and Q have even parity or if two of them have odd parity and one even.

Only odd l if all parities are odd, or two even and one odd.

An application of this parity rule is the excited state of Ne^{20} formed in the proton capture by F^{19} (§81). Other applications are connected with the γ-ray emission (§90). A further application may be the capture of slow neutrons by B^{10}, according to the scheme

$$B^{10}+n^1=B^{11*}=Li^7+He^4 \quad (693a)$$

for which the experimental determinations (cf. §102) indicate that Li^7 is always formed in its excited state of 0.44 MV excitation energy (cf. Table LXXIV).

Another important consequence of the even-odd rule for the orbital momentum is, of course, that the angular distribution of the disintegration products must be symmetrical with respect to the plane perpendicular to the direction of the incident beam. This was actually found by Kempton, Browne and Maasdorp (K2) for the disintegration

$$H^2+H^2=H^3+H^1,$$

which may be considered as evidence that even this disintegration has resonance character.

§84. Absolute Probability of Disintegration Processes; Angular Distribution of Disintegration Products

The absolute probability of processes showing pronounced resonance effects was already discussed in §81, 82 and was used to deduce the partial widths for γ-rays and α-particles of the levels concerned. The absolute probability of processes showing no marked resonances is a much less clean-cut problem.

[66] This seems to be the case, e.g., in the reactions $2H^2=H^3+H^1$ and $2H^2=He^3+n$ where the outgoing particles are fast enough for orbital momentum 1 or even 2. We therefore do not believe that Goldhaber's conclusion of zero intrinsic orbital momentum for H^3 and He^3 from these reactions is legitimate. From purely theoretical reasons (Chapter IV) zero orbital momentum seems, of course, very probable.

A. Light nuclei, total cross section

It is very probable that these processes should also be described in the general scheme of the formation and disintegration of a compound nucleus. From this point of view, even the processes showing no marked resonance are in reality resonance effects; but the resonance levels are so broad compared to their spacing that the maxima and minima are leveled out. Indications of a very weak resonance effect are found in the reaction most closely investigated, *viz.*

$$Li^7+H^1=2He^4. \tag{694}$$

The disintegration function for this reaction was observed experimentally by various authors (D16, H6, H27, H20, O6) for energies from 8 to 1400 kv, with thin and thick targets. Breit and his collaborators (O14, 15, 16) worked out the theoretical disintegration function with a special model, *viz.* assuming that the proton moves in the "potential" created by the Li^7 nucleus, and that this potential is a simple rectangular hole. They found that for certain values of the depth of this hole, agreement could be obtained with experiment. Furthermore they found that the relatively high yield of the reaction at low proton energies is irreconcilable with an orbital momentum 1 of the incident proton unless one assumes a resonance level fairly near zero proton energy. The value 1 for the proton orbital momentum is required by the selection rules for the parity (cf. §83A) which should be strictly obeyed. The conclusion concerning the necessity of a resonance for slow protons is independent of the special rectangular hole potential assumed but is simply a consequence of the penetration of p protons ($l=1$) through the Coulomb potential barrier.

Thus the calculations and experiments may be regarded as evidence for a resonance level of Be^8 at about 17 MV excitation energy, having even parity and probably angular momentum zero[67] (in contrast to the level responsible for the radiative capture of protons by Li^7, §81) and governing the probability of the reaction (694). This level probably has a width of $\frac{1}{2}$ MV or more, which is presumably mainly due to the disintegration into two α-particles. There can be little doubt that other processes, if investigated as carefully as $Li^7+H^1=2He^4$, will exhibit similar weak indications of resonance phenomena.

TABLE XXXXI. *Absolute cross sections.*

Reaction / Target	D+D= He^3+n^1 D_2O	$Li^7+D=$ $2He^4+n^1$ Li	$Be^9+D=$ $B^{10}+n^1$ Be	$Li^7+H=$ $2He^4$ Li
10^7p (800 kv)	8.0	17.1	9.5	
10^7p (700 kv)	6.2	8.6	5.9	
ϵ(750 kv)(in 10^{-15} cm² volt)	12	10	11	
Cross sect. σ(in 10^{-26} cm²)	2.2	8.5	4	0.33
$\lambda\!\!\!^{-}$ (in 10^{-13} cm)	7.4	4.8	4.5	6.0
Penetrability P	1	0.5	0.4	0.7
Sticking prob. ξ	0.013	0.25	0.16	0.004

Independent of the description of the processes as resonance effects, the cross section for probable processes must be of the order of

$$\sigma^P{}_Q=\pi\lambda\!\!\!^{-}{}_P{}^2P_P\xi_P, \tag{695}$$

where $\lambda\!\!\!^{-}{}_P$, ξ_P and P_P are wave-length, sticking probability and penetrability for the incident particle. The partial width Γ_Q for the outgoing particle has been assumed equal to Γ which is approximately true for "probable" processes. The quantity usually measured is the total number p of disintegrations per particle of a given energy E; from this σ can be deduced with the help of the energy loss cross section ϵ (cf. (688) and §95), *viz.*,

$$\sigma=\epsilon dp/dE. \tag{696}$$

Table XXXXI gives the absolute yield for four reactions for which the yield has been investigated with special care by Amaldi, Hafstad, Heydenburg and Tuve (H6, A11a). The first and second row give the total yield p (disintegrations per incident particle) at two energies (700 and 800 kv) of the incident particle, the third line gives the energy loss constant ϵ for the target substance, the fourth the cross section (at 750 kv) derived from these data, the next two lines give wave-length and penetrability for the incident particle and the last the sticking probability (cf. (695)).

The "sticking probabilities" obtained are of the order unity for the processes $Li^7+H^2=2He^4+n^1$ and $Be^9+H^2=B^{10}+n^1$, as should be expected. This justifies the assumption that these

[67] J must be even because disintegration into two α-particles is possible (§81). It cannot be greater than $l+i+s$ where $l=1$ is the orbital momentum, $i=3/2$ the spin of Li^7 and $s=1/2$ that of the proton. This reduces the possible values for J to 0 and 2. The angular distribution of the α-particles is spherically symmetrical which makes $J=0$ probable.

processes are "probable" processes. For the reaction $Li^7+H=2He^4$, the sticking probability comes out to be very small, about 1/50 of that for the two first named processes, in accord with the characterization of this process as, in first approximation, "forbidden" (cf. §83). The probability of the disintegration of deuterons by deuterons also comes out to be small. In this case, it must of course be considered that there is the alternative mode of disintegration into H^3 and a proton, which is experimentally (and theoretically for symmetry reasons) about as probable as the disintegration into He^3+n^1. Even so, the sticking probability would be only 1/10 of that for Li^7+D, and only 0.025 in absolute value. The reasons for this small internal disintegration probability are unknown; calculations using special models (D15) gave even greater theoretical probabilities (and therefore a greater discrepancy) than we would obtain for $\xi=1$. For a possible explanation cf. B.

B. Light nuclei, angular distribution

No theoretical calculations on the angular distribution of disintegration particles have yet been made. Except in the resonance case, such calculations would require the use of a special model.

Experimentally, the angular distribution has been investigated for the α-particles produced in the reaction

$$Li^7+H^1=2He^4 \tag{697}$$

(G14a), for the protons and neutrons from the processes

$$H^2+H^2=H^3+H^1, \tag{697a}$$

$$H^2+H^2=He^3+n^1, \tag{697b}$$

(K3, N4a) and for the α-particles from

$$B^{11}+H^1=Be^8+He^4 \tag{697c}$$

(N4a; homogeneous group of alphas of 4 cm range). In the first case, spherical symmetry was found with protons of 200 kv, in the other three cases, marked maxima in the forward and backward direction (about 60 percent more than at right angles) were observed at 100–200 kv deuteron energies. In all cases, the distribution is symmetrical about the "equatorial plane" as is required for resonance disintegrations (§83C, end).

In the Li case, we assume (§83) that the incident protons have orbital momentum one and might therefore expect an anisotropic distribution of the emitted α-particles; we conclude that the compound state of Be^8 must have $J=0$ (cf. A). In the deuteron case, at low energies primarily deuterons of zero orbital momentum should be effective, giving a spherically symmetrical distribution, unless $l=0$ is forbidden by selection rules. The most tempting hypothesis is to assume that the compound state of He^4 involved in reactions (697a, b) has odd parity. Since deuterons have even intrinsic parity, this means that only odd orbital momenta would be allowed for their relative motion, so that $l=1$ gives the main contribution. At the same time, this would explain the smallness of the absolute cross section (cf. A). Moreover, an odd state (^1P) of He^4 is expected to lie near the He^4 dissociation energy according to calculations of Feenberg (F9a). It only seems difficult to understand why even compound states should give no contribution at all.

The nonspherical distribution of the alphas in (697c) shows again that $l=0$ is probably forbidden by selection rules. The simplest assumption is that the state of C^{12} involved has even parity; then, since B^{11} has odd intrinsic parity, only odd l will be allowed for the incident proton. The outgoing α must then have even l', and since both Be^8 and He^4 have spin zero, J must be even for the compound state. Since $J=0$ is excluded on account of the nonspherical symmetry, and $J\leqslant l+i+s=2$, we must have $J=2$.

C. Heavy nuclei

In §77, we have derived a general formula for the probability of disintegration processes produced in heavy nuclei which was supplemented in §80 by a discussion of the special case of reactions of the *dp* type. Thus far, quantitative studies of the yield are only available for the disintegration of Pt by deuterons of 4–5 MV. This energy, though large, is decidedly lower than the potential barrier of Pt (~9 MV). Thus we may apply formulae (650), (633), *viz.*

$$\sigma^{P_P}{}_Q=\pi\lambda\!\!\!^{-2}\frac{g}{(1-x)^{\frac{1}{2}}}P_P\xi_P\frac{\Gamma_Q}{\Gamma}. \tag{698}$$

Here the wave-length is about $1.5\cdot10^{-13}$ cm (for

4.5 MV deuterons), $g=11.6$ for deuterons on Pt, $x=$deuteron energy/height of potential barrier $=\frac{1}{2}$. ξ_P is the sticking probability for the incident particles, P_P the penetrability of the barrier for them. Γ_Q is the width corresponding to the produced particles, Γ the total width. According to §79, the main contribution to the latter will probably come from neutron emission, a smaller part from emission of protons, a still smaller part from α-particles and a negligible fraction from deuteron emission. We shall in the following assume neutron emission; as was shown in §80, the emission of protons, according to the Oppenheimer-Phillips mechanism, is about as probable as that of neutrons according to the ordinary mechanism. Then we may put $\Gamma_Q=\Gamma$.

With these assumptions,

$$\sigma^{P_P}{}_Q=1.0\cdot 10^{-24}P_P\xi_P. \tag{698a}$$

We assume further that $\xi=1$ because of the high excitation energy of the compound nucleus formed (cf. §54). The penetrability P_P depends on the nuclear radius. With $R=12\cdot 10^{-13}$ cm we obtain

$$\begin{aligned} P&=5\cdot 10^{-5},\\ \sigma&=5\cdot 10^{-29}. \end{aligned} \tag{698b}$$

If we take account of the rather large size of the deuteron by assuming an effective radius of $15\cdot 10^{-13}$ cm, we have instead

$$\begin{aligned} P&=1.2\cdot 10^{-3},\\ \sigma&=1.2\cdot 10^{-27}. \end{aligned} \tag{698c}$$

This last figure agrees as to order of magnitude with the observed cross sections for the production of radioactivity in Pt which are of the order 10^{-28} to 10^{-27} cm^2 (C35).

In all reactions produced in Pt by deuterons, fluctuations of the yield with energy have been observed which resemble resonances. It seems practically certain that they must not be interpreted as such. The excitation energy of the compound nucleus formed by adding a deuteron to a heavy nucleus, was estimated in §78 to be about 12 MV plus the kinetic energy of the deuteron, i.e. more than 16 MV in our case. The spacing of levels at such an energy for a nucleus as heavy as Pt is probably of the order of a millivolt or less (§53). The observed fluctuations of the yield have maxima and minima spaced by about $\frac{1}{2}$ MV. They can therefore not be due to resonance. A possible explanation might be secular fluctuations in the matrix elements over large energy regions, or competition of various modes of disintegration.

§85. Three-Particle Disintegrations

As was already mentioned in the general discussion in §76, the residual nucleus may be left in the primary nuclear reaction in a state above the dissociation energy. Then it will in general disintegrate further, emitting a second heavy particle. A "cascade reaction" of this type will appear experimentally as a disintegration in which three nuclei, including the final nucleus, are produced.

The best investigated reaction of this type is

$$B^{11}+H^1\rightarrow C^{12*}\rightarrow Be^{8*}+He^4\rightarrow 3He^4. \tag{699}$$

The fact that the mechanism of this reaction is actually as indicated in (699), was first established by Dee and Gilbert (D9) from cloud chamber experiments. Similar to (699) is the reaction

$$B^{10}+H^2\rightarrow C^{12*}\rightarrow Be^{8*}+He^4\rightarrow 3He^4. \tag{699a}$$

Other three-body reactions with light nuclei are:

$$\begin{array}{ccccc} & & Be^{8*}+n^1 & & \\ & \nearrow & & \searrow & \\ Li^7+H^2\rightarrow Be^{9*} & & & & 2He^4+n^1, \\ & \searrow & & \nearrow & \\ & & He^{5*}+He^4 & & \end{array} \tag{699b}$$

$$\begin{array}{ccccc} Be^9+He^4 & & Be^{9*}+He^4 & & \\ & \searrow\ \nearrow & & \searrow & \\ B^{11}+H^2 & \rightarrow C^{13*} & & & 3He^4+n^1, \\ & \nearrow\ \searrow & & \nearrow & \\ C^{12}+n^1 & & C^{12*}+n^1 & & \end{array} \tag{699c}$$

$$N^{14}+H^2\rightarrow O^{16*}\rightarrow C^{12*}+He^4\rightarrow 4He^4, \tag{699d}$$

$$Be^9+n^1\rightarrow Be^{10*}\rightarrow Be^{9*}+n^1\rightarrow 2He^4+2n^1. \tag{699e}$$

Of these, (699b) has been known for a fairly long time and the continuous distributions of both neutrons (B37) and alphas (O2, K2, W21c) have been studied extensively. For the other reactions, similar but less complete evidence has been given (cf. Chapter XVII for discussion and references).

With heavy nuclei, evidence has been presented by Heyn (H33) for processes in which an incident fast neutron knocks out another neutron from a nucleus, thus decreasing the latter's atomic weight by one unit. An example of this type of reaction is

$$Cu^{63}+n^1\rightarrow Cu^{64*}\rightarrow Cu^{63*}+n^1\rightarrow Cu^{62}+2n^1 \quad (700)$$

detected by the positron activity of Cu^{62}. Pool, Cork and Thornton (P11b) have confirmed reaction (700) and found similar reactions with Cu^{65}, Ag^{107}, Ag^{109}, N^{14} and O^{16} giving Cu^{64}, Ag^{106}, Ag^{108}, N^{13} and O^{15}, respectively. Other examples are probably found in uranium and other very heavy nuclei (H7, M15).

Another three particle disintegration has been reported by Cork and Thornton (C35, 36) who found that deuteron bombardment of Au produced a radioactive isotope of Ir. This means the emission of three units of charge from the compound nucleus so that they suggested the process

$$Au^{197}+H^2=Au^{198}+H^1=Ir^{194}+H^1+He^4. \quad (700a)$$

Such a process is very probable with the Oppenheimer-Phillips disintegration as the first step (cf. §80). We have also given arguments (§79E) for assuming that a similar process with emission of neutrons and alpha-particles is very probable in the reaction of deuterons with heavy nuclei, e.g.,

$$Pt^{195}+H^2=Au^{196}+n^1=Ir^{192}+n^1+He^4. \quad (700b)$$

According to the general theory, three-particle reactions will have probabilities of the same order as two-particle disintegrations provided the necessary energy is available and no potential barriers prevent the escaping of the emissible particles. With large surplus energy, it may even happen that three-particle disintegrations become more important than two-particle ones, because of the frequently mentioned tendency of the residual nucleus to retain a large fraction of the available energy (evaporation model, §§54, 65, 79).

With the now accepted mechanism of three body reactions, it is, of course, not possible to deduce the reaction energy from the upper limit of the energy spectrum of the disintegration products as has formerly been done (O2, O3). The maximum energy of the product particles depends on the particular energy levels of the nucleus formed in the intermediate stages of the decay of the compound nucleus.

A. The disintegration of B^{11} by protons

After considerable controversy (K15, O2, O9, L5) it has been shown (D9) that the α-particles emitted after proton bombardment of boron fall into three groups, *viz.* (for 0.2 MV proton energy):

(1) A group of homogeneous energy of about 5.7 MV;

(2) A fairly homogeneous group at about 3.85 MV;

(3) A continuous distribution extending from very low energies to about 5 MV.

Group 3 contains roughly twice as many α-particles as 2, while group 1 contains, at low proton energies ($\sim$200 kv) only about 1 percent of all the particles produced.

Group 1 is to be attributed to the reaction

$$B^{11}+H^1=Be^8+He^4 \quad (701)$$

as was first pointed out by Kirchner and Neuert (K15). The Be^8 nucleus is, in this case, formed in the ground state. Its mass is almost identical with that of two α-particles (§108). The energy evolution in the reaction (701) is 8.5 MV, of which the α-particle receives $\frac{2}{3}$, the Be^8 nucleus $\frac{1}{3}$, according to the law of conservation of momentum.

Group 2 should be attributed to the first part of reaction (699), i.e.,

$$B^{11}+H^1=Be^{8*}+He^4. \quad (701a)$$

Since the average energy of the α-particles in this group is 3.85 MV, and the Be^8 recoil energy is one-half of this amount, the energy evolution turns out to be 5.77 MV. Several corrections to this figure have to be applied (§96, 97) but they happen just to cancel. The total reaction energy is about 8.57 MV (from the distribution of group 3, see below). This gives for the excitation energy of Be^{8*} a value $U=2.80$ MV.

The particles of group 2 do not all have the same energy, but their energy distribution corresponds about to the dispersion formula, with a width at half-maximum[68] of 0.51 MV. Since the

[68] The most accurate way of determining the width from the experimental data of Oliphant, Kempton and Rutherford seems to be the following: The total number of particles counted by these authors (O9, Fig. 7) is 2245. Of these, 30 belong to group 1. 25 particles should be subtracted because O. K. R. overestimated the number of very slow particles, assuming the distribution to be homogeneous down to zero energy, while this is not the case according to the theoretical distribution (Fig. 25). Of the remaining 2190 particles, two-thirds = 1460 belong to group 3. Since this group extends almost uniformly from 0 to 4.7 MV energy, the number of group 3 particles

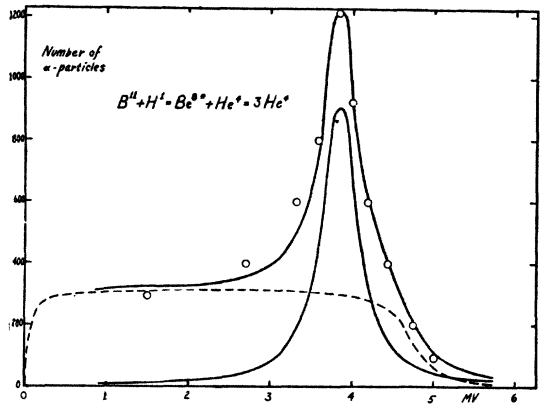

FIG. 25. Energy distribution of the α-particles emitted in the proton bombardment of B^{11} (without the discrete group corresponding to the reaction $B^{11}+H^1=Be^8+He^4$). Lower solid curve: theoretical distribution of the α-particles emitted in the primary process, *viz.* $B^{11}+H^1=Be^{8*}+He^4$. Broken curve: α-particles emitted in the break-up of the excited Be^8 (secondary reaction). Upper solid curve: total distribution, theoretical. Circles: experimental points (Oliphant, Kempton and Rutherford, O9).

kinetic energy of the recoiling Be^8 nucleus must always be one-half of the energy of the α-particle, the width of the Be^8 level turns out to be

$$\Gamma=\tfrac{3}{2}\cdot 0.51=0.77 \text{ MV}. \tag{702}$$

This means a lifetime of the excited Be^8 nucleus of

$$\tau=\hbar/\Gamma=1.04\cdot 10^{-27}/1.60\cdot 10^{-6}\cdot 0.77 =0.85\cdot 10^{-21} \text{ sec}. \tag{702a}$$

Group 3 is due to the disintegration of the excited Be^8 nuclei:

$$Be^8\rightarrow 2He^4+U. \tag{703}$$

If Q (=8.57 MV) is the total energy evolution in the whole reaction

$$B^{11}+H^1=3He^4+Q \tag{703a}$$

and U that in the disintegration of the excited Be^8, we have for the velocity of the center of gravity of the Be^{8*} nucleus

$$v_8=(Q-U/3M_\alpha)^{\frac{1}{2}} \tag{704}$$

and for the velocity of each α-particle relative to that center of gravity

$$v_4=(U/M_\alpha)^{\frac{1}{2}}. \tag{704a}$$

If we assume that there is no correlation between the directions of the two velocities, we find that the energies of the α-particles of group 3 will be uniformly distributed between the limits

$$E_{\min}^{\max}=\tfrac{1}{6}[(Q-U)^{\frac{1}{2}}\pm(3U)^{\frac{1}{2}}]^2. \tag{705}$$

With the numerical values given above,

$$E_{\min}=0.04 \text{ MV} \quad \text{and} \quad E_{\max}=4.68 \text{ MV}.$$

This distribution, when modified by the width of the level of Be^8, is well confirmed by the experimental data of Oliphant, Kempton and Rutherford (O9, cf. D9).

The theoretical distributions for groups 2 and 3 and the total distribution are given in Fig. 25. The agreement with experiment is excellent on the high energy side of the main maximum, but not so good on the low energy side. Part of this may be due to uncertainties in the range-energy relation.

The same order of magnitude ($\sim$3 MV) for the energy of the first excited level of Be^8 was obtained from theoretical calculations by Wigner and Feenberg (F10). The level is, theoretically, a ^1D level which is compatible with the fact that it can disintegrate into two α-particles.

B. Other three-particle disintegrations of light nuclei

The reaction

$$B^{10}+H^2=Be^{8*}+He^4=3He^4 \tag{699a}$$

follows undoubtedly the same mechanism as (699). The total reaction energy is considerably higher, *viz.* 18.0 MV. If Be^8 is left in the 3 MV state, the first α-particle (corresponding to group 2 in the disintegration of B^{11} by protons) should receive an energy of $\frac{2}{3}\cdot 15=10$ MV, corresponding to a range of 10.5 cm. An almost homogeneous group of this range seems to exist (C28) but its width seems to be larger than in reaction (709). It is very likely that, owing to the larger energy available, a number of higher excited levels of Be^8 can also be formed in this reaction which makes the observable energy

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per MV energy interval will be 1460/4.7=310. At the maximum, the number of particles observed is 1220 per MV. Thus 910 of these have to be attributed to group 2, and generally the number of particles of energy E in group 2 will be $910/[1+4(E-E_r)^2/\gamma^2]$ (dispersion formula, γ=width at half-maximum). On the other hand, the total number in group 2 must be $\frac{1}{3}\cdot 2190=730$. Therefore we have

$$910\cdot\tfrac{1}{2}\pi\gamma=730$$

from which $\gamma=0.51$ MV.

distribution of the α-particles more uniform and therefore the interpretation more difficult.

The reaction

$$Li^7+H^2=Be^{8*}+n^1=2He^4+n^1 \qquad (706)$$

or

$$Li^7+H^2=He^{5*}+He^4=2He^4+n^1 \qquad (706a)$$

has been studied by Bonner and Brubaker (B37, neutron distribution) and by Kempton, Browne and Maasdorp (K2, α-particles). The neutrons consist of a homogeneous group of about 13.2 MV energy, corresponding to the reaction

$$Li^7+H^2=Be^8+n^1 \qquad (706b)$$

and a practically continuous distribution extending at least up to 12.4 MV. If we wanted to explain the occurrence of neutrons of 12.4 MV on the basis of the mechanism (706a), we should have to assume an excited state of Be^8 at $9/8\ (13.2-12.4)=0.9$ MV. This is extremely improbable in view of the fact that no trace of such a state was observed in the process $B^{11}+H^1=3He^4$, and also from the theoretical calculations of Feenberg and Wigner. Moreover, in order to explain the continuous distribution of neutrons on this basis, it would be necessary to assume a continuous spectrum for Be^8. This is again quite contrary to the result from the $B^{11}+H^1$ reaction, and also contrary to the fundamental assumptions of Bohr's theory of nuclear processes. This shows that the mechanism indicated in (706a), with He^5+He^4 as intermediate products, besides the mechanism (706), must certainly play a part in the reaction.[69]

The reaction

$$Be^9+He^4=3He^4+n^1 \qquad (707)$$

was suggested by Bohr (B33) in order to explain the large number of very slow neutrons among the neutrons produced by α-bombardment of Be^9. According to the evaporation model, the neutrons formed in the reaction

$$Be^9+He^4=C^{12}+n^1 \qquad (707a)$$

should have, in the average, energies of the order of the "temperature" of the C^{12} nucleus formed, i.e., about 2–3 MV. In reality, most of the neutrons have less than 1 MV energy (§99, and D21). The mechanism of reaction (707) is one of the following:

$$\begin{array}{l} Be^9+He^4\rightarrow C^{13*}\rightarrow \\ \qquad\qquad Be^{8*}+He^4+n^1 \\ \qquad\nearrow\qquad\qquad\searrow \\ Be^{9*}+He^4 \qquad\qquad 3He^4+n^1, \\ \qquad\searrow\qquad\qquad\nearrow \\ \qquad\qquad He^{5*}+2He^4 \end{array} \qquad (707b)$$

so that the neutrons are emitted either in the break-up of an excited Be^9 or an excited He^5 nucleus.

The reaction (707) is confirmed by the fact that the compound nucleus C^{13} when formed from other initial particles, is known to break up into 3 α-particles and one neutron. E.g., the reaction

$$B^{11}+H^2=3He^4+n^1 \qquad (707c)$$

has been established both by observing large numbers of very slow neutrons (B40) and large numbers of slow α-particles (C28), in both cases forming a continuous energy distribution. The reaction

$$C^{12}+n^1=3He^4+n^1 \qquad (707d)$$

has also been observed, with fast neutrons in a cloud chamber (C8).

The process

$$Be^9+n^1=Be^9+2n^1 \qquad (708)$$

is established by observing an increase in the number of neutrons when neutrons are allowed to fall on beryllium (R20). The evidence for

$$N^{14}+H^2=4He^4 \qquad (709)$$

is the observation of very many slow α-particles, forming a continuous distribution in energy (C29).

[69] This is in contrast to the interpretation of Kempton, Browne and Maasdorp. *Note added in proof:* Our view is strongly confirmed by a recent paper of Williams, Shepherd and Haxby (W21c). These authors have investigated the distribution of the α-particles more accurately and find, superposed upon the continuous distribution, a fairly homogeneous group of energy 7.8 MV which they attribute to reaction (706a) with He^5 being left in the ground state. This gives for He^5 the very reasonable mass value of 5.0137, unstable by 0.8 MV against disintegration into an α-particle and a neutron. The width of the α-group is ~ 0.3 MV; therefore the width of the He^5 ground state $(9/5)\cdot 0.3=0.5_5$ MV, corresponding to a lifetime of $1.2\cdot 10^{-21}$ sec. which again appears very plausible. It is possible that He^5 can also be formed in excited states.

XIV. γ-Rays

§86. Classification of γ-Rays

The γ-rays emitted in nuclear processes fall into two main classes:

1. γ-rays emitted during the nuclear process itself.

2. γ-rays emitted in a secondary process following the proper (primary) nuclear process.

The most important case[70] for class 1 are the γ-rays emitted in simple capture processes. Such γ-rays have been observed from the capture of slow neutrons by heavy nuclei and of protons by light nuclei, and their energy has been measured in a number of cases.

The γ-rays of class 2 are emitted whenever a primary nuclear process of any kind leads to an excited state of the resultant nucleus. The primary nuclear process may be:

(a) Any nuclear transmutation process, produced by any kind of incident particle and with emission of any kind of outgoing particle (proton, neutron, deuteron, α-particle, γ-ray, etc.). Incident and outgoing particle may be of the same kind (inelastic scattering).

(b) A β-transformation.

(c) A natural α-decay.

Since it is the rule rather than the exception that nuclear disintegrations lead to excited states of the resultant nucleus, γ-rays of class 2 are an extremely frequent phenomenon and accompany practically every nuclear transmutation in which any nucleus of atomic weight greater than 6 is formed.

The probability of the emission of γ-rays of class 2 is given by the probability of the primary process which usually is connected with emission of material particles. Since we know that ordinarily the emission of material particles in the primary process is more probable than that of γ-rays, it follows that in general the γ-rays of class 2 are much more intense than those of class 1. Therefore the capture γ-rays can only be observed:

(a) When, contrary to the rule, simple capture is the only possible process. This case is realized for slow neutrons with heavy nuclei (§57, 60–62), for the capture of protons by C^{12} (§81) and probably by a few other light nuclei.

(b) When a primary process with particle emission is possible but never leads to an excited state of the residual nucleus. This is true for Li^7+H^1 since the α-particles emitted apparently can not be formed in excited states.

(c) When the capture is for certain energies enhanced by resonance while the class 2 γ-rays are not. This is true for $F^{19}+H^1$, Li^7+H^1, $C^{12}+H^1$ and perhaps other nuclei.

(d) When the energy of the capture γ-rays is much higher than the maximum possible energy of class 2 γ-rays.

In all other cases, the capture γ-rays will be swamped by the much more intense (by a factor 10,000 for light nuclei) γ-rays from the residual nucleus ("class 2").

Even in obvious capture processes such as the capture of slow neutrons by heavy nuclei, not all the γ-radiation observed is actually capture radiation, i.e., emitted in the capture process itself. For we know that any primary nuclear process will in general lead to an *excited* state of the residual nucleus especially if a great number of excited states are available as is the case for heavy nuclei. This is, of course, also true for the simple capture (§90). Therefore the capture process proper will be followed by secondary processes in which the residual nucleus loses its excitation energy by emission of one or more further γ-quanta of class 2. Thus the observed "capture radiation" is actually a mixture of γ-rays of classes 1 and 2.

§87. The γ-Ray Width

It has long been known from the study of the internal conversion of "natural" γ-rays (§88) that nuclear γ-radiation has about the same intensity whether it is dipole or quadrupole radiation. This is very surprising indeed since we should expect that the ratio of the two kinds of radiation is approximately as $(R/\lambda\!\!\!^{-})^2$ where $2\pi\lambda\!\!\!^{-}$ is the wave-length of the emitted

[70] Apart from capture processes, γ-rays may be emitted in a transition of the compound nucleus from its initial state to another state still above the dissociation energy. Then the later state may afterwards disintegrate with emission of a particle. These "noncapture" γ-rays emitted in the nuclear process itself have not yet been observed and are presumably very rare.

γ-ray and R the radius of the nucleus. Now for γ-rays of 1 MV energy, we have $\lambda\!\!\!^{-}=2\cdot10^{-11}$ cm whereas, even for the heaviest nuclei, $R=1.3\cdot10^{-12}$ cm. This gives $(R/\lambda\!\!\!^{-})^2=1/250$. Thus the quadrupole radiation should be about 250 times weaker than the dipole radiation, in striking contrast to experiment.

A. Elementary calculation of width

This discrepancy is emphasized by considerations of the absolute probability of emission of γ-rays. This absolute probability has been measured for γ-rays from natural radioactive nuclei (§88), for γ-rays from neutron capture (§61, 90) and from proton capture (§81, 90). In all cases, much smaller probabilities are found experimentally than would be expected from a simple theoretical consideration.

Generally, the probability of a radiative transition from a state m to a state n is per unit time

$$\frac{\Gamma_\gamma}{\hbar}=\frac{4}{3}\frac{\omega^3}{\hbar c^3}D_{mn}^2, \tag{710}$$

where D_{mn} is the matrix element of the electric dipole moment corresponding to the transition in question, and ω the frequency of the emitted γ-ray

$$\omega=(E_n-E_m)/\hbar. \tag{710a}$$

The dipole moment may be expressed by the oscillator strength which we define by

$$f_{mn}=2M\omega\hbar^{-1}(D_{mn}/e)^2, \tag{710b}$$

where M and e are mass and charge of the proton. Then

$$\Gamma_\gamma/\hbar=(2e^2\omega^2/3Mc^3)f. \tag{711}$$

If we express the quantum energy $\hbar\omega$ in MV, we obtain

$$\Gamma_\gamma/\hbar=8.0\cdot10^{15}(\hbar\omega)^2 f \text{ sec.}^{-1}, \tag{711a}$$

$$\Gamma_\gamma=5.24(\hbar\omega)^2 f \text{ volts}. \tag{711b}$$

For strong radiative transitions, we might expect f to be of the order unity, as in atoms. Or more generally, we may expect the sum of f over all the lines starting from a given nuclear level to have a value of the order unity.[71]

[71] A still larger value for Γ_γ is obtained by assuming the dipole moment D_{mn} in (710) to be of the order of a single elementary charge times the nuclear radius.

Actually, the observed lifetimes of excited states of natural radioactive nuclei are of the order 10^{-12} sec. (§69, 88), i.e. 10,000 times longer than would follow from (711a) with $\hbar\omega=1$ and $f=1$. Similarly, the γ-ray width of nuclear levels Γ_γ which is simply $\hbar$ times the reciprocal lifetime, is much smaller than the values obtained from (711b). One of the largest widths measured thus far is that of the excited state of Be^8 formed in the capture of protons by Li^7 (§81): it is 4 volts at a quantum energy of 17 MV, as compared to 1500 volts from (711b) with $f=1$. Again, the widths observed in the capture of slow neutrons (§61) and of protons by F^{19} (§81) are of the order of 0.1 to a few volts, for quantum energies of about 4 to 6 MV (§90) for which we should expect about 100 volts from (711b).

In all these cases, the probability of γ-ray emission is "too small" by a factor of about 100 for dipole radiation and therefore would be about right for quadrupole radiation if we use a similarly simple estimate for the intensity of the latter as in (711b). This fact, in combination with the observation mentioned above that quadrupole radiation is of the same order of intensity as dipole radiation, confronts us with the problem of explaining why dipole radiation is relatively so weak in nuclei.

B. Dipole and quadrupole radiation

A clue to the solution may be found in the fact that a system of particles *all having the same specific charge*, will not emit any dipole radiation but will emit quadrupole and higher multipole radiation. The general expression for the "effective electric moment" corresponding to a radiative transition $m\to n$ is (cf. B16, Eq. (38.10))

$$\mathbf{D}_{mn}=\hbar\omega^{-1}\int\Psi_n^*\sum_j\exp(i\mathbf{k}\cdot\mathbf{r}_j)(e_j/m_j)\nabla_j\Psi_m d\tau, \tag{712}$$

where e_j and m_j are charge and mass of particle number j, $\mathbf{r}_j$ its position, ∇_j the gradient operator with respect to r_j, $\mathbf{k}$ is the wave vector of the light quantum, ω its frequency, and $\Psi_m\Psi_n$ initial and final wave function of the radiating system. For dipole radiation, the retardation factor $\exp(i\mathbf{k}\cdot\mathbf{r}_j)$ should be replaced by unity:

$$\mathbf{D}_{mn}=\hbar\omega^{-1}\int\Psi_n^*\sum_j(e_j/m_j)\nabla_j\Psi_m d\tau. \quad (712a)$$

Quadrupole radiation is obtained by taking the second term in the expansion of the exponential. With $k=\omega/c$, we have then[72]

$$\mathbf{D}_{mn}=i\hbar c^{-1}\int\Psi_n^*\sum_j(e_j/m_j)(\mathbf{r}_j)_k\,\nabla_j\Psi_m d\tau. \quad (712b)$$

$(\mathbf{r}_j)_k$ is the component of $\mathbf{r}_j$ in the direction $\mathbf{k}$ of propagation of the light wave. Since the direction of the dipole moment $\mathbf{D}$ is identical with the direction of polarization of the light, $\mathbf{k}$ must always be taken perpendicular to $\mathbf{D}$ (i.e., to the direction of the gradient ∇_j).

We must now consider that the wave functions $\Psi_m\Psi_n$ will, of course, depend only on the relative coordinates of the particles with respect to the center of gravity, and not on the absolute coordinate of the latter. If the $\mathbf{r}_j$ denote *absolute* coordinates, the center of gravity is

$$\mathbf{R}=\sum_j m_j\mathbf{r}_j/M, \quad M=\sum_j m_j \quad (713)$$

and the relative coordinates

$$\varrho_j=\mathbf{r}_j-\mathbf{R}. \quad (713a)$$

Let x_j, X and ξ_j be the x components of $\mathbf{r}_j$, $\mathbf{R}$ and ϱ_j, respectively. Then

$$\frac{\partial\Psi_m}{\partial x_j}=\sum_i\frac{\partial\Psi_m}{\partial\xi_i}\frac{\partial\xi_i}{\partial x_j}=\frac{\partial\Psi_m}{\partial\xi_j}-\frac{m_j}{M}\sum\frac{\partial\Psi_m}{\partial\xi_i} \quad (713b)$$

and

$$\sum_j\frac{e_j}{m_j}\frac{\partial\Psi_m}{\partial x_j}=\sum_j\left(\frac{e_j}{m_j}-\frac{\sum e_i}{M}\right)\frac{\partial\Psi_m}{\partial\xi_j}. \quad (714)$$

Thus the transformation to relative coordinates makes the apparent charge of particle j different from e_j, *viz.*

$$e_j'=e_j-m_j\epsilon/M, \quad (715)$$

where

$$\epsilon=\sum_i e_i \quad (715a)$$

is the total charge of the system. In the particular case of nuclei, all the particles (neutrons and protons) have the same mass m_i, and $M=m_iA$ where A is the number of particles. Thus

$$e_j'=e_j-eZ/A, \quad (716)$$

which gives for protons ($e_j=e$)

$$e_j'=e(1-Z/A) \quad (716a)$$

and for neutrons

$$e_j'=-eZ/A. \quad (716b)$$

For the problem of emission of dipole radiation, neutrons should therefore be considered as having a negative effective charge equal to about half an elementary charge whereas protons have a positive effective charge which is only about half their true charge.

For a system whose particles have *all the same specific charge* (*i.e., ratio of charge to mass*), we obtain from (715)

$$e_j'\equiv 0. \quad (717)$$

This means that for such a system *the dipole moment would vanish identically whatever the wave functions* Ψ_m and Ψ_n. Thus we may understand the smallness of the dipole radiation if we can show that the nucleus, though actually composed of particles of different charge (neutrons and protons), acts like a system of particles all having the same charge. E.g., if the nucleus contains as many neutrons as protons ($A=2Z$), this will be true if we assume the nuclear particles to be combined in α-particles as subunits. Quite generally, the dipole moment will be zero if the centers of gravity of the neutrons alone and of the protons alone coincide so that the matrix element of their difference vanishes for all transitions. Owing to the large forces between neutrons and protons, this does not seem unlikely.

An alternative explanation due to Wigner would attribute the absence of dipole radiation in the spectra of natural radioactive elements to the fact that all low energy levels of a nucleus have the same "partition" (W17). This explanation would indeed seem much more attractive than the vague statement that the neutrons and protons will probably be distributed in a similar way. However, it does not explain why the γ-ray widths of the *high* excited states observed in neutron and proton capture are so much smaller than would be expected from the elementary calculation in A.

[72] D as defined in (712b) is k times the "quadrupole moment" corresponding to the transition, cf. Section C.

Whatever the explanation for the smallness of the dipole radiation may be, there will be no factors reducing the quadrupole and higher multipole radiation below the value expected from elementary considerations. E.g., if we repeat the calculations leading to (714) for quadrupole radiation propagated in the z direction, we find

$$ik\sum_i\left(\frac{e_j}{m_j}\zeta_j-\frac{\sum e_i\zeta_i}{M}\right)\frac{\partial\Psi_m}{\partial\xi_j}. \tag{718}$$

The expression in the parenthesis does *not* vanish when all particles have the same specific charge because it contains the coordinates of the particles.

This shows, at the same time, that multipole radiation higher than quadrupole will be improbable compared to quadrupole radiation by about the factors expected from elementary consideration, i.e., by a factor $(R/\lambda\!\!\!^{-})^2$ for each successive multipole. Higher multipole radiation will therefore, in general, not occur except from metastable states.

C. The radiation width from the liquid drop model

Bohr has suggested the use of the liquid drop model (§53C) to calculate the probability of emission of radiation. Since in this model only the density of nuclear matter is considered and not the density of neutrons and protons separately, the model can obviously only give quadrupole radiation (cf. B). Moreover, at low "temperature" of the nucleus, only surface vibrations are excited and therefore only the emission of quadrupole radiation by these modes of vibration is important.

The intensity of radiation I from a system of currents periodic in time

$$\mathbf{j}(\mathbf{r},t)=\mathbf{j}_0(\mathbf{r})\cos\omega t \tag{719}$$

is, according to classical electrodynamics,

$$I=\frac{1}{8\pi}\frac{\omega^2}{c^3}\int d\sigma|\mathbf{n}\times\mathbf{j}|^2, \tag{720}$$

where

$$\mathbf{j}=\int\mathbf{j}_0(\mathbf{r})\exp(i\omega\mathbf{r}\cdot\mathbf{n}/c)d\tau \tag{720a}$$

and the integration $\int d\sigma$ extends over all directions of the vector $\mathbf{n}$ (direction of propagation of the light). Quadrupole radiation corresponds to the second term in the Taylor expansion of the exponential in (720a) so that

$$|\mathbf{n}\times\mathbf{j}|=(\omega/c)\left|\int\mathbf{r}\cdot\mathbf{n}\;\mathbf{n}\times\mathbf{j}_0\,d\tau\right|. \tag{721}$$

Here the current density $\mathbf{j}_0$ of the radiating system may be expressed in terms of the charge density ρ; we have from (719) and the continuity equation:

$$\rho=\rho_0\sin\omega t,\quad \rho_0=\operatorname{div}\mathbf{j}_0/\omega. \tag{721a}$$

Neglecting the magnetic dipole radiation, i.e., assuming

$$\int(j_{0x}y-j_{0y}x)d\tau=0, \tag{721b}$$

we find for (721)

$$|\mathbf{n}\times\mathbf{j}|=(\omega^2/2c)\left|\int\rho_0\,\mathbf{r}\cdot\mathbf{n}\;\mathbf{r}\times\mathbf{n}\,d\tau\right|. \tag{722}$$

Taking the direction of emission $\mathbf{n}$ parallel to x, we have

$$|\mathbf{n}\times\mathbf{j}|=(e\omega^2/2c)([xy]^2+[xz]^2)^{\frac{1}{2}}, \tag{722a}$$

where

$$e[xy]=\int\rho_0\,xy\,d\tau \tag{722b}$$

is the quadrupole moment of the emitting system. If we assume that all quadrupole moments, *viz.* $[xy]$, $[xz]$, $[yz]$ etc., are equal in size (statistical equilibrium), the total radiation intensity (720) becomes

$$I=e^2(\omega^6/4c^5)[xy]^2. \tag{723}$$

The γ-ray width is $\hbar$ times the number of quanta emitted:

$$\Gamma_\gamma=\hbar I/\hbar\omega=\tfrac{1}{4}(\omega/c)^5e^2[xy]^2. \tag{723a}$$

For surface vibrations, the density of the nuclear liquid is constant (charge density $=Ze/\Omega$ where Ω is the total volume of the nucleus), and only the shape changes. If R is the normal radius of the nucleus, and

$$\zeta(\vartheta,\varphi)=\zeta_0(\vartheta,\varphi)\sin\omega t, \tag{724}$$

the displacement of the surface at a given point ϑ, φ, then

$$e[xy]=(Ze/\Omega)\int \sin\vartheta d\vartheta d\varphi \times\int_0^{R+\zeta_0} r^2\cos\vartheta\sin\vartheta\cos\varphi\, r^2dr, \quad (724a)$$

assuming the polar axis $\vartheta=0$ to be in the x direction. Integration over r gives

$$[xy]=(3Z/4\pi R^3)\int \sin^2\vartheta\cos\vartheta\cos\varphi d\vartheta d\varphi \times[\tfrac{1}{5}R^5+R^4\zeta_0+0(\zeta_0^2)]. \quad (724b)$$

The term R^5 vanishes upon integration over the angles, the terms containing higher powers of ζ_0 may be neglected so that

$$[xy]=(3ZR/4\pi)\int \zeta_0\sin^2\vartheta\cos\vartheta \times\cos\varphi d\vartheta d\varphi. \quad (725)$$

For each normal mode of surface vibration of a sphere, the displacement ζ depends on the angles ϑ and φ as a spherical harmonic Y_{lm}. Higher l corresponds to higher frequencies. (725) shows that of all the normal modes only one will give a contribution to the quadrupole moment, *viz.* that for which ζ depends on the angles as

$$\zeta_0=b\cos\vartheta\sin\vartheta\cos\varphi. \quad (725a)$$

This vibration corresponds to the deformation of the sphere into an ellipsoid; it is the vibration of lowest frequency possible. The calculation of the frequency is similar to that given in §53C; the index l of the spherical harmonic corresponds, for high l, to kR in the old calculation. For any l, (311) must be replaced by

$$\omega^2=(G/\rho S)(l+2)l(l-1)R^{-3}, \quad (726)$$

so that, with (313) to (314), we have for $l=2$

$$\omega^2=8\Gamma/(3Mr_0^2A), \quad (726a)$$

$$\hbar\omega=(8/3)^{\frac{1}{2}}(\Gamma P)^{\frac{1}{2}}A^{-\frac{1}{2}}. \quad (726b)$$

The potential energy is given by an expression similar to (307a), *viz.*

$$V=(G/2S)\int \sin\vartheta d\vartheta d\varphi \times\left[\left(\frac{d\zeta}{\partial\vartheta}\right)^2+\frac{1}{\sin^2\vartheta}\left(\frac{\partial\zeta}{\partial\varphi}\right)^2-2\zeta^2\right]. \quad (727)$$

For a given l, the integrand reduces to $(l+2)\times(l-1)\zeta^2$; and if ζ is given by (724, 725a), the total energy in the normal mode, which is twice the time average of V, becomes

$$E=2b^2G/15R^2=2b^2\Gamma/15r_0^2 \quad (727a)$$

(cf. (314)). Since the frequency of the vibration is rather low, its excitation energy will in general be equal to the nuclear temperature[73] τ, so that

$$b=(15/2)^{\frac{1}{2}}r_0(\tau/\Gamma)^{\frac{1}{2}}. \quad (727b)$$

Inserting (725a) (727b) into (725), we have

$$[xy]=(1/5)ZRb=(3/10)^{\frac{1}{2}}Zr_0^2\Gamma^{-\frac{1}{2}}A^{\frac{1}{2}}\tau^{\frac{1}{2}} \quad (728)$$

and, with (723a) (726b)

$$\Gamma_\gamma=\frac{2^{5/2}}{3^{3/2}\cdot 5}\frac{e^2}{\hbar c}\frac{\Gamma^{3/2}P^{1/2}}{(Mc^2)^2}\frac{(2Z)^2}{A^{11/6}}\tau. \quad (729)$$

Inserting the numerical values $\Gamma=P=10$ MV, $Mc^2=930$ MV, we have

$$\Gamma_\gamma=2\cdot 10^{-7}\tau(2Z)^2A^{-11/6}. \quad (729a)$$

The factor $(2Z)^2A^{-11/6}$ practically does not change with the nuclear mass; it increases only from 1 at low atomic weight to 1.5 at high A. The nuclear temperature τ is, for excitation energies of the order of 10 MV, about 1 to 2 MV. Therefore generally

$$\Gamma_\gamma=0.2 \text{ to } 0.5 \text{ volt}. \quad (729b)$$

This agrees surprisingly well with the average radiation width observed in the capture of slow neutrons by heavy nuclei (§61). For proton capture by light nuclei, larger widths up to 5 volts have been observed, probably due to dipole transitions which are not so rare in this case.

The radiation widths of low states of radioactive nuclei (§88) are much smaller, *viz.* of the order of millivolts. In our model, this may be understood by assuming the excitation energy

[73] This will not be true for low excitation energy of the whole nucleus, e.g., for the γ-rays emitted by natural radioactive nuclei. In this case, the average excitation energy per normal mode will be lower.

per normal vibration (727a) to be much smaller than at higher temperatures. A more elementary and more satisfactory treatment seems to be to deduce the quadrupole moment $[xy]$ from the observed width by means of (723a). If we measure ω in MV and the quadrupole moment $[xy]$ in 10^{-24} cm^2, we have

$$\Gamma_\gamma = 0.013[xy]^2\omega^5 \text{ volts.} \qquad (730)$$

The quadrupole moments of the transitions listed in Table XXXXIV are of the order 10^{-24} cm^2 which seems plausible.

For all but these very low states, the γ-ray width (729a) depends only very slightly on the excitation energy of the nucleus and on its size. This fact is in agreement with the observations (see above) and in striking contrast to the particle width which, for any kind of particle, increases enormously with increasing excitation energy (cf., e.g., §65, 78, 79) and decreases very much with increasing atomic weight (§54, 81). This means that the emission of radiation (simple capture) can in general only compete with that of particles (disintegration) if the nucleus concerned is heavy and the excitation energy low. The best known example is the capture of slow neutrons which is only probable for heavy nuclei (§57ff.). The capture of fast neutrons (§65) is very improbable compared to inelastic scattering, the capture of particles by light nuclei (§64, 81) improbable compared to particle disintegrations.

The γ-ray spectrum would, according to our liquid drop model, consist of a single line with a frequency given by (726b), i.e., about 1 MV for heavy, 2–3 MV for lighter nuclei. Of course, this consequence of the model should not be taken seriously; but perhaps the spectrum has actually an intensity maximum near the frequency (726b).

The formulae given (especially 729 to 729b) refer to the total γ-ray width. The width corresponding to a given γ-ray transition is smaller by a factor of the order of the number of possible final states. An estimate of the partial γ-ray width will be given in §90.

D. Metastable states

Weizsäcker has pointed out (W9) that some nuclei may have low excited states whose angular momentum is very different from that of the ground state, and that such states would be "metastable" and may have a very long lifetime against γ-emission. As we shall show below, lifetimes of several seconds, or even of years, may occur. This would explain the "isomerism" observed with various β-emitting nuclei. The best established case of this kind is a radioactive nucleus formed in Br by capture of slow neutrons. Bromine possesses two stable isotopes, Br^{79} and Br^{81} so that two radioactive nuclei (Br^{80} and Br^{82}) would be expected from slow neutron capture. Actually, three different radioactive nuclei with three different periods (18 min., 4.2 hr., 36 hr.) have been observed (K34), all of them chemically indentical with Br. Since no other process can occur but neutron capture, and since no stable isotope of bromine has been found despite very careful search (B28), we are forced to conclude that either Br^{80} or Br^{82} exists in two modifications with different half-lives.

A similar case may be indium which also has two stable isotopes (In^{113} and In^{115}) and gives three different periods under neutron bombardment (S28). However, it is possible (L26) that one of these activities ($3\frac{1}{2}$ hr.) is produced by fast neutrons through a np reaction and is in reality an isotope of Cd. Another case of isomerism may be the natural β-emitters UX_2 and UZ (G4).

The existence of "isomeric" β-emitting nuclei violates the rule that β-emission is in general slower than any other nuclear process, especially γ-radiation. As a rule, a nucleus formed in any way, e.g., by neutron capture, natural α-decay etc., will first emit γ-radiation until it arrives in its ground state, and the time required for that is usually of the order of 10^{-12} sec. (cf. Table XXXXIV). The β-emission which takes several seconds or more, can ordinarily not at all compete with γ-emission so that, in general, β-particles cannot be emitted from excited states of nuclei. From this consideration, any nucleus should have *one* characteristic lifetime with respect to β-decay, and isomeric nuclei could never exist.

This will be different only if there are selection rules making the emission of γ-radiation from a certain (in general the first) excited state much less probable than it is ordinarily. If this is the case, the metastable state will have its own decay period. Its decay may occur in two different ways, *viz.:*

(1) The metastable state may go over into the ground state with emission of γ-radiation, whereupon the ground state will decay further with β-emission. The half-life of the activity due to the metastable state is then determined by the probability of the forbidden γ-transition. It can only be observed as a separate period if it is longer than the natural β-period of the ground state. The energy spectrum of the β-rays is exactly the same for the two periods.

(2) The metastable state may emit a β-particle directly. This will be the case if the probability of γ-emission is smaller than that of β-emission for the given metastable state. The lifetime of the metastable state may in this case be longer or shorter than that of the ground state. The β-emissions from the metastable and from the ground state will in general lead to different states of the final nucleus because the angular momenta are very different (cf. below). Thus at least one of the β-transformations will be followed by a γ-ray from the residual nucleus, and the upper limits of the two β-spectra will not be the same. In the case of Br, the 18 min. period is also produced by photoelectric disintegration of Br (§91, 103) and should therefore belong to Br^{80}. Neither of the other two periods has been found as yet in the photoelectric process, thus we may provisionally ascribe them to Br^{82} which would then be the nucleus possessing the metastable level. One of the periods (35 hr.) gives β-rays of 0.8 MV (A1) and gammas (S16a), the other (4.2 hr.) gives betas of 2.05 MV and no gammas. If our interpretation is correct, the γ-rays should have an energy of about 1.25 MV. An exact measurement of the γ-ray energy should decide whether the 35 hr. or the 4.2 hr. activity belongs to the metastable level of Br^{82}.

The probability of emission of multipole radiation of order l is obtained by taking in (720a) the lth term in the Taylor expansion of the exponential. With the direction of observation $\mathbf{n}$ parallel to the x axis, we have

$$|\mathbf{n}\times\mathbf{j}| = \frac{(\omega/c)^{l-1}}{(l-1)!}\left|\int x^{l-1}\mathbf{n}\times\mathbf{j}_0 d\tau\right|. \quad (731)$$

It seems rather difficult to replace the current distribution $\mathbf{j}_0$ by the charge distribution ρ_0 in as rigorous a way as that used for the quadrupole radiation in (722a). However, it seems to be a fair approximation to replace (731) by

$$|\mathbf{n}\times\mathbf{j}| = (\omega^l/l!c^{l-1})\int x^{l-1}\mathbf{n}\times\mathbf{r}\,\rho_0 d\tau \quad (731a)$$

$$= [e\omega^l/l!c^{l-1}]([x^{l-1}y]^2+[x^{l-1}z]^2)^{\frac{1}{2}}. \quad (731b)$$

Roughly, we may estimate

$$[x^{l-1}y]\sim\kappa R^l, \quad (731c)$$

where κ is of the order unity. Then the probability of γ-emission per second becomes, in analogy to (723a),

$$\Gamma_\gamma/\hbar = (\omega/c)^{2l+1}(e^2/\hbar)\kappa^2 R^{2l}/l!^2. \quad (732)$$

Inserting the numerical values $\kappa=1$, $R=10^{-12}$ cm etc., we find for the lifetime against γ-emission

$$\tau = \hbar/\Gamma_\gamma = 5\cdot 10^{-21} l!^2 (20/\hbar\omega)^{2l+1}\text{ sec.}, \quad (733)$$

where $\hbar\omega$ is measured in MV.

Table XXXXII gives the lifetimes calculated from (733) for various l's and excitation energies. 200 kv is about the average excitation energy of the first excited state in a heavy nucleus; 50 kv occurs frequently, e.g., in 3 of the naturally radioactive nuclei; and 10 kv may occur occasionally. Differences of the angular momentum of as much as 4 units should not be very rare considering that several nuclei are known with an angular momentum of 9/2 in their ground state. Thus it seems possible that some nuclei may possess metastable states with lifetimes of a year or more against γ-emission which, of course, is ample for the explanation of two "lifetimes" of the same radioactive isotope. If the lifetime against γ-emission is as long as this, the excited state of a β-active nucleus will in general decay by direct emission of a β-particle (mode 2 above).

From our considerations, it is obvious that sufficiently metastable states will in general not exist for light nuclei, firstly because of the small

TABLE XXXXII. *Lifetime of metastable states.*

Excitation Energy (kv)	Change of Angular Momentum in Transition to Ground State			
	$l=2$ (Quadrupole)	$l=3$ (Octopole)	$l=4$	$l=5$
10	$6\cdot10^{-4}$ sec.	7 hrs.	$5\cdot10^{4}$ yrs.	$5\cdot10^{12}$ yrs.
50	$2\cdot10^{-7}$ sec.	0.3 sec.	10 days	10^{5} yrs.
200	$2\cdot10^{-10}$ sec.	$2\cdot10^{-6}$ sec.	3 sec.	10 days

density of levels for such nuclei and secondly because of the relatively small angular momenta. For heavy nuclei, we may expect metastable states roughly to be as frequent as "stable" neighboring isobars (§43), of which there are perhaps 10 examples altogether. Metastable states will, of course, not be restricted to β-radioactive nuclei but will also exist in stable nuclei: In this case, their existence is difficult to observe because the transition to the ground state will be accompanied by emission of very soft γ-rays only. Finally, there is a (very small) chance that some stable nucleus may possess a metastable state with a lifetime of the order of 10^{10} years; such a nucleus might then appear under normal conditions in two modifications, distinguishable by their spins.

§88. γ-Rays from Natural Radioactive Substances

A. Frequency of natural γ-rays and internal conversion

More accurate data are available on natural γ-rays than on those emitted during or after transmutations. This is partly due to the fact that the natural γ-rays have been studied over a longer period of time. But the main reason is that there is a method of study available for natural γ-rays which is not applicable to the γ-rays from transmutations of the light elements, *viz.* the internal conversion.

Crudely speaking, the internal conversion may be regarded as a photoelectric effect which the γ-ray produces in the same atom from whose nucleus it was emitted. This internal photoelectric effect is very probable because of the high intensity of the γ-radiation in the emitting atom itself. The process will lead to the emission of one of the external electrons[74] of the atom instead of the γ-ray. If the conversion electron is ejected from the nth electron shell whose ionization potential is E_n, its kinetic energy will be

$$E = h\nu - E_n. \tag{734}$$

By magnetic deflection of the conversion electron its kinetic energy may be measured quite accurately. If the origin of the electron is known the quantum energy $h\nu$ can be inferred from the kinetic energy by means of (734). The determination of $h\nu$ will be quite unambiguous when conversion electrons are ejected from several shells by the same γ-ray. This can be recognized by the fact that the difference between the energies of two groups of conversion electrons is equal to the known energy difference between two electronic levels of the atom. Then the assignment of the level of origin to each electron group is straightforward.

In one case (RaB), 8 groups of conversion electrons have been observed all originating from the same γ-ray (E8) by internal conversion in the electron shells[75] L I, L II, L III, M I, M II, M III, N I and O. From each of the groups, the same energy for the γ-quantum (52.91 kv) can be inferred, with a mean deviation of the individual determinations of about 0.1 percent. In most other cases, the internal conversion in outer electron shells is too weak to be observed because these electrons are too far from the source of the γ-radiation (nucleus). However, for a great number of γ-rays at least 3 groups of conversion electrons have been observed, *viz.* in the K, L I and M I shell. Only for very weak γ-ray lines, the number of conversion electron groups is reduced to one. In this case, there is of course no check on the γ-ray energy as in the case of several groups, but it is safe to assume that a single group of conversion electrons always originates from the K shell[76] because the probability of internal conversion decreases approximately as the inverse third power of the principal quantum number n of the ejected electron (for s electrons).

In this way, about 50 γ-ray lines originating from 12 different nuclei have been observed. A list of the results for the quantum energies $h\nu$, and of the conversion groups observed in each case, is found in Rasetti's book on *Nuclear Physics*, p. 124, etc.

[74] From the historical development of the subject (cf., e.g., Rasetti, *Nuclear Physics*, p. 121) it is still customary to call these secondary electrons "β-particles" and to speak of them as "discrete β-spectrum" in distinction from the continuous, true β-spectrum. In the light of our present knowledge, this must be regarded as a misnomer. The terms α-, β-, γ-particles should be reserved for particles coming from the nucleus itself. The internal conversion electrons originate from the electron shells of the atom and should be named in a way indicating their origin, e.g., conversion electrons.

[75] There is no conversion in the K shell because the energy of the γ-ray is not sufficient.

[76] This conclusion is quite safe if the energy of the conversion electron is greater than the difference of the ionization potentials of the K and L shell. If this is not the case, the electron may have originated from the L I shell.

The measurements of the electronic energies are accurate enough to determine the atomic number of the element from which the conversion electrons, and therefore the γ-rays, originate. The γ-rays observed in a radioactive transformation $A \rightarrow B$ must in almost all cases be ascribed to the product nucleus B, i.e., an emission *after* the α- or β-particle characteristic of the main transformation. This is in accord with theoretical expectation (cf. §69).

The γ-rays observed have been compared with the groups of α-particles found in α-disintegrations (§69) and gratifying agreement has been obtained between the $h\nu$ of the γ-rays and the differences between the energies of various α-groups (Table XXXII). Complete level schemes have been worked out for various nuclei from α- and γ-ray data (cf. §69, end).

B. Intensity of γ-rays. Dipole and quadrupole radiation

From the observed number of internal conversion electrons, the number of emitted γ-rays may be deduced if the coefficient of internal conversion, α, is known. This coefficient is defined as the ratio of the number of conversion electrons to the number of quanta emitted. The theory of internal conversion has been given by Taylor, Mott, Hulme and F. Oppenheimer (H39, T4, T5). It is found that the internal conversion coefficient decreases rapidly with increasing quantum energy and with increasing principal and azimuthal quantum number of the electron shell from which the conversion electron originates. The first fact is mainly due to the rapid increase of the probability of radiation with increasing energy. The dependence on the quantum numbers is explained by the smaller probability of coming near the nucleus for electrons with higher n and l. The conversion coefficient depends also on the nature of the radiation, being about 3 times larger for quadrupole than for dipole radiation. The theoretical dependence of the conversion coefficient in the K shell, α_K, on the quantum energy $h\nu$ for dipole and quadrupole radiation is shown in Fig. 26.

The expressions for the internal conversion coefficient are rather complicated owing to the complicated mathematical form of the Dirac wave functions. An approximate expression may

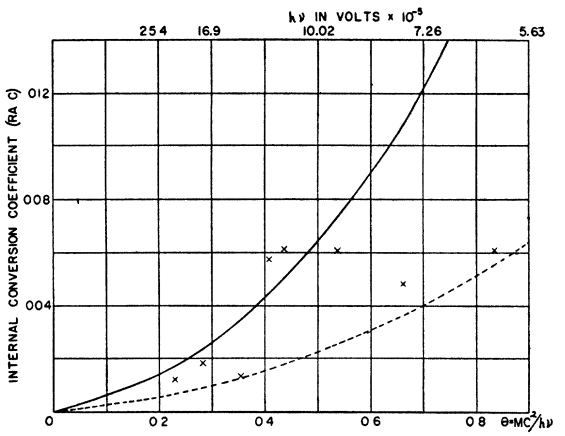

FIG. 26. Internal conversion coefficient for γ-rays in heavy nuclei ($Z=84$), according to Taylor and Mott. Solid curve: quadrupole radiation, broken curve: dipole radiation. Crosses represent experimental values determined by Ellis and Aston.

be obtained (R5a, p. 139) by neglecting relativity effects and at the same time assuming the frequency of the γ-ray to be large compared to the K absorption limit of the element in question. The result is, if $\alpha_K \ll 1$:

$$\alpha_K = \pi \left(\frac{\hbar c}{4 e^2}\right)^3 \left(\frac{Ry}{\hbar\omega}\right)^4 Z^4 \frac{e^{-4n \text{ arc cot } n}}{e^{2\pi n}-1}, \qquad (735)$$

where Ry is the Rydberg energy ($=13.54$ volts), $\hbar\omega$ the quantum energy of the γ-ray, Z the nuclear charge and

$$n^2 = Z^2 Ry/(\hbar\omega - Z^2 Ry). \qquad (735a)$$

The values of the conversion coefficient obtained from (735) are, for natural radioactive nuclei, about 5 times too small, owing to the unjustified approximations made.[77]

Experimentally, the internal conversion coefficient has been measured for a few γ-ray lines by comparing the number of internal conversion electrons with the number of photoelectrons produced in some material (usually lead) exposed to the γ-rays. Since the coefficient of photoelectric absorption is approximately known, both theoretically and experimentally, the absolute intensity of the γ-rays can be deduced from the number of photoelectrons observed. These experi-

[77] However, the formula would be correct for small nuclear charge Z. It shows that in this case the internal conversion coefficient would be exceedingly small so that there seems to be no chance of observing internal conversion of γ-rays in light atoms.

TABLE XXXXIII. *Intensities of γ-ray lines from ThC″.*

LINE	$10^4 p\alpha_K$	THEORETICAL α_K(%) DIPOLE	THEORETICAL α_K(%) QUADRUPOLE	CALCULATED p(%) DIPOLE	CALCULATED p(%) QUADRUPOLE	EXCITATION FROM α-GROUPS
γ_1	2.2	0.95	2.9	2.3	0.76	1.1
γ_2	2.2	1.02	3.2	2.2	0.69	0.16
γ_3	0.9	0.88	2.7	1.0	0.33	
γ_4	28	1.76	11.3	16	2.5	1.8
γ_5	6.1	1.49	7.5	4.1	0.81	

ments are not very accurate and give values for the internal conversion coefficient which are slightly (about 20–30 percent) higher than the theoretical values. The agreement is, however, sufficient to decide in most cases whether the γ-radiation investigated is dipole or quadrupole radiation (cf. Fig. 26).

Accepting the theory as correct, the internal conversion may be used for finding out the nature of the radiative transition when the correspondence between α-ray groups (§69) and γ-rays has been established and the intensities of the α-groups are known. This is, e.g., possible for ThC″ (cf. Table XXXII, §69). We denote by $p_1 \cdots p_6$ the intensities of the six γ-lines $\gamma_1 \cdots \gamma_6$ observed, in number of quanta per α-disintegration, and by $q_1 \cdots q_4$ the "excitations" of the four excited states of ThC″, i.e., the number of α-particles from the parent nucleus ThC corresponding to transitions to the four excited levels of ThC″. Then we have (cf. Table XXXI and Table XXXII)

$$\begin{aligned} q_4 &= p_1 &&= 1.10\%, \\ q_3 &= p_2 + p_3 &&= 0.16\%, \\ q_2 &= p_4 + p_5 &&= 1.80\%, \\ q_1 &= -p_1 - p_2 - p_4 + p_6 &&= 69.8\ \%. \end{aligned}$$

The observed number of conversion electrons from the K shell, $p\alpha_K$, is listed in the second column of Table XXXXIII for the five[78] γ-rays $\gamma_1 \cdots \gamma_5$. In the next two columns, the theoretical conversion coefficients for the known frequencies of the γ-rays are listed, for dipole and quadrupole radiation. Dividing the observed $p\alpha$ by these figures, the intensities p of the γ-rays are obtained (next two columns). It is seen that for $p_4 + p_5$ reasonable agreement with the value derived above (1.80 percent) is obtained by assuming both the γ-lines γ_4 and γ_5 to be quadrupole radiation while the assumption of dipole radiation would give about 10 times too much γ-radiation. For level 3 ($\gamma_2 + \gamma_3$), no agreement is obtained either way but the assumption of quadrupole radiation gives a less serious discrepancy. For the line γ_1, either dipole or quadrupole radiation may be assumed. Thus the evidence shows that either all the five γ-lines from ThC″ or at least four of them are due to quadrupole transitions.

In one case, *viz.* the third excited level of RaC′ (cf. Table XXXI) (excitation energy 1412 kv), no actual γ-ray has been observed but only an internal conversion electron. This is interpreted by assuming that the radiative transition from level 3 to the ground state is completely forbidden, e.g., because the two states both have angular momentum zero. Then the internal conversion is the only way in which the nucleus can go from state 3 to the ground state.

C. Absolute probability of γ-emission

The lifetime of nuclear states against γ-emission may be estimated by comparing the intensity of the long range α-particle groups from the excited levels of ThC′ and RaC′ with that of the γ-rays from the same levels. Table XXXXIV gives, for two levels each of RaC′ and ThC′, the observed number N_γ of γ-rays (per normal

[78] γ_6 does not have sufficient quantum energy to eject a K electron.

TABLE XXXXIV. *Lifetimes of excited states of RaC′ and ThC′.*

NUCL.	LEVEL	$\hbar\omega$ (MV)	N_γ	N_α	E_d(MV)	τ_α(SEC.)	τ_γ(SEC.)	Γ_γ (MILLIVOLTS)	f_0	$[xy]$ (10^{-24} CM²)
RaC′	1	0.61	0.4	$0.43 \cdot 10^{-6}$	8.437	$2.7 \cdot 10^{-7}$	$3 \cdot 10^{-13}$	2	$1 \cdot 10^{-3}$	1.8
	3*	1.41	0.0025	$22 \cdot 10^{-6}$	9.242	$9.5 \cdot 10^{-9}$	$8 \cdot 10^{-11}$	0.01	—	—
ThC′	1	0.73	0.14	$34 \cdot 10^{-6}$	9.674	$1.6 \cdot 10^{-9}$	$4 \cdot 10^{-13}$	1.5	$6 \cdot 10^{-4}$	0.6
	2	1.80	0.02	$190 \cdot 10^{-6}$	10.745	$6 \cdot 10^{-11}$	$6 \cdot 10^{-13}$	1	$6 \cdot 10^{-5}$	0.004

* Emits only conversion electrons.

α-disintegration), the observed number N_α of long range α-particles, the disintegration energy E_d for these α-particles, the lifetime against α-decay τ_α calculated from the energy by formulae (594, 600) with $G_\alpha=1$ volt, the lifetime against γ-emission

$$\tau_\gamma=(N_\alpha/N_\gamma)\tau_\alpha, \tag{736}$$

the γ-ray width $\Gamma_\gamma=\hbar/\tau_\gamma$, the oscillator strength f_0 calculated from (711b), and the quadrupole moment $[xy]$ associated with the transition according to (730).

The numbers of the excited levels are the same as in Table XXXI. N_γ is as estimated by Ellis and Aston (E8, E9). The estimates are based primarily on observations of the number of photoelectrons ejected from a Pt foil by the γ-rays (E9) supplemented by the intensities deduced from internal conversion using the theoretical value of the internal conversion coefficient. For the level 3 of RaC′ the figure given is the number of internal conversion electrons itself because for this transition no actual γ-ray is emitted (end of Section B). The calculated lifetimes are all of the same order, *viz.* a few times 10^{-13} sec., except for the highly forbidden transition 3 of RaC′ for which it is, of course, much larger. The γ-ray widths of the excited levels are of the order of a millivolt; they are, of course, smaller than the widths of neutron capture levels (§61) because from the levels considered here only *one* radiative transition can occur while transitions to many different final levels are possible from compound states responsible for neutron capture. The quadrupole moments are of the order 10^{-24} cm^2 as should be expected.

§89. γ-Rays from Residual Nuclei Produced by Transmutations

Not much experimental material is yet available on γ-rays from nuclear transmutations. Accounts of the experimental methods and results are given in §99–102. The most important points in the study of γ-rays are the following:

1. Nuclear energy levels. The levels of the residual nucleus can be deduced from the groups of outgoing particles emitted in the primary nuclear reaction (§§99–101, 109). The differences between the energy levels so deduced must agree with the energies of the γ-rays emitted by the residual nucleus. Thus the γ-rays provide an important check on the energy levels derived from particle groups. With light nuclei, it seems also possible to compare the experimental energy levels with theoretical expectation (F10) and to assign quantum numbers to them.

Checks between the data from particle groups and the γ-rays observed have been obtained for the nuclei B^{10} (produced in the reaction $Be^9+D=B^{10}+n^1$), B^{11} (reaction $B^{10}+D=B^{11}+H$), N^{15} (from $N^{14}+D=N^{15}+H$) and especially C^{12} (reactions $B^{11}+D=C^{12}+n^1$, $Be^9+He^4=C^{12}+n^1$ and $N^{14}+D=C^{12}+\alpha$). The particles produced in the reactions are neutrons, protons, and α-particles. The particle energies can in most cases be determined more accurately than the energy of the γ-rays, but the agreement obtained in the cases mentioned is satisfactory (cf. §99, 101).

2. Transition probabilities. A determination of the *absolute* radiative transition probabilities between the discrete levels of light nuclei does not seem feasible because, in contrast to natural radioactive nuclei (§88C), there is no alternative process of known probability (α-decay!) with which the γ-ray intensity could be compared.

Measurements of the relative intensities of the γ-ray lines starting from the same excited level of the nucleus will, however, give the relative transition probabilities. It seems feasible for light nuclei to calculate these transition probabilities theoretically from nuclear wave functions such as those given by Wigner and Feenberg (F10), using the Hartree model.

The sum of the intensities of all γ-ray lines starting from a given level of the residual nucleus, minus the intensities of the γ-ray lines ending at that level, must be equal to the intensity of the particle group in the primary reaction which corresponds to the formation of the nuclear level in question. Such comparisons of the intensity would serve in the first place to confirm the assignment of γ-rays to given transitions between nuclear energy levels. In fact, such assignments should not be called definite before a comparison of the intensities has been made. Moreover, intensity comparisons of the kind described might give more information about the efficiency of the apparatus used for detecting heavy particles as well as γ-rays.

A special kind of "γ-rays from the residual

nucleus" are the γ-rays emitted after an inelastic collision. Inelastic collisions have thus far only[79] been observed with fast neutrons (§65). The γ-rays emitted after the collision have been observed by Lea (L19) and by Kikuchi and collaborators (K7, K9, K5) for a great number of substances throughout the periodic table. The cross section for the production of the γ-rays was found to correspond approximately to the known scattering cross section for fast neutrons as should be expected. Measurements of the frequency are not yet available, but in view of the many levels of heavy nuclei there can be no doubt that the γ-spectra must be highly complex. Just as in capture processes (§90), the γ-emission will ordinarily take place in steps. It would be of interest to determine the average number of quanta emitted per inelastically scattered neutron.

Strictly speaking, all the γ-rays from capture processes but the one emitted first fall under the heading of γ-rays from the residual nucleus. This distinction is, however, not of much practical use.

§90. Capture γ-Rays

A. Light nuclei

The γ-rays from the capture of protons have been observed for the processes:

$$Li^7+H^1=Be^8+\gamma, \quad (I)$$
$$B^{10}+H^1=C^{11}+\gamma, \quad (II)$$
$$B^{11}+H^1=C^{12}+\gamma, \quad (III)$$
$$F^{19}+H^1=Ne^{20}+\gamma. \quad (IV)$$

I. The γ-rays from the proton bombardment of lithium were previously reported as very complex (C51). More recent experiments (D13) have shown that there is only one strong line at 17.1 MV and probably one or more weak ones between 10 and 17 MV. The first line corresponds to the transition to the ground state of Be^8. From the masses given in Table LXXIII (§108) the mass difference $Li^7+H^1-Be^8$ is found to be 17.0 MV. To this we must add $\frac{7}{8}$ (relative motion!) of the energy of the incident proton of 440 kv, giving a total of 17.4 MV in good agreement with the observed 17.1 MV. A line at 14 MV may be expected from a transition to the excited state of Be^8 of 2.8 MV excitation energy which is known from the study of the disintegration $B^{11}+H^1=Be^{8*}+He^4=3He^4$ (§85). If this is true, the line should be rather broad since the state at 2.8 MV has a width of about 0.8 MV (§85).

The occurrence of radiative transitions to excited states of Be^8 has been confirmed by Lauritsen and his co-workers (private communication) by the observation of groups of short range α-particles when Li^7 was bombarded by protons of 440 kv (resonance energy). The observation of these α-particles would give the most direct evidence on the position and width of the excited states of Be^8.

From the intensity of the γ-rays and simple theoretical considerations, we can obtain rather definite information on the angular momentum and parity of the resonance level of Be^8 responsible for the capture process. It seems certain theoretically that the ground state of Be^8 is a ^1S state of even parity. Since the γ-ray width of the resonance level is fairly large and the transition to the ground state the most intense, this transition is almost certainly "allowed"; it may still be either dipole or quadrupole. If the transition is a dipole transition, the resonance level must have odd parity and $J=1$. For a quadrupole transition, we must have even parity and $J=2$.[80] However, this is impossible because then the resonance level could disintegrate into two α-particles (§81). Therefore the resonance level must have $J=1$ and odd parity, and the γ-ray emitted must be dipole radiation. The first excited state of Be^8 is presumably an even ^1D state (F10); therefore the γ-ray transition to this state is also dipole radiation.

II and III. The γ-rays from these two capture processes are both observed when the mixed element boron is bombarded by protons (C50). The data do not yet allow definite conclusions but it seems certain that there is one γ-ray line corresponding to a transition into the ground state of C^{12} (energy about 15 MV) and several lines corresponding to transitions to excited states of that nucleus (cf. §101B).

IV. The γ-radiation from the capture of protons by fluorine is especially interesting since it is an excellent example of the working of selection rules. There seems to be a single γ-ray line having

[79] Except for the isolated example $Li^7+He^4=Li^{7*}+He^4$, cf. §99.

[80] While in general J may change by one unit in quadrupole radiation, the transition $1\rightarrow 0$ is forbidden.

a quantum energy of 6.0±0.2 MV (D14). The energy evolution from nuclear masses is 12.9 MV, to which the kinetic energy of the incident protons (0.33 MV resonance energy) has to be added. It seems therefore that the transition to the ground state is forbidden. This conclusion is even more stringent because the method used for investigating the γ-rays, *viz.* the observation of electron pairs, is most sensitive to high energy γ-rays.

The fact that the transition to the ground state must be forbidden, may be deduced directly from the known properties of the resonance level responsible for the capture of the protons. We know that this resonance level must have either even J and odd parity, or *vice versa*, because otherwise it would disintegrate into $O^{16}+He^4$ which would be irreconcilable with its small width (§81). Now the ground state of Ne^{20} is, again, almost certainly a ^1S state of even parity. Such a state combines optically only with states of $J=1$ and odd parity (dipole transitions) or of $J=2$ and even parity (quadrupole transition). Both these possibilities are excluded for the resonance level; therefore there can be no strong γ-ray corresponding to the direct transition into the ground state, in agreement with observation.[81]

The observed γ-radiation of 6.0 MV leads to a state of Ne^{20} with an excitation energy of 7.2 MV. This is 2.5 MV more than the dissociation energy of Ne^{20} into an α-particle and an O^{16} nucleus. Unless there are again selection rules forbidding it, this state of Ne^{20} will probably disintegrate into $O^{16}+He^4$ rather than emit a further γ-ray, because the potential barrier of O^{16} for α-particles is not very high (4.5 MV). It would be interesting to find these slow α-particles (2.0 MV)

The absolute probability of radiative transitions can be found from the γ-ray width of resonance levels responsible for proton capture (§81). The results are given in Table XXXXV. Besides the radiation width Γ_γ observed, and the lifetime of the excited state against γ-radiation, $\hbar/\Gamma_\gamma$, we have given the effective oscillator strength f of the γ-ray line emitted in the capture process as calculated from (711b). The values obtained ($3\cdot10^{-3}$ to 0.08) are larger than for radioactive nuclei (§88C). This is presumably due to the fact that the number of γ-lines starting and ending at a given level is very much greater in a heavy (naturally radioactive) than in a light nucleus, and therefore the f value for any single line smaller.

[81] The difference between the Ne^{20} and the Be^8 case (cf. I) is that the resonance level of Be^8 is allowed to be an odd level with $J=1$ while that of Ne^{20} is not. The reason for this is that Be^8 may only disintegrate into two like particles (α-particles) which are not capable of existing in a state of odd parity while Ne^{20} disintegrates into unlike particles ($O^{16}+He^4$) which may have a wave function of odd parity if only the angular momentum is also odd.

B. Heavy nuclei

The γ-ray spectrum emitted in the capture of neutrons will no doubt be enormously complex because of the extremely large number of energy levels between the ground state and the compound state responsible for the capture of the neutron (resonance level). Because of this complexity, it will, of course, not be possible to deduce the binding energy of the captured neutrons from the frequency of the emitted γ-rays. However, a study of the frequency distribution will give very valuable information on the distribution of energy levels over the region between the ground state and the dissociation energy of the nucleus.

We assume that the dipole moment D_{rs} will, in the average, be of the same order for transitions between any two states r and s. More accurately, D_{rs} shall not show any general trend with the excitation energies U_r and U_s of the two states or with their difference but shall only vary irregularly from level to level. Then the partial width of a level r due to a radiative transition to level s, will be in the average (cf. (710))

$$\gamma_{rs}\approx c\omega^3=c'(U_r-U_s)^3, \tag{737}$$

where c and c' are constants.

With this assumption, we can easily calculate the primary γ-ray spectrum emitted in the capture of a slow neutron. In this case, the excitation energy U_r of the initial state is equal to the dissociation energy Q of the nucleus. Let $D(U)$ be the spacing of levels at the excitation energy U. Then the number of quanta of energy between $\hbar\omega$ and $\hbar(\omega+d\omega)$ emitted is given by

TABLE XXXXV. *Absolute probabilities of capture of protons.*

	Process	Radiation Width (Volts)	Lifetime sec.	$h\nu$ MV	$10^3 f$
(I)	$Li^7+H^1=Be^8+\gamma$	4	$1.6\cdot10^{-16}$	17	3
(V)	$C^{12}+H^1=N^{13}+\gamma$	0.08	$8\cdot10^{-15}$	2.4	3
(IV)	$F^{19}+H^1=Ne^{20}+\gamma$	0.6	$1\cdot10^{-15}$	6.0	3
		18	$3.5\cdot10^{-17}$	6.5?	80
		8	$8\cdot10^{-17}$	6.6?	40

$$I(\omega)d\omega = \gamma_{rs}\hbar d\omega/D(Q-\hbar\omega) \sim \omega^3 d\omega/D(Q-\hbar\omega). \quad (738)$$

We assume for D a formula similar to that derived in §53, *viz.*

$$D(U) = Q\cdot\exp(-kU^n). \quad (738a)$$

($n = \frac{1}{2}$ for free particle model, 4/7 to $\frac{3}{4}$ for liquid drop model.) Then we have

$$I(\omega) \sim (\hbar\omega)^3 \exp[k(Q-\hbar\omega)^n]. \quad (739)$$

This intensity has a maximum at $\hbar\omega = Qx_0$ where x_0 is determined by

$$(1-x_0)^{1-n}x_0^{-1} = \tfrac{1}{3}nkQ^n = \tfrac{1}{3}n\log(Q/D(Q)). \quad (739a)$$

The intensity falls to half its maximum value for $\hbar\omega = Q(x_0 \pm \delta x)$ where (approximately)

$$\delta x \approx x\left[\frac{1-x}{1-xn}\right]^{\frac{1}{2}}\left(\frac{\log 2}{3}\right)^{\frac{1}{2}}. \quad (739b)$$

The position of the intensity maximum and the half-width are given in Table XXXXVI for $n = \frac{1}{2}$ and $\frac{3}{4}$ and for various values of the spacing of levels D at dissociation energy, i.e., for various atomic weights (cf. table XXI). It is seen that the maximum should occur at a quantum energy of between $\frac{1}{4}$ and $\frac{1}{2}$ of the dissociation energy while the half-width should be somewhat less than one-half of the quantum energy at the intensity maximum. These theoretical figures compare fairly well with the experimental determinations of the quantum energy by Rasetti (R3) and by Fleischmann (F25) who found approximately 4–5 MV, i.e., about half the dissociation energy, by a method which favors the harder components of the γ-radiation.

The intensity maximum should lie at a relatively higher quantum energy if the spacing of levels is large, i.e., for lighter nuclei. This tendency is supported by the increase of the dissociation energy itself with decreasing atomic weight. Evidence for this trend of the quantum energy with mass number was also found by Fleischmann. The apparent quantum energy of the neutron capture radiation was found to be 7.7, 4.1, and 4.2 for Fe, Cd and Pb, respectively.

TABLE XXXXVI. *Position of intensity maximum x_0 and half-width δx of capture spectrum from theoretical formula (approximate).*

Spacing of Levels at Dissoc. Energy (Volts)	$n=0.5$		$n=0.75$	
	x_0	δx	x_0	δx
1	0.31	0.13	0.23	0.10
10	0.35	0.15	0.27	0.12
100	0.41	0.17	0.32	0.14
1000	0.48	0.20	0.39	0.17

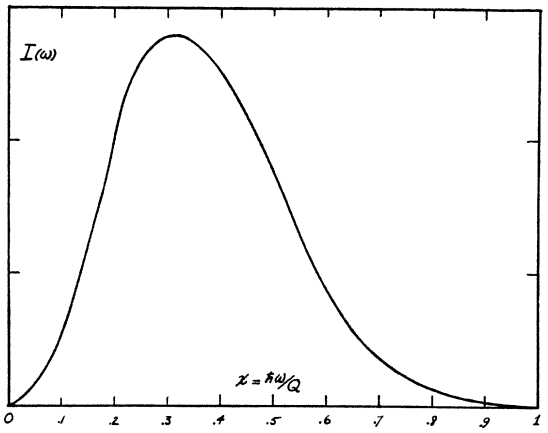

FIG. 27. Theoretical frequency distribution of the γ-rays emitted in the capture of a neutron by a heavy nucleus. The curve refers to the radiation emitted in the capture process itself, not including the γ-rays emitted afterwards by the residual nucleus. It has been assumed that the optical dipole moment is the same for the transition to each state of the final nucleus.

The expected γ-ray spectrum for $n = 0.6$ and $D(Q) = 10$ volts is shown in Fig. 27. The very small intensity near the maximum possible quantum energy ($\hbar\omega = Q$) should be noticed.

As the γ-ray emitted in the capture process itself (primary γ-ray) uses up only one-quarter to one-half of the energy available, several γ-rays will in general be emitted after this first one. This secondary γ-radiation will ordinarily be softer than the primary radiation. The number of γ-rays emitted in each capture process will in the average be between three and ten. Experimental evidence for this fact was obtained by Griffiths and Szilard (G23).

The total radiation width of neutron resonance levels has been determined from neutron experiments (§61) and theoretically (§87C) to be of the order of 0.1 to 1 volt. From this value, we may determine the average dipole moment $r_0^2 = (r_{rs}^2)_{Av}$ corresponding to transition to *one* of the final states. With the same assumptions as above, we have for the total radiation width

$$\Gamma^r_\gamma = \frac{4}{3}\sum_s \frac{e^2\omega_{rs}^3}{c^3} r_0^2 = \frac{4}{3}\frac{e^2r_0^2}{\hbar^3c^3}\int_0^Q \frac{(Q-U)^3}{D(U)}\,dU \quad (740)$$

$$= (\Gamma^r_{\gamma 0})_{Av} N_{\text{eff}}(Q), \quad (740a)$$

where

$$(\Gamma^r_{\gamma 0})_{Av} = \tfrac{1}{3}e^2r_0^2Q^3/\hbar^3c^3 \quad (740b)$$

is the partial γ-ray width for the transition to the ground state *if* the dipole moment for this transition has the "normal" value r_0, and

$$N_{\text{eff}}(Q) = \int_0^Q \left(\frac{Q-U}{Q}\right)^3 \frac{dU}{D(U)} = \int_0^1 x^3 e^{kQ^n(1-x)^n}dx \quad (741)$$

may be called the "effective" number of quantum states of the nucleus below dissociation energy, in distinction from the true number of levels

$$N(Q)=\int dU/D(U)=\int_0^1 \exp\,[kQ^n(1-x)^n]dx. \quad (741a)$$

The quantities mentioned are listed in Table XXXXVII, for $n=0.6$ (approximately corresponding to the liquid drop model, §53) and various values of the spacing of levels at dissociation energy, $D(Q)$.

The "effective" number of levels is seen to be much smaller than the true number because most of the levels lie very near the dissociation energy and are therefore unimportant as final states for γ-ray emission. The partial width due to radiative transitions to the ground state alone turns out to be of the order of a few millivolt for spacings of the order 10 to 100 volts as they are found for most elements for which the capture of neutrons has been investigated. This is just the same order of magnitude as for the low excited states of radioactive nuclei (§88C). The corresponding oscillator strength (cf. (711)) comes out between 10^{-5} and 10^{-4} (for natural radioactive nuclei 10^{-4} to 10^{-3}), and the average dipole moment for each transition is of the order of one thousandth of the nuclear radius.

§91. Nuclear Processes Produced by γ-Rays

Two kinds of processes are possible when γ-rays fall on nuclei: (a) The γ-ray may be absorbed and a material particle emitted from the nucleus (photodissociation). (b) The γ-ray may be scattered, either with or without giving part of its energy to the nucleus (Raman and Rayleigh scattering).

Since the emission of material particles is in general more probable (§65, 76), photodissociation will ordinarily be the more frequent process if it can occur energetically. Exceptions will be found (1) when the emission of the material particle is improbable because of the potential barrier and (2) when the emitted particle is very slow so that the factor v in the particle width (§52) makes the emission probability small. Scattering will be the only process occurring if the energy of the γ-ray is not sufficient to dissociate the nucleus.

Table XXXXVII. *Data on the capture γ-ray spectrum for heavy nuclei (Total γ-ray width $\Gamma_\gamma=0.5$ volt, dissociation energy $Q=8$ MV, $n=0.6$).*

Spacing at dissociation energy, $D(Q)$ (volts)	1	10	100	1000
Total number of levels below Q, $N(Q)$	$8\cdot10^6$	10^6	$1.2\cdot10^5$	$1.5\cdot10^4$
"Effective" number of levels, $N_{\text{eff}}(Q)$	3000	500	90	20
Partial width for ground st. transition $(\Gamma_{\gamma 0})_{Av}$ (millivolts)	0.15	1	5	25
Oscillator strength for trans. to ground st. f_0	$5\cdot10^{-6}$	$3\cdot10^{-5}$	$1.5\cdot10^{-4}$	$7\cdot10^{-4}$
Average dipole moment r_0 (cm)	$4\cdot10^{-16}$	$9\cdot10^{-16}$	$2\cdot10^{-15}$	$4\cdot10^{-15}$

The photodissociation has been observed for the deuteron (§16), for Be^9, and, more recently, for a great number of heavier nuclei (B47b, c, d). In all cases, one of the particles into which the nucleus dissociates is a neutron. There can be no doubt that, in principle, all nuclei can be disintegrated by γ-rays of sufficiently high quantum energy. The required dissociation energies are listed in Table LXVII (§103); they vary from 1.5 to over 10 MV for light nuclei. It must also be expected that particles other than neutrons can be split off a nucleus by γ-rays, e.g. protons, α-particles or deuterons. The relative probability of these various types of photodissociation will be determined by the respective "partial widths" of the levels of the compound nucleus, i.e. primarily by the penetrability of the potential barrier for the particle to be emitted; the probability will therefore be small for heavier nuclei.[81a]

If the emitted particle is a slow neutron, the corresponding partial width will be proportional to the neutron velocity (cf. §52, 265a). Therefore the cross section for a photodissociation into a slow neutron and a residual nucleus will be

$$\sigma=\text{const}\cdot E^{\frac{1}{2}}, \quad (742)$$

where

$$E=h\nu-Q \quad (742a)$$

is the kinetic energy of the neutron and Q the dissociation energy of the initial nucleus. The validity of (742) is, of course, restricted in a similar way as the $1/v$ law for neutron capture, *viz.* to neutron energies small compared to the energy of the first resonance level (§58b). Moreover, (742) holds only if the emitted neutron can have orbital momentum zero. This means that the angular momenta of initial and final nucleus must not differ by more than 3/2, i.e., the sum of the neutron spin (1/2) and the angular momentum of a γ-quantum (1 for dipole radiation).[82]

[81a] Photodissociation with emission of an α-particle will be probable when the energy of the γ-ray is insufficient for neutron-dissociation but sufficient to make the α-width larger than the γ-width (cf. §§79E, 80B).

[82] A stricter selection rule holds if Russell-Saunders coupling is valid in the initial and final nucleus. Then the internal *orbital* momenta of the two nuclei may differ by 0 or 1 but must not both be zero while the "multi-

For light nuclei, resonance effects will be observable in the photodissociation. E.g., if the γ-rays emitted in the capture of protons by B^{11} fall on a C^{12} nucleus, this nucleus will be dissociated into $B^{11}+H^1$. The cross section for this dissociation σ_{ph} is connected to the cross section for the emission of γ-rays in the proton bombardment of B^{11} by

$$\sigma_{ph}=\frac{(2i+1)(2s+1)}{(2i'+1)(2s'+1)}\frac{\lambda\!\!\!^{-}_{\gamma}{}^2}{\lambda\!\!\!^{-}_{p}{}^2}\sigma_{c0}. \qquad (743)$$

Here i' is the angular momentum of the dissociated nucleus (C^{12}, $i'=0$); i and s those of the dissociation products, *viz.* B^{11} and H^1 so that $i=\frac{1}{2}$ (probably, cf. R10a) and $s=\frac{1}{2}$; $2s'+1$ is the "statistical weight" for radiation, i.e., 2 (two directions of polarization); $\lambda\!\!\!^{-}_{\gamma}$ and $\lambda\!\!\!^{-}_{p}$ are the wave-lengths of γ-ray and proton, and σ_{c0} is that part of the capture cross section which corresponds to the formation of C^{12} in the ground state. Let us, e.g., assume that there is a resonance level for proton capture at 0.5 MV proton energy, corresponding to $\lambda\!\!\!^{-}_{p}=6.5\cdot10^{-13}$ cm, and that the corresponding cross section is 10^{-27} cm^2, as for the capture of protons by Li^7. The quantum energy of the γ-rays emitted in our process is 15 MV according to the masses of the nuclei concerned, so that $\lambda\!\!\!^{-}_{\gamma}=1.3\cdot10^{-12}$ cm. With these data, σ_{ph} would be $8\cdot10^{-27}$ cm^2 which should be observable without too great difficulty.

For heavy nuclei, it will, of course, be impossible to observe resonance maxima in the cross section for photodissociation. All we can hope to observe is the average cross section, averaged over energy regions large compared to the spacing between energy levels. Unlike that of material particles, the wave-length of γ-rays will be larger than the nuclear radius for quantum energies below about 20 MV. Therefore only dipole and quadrupole radiation will ever be of any importance. For dipole radiation, only compound states of angular momentum $J=i-1$, i and $i+1$ will be important where i is the angular momentum of the ground state of the nucleus to be dissociated. The average cross section for photodissociation becomes thus (cf. (405))

$$\sigma_{ph}=\frac{\pi^2\lambda\!\!\!^{-}_{\gamma}{}^2}{(2i+1)}\sum_J(2J+1)\frac{\Gamma^J{}_{\gamma0}\Gamma^J{}_Q}{D_J\Gamma_J}, \qquad (744)$$

where $\lambda\!\!\!^{-}_{\gamma}$ is the "wave-length" of the incident γ-ray, $\Gamma^J{}_{\gamma0}$ the partial width corresponding to emission of a γ-ray which leaves the residual nucleus in the ground state, averaged over all levels in the energy region in question, $\Gamma^J{}_Q$ the partial width corresponding to emission of particale Q (photodissociation), Γ_J the total width and D_J the average spacing of the levels of angular momentum J. Assuming these quantities to be independent of J and summing over the three possible values of J ($=i-1$, i and $i+1$), we find

$$\sigma_{ph}=3\pi^2\lambda\!\!\!^{-}_{\gamma}{}^2\frac{\Gamma_{\gamma0}\Gamma_Q}{D\Gamma}. \qquad (744a)$$

If the particle width Γ_Q is large compared to the γ-ray width Γ_γ which will be true for large kinetic energies of the emitted particle (cf., e.g., §65), we may put $\Gamma_Q=\Gamma$. Moreover, the partial width $\Gamma_{\gamma0}$ is equal to the total γ-ray width, Γ_γ, divided by the "effective number of quantum states $N_{eff}(h\nu)$" of the compound nucleus below the excitation energy $h\nu$, N_{eff} being defined as in the preceding section. We may write

$$N_{eff}(U)=U/D(U)\kappa(U), \qquad (745)$$

where κ is a function which varies slowly with the excitation energy U (slowly compared to $D(U)$). According to the values given in Table XXXVII, κ will be of the order of one to a few thousand. Inserting (745) into (744a), we obtain

$$\sigma_{ph}=3\pi^2\kappa\lambda\!\!\!^{-}_{\gamma}{}^2\Gamma_\gamma/h\nu. \qquad (745a)$$

From neutron experiments we know that the total γ-ray width of the levels of compound nuclei of atomic weight about 100, is, in the average, of the order $\frac{1}{2}$ volt. $h\nu$ may be estimated to be about 10 MV, corresponding to $\lambda\!\!\!^{-}_{\gamma}=1.9\cdot10^{-12}$

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plicities" must differ just by one. According to this rule, the photodissociation of the deuteron is forbidden, because the internal orbital momentum of the deuteron is zero, and that of the "residual nucleus," *viz.* the proton, is, of course, also zero. Indeed, the photodissociation of the deuteron does not lead to a state of the dissociated system of zero orbital momentum (s state) but to a p state ($l=1$), as was shown in §16. Correspondingly, the cross section is, in this case, proportional to $E^{\frac{1}{2}}$ rather than $E^{\frac{1}{2}}$ (cf. (265a)). This will no longer be true if magnetic dipole radiation is taken into account (§17).

cm. With $\kappa = 2000$, we obtain

$$\sigma_{ph} = 10^{-26}\ \text{cm}^2. \qquad (746)$$

The cross section observed by Bothe and Gentner in their experiments on the photodissociation of heavy nuclei (B47b, c, d) is of the order 10^{-27} cm². The agreement is satisfactory, in view of the crudeness of both the theoretical and the experimental estimate.

If the energy of the γ-ray is only just sufficient to dissociate the nucleus, the cross section for photodissociation will be small because of the factor $E^{\frac{1}{2}}$ in the particle width. This decrease of the cross section will set in when the particle width becomes smaller than the γ-ray width (cf. §65) which will occur for a kinetic energy of the particle of the order of a few hundred thousand volts.

When the γ-ray energy is insufficient for dissociation, the only process which may occur is the scattering of the γ-rays by the nucleus. Since, in this case, Γ_γ is identical with the total width Γ, (745a) now holds for the *scattering* cross section. The cross section for the nuclear scattering of γ-rays is thus of the order of[83] 10^{-26} cm². This is very much less than the ordinary Klein-Nishina scattering which amounts to $5 \cdot 10^{-23}$ cm² for atomic number $Z = 50$ and a γ-ray energy of 8 MV. Thus the nuclear scattering of γ-rays will probably be unobservable.

[83] Above the dissociation energy, the scattering cross section will be smaller by a factor Γ_γ/Γ.

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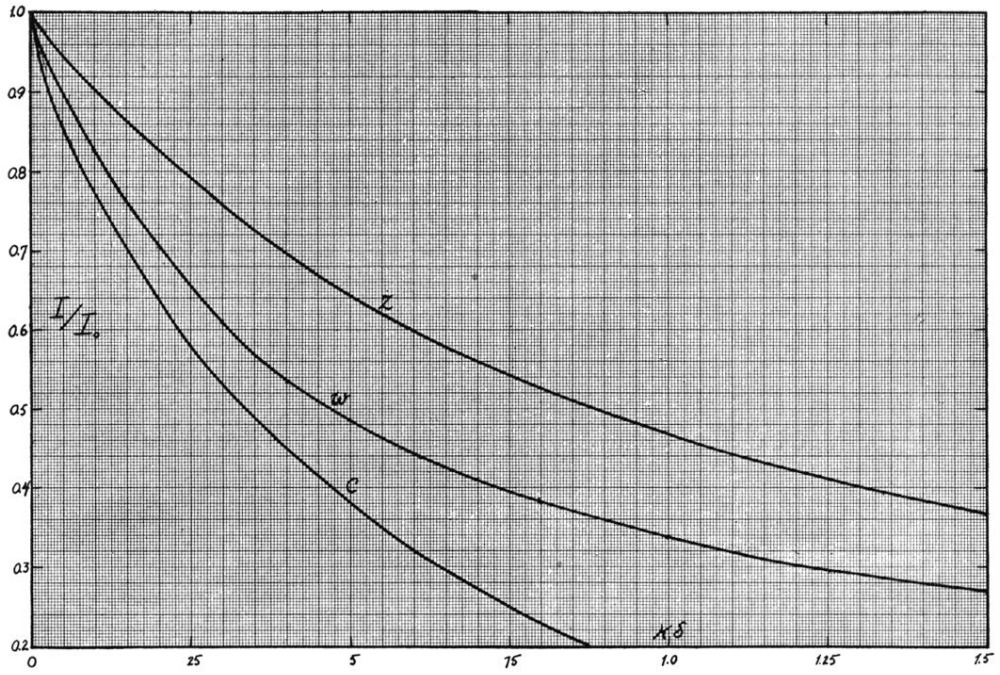

FIG. 16a. Absorption of resonance radiation. Abscissa: $K_r\delta$ where K_r is the average absorption coefficient of the resonance radiation for self-indication and small absorber thickness δ. (The figures on the abscissa should read 0.25, 0.5, 0.75, 1.0 etc.) Ordinate: Transmitted intensity. Curve z corresponds to a collimated incident beam of neutrons, curve w to an angular distribution $\cos\vartheta + \sqrt{3}\cos^2\vartheta$. For comparison, curve c gives the absorption for the same angular distribution with *constant* absorption coefficient, according to Amaldi and Fermi (A11, Fig. 2).

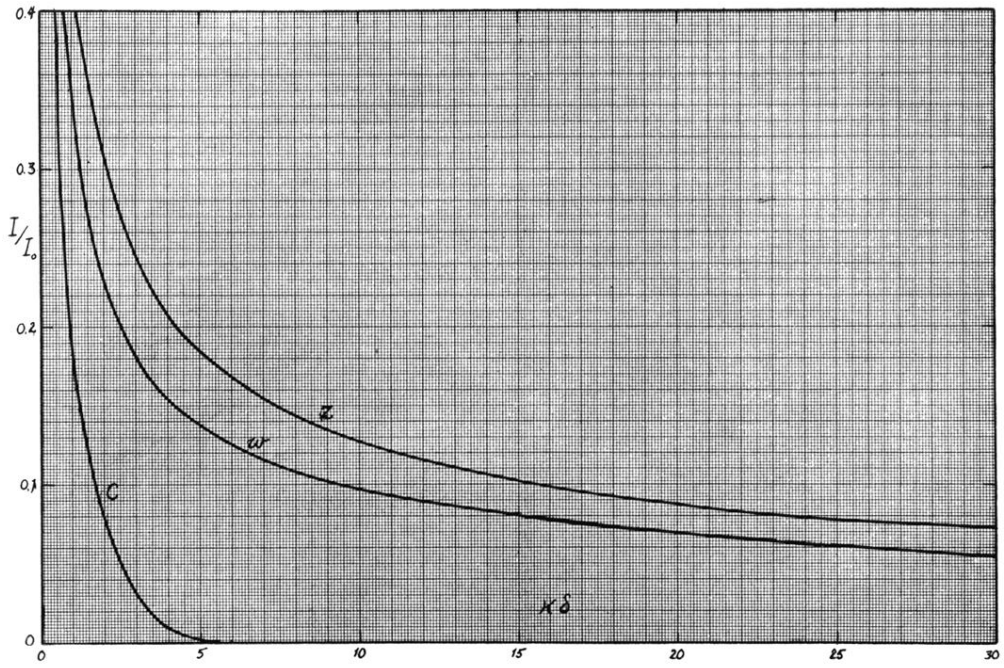

FIG. 16b.

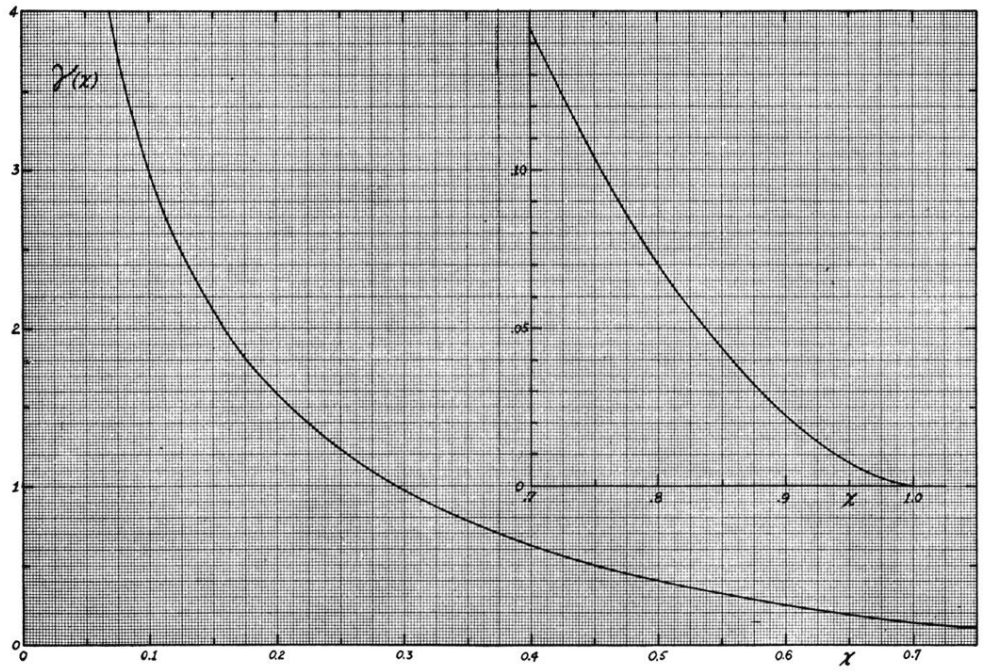

FIG. 18. The function $\gamma(x)$ determining the penetrability of the potential barrier (cf. (600)). x is the ratio of particle energy to height of barrier.

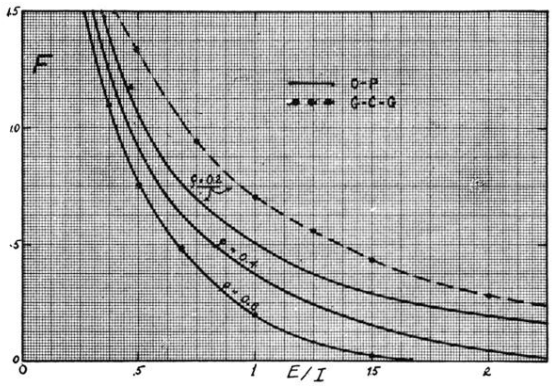

FIG. 24. The penetration function F in the Oppenheimer-Phillips theory. Curves are given for various ratios $\rho = IR/Ze^2$ of the binding energy I of the deuteron to the height of the potential barrier. $\rho = 0.6$ corresponds approximately to $Z = 17$, 0.4 to $Z = 35$, 0.2 to $Z = 100$. For $\rho = 0.2$ the penetration function following from the Gamow theory is given by the broken line. For $\rho = 0.6$, the Gamow points fall on the O-P curve.

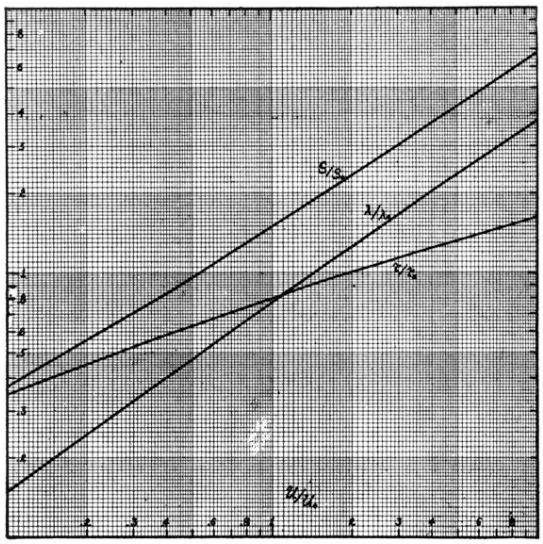

FIG. 9. Nuclear temperature τ, entropy S, and λ (cf. (281)) as functions of the excitation energy U in the liquid drop model. The constants U_0, S_0, τ_0, λ_0 are given in formulae ((324a), (326b), (326c), (327a)).