

Quantum Hall physics: Hierarchies and conformal field theory techniques

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The fractional quantum Hall effect, being one of the most studied phenomena in condensed matter physics during the past 30 years, has generated many ground-breaking new ideas and concepts. Very early on it was realized that the zoo of emerging states of matter would need to be understood in a systematic manner. The first attempts to do this, by Haldane and Halperin, set an agenda for further work which has continued to this day. Since that time the idea of hierarchies of quasiparticles condensing to form new states has been a pillar of our understanding of fractional quantum Hall physics. In the 30 years that have passed since then, a number of new directions of thought have advanced our understanding of fractional quantum Hall states and have extended it in new and unexpected ways. Among these directions is the extensive use of topological quantum field theories and conformal field theories, the application of the ideas of composite bosons and fermions, and the study of non-Abelian quantum Hall liquids. This article aims to present a comprehensive overview of this field, including the most recent developments.

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I. INTRODUCTION

A. Manifesto

The aim of this article is to give a systematic account of several of the main approaches to quantum Hall (QH) physics. In particular, we have aimed at a comprehensive review of conformal field theory (CFT) techniques and the various approaches to quantum Hall hierarchies. We describe the original ideas of Haldane and Halperin, the composite-fermion (CF) approach, and the various field theory descriptions based on composite bosons or fermions. We also, for the first time, give a comprehensive review of the CFT approach to both Abelian and non-Abelian hierarchy states.

We have tried to make this paper self-contained. We carefully introduce the main ideas and results relating to hierarchy states and also the CFT approach. For brevity, the main text assumes a basic knowledge of quantum Hall physics. For those needing a review of basics, a brief introduction is provided in Appendix A. We similarly assume some familiarity with a few concepts and techniques of CFT (Belavin, Polyakov, and Zamolodchikov, 1984; Di Francesco, Mathieu, and Sénéchal, 1997), so for the nonexpert we provide a brief introduction to these ideas in Appendix B. There are naturally several comprehensive texts that cover part of the material in this review, and we particularly mention Jain (2007), Ezawa (2008), and Fradkin (2013).

B. Brief historical introduction

It is hard to overemphasize the importance of the discoveries of the integer (Klitzing, Dorda, and Pepper, 1980) and fractional (Tsui, Stormer, and Gossard, 1982) quantum Hall effects. The integer quantization of the Hall conductance, while being crucial for modern metrology¹ (Weis and von Klitzing, 2011), is perhaps even more important for its theoretical implications. The precise quantization of the integer quantized Hall conductance was initially understood as a reflection of gauge invariance (Laughlin, 1981; Halperin, 1982) or as a measurement of the quantized electron charge. However, soon thereafter it was realized that the quantized conductance could also be understood as a robust topological invariant (Thouless *et al.*, 1982; Niu, Thouless, and Wu, 1985), which led to our modern understanding of topological insulators and superconductors (Hasan and Kane, 2010). Perhaps even more revolutionary from a theoretical point of view was the discovery of the fractional quantum Hall effect (FQHE) by Tsui, Stormer, and Gossard (1982), and the

¹The quantum Hall effect defines the standard of electrical resistance. The uncertainty of metrological quantum Hall measurements has been determined to be less than 1 part in 10^{10} despite the fact that the samples have substantial disorder. This is like measuring the distance from Stockholm to San Francisco to within 1 mm.

resulting realization that quantum liquids made of electrons could support excitations with fractional charge (Laughlin, 1983) and fractional statistics (Arovas, Schrieffer, and Wilczek, 1984; Halperin, 1984). This led to the concept of topological order, which is central to the modern classification of phases of matter (Wen, 2004).

Topological order is a way to characterize phases of matter that cannot be distinguished by the pattern of spontaneous symmetry breaking and the associated expectation values of local order parameter fields. A topologically ordered state of matter has an energy gap to bulk excitations, some of which carry fractional quantum numbers. In two spatial dimensions, the fractionalized particles are anyons, obeying fractional statistics that can be Abelian or non-Abelian. The notion of non-Abelian fractional statistics² opened up the possibility of using non-Abelian quasiparticles as a resource in quantum information processing (Kitaev, 2003; Nayak *et al.*, 2008). When defined on a topologically nontrivial closed manifold, a topologically ordered state exhibits a characteristic ground state degeneracy. On a manifold with boundaries, there are—at least in the cases we will be concerned with—gapless edge modes.³ The robust connection between the bulk topological order and the existence and properties of edge states is made most clearly through the connections forged by conformal field theory (Moore and Read, 1991).

The first observed fractional quantum Hall liquid (Tsui, Stormer, and Gossard, 1982) was the state at filling fraction⁴ $\nu = 1/3$ which was soon understood by Laughlin's seminal theory (Laughlin, 1983). This theory predicted states of the form $\nu = 1/m$ with m odd. A few additional states were experimentally observed that could be understood as simple generalizations of this, including particle-hole conjugates of these simple Laughlin states and analogous fractions in partially filled higher Landau levels. However, soon thereafter, new states were found experimentally which did not fit this framework (Stormer *et al.*, 1983). In particular, many fractions were observed in the lowest Landau level (LLL), generally of the form $\nu = n/(2mn \pm 1)$, which are now known as “Jain fractions” (Jain, 2007). Additional fractions, not fitting this Jain form, were later discovered in the LLL (Pan *et al.*, 2003) as well as in partially filled higher Landau levels (Xia *et al.*, 2004). See Figs. 1 and 2 for high quality experimental data in the lowest and second (or “first excited”) Landau levels.

Clearly it was necessary to find an organizing principle in order to understand the plethora of experimentally observed states. The first idea of this kind was due to Haldane (1983) and Halperin (1983, 1984), who suggested that the states in the LLL are hierarchically ordered. Moving away from the

²The first theoretical works on non-Abelian statistics of quasiparticles were by Bais (1980), Fredenhagen, Rehren, and Schroer (1989), Witten (1989), Fröhlich and Gabbiani (1990), and Moore and Read (1991), where the work of Moore and Read (1991) first suggested that such statistics may occur in quantum Hall systems.

³This list of properties provides a working definition of topological order, but there are also formal definitions in terms of modular tensor categories (Kitaev, 2003; Bonderson, 2007; Wang, 2010).

⁴The filling fraction ν is the number of complete Landau levels filled; see Appendix A.

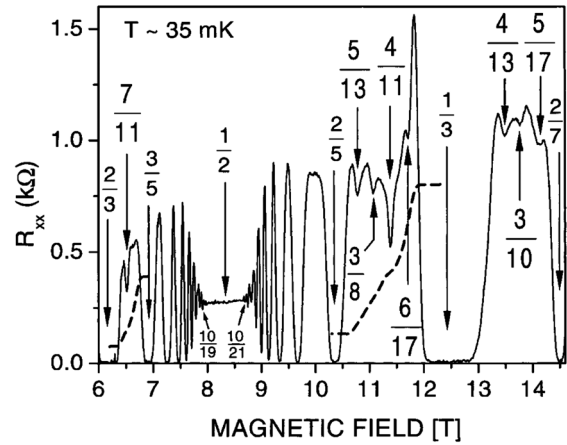


FIG. 1. Very high quality quantum Hall data showing many plateaus in the lowest Landau level. From Pan *et al.*, 2003.

center of a quantum Hall plateau by changing the filling fraction (by changing magnetic field or electron density) amounts to creating quasielectrons or quasiholes. These are then assumed to condense in a Laughlin-like state, thus forming a daughter state. In this way one can obtain any fraction with an odd integer denominator. A further major step was taken by Jain (Jain, 1989, 1990, 2007; Heinenon, 1998) who constructed trial wave functions for the fractional quantum Hall effect at $\nu = q/(2qp \pm 1)$ as being the integer quantum Hall effect (IQHE) of “composite fermions.” The latter can roughly be thought of as electrons with $2p$ quanta of vorticity (or “flux”) attached, which fill q effective Landau levels. These hierarchies (and other generalizations of the same ideas) are a main focus of this article.

The theoretical understanding of the fractional quantum Hall effect has progressed on several fronts. The first approach, starting with the seminal work of Laughlin, and

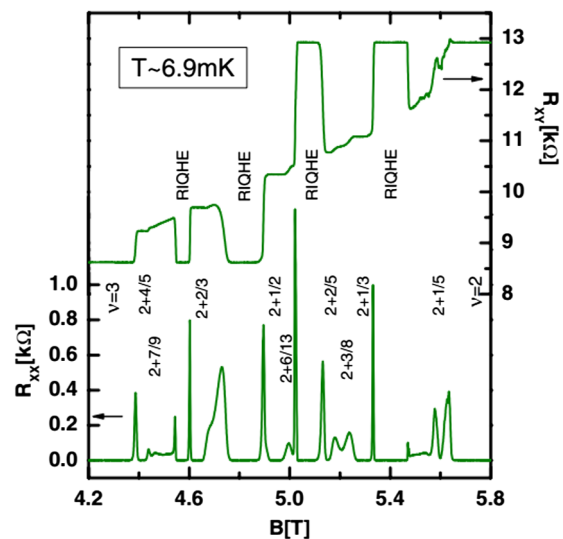


FIG. 2. Very high quality quantum Hall data showing plateaus in the second Landau level. Note that many of the plateaus labeled RIQHE (reentrant integer quantum Hall effect) are not fractional quantum Hall states but are believed to be some sort of charge density wave. From Kumar *et al.*, 2010.

later that of Jain, has come from the analysis of cleverly guessed, and numerically highly accurate, trial wave functions. Later, CFT was used to engineer model wave functions with interesting properties (Moore and Read, 1991), which allowed the first construction of non-Abelian quantum Hall states such as the Moore-Read (MR) (Moore and Read, 1991), Read-Rezayi (RR) (Read and Rezayi, 1999), and non-Abelian spin-singlet (Ardonne and Schoutens, 1999) states. A second front of attack was the development of effective field theories. These are of two different types. The first amounts to a rewriting of the original microscopic field theory of electrons moving in a strong magnetic field, as a field theory of composite bosons in zero field (Zhang, Hansson, and Kivelson, 1989), or composite fermions in a weaker magnetic field (Lopez and Fradkin, 1991). In principle these theories describe the microscopic physics of Abelian quantum Hall states, but they can be solved only by using mean-field methods. The second type of field theories are topological quantum field theories (TQFT) based on Chern-Simons (CS) gauge fields. These theories can be formulated for both Abelian and non-Abelian states, but, as the name indicates, only encode topological information, such as filling fractions and charge and statistics of the quasiparticles (Wen, 1995; Nayak *et al.*, 2008).

Connections between TQFTs and CFTs have been made extensively throughout the literature, starting with the pioneering work of Witten (1989). These same connections are extremely powerful in the quantum Hall context. The CFT approach (Moore and Read, 1991) gives a description of the dynamical [(1 + 1)-dimensional] theory of the quantum Hall edge (Wen, 1992; Kane and Fisher, 1996) and at the same time gives an explicit wave function for the [(2 + 0)-dimensional] quantum Hall bulk as a correlator of certain operators in the CFT. The primary operators of the CFT then define the fields of the corresponding TQFT and hence define the full statistics of quasiparticles present as excitations above the ground state.

Perhaps surprisingly, it has been hard to fit the prominently observed Abelian hierarchy states in the LLL into the CFT framework. The effective Chern-Simons description of these states is well developed, defining the universal properties of the edge CFT as well, but most work on trial wave functions has been in terms of composite fermions, with no obvious connection to a hierarchy formed by successive condensations of quasiparticles. This situation has changed over the past decade with the realization that CFT techniques can be used to construct representative wave functions for any state in both Abelian and non-Abelian hierarchies. Also, using these methods, the composite-fermion wave functions can be written as condensates of quasiparticles of a parent state and thus belong to the Haldane-Halperin hierarchy as well. A main objective of this article is to review these advances.

C. More detailed aims and organization of this article

In most of this review we will focus our attention on the simplest cases where there is a single, partially filled, Landau level with only one species of fermion (or boson). Note, however, that more complicated situations may be considered where each particle carries an additional nontrivial quantum number. This has been explored in many experiments, for

example, for a spin-unpolarized Landau level one must keep track of both spin species (Du *et al.*, 1995). Similarly in bilayer quantum Hall effect (Eisenstein *et al.*, 1992), the layer index may play the role of a pseudospin. The situation may be more complicated still in multivalley semiconductors such as graphene (Dean *et al.*, 2011), where one may have to keep track of both spin and valley indices.

This paper is organized as follows. In Sec. II we introduce the concept of quantum Hall hierarchies and then give a short review of the various theoretical approaches that have been employed to describe them. In Sec. III we discuss the current experimental status and relation of the proposed theoretical approaches, and in Sec. IV we review some important theoretical results with bearing on the connection between Chern-Simons theory, conformal field theory, and quantum Hall physics. Sections V and VI describe how to obtain explicit wave functions for Abelian and non-Abelian hierarchies, respectively. In particular, we explain the need for “quasilocal” CFT operators for describing quasiparticles and give a fairly detailed discussion of various approaches to non-Abelian hierarchies. In Sec. VII we generalize the construction of wave functions to the sphere and the torus, and explain why this is of interest. We conclude in Sec. VIII with a brief summary and outlook.

Several appendixes are included for those needing a bit more background. In Appendix A some basic facts about quantum Hall physics are elaborated. Appendix B gives basic facts about conformal field theory. Appendix C gives some details of derivations for one-dimensional edge theories.

D. Notation and conventions

For electron coordinates we use $\vec{r} = (x, y)$, $z = x + iy$, and $\bar{z} = x - iy$, and similarly $\vec{\eta}$, η , and $\bar{\eta}$ for quasiparticle coordinates. We sometimes use the abbreviation $\{z_i\}_a^b$ for the set $(z_a, z_{a+1}, \dots, z_b)$, etc., and just $\{z_i\}$ when the label i runs from 1 to N .

The term “quasiparticle” refers to either a quasidelectron or a quasihole. Up to a Gaussian factor, electronic wave functions will be holomorphic functions of the z_i ’s only and depend parametrically on the quasihole coordinates η_i and the quasidelectron coordinates $\bar{\eta}_i$ (see Appendix A). Note, however, that at an intermediate step, the hierarchy wave functions will be constructed as to have antiholomorphic components which will be either integrated out or projected onto the LLL. For sake of brevity, we often omit the Gaussian factors.

Late Greek letters μ, ν, σ , etc. will be used for Lorentz indices, and early Greek letters α, β, γ , etc. to label different CS gauge fields. In most of the paper we use a radial gauge $\vec{A} = (B/2)(-y, x)$, where a single-electron wave function in the LLL is $\sim f(z) \exp[-|z|^2/(4\ell^2)]$, and often we will put the magnetic length $\ell^2 = \hbar c/eB = 1$. The area element on the plane is denoted by $d^2r = dx dy = d^2z$. We will always put $c = 1$, often $\hbar = 1$, and always assume zero temperature.

II. QUANTUM HALL HIERARCHIES

A. Laughlin states: Plasma analogy and quasiparticles

In his original paper Laughlin (1983) argued that his wave function describes an incompressible liquid, and that the

quasihole excitations are fractionally charged. Soon thereafter it was shown that the quasiholes are anyons, i.e., particles obeying fractional quantum statistics (Leinaas and Myrheim, 1977; Wilczek, 1982). All these insights were based on using the so-called plasma analogy,⁵ which we briefly review using a modified version of the original argument given by Arovas, Schrieffer, and Wilczek (1984) and Halperin (1984). More details are presented in Sec. IV.A.1.

The Laughlin wave function (Laughlin, 1983) at filling fraction $\nu = 1/m$ with two quasiholes at positions η_1 and η_2 is given by

$$\Psi(\eta_1, \eta_2; z_1 \cdots z_N) = N(\eta_1, \eta_2) \prod_{i=1}^N (\eta_1 - z_i)(\eta_2 - z_i) \times \prod_{i < j} (z_i - z_j)^m e^{-\sum_i |z_i|^2 / (4\ell^2)}, \quad (1)$$

where $N(\eta_1, \eta_2)$ is a normalization constant. By writing

$$N(\eta_1, \eta_2) = \tilde{N} |\eta_1 - \eta_2|^{1/m} e^{-(|\eta_1|^2 + |\eta_2|^2) / (4m\ell^2)}, \quad (2)$$

$|\Psi|^2$ can be related to the Boltzmann factor $e^{-\beta U}$ of a two-dimensional Coulomb plasma, meaning that two point charges q_1 and q_2 repel each other by the potential $q_1 q_2 \ln |\vec{r}_1 - \vec{r}_2|^2$. The corresponding partition function is

$$Z(\eta_1, \eta_2) = \prod_{i=1}^N \int d^2 z_i |\Psi|^2 = |\tilde{N}|^2 \prod_{i=1}^N \int d^2 z_i e^{-\beta U}.$$

The inverse temperature of this analog plasma is $\beta = 1/m$, the charge of the z particles is m , the charge of the η particles is unity, and there is a quadratic confining background potential $m|r^2|/(4\ell^2)$ corresponding to a constant background charge density $2\pi\rho_{pl} = eB$. The crucial observation by Laughlin (1983) was that the classical plasma is in a screening phase, at least for $m < 70$.

From this follows a number of important results. First, in the absence of the quasiholes, a uniform density of z charges would (by Gauss's law) precisely screen the constant background charge. Thus, the wave function describes a system with uniform charge density, and since the plasmon in this system is gapped, the long-range density fluctuations are negligible and the quantum liquid is incompressible. Second, it is easy to see that the electrical charge associated with η must be exactly $-1/m$ of the charge of an electron and is spread over a distance given by the screening length of the plasma (on the order of ℓ). An important point is that these fractional charges are sharp in the sense that properly defined, the variance of their charge is essentially zero (Kivelson and Schrieffer, 1982) as discussed by Kjønsberg and Leinaas (1999). Third, the partition function Z is independent of the positions of the charges, and thus $|\tilde{N}|^2$ is a constant independent of η_1 and η_2 as long as they are far separated compared to the magnetic length (Halperin, 1984). From the properly

normalized wave function, one can verify that these particles obey fractional exchange statistics as explained later.

A heuristic argument for the exchange phase was first given by Halperin (1984), arguing that removing the absolute value of the relative factors of Eq. (2) amounts to a singular gauge transformation that leaves the electronic wave function “unchanged,” but allows us to interpret Eq. (1) as the proper wave function for anyonic particles in the LLL with exchange phase $\pm\pi/m$. Alternatively, one can compute the Berry phase factor $e^{i\gamma}$, or “holonomy,” associated with moving one quasiparticle at η_1 adiabatically around another at η_2 . The general expression for the Berry phase is

$$i\gamma = \oint d\tau \langle \Psi(\eta_1(\tau), \eta_2) | \frac{d}{d\tau} | \Psi(\eta_1(\tau), \eta_2) \rangle, \quad (3)$$

where τ is a parameter that takes the particle η_1 once around η_2 along some closed loop. Such an integral was first considered by Arovas, Schrieffer, and Wilczek (1984), who argued that the presence of a quasiparticle within the loop gives an additional $\pm 2\pi/m$ contribution to the Berry phase. Thus the Berry phase can be written as

$$\gamma = \frac{2\pi}{m} \frac{\Phi}{\phi_0} + 2\frac{\pi}{m}, \quad (4)$$

where $\phi_0 = h/e = 2\pi/e$ is the unit quantum of flux, and Φ is the flux enclosed by the loop. Thus the first term gives the usual Aharonov-Bohm phase for a particle of charge $e^* = e/m$ moving in a magnetic field, while the second term is twice the fractional exchange statistics phase for anyons with statistical angle $\theta = \pi/m$. For more details on this calculation see Stone (1992). We present an alternative derivation of the Berry phase in Sec. IV.A.1, which is based on the particular properties of the wave functions when written in terms of CFT correlators.

A more detailed derivation of this result was done by Kjønsberg and Leinaas (1997), who used the plasma analogy to map the expressions to the two-anyon system. This analytically derived result has been confirmed to be true numerically (Kjønsberg and Myrheim, 1999; Zaletel and Mong, 2012), so long as the quasiholes remain sufficiently far (on the order of ℓ) from each other. While there is no plasma analogy for handling quasielectrons (rather than quasiholes), it is often assumed that putting the factor $\prod_{i=1}^N (2\partial_{z_i} - \bar{\eta})$, proposed by Laughlin (1983), in front of the Laughlin wave function would amount to placing a quasielectron at position $\bar{\eta}$. This description is not unproblematic; we discuss quasielectrons in detail in Sec. V.C.

Laughlin's 1981 argument for the quantized conductance in case of the integer quantum Hall effect was based on adiabatic flux insertion in an annulus (Corbino) geometry where the edges are held at a fixed voltage difference V . Each inserted flux quantum can be shown to effectively move one unit of charge from one edge to the other giving an energy shift $\Delta E = eV$. On the other hand, the Hall current is given by $I = \Delta E / \phi_0$, so $I = (e^2/h)V = \sigma_0 V$ (Laughlin, 1981). Having established the presence of charge e/m quasiparticles,

⁵It is common nomenclature to call it an “analogy,” even though it is in fact a precise mapping.

Laughlin (1983) used the same argument to show that for the fractional effect the Hall conductance is σ_0/m .

This argument can in fact be used in “reverse” to establish that a quasihole created by adiabatic insertion of a unit flux in any incompressible quantum Hall state with filling fraction ν has a fractional charge $e^* = \nu e$ (Karlhede, Kivelson, and Sondhi, 1992). The argument, which does not invoke the plasma analogy, goes as follows: Imagine inserting a thin (radius $\ll \ell$) solenoid and then adiabatically turning on a flux $\Phi(t)$. This will, by Faraday’s law, induce an azimuthal electric field $E_\alpha = \partial_t \Phi(t)/(2\pi R)$, where R is the distance from the solenoid. Because of the quantum Hall response $\sigma_H = \nu e^2/h$ this results in a radial current density $j_r = \sigma_H E_\alpha$. So during a process which introduces one quantum of flux $\phi_0 = h/e$, the charge transported through the circle at R is $e^* = \int dt 2\pi R j_r(R) = \sigma_H \phi_0 = \nu e$. One can now invoke the Byers-Yang theorem which states that if a system consists entirely of particles with charge e , there is no physical consequence of having an integer number of elementary fluxes h/e added through a puncture in the plane (Byers and Yang, 1961). This is precisely the configuration we end up in after inserting the flux, so without changing any physical properties of the system, it can be gauged away leaving a local charge $Q = \nu e$ excitation. We do not know of any similarly clear and general argument for the statistics of the Laughlin quasiholes.

B. The Haldane-Halperin idea

If only a few quasiparticles are present, they will be pinned to impurities and remain inert, so that the system will have the same conduction properties as if the quasiparticles were not present (indeed, this is the origin of the finite width of the quantum Hall plateau). If many quasiparticles are present and are unpinned, they will form a system of itinerant charged particles in a magnetic field, which, at an appropriate filling fraction, is likely to condense into a Laughlin-like state. This is the heuristic picture behind the original proposal of a hierarchy due to Haldane (1983) and Halperin (1983, 1984). Given this idea, one can show that any filling fraction with an odd denominator can be obtained by successive condensations of quasiparticles into Laughlin-like states (see Secs. V.B and V.D). Since the charge of the elementary quasiparticle in a state $\nu = p/q$ is $\pm e/q$, the Coulomb gap is expected to be a decreasing function of q , and thus one expects states to be increasingly fragile, and thus harder to observe, as q increases. This tendency is clearly seen in experiments as illustrated in Fig. 3.

1. General form of an Abelian hierarchy wave function

Following Halperin, we refer to the wave functions for the quasiparticles $\Phi(\vec{\eta}_1 \cdots \vec{\eta}_M)$ as a *pseudo wave function* and write a general hierarchy state at level $n + 1$ as

$$\Psi_{n+1}(\vec{r}_1 \cdots \vec{r}_N) = \int d^2\vec{\eta}_1 \cdots \int d^2\vec{\eta}_M \Phi_n(\vec{\eta}_1 \cdots \vec{\eta}_M) \times \Psi_n(\vec{\eta}_1 \cdots \vec{\eta}_M; \vec{r}_1 \cdots \vec{r}_N), \quad (5)$$

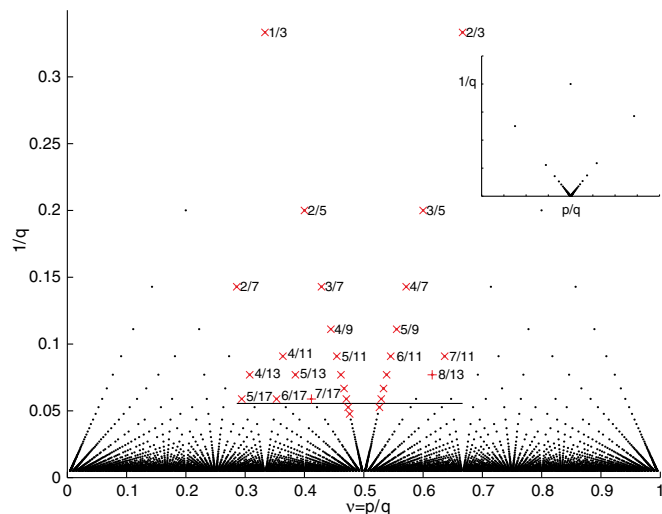


FIG. 3. Observed states and fractal structure: quantum Hall hierarchy states in the lowest Landau and their relative stability. For each rational $\nu = p/q < 1$, q odd, there is a unique state and its stability increases monotonically with $1/q$. The crosses denote observed states in the region $2/7 \leq \nu \leq 2/3$ (Pan *et al.*, 2003). The horizontal line marks the extent in ν of the experiment and is a line of constant gap. This line provides a good (although not perfect) approximate boundary for the observed states. The inset shows the structure of hierarchy states: At each $\nu = p/q$, q odd, there is a state with gap $\sim 1/q$ and quasiparticles with charge $\pm e/q$. When these condense, two sequences of states approaching p/q with decreasing gap are obtained. From Bergholtz *et al.*, 2007.

where Ψ_n is a state with M identical quasiparticle excitations in a parent state at level n . Recall that throughout this paper we use \vec{r}_i or z_i for the position of the i th electron and $\vec{\eta}_i$ for the i th quasiparticle.

One should be aware that the hierarchical constructions of Haldane and Halperin are slightly different. For instance, Halperin obtains the filling fraction of $3/7$ by condensing quasielectrons of the $2/5$ state into a Laughlin $1/2$ state, while Haldane condenses quasiparticles of the $1/3$ state into a $2/3$ quantum Hall state (the bosonic version of $2/5$). Also their arguments on how to determine the possible filling fractions differ. Despite these differences, the two constructions yield the same filling fractions with identical topological properties. In this review, we mostly follow Halperin’s construction; in particular, the pseudo wave function in Eq. (5) will always be of Laughlin form. Next we exemplify Haldane’s construction by giving a heuristic picture based on Haldane (1983) on how to determine the filling fraction and quasiparticle properties of a hierarchical daughter state.

Assume that M particles at positions ξ_i have condensed to form a parent quantum Hall state. (Note that these particles can be either electrons or the quasielectrons or quasiholes of some grandparent quantum Hall state.) A quasihole in this state at position $\vec{\eta}$ amounts to having a factor $\prod_{i=1}^M (\xi_i - \eta)$ and for a quasielectron we might take $\prod_{i=1}^M (2\partial_{\xi_i} - \vec{\eta})$. Effectively this means that each quasihole at position η “sees” each of the

particles ξ_i as if it were a single quantum of flux.⁶ We can now form multi-quasiparticle states simply by taking a product of such factors, and if we use the normalization (2) the wave function is symmetric in the η_i (or the $\bar{\eta}_i$ in case of quasielectrons), so the quasiparticles are bosons. In order for the η particles to condense they must see enough total flux to form a quantum Hall state. Since the quasiholes are bosons in this description, we must have an effective filling fraction $\nu^{\text{eff}} = 1/p$ with p even, meaning that there must be p times as many particles in the parent layer of the hierarchy as in the daughter.

Let us exemplify the hierarchical construction by starting from $\nu = 1/3$, with N electrons in $3N$ flux, and then adding an extra $N/2$ flux. This creates $N/2$ (bosonic) quasiholes, each experiencing one flux quantum per each of the original electrons, hence N fluxes. These $N/2$ quasiholes in N flux can condense in a $\nu = 1/2$ Laughlin liquid of bosons. We conclude that this first level daughter state would have N electrons in $3N + N/2$ flux, or a filling fraction $\nu = 1/(3 + 1/2) = 2/7$.

To determine the charge of the fundamental quasihole in this state we follow Haldane (1990) and imagine removing one electron at a fixed magnetic field. This creates three quasiholes of the original electron Laughlin state, leaving us with $N/2 + 3$ bosons in $N - 1$ flux. We must now figure out how many defects (quasiparticles) we obtain when we have $N/2 + 3$ bosons in $N - 1$ flux at filling fraction $\nu = 1/2$. Since there should be twice the number of flux quanta as bosons in the “ground” state of such a system, this would be $2(N/2 + 3) = N + 6$ flux. With $N - 1$ flux, we are clearly seven flux quanta short, and therefore there are seven quasiparticles present. Going back to our original system where we removed a charge of $-e$, we conclude that the elementary quasihole charge is $e/7$. Note that there are no rigorous arguments for deducing the fractional statistics of such quasiholes. However, Wen’s K -matrix expression in Sec. II.F predicts a statistical phase $-3\pi/7$ for the elementary quasihole $e/7$. Moreover, it has been shown that predictions based on clustering arguments as invoked by Su (1986) are generally consistent with those from the K -matrix formalism (Fulsetbakke, 2010).

Read (1990) suggested an alternative way to extract the topological content of hierarchy wave functions written as in Eq. (5). The basic assumption is that two such wave functions are orthogonal whenever the quasiparticle positions differ much more than a magnetic length. Using this orthogonality postulate the expectation values of operators, obtained by

⁶What we precisely mean here is that the wave function for the η particle is an analytic polynomial whose overall degree is M . This is exactly of the same form as we would have for a unit charge particle in a magnetic flux of $M\phi_0$. It is tempting to imagine that the η particles “see” a flux quantum at the position of each of the ξ particles. This is not correct, as a real flux quantum would just imply a phase winding (i.e., it is a singular gauge transformation). It is more correct to think of the ξ particles as having captured the vortices of the wave functions for the η particles. At a “mean-field” level, one can heuristically think of the ξ particles as providing a uniform magnetic field that the η particles experience; see Secs. II.D and II.E.

integrating over the electron coordinates, simplify to integrals that can be reinterpreted in terms of a multicomponent plasma, and arguments similar to the ones given previously for the Laughlin case will apply. We will discuss the status of the plasma analogy for hierarchy states further in Sec. V.E.

The hierarchy argument can be continued to higher levels to give filling fractions of the continued fraction form (Haldane, 1983; Halperin, 1984),

$$\nu = \frac{1}{m - \frac{1}{\sigma_1 p_1 - \frac{1}{\sigma_2 p_2 - \frac{1}{\ddots - \frac{1}{\sigma_{n-1} p_{n-1}}}}}} \quad (6)$$

where $1/m$ with m odd is the filling fraction of the 1st level state, $1/p_\alpha$ with p_α even is the filling fraction of the α th level daughter bosonic condensate of quasiparticles and $\sigma_\alpha = \pm 1$ indicates whether this condensate is formed by quasiholes (−) or quasielectrons (+). Note that with these conventions the Laughlin states are at level $n = 1$, and a general level n state is formed by $n - 1$ condensates determined by p_α and σ_α , $\alpha = 1, 2, \dots, n - 1$.

An often aired criticism of the hierarchy scheme is that it is based on a quasiparticle picture, which is valid only when the separation between quasiparticles is large compared to their radius. However, condensation into a daughter state occurs precisely when the quasiparticles are overlapping, and hence where a quasiparticle description loses its integrity. Thus the very notion of condensation is questionable (Jain, 2007). We believe that this point of view is based on demanding more than what can be expected from a “picture.” The real content of the hierarchy (in the Halperin sense) is embodied in Eq. (5), as well as in the general arguments about the relative stability and the properties of their excitations; see, e.g., Halperin (1984). The description in terms of condensation is only a (very suggestive) picture. It is worth mentioning here that ordinary electrons in the LLL also have a finite size ($\sim \ell^2$) and that they condense in a Laughlin state precisely when the distance between them becomes comparable with this length scale. Of course, electrons are in a sense “real” particles, but it is far from clear that this distinction should be very important.⁷

2. Difficulties and early numerical work

Although it is in principle straightforward to generate hierarchy wave functions using Eq. (5), in practice this formula turns out to be very difficult to work with. As a concrete example, consider the simplest case of a quasihole condensate in a $\nu = 1/3$ Laughlin state. The corresponding quasihole wave function is the generalization of Eq. (1) with the normalization factor $\prod_{\alpha < \beta}^M |\eta_\alpha - \eta_\beta|^{1/3}$, which assures that the normalization, up to a phase, is independent of the

⁷This is commented upon in Sec. V.C.3.

positions η_i , as long as these are far separated. The pseudo wave function is taken as

$$\Phi = \left[\prod_{\alpha < \beta}^N (\eta_\alpha - \eta_\beta)^2 \right] e^{-(1/3) \sum_\alpha |\eta_\alpha|^2 / (4\ell^2)},$$

which is a Laughlin state treating the quasiholes at positions η_α as bosons (Haldane, 1983; Halperin, 1984), and the factor of $1/3$ in the Gaussian exponent accounts for the quasiholes having charge $1/3$ and hence a longer magnetic length than the electrons. Choosing $M = N/2$ and using these expressions in the wave function Eq. (5) gives a quantum Hall wave function at $\nu = 2/7^8$:

$$\Psi = \int \prod_{\alpha=1}^M d^2\eta_\alpha e^{-(1/3) \sum_\alpha |\eta_\alpha|^2 / (4\ell^2)} \prod_{\alpha < \beta}^M (\bar{\eta}_\alpha - \bar{\eta}_\beta)^2 |\eta_\alpha - \eta_\beta|^{1/3} \\ \times \left[\prod_{\alpha=1}^M \prod_{i=1}^N (\eta_\alpha - z_i) \prod_{i < j}^N (z_i - z_j)^3 \right] e^{-\sum_i |z_i|^2 / (4\ell^2)}. \quad (7)$$

Even for this simplest case of a hierarchy wave function, there is unfortunately no known way to evaluate the integrals analytically, and numerical methods are feasible only for a small number of particles. Such calculations were done early on by Greiter (1994), who compared hierarchy states⁹ with results of exact diagonalization; mainly at level 2 (i.e., first level daughter) for up to eight particles, but also level 3 for up to six particles. The overlaps for these small systems are very good, but it is clearly questionable to draw conclusions from studying a level 3 hierarchy state of only six electrons.

The Laughlin wave function is the exact ground state wave function for a certain ultrashort-ranged electron-electron interaction (Haldane, 1983; Pokrovsky and Talapov, 1985; Trugman and Kivelson, 1985) which makes it very amenable to certain analytic approaches. Similarly, some of the more exotic wave functions such as the Moore-Read wave function (Moore and Read, 1991) are also the exact ground states of simple interactions (Greiter, Wen, and Wilczek, 1991; Read and Rezayi, 1996). Unfortunately, there is no known electron-electron interaction for which any of the hierarchy wave functions (those discussed here or later) are exact ground states, and this substantially complicates the detailed analysis of their properties.

If we relax the condition that the wave function should reside in the LLL, there are potentials for which the *unprojected* Jain state¹⁰ at $\nu = 2/5$ is the ground state (Jain, Kivelson, and Trivedi, 1990). This might have been

⁸Note that at this density there is no reason to believe that this wave function is correctly normalized. As mentioned in Sec. II.B.1, the real meaning of “condensation” is embodied in Eq. (5), and here we note that it must also come with a definite prescription for how to construct the pseudo wave function for the condensing particles.

⁹In order to evaluate Eq. (7), Greiter omitted the factor $|\eta - \eta|^{1/3}$, which amounts to a change in the short-distance behavior thought to be physically unimportant; cf. Sec. V.C.

¹⁰Jain’s composite-fermion states will be introduced in more detail in Sec. II.E.

a satisfactory situation if components in higher Landau levels had been relatively small, but in reality this is not the case.¹¹ Another argument for restricting the discussion to within a single Landau level is that the large B (or small m_e) limit provides a theoretically well-defined limit that is believed to capture the essence of quantum Hall physics. So it seems like a good strategy to first try to solve that problem and only later include effects of higher Landau levels (see the discussion at the end of Sec. III.A).

3. Why wave functions and which?

At this point it is fair to ask why we are at all interested in constructing explicit many-body wave functions? None of the model wave functions considered here are the exact ground states of any experimentally relevant Hamiltonian, and the overlap with the realistic ground state will always be zero in the thermodynamic limit. In addition, one may ask whether model wave functions can tell us anything beyond what we can deduce from the effective theories that are discussed in the following sections.

First we should make clear that the wave functions we are considering in this review are “representative wave functions” in the sense of Moore and Read.¹² This means that even though they might not be eigenstates of any physical Hamiltonian, they have the topological characteristics of some distinct phase of matter. These are typically the charge and statistics of quasiparticles, as well as properties of edge states and the ground state degeneracy on topologically nontrivial manifolds. It is these topological features we are interested in, rather than microscopic details of the actual wave functions. This, however, still begs the question of relevance—why are we not satisfied with an effective field theory description, since they are after all constructed precisely in order to capture the topological characteristics of a phase? There are at least three answers to this.

The first, which was already stressed in Sec. II.A, is that in many cases the construction of wave functions has preceded the effective field theories. This was true for both the Laughlin states, the Jain states, and the states in the Read-Rezayi series. In fact, for the latter there is to this day no generally accepted low-energy field theory. Thus, constructing explicit model wave functions has been an important step in understanding many topological phases, and this might well continue to be true in the future.

Second, one should be aware that the effective field theories can rarely be *derived* from the microscopic physics. In fact, even with rather low standards of rigor, this can be done only for the Laughlin (Lee and Zhang, 1991) and the Jain states (Lopez and Fradkin, 1992). Thus it is not at all guaranteed that

¹¹Already for relatively small systems, such as six particles in the (bosonic) $\nu = 2/3$ Jain state on the torus, only about 32% of the unprojected state resides within the LLL, and this percentage decreases rapidly with increasing system size. Similar numbers for the LLL weight have been reported for the spherical geometry, for both bosonic and fermionic Jain states (N. Regnault, private communication).

¹²As far as we know this concept was first introduced by Moore and Read (1991).

given an effective field theory, there will be *any* microscopic wave function with the same topological properties. In this case the construction of trial wave functions provides a sanity check of the predictions based on the effective field theory, provided one can determine the topological properties, by either analytical or numerical methods.¹³ The latter has often proven hard, especially when dealing with composite fermions. Here the CFT methods detailed in later sections have proven useful, as their direct connection to TQFTs allows us to conjecture their topological properties. In this sense, rewriting the composite-fermion wave functions using CFT is more than just an academic exercise, but allows us more direct access to the important properties of the representative wave functions.

Third, as discussed many times later, explicit wave functions for a small number of particles can be compared with wave functions obtained by numerically solving the Schrödinger equation for a realistic interaction. Often good agreements (typically as measured by overlaps of wave functions) are taken as a proof of the validity of the model wave function, but here we should add a note of caution. First, it is clear that any conceivable model wave function will have zero overlap with a numerically obtained one in the thermodynamic limit.¹⁴ Moreover, for a too small number of particles, the relevant Hilbert space, taking symmetries into account, might not be very large, so that any model wave function that is constructed as to keep the particles well separated, typically by including Jastrow factors, will have a good chance to have a large overlap. Thus it is to some extent a matter of judgment and experience to decide upon the success of a trial wave function by only studying overlaps for a small, but not too small, number of particles. The Laughlin state is often considered as something of a gold standard for overlaps. For example, for $N = 9$ electrons at $\nu = 1/3$ on a spherical geometry (see Sec. VII.A), the dimension of the symmetry-reduced Hilbert space is 84 and the squared overlap of the trial state with the exact Coulomb ground state is 0.988 (Fano, Ortolani, and Colombo, 1986). The Jain wave functions are similarly (if not more) impressive. For example, for $\nu = 2/5$ with $N = 10$ electrons on a sphere, the symmetry-reduced Hilbert space has a dimension of 52, and the squared overlap with the exact Coulomb ground state is 0.9956 (Tóke and Jain, 2009). A model wave function that gives overlaps this large for comparably large Hilbert space is usually considered quite good. However, overlaps must be used with caution. For example, the Gaffnian wave function mentioned in Sec. II.J (which, as discussed there, is gapless and therefore not a valid quantum Hall state) also has very high overlaps: for the same

$\nu = 2/5$ system the squared overlap is 0.954. Because of this difficulty in interpreting overlaps, other methods for determining the validity of wave functions have become important. A particularly useful one is the entanglement spectrum (Li and Haldane, 2008), which is discussed in Sec. II.G.4.

C. Bosonic Chern-Simons theory and the quantum Hall phase diagram

It took some time after the discovery of the Laughlin wave functions to unravel the physical nature of the quantum Hall liquids. In modern terminology, they are topologically ordered states (Wen, 2004). An important step was taken by Girvin and MacDonald (1987), who proposed an order parameter and showed that the Laughlin states are characterized by algebraic off-diagonal long-range order (ODLRO). In essence, by removing all phases from the Laughlin wave function using a singular gauge transformation (see Sec. II.D) the remaining wave function has algebraic long-ranged order. Girvin and MacDonald also suggested that this order pointed toward a field theory description in terms of composite objects consisting of charge and flux (Girvin and MacDonald, 1987; Girvin, 1990).

This idea was later exploited by Zhang, Hansson, and Kivelson (1989) who proposed the Lagrangian for the $\nu = 1/m$ Laughlin state

$$\mathcal{L} = \phi^*(i\partial_0 - a_0)\phi - \frac{1}{2m_e} |(\vec{p} - e\vec{A} + \vec{a})\phi|^2 - V(\rho) + \frac{1}{4\pi m} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma, \quad (8)$$

where m_e is the electron band mass. Here ϕ is a complex scalar order parameter field that is minimally coupled to the “statistical” gauge field a_μ , A_μ is the external electromagnetic field, and $\rho = \phi^*\phi$ is the particle density. It had already been shown that coupling a gauge field with a CS term in the action will effectively change the statistics of the original bosonic or fermionic matter field (Wilczek and Zee, 1983; Pisarski and Rao, 1985; Wilczek, 1990). The choice of coefficient in front of the Chern-Simons term in Eq. (8) (the final term) corresponds to a change from fermions to bosons when m is odd, i.e., the original underlying electrons are fermions, but the field ϕ is bosonic. When the statistical gauge field is treated in a mean-field approximation, where $\vec{a} = e\vec{A}$, this Ginzburg-Landau-Chern-Simons theory (GLCS)¹⁵ thus describes interacting charged bosons without any magnetic field.¹⁶ In this

¹³Note that some properties, such as the edge theory or the topological ground state degeneracy, cannot be probed using the sphere geometry, so constructing model wave functions on different geometries is an important aspect in numerics.

¹⁴An amusing exception to the rule that overlaps decrease with increasing system size is the composite-fermion trial wave function for the so-called single vortex state of bosons in the LLL with contact interaction. Its overlap with the exact ground state *increases* with the number of particles and approaches 1 in the limit $N \rightarrow \infty$ (Korslund and Viefers, 2006).

¹⁵This name has stuck, although it is an exact rewriting of the underlying fermionic theory and as such a proper microscopic theory with the extra condition of the bosons having hard cores. The usual GL theory for superconductors are more like the hydrodynamic theory discussed in Sec. II.F.

¹⁶There is a precursor to this picture where the physical magnetic field is “canceled” by a collective effect due to a condensation of bosons. Starting from a Wigner crystal, Kivelson *et al.* (1987) used semiclassical methods to study instabilities that occur at the Laughlin filling fraction, due to coherent tunneling events that can be viewed as a condensation.

effective description, the system is essentially a charged superfluid and captures many topological properties of the Laughlin states, including the algebraic ODLRO previously discussed. Later it was also shown that the Laughlin wave functions can be derived by including fluctuations around the mean-field solution in a random phase approximation (RPA) (Kane *et al.*, 1991; Zhang, 1992). The GLCS approach was further developed to include effects of orbital spin and the response to curvature (Abanov, 2013; Son, 2013); see also Sec. VII.B.

Applying the same ideas to the hierarchy states, Kivelson, Lee, and Zhang (1992) used the GLCS approach to generate the full Abelian quantum Hall hierarchy and to study the resulting phase diagram and associated phase transitions. An important tool in this analysis was the concept of “corresponding states” (Jain, Kivelson, and Trivedi, 1990), which relates transport properties of quantum Hall liquids at different filling fractions. Kivelson, Lee, and Zhang (1992) also proposed the existence of another phase of a two-dimensional electron gas in a magnetic field, which they named a “Hall insulator.” In this phase $\sigma_{xy} \sim \sigma_{xx}^2 \rightarrow 0$ as $T \rightarrow 0$, so ρ_{xy} remains constant at very low temperatures. Early experiments on heterojunctions giving evidence for this phase were made by Shahar *et al.* (1995), and recently it was also found in disordered indium-oxide films (Breznay *et al.*, 2016). Lütken and Ross (1992, 1993) analyzed the quantum Hall phase diagram using an effective field theory and certain, rather strong, symmetry assumptions; see also Fradkin and Kivelson (1996).

The bosonic Chern-Simons approach correctly obtained much of the topological data about each of the hierarchy states (including Hall conductivity, quasiparticle charge, and statistics) and set the stage for field theoretic evaluation of many additional quantities. While the construction of the Chern-Simons Lagrangian Eq. (8) is in principle exact, the evaluation of any quantity beyond mean-field or random phase approximation level is quite challenging, and the lack of any small parameter to control perturbation theory makes it essentially impossible to confirm the correctness of any such calculation without support from other approaches such as numerics. Particularly problematic is the issue of obtaining the proper energy scale. While the bare band mass of the electron m_e is present in the Lagrangian Eq. (8), all low-energy physics (which describes physics of a partially filled single Landau level) should be independent of the value of this parameter, particularly in the limit of large Landau level spacing. In principle, this result should emerge were calculations done exactly, but all approximations so far attempted have failed to achieve this.

D. Composite bosons

In a first quantized language, the wave function for the charge-flux bosons described by Eq. (8) is related to the wave function of the original fermions via¹⁷

¹⁷In the context of anyon superconductivity, attachment of phase factors was already used by Laughlin (1988) to describe fractional statistics particles in terms of fermion wave functions.

$$\Psi_F(\vec{r}_1, \dots, \vec{r}_N) = \prod_{i < j} \left(\frac{z_j - z_i}{|\vec{z}_j - \vec{z}_i|} \right)^m \Psi_B(\vec{r}_1, \dots, \vec{r}_N),$$

i.e., they differ by a phase factor which can be interpreted as attaching singular (infinitely thin) flux tubes each carrying m flux quanta at the position of the particles. An alternative order parameter was introduced by Read (1989),

$$\begin{aligned} \phi_R^\dagger(\vec{r}) &= \psi^\dagger(\vec{r}) U_m(\vec{r}) \\ &= \psi^\dagger(\vec{r}) \exp \left[-\frac{|z|^2}{4\ell^2} + m \int d^2 r' \ln(z' - z) \rho(\vec{r}') \right], \end{aligned} \quad (9)$$

who showed that it exhibits true long-range order. Here $\psi^\dagger(\vec{r})$ creates an electron, $\rho = \psi^\dagger \psi$ is the electron density operator, and U_m is a vortex of strength m .¹⁸ In a first quantized formalism, such a vortex, centered at the particle position \vec{r} , is $U_m(\vec{r}) = \prod_i (z_i - z)^m e^{-|z|^2/(4\ell^2)}$.

The Read operator ϕ_R is bosonic, but it is not related to the original fermion by a unitary transformation, and it is therefore more difficult to derive an effective GLCS theory (Read, 1989; Rajaraman and Sondhi, 1996). The difference, compared to the pure phase transformation leading to Eq. (8), is that the Read operator builds in the characteristic vanishing property $\lim_{z_i \rightarrow z_j} \Psi_L(z_1 \dots z_N) \sim (z_i - z_j)^m$ and exhibits true (i.e., not algebraic) ODLRO. The Laughlin state can be constructed in a way similar to a Bose condensate, either as a coherent state where the ϕ_R has a finite expectation value or as the state

$$|1/q; N\rangle_L = \left[\int d^2 r \phi_R^\dagger(\vec{r}) \right]^N |0\rangle$$

at a fixed number of particles, which directly yields the $\nu = 1/m$ Laughlin wave function for N particles.

E. Composite fermions

The idea of composite fermions (Heinonen, 1998; Jain, 2007) is most easily understood as the fermionic version of the transformation (9), where an even number of vortices is attached to the original fermions in order to form a composite fermion. The transformation between the fermion wave function Ψ_F and the CF wave functions Ψ_{CF} is thus

$$\Psi_F(\vec{r}_1, \dots, \vec{r}_N) = \Phi_0^{2p}(\vec{r}_1, \dots, \vec{r}_N) \Psi_{CF}(\vec{r}_1, \dots, \vec{r}_N), \quad (10)$$

where Φ_0 is the wave function for a filled LLL. This formula can be written in an arbitrary geometry and specializing to the plane we have $\Phi_0 = \prod_{i < j} (z_j - z_i) e^{-1/(4\tilde{\ell}^2) \sum_j |z_j|^2}$, where $\tilde{\ell}$ is not the physical magnetic length, but an effective one set by $\tilde{N}_\phi = N_e$ corresponding to a single filled Landau level (similar for Ψ_{CF}). This convention has the advantage that it gives the correct Gaussian factors (i.e., the correct magnetic

¹⁸Note that ψ^\dagger is the full electron creation operator. In Read’s original formulation, ϕ_R^\dagger is defined in terms of the electron operator projected on the LLL.

length) for the final wave function independent of the geometry or gauge choice. Note that the polynomial part of $(\Phi_0)^2$ is identical to that of $\prod_i U_2(z_i)$ which provides the link to the composite-boson transformation (9).

1. The Jain wave functions

The composite-fermion transformation (10) was used by Jain to establish a powerful correspondence between fractional and integer quantum Hall states. His basic insight was that Eq. (10) allows us to think of the $\nu = 1/(2p + 1)$ Laughlin state as a filled LLL of a composite fermion. One may then consider other homogeneous CF wave functions in order to form new model wave functions. In particular, filling q Landau levels of composite fermions—which we call Λ levels following Jain’s terminology—gives a wave function at $\nu = q/(2pq + 1)$ in the positive Jain series (Jain, 1989, 1990),

$$\Psi_{q/(2pq+1)}(\{\vec{r}_i\}) = \mathcal{P}_{\text{LLL}}[\Phi_0^{2p}(\{\vec{r}_i\})\Phi_q(\{\vec{r}_i\})], \quad (11)$$

where \mathcal{P}_{LLL} projects on the LLL. The filling fraction follows from a simple counting argument: Since the state is expected to be homogeneous (up to edge effects), the filling fraction is given by $\nu = N/L_{\text{max}} = N/(2pN + N/q) + O(1/N)$, where we used that each Landau level contains N/q particles and that all filled Landau levels have the same maximum angular momentum L up to boundary effects; cf. Appendix A. Similarly, the states in the negative Jain series $\nu = q/(2pq - 1)$ are obtained by attaching an even number of antivortices to the electrons (Jain, 2007).

Quasiparticle excitations amount to putting composite fermions in an unfilled Λ level (usually the lowest), or holes in any of the filled Λ levels, entirely analogous to how we would make the corresponding low-energy excitations for integer quantum Hall states. This simple mapping between fractional quantum Hall states and systems of free, or at most weakly interacting, fermions gives a fruitful intuitive picture that extends far beyond the previous example. Another example is paired states of composite fermions, which will be discussed later in the context of non-Abelian quantum Hall states. The ground state wave functions (11), as well as those for its quasiparticle excitations, have an excellent overlap with the states found by exact diagonalization of realistic Hamiltonians, thus providing a strong *a posteriori* justification for the CF ansatz. There is a large and successful body of work based on the composite-fermion wave functions which is reviewed by Jain (2007) and Heinonen (1998).

2. The Lopez-Fradkin and Halperin-Lee-Read effective field theories

Just as the GLCS theory (8) can be considered as a field theory for bosonic flux-charge composites, there is also a field theory for fermionic flux-charge composites that was proposed and analyzed by Lopez and Fradkin (1991). The only difference in the Lagrangian from Eq. (8) is that the integer in the coefficient in front of the CS term is now even, $m = 2p$, which means that the composite object ϕ is still a fermion. The mean-field solution does, however,

differ from the one in the GLCS theory in that the statistical mean field only partially cancels the external magnetic field. This leaves the composite objects moving in a weaker effective field. For special filling fractions, these composite objects completely fill a number of Landau levels and form a gapped state. The effective mean field seen by the composite fermions is

$$B^{\text{eff}} = B^{\text{ext}} - 2p\phi_0\rho,$$

where ρ is the density of electrons, ϕ_0 is the flux quantum, and B^{ext} is the externally applied field. Thus, the effective filling fraction is

$$\nu^{\text{eff}} = \frac{\rho\phi_0}{B^{\text{eff}}} = \frac{\rho\phi_0}{B^{\text{ext}} - 2p\phi_0\rho} = \frac{\nu}{1 - 2p\nu},$$

which is an integer q exactly when $\nu = q/(2pq + 1)$. At mean-field level, the fractional quantum Hall gap is given by the effective cyclotron energy $eB^{\text{eff}}/m_e = \phi_0\rho e/(qm_e)$. As with the Chern-Simons boson theory, the (exact) derivation of the Chern-Simons fermion Lagrangian results in the bare electron band mass m_e appearing in the kinetic term (and this sets the size of the mean-field gap). However, since the low-energy physics occurs within a single Landau level, m_e should not be an important parameter of the problem (and should vanish altogether in the limit of large Landau level spacing). As with the bosonic case, this problem occurs since the Chern-Simons theory can only be evaluated using rather crude methods such as mean-field or random phase approximation. Presumably a more precise (or exact) calculation would result in the dependence on m_e vanishing. As with the bosonic case, there is, however, serious difficulty in achieving any more precise calculation, since there is no natural small parameter in the theory.

The composite-fermion field theory approach was further developed by Halperin, Lee, and Read (1993) who considered the case of $\nu = 1/2$ (or any $1/m$ with m even) where the composite fermions move in zero effective field (i.e., $\nu^{\text{eff}} = \infty$) and thus can form a Fermi liquid. In the Fermi liquid context, a natural solution to the problem of energy scales results from viewing the strongly interacting Fermi liquid via the phenomenological Landau Fermi liquid theory where an energy functional is proposed to describe the deformation modes of the Fermi surface. In this framework the electron mass can be renormalized to an effective mass m^* set by the interaction strength of the electrons so as to correctly give a fractional quantum Hall gap (the composite-fermion cyclotron energy eB^{eff}/m^*) on the interaction scale (Halperin, Lee, and Read, 1993; Simon and Halperin, 1993). As is usual in Landau theory, the mass renormalization is linked to a certain Fermi surface deformation parameter so as to preserve Galilean invariance of the system (Nozieres and Pines, 1999).

A feature of the composite-fermion field theory, as well as the composite-fermion wave functions, is that they do not manifestly obey particle-hole conjugation. In the limit of infinite Landau level spacing, the lowest Landau level is

perfectly particle-hole symmetric,¹⁹ so, for example, the energy gap for $\nu = 1/3$ should be identical to the energy gap at $\nu = 2/3$. In the wave function approach this symmetry is very close to true, but not exactly true (Möller and Simon, 2005). In the field theory approach, since the Lagrangian (8) is in principle exact, the particle-hole symmetry should emerge if calculations are performed exactly. However, any practical calculation involves approximations, and no known approximation preserves the particle-hole symmetry [although RPA methods come very close (Simon and Halperin, 1993)], similar to the wave function approach.

The composite-fermion field theory gave rise to other field theory approaches to composite fermions which emphasized the dipolar nature (Read, 1994) of the composite object (Pasquier and Haldane, 1998; Read, 1998; Lee, 1999; Stern *et al.*, 1999; Murthy and Shankar, 2003). There is a recent surge in interest in constructing a particle-hole symmetric description of the composite-fermion liquid in terms of neutral Dirac particles (Barkeshli, Mulligan, and Fisher, 2015; Son, 2015; Wang and Senthil, 2015; Geraedts, Zaletel, Mong *et al.*, 2016; Murthy and Shankar, 2016). This approach has the notable advantage that it is manifestly particle-hole symmetric at all orders.

Note that just as in the GLCS theory, in the composite-fermion field theory approach, the Laughlin wave function does not appear at the mean-field level. This is because in both cases the composite object is formed by a charge and a pointlike statistical flux, so the extra “correlation hole” $\sim \prod_{j \neq i} (z_j - z_i)^{2p}$ around the electron at position z_i must arise from fluctuations around the mean field (Lopez and Fradkin, 1992).

Sometimes the wave functions (11) are also motivated in terms of flux attachment—the cartoon picture being that each electron grabs $2p$ flux quanta, leaving a weaker effective magnetic field. This picture is somewhat misleading since in the approaches by Read and Jain the charge binds to a *vortex* that includes the correlation hole and directly yields good wave functions with filling fractions following from a counting argument without any appeal to effective fields.

F. The Wen-Zee effective Chern-Simons theory

In the Chern-Simons approaches discussed previously one can imagine integrating out the particles of the theory to leave an effective Lagrangian for the longest length scales of the system. Since the quantum Hall states are gapped, one might naively expect that this leaves no physics at all remaining. This is, however, not true. First, as discussed in some detail in the next section, in realistic geometries, there are low-energy modes propagating along the edges of the sample. Second, even in closed geometries the topological data remain at arbitrarily long length scales and low energy.

Before presenting the general low-energy Lagrangian first proposed by Wen and Zee (1992) we make some general

comments about its structure and status. It is useful to think of this kind of effective theories in terms of hydrodynamics, implying that the strategy for deriving them is not to keep the original degrees of freedom below some cutoff scale, but rather to introduce new ones. In hydrodynamics, the conserved currents are natural variables, and in two dimensions they are conveniently parametrized as $j^\mu = \epsilon^{\mu\nu\sigma} \partial_\nu \tilde{a}_\sigma$. In this way, current conservation is built in, at the expense of introducing a gauge symmetry. The hydrodynamical gauge field \tilde{a} is not to be confused with the statistical gauge field a in the previous sections. In the case of the Laughlin states where we have a GLCS theory, or a composite-fermion field theory, we can derive the hydrodynamical theory by directly integrating over the microscopic field ϕ and the statistical gauge field a . For general QH states this procedure is not possible, and one must revert to the usual philosophy of effective low-energy theories and include in principle all terms allowed by symmetry. In practice one is usually keeping only the lowest derivative terms. In the QH context, these are Chern-Simons terms $\sim \epsilon^{\mu\nu\sigma} \tilde{a}_\mu \partial_\nu \tilde{a}_\sigma$, which are linear in derivatives and preserve rotation invariance but break parity and time-reversal symmetry. These terms are also Lorentz invariant, which is however a consequence of gauge symmetry. Higher derivative terms are, in general, not Lorentz invariant. What is special with the Wen-Zee effective action is that the coefficients in front of the various terms have a direct topological meaning and thus can take only discrete values.

In fact, all topological information about an Abelian quantum Hall state at the n th level of hierarchy can be coded in the following effective Lagrangian based on n Chern-Simons gauge fields (Wen and Zee, 1992; Wen, 1995):

$$\begin{aligned} \mathcal{L}_{\text{WZ}} = & -\frac{1}{4\pi} K_{\alpha\beta} \epsilon^{\mu\nu\sigma} a_\mu^\alpha \partial_\nu a_\sigma^\beta - \frac{e}{2\pi} t_\alpha A_\mu \epsilon^{\mu\nu\sigma} \partial_\nu a_\sigma^\alpha \\ & - \frac{s_\alpha}{2\pi} \omega_i \epsilon^{i\nu\sigma} \partial_\nu a_\sigma^\alpha + a_\mu^\alpha l_\alpha^i j_\mu^i, \end{aligned} \quad (12)$$

where we suppressed the tilde on the field a for ease of notation, where μ, ν , and σ denote space-time indices, and while the index i runs over spatial indices only. A_μ is an external electromagnetic field, and a_μ^α are the n internal Chern-Simons fields. The indices α, β run from $1, \dots, n$, where n is the rank of $K_{\alpha\beta}$ which is a symmetric matrix with odd integers along the diagonal and with all other entries even integers, and t_α is a vector of n integers defining the electric charge of the currents $\epsilon^{i\nu\sigma} \partial_\nu a_\sigma^\alpha$. The coupling of the n quasiparticle currents j_μ^i to the Chern-Simons fields is determined by the integer vector l_α^i , and taking $l_\alpha^i = \delta_{q\alpha}^i$ corresponds to the “elementary” currents j_μ^i that couple minimally to the gauge fields. In Eq. (12) s_α is a vector of spins associated with the different Chern-Simons fields and ω_i is a spin connection, such that the corresponding field strength $\mathcal{R} = \epsilon^{ij} \partial_i \omega_j$ is the Gaussian curvature of the surface on which the quantum Hall state lives.²⁰ We return to this in Sec. VII.A where we study quantum Hall states on a sphere. On a flat surface, however, ω can be taken to be zero and the spin does not enter.

¹⁹In the presence of particle-hole symmetric disorder one can also show that σ_{xy} should be exactly $e^2/(2h)$ when the Landau level is half filled (Kivelson *et al.*, 1997).

²⁰The Ricci curvature R is twice the Gaussian curvature $R = 2\mathcal{R}$.

The full topological information can be extracted from the Lagrangian (12). For example, using a vector notation, the filling fraction is given by (Wen and Zee, 1992)

$$\nu = \mathbf{t}^T \mathbf{K}^{-1} \mathbf{t} \quad (13)$$

and the ground state degeneracy on a manifold with genus g is given by

$$(\det \mathbf{K})^g. \quad (14)$$

Another important topological quantum number is the shift S defined by

$$N_e = \nu(N_\Phi + S), \quad (15)$$

where N_Φ denotes the total number of flux quanta (on the sphere) or the highest occupied angular momentum orbital (on the disk), respectively. Its value is determined by the topological data as

$$S = 2\bar{s} = \frac{2}{\nu} \mathbf{t}^T \mathbf{K}^{-1} \mathbf{s}, \quad (16)$$

where \bar{s} is known as the average orbital spin. The electric charge of the quasiparticle of the q th type is

$$Q_q = -e \mathbf{t}^T \mathbf{K}^{-1} \mathbf{l}_q, \quad (17)$$

and clockwise braiding of quasiparticle type q around one of type r gives a phase of

$$\theta_{q,r} = 2\pi \mathbf{l}_q^T \mathbf{K}^{-1} \mathbf{l}_r, \quad (18)$$

whereas clockwise exchange of two quasiparticles of type q gives

$$\theta_q = \pi \mathbf{l}_q^T \mathbf{K}^{-1} \mathbf{l}_q. \quad (19)$$

A given topological order does not uniquely define the necessary data $(\mathbf{K}, \mathbf{t}, \mathbf{s})$ in the Lagrangian (12). In fact, one is free to make the transformation (Read, 1990; Wen and Zee, 1992)

$$\begin{aligned} \mathbf{K} &\rightarrow \mathbf{W}^T \mathbf{K} \mathbf{W} & \mathbf{s} &\rightarrow \mathbf{W}^T \mathbf{s} \\ \mathbf{t} &\rightarrow \mathbf{W}^T \mathbf{t} & \mathbf{l}_q &\rightarrow \mathbf{W}^T \mathbf{l}_q, \end{aligned} \quad (20)$$

where \mathbf{W} is an integer matrix with unit determinant, without changing the values obtained from Eqs. (13), (14), and (16)–(19). There are two very natural bases to work in. The first, stemming from the Haldane-Halperin hierarchy and thus called hierarchical basis, has $t_1 = 1$ and $t_\alpha = 0$ for $\alpha > 1$. The second basis is the symmetric, or composite-fermion basis, which has $t_\alpha = t = 1$ for all α . The transformation that maps the symmetric to the hierarchical basis, i.e., $\mathbf{K}^h = \mathbf{W}^T \mathbf{K}^s \mathbf{W}$, is given by $W_{ij} = \delta_{i,j} - \delta_{i+1,j}$.

Wen argued that for a state at level n in the hierarchy, there must be n linearly independent combinations with charge e , or in other words n independent electron currents (Wen, 1995).

These are described by the n vectors \mathbf{l}_e^α . In the symmetric basis these take the simple form $l_{e,r}^\alpha = K_{r\alpha}$, where α labels the vector and r its component, resulting in $Q_e = -et = -e$ as expected. This will be important when we later use the topological data $(\mathbf{K}, \mathbf{t}, \mathbf{s})$ to construct explicit electronic wave functions. Finally we give the explicit form, in the symmetric basis, for the K matrix and the spin vector that describe the full hierarchy obtained by successive condensations of quasiparticles:

$$\begin{aligned} K_{\alpha\beta} &= \begin{cases} m + \Sigma_\alpha & \text{if } \alpha = \beta, \\ -\sigma_\alpha + m + \Sigma_\alpha & \text{if } \alpha < \beta, \end{cases} \\ \Sigma_\alpha &= \sum_{j=1}^{\alpha-1} (p_j - 2)\sigma_j, \\ s_\alpha &= \frac{1}{2} K_{\alpha\alpha} + \alpha - 1, \end{aligned} \quad (21)$$

with m odd and $K_{\alpha\beta} = K_{\beta\alpha}$, if $\alpha > \beta$. The even integers p_j specify the density of the quasiparticle condensates, and $\sigma_\alpha = +1$ (-1) for condensing quasielectrons (quasiholes) [cf. Eq. (6)].

G. Physics at the quantum Hall edge

On a closed surface (such as a sphere or torus, see Sec. VII), the Lagrangian (12) is completely self-consistent. However, for a system with edges, such as a disk, the action turns out to not even be gauge invariant, as discussed further in Sec. II.G.2. The reason for this failure is that quantum Hall systems always have nontrivial edge states, which must be included in the theory. We thus briefly turn our attention now to the physics of quantum Hall edge states, and, in particular, edge states of the hierarchy. There are a number of papers which give more detail about quantum Hall edge physics (Wen, 1992; Kane and Fisher, 1996; Chang, 2003). Here we give a very brief review just to indicate how these ideas fit into the overall framework of hierarchies.

1. Introduction to edge physics

The existence of edge currents in a quantum Hall system is virtually guaranteed by the presence of crossed magnetic field \mathbf{B} and electric field \mathbf{E} (the electric field being caused by the confining potential keeping the electrons inside the sample). Classically one expects a drift velocity $\mathbf{v} = \mathbf{E} \times \mathbf{B} / |\mathbf{B}|^2$. Thus a perturbation in density will propagate along the edge, in only one direction, at this velocity, thus giving a chiral hydrodynamics. Another semiclassical picture of these edge states is that of electrons executing so-called “skipping orbits” that bounce off the edge and travel in one direction. What is less obvious is how to quantize this sort of motion. An argument by Wen (1992) gives a proper quantum mechanical treatment for simple Laughlin states, which we now review.

Let us imagine a quantum Hall system filling the lower half plane, with an edge running in the x direction at position $y = 0$. Given that the quantum Hall system is incompressible, any perturbation of the edge must come from a physical displacement of the edge by some distance $h(x)$ in the y direction so that the excess density along the edge is

$\rho(x) = \bar{n}h(x)$, where $\bar{n} = \nu/(2\pi\ell^2)$ is the electron density of the bulk quantum Hall state. The energy of the displacement should be given simply in terms of the electrostatic potential $V(x, y) = Ey$ on the edge.²¹ We expect (Wen, 1992)

$$\begin{aligned} H &= \int dx \int_0^{h(x)} dy eV(x, y)\bar{n} = \frac{Ee\bar{n}}{2} \int dx h(x)^2 \\ &= \frac{\pi v}{\nu} \int dx \rho(x)^2 = \frac{2\pi v}{\nu} \sum_{k>0} \rho_k \rho_{-k} \end{aligned} \quad (22)$$

with the Fourier transform convention

$$\rho_k = \frac{1}{\sqrt{L}} \int dx e^{-ikx} \rho(x).$$

Now, given that any density perturbation should move along the edge at the drift velocity, we expect dynamical equations $(\partial_t - v\partial_x)\rho(x) = 0$ or equivalently

$$(\partial_t - ivk)\rho_k = 0. \quad (23)$$

From the Hamilton equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, it follows that the momentum conjugate to the coordinate $q = \rho_k$ for $k > 0$ must be taken as $p = 2\pi\rho_{-k}/(ik\nu)$ in order for the dynamical equation (23) to be consistent with the Hamiltonian (22). Canonical quantization is now straightforward and one obtains the so-called $U(1)$ Kac-Moody algebra,

$$[\rho_k, \rho_{k'}] = \frac{\nu k'}{2\pi} \delta_{k+k'}. \quad (24)$$

This well-known, but nevertheless quite subtle result is derived and commented upon in Appendix C.

Next we introduce a bose field $\varphi(x)$ related to the density by

$$\rho(x) = \frac{1}{2\pi} \partial_x \varphi(x), \quad (25)$$

where the field φ has equal-time commutation relations

$$[\varphi(x), \varphi(x')] = i\nu\pi \text{sign}(x - x'). \quad (26)$$

For the normalization convention, see Appendix B.2. Using these commutation relations together with Eq. (25) for $\rho(x)$ and Fourier transforming we recover the Kac-Moody commutators (24).

From Eqs. (25) and (26) it is easy to see that $\partial_x \varphi/(2\pi\nu)$ is canonically conjugate to $\varphi(x)$ (their commutator is proportional to a delta function). Using this information with the Hamiltonian previously given, Wen (1992) wrote the Lagrangian for the edge as

$$\mathcal{L}_{\text{edge}} = \frac{1}{4\pi\nu} (\partial_x \varphi)(\partial_t \varphi) - \frac{v}{4\pi\nu} (\partial_x \varphi)(\partial_x \varphi), \quad (27)$$

which is that of a one-dimensional chiral boson.

²¹With a sufficiently abrupt edge, this approach breaks down and very different edge physics can result (Fern and Simon, 2016).

One can write a creation operator for a bump of charge α at position x as

$$V_\alpha(x) = :e^{i\alpha\varphi(x)}:, \quad (28)$$

with colons denoting normal ordering. It is easy to check that this inserts charge $\alpha\nu$ by noting that

$$[\rho(x'), V_\alpha(x)] = \nu\alpha V_\alpha(x)\delta(x - x').$$

Crucially, we can create a charge e using the operator $\Psi_e = V_{1/\nu}$. Note, however, that this operator has proper fermionic commutations $\Psi_e(x)\Psi_e(x') = -\Psi_e(x')\Psi_e(x)$ only when ν is one over an odd integer, corresponding to half integer conformal spin (see Appendix B). This implies that only these simple Laughlin states can be described by a single edge mode.

2. Chern-Simons approach and Wen-Zee edge

Another way to understand the edge physics is via the Chern-Simons approach. In the case of a Laughlin state, the Wen-Zee effective Chern-Simons Lagrangian (12) is very simple:

$$\mathcal{L} = -\frac{1}{4\pi\nu} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma - \frac{e}{2\pi} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu A_\sigma, \quad (29)$$

where we have left out the coupling to external currents and possible curvature of the two-dimensional surface.

On a closed manifold, it follows by integration by parts that the corresponding action is gauge invariant. However, as mentioned, the action is not gauge invariant in the presence of a boundary. Let again our fluid occupy $y < 0$, and our edge run along the $y = 0$ axis. Now we cannot integrate by parts in the y direction without getting a boundary term. What happens is that the would-be gauge mode at the boundary becomes a physical degree of freedom, which is shown as follows: First notice that a_0 is not dynamical, but a Lagrange multiplier implementing the constraint $\epsilon^{ij}\partial_i a_j + \nu e \epsilon^{ij}\partial_i A_j = b + \nu e B = 0$, which is solved by $a_i = -\nu e A_i + \partial_i \varphi$. Substituting this back in the action (29) and picking the $a_0 = 0$ gauge, we get after partial integration and some rearrangements:

$$\begin{aligned} S &= \int dt d^2x \mathcal{L}_0 \\ &= \frac{1}{4\pi\nu} \int dt dx (\partial_x \varphi - \nu e A_x)(\partial_t \varphi - \nu e A_0) \\ &\quad + \frac{\nu e^2}{4\pi} \int dt d^2x \epsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma - \frac{e}{4\pi} \int dt dx \varphi E_x, \end{aligned} \quad (30)$$

where $\varphi(x, t) \equiv \varphi(x, 0, t)$ is a scalar field with support along the boundary $y = 0$ and $E_x = \partial_t A_x - \partial_x A_0$. Written in this form it is clear that the only dynamical degrees of freedom reside on the edge, and from the ‘‘symplectic’’ term $\sim \partial_x \varphi \partial_t \varphi$ we immediately get the canonical commutation relation (Faddeev and Jackiw, 1988)

$$[\varphi(x), \partial_x \varphi(x')] = 2\pi i \nu \delta(x - x'), \quad (31)$$

which again means that $\partial_x \varphi / (2\pi\nu)$ is canonically conjugate to $\varphi(x)$ as was inferred from Eq. (24). For the action (30) to be invariant under gauge transformations of this background field, $A_\mu \rightarrow A_\mu + \partial_\mu \xi / e$, the scalar field φ must transform as $\varphi \rightarrow \varphi + \nu \xi$ which identifies it as a phase field. Also note that the edge action on the first line is separately invariant under the gauge transformation, while the ‘‘anomaly’’ term $\sim E_x$ on the second line exactly cancels the gauge variation of the Chern-Simons term for the background field (Hansson and Viefers, 2000).

Neglecting the background field, the Lagrangian (30) matches Eq. (27) except that it has zero edge velocity. This is perhaps not surprising since the velocity is a nonuniversal parameter of the edge, depending on the details of the edge confinement. In order to precisely obtain the edge Lagrangian (27), one can choose a different gauge fixing $a_0 - \nu a_x = 0$, as discussed by Wen (1992), or alternatively one can simply add the gauge invariant potential term $-(v/4\pi\nu)(\partial_x \varphi - \nu e A_x)^2$ by hand.

Calculating the edge charge and current by varying the action (30) with respect to A_0 and A_x we get $\rho_{\text{edge}} = (e/2\pi)(\partial_x \varphi - \nu e A_x)$ and $j_{\text{edge}} = 0$. Using the equation of motion for φ we get $\dot{\rho}_{\text{edge}} = -(\nu e^2/2\pi)E_x$, while a direct variation of the bulk action with respect to A_y gives the outgoing bulk current perpendicular to the edge, and we find $J_y = \dot{\rho}_{\text{edge}}$. This ‘‘anomaly cancellation’’ which was originally pointed out in the quantum Hall context by Stone (1991), based on earlier work by Callan and Harvey (1985), is a general phenomenon. Charge nonconservation in a CS theory, which physically corresponds to a current perpendicular to the boundary, is exactly compensated by the charge not being conserved in the edge theory since the corresponding current is anomalous. Here we considered the charge current, but we later also discuss the energy-momentum current related to heat transport.

It is quite easy to generalize to arbitrary hierarchy states. One starts with the first term of the Lagrangian (12) and similarly enforces gauge invariance to obtain an edge Lagrangian generalizing Eq. (30) to hierarchy states,

$$\mathcal{L}_0 = \frac{1}{4\pi} K_{\alpha\beta} (\partial_x \varphi^\alpha) (\partial_t \varphi^\beta),$$

where we suppressed the external field A_μ , and where φ^α are a set of Bose fields with commutations

$$[\varphi^\alpha(x), \varphi^\beta(x')] = i\pi K_{\alpha\beta}^{-1} \text{sign}(x - x').$$

As in the case of Eq. (27) there is also a nonuniversal interaction term \mathcal{H} , which depends on details of the edge structure as well as the strength of the interaction between the different edge modes. We generally write the full Lagrangian as $\mathcal{L}_{\text{edge}} = \mathcal{L}_0 - \mathcal{H}$ with the Hamiltonian

$$\mathcal{H} = \frac{1}{4\pi} V_{\alpha\beta} (\partial_x \varphi^\alpha) (\partial_x \varphi^\beta),$$

where V is a positive definite matrix.

Given the coupling of the gauge field a in Eq. (12) to the external electromagnetic field A , it follows that the charge density associated with an edge excitation must be

$$\rho(x) = \frac{1}{2\pi} t_\alpha (\partial_x \varphi^\alpha),$$

and a quasiparticle described by a vector \mathbf{l} with entries l_α as in Eq. (17) can be created using the operator

$$\Psi_{\mathbf{l}}^\dagger(x) = :e^{i \sum_\alpha l_\alpha \varphi^\alpha(x)}:.$$

The emerging picture of the edge of a hierarchy state is one involving multiple chiral Bose modes, one for each level of the hierarchy. However, one should be aware that modes corresponding to any negative eigenvalue of the K matrix will be reverse moving. One should also remember that there may be processes that can remove an electron from one of the edge modes and put it back into a different edge mode. In composite-fermion language this corresponds to removing an electron from one Λ level edge and placing it in another.

3. The conformal edge

Whether considering a single edge mode or a system with multiple edge modes, one obtains a gapless (massless) one-dimensional system, which is necessarily described by a conformal field theory. There are, however, several added features. To begin with, multiple edge modes will generically have different velocities, whereas in a true CFT there should be only one velocity of light. More generally the edge can be a perturbed CFT where the perturbations may be irrelevant (vanishing at low energy), marginal, or relevant. In some cases, a relevant perturbation can drive the edge to another gapless fixed point (Haldane, 1995; Read, 2009a; Cano *et al.*, 2014). Nonetheless, even when we consider more complex quantum Hall states (such as non-Abelian states), the edges will always be described by (possibly perturbed) CFTs.

The connection to CFT turns out to be far more significant than it might appear so far. As seen in Sec. IV.C.2, a so-called bulk-edge correspondence allows us to describe the bulk of the quantum Hall state with the same CFT that we use for a minimal description of the edge.

4. Edge CFT and entanglement spectrum

For quantum Hall wave functions that are the exact zero-energy ground states of a model Hamiltonian, one can explicitly derive the edge excitations and show that the state counting (as a function of angular momentum) is the same as in the associated CFT (Read and Rezayi, 1996; Ardonne, Kedem, and Stone, 2005; Read, 2006). For other quantum Hall states, one has to resort to numerical simulations. While the study of quantum Hall edges potentially gives a direct way to study the spectrum of the $(1+1)$ -dimensional CFT associated with a quantum Hall state, it can also be complicated by the fact that these spectra depend on nonuniversal properties of the edges (such as the slope of the potential at the edge) and can also be further complicated by so-called ‘‘reconstruction’’ where edges become nonmonotonic in

density (MacDonald, 1990; Johnson and MacDonald, 1991; de C. Chamon and Wen, 1994).

Recently it was realized that one can learn about the *excitations* by looking at the entanglement properties of the representative *ground state* wave function. The entanglement entropy, independently introduced by Kitaev and Preskill (2006) and Levin and Wen (2006), is currently the most commonly used entanglement measure.²² It contains, in principle, all the information about the S matrix, and thus the quasiparticle excitations (Zhang *et al.*, 2012), and gives direct access to the total quantum dimension of the underlying field theory. The numerical evaluation is, however, very difficult for FQH systems (Haque, Zozulya, and Schoutens, 2007), and only recently was it possible to obtain reliable results (Zaletel, Mong, and Pollmann, 2013). A numerically more accessible quantity is the “entanglement spectrum” (Li and Haldane, 2008), which allows one to examine the CFT spectrum of a virtual edge and has quickly become an indispensable tool of the community.

We consider a system bipartitioned into two subsystems A and B (we can choose A and B to be the top and bottom halves of a sphere divided along the equator, although many other geometries are possible). The ground state wave function $|\Psi\rangle$ is contained in the tensor product of the Hilbert spaces of the A subsystem and the B subsystem. We make a Schmidt decomposition of the state as

$$|\Psi\rangle = \sum_n e^{-\xi_n/2} |\psi_A^n\rangle \otimes |\psi_B^n\rangle, \quad (32)$$

where $|\psi_A^n\rangle$ and $|\psi_B^n\rangle$ are states forming orthonormal sets completely contained in the A and B subsystems, respectively. The ξ_n are known as “entanglement energies,” but one should realize that they are properties of the ground state decomposition and have nothing to do with the physical energies of the quantum Hall state. The conjecture by Li and Haldane (2008) [see also Kitaev and Preskill (2006)] is that the spectrum of entanglement energies should match that of the $(1+1)$ -dimensional CFT defining the quantum Hall state.

In order to examine these spectra more closely, one typically divides the system into A and B so as to conserve several quantum numbers. For example, if we divide the sphere into an upper and lower part along a latitude, the division conserves the total L_z angular momentum, and, assuming a rotationally invariant interaction, it will conserve the number of particles in each part. In this case one can plot the entanglement energies as a function of these conserved quantities, for example, $\xi(N, L_z)$. One can choose to bipartition the system in several different ways: an orbital cut, where certain single-electron orbitals are placed in subsystem A and others in subsystem B (Li and Haldane, 2008); a real-space cut, where particles in one physical region are placed in subsystem A and others in subsystem B (Dubail, Read, and Rezayi, 2012b; Rodríguez, Simon, and Slingerland, 2012; Sterdyniak *et al.*, 2012); or a particle cut, where certain

particles are placed in subsystem A and others in subsystem B (Zozulya *et al.*, 2007; Sterdyniak, Regnault, and Bernevig, 2011).

For topologically ordered states, both the orbital and the real-space cut can be used to extract important information about the edge spectrum, viz. the type of CFT that describes the edge spectrum and the compactification radius of the charge field, which is directly related to the filling fraction (Hermanns *et al.*, 2011; Dubail, Read, and Rezayi, 2012b). Both cuts should also give the same topological entanglement entropy (Zozulya *et al.*, 2007; Dubail, Read, and Rezayi, 2012a; Zaletel, Mong, and Pollmann, 2013), even though the finite-size corrections in numerical simulations can be quite substantial for the currently accessible system sizes (Sterdyniak *et al.*, 2012). There are, however, differences which reflect that the orbital cut strictly speaking does not mimic a spatial edge, as can be easily seen in the case of the IQHE.²³ From practical point of view, the orbital cut is the favored tool since it is much easier to evaluate numerically.

The particle cut provides information about bulk excitations. More precisely, for trial wave functions that are zero-energy ground states of model Hamiltonians, the particle entanglement spectrum reproduces the quasihole state counting: the number of noninfinite eigenvalues equals the number of zero-energy states for the appropriate number of particles and flux. For other trial states, where no model Hamiltonian exists, the particle entanglement spectrum may instead be used to “define” the quasihole state counting. This “definition” is consistent both with the counting obtained using the composite-fermion picture, as long as one considers holes in *any* of the filled Λ levels, and with that obtained using CFT techniques, as long as *all* quasihole operators (see Sec. V) are taken into account.

Even though entanglement spectra of strongly correlated states are very complicated and rigorous proofs are rare, there have been some advancements recently. Several works have formalized the connection between the CFT spectrum and entanglement spectrum (Dubail, Read, and Rezayi, 2012a, 2012b; Qi, Katsura, and Ludwig, 2012; Swingle and Senthil, 2012) at various levels of rigor. In addition, one can derive a bulk-edge correspondence in the entanglement spectra, analogously to the well-established bulk-edge correspondence in the physical system (Chandran *et al.*, 2011).

The “simple” quantum Hall states, i.e., those that can be written as correlators of primary fields of a CFT, have correspondingly simple entanglement spectra. The spectra for hierarchy states are somewhat more complicated and have been studied to a much lesser extent. At least for $\nu = 2/5$ and $3/7$ the counting of low entanglement energy modes matches that of two or three chiral bosons, respectively (Regnault, Bernevig, and Haldane, 2009; Rodríguez *et al.*, 2013). It has also been suggested that these states can be described fairly simply in terms of composite fermions (Davenport *et al.*, 2015) although this has only been explicitly tested for the

²²For a mathematical perspective on topological entanglement entropy in Chern-Simons theories and quantum Hall states, see Dong *et al.* (2008) and Hikami (2008).

²³Being a single Slater determinant, this is trivially a short-range entangled state, and an orbital cut gives an entanglement spectrum with a single state. A real-space cut reveals that the edge supports a gapless fermion, i.e., a $c = 1$ CFT with unit charge radius.

fractional quantum Hall effect of bosons at $\nu = 2/3$ (which is presumably similar to fermions at $\nu = 2/5$).

H. Effective response action

Since the Wen-Zee action (12) is quadratic in the topological gauge fields a_μ^α , these can be integrated to give an effective action that describes the response to external fields. Neglecting the quasiparticle currents,²⁴ a (nontrivial) calculation and some rearrangement gives (Gromov *et al.*, 2015b)

$$S_{\text{eff}} = \frac{\nu}{4\pi} \int d^3x e^{\mu\nu\sigma} [(eA + \bar{s}\omega)_\mu \partial_\nu (eA + \bar{s}\omega)_\sigma + \beta \omega_\mu \partial_\nu \omega_\sigma] - \frac{c}{48\pi} \int d^3x e^{\mu\nu\sigma} \omega_\mu \partial_\nu \omega_\sigma. \quad (33)$$

Note that the first term is what we naively get from the Gaussian integration. The filling fraction ν and the average orbital spin \bar{s} are given by Eqs. (13) and (16) respectively, and the orbital spin variance β is $\beta = \nu(\bar{s}^2 - \bar{s}^2)$ with $\nu\bar{s}^2 = \mathbf{s}^T \mathbf{K}^{-1} \mathbf{s}$. For a $\nu = 1/q$ Laughlin state $\bar{s} = q/2$ and $\beta = 0$, since in a one-component state all electrons have the same orbital spin so the variance vanishes. For general Abelian hierarchy states, as the Jain states, $\beta \neq 0$.

In the last term, the chiral, or topological, central charge c equals the signature of the K matrix which is the difference between the number of positive and negative eigenvalues and determines the thermal Hall coefficient, κ_H , by (Kane and Fisher, 1997; Read and Green, 2000)

$$\kappa_H = c \frac{\pi k_B^2 T}{6}.$$

Although the first term in Eq. (33) goes back to the original work of Wen and Zee (1992), the history of the second term is more involved. It was introduced in the condensed matter context by Volovik (1990) in his work on liquid ³He films, and later Read and Green (2000) argued that it was needed to cancel the anomaly in the energy-momentum current conservation on the boundary, in analogy with the anomaly cancellation in the charge current described in Sec. II.G.2. Later it was found by an explicit calculation using free fermions (i.e., the IQHE) by Abanov and Gromov (2014), and subsequently by Gromov *et al.* (2015a, 2015b) by carefully performing the integral over topological gauge fields in Eq. (12). The subtlety is that although the action is independent of the metric, performing the integral requires a gauge fixing which will depend on the metric. As shown by Witten (1989), this generates an extra term, which essentially is the second one in Eq. (33). Bar-Natan and Witten (1991) referred to this as the “framing anomaly.” In a parallel development, Bradlyn and Read (2015a) determined the general form of the effective low-energy action obtained by integrating out the matter fields. Later they calculated the second term in Eq. (33) directly from particular CFT wave functions (Bradlyn and Read, 2015b). They also argued, based

on anomaly cancellation [for this see also Stone (2012)], that the orbital spin variance β should vanish in a general one-component (non-Abelian) quantum Hall state. Related work on the gravitational response of Hall fluids was done by Hoyos and Son (2012), Son (2013), Can, Laskin, and Wiegmann (2014, 2015), and Cho, You, and Fradkin (2014).

I. Non-Abelian quantum Hall states and non-Abelian hierarchies

The defining features of a non-Abelian quantum Hall state are that the quasiparticles are described by multicomponent wave functions and obey non-Abelian fractional statistics due to the appearance of non-Abelian Berry phases (Wilczek and Zee, 1984; Moore and Read, 1991) during braiding. The first concrete proposal was the Moore-Read Pfaffian state discussed in detail in Sec. IV.C.5. Several more filling fractions, all of them in the second Landau level (see Fig. 2), have been conjectured to be non-Abelian. It is natural to wonder whether there is a single governing principle that can explain all the observed states as well as their properties and relative stability, similar to the Haldane-Halperin hierarchy in the LLL. Another question is whether it is possible to form hierarchy states by condensing non-Abelian quasiparticles. Several proposals in these directions are discussed in Sec. VI.

J. Other approaches

There has been no shortage of attempts to construct new representative wave functions for the quantum Hall effect. Ideally we want these to be based on some physical principles, and we want them to be gapped ground states of some (even slightly realistic) Hamiltonians. This combination of demands makes successful wave function building extremely difficult. Establishing that a ground state is gapped is particularly difficult and is only ever really confirmed by extensive numerics, requiring large system size extrapolation, which is always open to at least some degree of doubt. Even the conventional hierarchies previously discussed are not beyond question. For example, a detailed analysis of the conventional hierarchy was given by Quinn *et al.* (2009). Further it was proposed that certain experimentally observed fractional quantum Hall states may not be of the hierarchy or composite-fermion form (Mukherjee *et al.*, 2014a, 2014b), although it is not at all clear if the proposed competing states would be viable gapped states in the thermodynamic limit.

One straightforward way to build new wave functions is to perform one of several well-established transformations on a known wave function. One such transformation is to build the same wave function in a higher Landau level, which should be a simple alteration; indeed, given a Hamiltonian with a corresponding ground state in one Landau level, it is easy to produce another Hamiltonian that will give precisely the same ground state within another Landau level (Haldane, 1990; Simon, Rezayi, and Cooper, 2007b). Another transformation is particle-hole conjugation, which can potentially produce new topological orders (Jolicoeur, 2007) even for a half filled Landau level (Lee *et al.*, 2007; Levin, Halperin, and Rosenow, 2007). A third transformation is flux attachment, which has

²⁴These can be included, but will give a nonlocal action, which is needed in order to encode the statistical phases related to the braiding of the quasiparticles.

seen great success in the composite-fermion (Jain, 2007) and composite-boson approaches. Finally, one can consider simply multiplying together two known wave functions, which can be interpreted as a parton construction (Wen, 1999); see also Sec. IV.D.2. Combinations of these techniques have been used to generate a number of novel filling fractions outside of the conventional hierarchy such as $\nu = 3/8$ (Töke, Shi, and Jain, 2008; Mukherjee *et al.*, 2012; Hutasoit *et al.*, 2016) and an alternative candidate for $4/11$ that is distinct from the hierarchical one (Mukherjee *et al.*, 2014a).

There have also been approaches proposed that precisely reproduce known conventional hierarchies from different physical intuition. In two interesting papers using a formalism entirely in the LLL, Haxton and co-workers (Ginocchio and Haxton, 1996; Haxton and Haxton, 2015) provided a quite different way to understand why the hierarchy and composite-fermion states are energetically favorable. Another work by Jain and Kamilla (1997a) physically motivated the fully chiral subset of the CF hierarchy without appeal to higher Landau levels. This is also the case for the CFT approach (Bergholtz *et al.*, 2008; Hansson, Hermanns, and Viefers, 2009; Suorsa, Viefers, and Hansson, 2011a, 2011b), which is discussed in detail in Sec. V.A.

Aside from the previously discussed methods, there have been attempts to unify fractional quantum Hall states, and to construct new states, by focusing on clustering patterns. There are several closely related ways of approaching this which we will review. Generalizing the clustering patterns of the Laughlin and Moore-Read states leads to the Read-Rezayi series of quantum Hall states (Read and Rezayi, 1999), which can also be defined as correlators of primary fields of CFTs. In addition to the Read-Rezayi series, several other wave functions can be described in this clustering language including the Haffnian wave function (Green, 2001) describing electrons at $\nu = 1/3$ and the Gaffnian wave function (Yoshioka, MacDonald, and Girvin, 1988; Simon *et al.*, 2007) describing electrons at $\nu = 2/5$. Unfortunately, despite having good overlaps with exact diagonalizations for small system sizes, the Gaffnian and Haffnian wave functions are defective in certain ways.

The Haffnian turns out to be a correlator of a nonrational conformal field theory which invalidates it as a candidate for a sensible quantum Hall state as discussed in detail by Read (2009b). Some of its resulting defects are that it has an infinite number of particle types and correspondingly an infinite ground state degeneracy on the torus in the thermodynamic limit, both of which strongly suggest that it is gapless in this limit (Read, 2009b). Indeed, arguments have been made that this wave function represents a particular critical point (Green, 2001).

Similarly, the Gaffnian is a correlator of a nonunitary conformal field theory (Simon *et al.*, 2007), and Read (2009a, 2009b) has given a series of arguments demonstrating that no nonunitary CFT can represent a gapped phase of matter. Recently, numerical studies on the Gaffnian verified that its excitation spectrum is gapless (Jolicoeur, Mizusaki, and Lecheminant, 2014; Papić, 2014; Weerasinghe and Seidel, 2014) and that charge is indeed not screened (Bernevig, Bonderson, and Regnault, 2012; Wu *et al.*, 2014; Estienne, Regnault, and Bernevig, 2015). This is an unfortunate feature of

the clustering approaches—focusing on the single-component case for simplicity, none of these clustering approaches have convincingly yielded any new gapped phases of matter, beyond the Read-Rezayi series.

One of the clustering approaches to building new wave functions is to identify polynomials (wave functions) with certain clustering or vanishing properties. For example, the Laughlin $\nu = 1/3$ wave function is uniquely identified by being the lowest degree antisymmetric polynomial in z_1, \dots, z_N which vanishes as three or more powers when any two coordinates z_i and z_j approach the same point. This direction of thought has resulted in the detailed study of Jack polynomials (Greiter, 1993; Bernevig and Haldane, 2008a), which are a family of special polynomials having well-defined vanishing properties, where each polynomial can be identified with a particular CFT (Feigin *et al.*, 2003; Bernevig, Gurarie, and Simon, 2009; Estienne and Santachiara, 2009). While this Jack approach successfully describes the Read-Rezayi series and is very powerful for both numerical and analytic computation (Bernevig and Haldane, 2008b; Bernevig and Regnault, 2009), all of the other CFTs it describes (including the Gaffnian, for example) are nonunitary and therefore cannot describe gapped phases of matter (Read, 2009a, 2009b).

A closely related approach is the so-called “pattern of zeros” (Wen and Wang, 2008). In this case, one tries to fully describe polynomials by the way in which zeros emerge as various numbers of particles approach each other. Again this works well for the Read-Rezayi series and can also define wave functions such as the Gaffnian and Haffnian (which as discussed are not acceptable quantum Hall states). However, the approach runs into trouble in many other cases since the pattern of zeros is not generically sufficient to uniquely define a polynomial (Simon, Rezayi, and Cooper, 2007b; Simon, Rezayi, and Regnault, 2010; Jackson, Read, and Simon, 2013) and must be supplemented by further information (Lu *et al.*, 2010) to fully define a wave function, which ends up being almost the same as simply defining a full CFT.

Another closely related approach is to examine quantum Hall states in the limit of a thin torus or cylinder geometry (Bergholtz and Karlhede, 2005; Bergholtz *et al.*, 2006; Seidel and Lee, 2006). In this “TT limit”²⁵ the problem of finding the ground state and low-energy excitations of strongly interacting quantum particles is reduced to finding the lowest energy configurations of classical, static charges; see, e.g., Bergholtz and Karlhede (2008) for a detailed discussion on this. In the TT limit, the quantum Hall ground states become simple charge density waves,²⁶ where the pattern of occupied and unoccupied orbitals maps to the pattern of zeros or vanishing properties in the clustering approaches. Indeed, each Jack polynomial can be directly mapped to a TT pattern

²⁵Here TT can stand for either thin torus or Tao and Thouless, who first considered this limit (Tao and Thouless, 1983).

²⁶Note that there is a close relation to the charge density waves occurring in one-dimensional coupled electron-phonon systems as studied by Su, Schrieffer, and Heeger (1979) and later works (Goldstone and Wilczek, 1981; Jackiw and Schrieffer, 1981; Su and Schrieffer, 1981). These systems also harbor fractionally charged excitations on the domain walls between different ground states.

of occupancies, known as its “root state.” Furthermore the domain walls in these charge density wave states map precisely to the operators of the corresponding CFT (Ardonne, 2009) and give a simple way to understand many of the complicated quasiparticle properties of these quantum Hall wave functions. Nonetheless, with the exception of the Abelian hierarchy states and the Read-Rezayi series, the CFTs are all nonunitary.

There is an interesting alternative way of looking at the TT limit that is also explained by Bergholtz and Karlhede (2008). Rather than thinking of making a torus very thin, one can equivalently keep the geometry fixed, but change the interaction to become very anisotropic. Viewed in this way, the TT limit amounts to a renormalization group (RG) flow, toward (an admittedly unphysical) fixed point Hamiltonian. Conversely, moving from the TT limit to a realistic Hamiltonian, the topological properties, that are manifest in the TT limit, will persist as long as the excitation gap does not close.

Yet another clustering approach is to define model Hamiltonians that enforce a particular clustering behavior (Simon, Rezayi, and Cooper, 2007a, 2007b). Quite similarly to the approaches discussed previously one can design a Hamiltonian to forbid clusters of p particles having relative angular momentum less than some number m . Again this works well for the Read-Rezayi series but beyond this appears to generate gapless states.

Given that there are many CFTs that are unitary (in fact an infinite number), it seems that one should be able to define a Hamiltonian whose ground state is the correlator of some *rational, unitary* CFT that would evade the arguments given by Read (2009a). Indeed, in several cases a Hamiltonian has been constructed so that its ground state is uniquely given by the correlator of a unitary CFT which is different from those defining the Read-Rezayi series (Simon, Rezayi, and Regnault, 2010; Jackson, Read, and Simon, 2013). Unfortunately, unitarity of a CFT still does not guarantee that the system is gapped. Indeed, in no case considered so far, except the Read-Rezayi series, has anyone written a single-component quantum Hall wave function which is a simple correlator of primary fields of a CFT and which appears to be gapped. Nonetheless, there have been many varieties of hierarchies attempted from the field theoretical or CFT viewpoints (Fröhlich and Zee, 1991; Fröhlich, Studer, and Thiran, 1994; Cappelli and Zemba, 1997; Fröhlich *et al.*, 2001; Flohr and Osterloh, 2003; Estienne, Regnault, and Santachiara, 2010). It remains to be seen if any of these will actually result in gapped ground states of some (even slightly realistic) Hamiltonian.

III. RELATION TO EXPERIMENT

While much of the theoretical study of hierarchies (and fractional quantum Hall effect in general) focuses on the construction of trial wave functions, at the end of the day one is inevitably interested in explaining some physical experiment (whether it be real or hypothetical). As mentioned at the end of Sec. II.B there are special short-range interactions for which the Laughlin states are exact ground states (Haldane, 1983; Pokrovsky and Talapov, 1985; Trugman and Kivelson, 1985), and similarly for the Moore-Read state (Greiter, Wen,

and Wilczek, 1991; Moore and Read, 1991) as well as the Read-Rezayi series (Read and Rezayi, 1999). Yet there is no known special interaction which yields the ground state for any other spin polarized gapped quantum Hall state (including all of the hierarchy or Jain states except the Laughlin states). While such short-range interactions are perhaps natural in a cold-atom experimental context (Cooper, 2008; Viefers, 2008; Cooper and Dalibard, 2013; Yao *et al.*, 2013) or lattice-based quantum Hall realizations,²⁷ such interesting systems remain proposals and have not yet been realized experimentally.²⁸ For real physical systems where the fractional quantum Hall effect has been realized, interactions are much more complicated, and it is usually impossible to make any exact statements.

A. Connection of theory to real experiments

If we focus on real experiments, then the only systems ever to show the FQHE are two-dimensional electron gases in semiconductors. While the vast majority of fractional quantum Hall experiments have been performed in some variety of GaAlAs heterostructures, a few other semiconductor systems have also allowed observation of FQH physics, including Si MOSFETS (metal-oxide semiconductor field-effect transistors) (Furneaux *et al.*, 1986), SiGe (Monroe *et al.*, 1992; Lai *et al.*, 2004; Lu *et al.*, 2012), ZnMnO (Tsukazaki *et al.*, 2010), and particularly graphene (Bolotin *et al.*, 2009; Du *et al.*, 2009; Dean *et al.*, 2011).

In any given case of experimental interest, the Hamiltonian should reflect the particular electron-electron interaction relevant to the physical system in question. While one might naively guess that the physical interaction between electrons is simply the Coulomb interaction, the situation is actually quite a bit more complicated. Wave functions of electrons in two-dimensional electron systems typically have a substantial width in the direction transverse to the two-dimensional plane. As a result, one typically models the Coulomb interaction as being softened at short distances depending on the shape of the transverse wave function (Stern and Howard, 1967; Girvin, MacDonald, and Platzman, 1986; Zhang and Das Sarma, 1986). There may also be some amount of screening due to nearby metallic layers, such as electrostatic gates, which must be taken into account. When all of these things are done, one obtains an electron-electron interaction within a single Landau level which can be studied to try to predict what FQH states will be seen in a given experiment and what their properties will be.

As mentioned in Appendix A, independent of which Landau level is being studied, it is always possible to map the problem to an effective interaction within the LLL (Haldane, 1990), which is often done for simplicity. We emphasize, however, that the effective interaction resulting

²⁷Fractional quantum Hall effects of particles on a lattice, also known as “fractional Chern insulators,” are a large subfield. We do not discuss any of these interesting proposals, but we refer the interested reader to Bergholtz and Liu, 2013.

²⁸An extremely interesting preprint (Gemelke, Sarajlic, and Chu, 2010) suggests that one can get small numbers of cold atoms into the Laughlin regime.

from the physical problem of a partially filled n th Landau level is extremely dependent on the value of n . For typical GaAs samples, in the LLL ($\nu < 2$ since there are two spin species), the effective interaction is fairly strong at short range, which tends to favor the realization of the Jain series of quantum Hall states (Pan *et al.*, 2008); see Fig. 1. On the other hand, in high Landau levels, the effective interaction is longer ranged, as well as oscillatory. This type of interaction favors charge density wave ground states, such as stripes and bubbles (Fogler, Koulakov, and Shklovskii, 1996; Moessner and Chalker, 1996). This is in agreement with experiment: no quantum Hall state has been observed²⁹ for Landau levels greater than the first excited level (i.e., $\nu > 4$), whereas anisotropic compressible states [presumed to be striped or nematic (Fradkin and Kivelson, 1999)] are observed in ultraclean samples for filling fractions between $9/2$ and $15/2$ (Du *et al.*, 1999; Lilly *et al.*, 1999). The intermediate case of the partially filled second Landau level ($2 < \nu < 4$ for typical GaAs samples) is potentially the most interesting having both fractional quantum Hall physics and some charge density wave physics (Eisenstein *et al.*, 2002). It is in this regime that the potentially non-Abelian quantum Hall states are experimentally observed at $\nu = 5/2$ and $12/5$ (Willett *et al.*, 1987; Pan *et al.*, 1999, 2008; Kumar *et al.*, 2010); see also Fig. 2. The reason that this regime is special stems from the form of the intermediate range of the Coulomb interaction projected to the second Landau level, which promotes clustering of electrons, which in turn appears to favor non-Abelian quantum Hall states (see also the discussion at the beginning of Sec. VI).

Theoretically once an interaction is determined for a physical system of interest there are several common methods of analysis. The simplest is to choose several trial wave functions and compare their interaction energies (i.e., their ground state energies within the Landau level) using Metropolis Monte Carlo integration (Chakraborty and Pietiläinen, 1995). Although this method is quite crude (since one is simply guessing the ground state wave function) it has the advantage that such Monte Carlo calculations can be performed for extremely large systems, with numbers of electrons possibly over 100.

A second method of analysis (and, indeed, the gold standard of fractional quantum Hall theory for a generation) is to perform exact diagonalization of a system for a small number of interacting electrons [typically on a closed surface such as a sphere (Haldane and Rezayi, 1985a) or torus (Haldane and Rezayi, 1985b) to avoid edge effects]. Once one obtains the ground state, one usually attempts to identify which phase of matter it belongs to by examining its properties, such as its shift [see Eq. (15)] or its excitation spectrum. More recently the entanglement spectrum (see Sec. II.G.4) was used as a fingerprint for phases of matter.

A hybrid approach, known as composite-fermion diagonalization, was pursued by Jain (2007). In this work one

chooses a small basis of trial wave functions corresponding to states that would be low energy in composite-fermion language. Using the Monte Carlo method, one can calculate matrix elements in this basis and then diagonalize the resulting matrix. This approach has the advantage of allowing the study of larger systems than exact diagonalization, but presumably introduces a bias toward composite-fermion states.

Recently another set of extremely powerful tools have been added to the toolbox of numerical techniques. Using CFT ideas, it is now possible to express certain quantum Hall states, such as the Laughlin, Moore-Read, and Read-Rezayi states, in matrix product representation (Zaletel and Mong, 2012; Estienne *et al.*, 2013; Estienne, Regnault, and Bernevig, 2013). This highly efficient encoding of the wave function enables numerical calculations on very large systems (infinitely long cylindrical geometry with circumference of 30 magnetic lengths) for these particular wave functions obtaining quantities such as quasiparticle braiding statistics, correlation length, pair-correlation function, and so forth. Closely related to this technique is the density matrix renormalization group (DMRG) method (White, 1992), which is effectively a variational matrix product approach and can be used as a replacement for exact diagonalization but for much larger systems, similar to that for matrix product calculations (Bergholtz and Karlhede, 2003; Shibata and Nomura, 2009; Kovrizhin, 2010; Zaletel, Mong, and Pollmann, 2013; Geraedts, Zaletel, Papić, and Mong, 2015; Mong *et al.*, 2015; Zaletel *et al.*, 2015).

For simplicity one typically views the fractional quantum Hall effect as existing within a single partially filled Landau level. However, in real semiconductor systems, the Landau level mixing parameter, the ratio of the interaction energy $E_{\text{Coulomb}} = e^2/\epsilon\ell$ to the cyclotron energy $\hbar\omega_c = eB/m^*c$ is often of order unity, meaning that virtual transitions between Landau levels can potentially be important. While often it is assumed that such transitions have only a small effect on the physics, in cases where two possible states of matter are very close in energy, the small effects of Landau level mixing can be crucial in determining which one is the ground state. One case where Landau level mixing terms are crucial is in determining³⁰ whether the quantized Hall state experimentally observed at $\nu = 5/2$ is the Moore-Read Pfaffian (Moore and Read, 1991), or its particle-hole conjugate, known as the anti-Pfaffian (Lee *et al.*, 2007; Levin, Halperin, and Rosenow, 2007)—the two states being energetically equivalent for any two-body interaction without Landau level mixing. There are two controlled methods of handling Landau level mixing theoretically, the first being to integrate out inter-Landau level transitions (Rezayi and Haldane, 1990; Peterson and Nayak, 2013; Simon and Rezayi, 2013; Sodemann and MacDonald, 2013; Wooten, Macek, and Quinn, 2013) at leading order in the Landau level mixing parameter ($E_{\text{Coulomb}}/\hbar\omega_c$) to obtain a modified interaction

²⁹Possible exceptions are filling fractions $4 + 1/5$ and $4 + 4/5$ which look like FQHE states at intermediate temperature (Gervais *et al.*, 2004).

³⁰Landau level mixing also appears quite important in determining the nature of the experimentally observed $\nu = 12/5$ fractional quantum Hall state and the apparent charge density wave at $\nu = 13/5$ (Mong *et al.*, 2015; Pakrouski *et al.*, 2016).

within a single Landau level.³¹ A second approach is to include multiple Landau levels within an exact diagonalization or DMRG calculation and attempt an extrapolation to a system that includes all the Landau levels, as well as the usual extrapolation to large system size (Rezayi and Simon, 2011; Mong *et al.*, 2015; Zaletel *et al.*, 2015). In principle the two approaches should agree with each other at least for small values of $E_{\text{Coulomb}}/\hbar\omega_c$. However, there are cases where they do not agree (see, in particular, the discussion regarding $\nu = 5/2$ next), and the cause of the conflict remains undetermined.³²

B. Status of real experiments

Perhaps the most surprising thing about the fractional quantum Hall effect is that it is so definitively observed in experiment. [Indeed, we should not forget that it was discovered in experiment (Tsui, Stormer, and Gossard, 1982) before it was explained theoretically.³³] In this section we discuss what is known from some key experiments. It is obviously impossible for us to discuss all of the experiments that have been performed on FQH systems so we instead focused on those that are most interesting in the context of this review.

1. The nature of plateaus

The observation of any fractional quantum Hall plateau always raises the question of what the properties of that plateau are, that is, what phase of matter are we observing? Focusing on high mobility GaAs heterostructures within the LLL ($\nu < 2$) over 60 FQH plateaus have been observed (Pan *et al.*, 2003, 2008). Of these, all but a very few are of the Jain form $\nu = q/(2pq \pm 1)$ or are closely related (particle-hole conjugate, for example) and are assumed to be topologically equivalent to simple Jain or Halperin-Haldane hierarchy states. A few fractions observed within the LLL do not fit this Jain form, including 4/11, 7/11, 4/13, 5/13, 5/17, 6/17, and 4/19. None of these are fully formed fractional quantum Hall plateaus, but the evidence for 4/11 (and possibly 5/13), in particular, appears fairly good (with higher denominators getting successively weaker as suggested by Fig. 3). These anomalous states could potentially all be explained within the hierarchy construction. Nonetheless there have been a number of other proposals for the nature of these states which are discussed in Mukherjee *et al.* (2014a). The most recent study (Mukherjee *et al.*, 2014a, 2014b) has been interpreted as suggesting quantum Hall states arising from an exotic mechanism proposed by Wójs, Yi, and Quinn

(2004),³⁴ although this conclusion is certainly not generally agreed upon.

In the second Landau level ($2 < \nu < 4$), where roughly a dozen FQH plateaus have been observed, the situation is potentially even more interesting. There have been many proposals that many of these states are nonconventional in one way or another.

5/2: The most prominent of the states in this range is the 5/2 state (and its sister the 7/2 state). Since the seminal work of Morf (1998), it has been increasingly clear (and now fairly solidly established numerically) that this state is either in the Moore-Read Pfaffian phase (Moore and Read, 1991) or in its particle-hole conjugate phase, known as the anti-Pfaffian (Lee *et al.*, 2007; Levin, Halperin, and Rosenow, 2007). As mentioned, in the absence of Landau level mixing, the two possibilities are equal in energy. Deciding which is realized has been difficult. While Rezayi and Simon (2011) and Zaletel *et al.* (2015) found the anti-Pfaffian is favored, the work of Pakrouski *et al.* (2015) found that the Pfaffian is favored. It is not yet clear why these do not agree (see, however, footnote 32). The only experiment that attempts to directly distinguish the two possibilities (Radu *et al.*, 2008) was found to be more consistent with the anti-Pfaffian.³⁵ The experimental observation by Dolev *et al.* (2011) of upstream neutral edge currents (flowing opposite the direction of charge transport) seems more natural for the anti-Pfaffian, but could also occur for the Pfaffian if there were edge reconstruction. Finally we mention that recent interferometry experiments [see Willett *et al.* (2013), for example] have been interpreted in a simple picture in terms of the anti-Pfaffian, but not the Pfaffian (von Keyserlingk, Simon, and Rosenow, 2015). This conclusion should not be taken as definitive given the controversy over interpretation of these experiments (see Sec. III.B.3).

7/3, 8/3: The overlap of the Coulomb ground state with the conventional Laughlin wave function is found to be quite low for 7/3 and 8/3 (Balram *et al.*, 2013). It was later suggested by Peterson *et al.* (2015) that (when Landau level mixing is included) the 8/3 state might be the Z_4 Read-Rezayi phase,

³⁴This work stems from the proposal that a different kind of quantum Hall state for $\nu = 1/3$ can be produced by having an effective hollow-core-like interaction, generating a rotationally invariant ground state on the sphere for $\nu = 1/3$ at a shift $S = 7$ instead of the usual Laughlin value $S = 3$. Using the corresponding $\nu = 4/3$ wave function as Ψ_{CF} in Eq. (10) (with $p = 1$) then results in a trial wave function for $\nu = 4/11$ and similarly using 5/3 gives a trial state for $\nu = 5/13$. A toy model of this type of wave function can be built at $\nu = 1/5$ by using an interaction that forbids pairs with relative angular momentum 3 but allows pairs with relative angular momentum 1 (Jolicœur, 2017). This model interaction generates a rotationally invariant ground state at shift $S = 9$ rather than the Laughlin value of $S = 5$. Such a ground state is unfortunately gapless suggesting that all such states in this family might be also. Examining the data in Mukherjee *et al.* (2014a) it looks likely that the 5/13 state considered there is indeed gapless, but it is less clear for 4/11.

³⁵Repeats of this experiment (Lin *et al.*, 2012; Baer *et al.*, 2014) found that the 331 state was favored, which is hard to understand, being completely contradictory to a large body of numerical work, and potentially casts doubt on the interpretation of the experiment.

³¹Beyond leading order one obtains retarded terms, which make it impossible to represent the resulting interaction terms as simply an equal-time interaction within a single Landau level.

³²This conflict appears to have recently been resolved by Rezayi (2017), with the conclusion being that the anti-Pfaffian is favored at low values of $E_{\text{Coulomb}}/\hbar\omega_c$.

³³It is interesting to debate whether anyone would have believed quantum Hall effect (integer much less fractional) theory if it had been provided before the experimental observation.

and this has been supported using exact diagonalization of up to 12 electrons. However, more recent DMRG calculations for much larger systems, including Landau level mixing terms, conclude that the $8/3$ state is of Laughlin type (Zaletel *et al.*, 2015). The experiment of Baer *et al.* (2014) concluded that both $7/3$ and $8/3$ are of Laughlin type (although there is some question of the reliability of this type of experiment, see footnote 35).

$12/5$: The observation of a $12/5$ plateau (Pan *et al.*, 1999) is now beyond any doubt (Kumar *et al.*, 2010). Because of the small size of the gap (around 80 mK in the only experiments that have observed this fraction), it has been impossible to perform many experiments beyond simply the observation of the plateau. Theoretically, the most recent works by Mong *et al.* (2015) and Pakrouski *et al.* (2016) confirmed that the $12/5$ state is (the particle-hole conjugate of) the Z_3 Read-Rezayi state, and further that Landau level mixing suppresses the (experimentally unobserved) $13/5$ state.

$2+6/13$: Perhaps the most intriguing fraction yet observed is at filling fraction $2 + 6/13$ (Kumar *et al.*, 2010). While this could potentially fit into a conventional hierarchy, it would be an extremely unusual creature corresponding to $q = 6$ of the Jain series $q/(2q + 1)$, where $q = 3, 4, \text{ and } 5$ are not observed. A natural explanation of this state would be a first order daughter state of the anti-Pfaffian at $5/2$ (Levin and Halperin, 2009). The physics of this hierarchy are discussed in Sec. VI.B. The experimentally measured activation gap for this state is exceedingly small at only about 10 mK.

$2+3/8$: In the LLL there is possibly some (albeit very weak) evidence for a $\nu = 3/8$ plateau (Pan *et al.*, 2003) which was discussed theoretically by Scarola, Jain, and Rezayi (2002) and Mukherjee *et al.* (2012). (If verified this would be the first even denominator state in the LLL.) However, in the second Landau level, the evidence for a $2 + 3/8$ state is somewhat stronger, albeit with a small measured activation gap of roughly 10 mK. Recent theoretical work by Hutasoit *et al.* (2016) comparing possible trial wave functions suggested that this state is of a Bonderson-Slingerland type (Bonderson and Slingerland, 2008), discussed in Sec. VI.A.

2. Quasiparticle charge experiments

One of the particularly interesting features of FQHE physics is the existence of fractionally charged quasiparticles. Although in some sense the fractionally quantized Hall conductance is indirect evidence of fractionally charged quasiparticles, it has nonetheless been a key experimental quest to directly measure these fractional charges.

The first clear measurements were made via shot noise obtaining charges of $e^* \approx e/3$ at $\nu = 1/3$ (de Picciotto *et al.*, 1997; Saminadayar *et al.*, 1997) compatible with the Laughlin prediction, as well as $e^* \approx e/5$ at $\nu = 2/5$ and $e^* \approx e/7$ at $\nu = 3/7$ (Heiblum, 2010). These latter charges are compatible with the prediction of the Halperin-Haldane hierarchy. More recent shot noise experiments by Dolev *et al.* (2008) established charges of $e^* \approx e/4$ at $\nu = 5/2$ compatible with the

Moore-Read state or the anti-Pfaffian,³⁶ as well as establishing charge $e/3$ at filling $\nu = 8/3$ and $5/3$. Unfortunately, when the same experiments were performed at extremely low temperature and low voltage other values of charge were sometimes measured, and these results remain not fully understood (Dolev *et al.*, 2010; Heiblum, 2010).

A second approach has been to measure charge motion with scanning tips or stationary single-electron transistors. Such experiments similarly give very clear evidence (Martin *et al.*, 2004) of $e^* \approx e/3$ at $\nu = 1/3$ and $2/3$. The same experiments attempted at $\nu = 5/2$ were far more challenging for a number of technical reasons. Although the expected result of $e^* \approx e/4$ was indeed measured (Venkatachalam *et al.*, 2011), the result is a bit less clear than for the same experiments at $\nu = 1/3$.

3. Quasiparticle statistics and quantum Hall interferometers

Perhaps the single most theoretically exciting quantity to be measured experimentally would be exotic braiding statistics (fractional or non-Abelian). Unfortunately, despite many attempts over the years there is still no clear demonstration of quasiparticle statistics. The main proposals for achieving such a demonstration (de C. Chamon *et al.*, 1997) revolve around building Fabry-Pérot interferometers out of two-point contacts.³⁷ Early experiments reported a number of successes [see, for example, Camino, Zhou, and Goldman (2007)] but it was later realized by Rosenow and Halperin (2007) [with later theoretical extensions by Halperin *et al.* (2011) and von Keyserlingk, Simon, and Rosenow (2015)] that Coulomb charging effects may dominate the physics and as a result the statistical phase may be hidden. Further interferometry experiments confirmed the picture of ‘‘Coulomb domination’’ in some detail (Zhang *et al.*, 2009; Ofek *et al.*, 2010; Choi *et al.*, 2011; McClure *et al.*, 2012).

Similar interferometry experiments were predicted to give particularly clear evidence of non-Abelian statistics for the $\nu = 5/2$ state (Bonderson, Kitaev, and Shtengel, 2006; Stern and Halperin, 2006) given a number of simplifying assumptions [see also the discussion by Rosenow *et al.* (2009) and von Keyserlingk, Simon, and Rosenow (2015)]. A series of detailed experiments were conducted attempting to see this physics [see Willett *et al.* (2013) and references therein]. However, the published data actually disagree with predictions as detailed in von Keyserlingk, Simon, and Rosenow (2015) and the situation remains controversial. Potentially, further experiments will clarify the situation.³⁸

³⁶In fact, Levin and Stern (2009) argued that the additional charge fractionalization of $e/2$ to $e/4$ has to occur for any valid representative wave function, simply because of the even denominator filling fraction.

³⁷There have also been impressive experiments with Mach-Zehnder interferometers in the integer quantum Hall regime, but not in the fractional regime. See, for example, Neder *et al.* (2006).

³⁸Another experiment to see this statistical physics with an interferometer by examining phase slips claimed to observe both fractional and non-Abelian statistics (An *et al.*, 2011). However, as of yet, this work has not been reproduced.

4. Composite-fermion experiments

There have been quite a few experiments demonstrating the detailed physics of composite fermions near the half filled (often lowest) Landau level. In particular, these have shown that there is indeed a fermionic quasiparticle which moves with a cyclotron radius $R_c^* = \hbar k_F / e \Delta B$, where k_F is set by the density of particles and $\Delta B = B - B_{1/2}$ is the deviation of the magnetic field from the half filled Landau level. The existence of this length scale is exactly as predicted by [Halperin, Lee, and Read \(1993\)](#). We refer the interested reader to [Jain \(2007\)](#) as well as to [Heinonen \(1998\)](#) for detailed discussions of these experiments.

IV. QUANTUM HALL LIQUIDS FROM CFT

A. Chern-Simons theory and CFT: Abelian quantum Hall states

The presence of fractional statistics particles suggests there should be a Chern-Simons description of the topological properties of the system. Indeed, this is exactly what is embodied in the Wen-Zee Lagrangian (12) discussed earlier. Once the connection to Chern-Simons theories is made, one can exploit the many connections and mathematical results relevant to these well-studied theories.

The Chern-Simons term was first considered in a physics context by [Deser, Jackiw, and Templeton \(1982\)](#), who showed that when added to a Maxwell, or in the non-Abelian case a Yang-Mills, Lagrangian it gives mass to the gauge field without breaking the gauge invariance. Soon afterward it was realized that coupling nonrelativistic particles minimally to a gauge potential with a Chern-Simons term effectively changes their statistics. Starting with the work of [Witten \(1989\)](#), it was realized that there is an intimate connection between Chern-Simons theories and CFT. It is perhaps then not surprising that quantum Hall states can be written in terms of CFT correlators.

As the connection between Chern-Simons theory and CFT was developed, it was noticed that the polynomial part of the Laughlin wave functions can be interpreted as holomorphic conformal blocks ([Fubini and Lütken, 1991](#); [Moore and Read, 1991](#)). For example, the wave function for two quasipoles in Eq. (1) can be written as

$$\begin{aligned} \Psi(\eta_1, \eta_2; \{z_i\}) &= \langle H(\eta_1) H(\eta_2) V(z_1) \cdots V(z_N) \mathcal{O}_{bg} \rangle \\ &= (\eta_1 - \eta_2)^{1/m} \prod_{i=1}^N (\eta_1 - z_i)(\eta_2 - z_i) \\ &\quad \times \prod_{i < j} (z_i - z_j)^m e^{-1/(4m\ell^2)(|\eta_1|^2 + |\eta_2|^2)} \\ &\quad \times e^{-(1/4\ell^2) \sum_i |z_i|^2}, \end{aligned} \quad (34)$$

where the average is taken with respect to the action of a massless chiral boson ϕ , normalized as

$$\langle \phi(z) \phi(z') \rangle = -\log(z - z').$$

The vertex operators are given by

$$\begin{aligned} V(z) &= :e^{i\sqrt{m}\phi(z)}:, \\ H(\eta) &= :e^{i\sqrt{1/m}\phi(\eta)}:, \end{aligned} \quad (35)$$

and \mathcal{O}_{bg} provides a neutralizing background charge that is discussed in more detail later. Although closely related, ϕ is distinct from the φ used in Sec. II.G. In particular, their normalizations differ. More details on this chiral boson CFT are found in Appendix B. The value of the correlator in Eq. (34) is almost precisely that of Eq. (1) except for the $(\eta_1 - \eta_2)^{1/m}$ factor in Eq. (34), which is holomorphic (although with a branch cut), while Eq. (1) has a factor $|\eta_1 - \eta_2|^{1/m}$. The difference is just a choice of gauge; the one used in Eq. (34) is often called ‘‘holomorphic’’ or ‘‘fractional statistics’’ gauge. We should keep in mind that the z variables are actual positions of the physical electrons in the system, whereas the η variables are just parameters in the wave function, and we are free to multiply the wave function with any η -dependent phase factor.

As mentioned in Appendix B, correlators of vertex operators vanish if the fields inside the correlator do not fuse to the identity. This ‘‘charge neutrality’’ is ensured by the insertion of the background charge operator \mathcal{O}_{bg} . In principle, there is a freedom of where to put this neutralizing charge. However, it turns out that if the operator is chosen as ([Moore and Read, 1991](#))

$$\mathcal{O}_{bg} = \exp\left(-i\rho\sqrt{m} \int d^2z \phi(z)\right), \quad (36)$$

where ρ is the (constant) fermion density, it provides the Gaussian factors needed to directly interpret the correlators as LLL wave functions. Making this operator well defined is a bit tricky, as discussed in some detail by [Hansson, Hermanns, and Viefers \(2009\)](#). On geometries other than the plane, the Gaussian factors look different, but, *mutatis mutandis*, the same recipe will work (see Sec. VII.A).

In general, primary fields of the CFT correspond to the Wilson line operators of the corresponding Chern-Simons theory. In the present case, the gauge group is a single $U(1)$ at ‘‘level’’ $m/2$. The operator content of the CFT contains all clusters of multiple elementary quasipoles H and each of these should correspond to a different species of Wilson line. The Wilson lines generally braid nontrivially with each other, but the electron operator (which topologically amounts to a cluster of m elementary quasiparticles) should actually be topologically trivial. This statement is precisely true if the ‘‘electron’’ is a boson, which is true for even m . For odd m , braiding is trivial, but exchange of two fermions accumulates a statistical sign.

1. Quasiparticle braiding and monodromies

We now revisit the question of fractional charge and statistics and use the wave function (34) to derive the Berry phase in Eq. (4) for braiding one quasipole at position η_1 around another at η_2 . Using Eq. (34) [rather than Eq. (1)] for the wave function Ψ , and the decomposition

$$\frac{d}{d\tau} = \left(\frac{d\eta_1}{d\tau}\right)\partial_{\eta_1} + \left(\frac{d\bar{\eta}_1}{d\tau}\right)\partial_{\bar{\eta}_1},$$

it is particularly easy to calculate the Berry phase (Nayak and Wilczek, 1996; Gurarie and Nayak, 1997). In $\langle\Psi|\partial_\tau|\Psi\rangle$ in Eq. (3) the $\partial_{\bar{\eta}_1}$ acts only on the exponent, bringing down a factor of $-\eta_1/4m\ell^2$. To handle the ∂_{η_1} we integrate by parts so that $\langle\Psi|\partial_{\eta_1}|\Psi\rangle$ is replaced by $(-\partial_{\eta_1}\langle\Psi|)|\Psi\rangle$ noting that $\langle\Psi|\Psi\rangle = 1$ so the total derivative vanishes. Once in this form, the ∂_{η_1} again operates only on the exponent to give $\bar{\eta}_1/4m\ell^2$. Thus we obtain

$$\begin{aligned} i\gamma &= \frac{1}{4m\ell^2} \oint d\tau \left[\bar{\eta}_1 \frac{d\eta_1}{d\tau} - \eta_1 \frac{d\bar{\eta}_1}{d\tau} \right] \\ &= \frac{iA}{m\ell^2} = i\frac{2\pi}{m}\Phi, \end{aligned} \quad (37)$$

with A being the (signed) area enclosed by the path of η_1 and Φ the corresponding enclosed flux in units of the flux quantum. This is simply the Aharonov-Bohm phase associated with moving a charge $+e/m$ around the area A .

At this point, it appears that there is no effect of the other quasihole η_2 . Indeed, the position of η_2 does not enter the Berry phase calculation here. However, examining the holomorphic form of the wave function (34) we see that the wave function is actually multivalued; mathematically this amounts to having a branch cut in the complex η_2 plane, or equivalently defining this variable on a Riemann surface with m sheets. The wave function itself accumulates a phase of $e^{2\pi i/m}$ when η_1 is moved around η_2 . This explicit phase in the wave function is known as the *monodromy*. Thus the total phase accumulated as one particle is moved around the other is the sum of the Berry phase and the monodromy. The latter gives twice the expected statistical angle $\theta = \pi/m$, precisely the same as that obtained from Eq. (4). In the current calculation, the fractional statistics is explicit in the phase of the wave function and the Berry phase gives only the Aharonov-Bohm phase, whereas if one works in the gauge of Eq. (2) there is no monodromy and both the statistical phase and the Aharonov-Bohm phase are parts of the Berry phase (Kjønsgberg and Leinaas, 1997). Both of these approaches are correct, in that they give the same total phase. However, the holomorphic gauge appears much simpler for most purposes, and it will generalize very easily to the non-Abelian case treated in Sec. IV.C.4.

B. Multicomponent Abelian states

Although the electrons are spin half particles, the spin degree of freedom can often be neglected in the context of the quantum Hall effect, since it is quenched by the strong magnetic field. However, this is not always the case (Girvin and MacDonald, 1995), and in general one must consider both components (Halperin, 1983; Du *et al.*, 1995). Other situations that require a multicomponent, or multilayer, description are bilayers (Yoshioka, MacDonald, and Girvin, 1989; Eisenstein *et al.*, 1992; Lopez and Fradkin, 1995), where the layer index is a new quantum number, and semiconductors, such as graphene, where there is a valley degeneracy (Dean *et al.*, 2011).

All these cases can be described in the K -matrix formalism, and the recipe for finding the corresponding CFT wave functions is a straightforward generalization of the Laughlin case: For a state with n components, or layers, the integer-valued positive definite K matrix is symmetric and has rank n , and it can always be factorized as

$$\mathbf{K} = \mathbf{Q}\mathbf{Q}^T, \quad (38)$$

where \mathbf{Q} is a $n \times k$ matrix and $k \geq n$, since the rank of \mathbf{K} is n .³⁹ We can then form n electron operators, using k chiral boson fields⁴⁰:

$$V_\alpha = :e^{iQ_{\alpha\beta}\phi^\beta}:, \quad (39)$$

where the chiral bosons satisfy

$$\langle\phi^\alpha(z)\phi^\beta(z')\rangle = -\delta_{\alpha\beta}\log(z-z').$$

To neutralize a correlator with N_α particles in the α th layer, we insert a background charge

$$\mathcal{O}_{bg} = \exp\left(-i\int d^2r\rho(\vec{r})n_\alpha Q_{\alpha\beta}\phi^\beta(z)\right), \quad (40)$$

where ρ is the fermion density (which usually is assumed to be constant), so $\int d^2r\rho(\vec{r}) = N$. The neutrality condition for the field ϕ^β becomes

$$N_\alpha Q_{\alpha\beta} = n_\alpha N Q_{\alpha\beta}$$

from which follows $N_\alpha = n_\alpha N$, so n_α is just the relative density of the layers.⁴¹ The relative filling fraction ν_α is given by

$$\nu_\alpha = N_\alpha/N_\Phi = n_\alpha\nu = K_{\alpha\beta}^{-1}t_\beta. \quad (41)$$

In most cases one takes $k = n$, i.e., uses as many fields as there are layers, but this is not necessary. Even for a given k the factorization is not unique, as shown in the following

³⁹An alternative notation is to write $K_{\alpha\beta} = \vec{Q}^{(\alpha)} \cdot \vec{Q}^{(\beta)}$, where $\vec{Q}^{(\alpha)}$ are n different k -dimensional ‘‘charge vectors,’’ with elements $Q_\beta^{(\alpha)} = Q_{\alpha\beta}$. The vertex operators are then written as $V_\alpha = : \exp(i\vec{Q}^{(\alpha)} \cdot \vec{\phi}) :$, where $\vec{\phi} = (\phi^1, \dots, \phi^k)$.

⁴⁰While we need at least n chiral bosons to represent the state, in some cases it is advantageous to use more, for instance when representing the Jain series on the sphere (Kvornjig, 2013).

⁴¹Note that there is a $U(1)^n$ symmetry, and thus n conserved Abelian charges (the electric charge is $-e\rho$). In many physical situations this charge is not precisely conserved and strictly speaking the wave function should be a superposition of sectors having different numbers of particles in the different ‘‘layers.’’ For example, the bilayer 111 state can be more properly written as $\prod_m (c_{m\uparrow}^\dagger + c_{m\downarrow}^\dagger)|0\rangle$. However, for many purposes it is sufficient to think about a single sector with fixed particle number in each layer. Note that the K matrix of such a state is 1, not $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, as we always require it to be invertible; see Eq. (13).

example. One can think of \mathbf{Q} as the matrix square root of the \mathbf{K} ; in the Laughlin case where \mathbf{Q} is a scalar, we have $\mathbf{Q} = \sqrt{\mathbf{K}} = \sqrt{m}$. The multiparticle wave functions will be built by products of factors such as

$$\langle V_\alpha(z_i) V_\beta(z_j) \rangle = (z_i - z_j)^{K_{\alpha\beta}}$$

and of course also the proper Gaussian factors. As an example, we show how this works⁴² for the Halperin (m, m, n) states (Halperin, 1983) with $m > n$. The corresponding \mathbf{K} can be factorized as

$$\begin{aligned} \mathbf{K} &= \begin{pmatrix} m & n \\ n & m \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{m} & 0 \\ \frac{n}{\sqrt{m}} & \sqrt{\frac{m^2-n^2}{m}} \end{pmatrix} \begin{pmatrix} \sqrt{m} & \frac{n}{\sqrt{m}} \\ 0 & \sqrt{\frac{m^2-n^2}{m}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{m+n}{2}} & \sqrt{\frac{m-n}{2}} \\ \sqrt{\frac{m+n}{2}} & -\sqrt{\frac{m-n}{2}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{m+n}{2}} & \sqrt{\frac{m+n}{2}} \\ \sqrt{\frac{m-n}{2}} & -\sqrt{\frac{m-n}{2}} \end{pmatrix}, \quad (42) \end{aligned}$$

where we showed two different ways to factorize. In both cases the ground state wave function is obtained by taking an equal number of the operators V_1 and V_2 in the correlator, and it is easy to show that using the second, symmetric, factorization, only the field ϕ_1 is related to the electric charge and requires a background. As a concrete example, the vertex operators corresponding to the first factorization are

$$\begin{aligned} V_1 &= :e^{i\sqrt{m}\phi_1}:, \\ V_2 &= :e^{i(n/\sqrt{m})\phi_1 + i\sqrt{(m^2-n^2)/m}\phi_2}:, \end{aligned}$$

giving the Halperin wave function

$$\psi_{mmn} = \prod_{i < j} (z_i^\uparrow - z_j^\uparrow)^m \prod_{k < l} (z_k^\downarrow - z_l^\downarrow)^m \prod_{r,s} (z_r^\uparrow - z_s^\downarrow)^n \quad (43)$$

with \uparrow and \downarrow denoting the (pseudo)spin up and down species, respectively.

It is also possible to generalize Laughlin's plasma analogy to the multicomponent case, and thus to extract the topological properties of the states, as long as the equivalent multicomponent Coulomb plasma is in a screening phase (Qiu, Joynt, and MacDonald, 1989).

C. Non-Abelian quantum Hall states and the Moore-Read conjecture

That the Laughlin wave functions can be represented as correlators in a CFT suggests that other CFTs might be used to generate valid quantum Hall wave functions. Indeed, Moore and Read (1991) suggested that quantum Hall states might generally be constructed in this way. The close relationship between CFT and TQFT both allows an immediate identification of the

⁴²The CFT vertex operators for the Halperin states were first constructed by Moore and Read (1991).

topological properties associated with that quantum Hall state and provides a natural way to describe the edge of both Abelian and non-Abelian states. Not every CFT will produce acceptable quantum Hall states. One important condition is that the CFT is rational, which in this context means that there is a finite number of inequivalent particle types, and that the ground state on a torus has a finite degeneracy in the thermodynamic limit. Further the CFT should be unitary (Read, 2009b), as a nonunitary theory will inevitably result in a gapless state. However, it seems unlikely that these two conditions alone are sufficient to guarantee a well-behaved quantum Hall state in many cases (Jackson, Read, and Simon, 2013).

The physics of the quantum Hall and CFT connection, and the resulting study of a number of non-Abelian quantum Hall states, was reviewed in some depth by Nayak *et al.* (2008). In this review we mainly restrict ourselves to the simplest of these states, namely, the Moore-Read, or Pfaffian state.

1. Electronic wave functions as conformal blocks

In order to implement the Moore-Read scheme, one starts by choosing a CFT to work with. There will always be a $U(1)$ charge sector given by a chiral boson, as we explored for the case of the Laughlin wave function. Multiple $U(1)$ sectors can be present if there are multiple species of electrons, as in the multicomponent liquids in Sec. IV.B, or the hierarchies to be discussed in Sec. V. For now, we assume only one $U(1)$ sector. The electrical charge of a particle is determined entirely by its $U(1)$ charge. In addition there may be a neutral sector of the CFT which might be a more complicated CFT, such as the Ising CFT in the case of the Moore-Read state. Those who find the following general discussion too abstract are referred to Sec. IV.C.5, where the example of the Moore-Read state is treated explicitly.

Within the CFT, we must choose some simple current field, which we call ψ_e to represent the electron (see Appendix B for a definition of simple current). The electron must also be a fermion, meaning that it has half-odd integer conformal weight (or scaling dimension). Or, if one is considering the quantum Hall effect of bosons, the ‘‘electron’’ must be a boson, meaning that it has integer conformal weight. The ground state wave function is given as the holomorphic chiral correlator or *conformal block*,

$$\Psi(z_1, \dots, z_N) = \langle \psi_e(z_1) \psi_e(z_2) \cdots \psi_e(z_N) \mathcal{O}_{bg} \rangle,$$

where \mathcal{O}_{bg} is the appropriate background field operator such that the correlator does not vanish. Except for the Gaussian factors, this wave function is fully holomorphic and is appropriate as a LLL wave function. For fermionic fields ψ_e the wave function is totally antisymmetric and for bosonic fields it is symmetric.

We can further consider fields ψ_{qh} , yielding wave functions such as

$$\begin{aligned} \Psi(\{\eta_i\}_1^M; \{z_i\}) &= \langle \psi_{qh}(\eta_1) \psi_{qh}(\eta_2) \cdots \psi_{qh}(\eta_M) \\ &\quad \times \psi_e(z_1) \psi_e(z_2) \cdots \psi_e(z_N) \mathcal{O}_{bg} \rangle \end{aligned}$$

which represent quasiholes at positions η_1, \dots, η_M inserted into the prior ground state wave function. In general there may

be several different species of quasiholes, represented by distinct operators.

In order to be a valid electron wave function, these correlators must be single valued, i.e., the electron operators must braid trivially with the quasihole operators and no singularities in the wave function must result. This condition corresponds to the following. Consider taking a quasihole to the same position as an electron. The combination will form some other type of quasiparticle excitation ψ_f , which we call the *fusion* of ψ_e and ψ_{qh} . (In CFT language, we say $\psi_e \times \psi_{qh} = \psi_f$.) If the scaling dimensions of these operators are h_e , h_{qh} , and h_f , respectively, then examining the operator product expansion in Eq. (B4) tells us that we must have

$$h_f - h_e - h_{qh} = \text{integer} \geq 0 \quad (44)$$

for the wave functions to remain single valued (no branch cuts) and nonsingular (no poles) as an electron approaches the position of the quasihole. This provides a strong constraint on the type of quasihole operator that can exist in our theory. The simplest example is that of a hole in a $\nu = 1/m$ Laughlin state. The operators (35) are fused using the operator product expansion $\lim_{\eta \rightarrow z} V(z) \times H(\eta) \sim (z - \eta) : e^{i(m+1)\phi(z)/\sqrt{m}} :$, so, recalling that $V_\alpha = : e^{i\alpha\phi} :$ has the conformal dimension $h_\alpha = \alpha^2/2$ (cf. Appendix B), we get $h_f - h_e - h_{qh} = \frac{1}{2}(m+1)^2/m - \frac{1}{2}m - \frac{1}{2}1/m = 1$ which fulfills Eq. (44).

2. Bulk-edge correspondence

The massless boson CFT that describes the bulk of the Laughlin state is in fact the same massless bose theory that describes the Laughlin quantum Hall edge that we considered earlier in Sec. II.G.. Even the operators that create charge are, in this case, identical [compare the quasiparticle creation operators (35) to (28)]. This similarity, which is a manifestation of what is known as bulk-edge correspondence, is not a coincidence, but follows naturally from deep mathematical results on the quantization of Chern-Simons theories (Witten, 1989; Cappelli *et al.*, 1993). The general idea is that the Chern-Simons description of the quantum Hall state is essentially an isotropic ($3 = 2 + 1$)-dimensional theory. If we “cut” the 3-manifold in a spacelike slice (at fixed time) we end up with the $(2 + 0)$ -dimensional CFT that describes the ground state wave function.⁴³ On the other hand, if we cut the

⁴³In the CS description, particles are represented by Wilson lines and the statistical phase factors, Abelian or non-Abelian, can be extracted from the correlation functions of such lines. At a fixed time surface, the Wilson lines correspond to a number of points or “punctures.” Witten showed how to canonically quantize the CS theory in the presence of such punctures (Witten, 1989). The outcome is that the corresponding finite-dimensional Hilbert space is precisely the space spanned by the blocks of a conformal field theory. Thus, identifying the coordinates in the conformal block as the (holomorphic) coordinates of the particles it is quite natural to interpret the blocks themselves as bona fide representative wave functions for the state in question, since the monodromies of these functions will properly represent the statistics of the particles, be they electrons or Abelian or non-Abelian anyonic quasiparticles.

3-manifold in a timelike slice (a fixed one-dimensional curve crossed with time) we obtain a $(1 + 1)$ -dimensional CFT describing an edge.

Also in the general case of hierarchy and/or non-Abelian states, there is a correspondence between the bulk CFT, whose correlators describe the stationary $(2 + 0)$ -dimensional wave function of the state, and the edge CFT that describes the dynamics of the $(1 + 1)$ -dimensional quantum Hall edge. As discussed in Sec. II.G., there is always at least one chiral boson mode describing the dynamics of the charged edge modes. However, in addition there can also be dynamical neutral modes corresponding to the nonboson part of the CFT. While there is often no simple representation of the Lagrangian, the mathematical power of CFT can be brought to bear on this part of the problem (Di Francesco, Mathieu, and Sénéchal, 1997). For example, using CFT methods one can determine fine details of the dynamical neutral mode spectrum without ever having to write a Lagrangian (Ilan *et al.*, 2009; Bonderson, Nayak, and Shtengel, 2010; Cappelli, Viola, and Zemba, 2010; Stern *et al.*, 2010).

3. Vector space of conformal blocks, braiding, and monodromy

As in the case of Eq. (34), the wave functions need not be single valued in the coordinates η of the quasiholes, since these are simply parameters of the wave function, rather than physical electron coordinates. We already discussed the Laughlin case where fixing the hole position η_2 , the single-component wave function $\Psi_{1/m}(\eta_1, \eta_2)$ defines a function of η_1 on a Riemann surface with m sheets. In the case of a more complicated CFT, and multiple quasiholes, the wave function has many components, each with a complicated analytic structure with multiple branch cuts. During a general braiding operation (that also can amount to the exchange of identical quasiparticles) the components transform into each other by a unitary matrix. To make this more concrete, let us consider the Moore-Read state at filling fraction $\nu = 1/2$ that we discuss at length in Sec. IV.C.5. The correlator of four quasiholes only (no electrons) can be calculated by standard methods (Zuber and Itzykson, 1977; Belavin, Polyakov, and Zamolodchikov, 1984) with the result

$$\begin{aligned} & \lim_{\omega \rightarrow \infty} \langle \psi_{qh}(0) \psi_{qh}(\eta) \psi_{qh}(1) \psi_{qh}(\omega) \mathcal{O}_{bg} \rangle \\ & \sim a_+ \Psi_+(\eta, \omega) + a_- \Psi_-(\eta, \omega), \end{aligned}$$

which holds for $|\omega| \gg |\eta|, 1$, where a_\pm are arbitrary coefficients, and

$$\Psi_\pm \sim \omega^{1/4} \sqrt{1 \pm \sqrt{1 - \eta}}.$$

Put differently, the wave function, at fixed positions of the quasiholes, is given by the vector with coordinates (a_+, a_-) in the basis of the two conformal blocks Ψ_\pm . Letting the quasiparticle at ω encircle the three others clockwise, while keeping η fixed, the wave function picks up an Abelian phase $e^{i\pi/2}$ in close analogy to the Laughlin case. The interesting effect occurs when ω is kept fixed, and the quasiparticle at η encircles the one at 1. This amounts to $\Psi_\pm(\eta, \omega) \rightarrow \Psi_\mp(\eta, \omega)$,

or equivalently $a_{\pm} \rightarrow a_{\mp}$, i.e., to a unitary rotation σ_x of the two-dimensional state vector when thought of as a spinor.

Including the electrons to get the full quantum Hall wave function is more complicated and was originally done for four quasipoles by [Nayak and Wilczek \(1996\)](#) and later for the general case of M holes and N electrons by [Ardonne and Sierra \(2010\)](#). The explicit expressions are complicated, but schematically the result is

$$\Psi(\eta_1, \eta_2, \dots, \eta_M; \{z_i\}) = \sum_{\alpha} c_{\alpha} \Psi_{\alpha}(\eta_1, \eta_2, \dots, \eta_M; \{z_i\}),$$

where the conformal blocks, labeled by α , form a degenerate vector space. Moving the quasiparticles around each other and then returning to the same positions leaves the Ψ_{α} unchanged, but transforms the coefficients c_{α} by a unitary matrix,

$$c_{\alpha} \rightarrow U_{\alpha\beta}^{(\text{braid})} c_{\beta},$$

where the monodromy matrix $U^{(\text{braid})}$ depends on the particular braid performed on the coordinates. This is the essence of non-Abelian statistics. By braiding quasiparticles around each other, the wave function is transformed unitarily within the degenerate vector space spanned by the Ψ_{α} . The term ‘‘non-Abelian’’ stems from the fact that the unitary matrices, corresponding to different braid operations, will generically not commute with each other ([Moore and Read, 1991](#); [Nayak *et al.*, 2008](#); [Rowell and Wang, 2016](#)).

4. Holonomy equals monodromy and the Moore-Read conjecture

In the Abelian case, as discussed in Secs. [II.A](#) and [IV.A.1](#), the result of adiabatically moving one quasiparticle around another is given by the sum of the phase due to the monodromy and the Berry phase (3). Furthermore, for wave functions extracted from conformal blocks, the Berry phase solely comes from the Gaussian factor which gives the Aharonov-Bohm phase due to the magnetic field. In the present case where we have a degenerate vector space, the Berry phase becomes matrix valued. Following the notation in [Read \(2009b\)](#) we write for the holonomy

$$B = M\mathcal{P} \exp \left(\oint d\tau \langle \Psi_{\alpha} | \frac{d}{d\tau} | \Psi_{\beta} \rangle \right), \quad (45)$$

where we denoted the monodromy matrix with M , and where \mathcal{P} is a path ordering operator which is necessary since the matrices in the exponent do not commute at different times.

Assuming that the charge sector can be ignored, [Read \(2009b\)](#) explicitly calculated the non-Abelian monodromy matrices for two and four quasiparticles in the Moore-Read state. He furthermore argued that the Berry phase in Eq. (45) just gives the Aharonov-Bohm phase times a trivial identity matrix.

In general, the path ordered exponential in Eq. (45) can be calculated by the methods given in Sec. [IV.A.1](#), assuming the

conformal blocks to be holomorphic. In this case the Berry matrix will again just give the Aharonov-Bohm phase times the identity, *so long as we have* orthonormality of the conformal blocks

$$\langle \Psi_{\alpha} | \Psi_{\beta} \rangle = C \delta_{\alpha,\beta}. \quad (46)$$

Thus, assuming this, the holonomy B (the physical result of moving quasiparticles around each other) is given by the monodromy M (how the wave function explicitly transforms due to its branch cuts as discussed in Sec. [IV.C.3](#)) multiplied by the Abelian Aharonov-Bohm phase ([Gurarie and Nayak, 1997](#); [Read, 2009b](#)).

While for well-behaved quantum Hall states it appears that Eq. (46) is true (and this is always assumed), proving this fact turns out to be quite difficult. Equivalently we might ask whether there are ways to prove that the holonomy is the monodromy. The latter has been shown to be true numerically for the Moore-Read state several times ([Tserkovnyak and Simon, 2003](#); [Baraban *et al.*, 2009](#)), and then more recently for the Z_3 Read-Rezayi state as well ([Wu *et al.*, 2014](#)). Direct numerical tests of Eq. (46) have been done by [Baraban \(2010\)](#) for the Moore-Read state. On the analytical side, we already mentioned the calculation by [Read \(2009b\)](#) for two and four quasipoles in the Moore-Read state. An alternative derivation based on a mapping to a more complicated plasma was given by [Bonderson, Gurarie, and Nayak \(2011\)](#).

[Read \(2009b\)](#) also gave some very general arguments for the assertion holonomy equals monodromy, or equivalently the orthogonality relation (46), for quasiparticle wave functions constructed from holomorphic conformal blocks. The main idea is to consider expressions such as

$$\langle e^{i\lambda \int d^2z \bar{\psi}_e(\bar{z}) \psi_e(z)} H(\eta_1) \cdots H(\eta_M) \mathcal{O}_{bg} \rangle,$$

where the electron operator has been factorized in a charge and a topological part viz. $\psi_e = e^{(i/\sqrt{D})\phi} \psi$. For $\lambda = 0$ this formula is just a CFT correlator of some primary fields and will have power-law dependence on the quasiparticle coordinates η_i . For nonzero λ , expanding the exponential, we see that the expression resembles the grand canonical sum over different numbers of electrons, in addition to the quasiparticles. With a suitably chosen background, this will describe an N electron state with negligibly small charge fluctuations. Alternatively, we can consider the operator $\bar{\psi}_e \psi_e$ as a perturbation on the original CFT. For the charged sector, one can use the Laughlin plasma analogy to infer screening of the electric charge.

To demonstrate ‘‘generalized screening,’’ i.e., that all correlations fall exponentially, which implies holonomy equals monodromy, one must show that the topological sector is also in a massive phase. [Read \(2009b\)](#) gives a series of arguments for this assertion, but also points out that it is hard to draw firm conclusions, since scaling arguments based on the conformal dimensions of the perturbing operators are not to be trusted for large (~ 1) values of λ .

We now summarize the results of the preceding sections in what we refer to as the Moore-Read conjecture.

Representative wave functions for quantum Hall ground states, as well as their quasihole excitations, can be expressed in terms of conformal blocks of primary fields in (rational, unitary⁴⁴) CFTs. The holonomies, or statistical braiding matrices, equal the monodromies of these blocks. Furthermore, the same CFTs yield a minimal⁴⁵ dynamical theory for the edge of the quantum Hall liquid.

This conjecture, which has been a crucial guiding principle for later work, applies directly to all one-component states, as well as to multicomponent states of the Halperin type. As formulated, it however does not apply to hierarchy states where some of the electrons are described by descendant fields. This is a main topic in Sec. V.

5. Example: The Moore-Read state

As a well-known example, we consider in detail the extensively studied Moore-Read state (Moore and Read, 1991). The corresponding CFT is based on the Ising CFT,⁴⁶ which has three primary fields: $\mathbf{1}, \psi, \sigma$ with conformal weights 0, 1/2, and 1/16, respectively. The fusion rules (see Appendix B for a crash course in CFT) are given by

$$\begin{aligned} \mathbf{1} \times \mathbf{1} &= \mathbf{1} & \mathbf{1} \times \psi &= \psi & \mathbf{1} \times \sigma &= \sigma \\ \psi \times \psi &= \mathbf{1} & \psi \times \sigma &= \sigma & & \\ & & \sigma \times \sigma &= \mathbf{1} + \psi. & & \end{aligned} \quad (47)$$

That there is a fusion rule with two fields on the right-hand side (the last rule) indicates that this is a non-Abelian theory. This Ising CFT is particularly simple in that ψ is just a free Fermi field, so Wick's theorem applies, and we can evaluate correlators such as

$$\begin{aligned} \langle \psi(z_1) \cdots \psi(z_N) \rangle &= \mathcal{A} \left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N} \right) \\ &\equiv \text{Pf} \left(\frac{1}{z_i - z_j} \right), \end{aligned}$$

where N is even and \mathcal{A} denotes an antisymmetric sum over all pairings of coordinates. This expression defines a Pfaffian and we denote it as Pf. Correlators including σ fields (which are not free so Wick's theorem does not apply) can be calculated analytically as well, although the results are far more complicated (Belavin, Polyakov, and Zamolodchikov, 1984; Nayak and Wilczek, 1996; Ardonne and Sierra, 2010; Bonderson, Gurarie, and Nayak, 2011). This Ising CFT must be supplemented with a $U(1)$ CFT to make a proper quantum Hall state. We consider the $\nu = 1/2$ state of fermions and correspondingly define

$$\psi_e(z) = \psi(z)V(z) = \psi(z) : e^{i\sqrt{2}\phi(z)} : , \quad (48)$$

where $V(z)$ is a bosonic vertex operator in the $U(1)$ sector. Recalling that the scaling dimension of $: e^{i\alpha\phi} :$ is $\alpha^2/2$ (see Appendix B), we note that the scaling dimension of the electron field is $h_e = h_\psi + h_V = 1/2 + 1 = 3/2$ which is a half integer and therefore appropriate for a fermionic wave function.⁴⁷ The ground state wave function is then obtained by multiplying together the results of the Ising and $U(1)$ sectors

$$\Psi_{MR} = \langle \psi_e(z_1) \cdots \psi_e(z_N) \mathcal{O}_{bg} \rangle = \text{Pf} \left(\frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j)^2.$$

Note that this wave function is nowhere singular. However, the Pfaffian allows particles to “pair,” i.e., the wave function vanishes only as a single power, when two particle positions coincide.

Consider now the rather surprising fact that we constructed an interesting wave function based on noninteracting fermion operators (obeying Wick's theorem) and noninteracting free boson operators, and yet this wave function has built into it some extremely complicated correlations.

We can also construct quasihole operators by

$$\psi_{qh} = \sigma(\eta)H(\eta) = \sigma(\eta) : e^{i\phi(\eta)/(2\sqrt{2})} : . \quad (49)$$

The coefficient in the exponent of the vertex operator is chosen to give the lowest possible charge while still satisfying Eq. (44). To see this, note that $\psi_{qh} \times \psi_e = \sigma : e^{i5\phi/(2\sqrt{2})} : = \psi_f$. The scaling dimensions are $h_{qh} = h_\sigma + h_H = 1/16 + 1/16 = 1/8$ and similarly $h_f = 1/16 + 25/16 = 13/8$. Thus with $h_e = 3/2$ we obtain $h_f - h_e - h_{qh} = 0$ which assures no singularity in the electron wave function. There are also other operators that are nonsingular but they can all be obtained by fusing a number of elementary holes (49).

We get for the Moore-Read wave function with N electrons and M quasiholes,

$$\begin{aligned} \Psi_\alpha &= \langle \sigma(\eta_1) \cdots \sigma(\eta_M) \psi(z_1) \cdots \psi(z_N) \rangle_\alpha \\ &\quad \times \langle H(\eta_1) \cdots H(\eta_M) V(z_1) \cdots V(z_N) \mathcal{O}_{bg} \rangle \\ &= \langle \sigma(\eta_1) \cdots \sigma(\eta_M) \psi(z_1) \cdots \psi(z_N) \rangle_\alpha \\ &\quad \times \prod_{\gamma < \mu}^M (\eta_\gamma - \eta_\mu)^{1/8} \prod_{i=1}^N \prod_{\gamma=1}^M (\eta_\gamma - z_i)^{1/2} \prod_{i < j} (z_i - z_j)^2. \end{aligned}$$

For the correlator not to vanish, all the Ising fields have to fuse to the identity, and from Eq. (47) it follows that M must be even. As described in Sec. IV.C.3, correlators involving multiple quasiholes (and thus multiple σ fields) will generally be multivalued and span a vector space rather than giving a single wave function. For $2n$ quasiholes on a spherical or planar geometry, the corresponding vector space is 2^{n-1} dimensional. The most suitable basis for this vector space

⁴⁷Indeed, the scaling dimension of 3/2 indicates that this is actually a superconformal field theory, but we do not need this fact (Di Francesco, Mathieu, and Sénéchal, 1997).

⁴⁴The importance of unitarity was pointed out by Read (2009b).

⁴⁵Many edge CFTs can correspond to a single 2D topological bulk. This nonuniqueness is discussed in detail by Cano *et al.* (2014).

⁴⁶More correctly, it is based on the chiral part of the Ising CFT. For a pedagogical discussion of the full Ising theory, see Belavin, Polyakov, and Zamolodchikov (1984) and Di Francesco, Mathieu, and Sénéchal (1997).

is given by the conformal blocks (Di Francesco, Mathieu, and Sénéchal, 1997), labeled by the index α in the earlier equation. The charge of the quasihole is set by the repulsion of the electrons from the quasihole position, i.e., the $(\eta - z)^{1/2}$ factor. Note that although this has a fractional power, the product of all terms on the right is guaranteed, by Eq. (44), to be nonsingular and without branch cuts for electron coordinates. The half power can be viewed as 1/2 of a Laughlin quasihole, which is obtained by inserting a flux quantum and has electrical charge $+e\nu$. Thus, the elementary quasihole (49) has charge $\nu/2 = +e/4$. For a discussion of the corresponding braiding properties see Nayak *et al.* (2008).

D. Non-Abelian effective Chern-Simons theory

Just as for the Abelian states, there are effective low-energy gauge theories also for the non-Abelian states, and not surprisingly they contain non-Abelian Chern-Simons terms. These theories can be derived in different ways, and there can be different candidate theories for the same state.

1. Construction based on the anomaly

Starting from the bosonic version of the Moore-Read state at $\nu = 1$, the edge theory consists of a Majorana fermion ψ and a chiral boson $:e^{i\phi(z)}:$. This boson can be fermionized to a Dirac fermion, or equivalently two Majorana fermions. Consequently the edge theory has three Majorana fermions. From these one can form three current operators that transform as a spin one representation of an $SU(2)_2$ Kac-Moody algebra, where the subscript denotes the level. This algebra is also generated by a chiral $SU(2)_2$ Wess-Zumino-Witten model, which is thus the purely bosonic incarnation of the edge theory. To proceed from here, we recall the anomaly approach to the bulk-edge correspondence described in Sec. II.G.2. Starting from the bulk theory, we derived an anomalous edge theory where the nonconservation of the edge charge was due to inflow from the bulk. Here we turn the argument around. Knowing the edge theory we can find the anomaly and then ask what bulk theory will have a nongauge invariant boundary term that exactly cancels this anomaly. The answer is a non-Abelian $SU(2)_2$ CS theory. See Fradkin *et al.* (1998) for details of the derivation, and also for a similar, albeit more complicated, construction for the $\nu = 1/2$ fermionic Moore-Read state. [In a later paper, a Ginzburg-Landau approach, similar to the one described in Sec. II.C, is constructed for the bosonic case (Fradkin, Nayak, and Schoutens, 1999).]

2. Parton construction

An alternative way to derive effective Chern-Simons theories for the non-Abelian states is the *parton construction* due to Wen (1999). The main idea is to introduce a redundant set of free fermions, or partons, that transform under an $SU(N)$ ‘‘color’’ group. The electron operator is a singlet under this group, and the projection on the singlet sector is achieved by gauging the symmetry. By integrating out the redundant parton degrees of freedom one is left with an effective non-Abelian theory for the gauge field. We illustrate this with an example given by Wen (1999). We start from a set of fermionic, charge $e/2$ parton fields $\psi_{\alpha\alpha}$ with the Lagrangian,

$$\mathcal{L}_{\text{part}} = i\psi_{\alpha\alpha}^\dagger \partial_t \psi_{\alpha\alpha} + \frac{1}{2m} \psi_{\alpha\alpha}^\dagger \left(\partial_i - i\frac{e}{2} A_i \right)^2 \psi_{\alpha\alpha},$$

where α is a spin half index and $a = 1, 2$ an $SU(2)$ color index, also in the spin half representation. By combining two partons we can form a charge e spin one bosonic electron,

$$\Psi_m(z) = \psi_{\alpha\alpha}(z) \psi_{b\beta}(z) \epsilon_{ab} C_{\alpha\beta}^m,$$

where m is a spin one index and $C_{\alpha\beta}^m$ is the pertinent Clebsch-Gordan recoupling coefficients. Next we assume that the only physical excitations are bosons with spin one and unit electric charge. This can be achieved by coupling the color current to a dynamical $SU(2)$ gauge field a_μ^k ,

$$\begin{aligned} \mathcal{L}_{\text{part}}^{(\text{Proj})} &= i\psi_{\alpha\alpha}^\dagger (\delta_{ab} \partial_t - ia_{0^k}^{ab}) \psi_{ba} \\ &+ \frac{1}{2m} \psi_{\alpha\alpha}^\dagger \left[\left(\partial_i - i\frac{e}{2} A_i \right) \mathbf{1} - i\vec{a}_i \cdot \vec{\mathbf{t}} \right]_{ab}^2 \psi_{ba}, \end{aligned} \quad (50)$$

where $\vec{\mathbf{t}} = (1/2)\vec{\sigma}$ with σ^k being the Pauli matrices. In this theory only the color singlet operators are physical. This can be seen in different ways. First, variation with respect to the gauge fields shows that the color currents vanish identically. Alternatively one can think of adding a Maxwell term $(1/4g^2)\text{Tr}F_{\mu\nu}^k F^{k,\mu\nu}$ to the action and note that Eq. (50) corresponds to the strong coupling (confining) limit $g \rightarrow \infty$. Since the color singlet is antisymmetric the spin wave function must be symmetric by the Pauli principle, so the theory indeed describes spin one bosons. To establish that the state is non-Abelian, one integrates the parton field to get a low-energy effective action. Assuming that the partons are at a density corresponding to two filled Landau levels, the resulting low-energy action is again a non-Abelian $SU(2)_2$ CS theory, which is the spin one counterpart of the spin zero theory derived using the anomaly approach.

In a later work Barkeshli and Wen (2010) carried out a parton construction for the Z_k Read-Rezayi parafermion states at filling factor $\nu = k/(kM + 2)$, for which they found the non-Abelian fermionic bulk theory to be $[U(M) \times Sp(2k)]_1$.

E. CFT quasiparticles and hierarchies: A summary

The task of generalizing the CFT construction to even the Abelian hierarchies has turned out to be quite difficult, and we pursue this in more depth in the coming sections. Since these by necessity will be somewhat more technical, we provide here a short summary of the main results with references to the proper sections.

Before doing so, however, we note that as long as we are concerned only with universal topological properties (and do not aim at obtaining accurate wave functions) there is a relatively straightforward path to be followed (Moore and Read, 1991). Following the idea of a pseudo wave function [Eq. (5)], we can, for example, construct a $\nu = 2/7$ wave function as follows [cf. Eq. (7)]:

$$\Psi = \int \prod_{\alpha=1}^M d^2\eta_\alpha \langle \hat{V}(\bar{\eta}_1) \cdots \hat{V}(\bar{\eta}_M) \mathcal{O}_{bg} \rangle \\ \times \langle H(\eta_1) \cdots H(\eta_M) V_1(z_1) \cdots V_1(z_N) \mathcal{O}_{bg} \rangle$$

with $M = N/2$. Here $V(z)$ and $H(\eta)$ were defined in Eq. (35) (with $m = 3$ in this case), and

$$\hat{V}(\bar{\eta}) = : e^{i\sqrt{2+1/3}\phi(\bar{\eta})} :.$$

Note that the correlator of the \hat{V} is actually antiholomorphic and corresponds to fields with opposite chirality in order to properly cancel the phases in the holomorphic factors $(\eta_\alpha - \eta_\beta)^{1/3}$ that arise from the second correlator. Ignoring, for simplicity, the background charges and the resulting Gaussian exponential factors, we obtain almost the same wave function as in Eq. (7)—the only difference is that the fractional power of the factors $|\eta_\alpha - \eta_\beta|$ changes from $1/3$ to $2/3$. In Sec. V.C we argue that this is a short-distance effect that does not influence the topological properties.

Topologically this construction is valid, but, as stressed in Secn. II.B.2, it is difficult to evaluate the integrals for a reasonable number of particles. Also, if one tries to describe condensates of quasielectrons by naively replacing the hole operators $H(\eta)$ with their inverse $H^*(\bar{\eta}) = e^{-i\phi(\bar{\eta})}$, the correlators will have singularities $\sim (z_i - \bar{\eta})^{-1}$. Again, this should not change the long distance topological properties, but to get acceptable wave functions one would need not only to handle these singularities in the integrals over the positions $\bar{\eta}_i$, but also to project the result on the LLL for the electrons.

Physically the singularities come about since introducing a vortex of strength m changes the angular momentum of all the electrons by m units with respect to the position of the vortex. The problem occurs when one tries to make an antivortex, i.e., a vortex with negative m , at a position where there is already an electron. This electron cannot lower its angular momentum while still staying in the LLL, and this is what is manifested in the singularities $\sim (z_i - \bar{\eta})^{-1}$ encountered when naively trying to form a quasielectron using the operator H^* . Leaving the LLL, the angular momentum can be lowered by including a factor $\bar{z}_i - \bar{\eta}$ which after projection onto the LLL provides the rationale for Laughlin's proposal that going from a quasihole to a quasielectron amounts to $\prod_i (z_i - \eta) \rightarrow \prod_i (2\partial_i - \bar{\eta})$. The resulting multi-quasielectron wave functions are, however, equally difficult to handle in the hierarchy integrals as their quasihole counterparts.

Fortunately, there is a way out of these problems, and the key insight is that calculating correlators of H , or H^* , is not the only way to represent quasiparticles using the CFT formalism. To explain the alternative method, we first notice that in the Laughlin ground state, the i th electron is surrounded by a correlation hole $\prod_j (z_i - z_j)^m$ that we think of as m elementary holes on top of each other. The charge counting, with respect to the ground state, now works as follows. Introducing m elementary holes amounts to electric charge $+e$, and the actual electron at position z_i adds charge $-e$, meaning that the electron together with its correlation hole is neutral. This is as it should be, since adding an electron to the incompressible quantum Hall liquid does not change the local

charge density, but just expands the droplet a tiny bit at the edge. This provides the crucial clue for how to introduce quasiholes or quasielectrons—we just modify a number of electron operators by contracting or expanding their correlation holes by one unit. These words can be translated into CFT equations: contracting amounts to fusing an electron operator $V(z)$ with an inverse hole, i.e., the operator H^* . Just as in Sec. IV.C.1 fusion means making an operator product expansion, but in this case the first term turns out to vanish, so the leading term in the expansion will be a *descendant* which means that it includes a derivative. In the simplest case of the $\nu = 1/(2p + 1)$ Laughlin state, fusing the electron operator (35) with the conjugate of the hole (35) we get $\hat{V}(z) = H^* \otimes V(z) \sim : \partial_z e^{i2p/\sqrt{2p+1}\phi(z)} :$. We can now form states with an arbitrary number of quasielectrons simply by replacing a number of the original electron operators V with the new operators \hat{V} .⁴⁸ This will give rise to a state with increased density at the second level of the hierarchy. In particular, if half of the original electrons are replaced by \hat{V} , one obtains the $\nu = 2/(4p + 1)$ Jain state. By constructing the relevant hole operators in this new state the procedure can be iterated.

In complete analogy one can introduce quasiholes by expanding the correlation holes around the electrons; this is done by fusing V with an appropriate hole operator.⁴⁹ In this case, modifying a fraction of the original electrons results in a second level hierarchy state with charge density lower than the parent state. In particular, this procedure will give the states of the negative Jain sequences. That some of the Jain states were obtained in these simple cases is not a coincidence; we show that all the Jain composite-fermion states are exactly reproduced by this procedure.

Note that in this discussion there was no reference to the hierarchy integrals or even states of several localized quasiparticles. Such states can however be constructed by inserting a number of quasilocal operators $\mathcal{H}(\eta)$ and $\mathcal{P}(\bar{\eta})$, representing quasiholes and quasielectrons, respectively, into the correlators. The exact definition and properties of these operators are discussed in Sec. V.C. A most remarkable property of the resulting multi-quasiparticle wave functions is that when inserted in the integral expressions for the hierarchy wave functions, the integrals over the quasiparticle coordinates can be carried out exactly, thus giving closed form expressions for the wave function of the daughter state. These are just the wave functions discussed previously. The details of this are explained in Sec. V.F. An upshot of this is also that all composite-fermion wave functions can be exactly written in a manifestly hierarchical form.

In Sec. IV.C.4 we emphasized the importance of the plasma analogy for proving the monodromy equals holonomy conjecture that the topological properties of the quantum Hall states could be obtained directly from the monodromies of the

⁴⁸As discussed in detail in Sec. V.C this is not entirely true; as written the operator \hat{V} is anyonic and has to be augmented by another vertex operator to give fermionic commutation relations.

⁴⁹Perhaps surprisingly, the pertinent hole operator is not the naive Laughlin hole H ; we return to this technical detail in Sec. V.C.

conformal blocks. This turns out to be more difficult in the case of the hierarchy states, which is discussed in Sec. V.E.

V. EXPLICIT HIERARCHY WAVE FUNCTIONS FOR ABELIAN QUANTUM HALL STATES

In Sec. II.B we gave the general form of a hierarchy wave function expressed as integrals over quasiparticle coordinates and in Sec. II.E the explicit composite-fermion wave functions in the Jain series. As mentioned, the latter have successfully been compared with experiments, while the former are usually too hard to evaluate. In this section we tie the two pictures together using the CFT approach. Following Bergholtz *et al.* (2008) and Suorsa, Viefers, and Hansson (2011a, 2011b), we first present explicit vertex operators whose correlators produce trial wave functions for any Abelian hierarchy state. These are consistent with Wen’s topological classification and reduce to Jain’s wave functions whenever these exist. We then proceed to explain how to construct excited states with quasielectrons and quasiholes. Finally we show that these wave functions do form a hierarchy in that they can all be written in the Halperin form (5), or in other words they can all be obtained by successive condensations of quasiparticles. For readability, we often omit the background charge operator (40) from the correlators in this section.

A. The chiral hierarchy

For simplicity, we first consider states with a positive semidefinite K matrix which describe fully chiral states, where all the modes of the minimal, i.e., unreconstructed, edge move in the same direction. These states form a subhierarchy within the full hierarchy that will be considered in Sec. V.B.

1. The need for many electron operators

In order to describe a state at level n in the hierarchy, we use n different electron operators (Read, 1990; Wen, 1995); see also Sec. II.F. In the edge theory, these are CFT vertex operators. A natural assumption, consistent with the Moore-Read conjecture, is that the bulk should be described by the same CFT [cf. Sec. II.A in Suorsa, Viefers, and Hansson (2011b)]. Given this, it is natural that the description of the bulk states involves one electron (vertex) operator for each level of the hierarchy.

The n electron operators will differ by their conformal spin, which has a natural interpretation as an orbital spin (cf. Secs. II.F, II.H, and VII). This however does not mean that the electrons come in different types. They are indistinguishable, and the electronic wave functions are fully antisymmetric as required by the Pauli principle. For heuristic purposes, it is, however, sometimes useful to think in terms of distinct types of electrons. For instance, we will show that in all cases where composite-fermion wave functions exist, the CFT approach leads to identically the same wave functions. In these cases, the existence of n different “kinds” of electrons finds its natural counterpart in the CF language, where the different electron operators correspond to composite fermions in different effective “ Λ levels.”

2. Chiral ground states: Wave functions from topological data

We now present the general form of ground state wave functions for chiral states. These are hierarchy states resulting from quasielectron condensation only. They can be expressed as holomorphic correlators of local chiral operators, and their edge states all have the same chirality. These wave functions can be derived via an explicit hierarchical condensation procedure, discussed in Sec. V.D. Here we directly present the upshot—a set of vertex operators whose correlators give wave functions that are consistent with Wen’s topological classification (Bergholtz *et al.*, 2008; Suorsa, Viefers, and Hansson, 2011a).

The vertex operators at level n of the hierarchy closely resemble those of the multicomponent states,

$$V_\alpha(z) = : \partial_z^{\alpha-1} e^{i \sum_\beta Q_{\alpha\beta} \phi^\beta(z)} : \quad \alpha = 1, 2, \dots, n, \quad (51)$$

where we recall from Eq. (38) that $\mathbf{K} = \mathbf{Q}\mathbf{Q}^T$. V_α has conformal spin $s_\alpha = (1/2)K_{\alpha\alpha} + \alpha - 1$, where the two last terms come from the derivatives, and the first term is just the conformal dimension of the corresponding primary field. These are exactly the entries in the hierarchy spin vector (21), which should be interpreted as the orbital spins of the electrons. The derivatives appear naturally from the condensation construction presented later and are necessary for the resulting wave functions not to vanish under antisymmetrization. In our scheme we divide the electrons into n groups I_1, \dots, I_n , assigning a different electron operator V_α to each of the groups, and then sum over all ways of choosing the groups. The ground state wave functions become

$$\begin{aligned} \Psi &= \mathcal{A} \left\langle \prod_{\alpha=1}^n \prod_{i_\alpha \in I_\alpha} V_\alpha(z_{i_\alpha}) \mathcal{O}_{bg} \right\rangle \\ &= \mathcal{A} \{ (1-1)^{K_{11}} \partial_2 (2-2)^{K_{22}} \dots \partial_n^{n-1} (n-n)^{K_{nn}} \\ &\quad \times (1-2)^{K_{12}} (1-3)^{K_{13}} \dots ((n-1)-n)^{K_{n-1,n}} \} \\ &\quad \times e^{-\sum_i |z_i|^2 / (4\ell^2)}, \end{aligned} \quad (52)$$

where \mathcal{A} is the antisymmetrization operator. The derivatives act on the Jastrow factors, but not on the Gaussian, and we use the shorthand notation,

$$\begin{aligned} \partial_\alpha^k &\equiv \prod_{i_\alpha \in I_\alpha} \partial_{z_{i_\alpha}}^k, \\ (\alpha - \alpha)^{K_{\alpha\alpha}} &\equiv \prod_{i < j \in I_\alpha} (z_i - z_j)^{K_{\alpha\alpha}}, \\ (\alpha - \beta)^{K_{\alpha\beta}} &\equiv \prod_{i_\alpha \in I_\alpha} \prod_{i_\beta \in I_\beta} (z_{i_\alpha} - z_{i_\beta})^{K_{\alpha\beta}}. \end{aligned} \quad (53)$$

The groups I_α have the same partial filling fractions $\nu_\alpha = K_{\alpha\beta}^{-1} t_\beta$, as the multicomponent states in Sec. IV.B, and the neutralizing background (40) is also the same. The size of each of the groups is determined by requiring homogeneity. For details about how to impose the homogeneity condition, see Bergholtz *et al.* (2008). The vertex operators in Eq. (51) are by no means unique. In principle, one needs to consider

more general expressions, where the derivatives act only on parts of the vertex operator, such as

$$:e^{ia\phi^a} \partial^n e^{ib\phi^b} :. \quad (54)$$

The freedom in placing the derivatives reflects the various possibilities to regularize operator products between electron and quasiparticle operators; see, e.g., [Hansson, Hermanns, and Viefers \(2009\)](#) and [Suorsa, Viefers, and Hansson \(2011a, 2011b\)](#). Numerical studies indicate that on the plane the precise placement of the derivatives does not matter, i.e., it does not change any of the topological features of the state. This is widely used in numerical computations in order to find alternative descriptions of the state that can be implemented more efficiently on a computer ([Jain and Kamilla, 1997a, 1997b](#); [Möller and Simon, 2005](#); [Bonderson and Slingerland, 2008](#); [Rodríguez *et al.*, 2012](#)), even though there is no rigorous proof of the equivalence of these various alternatives as of yet. However, not all choices are allowed in Eq. (54), as some would give singular wave functions by having too many derivatives acting on a Jastrow factor to a broken power like $(z_i - z_j)^{p/q}$. On the plane, a simple prescription that avoids this is to always move all derivatives to the left so that they act on fully holomorphic functions. For instance, the operators (51) as well as \mathcal{P} and \mathcal{H} in Sec. V.C are defined such as to give this prescription when used in a hierarchy construction. However, it was noted by [Kvorning \(2013\)](#) that it does not give correct vertex operators for the spherical geometry, and one is forced to use the more general expression (54) in order to obtain wave functions that are nonvanishing; see Sec. VII.A.

By evaluating the wave functions (52) on a thin cylinder, [Bergholtz *et al.* \(2008\)](#) exactly reproduced the same charge density wave patterns as those found by [Bergholtz and Karlhede](#) in the TT limit (cf. Sec. II.J). It is both reassuring and encouraging that the CFT wave functions precisely reproduce the exact ground states found in a nontrivial model of interacting electrons and constitutes an important consistency check of the former.

To make this discussion less abstract, let us consider the example of the quantum Hall state at filling $\nu = 2/5$ and factorize \mathbf{K} as in the first alternative in Eq. (42),

$$\mathbf{K} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 \\ \sqrt{4/3} & \sqrt{5/3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \sqrt{4/3} \\ 0 & \sqrt{5/3} \end{pmatrix}.$$

Equation (51) then gives the two vertex operators,

$$V_1 = :e^{i\sqrt{3}\phi_1} :., \quad V_2 = :e^{i(\sqrt{4/3}\phi_1 + \sqrt{5/3}\phi_2)} :.$$

Homogeneity requires the two sets I_1 and I_2 to be of equal size, i.e., $N/2$, where N is the number of electrons, and the wave function becomes

$$\Psi_{2/5} = \mathcal{A}\{(1-1)^3 \partial_2 (2-2)^3 (1-2)^2\} e^{-\sum_i |z_i|^2 / (4\ell^2)}. \quad (55)$$

This wave function is identical to the one obtained in Eq. (71) by condensing quasielectrons on top of the

Laughlin $1/3$ state, and in fact also identical to the $\nu = 2/5$ Jain wave function.

At level n of the hierarchy there are n elementary quasiholes,

$$H_\alpha(\eta) = :e^{i\sum_\beta Q_{\beta\alpha}^{-1} \phi^\beta(\eta)} :.,$$

where \mathbf{Q}^{-1} is the inverse of the matrix \mathbf{Q} in the electron operators (51).⁵⁰ In Wen's language, a generic quasihole is some composite of these, as encoded in the \mathbf{I} vector defined in Sec. II.F. For example, a fundamental hole created by the operator H_α corresponds to $l_\alpha^q = \delta_{q\alpha}$. A Laughlin hole, on the other hand, amounts to a unit vortex in all condensates and is created by the operator $H_L = \prod_\alpha H_\alpha$ and represented by the \mathbf{I} vector $(1,1,1,\dots,1)$. The simple connection between the K matrix and the hierarchy wave functions is manifest only in the "symmetric basis" where the \mathbf{t} vector equals $(1,1,\dots,1)$ (see Sec. II.F). The statistics of any quasihole follows from Eq. (18).

The conformal spin of the operator H_α is calculated as $2s_\alpha = \sum_\beta Q_{\beta\alpha}^{-1} Q_{\beta\alpha}^{-1} = \sum_\beta (Q^{-1})_{\alpha\beta}^T Q_{\beta\alpha}^{-1} = K_{\alpha\alpha}^{-1}$ in accordance with Eq. (21), if we again identify the conformal spin with the orbital spin in the Wen-Zee formalism. The holes obtained using H_α do not in general have the same charge, but there is always one with the minimal charge e/q for a state with $\nu = p/q$. We believe that at level n in the hierarchy one can always find n distinct hole operators with the same charge and that only differ by integer units of conformal spin.⁵¹ Since the n holes are topologically the same, it is fair to ask if all are needed. The tentative answer is yes, but is based on arguments that go beyond topology. Just as the n electron operators are needed in order to construct a ground state wave function at level n in the hierarchy, we presumably need all n quasihole operators to describe a general quasihole wave function. This belief is based on calculations of the particle entanglement spectrum and is briefly outlined in Sec. II.G.4.

While this description of quasiholes was rather straightforward, the situation is more complicated for quasielectrons. A naive approach leads to singular wave functions and there is no local operator that creates acceptable quasielectron wave functions. A heuristic way to understand what goes wrong was given in Sec. IV.E. In Sec. V.C we explain how this problem can be circumvented ([Hansson *et al.*, 2009](#); [Hansson, Hermanns, and Viefers, 2009](#)) and discuss in some detail how to construct operators for quasielectrons.

B. Ground states of the full Abelian hierarchy

Having positive definite K matrices is a special case. For general hierarchy states, containing quasihole condensates and, correspondingly, antichiral edge modes, a description in terms of purely holomorphic conformal blocks is not possible. [Suorsa, Viefers, and Hansson \(2011a, 2011b\)](#) generalized the

⁵⁰If \mathbf{Q} is not a square matrix, one instead chooses \mathbf{Q}^{-1} as the Moore-Penrose pseudoinverse, which (as n is the rank of \mathbf{K}) fulfills $\sum_\beta Q_{\alpha\beta} Q_{\beta\gamma}^{-1} = \delta_{\alpha\gamma}$.

⁵¹We have not tried to prove this in general, but we have tested several nontrivial cases of K matrices of rank two and three.

construction in the previous section to general hierarchy states. The idea is to split the K matrix as

$$\mathbf{K} = \kappa - \bar{\kappa}, \quad (56)$$

where both κ and $\bar{\kappa}$ are positive semidefinite and are associated with the chiral and antichiral sectors, respectively. This decomposition is in general not unique, but all such choices are believed to represent the same topological phase. In particular, filling fraction, quasiparticle charges, and statistics depend only on the full K matrix; cf. Sec. II.F. There might be ways to restrict the freedom in splitting the K matrix, for instance by demanding that the resulting wave function should have the correct TT limit just as the chiral states discussed earlier. The freedom of decomposing the K matrix will be illustrated in later examples.

Given a splitting of the K matrix, we parallel the decomposition (38) in Sec. IV.B and write $\kappa = qq^T$ and $\bar{\kappa} = \bar{q}\bar{q}^T$.⁵² With this notation, the background operator (40) generalizes to

$$\mathcal{O}_{bg} = \exp\left(-i \int d^2r \rho(\vec{r}) n_\alpha (q_{\alpha\beta} \phi^\beta + \bar{q}_{\alpha\beta} \bar{\phi}^\beta)\right), \quad (57)$$

and the generalized version of the vertex operators (51) is

$$V_\alpha = : \partial_{z^\alpha}^{\sigma_\alpha} \partial_{\bar{z}^\alpha}^{\bar{\sigma}_\alpha} e^{i \sum_\beta q_{\alpha\beta} \phi^\beta} e^{i \sum_\beta \bar{q}_{\alpha\beta} \bar{\phi}^\beta} :, \quad (58)$$

where the powers of derivatives are related to the spin vector $s_\alpha = (1/2)K_{\alpha\alpha} + \sigma_\alpha - \bar{\sigma}_\alpha$. In a minimal hierarchy construction we get a single derivative at each level, so $\sigma + \bar{\sigma} = \alpha - 1$. By adding extra derivatives, it can be generalized to states with a shift (15) that differs from the minimal one obtained by the condensation procedure described in Sec. V.D, but this has so far no application to experimentally or numerically observed quantum Hall states.

The properly antisymmetrized correlators of products of these operators give the functions

$$\begin{aligned} \Psi(\{\xi_i\}, \{\bar{\xi}_i\}) &= \mathcal{A} \left\{ \prod_\alpha \partial_\alpha^{\sigma_\alpha} \partial_{\bar{\alpha}}^{\bar{\sigma}_\alpha} (\alpha - \alpha)^{\kappa_{\alpha\alpha}} (\bar{\alpha} - \bar{\alpha})^{\bar{\kappa}_{\alpha\alpha}} \right. \\ &\quad \left. \times \prod_{\alpha < \beta} (\alpha - \beta)^{\kappa_{\alpha\beta}} (\bar{\alpha} - \bar{\beta})^{\bar{\kappa}_{\alpha\beta}} \right\} e^{-\sum_i |\xi_i|^2 / (4\ell^2)} \\ &= \Psi_\kappa(\{\xi_i\}) \times \Psi_{\bar{\kappa}}(\{\bar{\xi}_i\}), \end{aligned} \quad (59)$$

where we used the same shorthand notation as in Eq. (53), except that the z 's are replaced by ξ 's and the bar denotes complex conjugation. They nicely factorize into a chiral and antichiral part (except the Gaussian, which is distributed equally between these two parts). Clearly the expressions in Eq. (59) are not valid (holomorphic) LLL wave functions, but they can be interpreted as wave functions in a coherent state representation, i.e., the ξ 's are considered as coordinates for the guiding centers of the electrons. The coherent state wave functions are related to LLL wave functions by the transformation

$$\Psi(\{z_i\}) = \int [d^2\xi_i] \langle \{z_i\} | \{\xi_i\} \rangle \Psi_\kappa(\{\xi_i\}) \Psi_{\bar{\kappa}}(\{\bar{\xi}_i\}). \quad (60)$$

Here $\langle \{z_i\} | \{\xi_i\} \rangle = \prod_{i=1}^N \langle z_i | \xi_i \rangle$, with $\langle z | \xi \rangle = \exp[-(|\xi|^2 - 2\bar{\xi}z + |z|^2)/4\ell^2]$ is the coherent state kernel, describing N particles maximally localized to the points (ξ_1, \dots, ξ_N) ; up to a phase, $\langle z | \xi \rangle = \exp(-|z - \xi|^2/4\ell^2)$. Technically, this transformation can also be interpreted as a LLL projection, as the coherent state kernel and the Gaussian from Ψ combine to a LLL delta function,

$$\delta_{\text{LLL}} = e^{-(|\xi|^2 - \bar{\xi}z)/(2\ell^2)}. \quad (61)$$

A third interpretation is that the wave function $\Psi(z_1, z_2, \dots, z_N)$ can be viewed as a CFT correlator of quasiloc electron operators

$$\mathcal{V}(z) = \int \frac{d^2\xi}{2\pi} e^{-(|\xi|^2 - 2\bar{\xi}z + |z|^2)/(4\ell^2)} \mathcal{V}(\xi, \bar{\xi}). \quad (62)$$

This latter view is appealing in that it displays the inherent ‘‘fuzziness’’ of particles in the LLL, i.e., they cannot be localized more precisely than the magnetic length. The concept of quasilocality will be central when we discuss how to explicitly build a hierarchy of quasielectron and quasihole condensates.

The ‘‘sewing together’’ of the chiral and antichiral sectors amounts to a two-fluid picture in which the chiral component consists of the bare electrons and all quasielectron condensates, and the antichiral sector consists of the hole condensates. The actual particles are built as composites of the positively and negatively charged parts, with their coordinates identified (Suorsa, Viefers, and Hansson, 2011b).

With this background, we now propose a generalized version of the Moore-Read conjecture.

Representative wave functions for ground states of hierarchical quantum Hall phases, as well as their quasiparticle excitations, can be constructed from antisymmetrized products of chiral and antichiral conformal blocks of primary and descendant fields in a CFT, in the basis of coherent states $|\xi_1, \dots, \xi_N\rangle$. These wave functions obey generalized screening and have minimal edge theories given by the same CFTs that describe the bulk wave functions.

The assumption about generalized screening, which is equivalent to postulating that holonomy equals monodromy, is discussed further in Sec. V.E. So far we considered only Abelian phases and presented a concrete construction that yields explicit wave functions for the full hierarchy. In Sec. VI we generalize to non-Abelian states some of which again have explicit wave functions, while others can be expressed only in terms of integrals over quasiparticle coordinates.

1. Examples

The coherent state formulation and the freedom in decomposing the K matrix can be illustrated already at the level of

⁵²Recall that q and \bar{q} are matrices.

the Laughlin wave function. The K matrix for the standard $\nu = 1/m$ state is simply $\mathbf{K} = m$. However, one can imagine a more general form

$$\mathbf{K} = \kappa - \bar{\kappa} = (m + k) - k.$$

This corresponds to the modified Laughlin wave functions introduced by Girvin and Jach (1984),

$$\begin{aligned} \Psi(\{z_i\}) &= \int [d^2\xi_i] \langle \{z_i\} | \{ \xi_i \} \rangle \prod_{\alpha < \beta} (\bar{\xi}_\alpha - \bar{\xi}_\beta)^k \\ &\times \prod_{\alpha < \beta} (\xi_\alpha - \xi_\beta)^{k+m} e^{-\sum_i |\xi_i|^2 / (4\ell^2)} \\ &= \int \prod d^2\xi_i \delta_{\text{LLL}}(z_i, \xi_i) \prod_{\alpha < \beta} |\bar{\xi}_\alpha - \bar{\xi}_\beta|^{2k} \\ &\times \prod_{\alpha < \beta} (\xi_\alpha - \xi_\beta)^m e^{-\sum_i |\xi_i|^2 / (4\ell^2)}. \end{aligned} \quad (63)$$

The second line emphasizes that this can be considered as the LLL projection of the Laughlin wave function with additional correlation factors $\prod_{\alpha < \beta} |\bar{\xi}_\alpha - \bar{\xi}_\beta|^{2k}$. For integer k , the explicit form of the LLL projected wave function is

$$\Psi(\{z_i\}) = \prod_{i < j} (\partial_i - \partial_j)^k \prod_{i < j} (z_i - z_j)^{k+m} e^{-\sum_i |z_i|^2 / (4\ell^2)}.$$

The pertinent vertex operator for obtaining this generalized Laughlin wave function as a CFT correlator is

$$V_{(m,k)}(\xi, \bar{\xi}) = : e^{i\sqrt{m+k}\phi(\xi)} e^{i\sqrt{k}\bar{\phi}(\bar{\xi})} :.$$

Girvin and Jach proposed Eq. (63) as a particular scheme to make Laughlin's wave function variational with k as the variational parameter. The additional correlation factor in Eq. (63) or more generically a function $f(\prod_{\alpha < \beta} |\bar{\xi}_\alpha - \bar{\xi}_\beta|^{2k})$ can be used to change the short-distance behavior of the wave function, and thus lower the ground state energy, without affecting the topological properties (Fremling *et al.*, 2016).

One might wonder whether such a nonminimal description of the Laughlin states would have implications for the edge physics. To our knowledge this question has not been investigated numerically, but the simplest guess is that the edge still just supports a single, charged mode. To see this, we use a nonminimal vertex operator

$$\tilde{V}_{(m,k)}(\xi, \bar{\xi}) = : e^{i\sqrt{m}\phi(\xi)} e^{i\sqrt{k}\phi_1(\xi)} e^{i\sqrt{k}\bar{\phi}_1(\bar{\xi})} :,$$

where ϕ is the same field as in the original description of the Laughlin state; cf. Eq. (35). The corresponding edge Lagrangian contains kinetic terms for the left-moving charged field ϕ as well as for the neutral right- and left-moving fields ϕ_R and ϕ_L . However, it generically also contains a term $\sim \cos(\phi_R - \phi_L)$ which opens a gap in the neutral sector, while leaving the edge unchanged, up to microscopic details.

As a second example, consider $\nu = 2/3$. It can be viewed as the particle-hole conjugate of the $1/3$ Laughlin state or alternatively as a hole condensate in the $\nu = 1$ quantum

Hall state. In the language of composite fermions it is described as a reverse flux attachment CF state. Its K matrix is

$$\mathbf{K} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

There is no consistent purely chiral description of this state, since \mathbf{K} has a negative eigenvalue, corresponding to a counterpropagating edge mode. This can also be seen at the level of wave functions, in that the naive (112) state has a very poor overlap with the exact Coulomb state, and it does not have the correct thin torus limit (Bergholtz *et al.*, 2008; Suorsa, Viefers, and Hansson, 2011a, 2011b). So one has to introduce an antichiral component. Again there are many possible ways of splitting \mathbf{K} . In particular, all integer-valued splittings

$$\mathbf{K} = \kappa - \bar{\kappa} = \begin{pmatrix} 1+k & 2+l \\ 2+l & 1+k \end{pmatrix} - \begin{pmatrix} k & l \\ l & k \end{pmatrix}$$

are ‘‘allowed’’ in the sense that κ and $\bar{\kappa}$ are both positive (semi) definite, provided $k \geq l + 1$ ($k, l \geq 0$). The special case ($l = 0, k = 1$) reproduces Jain's $\nu = 2/3$ reverse flux attachment wave function (Wu, Dev, and Jain, 1993),

$$\begin{aligned} \psi_{2/3}^{\text{Jain}} &= (\bar{1} - \bar{1})\bar{\partial}_2(\bar{2} - \bar{2})(1 - 1)^2(2 - 2)^2(1 - 2)^2 \\ &\times e^{-\sum_j |z_j|^2 / 4\ell^2}. \end{aligned}$$

This wave function has been tested numerically (in spherical geometry) and is very close to the exact Coulomb ground state (Wu, Dev, and Jain, 1993). It would be interesting to perform a more systematic numerical comparison of states with different splittings of the K matrix. Again note that the derivatives do not follow from the form of the K matrix itself, but rather from the explicit condensation procedure explained later, and they are needed in order to get the correct shift.

C. Local and quasilocaI quasiparticle operators

Our ‘‘recipes’’ for writing down representative ground state wave functions for general Abelian quantum Hall ground states were stated in Secs. V.A and V.B, without further explanation. In this section we discuss the CFT description of fractionally charged excitations, in particular, quasielectrons.

Recall Laughlin's wave function for a quasihole in a $\nu = 1/m$ quantum Hall state, localized at η ,

$$\psi_{Lqh}(\eta, z_1, \dots, z_N) = \prod_{i=1}^N (z_i - \eta) \psi_L(z_1, \dots, z_N)$$

and correspondingly for quasielectrons,

$$\psi_{Lqe}(\bar{\eta}, z_1, \dots, z_N) = \prod_{i=1}^N (2\partial_i - \bar{\eta}) \psi_L(z_1, \dots, z_N). \quad (64)$$

While the former agrees with the composite-fermion quasi-hole, the CF quasielectron differs quite substantially from Laughlin's (Jain, 2007). The CF expression for a quasielectron localized at η is

$$\begin{aligned} \psi_{CFqe}(\bar{\eta}; \{z_i\}) &= \sum_i (-1)^i e^{-q_h(|\bar{\eta}|^2 - 2\bar{\eta}z_i)/(4\ell^2)} \\ &\times \prod_{j<k, j, k \neq i} (z_j - z_k)^m \partial_i \prod_{l \neq i} (z_l - z_i)^{m-1} e^{-\sum_j |z_j|^2/(4\ell^2)}. \end{aligned} \quad (65)$$

Numerical studies (Kjønsgberg and Leinaas, 1999; Jeon, Graham, and Jain, 2003) showed that the CF quasielectrons do better than Laughlin's in that the former exhibits well-defined fractional statistics for braiding of two well-separated quasielectrons, while this is not the case for the latter. This issue was revisited by Jeon and Jain (2010), who found that Laughlin's quasielectron, too, has the correct braiding statistics as long as the ratio $A_{\text{path}}/A \rightarrow 0$, where A_{path} is the area enclosed by the braiding path and A is the total area of the quantum Hall droplet. However, if A_{path} encloses a finite fraction of the droplet, then the braiding phase of Laughlin's quasielectron becomes ill defined, due to the $\mathcal{O}(1/N)$ corrections to the Aharonov-Bohm phase found by Kjønsgberg and Myrheim (1999).

There is an inherent asymmetry between the traditional descriptions of quasiholes and quasielectrons that is also reflected in the CFT formulation. When writing wave functions as CFT correlators, the expression for a Laughlin quasihole at η is reproduced by inserting a local hole operator $H(\eta) = \exp[i\phi(\eta)/\sqrt{m}]$. However, the naive guess for a quasielectron using the inverse hole operator $H^{-1}(\eta)$ does not work, as this leads to unwanted singularities of the form $\prod_j (z_j - \eta)^{-1}$ in the wave function. The physical reason behind this was discussed in Sec. IV.E, where it was also pointed out that the problem can be solved by noting that since every electron in the liquid is surrounded by a correlation hole (m fold in the $\nu = 1/m$ state), it is "safe" to contract the liquid by contracting the correlation hole around any electron. Formally, this is obtained by modifying one of the electron operators; in the simplest case of a $1/m$ Laughlin state one introduces

$$P(z) = \partial_z : e^{i(\sqrt{m}-1/\sqrt{m})\phi(z)} :. \quad (66)$$

This is essentially an inverse quasihole fused with an electron, using a particular regularization scheme to avoid the singularities mentioned previously, while retaining the correct fractional charge and anyonic statistics of the inverse hole. Replacing one of the electron operators $V(z)$ in the ground state correlator by $P(z)$ and antisymmetrizing over particle coordinates exactly reproduces the expression for a CF quasielectron at the center. It is then possible to, by hand, write an expression for a localized quasielectron as a coherent superposition of such a quasielectron in different angular momentum states. However, this is not fully satisfactory as one would wish for one single operator $\mathcal{P}(\bar{\eta})$ which, when inserted into the ground state correlator, directly produces such

a localized quasielectron state—in analogy to the quasihole operator $H(\eta)$. Ideally, such an operator should make the anyonic properties, fractional charge and anyonic braiding statistics, manifest. It turns out that requiring manifest braiding statistics, i.e., an anyonic monodromy factor and zero Berry's phase as discussed in Sec. IV.C.5, is quite nontrivial and has so far not been achieved. It is, however, possible to instead use bosonic or fermionic representations of the quasiparticles. The monodromies are then bosonic or fermionic, and the statistics is hidden in the Berry phase. Before presenting this operator, let us shortly explain why obtaining explicit anyonic monodromies is problematic for quasielectrons.

The root of the problem can already be seen from Eq. (66); trying to use it to create multi-quasielectron states results in factors $\sim (z_i - z_j)^{4/3}$ for each pair of P 's. This is unacceptable since the wave function, although being a many-quasielectron state, describes a fermionic many-body state in the LLL and thus must be analytic and antisymmetric. To remedy this problem, one introduces (chargeless) auxiliary fields that make the braiding of the quasielectrons trivial (either bosonic or fermionic), while leaving their braiding properties *vis-à-vis* the electrons and quasiholes unchanged. For instance, for a Laughlin state at filling $1/m$ one can use

$$P(z) = \partial_z : e^{i[(m-1)/\sqrt{m}]\phi(z) + i\sqrt{(2m-1)/m}\chi(z)} : \quad (67)$$

instead of the operator (66), which amounts to fusing a bosonized inverse quasihole,

$$H_b^{-1} = : e^{-i\sqrt{1/m}\phi + i\sqrt{(2m-1)/m}\chi} :,$$

with an electron operator. Even though this operator has fermionic monodromy, it can be used to create proper many-quasielectron states that exhibit the expected anyonic braiding phase.

To get a better feel for this, let us briefly recall the case of quasiholes. In their original form $: e^{i\phi/\sqrt{m}} :$, the quasihole operator produces factors $\sim (\eta_\alpha - \eta_\beta)^{1/m}$ for any pair of quasiholes. In this representation, the holonomy equals the monodromy, i.e., the fractional braiding phase is fully displayed in the many-hole wave function, as explained in Sec. IV.C. Such a representation, commonly believed to be connected to the "minimal" CFT description, is not the only viable choice. Alternatively, we can change the monodromy of H to bosonic ones, by introducing auxiliary bosonic fields as shown. This does not, of course, change the physics, but amounts to "shuffling" the fractional statistics from the monodromy into the Berry phase. For quasiholes, we can therefore freely change the monodromy factor, while for quasielectrons we are limited to either bosonic or fermionic monodromy in order to avoid branch cuts in the electron coordinates. In this latter case the different choices will not just be a matter of normalization, but in fact correspond to wave functions that differ by factors $(z_i - z_j)^n$, where the coordinates are those of the quasielectrons. Although such factors are important when the quasielectrons come close to each other, they do not influence the rest of the electrons. They are also negligible when the quasiparticles are far apart and thus do not change the topological properties.

Here we also recall two other sources of short-distance ambiguities related to the CFT wave functions that were mentioned earlier. The first occurred in the first example in Sec. V.B and amounts to introducing Jastrow factors such as $\prod_{i,j} |\xi_i - \xi_j|^k$ or even more general functions of the distance between the particles. Since such factors do not introduce any angular momentum, they will not influence the density of the liquid and most likely⁵³ neither will they influence any other topological property. The second source of short-distance ambiguities was discussed in Sec. V.B and relates to the particular ordering of the derivatives that are present in the LLL wave functions. We stress that this, rather large, freedom in changing the short-distance details of the wave functions without altering the topological properties is in fact a virtue. It might be used in a variational sense to find good wave functions with specified topological properties for realistic Hamiltonians. In this context we also recall that within the composite-fermion picture, similar short-distance ambiguities can occur. A prominent example is the approximation to the LLL projection that was devised by Jain and Kamilla (1997b) and shows good agreement with the exact projection.

1. Quasilocal quasiparticle operators

Let us now return to the question of how to actually write down an operator $\mathcal{P}(\bar{\eta})$ that directly creates a quasielectron at an arbitrary position $\bar{\eta}$. The idea is that the surplus of charge is generated by contracting the correlation holes of electrons in the vicinity of $\bar{\eta}$. This localization is accomplished by means of a Gaussian weight which localizes the quasielectrons on the scale of the magnetic length, thus the term quasilocal. The quasielectron operator has the same charge and conformal dimensions as H^{-1} . As indicated earlier, it involves “fusing” an inverse quasihole with an electron in an appropriate way. In order to avoid the branch cuts that rendered Eq. (66) unacceptable, we need to use a quasihole operator with a trivial—either fermionic or bosonic—monodromy factor. The two choices differ in their short-distance properties, i.e., how exactly the charge is localized. Depending on the task at hand, one or the other might be more convenient and we use both possibilities in the following discussion. In analogy with the quasihole case, a correlator of the type $\langle \mathcal{P}(\bar{\eta}) \prod_i V(z_i) \rangle$ gives a wave function for a quasielectron localized around $\bar{\eta}$. Moreover, multiquasielectron states can be generated by multiple insertions of $\mathcal{P}(\bar{\eta})$.

To be more concrete, the operator for a quasielectron with charge $-q_h e$ is given by⁵⁴

$$\mathcal{P}(\bar{\eta}) = \int d^2 w e^{-q_h/(4\ell^2)(|w|^2 + |\eta|^2 - 2\bar{\eta}w)} (H^{-1} \bar{\partial} J)_n(w). \quad (68)$$

Note the similarity to the quasilocal electron operators (62) discussed in Sec. V.B. The exponential factor indeed localizes the quasielectron around the position $\bar{\eta}$ as it equals the

Gaussian $\exp[(-q_h/4\ell^2)|w - \eta|^2]$ up to a pure phase. Note that this exponential factor is fully determined by this condition together with the requirement that the resulting Gaussian factors for the electrons should come out correctly. This is also equivalent to demanding that the Gaussian factor in $\bar{\eta}$ in Eq. (68) is precisely the one expected for a charge q_h particle. $J(w) = i\partial\phi(w)/\sqrt{m}$ is the holomorphic $U(1)$ current related to the electric charge. Its divergence $\bar{\partial}J(w)$ vanishes except at the position of the charges. Thus, it can effectively be replaced by a sum of δ functions centered at the electron positions (Hansson, Hermanns, and Viefers, 2009). $(\dots)_n$ denotes a generalized normal ordering, which we return to later. Its purpose is to regularize the fusion of the inverse hole with the electron operators.

An appealing property of the quasielectron operator (68) is its general applicability. It is not limited to Laughlin states, but can be used for any Abelian, or even non-Abelian, quantum Hall state, as long as one makes sure to use the appropriate choice for the inverse quasihole. For hierarchy states at level n , only one of the n quasiholes gives a nonvanishing result when inserted into Eq. (68). Thus, for any level there is only a single quasielectron, in accordance with findings by Jain (2007) for the CF series. The example of non-Abelian quasielectrons in the Moore-Read state is discussed in more detail in Sec. VI. Finally, we stress that in spite of the quasilocal nature of the operator \mathcal{P} , the resulting density profile of the corresponding quasielectron is only marginally larger than that of the quasiholes in the same state [see for instance Chapter 8.5 in Jain (2007)].

2. A case study: Quasielectrons in the $\nu = 1/3$ Laughlin state

To illustrate what happens when computing correlators of the type $\langle \mathcal{P}(\bar{\eta}) \prod_i V(z_i) \rangle$, we consider the simplest case, quasielectrons in the $\nu = 1/3$ Laughlin state, so that $V = : \exp(i\sqrt{3}\phi_1) :$. Without changing the topological properties, we can use a bosonized inverse quasihole,

$$H_b^{-1} = : e^{-i(1/\sqrt{3})\phi_1 + i\sqrt{5/3}\phi_2} :.$$

As mentioned, current conservation implies that $\bar{\partial}J(w)$ has support only at positions of charges. This leads to delta functions $\sum_i \delta^2(w - z_i)$, where z_i are the electron positions, and thus

$$\begin{aligned} \left\langle \mathcal{P}(\bar{\eta}) \prod_i V(z_i) \right\rangle &= \left\langle \int d^2 w e^{-q_h/(4\ell^2)(|w|^2 + |\eta|^2 - 2\bar{\eta}w)} \right. \\ &\quad \times (H_b^{-1} \bar{\partial} J)_n(w) \prod_i V(z_i) \left. \right\rangle \\ &= \sum_k (-1)^k e^{-q_h/(4\ell^2)(|z_k|^2 + |\eta|^2 - 2\bar{\eta}z_k)} \\ &\quad \times \left\langle (H_b^{-1} V)_n(z_k) \prod_{i \neq k} V(z_i) \right\rangle, \end{aligned}$$

where $q_h = 1/3$ denotes the quasielectron charge. In the simplest case of Laughlin states, the normal ordering is the conventional one,

⁵³In spite of this reservation, we strongly believe in this statement.

⁵⁴For simplicity we are here somewhat cavalier about the subtle issue of the background charge. See Suorsa, Viefers, and Hansson (2011a) for a precise discussion of this point.

$$\begin{aligned} (H_b^{-1}V)_n(z_k) &= \frac{1}{2\pi i} \oint \frac{dz}{z-z_k} H_b^{-1}(z)V(z_k) \\ &= \partial : e^{[i(2/\sqrt{3})\phi_1(z_k)+i\sqrt{5/3}\phi_2(z_k)]} :, \end{aligned} \quad (69)$$

where the operator product expansion of $H_b^{-1}V$ was carried out to next to leading order before integrating, thus keeping the first descendant (here just meaning first derivative) of the vertex operator. The result is nothing but the operator $P(z_k)$ in Eq. (67). Note that the derivative is a direct consequence of the normal ordering.⁵⁵ So, to summarize, one gets

$$\begin{aligned} \left\langle \mathcal{P}(\vec{\eta}) \prod_i V(z_i) \right\rangle &= \sum_k (-1)^k e^{-q_h/(4\ell^2)(|z_k|^2+|\eta|^2-2\vec{\eta}z_k)} \\ &\times \left\langle P(z_k) \prod_{i \neq k} V(z_i) \right\rangle, \end{aligned} \quad (70)$$

with the announced Gaussian-weighted sum over “shrinking the correlation hole” (i.e., turning V ’s into P ’s) at the various electron positions. This expression is identical to the one for a CF quasielectron localized at η in Eq. (65). However, the expressions for several quasielectrons do differ slightly between the CF and CFT construction (Hansson, Hermanns, and Viefers, 2009). We should also mention that for states where there are additional quasielectrons or quasiholes present, the position of the quasielectrons will in general be shifted away from their expected positions given by the arguments in the \mathcal{P} operators. This discrepancy, which was originally noticed for CF wave functions (Jeon, Graham, and Jain, 2003), can in the CFT framework be understood as a consequence of the field ϕ_2 not being properly screened. Adding appropriate screening charges will alleviate this problem (Ardonne *et al.*, 2017).

3. Quasilocal quasiholes and local quasielectrons

In fact, the description of quasielectrons and quasiholes is not as asymmetric as it seems at first sight. It is possible to reverse this logic and construct a *quasilocal quasihole* operator $\mathcal{H}(\eta)$ as a Gaussian-weighted sum of *expanding* the correlation holes around the electrons in the vicinity of some position η (Suorsa, Viefers, and Hansson, 2011a). Technically, this involves fusing the inverse of a *local* quasielectron operator [which in fact corresponds to Laughlin’s original quasielectron (64)] with a suitable representation of the electron operator. This is well defined within the coherent state formalism discussed in Sec. V.B. A single

⁵⁵For more general hierarchy states, involving several bosonic fields, one has to apply a generalized normal ordering [which coincides with Eq. (69) in the Laughlin case],

$$(AB)_n(w) \equiv \oint_w dy T(y) \oint_w dz A(z) B(w),$$

where T is the energy-momentum tensor. This normal ordering, including the definition of the integration contours, is explained by Hansson, Hermanns, and Viefers (2009). Here we just mention that the y integration amounts to taking the first descendant of the operators to its right, which again produces the pertinent derivatives.

insertion of a hole operator $\langle \mathcal{H}(\eta) \prod_i \mathcal{V}(z_i) \rangle$ produces an alternative to Laughlin’s quasihole wave function which actually is slightly better numerically than the latter (Jeon *et al.*, 2005). It is appealing that quasielectrons and quasiholes can be put on the same footing in this way. Moreover, it is this description of quasiholes, rather than local Laughlin holes, one has to resort to when carrying out an explicit hierarchy construction for condensates of quasiholes. Also note the similarity with the quasilocal electron operator in Eq. (62). This emphasizes the point made in passing at the end of Sec. II.B.1: Because of the cyclotron motion, all charged particles acquire a size in the LLL, so condensing quasiparticles into a hierarchy state might not be conceptually very different from condensing electrons into a Laughlin state.

D. CFT wave functions as hierarchy states

Hansson, Hermanns, and Viefers (2009) proved that the chiral states of Sec. V.A can be written in the Halperin form and thus form a hierarchy. The extension to the full hierarchy, including hole condensates, was given by Suorsa, Viefers, and Hansson (2011a). We give here the argument for the simplest case of the $\nu = 2/5$ state in the positive Jain series that is obtained by condensing quasielectrons as densely as possible in the $\nu = 1/3$ state. The starting point is Eq. (5)

$$\begin{aligned} \Psi_{n+1}(\vec{r}_1 \cdots \vec{r}_N) &= \int d^2\vec{\eta}_1 \cdots \int d^2\vec{\eta}_M \Phi_n^*(\vec{\eta}_1 \cdots \vec{\eta}_M) \\ &\times \Psi_n(\vec{\eta}_1 \cdots \vec{\eta}_M; \vec{r}_1 \cdots \vec{r}_N), \end{aligned}$$

describing the $\nu = 2/5$ wave function as a coherent superposition of quasielectrons in the parent Laughlin state Ψ_1 at filling $\nu = 1/3$. For this densest possible condensate, the number of quasielectrons M equals half the number of electrons in the parent state. In general, the relation between the number of quasielectrons and electrons is determined from demanding homogeneity of the final state (Hansson, Hermanns, and Viefers, 2009). One constructs multi-quasielectron states by multiple insertions of the quasielectron operator,

$$\Psi_1(\{\eta_i\}_1^M; \{\vec{r}_i\}) = \left\langle \mathcal{P}(\vec{\eta}_1) \mathcal{P}(\vec{\eta}_2) \cdots \mathcal{P}(\vec{\eta}_M) \prod_{i=1}^N V_1(z_i) \right\rangle,$$

where $V_1 = : e^{i\sqrt{3}\phi_1} :$. Evaluation of this correlator is a direct generalization of the derivation of Eq. (70), with one set of Gaussian factors for each quasielectron coordinate. Since the final polynomial has to be antisymmetric, one has to choose a symmetric pseudo wave function for the Laughlin-like state of the quasielectrons. Its generic form is

$$\Phi^*(\vec{\eta}_1, \dots, \vec{\eta}_M) = \prod_{i < j} (\eta_i - \eta_j)^{2k} e^{-q_h/(4\ell^2) \sum_i |\eta_i|^2},$$

where in this case $q_h = 1/3$, and $k = 0$ for the densest possible condensate. The Gaussian factors in the pseudo wave function combine with those from the many-quasielectron correlator into holomorphic delta functions (61), so that the integration over the η ’s becomes trivial.

The result is, for general k ,

$$\begin{aligned} \Psi_2(\{\vec{r}_i\}) &= \mathcal{A} \left[\left\langle \prod_{k=1}^M V_2(z_k) \prod_{l=M+1}^N V_1(z_l) \right\rangle \prod_{i<j=1}^M (z_i - z_j)^{2k} \right] \\ &= (1-1)^3 (2-2)^{2k} \partial_2 (2-2)^3 (1-2)^2 e^{-\sum_j |z_j|^2 / (4\ell^2)}, \end{aligned} \quad (71)$$

with $M = N/(2 + 2k)$ and

$$V_2 = : e^{i\sqrt{2k}\phi_3} \partial e^{i2/\sqrt{3}\phi_1 + i\sqrt{5/3}\phi_2} :.$$

Note that the precise form of the vertex operator depends both on the details of the normal ordering (69) and on the choice of the quasihole operator. For instance, using the simplest *fermionic* quasihole (combined with a fermionic pseudo wave function) yields a vertex operator

$$V_2 = : e^{i\sqrt{2k+1}\phi_3} \partial e^{i\sqrt{4/3}\phi_1 + i\sqrt{2/3}\phi_2} :.$$

For $k = 0$, Eq. (71) is simply the Jain 2/5 wave function, while the less dense condensate with $k = 1$ gives a trial wave function for filling fraction $\nu = 4/11$ (Bergholtz *et al.*, 2007). This procedure naturally generalizes to the entire chiral hierarchy and gives the wave functions shown in Eq. (52). Similarly, quasihole condensation can be carried out using multiple insertions of the quasilocal quasihole operator $\mathcal{H}(\eta)$ (discussed in Sec. V.C.3) to construct the required many-quasihole states (Suorsa, Viefers, and Hansson, 2011b).

E. Topological properties and plasma analogy

We already mentioned that it, perhaps surprisingly, has turned out to be quite difficult to extract the topological content from even explicitly given Abelian hierarchy wave functions. Superficially, the problem resembles the one we discussed in Sec. IV.C.4 in the context of non-Abelian states. There we saw that correlators describing states with many quasiparticles are not uniquely defined, but depend on the fusion channels of the quasiparticles, and are generally to be thought of as a vector in the space of conformal blocks. For the Abelian hierarchy wave functions there is no such degeneracy, but instead the wave functions are given by sums of conformal blocks; these sums are needed in order to satisfy the Pauli principle. In neither of the cases, there is a simple plasma analogy as in the case of the Laughlin states, or the multicomponent states discussed in Sec. IV.B. An additional complication in the hierarchy case is that the operators are not all primary fields, but descendants involving derivatives. At this point it is important to stress that although a hierarchy state is an antisymmetrized (in the z_i 's) sum of correlators $\langle \prod_k \mathcal{H}(\eta_k) \prod_l \mathcal{P}(\eta_l) \prod_i \mathcal{V}(z_i) \rangle$, each term has the same monodromy. Thus it is meaningful to talk about the monodromy of a hierarchy state and assuming generalized screening as discussed in Sec. IV.C.4 we posited that holonomy equals monodromy also for these states. We now discuss the status of this conjecture.

Most attempts to understand the topology of the Abelian hierarchy states boil down to the claim that they essentially

can be described by a multicomponent Coulomb plasma. In Sec. II.B we mentioned the arguments for this given by Read (1990). These are based on the assumption that quasiparticle wave functions are orthogonal if the particle positions differ much more than a magnetic length, and while quite reasonable, at least for quasiholes, it is not so easy to prove. In the case of quasielectrons, there are extra complications due to the need to regularize singularities such as $(z_i - \bar{\eta})^{-1}$.

In Sec. IV.C.4 we discussed various attempts to find plasma mappings also for non-Abelian states. Here we mention that although the main focus in Read (2009b) is on the non-Abelian states, the analysis is also applicable to Abelian multicomponent states, and the Halperin (m, m, n) state discussed in Sec. IV.B is in fact taken as an example. Because of the need of antisymmetrization between inequivalent groups of electrons, it is far from clear that this kind of analysis would apply to the hierarchy states, but it is certainly worth further consideration. Bonderson, Gurarie, and Nayak (2011) also considered hierarchy states, but again it is not clear that their analysis properly takes into account the need for antisymmetrization.

To appreciate the problem, consider the simplest possible case—the normalization integral for the $\nu = 2/5$ ground state wave function (55)

$$\Psi = \mathcal{A} \{ (1-1)^3 \partial_2 (2-2)^3 (1-2)^2 \} e^{-\sum_i |z_i|^2 / (4\ell^2)}.$$

For a $2N$ -particle state, the two groups each contain N particles, and recalling that the function is fully antisymmetric within the groups, the sum due to the antisymmetrizer \mathcal{A} contains $(2N)!/(N!)^2$ distinct terms. Up to an overall constant, the normalization integral is

$$\sum_P \int \prod d^2 z_i \Psi^*(z_1 \cdots z_N) \Psi(z_{P(1)} \cdots z_{P(N)}), \quad (72)$$

where P denotes a permutation of the coordinates. Let us first consider those permutations that reshuffle the coordinates only within the groups. Disregarding, for the time being, the derivatives, the integrand is

$$\prod_{i<j=1}^N |z_i - z_j|^6 \prod_{i<j=N+1}^{2N} |z_i - z_j|^6 \prod_{i=1}^N \prod_{j=N+1}^{2N} |z_i - z_j|^4$$

and can directly be written as the partition function for a two-component Coulomb plasma, just as in the multicomponent case discussed earlier.

However, for general terms in the sum (72) the coordinates will not match, since a particle at z_i belonging to group 1 together with another at z_j belonging to group 2 will give the complex factor $(\bar{z}_i - \bar{z}_j)^3 (z_i - z_j)^2$. Thus, the Hamiltonian for the equivalent plasma will, in addition to the usual “electric” interaction $\sim \ln |z_i - z_j|$, contain a term $\sim i \text{Arg}(z_i - z_j)$ which is interpreted as the interaction between a charge and a vortex (Nienhuis, 1984). Thus, still disregarding the derivatives, the terms in the sum in Eq. (72) have the interpretation of a Coulomb plasma with electric charges and a number of vortices. It is rather straightforward to include quasiparticles, so the question is whether all of the terms (or at least the dominant ones) correspond to a screening plasma. Also

including the derivatives is far from trivial, but the hope is that they can be mapped onto impurities in the plasma. It remains an open question whether or not these expressions can be understood well enough to be useful in the quantum Hall context.

F. Composite fermions and the hierarchy

Recall that a strength of the hierarchy construction is that it encodes the topological properties of a hierarchy state, while these are less obvious in the CF description. On the other hand, the CF formulation gives an appealing, and phenomenologically successful, picture of the prominent states in the Jain sequences as systems of essentially free fermions, while the hierarchy picture of these states is rather complicated ($\nu = 4/9$, for example, amounts to three successive quasielectron condensates on top of a Laughlin $\nu = 1/3$ state). It is also far from clear why any of the schemes should lead to good candidates for ground states. The argument in the CF case is that the introduction of a “flux attachment” factor that keeps the electrons apart will vouch for a low interaction energy; but this is true only provided that the projection to the LLL is not too important, which is by no means intuitively obvious. In the hierarchy picture one expects low-energy states since the daughter states are built by correlating the low-energy excitations of the parent state, viz., the quasiparticles, in an energetically favorable Laughlin-like configuration. It is, however, not clear why the good correlations in the original parent liquid are not totally destroyed by the sequence of condensations.

Thus, at the level of heuristics, both pictures have their pros and cons, and the precise relation between them has been a topic of debate. Read (1990) argued that both schemes have the same topological content and thus essentially describe the same physics,⁵⁶ while Jain (1990, 2007) argued that the two schemes are distinct. This issue has been difficult to settle; while the CF approach has provided explicit wave functions that can be tested numerically to a great precision, there has been no viable numerical scheme to evaluate the hierarchy wave functions.

The results of the previous sections shed considerable light on the relation between the hierarchy and the composite-fermion states. First, it shows that Jain’s successful wave functions fit beautifully into the hierarchy scheme, which strongly suggests that they have the expected topological properties. Second, it shows that the hierarchy construction of “condensates within condensates” in the end produces wave functions which intuitively have very low interaction energy since they build in a large number of Jastrow factors in the electron coordinates, in a way that is compatible with the filling fraction and the shift. Third, and for the same reason, it provides an intuitive understanding of why the Jain states are so good in spite of the LLL projection.

VI. CFT DESCRIPTION OF NON-ABELIAN HIERARCHIES

While the physics pertinent to the LLL is well understood in terms of the Haldane-Halperin hierarchy described in the previous sections, this is not true for the second LL. The latter

is potentially very interesting, as numerical studies have shown that certain non-Abelian quantum Hall states could be stabilized. One of the underlying reasons is the nodal structure of single-particle orbitals within the second LL. This causes the matrix elements, even for purely repulsive interactions, in the second Landau level to have a “hollow core,” meaning that the effective interaction between two electrons can be attractive at certain length scales. This favors pairing or clustering of electrons, which is a common feature of non-Abelian candidate states. From numerical work, e.g., by Wójs (2001), one expects clustering to be important in the second LL, which in turn may stabilize non-Abelian quantum Hall liquids (Greiter, Wen, and Wilczek, 1991; Moore and Read, 1991; Read and Rezayi, 1999). One naturally may wonder if there is an underlying, governing principle for the fractions in the second LL, similar to the Haldane-Halperin hierarchy in the LLL. There have been several attempts to generalize the Haldane-Halperin hierarchy to non-Abelian states. The main difficulty in such an endeavor is the non-Abelian statistics of the quasiparticles. It is far from clear how to make sense of the integration over quasiparticle positions in Eq. (5), as the many-quasiparticle states form a vector space and moving quasiparticles around changes not just an overall phase, but the actual state.

In this section we review three different possible solutions to this conundrum that lead to quite different predictions as to what filling fractions can be obtained and what properties the resulting quantum Hall liquids have. For simplicity, we restrict the discussion to using the Moore-Read state as the parent state. But all the hierarchies can be applied to general non-Abelian states, in particular, the Read-Rezayi series (Read and Rezayi, 1999). Note that hierarchy states on top of the anti-Pfaffian (Lee *et al.*, 2007; Levin, Halperin, and Rosenow, 2007), the particle-hole conjugate of the Moore-Read Pfaffian state, can be obtained by simply taking the particle-hole conjugate of hierarchy states on top of the Moore-Read state. While we are of course interested in these hierarchies because of their relevance for the second LL, we phrase the wave functions (and filling fractions) within the LLL. Note that you can always map between different LLs as long as the interaction Hamiltonian is properly adjusted (see Appendix A).

A. The Bonderson-Slingerland hierarchies

The hierarchy by Bonderson and Slingerland (2008) (BS) is essentially an Abelian hierarchy on top of a non-Abelian state. The basic idea behind this hierarchy is that for a sufficiently large density of the non-Abelian quasiparticles, the interactions among them lift the degeneracy between the different fusion channels. Thus, it may be energetically favorable that pairs of quasiparticles form essentially Abelian excitations, which may then condense in the same fashion as for Abelian quantum Hall states (5). Using the CFT description of the Moore-Read state in Sec. IV.C.5, these Abelian quasihole operators can be written either as

$$\psi_{qh} = \mathbf{1} : e^{i\sqrt{1/2}\phi_1(n)} : \quad (73)$$

or as

⁵⁶The same conclusion was reached by Bergholtz *et al.* (2008) by studying the TT limit of the hierarchy and also by Bonderson (2012).

$$\psi_{qh} = \psi(\eta) : e^{i\sqrt{1/2}\phi_1(\eta)} :, \quad (74)$$

i.e., they combine the usual Laughlin quasihole of an Abelian, bosonic $\nu = 1/2$ state with the fields 1 or ψ depending on which of the fusion channels is energetically preferred. For quasielectrons, one instead combines a Laughlin or CF quasielectron with 1 or ψ . The two possibilities for the preferred fusion channel give distinct hierarchies. In particular, the series of filling fractions as well as the shift at a given filling fraction differs for the two cases. Note that condensing Abelian quasiparticles cannot change the non-Abelian part of the CFT, but only the charge part. In particular, daughter states in the BS hierarchy on top of the Moore-Read state always have quasiparticles of Ising type. Consequently, the torus ground state degeneracy for such hierarchy states is always given by $3q$ for an even number of electrons and q for an odd number of electrons, where q denotes the denominator of the filling fraction. The first factor 3 is due to the Pfaffian, while the second factor q encodes the (trivial) center-of-mass degeneracy of fractional quantum Hall states on the torus; see Eq. (14) or alternatively Sec. VII.B.

Let us first focus on the condensation of pairs in fusion channel 1, where the discussion is slightly simpler. As we are essentially condensing Laughlin quasiholes or quasielectrons, we can use the same pseudo wave function as for the Abelian hierarchy. The resulting daughter wave functions have a particularly simple form,

$$\Psi_{BS} = \text{Pf} \left[\frac{1}{z_i - z_j} \right] \times \Psi_\nu^{(b)}(z_1, \dots, z_N), \quad (75)$$

where $\Psi_\nu^{(b)}$ is a bosonic state in the Abelian hierarchy over the Laughlin $1/2$ wave function, and the possible filling fractions for Ψ_{BS} are given by Eq. (6) with $m = 2$. Its shift S differs from the one for $\Psi_\nu^{(b)}$ due to the Pfaffian, which reduces the highest possible angular momentum by 1. Thus, $S = S_\nu + 1$, where S_ν denotes the shift of $\Psi_\nu^{(b)}$ as given in Eq. (16). Equation (75) is an example of building a hierarchy by multiplying two known wave functions, as discussed in Sec. II.J.

To give a concrete example we consider the first hierarchy level on top of the Moore-Read state. In this case, we can use the techniques of the Abelian hierarchy in Sec. V to obtain explicit wave functions. For instance, condensing quasielectrons with maximal possible density gives a model wave function for filling fraction $\nu = 2/3$,

$$\begin{aligned} \Psi_{BS} &= \text{Pf} \left[\frac{1}{z_i - z_j} \right] \times \Psi_{2/3}^{(b)}(z_1, \dots, z_N) \\ &= \text{Pf} \left[\frac{1}{z_i - z_j} \right] \times \mathcal{S} \{ (1-1)^2 \partial_2 (2-2)^2 (1-2) \} e^{-\sum_i |z_i|^2 / (4\ell^2)}, \end{aligned} \quad (76)$$

where \mathcal{S} denotes an overall symmetrization. Note that $\Psi_{2/3}^{(b)}$ is simply the bosonic version of the wave function (55). This state has two electron operators, both of which are combinations of ψ with the vertex operators of the $\nu = 2/3$ bosonic CF state,

$$\psi_{e1}(z) = \psi(z) : e^{i\sqrt{2}\phi_1(z)} :,$$

$$\psi_{e2}(z) = \psi(z) \partial_z : e^{i\sqrt{1/2}\phi_1(z) + i\sqrt{3/2}\phi_2(z)} :.$$

The quasihole spectrum is generated by a quasihole operator with charge $e/3$,

$$\psi_{qh}(\eta) = \sigma(z) : e^{i\sqrt{1/8}\phi_1(\eta) + i/(2\sqrt{6})\phi_2(\eta)} :,$$

which is the straightforward analog to the Moore-Read quasihole. It is determined by requiring it to reproduce $(z - \eta)^{1/2}$ in correlation functions with the vertex operators of both ψ_{e1} and ψ_{e2} , i.e., the quasihole-electron correlations are identical to the ones in the Moore-Read state.

Condensing Abelian quasiparticles of the type $\psi(\eta) : e^{i\sqrt{1/2}\phi_1(\eta)} :$ is not conceptually more difficult, but the expressions become much more complicated, as the quasiparticle coordinates enter in the Pfaffian. Also note that the extra fermion field ψ in Eq. (74) changes the statistics of the quasiholes compared to that in Eq. (73). To compensate for this, the pseudo wave function acquires an extra Jastrow factor in the quasihole positions, which in turn alters the K matrix of the daughter states. As the filling fractions in the BS hierarchies are solely determined by the K matrix, one finds a different series of filling fractions compared to the case where the condensed quasiparticle pairs are in the trivial fusion channel. The shifts of the daughter states also differ and are given by $S = S_\nu + 2 - \sum_j (K^{-1})_{1j}$, where S_ν is determined by the K matrix via Eq. (16). To give a concrete example, let us again consider the maximal density quasielectron condensate at filling $\nu = 3/5$, which is described by the K matrix

$$\mathbf{K} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

This state has shift $S = 13/3$ and a torus ground state degeneracy of 15 for an even number of electrons. Note that even though the shift is fractional, $\nu(N_\Phi + S)$ in Eq. (15) is an integer.

B. The Levin-Halperin hierarchies

The hierarchy by Levin and Halperin (2009) is simplest to describe for condensing non-Abelian quasiholes. As already mentioned, this is in general not possible because of the branch cuts at the quasiparticle positions that make the integrals (5) ill defined. The way out devised by Levin and Halperin is to choose a pseudo wave function which combines with the original multiquasiparticle wave function to form a correlator of full CFT fields, i.e., fields containing both holomorphic and antiholomorphic parts. Such a correlator is guaranteed not to contain any branch cuts and can thus be integrated over with impunity. The pseudo wave function can, in fact, be interpreted as the proper many-body wave function for the non-Abelian anyons and is given by

$$\Phi_\alpha^*(\vec{\eta}_1, \dots, \vec{\eta}_M) = \left\langle \prod_{j=1}^M \bar{\sigma}(\vec{\eta}_j) e^{i\sqrt{2m+1/8}\bar{\phi}(\vec{\eta}_j)} \mathcal{O}_{bg} \right\rangle_\alpha$$

for the Moore-Read state. Here M is the number of anyons, α labels the conformal block of the σ correlator, and m is a positive integer that sets the density of quasiholes in Eq. (5). Note that the bosonic field $\bar{\phi}$ in the pseudo wave function is independent of the original bosonic field in Eq. (48). The product of the pseudo wave function and the many-quasihole wave function in Eq. (5) now becomes an inner product, as both wave functions are vectors in the $2^{M/2-1}$ -dimensional vector space,

$$\Psi_1(\vec{r}_1 \cdots \vec{r}_N) = \int d^2\vec{\eta}_1 \cdots \int d^2\vec{\eta}_M \sum_\alpha \Phi_\alpha^*(\vec{\eta}_1 \cdots \vec{\eta}_M) \times \Psi_{MR}^\alpha(\vec{\eta}_1 \cdots \vec{\eta}_M; \vec{r}_1 \cdots \vec{r}_N). \quad (77)$$

As promised, the integrand is well defined in the $\vec{\eta}$ coordinates, i.e., it no longer has branch cuts. This can readily be seen from the decomposition of the full σ correlator into holomorphic and antiholomorphic blocks,

$$\left\langle \prod_{j=1}^M \sigma(\eta_j, \bar{\eta}_j) \cdots \right\rangle = \sum_\alpha \left\langle \prod_{j=1}^M \bar{\sigma}(\bar{\eta}_j) \cdots \right\rangle_\alpha \left\langle \prod_{j=1}^M \sigma(\eta_j) \cdots \right\rangle_\alpha, \quad (78)$$

and using that the full correlator has no branch cuts. This hierarchical construction yields candidate wave functions for filling fractions $\nu = 8m/(16m+1)$. It turns out that these states are Abelian, despite being daughters of a non-Abelian state.

Evaluating Eq. (77) is in principle possible for small numbers of electrons, but there is no closed expression for an arbitrary number of electrons. We can, however, still deduce the quasihole properties by requiring that the corresponding operators are local with respect to the electron operators and the full quasihole operator, i.e., the combination of both the holomorphic and antiholomorphic parts given by

$$\sigma(\eta, \bar{\eta}) : e^{i\phi(\eta)/(2\sqrt{2}) + i\sqrt{2m+1/8}\bar{\phi}(\bar{\eta})} :$$

The locality conditions ensure that the full expression in Eq. (77) has no branch cuts or poles, and, thus, gives a valid electronic wave function. From this constraint one can deduce that the quasihole spectrum is generated by a single quasihole operator with electric charge $e/(16m+1)$ and Abelian exchange phase $\theta = \pi(16m-1)/(16m+1)$. As the first level daughter states are Abelian, further condensation of quasiholes will, of course, yield only Abelian candidate states. Interestingly, it was pointed out by [Levin and Halperin \(2009\)](#) that the series of states (77) can alternatively be obtained by an Abelian hierarchy starting from the strong-pairing $\nu = 1/2$ state, which is a $\nu = 1/8$ Laughlin state of strongly bound electron pairs that was originally discussed by [Halperin \(1983\)](#). The two approaches yield wave functions with the same topological data and should therefore be in the same universality class.

The simplest example for the Levin-Halperin hierarchy is the state with the highest density of quasiholes, i.e., $m = 1$, which yields a candidate state at filling $\nu = 8/17$. Even though this state is Abelian, it is distinct from the Abelian Haldane-Halperin hierarchy state at this filling fraction; the latter is obtained with $q = 8$ and $p = 1$ in the Jain series in Eq. (11). This can most easily be seen from their different shifts; the Levin-Halperin hierarchy state has shift $S = 5/2$, while the Abelian Haldane-Halperin hierarchy state has shift 8. Candidate states for this filling can also be obtained by the other hierarchies, but only at higher levels.

Levin and Halperin also suggested model states for condensing non-Abelian quasielectrons, which in the simplest case yields a candidate state at $\nu = 7/13$. This does not seem as natural as the quasihole condensates. In particular, the quasielectron wave functions contain unphysical poles, and it is not clear if these can be regularized while keeping the non-Abelian braiding statistics explicit. The proposal for a state at filling fraction $7/13$ is still interesting because its particle-hole conjugate state at $\nu = 2 + 6/13$ has been seen in experiments by [Kumar *et al.* \(2010\)](#), suggesting that this quantum Hall state may be a daughter state of the anti-Pfaffian (see also the discussion in Sec. III.B.1). Note that this filling fraction also occurs in the BS and Hermanns hierarchies. However, neither of these provides a natural explanation for its occurrence. The BS hierarchy contains this state at the second level, but it also predicts a state at filling $4/9$ as more prominent. The latter was, however, not observed in [Kumar *et al.* \(2010\)](#). The Hermanns hierarchy discussed next contains this filling in the third level of hierarchy as a daughter state of $4/7$. The latter was again not observed. Thus, so far only the Levin-Halperin hierarchy provides a natural explanation for $\nu = 2 + 6/13$.

C. The Hermanns hierarchies

In contrast to the other non-Abelian hierarchies discussed in the previous sections, the daughter states in this hierarchy have non-Abelian statistics different from their parents. Before getting to the actual construction, we first devote some time to explain how non-Abelian quasielectrons can be represented in terms of CFT operators without unphysical singularities. We focus on the condensation of quasielectrons. One can also study quasihole condensation, but the wave functions will no longer have such a simple form and interpretation.

1. Quasielectrons in the Moore-Read state

Before discussing the non-Abelian quasielectrons in the Moore-Read state, let us first recall again the essential steps in the definition of the Abelian quasielectron (68). The quasielectron operator is obtained by attaching an inverse quasihole to the electrons in such a way that the resulting correlator has no branch cuts and no poles in the electron coordinates. The latter is ensured by regularizing the resulting operators using a generalized normal ordering. The former is achieved by using a quasihole operator which has trivial monodromies. In the Abelian case, the monodromies of the quasihole can be made trivial simply by adding auxiliary, chargeless, boson fields. This is not possible for the non-Abelian quantum Hall states, within the usual description in

terms of the Ising CFT, as the quasipoles are manifestly non-Abelian. Braiding quasipoles around each other does not solely give an additional phase factor, but in general also changes the state. As a result, the quasipole of the Ising CFT description cannot be used in the definition of the quasielectron operator, as it necessarily leads to branch cuts in the electron coordinates.

A way around this problem is to use an equivalent description of the Moore-Read state that does allow for trivial monodromies of the quasipoles. In the following, we use the multilayer description of Cappelli, Georgiev, and Todorov (2001). The bosonic Read-Rezayi state at level k is described by k layers of bosonic Laughlin $\nu = 1/2$ states with a subsequent symmetrization over the layer index. For example, the bosonic Moore-Read wave function can alternatively be written, using the shorthand notation (53), as

$$\Psi_{MR} = \mathcal{S}\{(1-1)^2(2-2)\} e^{-\sum_i |z_i|^2/(4\ell^2)},$$

where we divided the particle coordinates into two equal-sized groups I_1 and I_2 , and \mathcal{S} denotes the overall symmetrization over all such partitions. This faithfully reproduces the Moore-Read ground state. Quasipole excitations are obtained by inserting Laughlin quasipoles in each of the layers. At level k , one thus artificially introduces k quasipole operators, which are then identified by the symmetrization. The vector space of n quasipole states is spanned by the various orderings of the k types of operators at positions η_1, \dots, η_n and is identical to the space obtained using the Ising CFT description (Cappelli, Georgiev, and Todorov, 2001). However, in this description the non-Abelian part of the braiding statistics resides entirely in the Berry's phase. The monodromies can then be made trivial by introducing k additional bosonic fields in the same way as in the Abelian case.

As an example, the fermionic Moore-Read state, phrased in the two-layer description, has vertex operators

$$\begin{aligned} V(z) &= V_+(z) + V_-(z), \\ V_{\pm}(z) &= :e^{\pm i\phi(z)} e^{i\sqrt{2}\chi_{\pm}(z)}:, \\ H_{\pm}(\eta) &= :e^{\pm(i/2)\phi(\eta) + i\sqrt{1/8}\chi_1(\eta) - i\sqrt{3/8}\chi_2(\eta) \mp (i/2)\chi_3(\eta)}:, \end{aligned} \quad (79)$$

where \pm denotes the two layers (Cappelli, Georgiev, and Todorov, 2001) and the manifest monodromies of the quasipoles were chosen to be fermionic. Using the quasipoles of Eq. (79) in Eq. (68) one can construct the two corresponding quasielectron operators $\mathcal{P}_+(\vec{\eta})$ and $\mathcal{P}_-(\vec{\eta})$ with the correct quasipole excess charge $q_e = -q_h$. Just like the quasipoles, the quasielectrons must be inserted in pairs, otherwise the correlator vanishes because of charge neutrality in the ϕ sector. Inserting $2n$ quasielectrons, one finds 2^{n-1} linearly independent orderings of $\mathcal{P}_+(\vec{\eta})$ and $\mathcal{P}_-(\vec{\eta})$ in complete analogy to the quasipoles (Hansson *et al.*, 2009; Hansson, Hermans, and Viefers, 2009). An appealing, alternative interpretation of these operators is possible in the bosonic Moore-Read state. There they can be seen as the usual Abelian quasielectrons in each of the layers, and the non-Abelian properties arise due to the symmetrization over the layers. As shown by Rodríguez *et al.* (2012), this allows for a

numerically efficient way to study the properties of quasielectrons as well as of the neutral excitations.

2. Condensing quasielectrons

We now proceed to generalize the Haldane-Halperin hierarchy using the non-Abelian quasielectrons introduced previously. We restrict our discussion to the Moore-Read state as the parent state, but mention briefly how the construction can be adapted to other non-Abelian states, such as the Read-Rezayi series.

Consider a quasielectron wave function for an insertion of $2M$ quasielectrons using an equal number of \mathcal{P}_+ and \mathcal{P}_- at positions η_1, \dots, η_{2M} . Such a multiquasielectron state is for instance given by

$$\begin{aligned} \Psi_1(\vec{\eta}_1, \dots, \vec{\eta}_{2M}; \vec{r}_1, \dots, \vec{r}_N) \\ = \left\langle \prod_{j=1}^M \mathcal{P}_+(\eta_j) \prod_{j=M+1}^{2M} \mathcal{P}_-(\eta_j) \prod_{j=1}^N V(z_j) \mathcal{O}_{bg} \right\rangle, \end{aligned}$$

but any other ordering of \mathcal{P}_+ and \mathcal{P}_- gives an equally valid wave function. Note that the order of the quasielectron operators is, in general, important. In particular,

$$\begin{aligned} \langle \mathcal{P}_+(\eta_1) \mathcal{P}_+(\eta_2) \mathcal{P}_-(\eta_3) \mathcal{P}_-(\eta_4) \dots \rangle \\ \neq \langle \mathcal{P}_+(\eta_1) \mathcal{P}_-(\eta_2) \mathcal{P}_+(\eta_3) \mathcal{P}_-(\eta_4) \dots \rangle, \end{aligned}$$

even though \mathcal{P}_+ and \mathcal{P}_- are indistinguishable by any local measurement.

Only 2^{M-1} of these orderings are linearly independent, but that still leaves a macroscopic number of parent states to choose from. It turns out that it is unimportant which of these states is chosen in Eq. (5), as the integration over the quasielectron coordinates can be interpreted as an average over all positions, and the final result is independent of which of the multiquasielectron states was used as a starting configuration. As the monodromies of the quasielectrons are fermionic, the simplest appropriate pseudo wave function is a fermionic Laughlin state for each of the layers,

$$\begin{aligned} \Phi^*(\vec{\eta}_1, \dots, \vec{\eta}_{2M}) &= \prod_{i < j \in I_1} (\eta_i - \eta_j)^{2p-1} \\ &\times \prod_{a < b \in I_2} (\eta_a - \eta_b)^{2p-1} e^{-q_h \sum_{j=1}^{2M} |\eta_j|^2/(4\ell^2)}, \end{aligned}$$

where I_1 contains η_1, \dots, η_M , i.e., all quasipole positions appearing in \mathcal{P}_+ , and I_2 contains the rest. The integer p is fixed by requiring homogeneity of the daughter state $M/N = 1/(4p)$. The integral in Eq. (5) is straightforward to evaluate, and the daughter state at filling fraction $\nu = 4p/(8p-1)$ can be compactly written as

$$\begin{aligned} \Psi_n(\vec{r}_1, \dots, \vec{r}_N) &= \mathcal{S}\{\Psi_{\tilde{\nu}}(\vec{r}_1, \dots, \vec{r}_{N/2}) \\ &\times \Psi_{\tilde{\nu}}(\vec{r}_{N/2+1}, \dots, \vec{r}_N)\} \prod_{i < j} (z_i - z_j), \end{aligned} \quad (80)$$

where $\Psi_{\tilde{\nu}}$ is a bosonic hierarchy state at filling fraction $\tilde{\nu} = 2p/(4p-1)$ and $\prod_{i < j} (z_i - z_j)$ denotes a full Jastrow

factor containing all the electrons. Note that in writing the wave function in this form, we are cavalier about the exact placement of the derivatives, which now act solely within $\Psi_{\tilde{\nu}}$. While this changes the wave function slightly, it is widely believed not to have an impact on its non-Abelian properties [see also the discussion below Eq. (54)]. The benefit of writing Eq. (80) in this form lies in its natural interpretation as symmetrized products of hierarchy (or CF) wave functions (times a Jastrow factor), similar to the interpretation of the Moore-Read state as a symmetrized product of two Laughlin $1/2$ states (again times a Jastrow factor). Successive condensation of quasielectrons yields a hierarchy of model states with filling fractions $\nu = 2\tilde{\nu}/(1 + 2\tilde{\nu})$, where $\tilde{\nu}$ denote the filling fractions of the chiral (bosonic) hierarchy over the Laughlin $1/2$ state.

An exciting feature of this hierarchy is that the non-Abelian statistics changes from parent to daughter state, in contrast to the hierarchy described in Sec. VI.A and without making the non-Abelian statistics trivial as in Sec. VI.B. From Eq. (80), however, it is not apparent what the topological properties of the states are, as all the non-Abelian properties are hidden in the Berry phase. The easiest way to proceed is to consider the bosonic version of Eq. (80) and use the techniques introduced by Ardonne *et al.* (2008, 2009) for the thin torus limit (Bergholtz *et al.*, 2008). This yields the ground state degeneracy and the quasihole fusion rules, but not the braiding properties of the state. For instance, for the subset of wave functions obtained by successively condensing quasielectrons with maximal density,

$$\Psi_n(\vec{r}_1, \dots, \vec{r}_N) = \mathcal{S}\{\Psi_{n/(n+1)}(\vec{r}_1, \dots, \vec{r}_{N/2}) \times \Psi_{n/(n+1)}(\vec{r}_{N/2+1}, \dots, \vec{r}_N)\},$$

one finds a ground state degeneracy on the torus of $(n + 1)(n + 2)/2$ (for an even number of particles) and $su(n + 1)_2$ type fusion. The properties of the corresponding fermionic filling fractions can then be deduced by incorporating the Jastrow factor again. In fact, for the highest density condensate ($p = 1$) in Eq. (80) at filling fraction $\nu = 4/7$, one can even identify the CFT that makes braiding and fusion properties explicit in the monodromy factor by noting the close resemblance to the non-Abelian spin-singlet state proposed by Ardonne and Schoutens (1999) and Ardonne *et al.* (2001). This analysis determines the pertinent CFT to be $su(3)_2/u(1)^2$ (Gepner, 1987), as shown by Hermanns (2010). The generalization of this hierarchy to the Read-Rezayi states by taking k layers instead of two is straightforward. Numerical tests on these types of wave functions have been performed by Sreejith *et al.* (2011, 2013), showing their relevance as possible candidate states in the second LL.

VII. CFT HIERARCHY WAVE FUNCTIONS IN OTHER GEOMETRIES

There are several reasons for studying quantum Hall liquids on geometries other than the plane, and, in particular, on closed geometries. Although these are not experimentally realizable (we do not have magnetic monopoles at our disposal in the lab), they are interesting from both a theoretical

and a numerical point of view. In particular, closed geometries do not suffer from edge or boundary effects. On the disk geometry, these can be substantial for the small system sizes possible to probe with numerical techniques.

The simplest example of a closed manifold is the sphere, which after the pioneering work by Haldane (1983) has not only become the favorite geometry for numerical simulations, but also has been instrumental for understanding how the quantum Hall liquids respond to spatial curvature. The other example that we consider is the torus. Here the geometry is locally flat, but the manifold is topologically nontrivial which is manifested in a characteristic ground state degeneracy. In both these cases, a key point is to understand how the magnetic translations, which are the proper symmetries of an infinite plane penetrated by a constant magnetic field, are manifested in other geometries. In the following we outline how this is done and how hierarchy wave functions on the sphere and the torus can be obtained using CFT techniques.

A. The sphere

In a suitably chosen gauge (Dirac gauge with the string entering at the North Pole) solutions of the Landau problem on the sphere are monopole harmonics (Stone, 1992; Jain, 2007), and the multiparticle solutions can be organized in angular momentum multiplets, with homogeneous states, such as ground states, having $L = 0$. Stated differently, the state should be invariant under magnetic rotations. We already saw that in the CFT formulation the physical condition of homogeneity of the quantum Hall liquid translated into charge neutrality enforced by coupling to a background charge density. On a curved surface this is not sufficient for having a consistent description, since the electrons and the quasiparticles carry (orbital) spin and thus respond to curvature. As we shall see, what is needed is both an extra piece in the action and a modification of the electron operators.

Before turning to the general hierarchy wave functions, we stress that in many cases one does not have to worry about these complications. For the simple wave functions, i.e., those that can be written as correlators of primary fields only, it is straightforward to move from the expressions on the plane to those on the sphere. In essence, this recipe amounts to expressing the z coordinates in the polynomial part of the wave functions in terms of the spherical coordinates (θ, ϕ) , by the stereographic mapping (Read and Rezayi, 1996),

$$z \rightarrow 2R \tan(\theta/2) e^{i\phi}, \quad (81)$$

from the infinite plane to a sphere with radius R . At the same time, the Gaussian factors have to be changed as $\exp[-|z|^2/(4\ell^2)] \rightarrow 1/[1 + |z|^2/(4R^2)]^{1+N_\Phi/2}$. Choosing the number of electrons N_e and the number of flux quanta N_Φ as to implement the correct shift (15), one obtains a valid wave function (Haldane, 1983). A similar, although somewhat more complicated scheme for moving wave functions from the plane to the sphere has also been devised for the composite-fermion states in the Jain series (Jain, 2007). For the general hierarchy states, which cannot be written as a

projected version of a number of filled Landau levels, these simple methods do not work.

We now turn to the general case, and review the main results from [Kvornring \(2013\)](#). Naively, one would expect that expressing hierarchical wave functions on the sphere is straightforward: one simply has to evaluate the corresponding CFT correlators [see, e.g., Eq. (52) for the fully chiral states] on the sphere. It would indeed be that simple, if the correlators contained only proper CFT operators (primaries or descendants), but the background charge makes it more complicated. So far we referred to the insertion \mathcal{O}_{bg} in Eq. (57) as a background operator. An equivalent way, which is more appropriate for the present purpose, is to view it as a part of the action for the fields ϕ^β . [Kvornring \(2013\)](#) showed that using

$$S_{bg} = -i \frac{(K^{-1})^{\alpha\beta}}{2\pi} \int dS (e t_\alpha B + s_\alpha \mathcal{R})(q_{\beta\gamma} \phi^\gamma + \bar{q}_{\beta\gamma} \bar{\phi}^\gamma) \quad (82)$$

in the action when evaluating the correlators indeed gives wave functions that are invariant under magnetic rotations. Note that adding the term $\sim \mathcal{R}$ is the standard way to include curvature, while the term $\sim B$ describes the neutralizing background. Equation (82) is quite natural in light of the combination $t_\alpha e A_\mu \epsilon^{\mu\nu\sigma} \partial_\nu a_\sigma^\alpha + s_\alpha \omega_i \epsilon^{i\nu\sigma} \partial_\nu a_\sigma^\alpha$ appearing in the Wen-Zee Lagrangian (12). Making a partial integration to rewrite it in terms of the dual electromagnetic field strength $*F^\mu = \epsilon^{\mu\nu\sigma} \partial_\nu A_\sigma$ and the corresponding $*R^\mu = \epsilon^{\mu\nu\sigma} \partial_\nu \omega_\sigma$ where ω_i is the spin connection, the a_0^α coefficient is precisely the combination $e t_\alpha B + s_\alpha \mathcal{R}$ in Eq. (82).

The remaining difficulty for obtaining the hierarchy wave functions on the sphere is related to the precise form of the vertex operators, when they contain derivatives. In Sec. V we pointed out that there is a freedom in distributing the derivatives acting in the wave functions. On the disk, a valid (in fact the simplest) prescription is to move all derivatives to the left, which ensures that the derivatives always act on fully holomorphic functions (or antiholomorphic in the case of $\bar{\partial}$). However, this prescription does not work on the sphere (and likely not on any other closed surface).

The reason is that convoluting the resulting coherent state wave functions with the spherical version of the coherent state kernel, as in Eq. (60), gives zero, since it basically amounts to integrating a total derivative over a closed manifold. We must thus resort to consider more general expressions for the vertex operators of the type (54):

$$: e^{ia\phi^n} \partial^n e^{ib\phi^\beta} : = : D^n [b\phi^\beta] e^{ia\phi^n} e^{ib\phi^\beta} :,$$

where $D^n[f] = e^{-if} \partial^n e^{if}$, so $D^n[b\phi^\beta]$ is nothing but a polynomial in ϕ^β . For details on how to adapt the operator D to the sphere, and how to use it to construct a set of electron operators that give well-behaved representative wave functions for all states in the hierarchy, see [Kvornring \(2013\)](#).

B. The torus

The torus is interesting for several reasons. First, the ground state wave function is degenerate, where the degeneracy is

identical to the number of particle types in the CFT. This ground state degeneracy is in fact a smoking gun for a topologically ordered phase. From a computational point of view, the torus is interesting since candidate wave functions with different shifts can be directly compared with each other (on the sphere, wave functions with different shifts require a different number of flux quanta and, therefore, cannot be directly compared). In fact, at first it was not clear in what way states with different shift would differ on the torus. Through work by [Avron, Seiler, and Zograf \(1995\)](#) and [Read \(2009b\)](#) it became clear that the shift is related to the Hall viscosity, which is a transport coefficient characterizing a nondissipative flow possible only in 2D systems violating time-reversal symmetry. As explained in Sec. VII.B.4, the Hall viscosity can be probed by deforming the geometry of the torus and can be extracted from the CFT wave functions, again evoking assumptions about screening similar to those described in Sec. IV.C.4. Last, but not least, studying quantum Hall wave functions on the torus provides an independent test of the holonomy equals monodromy conjecture, as discussed in Sec. VII.B.5. Note that in this section we consider only chiral Abelian states.

1. QH wave function on the torus

There are various ways to obtain the proper torus wave functions for the simple quantum Hall states, i.e., those built from CFT correlators involving only primary fields. The torus version of the Laughlin wave function was first derived by [Haldane and Rezayi \(1985b\)](#), using the vanishing properties of the state as well as the constraints imposed by the magnetic translations. The latter implies that proper torus wave functions need to fulfill certain quasiperiodic boundary conditions. These two properties are sufficient to uniquely specify the set of m degenerate ground state wave functions for $\nu = 1/m$ Laughlin states. In fact, not only is the ground state m -fold degenerate, but so is the entire spectrum ([Haldane, 1985](#)). This center-of-mass degeneracy occurs because the magnetic translations along the two cycles of the torus do not commute.

A similar approach as described for the Laughlin states can be used for the Halperin mmn states ([43](#)) ([Wen, 2012](#)) and even the Read-Rezayi series; see, e.g., [Greiter, Wen, and Wilczek \(1991\)](#) and [Chung and Stone \(2007\)](#) for the Moore-Read ground state and quasihole states, respectively. Note that for non-Abelian states there is an extra ground state degeneracy in addition to the kinematic center-of-mass degeneracy ([Read and Rezayi, 1996](#)).

Another way to derive the pertinent wave functions is to compute the relevant CFT correlators on the torus. Then, just as on the sphere, one can extract wave functions by calculating correlators of vertex operators to get the conformal blocks and then impose the boundary conditions ([Hermanns *et al.*, 2008](#)). For technical reasons, one needs to consider the full fields (holomorphic and antiholomorphic components) and separate the correlator into a sum over conformal blocks ([Di Francesco, Mathieu, and Sénéchal, 1997](#)). When doing this, it is also important to correctly handle the Gaussian factor, which amounts to distributing it symmetrically. A detailed derivation of these wave functions, keeping the full τ dependence, was given for the Laughlin case by [Read \(2009b\)](#) and for the multilayer states by [Fremling, Hansson, and Suorsa \(2014\)](#). In

both cases, the ground state wave functions are fully determined by the topological data; in the Laughlin case this is only the filling fraction ν , while for the multilayer states it is both the K matrix and the spin vector.

One would naively assume that this approach has a straightforward generalization to the hierarchical states. However, the presence of derivatives in the wave functions makes the problem much harder, as they destroy the quasiperiodic boundary conditions of the holomorphic blocks. Thus, one needs to identify the proper torus analog of derivatives. To do this, we must understand the modular properties of the torus wave functions, which is the topic of the next section.

2. Modular properties of CFT wave functions

The geometry of a (flat) torus is determined by a complex number, the modular parameter τ , which defines the periods under which the coordinates are to be identified. We imagine starting with a complex plane and drawing a segment from 0 to 1 on the real axis and another segment from 0 to τ . Forming a parallelogram from these two segments and identifying opposite edges, we obtain a torus. The boundary conditions on the wave functions are defined by specifying how they transform under $z \rightarrow z + 1$ and $z \rightarrow z + \tau$. The relation between the geometry of the torus and the modular parameter is not one to one; different τ can describe the same torus (i.e., they enforce the same periodicity on the plane). The different values are related via modular transformations, which form a group that is generated by the two transformations,

$$\mathcal{T}: \tau \rightarrow \tau + 1, \quad \mathcal{S}: \tau \rightarrow -\frac{1}{\tau}.$$

It will be important in the following that the \mathcal{S} transformation involves a rotation of the torus. Since values of τ that differ only by a modular transformation describe the same physics, the space spanned by the ground states must be invariant, meaning that the states must transform into each other by a unitary transformation.

For the CFT hierarchy construction, we need the modular transformations of the multicomponent wave functions, which can be derived using the techniques employed by Read (2009b) for the Laughlin case. Schematically, they can be written as⁵⁷

$$\begin{aligned} \psi_s &\xrightarrow{\mathcal{S}} \left(\frac{\tau}{|\tau|}\right)^{h_{\text{tot}}} \sum_{s'=1}^m \mathcal{S}_{s,s'} \psi_{s'}, \\ \psi_s &\xrightarrow{\mathcal{T}} \sum_{s'=1}^m \mathcal{T}_{s+s'} \psi_{s'}, \end{aligned} \quad (83)$$

⁵⁷The complete formulas, which include gauge transformations, τ -independent phase factors, and offsets in the arguments of the modular \mathcal{S} and \mathcal{T} matrices, also involve changes in the boundary conditions. These can be periodic or antiperiodic in the cycles of torus. The \mathcal{S} transformation effectively swaps $x \leftrightarrow y$ and consequently also their boundary conditions; for details see Fremling, Hansson, and Suorsa (2014).

where ψ_s are the (degenerate) ground state wave functions, and h_{tot} is the total conformal weight of all the electron operators. In the \mathcal{S} transformation we also let $z \rightarrow z' = |\tau|z/\tau$, while in the \mathcal{T} transformation, z is unchanged. Note that the only τ dependence in Eq. (83) is in the \mathcal{S} transformation. This is easy to understand: If we write $\tau = |\tau|e^{i\alpha}$ the \mathcal{S} transformation is associated with a rotation of the coordinate z by an angle α in the negative direction. Thus, an operator with conformal dimension h_ψ , which here equals the conformal spin, acquires a phase factor $e^{i\alpha h_\psi}$. The factor $(\tau/|\tau|)^{h_{\text{tot}}}$ in Eq. (83) is precisely the product of these phase factors for all of the electrons.

We are now ready to treat the hierarchy states. But before showing how the general CFT formalism in Sec. V.A can be implemented on the torus, we describe some work specially aimed at the Jain states.

3. Composite-fermion wave functions on the torus

For obtaining the Jain series on the torus, one can take a different route by using the fact that the product of two single-particle states at fluxes N_{Φ_1} and N_{Φ_2} is a valid single-particle state at flux $N_{\Phi_1} + N_{\Phi_2}$, independent of the underlying geometry. Thus, the Jain wave functions (including the Laughlin states) on the torus can be computed by using the torus version of Eq. (11). Note that this expression contains only integer quantum Hall wave functions, which are known on the torus. The only remaining difficulty lies in the projection to the LLL, for which explicit expressions were derived by Hermans (2013). This approach is numerically inefficient and one can handle only small system sizes for two reasons. First, there is no analog of the fairly simple projection scheme of Girvin and Jach (1984) on the torus. Second, due to the periodic boundary conditions, momentum is not conserved when multiplying two single-particle wave functions (it is conserved only modulo N_Φ), which makes the evaluation of the projection much more numerically costly than on the sphere.

Note that evaluating Eq. (11) on the torus always gives a unique ground state wave function, in contrast to the expected ground state degeneracy on the torus. As momenta are defined only modulo N_Φ , the flux attachment of Eq. (11) does not conserve the total momentum of the state and the resulting wave function is, in general, a linear combination of the different degenerate ground states. The set of degenerate ground states can be recovered by acting with the center-of-mass translation operators or by allowing nontrivial boundary conditions around the two handles of the torus for the individual IQH wave functions. This construction even works well for the MR state, when using the ‘‘layer description’’ by Cappelli, Georgiev, and Todorov (1999, 2001). For the general RR Z_k series, however, the full set of degenerate ground (and excited) states can be generated only by amending this construction as discussed by Repellin *et al.* (2015).

4. Chiral hierarchical wave functions on the torus

The first attempt to use CFT methods for obtaining hierarchy states was made by Hermans *et al.* (2008), where

the derivatives were replaced by finite translations. While these were chosen to locally have the same effect as derivatives, the resulting wave functions did not transform properly under modular transformations and worked well only for tori of certain shapes. These difficulties were resolved by [Fremling, Hansson, and Suorsa \(2014\)](#) who conjectured that the transformations (83), which were derived for the multicomponent case, should hold true also for the hierarchy states.

The main achievement of [Fremling *et al.*](#) was to replace the derivatives with combinations of finite differences in a way such that Eq. (83) still holds, but with the total conformal dimension h_{tot} adjusted to include the contributions from the derivatives. The construction is rather technical, but the upshot is that the torus generalization of the chiral hierarchy wave functions (52) is given by

$$\tilde{\psi}_s = \mathcal{A} \prod_{\alpha=1}^n \mathbb{D}_{(\alpha)}^{\alpha-1} \psi_s, \quad (84)$$

where ψ_s is the toroidal wave function of the corresponding multicomponent state with the same K matrix. The product is over the n distinct electron operators, and the generalized shift operator $\mathbb{D}_{(\alpha)}$ is a sum over finite magnetic translations on a lattice with N_{Φ}^2 points.⁵⁸ The actual expression for $\mathbb{D}_{(\alpha)}$ is quite complicated but, up to a single choice of a sign, it is fully determined by the requirement that the wave function (84) transforms as (83) with the appropriate conformal spins. The simplest example of a wave function (84), namely, the second level Jain state at $\nu = 2/5$, shows an excellent agreement over the whole τ plane with the numerically obtained Coulomb wave function for eight electrons. We stress that the overlap, as a function of τ , is obtained without fitting any parameter.

5. The Hall viscosity

[Avron, Seiler, and Zograf \(1995\)](#) showed that the Hall viscosity, which is a transport coefficient that determines the nondissipative response to a strain rate, can be obtained by adiabatically changing the modular parameter τ , much like the Hall conductance σ_H can be obtained from an adiabatic change of the flux through the holes in the torus ([Niu, Thouless, and Wu, 1985](#)). In analogy with this case, we define the Berry potentials related to the modular parameter $\tau = \tau_1 + i\tau_2$ by $\mathcal{A}_\tau = i\langle \Psi(\tau) | \partial_\tau \Psi(\tau) \rangle$, and $\mathcal{A}_{\bar{\tau}} = i\langle \Psi(\tau) | \partial_{\bar{\tau}} \Psi(\tau) \rangle$, and the corresponding field strength $\mathcal{F}_{\tau\bar{\tau}} = i\partial_{\bar{\tau}} \mathcal{A}_\tau - i\partial_\tau \mathcal{A}_{\bar{\tau}}$. In terms of this quantity, the result of [Avron, Seiler, and Zograf \(1995\)](#) reads

$$\eta^H = -\frac{2\tau_2^2}{A} \mathcal{F}_{\tau\bar{\tau}}, \quad (85)$$

where A is the area of the system.

[Avron, Seiler, and Zograf \(1995\)](#) used Eq. (85) to calculate η^H for the filled LLL level where the wave function is known.

Later [Read \(2009b\)](#) applied the same methods to the Laughlin and Moore-Read states, using the representative CFT wave functions.⁵⁹ The result of these calculations could be summarized as

$$\eta^H = \frac{\hbar}{2} \rho \bar{s} = \frac{\hbar}{4} \rho S, \quad (86)$$

where ρ is the electron density, \bar{s} is the average conformal spin of the electrons, S is the shift on the sphere [see Eq. (16)], and where we reinserted \hbar . [Read](#) argued that Eq. (86) should hold also for general quantum Hall states.

a. Adiabatic calculation for the Laughlin states

Here we outline the main ingredients used by [Read \(2009b\)](#) to calculate η^H for the Laughlin states. To determine the Berry potentials \mathcal{A}_τ and $\mathcal{A}_{\bar{\tau}}$, and thus the field strength, one must determine the full τ dependence of the normalized wave functions. In this case this can be done by generalizing the plasma analogy described in Sec. II.A to the case of a torus with arbitrary τ . Technically it amounts to keeping the full τ dependence in the calculation of the conformal blocks used to construct the wave function as discussed. The calculation of \mathcal{A}_τ then proceeds similarly to the derivation of Eq. (37); using the fact that the normalization constant is τ independent, the only contribution comes from the nonholomorphic (in τ) piece in the wave function which turns out to be determined by the orbital spin. In fact this nonholomorphic piece is directly related to the conformal spin of the electron operators. So, as long as holonomy equals monodromy, Eq. (86) will hold for any wave function which is just a product of conformal blocks up to Gaussian factors.

b. Other approaches to the Hall viscosity

The most direct way to use Eq. (85) is to simply compute $\mathcal{F}_{\tau\bar{\tau}}$ numerically from a normalized wave function that can be either a model wave function or one obtained numerically from a model Hamiltonian. Such calculations for the $\nu = 1/2$ bosonic and $\nu = 1/3$ fermionic Laughlin and the numerically found Moore-Read state were done by [Read and Rezayi \(2011\)](#), who found good agreement with Eq. (86). They also checked the stability of the result for the $\nu = 1/2$ state by adding a perturbation to the exact model Hamiltonian.

It was pointed out by [Bradlyn, Goldstein, and Read \(2012\)](#) and [Hoyos and Son \(2012\)](#) that, assuming Galilean invariance, the Hall viscosity can be extracted from the q^2 term in the momentum dependent Hall conductance $\sigma_H(q)$, which also can be read off from the term $\sim \bar{s} A d\omega$ in the effective response action (33) in Sec. II.H. This is perhaps not very surprising given that, according to Eq. (86), the Hall viscosity is proportional to the average orbital spin \bar{s} .

⁵⁸This lattice provides a natural way to define coherent states on the torus ([Haldane, 1985](#); [Fremling, 2013](#)).

⁵⁹This paper also discusses earlier work on the Hall viscosity and gives relevant references.

c. Hall viscosity for Abelian hierarchy states

Using the same techniques as Read and Rezayi (2011), Fremling, Hansson, and Suorsa (2014) calculated η^H for the $\nu = 2/5$ Jain state and found good agreement, both with Eq. (86) and with the corresponding results from the numerically obtained Coulomb wave functions. To appreciate the significance of this, let us again emphasize that the strong arguments for Eq. (86) were given under the assumption that our wave functions are normalized such that no extra Berry phases are incurred during the adiabatic change of τ , i.e., that all relevant τ dependence is explicit in the wave functions. For some of the states that can be written as correlators of primary fields, this can be convincingly argued using the exact mapping onto a classical plasma. Equation (86) will, however, apply to hierarchy states only if the reasoning based on the plasma analogy holds. Thus, the mere fact that the numerical results for η^H are in accordance with it gives strong support for the assertion that holonomy equals monodromy in the case of adiabatic changes of the τ parameters. Although this does not *per se* lend support to the corresponding claim for adiabatically moving quasiparticles, it certainly lends indirect support.

d. Hall viscosity in realistic systems

Here we comment on the significance of the Hall viscosity when we move away from an isotropic and clean system.

Concerning anisotropies, Haldane (2011) and Park and Haldane (2014) argued that when rotational invariance is broken, e.g., by a tilted magnetic field, Eq. (86) between Hall viscosity and orbital spin has to be modified. Also on a sufficiently deformed torus, i.e., coming close to the TT limit, Eq. (86) no longer holds (Fremling, 2015), in much the same way as braiding phases of quasiparticles are well defined only when they are sufficiently separated. Stated differently, we can only expect the topological properties to emerge in a suitable scaling limit (Fröhlich and Kerler, 1991).

As far as we know, there is no conclusive answer to whether or not the Hall viscosity has any meaning for weakly disordered systems, and the same holds for the shift on the sphere. Read and Rezayi (2011) claimed that in the presence of disorder “the shift ceases to have significance, due to the loss of rotational invariance on the sphere. The same will be true for the Hall viscosity.” In a strict sense, this is presumably true, but we believe that for sufficiently weak disorder, the shift ought to be robust, since changing the shift amounts to changing the number of particles in the ground state at given flux. Seen from the undisturbed system, this amounts to exciting quasiparticles that will be pinned to the impurity centers, but this will be energetically advantageous only if the strength of the impurity is comparable to the gap. We know of no similar argument for the Hall viscosity and find it quite likely that the quantization of this quantity is lost even for weak disorder. Even in this case the average value of the orbital spin of the constituent electrons might have a meaning, at least for weak disorder. However, it is completely unclear if this is an observable that could even in principle be measured.

Having said this, we stress that from both a conceptual and a computational point of view the Hall viscosity is a quite interesting quantity, as should have been clear from the previous discussion.

VIII. SUMMARY AND FUTURE DIRECTIONS

The main message of this review is that to address the plethora of observed states, and the multitude of theories, models, and schemes, there are powerful theoretical tools and deep physical principles that help us organize our knowledge of the vast and fascinating field of quantum Hall physics.

Hierarchy is such a principle. It comes in many varieties, but the basic idea of starting from a parent state and generating a sequence of topological liquids, by successive condensation of quasiparticles, is common to all of them. If the parent state is Abelian, so is the hierarchy, while a non-Abelian parent state can give rise to qualitatively different hierarchies depending on which quasiparticles condense, and the details of how it is done. In particular, certain types of condensations give rise to Abelian offspring, while others can result in non-Abelian states either similar to or different from that of the parent.

Conformal field theory has emerged as a unifying theoretical tool in quantum Hall physics. Originally its relation to quantum Hall physics was via Chern-Simons theory. Expectation values of the Wilson loops in these gauge theories, which encode the topological properties of quasiparticles, are closely related to the monodromies of the conformal blocks of the corresponding CFT. In the quantum Hall context, the CFT plays the dual role of providing representative bulk wave functions, and being the dynamical theory of the edge modes. In this review we also showed that the CFT techniques are not restricted to states that can be expressed as correlators of primary fields, such as the Laughlin, Moore-Read, and the Read-Rezayi states, but can also successfully be applied to hierarchy states. Although we did not mention it prominently, there is a very general, and in a sense minimal, mathematical description of topological phases, phrased in terms of tensor categories (Kitaev, 2003; Bonderson, 2007; Wang, 2010).

There are many intriguing questions concerning the relationships between different hierarchies, and between hierarchy constructions and other organizing principles. Approaches based on effective field theories for composite fermions and bosons naturally lead to hierarchy schemes, and effective field theories based on Chern-Simons gauge fields are general enough to encompass all the hierarchies discussed in this review. The composite-fermion wave functions also fit nicely in a hierarchy scheme, although they were originally organized using quite a different logic.

In spite of the successes of the various hierarchy schemes, there are many unanswered questions and several promising lines of research. Clearly there is still no comprehensive theoretical understanding of the non-Abelian hierarchies, and we are still waiting for a universally accepted experimental signature for any non-Abelian state. At a more detailed level, there are many remaining theoretical problems. We already mentioned the need for proofs of the expected topological properties of quasiparticles in the hierarchy states. At the

theoretical level this would require new tools, such as much more refined plasma analogies, while convincing numerical work would require developing novel methods. The extension of the matrix product state methods (Zaletel and Mong, 2012; Wu *et al.*, 2015) to hierarchy states might be the way to go forward. Another challenging problem is to find a direct and precise link between the microscopic physics of the quantum Hall effect and the description in terms of CFT correlators, which is not based on case-by-case numerical verifications of the Moore-Read conjecture.

The biggest question is, perhaps, to what extent the methods discussed in this work will pertain to other systems. The quantum Hall effect is clearly quite special as the only (for dimensions $d > 1$) experimental system that has convincingly demonstrated topological properties, such as charge fractionalization. Nonetheless, there is an enormous amount of research activity exploring other systems that have nontrivial topological properties (Hasan and Kane, 2010; Kallin and Berlinsky, 2016; Savary and Balents, 2017) and many such materials have recently been found. While no experiment has observed any hierarchy physics in systems that are not quantum Hall systems, there is nonetheless substantial influence of theoretical ideas from the quantum Hall world on these research topics (Lu and Vishwanath, 2012, 2016). It is very likely that our deep understanding of quantum Hall hierarchies will continue to enlighten a broad range of future research directions.

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APPENDIX A: SUPERABBREVIATED QUANTUM HALL BASICS

The bare basics of quantum Hall physics will be given here for reference. For further details see the various books on the subject (Prange and Girvin, 1990; Stone, 1992; Chakraborty and Pietiläinen, 1995; Ezawa, 2008).

We consider a particle in two dimensions, of mass μ and charge q in a magnetic field B perpendicular to the plane. The single particle energy spectrum breaks up into degenerate ‘‘Landau levels’’ with energies $E = \hbar\omega(n + 1/2)$, where $\omega = qB/\mu$ is the cyclotron frequency. Working in symmetric gauge so that the vector potential is $\vec{A} = (B/2)(y, -x, 0)$, we can construct a set of ladder operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(z/2 + 2\bar{\partial}), & a^\dagger &= \frac{1}{\sqrt{2}}(\bar{z}/2 - 2\partial), \\ b &= \frac{1}{\sqrt{2}}(\bar{z}/2 + 2\partial), & b^\dagger &= \frac{1}{\sqrt{2}}(z/2 - 2\bar{\partial}), \end{aligned}$$

with $z = x + iy$, and where $\partial \equiv \partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$ and $\bar{\partial} \equiv \partial/\partial \bar{z} = (1/2)(\partial/\partial x + i\partial/\partial y)$, and the magnetic length scale $\ell = \sqrt{\hbar/eB}$ has been set to unity (a convention that we adopt often). Note that $\partial^\dagger = -\bar{\partial}$. These operators satisfy $[a, a^\dagger] = [b, b^\dagger] = 1$ with other commutators vanishing. The single particle Hamiltonian and the z component of angular momentum take the form

$$H = \hbar\omega(a^\dagger a + 1/2), \quad L_z = \hbar(b^\dagger b - a^\dagger a). \quad (\text{A1})$$

Thus, a^\dagger is a Landau level raising operator, which changes the energy eigenvalue and b^\dagger increases the angular momentum one unit within the Landau level.

Defining a fiducial state $|0, 0\rangle$, which is annihilated by both a and b , we can construct a single particle eigenstate with energy eigenvalue $\hbar\omega(n + 1/2)$ and L_z eigenvalue $\hbar(m - n)$:

$$|n, m\rangle \sim (a^\dagger)^n (b^\dagger)^m |0, 0\rangle.$$

Explicitly, we have the real-space wave function form of the fiducial state $\langle z|\psi_{00}\rangle \sim e^{-|z|^2/(4\ell^2)}$ so that the LLL basis states become

$$\langle z|\psi_{0m}\rangle \sim (b^\dagger)^m e^{-|z|^2/(4\ell^2)} \sim z^m e^{-|z|^2/(4\ell^2)}.$$

Here the m th state is shaped like a ring of radius $r^2 = x^2 + y^2 = 2m\ell^2$. The density of single particle eigenstates (and hence the density of a filled Landau level) is thus $m/(\pi 2m\ell^2) = 1/(2\pi\ell^2)$. The most general LLL wave function for a single particle is $f(z)e^{-|z|^2/(4\ell^2)}$ for any analytic function f .

Unless otherwise stated, we will always be interested in the physics of a partially filled LLL. We define the filling fraction $\nu = \rho\phi_0/B$, where ρ is the particle density and $\phi_0 = 2\pi\hbar/e$ is the flux quantum. For fermionic particles, ν can also be understood as the fraction of the orbitals available in the Landau level that are occupied. When the LLL is partially filled there is an enormous degeneracy corresponding to the choice of which orbitals should be filled and which should be left empty.⁶⁰ This degeneracy is broken only by the interactions between the particles, since within a single Landau level, the kinetic energy H from Eq. (A1) is simply an uninteresting constant.

When there are N identical particles in a fractionally filled LLL, a many particle wave function must be of the form

$$\Psi = f(z_1, \dots, z_N) \prod_{i=1}^N e^{-|z_i|^2/(4\ell^2)},$$

⁶⁰For bosons, the degeneracy remains even for ν an integer.

where f is a symmetric (antisymmetric) analytic function when the particles are bosons (fermions). If the Hamiltonian is rotationally symmetric and the ground state is nondegenerate, then it should be an angular momentum eigenstate, which means the function f should be a homogeneous polynomial with an overall degree that is set by the total angular momentum of the system. Given a function f , one can determine the filling fraction, by finding the highest power L_{\max} of z_1 (or of any given z_i). This determines the maximum radial extent of the droplet. Thus, the filling fraction is given by

$$\nu \rightarrow N/L_{\max}$$

as both N and L_{\max} become large. For finite N we have $L_{\max} = N/\nu - S$, where S is a constant known as the *shift*; see Eq. (15).

Which analytic function f should be chosen is determined by the particular interparticle interaction so as to minimize the interaction energy at the appropriate filling fraction in order to find a ground state wave function. Laughlin's original ansatz for $\nu = 1/m$ given by

$$f = \prod_{i<j} (z_i - z_j)^m$$

was motivated by the intuition that forcing the wave function to vanish rapidly as two particles approach each other will essentially keep the particles farther apart and is therefore prone to lower the energy due to a short-ranged repulsive interaction.

In fact, the Laughlin wave function can be shown to be the exact ground state of a particular ultra-short-range interaction which gives positive energy when the wave function vanishes slower than m powers as two particles approach each other (Haldane, 1983; Pokrovsky and Talapov, 1985; Trugman and Kivelson, 1985). The Laughlin $\nu = 1/m$ ground state is the highest density zero energy state of this interaction. The quasihole states obtained by multiplying the wave function by factors $\prod_i (z_i - \eta_a)$ are also zero energy states of the Hamiltonian, but are at lower electron density. Similarly, edge excitations can be obtained by multiplying the Laughlin ground state by any symmetric polynomial.

Let us comment on the case of a partially filled higher Landau level. The easiest way to handle such higher Landau level wave functions is to use the one-to-one mapping to LLL wave functions, via

$$|n\text{th LL wave function}\rangle = \prod_i (a_i^\dagger)^n |\text{LLL wave function}\rangle,$$

where a_i^\dagger is the Landau level raising operator for the i th particle. Note that the application of a^\dagger operators puts \bar{z} factors in front of the Gaussian, and typically the n th Landau level wave functions will have terms with n such factors of each \bar{z} . A wave function can be projected (Girvin and Jach, 1984) to the LLL by moving all \bar{z} coordinates to the left and replacing them with $2\partial_{\bar{z}}$.

For many filling fractions, the system will have a gap in the excitation spectrum above the ground state—this is a sign of

incompressibility. If the system is also fluid (in the sense of not having broken translational symmetry) then this is known as a quantum Hall ground state. The stability of the quantum Hall state is determined by the size of the excitation gap. There are two types of collective excitations in quantum Hall systems. The low-energy excitations are intra-Landau-level quasiparticle-quasihole pairs, which are commonly called magnetorotons. Their energy gap is set by the Coulomb interaction, and it is this gap that is relevant for the stability of the quantum Hall state. The other collective mode is the cyclotron resonance mode of the center of mass, also called the “Kohn mode.” At zero momentum it is just the cyclotron motion of the center of mass, so the gap is $\hbar\omega_c$ independent of the interactions (Kohn, 1961).

In the absence of disorder, by Galilean invariance,⁶¹ the conductance matrix of a two-dimensional system in a magnetic field is always given by $\sigma_{xx} = \rho_{xx} = 0$ for the longitudinal component and $\sigma_H = \sigma_{xy} = 1/\rho_{yx} = \nu e^2/h$ for the Hall component, with h being Planck's constant and e the electron charge. If we consider a fractional quantum Hall ground state with filling fraction $\nu = p/q$, when disorder is added, the Hall conductance will be fixed at $\sigma_H = (p/q)e^2/h$ for a range of values of ν around p/q and σ_{xx} remains zero over this range as well. This is known as a quantum Hall plateau and typically the size of the plateau, i.e., the range of ν for which the conductance is quantized, is larger for quantum Hall states with larger gaps. If the disorder or the temperature becomes larger than roughly the gap scale, then the quantum Hall state will be destroyed: σ_{xx} becomes nonzero, and the quantization of σ_H is lost.

APPENDIX B: EVERYTHING YOU NEED TO KNOW ABOUT CONFORMAL FIELD THEORY

Conformal field theory (CFT) is a rich and interesting subject and there are many excellent introductions to the subject itself, most notably perhaps is the book by Di Francesco, Mathieu, and Sénéchal (1997). For a pedagogical review on how it is applied to quantum Hall states, see Nayak *et al.* (2008). Although the topic is quite complex,⁶² for our purposes we use only some of the very basic ideas.

While CFTs exist in higher dimensions, they are particularly powerful for describing the physics of gapless (1 + 1)-dimensional relativistic systems with spatial coordinate x , time coordinate t , and speed of light c which we usually set to unity. We often use the complex coordinates $z = x + it$ and $\bar{z} = x - it$, which naturally describe left-moving and right-moving particles. In the case of general-gapless (1 + 1)-dimensional systems the two sectors decouple almost perfectly, and one can treat the holomorphic (z) and antiholomorphic (\bar{z}) variables almost entirely independently of each other. In this Appendix we mainly focus on the holomorphic parts, so we are essentially considering a left-moving chiral theory. The generalization to the right-moving,

⁶¹The Galilean invariant result holds for strictly zero disorder, but not necessarily for the limit of disorder going to zero.

⁶²Pun intended.

or antiholomorphic sector, which is needed when considering general quantum Hall states, should be obvious.

The use of CFT in the quantum Hall context is twofold. As discussed in Sec. I.B in the main text, the CFT is used to describe not only the (1 + 1)-dimensional dynamics of the chiral quantum Hall edge, but also the bulk (2 + 0)-dimensional wave function, which essentially is a purely holomorphic polynomial. While CFTs in 1 + 1 dimension completely describe the physics of such systems, for our purposes it is easier to think of a particular CFT as being simply an operator algebra with certain properties, and which is used to generate holomorphic functions, and hence quantum Hall wave functions.

While one might attempt to build a quantum Hall state, both at the edge and in the bulk, from any (1 + 1)-dimensional CFT, it turns out that various subtle consistency conditions prevent most CFTs from describing sensible quantum Hall states. Most of them turn out to be described by some version of one of the simplest CFTs imaginable, the chiral boson, which we describe in some detail later.

1. General CFTs

Generally, a CFT contains a set of primary fields $\Phi_i(\bar{z}, z)$ (or sometimes just Φ_i) each with scaling dimensions Δ_i . Each primary field also has a tower of corresponding descendant fields (which are basically derivatives of the primary fields) whose scaling dimensions are higher by integer steps from the corresponding primary. The primary fields obey a set of fusion rules⁶³

$$\Phi_i \times \Phi_j = \sum_k N_{ij}^k \Phi_k, \quad (\text{B1})$$

where the N_{ij}^k 's are non-negative integers. A field Φ_i is known as a simple current if fusion with this field acts as a permutation on the fields, or in other words if for fixed i and any j we have N_{ij}^k being zero for all values of k except a single value of k for which $N_{ij}^k = 1$.

There always exists a special field known as the identity (written usually as 1) which fuses trivially with all fields $N_{1j}^k = \delta_{j,k}$. Each field i has a unique conjugate field \bar{i} , with which it can fuse to form the identity, i.e., such that $N_{i\bar{i}}^1 = 1$. Note that a field's inverse may be itself. A so-called charge conjugation matrix C_{ij} is defined to be unity for two fields that are conjugates $C_{i\bar{i}} = 1$ and zero otherwise. The matrix $N_{ijp} = \sum_k C_{pk} N_{ij}^k$ must be fully symmetric under permutation of i, j , and p .

The objects of particular interest for us are the conformal blocks, which are correlation functions of holomorphic or

antiholomorphic fields. In many cases there are techniques for calculating these blocks independently, while in general they can be extracted by factorizing the correlators of the full fields $\Phi(\bar{z}, z)$, which essentially are the products of the holomorphic $\Phi(z)$, and the antiholomorphic fields $\bar{\Phi}(\bar{z})$,

$$\begin{aligned} &\langle \Phi(z_1 \bar{z}_1), \dots, \Phi(z_N, \bar{z}_N) \rangle \\ &= \sum_{\alpha} \langle \bar{\Phi}(\bar{z}_1), \dots, \bar{\Phi}(\bar{z}_N) \rangle_{\alpha} \langle \Phi(z_1), \dots, \Phi(z_N) \rangle_{\alpha}, \end{aligned} \quad (\text{B2})$$

where α labels the different blocks. In simple cases the sum has only one term, but in general there are many blocks that will correspond to different quantum Hall wave functions, as discussed in Secs. IV.C.1 and VII.B. The scaling dimensions or “conformal weights” of the fields Φ_i and $\bar{\Phi}_i$ are denoted by h_i and \bar{h}_i , respectively.⁶⁴ The total scaling dimension is then $\Delta_i = h_i + \bar{h}_i$, and the conformal spin, which in the context of quantum Hall wave functions is identified with the orbital spin, is given by $s_i = h_i - \bar{h}_i$.

There is a “neutrality” condition on any correlator $\langle \Phi_1 \Phi_2 \dots \Phi_M \rangle$ that the fields inside the correlator must fuse to the identity, or the entire correlator will vanish. The correlator of two conjugate fields must take the form

$$\langle \Phi_i(z) \Phi_{\bar{i}}(w) \rangle = (z - w)^{-2h_i} \quad (\text{B3})$$

by dimensional analysis. The coefficient of unity (not written) on the right of this equation is a conventional normalization of the fields. There is a completely algebraic approach to computing the correlators $\langle \Phi_1 \Phi_2 \dots \Phi_M \rangle$, using fusion rules and conformal invariance, but in the cases where there is a Lagrangian description the correlators are just the usual vacuum expectation values of time (or in the Euclidian case radially) ordered products of field operators that can be calculated using standard field theory techniques. The main examples used in this review, namely, the chiral boson and the Ising CFT (used to construct the Moore-Read state), belong to this category.

The fusion rules and the scaling dimensions allow one to write the operator product expansion which describes what happens when coordinates of the fields approach each other

$$\Phi_i(z) \Phi_j(w) = \sum_k C_{ij}^k (z - w)^{h_k - h_i - h_j} \Phi_k(w) + \dots, \quad (\text{B4})$$

where the coefficients C_{ij}^k are constants that are only nonzero when N_{ij}^k is. There are also terms on the right-hand side corresponding to descendant fields,⁶⁵ but if we are considering only the leading terms in the expansion, it is sufficient just to write the primaries.

⁶³Fusion rules simply specify the possible outcomes when taking the short-distance product of two local fields. One can consider it a shorthand notation of the operator product expansion in Eq. (B4). When the integer N_{ij}^k is greater than 1, it means that when Φ_i fuses with Φ_j it results in several copies of the field Φ_k . This is analogous to tensor products of group representations; e.g., in $SU(3)$, $8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \bar{10} \oplus 27$, so 8 occurs twice.

⁶⁴Note that the conjugate of a field i has the same conformal weight as the original field, i.e., $h_i = \bar{h}_i$.

⁶⁵Descendants are obtained by acting with derivatives on primary fields.

2. The chiral boson

Perhaps the simplest example of a CFT is the chiral boson, described by the Lagrangian (27). We define a boson field $\phi(z)$ as a function of the holomorphic coordinate z . These are free boson fields so the bosonic version of Wick's theorem applies. The elementary two-point correlator is given by

$$\langle \phi(z)\phi(z') \rangle = -\frac{1}{k} \log(z - z').$$

Note that the constant k can be changed by renormalizing the fields. In Sec. II.G we have $k = 1/\nu$. In Sec. IV and onward, we will take $k = 1$. In the main text, chiral bosons with normalization $k = 1/\nu$ are usually denoted by φ , while those with normalization $k = 1$ are denoted by ϕ .

The vertex operators are defined as

$$V_\alpha(z) = :e^{i\alpha\phi(z)}:, \quad (\text{B5})$$

with the colons representing normal ordering. The fusion rules are given by

$$V_\alpha \times V_\beta = V_{\alpha+\beta}.$$

The key identity (which can be proven by use of Wick's theorem) is

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\cdots V_{\alpha_N}(z_N) \rangle = \prod_{i<j} (z_i - z_j)^{\alpha_i\alpha_j/k} \quad (\text{B6})$$

provided that $\sum_j \alpha_j = 0$, otherwise the correlator vanishes identically. The reason for this is that α is the charge of the operator V_α with respect to the $U(1)$ symmetry $\phi(z) \rightarrow \phi(z) + a$. For the correlator on the left-hand side of Eq. (B6) to be invariant under such $U(1)$ transformations, the resulting phase factor $\prod_j e^{i\alpha_j a}$ has to be 1, thus yielding the charge neutrality conditions. Frequently we can be sloppy about this final constraint on the sum of α as we can imagine placing an appropriate neutralizing charge at infinity to force it to be satisfied if we want. Or as discussed in the main text [see Eq. (36)] we can use a smeared background charge operator.

By dimension counting, the scaling dimension of the primary field V_α is given by $\alpha^2/(2k)$, and combining this with the operator product expansion (B4), we see that, taking $k = 1$, the vertex operators (B5) are bosonic or fermionic for α being an even or odd integer, respectively.

3. The compactified boson

In the quantum Hall context (say, for the Laughlin states) we do not want our Bose CFT to contain quasiparticle operators V_α for all possible α . As discussed in connection with Eq. (44), once we define our electron operator $\psi_e = V_{\alpha_0}$, we must insist that all other operators in the theory are mutually local with respect to this operator; that is, there should be no branch cuts in the electron wave function due to the insertion of quasiparticle operators. This limits the particle content of our theory to only contain operators $\psi_n = V_{n/\alpha_0}$ with integer n .

This constraint on the Bose vertex algebra is in fact equivalent to having what is known as a ‘‘compactified’’ boson. One begins with the same Bose Lagrangian (27), but imposes the periodicity relation

$$\phi(z) = \phi(z) + 2\pi R,$$

where R is known as the compactification radius, i.e., ϕ is defined on a circle with radius R . This immediately constrains the possible vertex operators since

$$V_\alpha = :e^{i\alpha\phi}: = :e^{i\alpha(\phi+2\pi R)}: = e^{2\pi i\alpha R} V_\alpha$$

which implies that $\alpha = n/R$, i.e., there is a discrete set of allowed values for α .

APPENDIX C: COMMUTATOR OF EDGE OPERATORS

In the text we motivated the fundamental anomalous commutator, Eq. (24), by a heuristic argument, based on the expected equation of motion for edge excitations. In this Appendix we give two complementary field theoretic derivations.

1. Approach 1: The UV perspective

Here we give a derivation which is essentially the original one by Mattis and Lieb (1965) but cast in the quantum Hall context. Working in Landau gauge, single-electron orbitals can be taken to be strips parallel to the edge, indexed by their momentum in the direction along the edge k and having position perpendicular to the edge $y = k\ell^2$. For $\nu = 1$, the ground state is simply described as a Slater determinant of k states filled up to a Fermi energy k_F (we imagine a half-infinite space so we consider only one edge). The density operator along the edge (with the same normalization as Sec. II.G.1) is then

$$\rho_q = \frac{1}{\sqrt{L}} \sum_k c_{k+q}^\dagger c_k. \quad (\text{C1})$$

Direct calculation of the commutator yields

$$[\rho_q, \rho_{q'}] = \frac{1}{L} \sum_{k,k'} [c_{k'+q+q'}^\dagger c_{k'} \delta_{k,k'+q} - c_{k+q+q'}^\dagger c_k \delta_{k',k+q}].$$

While it may look like the two terms on the right precisely cancel, consideration of the ultraviolet limit (k far from the Fermi surface) can give something nonzero. When we are far below the Fermi surface, since all states are filled, $c_{k+q+q'}^\dagger c_k$ is zero unless $q = -q'$. However, when $q = -q'$, then we are generally subtracting two nonzero terms from each other and we must be more careful since there are potentially an infinite number of such terms, and depending on how they are paired up, they can leave a finite result. Thus we focus on this particular case and obtain

$$[\rho_q, \rho_{-q}] = \frac{1}{L} \sum_k [n_{k+q} - n_k], \quad (\text{C2})$$

where $n_k = c_k^\dagger c_k$ which is exactly unity deep below the Fermi surface. Now suppose we try to cut off the infinite sum in Eq. (C1), say by including a regularization factor w_k inside the sum which is equal to unity near the Fermi surface, but drops to zero very far away from the Fermi surface. For any form of this cutoff function (we can consider a step function) we find that the difference in the two terms is nonzero and gives exactly $q(L/2\pi)$ which is just the number of k states that lie between $k + q$ and k , thus recovering exactly Eq. (24). In the case of FQHE, one need only modify this derivation by assuming that far below the Fermi surface n_k will be ν rather than 1.

2. Approach 2: The IR perspective

This derivation was contingent upon subtle noncancellations of potentially UV divergent terms and as such appears as a UV effect. Although formally correct, this looks strange, particularly from a condensed matter perspective where there is certainly no infinite Dirac sea. A similar conundrum is related to the axial anomaly in 1 + 1 dimensions. Defining left- and right-moving currents to be j_+ and j_- , respectively, the vector and axial currents are $j_V = j_+ + j_-$ and $j_A = j_+ - j_-$. We then have $\partial_\mu j_A^\mu = eE/\pi$, which for a chiral theory amounts to an anomaly in the fermion current $j = (j_A + j_V)/2$ given by $\partial_\mu j^\mu = eE/(2\pi)$. This can, by the following reasoning, be understood as an IR effect.

A constant electric field in the positive x direction will give rise to a spectral flow at the Fermi level, which corresponds to creating right-moving particles and left-moving holes in such a way that the total electric charge is conserved, while the axial charge, i.e., the difference between right and left movers, is not. For a more detailed description of this “infinite hotel”⁶⁶ derivation of the axial anomaly, see [Zahed and Brown \(1986\)](#).

We now give a direct derivation of the anomalous commutator using only IR arguments. Starting with an achiral one-dimensional system, we can construct a chiral theory by separating the two chiralities. To do this we start with the usual one-dimensional (nonchiral) current and density operators in first quantization

$$\rho(x) = \sum_{n=1}^N \delta(x - x_n)$$

and

$$j(x) = \frac{1}{2m} \sum_{n=1}^N [p_n \delta(x - x_n) + \delta(x - x_n) p_n]$$

from which we can directly derive

$$[\rho(x), j(x')] = -i(\hbar/m)\rho(x)\partial_x\delta(x - x') \quad (\text{C3})$$

along with

⁶⁶This name seems to have been coined by H. B. Nielsen in reference to Hilbert’s infinite hotel paradox.

$$[\rho(x), \rho(x')] = [j(x), j(x')] = 0.$$

We then assume small excitations of both the left-going and right-going Fermi surfaces. Then we can write $\rho = \rho_0 + \rho_+ + \rho_-$ and $j = v_F(\rho_+ - \rho_-)$ where $+$ and $-$ again indicate left and right movers, and ρ_0 is a constant background density which is much larger than the perturbations ρ_+ and ρ_- . We then have

$$\begin{aligned} [\rho_+(x), \rho_+(x')] &= \frac{1}{4}[\rho(x) + j(x)/v_F, \rho(x') + j(x')/v_F] \\ &= -i\hbar/(2v_F m)\rho(x)\partial_x\delta(x - x') \\ &= -i/(2\pi)\partial_x\delta(x - x'), \end{aligned} \quad (\text{C4})$$

where in going to the last line we substituted ρ_0 for $\rho(x)$ on the right, since it is assumed to be much larger than ρ_+ and ρ_- . We also used that for spinless one-dimensional fermions $\rho_0 = k_F/\pi = mv_F/(\pi\hbar)$. The chiral operators ρ_+ are what we called $\rho(x)$ in Sec. II.G.1, and Fourier transforming gives Eq. (24).

Note that just as in the infinite hotel derivation of the axial anomaly, we considered only effects close to the Fermi surface. The connection between the two calculations follows if we couple the right-moving charge to an electromagnetic potential by $H = e \int A_0 \rho_+$ and use Eq. (C3) and partial integration to get

$$\begin{aligned} \partial_t Q_+ &= \int dx dy [\rho_+(x), eA_0(y)\rho_+(y)] \\ &= e \int dx \partial_x A_0(x) \\ &= (1/\pi) \int dx eE(x) \end{aligned}$$

which is the integrate form of the fermion current anomaly for spatially constant fields.

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