REVIEWS OF MODERN PHYSICS, VOLUME 89, JANUARY–MARCH 2017

Colloquium: Status of α-particle condensate structure of the Hoyle state

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(published 26 January 2017)

The present understanding of the structure of the Hoyle state in ^{12}C is reviewed. Most of the theoretical approaches to the Hoyle state are shortly summarized. The corresponding results are analyzed with respect to whether they give evidence to the α-particle condensation structure of the Hoyle state (and other Hoyle-like states in heavier self-conjugate nuclei) or not.

DOI: 10.1103/RevModPhys.89.011002

CONTENTS

I. INTRODUCTION

The 0_2^+ state at 7.65 MeV in ^{12}C, known as the Hoyle state, is one of the most important states in nuclear physics. This stems from the fact that it is the gateway for the massive ^{12}C production in the Universe and is, thus, responsible for life on Earth. It was predicted by the astrophysicist Hoyle (1954) at practically the correct energy [found by W. A. Fowler *et al.* in 1957 (Cook *et al.*, 1957)]. There was some discussion in the past whether F. Hoyle predicted his state on anthropic grounds or not. Apparently, this was not the case; see Kragh (2010). Standard shell model calculations give the energy of the first 0^+ excited state in ^{12}C at over 20 MeV. Therefore, because of its unexpected low energy, the structure of the Hoyle state stayed mysterious for a long time. Morinaga (1956) postulated that it is a state where the three α particles are lined up straight, the so-called three α chain state. However, Horiuchi (1974, 1975) found with the semimicroscopic approach called the orthogonal condition model (OCM) (Saito, 1968, 1969) (see later) that the Hoyle state should be interpreted as a state of three weakly coupled α particles interacting pairwise in relative $0S$-wave states. This point of view was confirmed in 1977 by two ground-breaking works by Kamimura (1981) and Uegaki *et al.* (1977, 1978). Using a phenomenological nucleon-nucleon force of Gaussian type which was earlier adjusted independently [Volkov force (Volkov, 1965)], they reproduced with a fully microscopic 12 nucleon wave function all known properties of the Hoyle and other loosely bound α states above the Hoyle state. The achievement of the two works was, at that time, so outstanding that one is tempted to say that as usual after great exploits the subject of the Hoyle state stayed practically dormant for about a quarter of a century. Only in 2001 did the work appear of Tohsaki, Horiuchi, Schuck, and Röpke (THSR) which interpreted the Hoyle state (and other states in self-conjugate nuclei) as a condensate of α particles (Tohsaki *et al.*, 2001). That means that the α particles with their center of mass (c.m.) motion occupy all the lowest $0S$ orbits of their common mean field. This work triggered an intense new interest in the Hoyle state, both theoretically and experimentally; see Schuck *et al.* (2007), Funaki *et al.* (2007), Yamada *et al.* (2011a, 2011b), Freer and Fynbo (2014), and Funaki, Horiuchi, and Tohsaki (2015), and references therein. At present, this research culminates in works trying to explain the properties of the Hoyle state from *ab initio* and/or quantum Monte Carlo (QMC) approaches (Epelbaum *et al.*, 2012; Carlson *et al.*, 2015). Already in the work of THSR (Tohsaki *et al.*, 2001), it was predicted that in other $n\alpha$ nuclei Hoyle-analog states should exist around the $n\alpha$ disintegration threshold. For instance, ^{16}O is the subject of intense studies both theoretically and experimentally (Funaki *et al.*, 2008; Itoh *et al.*, 2010, 2014; Epelbaum *et al.*, 2014). The situation in these self-conjugate nuclei is now considered under a completely novel aspect, namely, that at energies close to the α disintegration threshold, there exist states of extended volume (3–4 times the volume of the ground state) where the nuclei are formed by a gas of loosely bound α particles which move in their own mean field. These bosonic states coexist with the standard fermionic ones, where individual nucleons move in a common mean field. This is an exciting new feature of nuclear physics of importance for nuclei and for astrophysical aspects.

Pairing is well known and well accepted as a useful approximation to describe two particle correlations. Pairing between like nucleons (n-n, p-p) is a useful concept not only in infinite matter, but also in finite nuclei (Ring and Schuck, 1980). Because the interaction between protons and neutrons (deuteron channel) is even stronger (a bound state can be formed), one should expect that in symmetric matter also proton-neutron pairing should appear. Calculations performed for the p-n channel give a critical temperature depending on density which can rise up to a value T_c near to 4.5 MeV (Jin, Urban, and Schuck, 2010) which is about 3 times larger than the maximum critical temperature in the isospin triplet channel. However, it competes with the formation of α particles and for these particles the transition to a Bose condensate at increasing density occurs prior to the quantum condensation in the deuteron channel as discussed later.

The next step is quartetting which is also a good approximation in special situations. The THSR wave function was an important step to be introduced in nuclear structure physics. Nuclear physics is now on the forefront of the studies on quartet and cluster formation, and their possible condensation, but there are also works in other fields with still ongoing interest. For example, there exist speculations that in semiconductors where excitons, that is bound states of a conduction electron and an electron hole, can bind to bi-excitons which may enter in competition with single excitons in a possible Bose-Einstein condensation (BEC) (Noziéres and Saint-James, 1982). There exist also theoretical works which predict that, once four different fermions can be trapped in cold atom devices, bound quartets can be formed with again the possibility of a BEC (Capponi *et al.*, 2007).

The purpose of this paper is to briefly review the present situation concerning the possibility of α-particle condensation and other α gas states in self-conjugate nuclei which was proposed for the first time 15 years ago by Tohsaki *et al.* (2001). It is important to clarify at this point that we understand the word "condensate" or "condensation" in the sense that the α particles with inert internal structure all move with their c.m. in the same lowest $0S$ orbit of their common mean field. We show in this article that these states can be considered as the precursors of a macroscopic α-particle condensate in homogeneous nuclear matter at low density. Of course, one should understand that in the following the term condensate for a handful of α particles is stretched to its limits.

Because of its outstanding importance, we mainly concentrate on ^{12}C and the Hoyle state but, at the end of this Colloquium, we also touch on the situation in other nuclei. We present in condensed form the various theoretical attempts which are used to describe the formation and existence of quartets in nuclei, that is the α gas states, induced by strong four nucleon correlations (two neutrons, two protons) at densities well below saturation. We discuss to which degree they give arguments against or in favor of the hypothesis that the Hoyle state can be considered to good approximation as a state where the three α particles are condensed into the c.m. $0S$ orbital.

Historically, the idea of α condensation is based on the study by Röpke *et al.* (1998), where the critical temperature of quartet condensation in infinite matter was investigated. This study was performed in analogy to the determination of the critical temperature for the onset of pairing, i.e., superfluidity or superconductivity, employing the in-medium two fermion equation as done by Thouless (1960) (now known as the Thouless criterion). For the α particle, the corresponding in-medium four fermion equation was used and solved by Röpke *et al.* (1998). Since the α particle is a very strongly bound quartet with a binding energy or particle of $\sim$7.5 MeV/particle which is about 7 times larger than the one of the deuteron and almost as large as in the strongest bound nucleus which is iron, the critical temperature turned out to be, at low density, over a factor of 6 higher than the one of neutron-neutron pairing. This finding was then logically transposed, in analogy with what had happened in the case of pairing to finite nuclei. The presentation of the physics involved in quartetting is the main subject of this Colloquium.

The paper is organized as follows. In Sec. II, we give a short summary of the THSR approach with the main focus on what it predicts with respect to the Hoyle state being an α-particle condensate state. In Sec. III, we revisit in a nutshell all other theories which may have some connection with the α condensate aspect. In Sec. IV we give a glimpse into the situation of ^{16}O and in Sec. V, eventual experimental evidences are discussed. Finally in the last section, we present some further discussions together with our conclusions and a short outlook.

II. THE THSR APPROACH AND THE HOYLE STATE

As mentioned, in 2001 a new aspect of the Hoyle state came to the forefront of the discussion. Tohsaki *et al.* (2001) proposed that the Hoyle state might not only be a gaslike state of three α particles but it was suggested that the three α's are, with their c.m. motion condensed into an identical $0S$ orbital, a situation reminiscent of what happens in cold atom physics where, however, a much larger number of bosons exists. In addition it was predicted that not only does ^{12}C contain such an α condensate but also several heavier self-conjugate nuclei such as ^{16}O, ^{20}Ne, etc., may exhibit analogous features. The idea of condensation was first investigated in nuclear matter (Röpke *et al.*, 1998) and then borne out by the use of a condensate type of wave function for finite self-conjugate nuclei, now known as the THSR wave function (Tohsaki *et al.*, 2001). The most basic form of THSR is for the case of three α particles a *single* wave function of the following structure:

$$\Psi_{\mathrm{THSR}} \propto \mathcal{A}\psi_1\psi_2\psi_3 \equiv \mathcal{A}|B\rangle \tag{1}$$

with

$$\psi_i = e^{-[(\mathbf{R}_i-\mathbf{X}_G)^2]/B^2}\phi_{\alpha_i} \tag{2}$$

and

$$\phi_{\alpha_i} = e^{-\sum_{k<l}(\mathbf{r}_{i,k}-\mathbf{r}_{i,l})^2/(8b^2)}. \tag{3}$$

In Eq. (1) the $\mathbf{R}_i$ are the c.m.coordinates of α particle i and $\mathbf{X}_G$ is the total c.m. coordinate of ^{12}C. $\mathcal{A}$ is the antisymmetrizer of the 12 nucleon wave function with ϕ_{α_i} the intrinsic translational invariant wave function of the α particle i. The whole 12 nucleon wave function in Eq. (1) is, therefore, translationally invariant. Note that we suppressed the scalar spin-isospin part of the wave function. The special Gaussian form given in Eqs. (2) and (3) was chosen by Tohsaki *et al.* (2001) to ease the variational calculation. The condensate aspect lies in the fact that Eq. (1) is a (antisymmetrized) product of 3 times the same α-particle wave function and is, thus, analogous to a number projected BCS wave function in the case of pairing. This 12 nucleon wave function has two variational parameters b and B. It possesses the remarkable property that for $B = b$ it is a pure harmonic oscillator Slater determinant [this aspect of Eq. (1) is explained by Bayman and Bohr (1958) and Yamada *et al.* (2008)], whereas for $B \gg b$ the α's are at low density so far apart from one another that the antisymmetrizer can be dropped and, thus, Eq. (1) becomes a simple product of three α particles, all in identical $0S$ states, that is, a pure condensate state. The minimization of the energy with a Hamiltonian containing a nucleon-nucleon force independently determined earlier (Tohsaki, 1994) allows one to obtain a reasonable value for the ground state energy of ^{12}C. Variation of energy under the condition that Eq. (1) is orthogonal to the previously determined ground state allows one to calculate the first excited 0^+ state, i.e., the Hoyle state. While the size of the individual α particles remains very close to their free space value ($b \simeq 1.37$ fm), the variationally determined B parameter takes on about 3 times this value. It is important to note that this so determined THSR wave function has about 98% squared overlap with the one of Kamimura (1981) (and practically 100% squared overlap with a slightly more general THSR wave function superposing several B values). We will shortly explain Kamimura's wave function in Sec. III. It can, even today, be considered as one of the most efficient approaches for the Hoyle state. In any case, as in the work of Kamimura (1981), so does the THSR approach reproduce very well all known experimental data about the Hoyle state. This concerns for instance the inelastic form factor, electromagnetic transition probability, and position of energy; for more details, see Schuck *et al.* (2007), Funaki *et al.* (2006, 2007), Yamada *et al.* (2011a, 2011b), Freer and Fynbo (2014), and Funaki, Horiuchi, and Tohsaki (2015). (The inelastic form factor is shown in Fig. 8.) At practically the same time Uegaki *et al.* (1977, 1978) published a very similar paper leading to almost identical results. In the following, we often refer only to Kamimura's work, since we were able to numerically compare THSR and Kamimura's wave functions. However, all that we say about Kamimura's work equally applies to the work of Uegaki *et al.*

The THSR wave function contains two limits: a pure Slater determinant and a pure Bose condensate below about a fifth of the saturation density ρ_0. To which end is the Hoyle state closest?

To this end it is instructive to consider the effect of the antisymmetrizer in Eq. (1) in more detail. In Fig. 1 we show the expectation value of the antisymmetrizer

$$N(B) = \frac{\langle B|\mathcal{A}|B\rangle}{\langle B|B\rangle} \tag{4}$$

in the Hoyle state. Indicated are the optimal values of the B parameter for the ground state and the Hoyle state.

For $B \to \infty$ the quantity in Eq. (4) tends to 1, since as mentioned the α particles are in this case so far apart from one another that antisymmetrization becomes negligible. The result for $N(B)$ is shown in Fig. 1 as a function of the width parameter B. We chose as optimal values of B for describing the ground and Hoyle states, $B = B_g = 2.5$ fm and $B = B_H = 6.8$ fm, for which the normalized THSR wave functions give the best approximation of the ground state 0_1^+ and the Hoyle state 0_2^+, respectively. We found that $N(B_H) \sim 0.62$ and $N(B_g) \sim 0.007$. These results indicate that

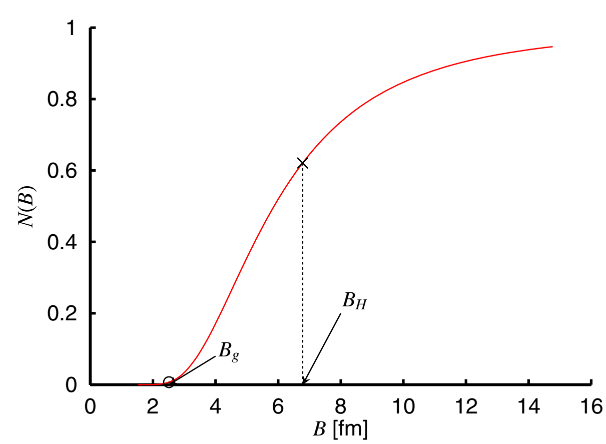

FIG. 1. Expectation value of the antisymmetrization operator in the product state $|B\rangle$. The values at the optimal B values, B_g for the ground state and B_H for the Hoyle state, are denoted by a circle and a cross, respectively. From Funaki *et al.*, 2009.

the influence of the antisymmetrization is strongly reduced in the Hoyle state compared with the ground state. This study gives us a first indication that the Hoyle state is quite close to the quartet condensation situation rather than being close to a Slater determinant. However, there is another quantity which tells us more directly whether the Hoyle state is close to a three α condensate or not. Matsumara and Suzuki (2004) evaluated the bosonic occupation numbers using a Gaussian representation of the c.m. part χ of the resonating group method (RGM) wave function (see Sec. III.B) to calculate the single α-particle density matrix $\rho_\alpha(\mathbf{R}, \mathbf{R}')$ and diagonalizing it. The bosonic occupation numbers were also calculated by Yamada *et al.* in Funaki *et al.* (2009) using, however, the already mentioned OCM approach; see Sec. III.A. Both calculations concluded that the three α's in the Hoyle state occupy about 70% the same $0S$ orbit whereas all other ones are reduced by at least a factor of 10; see Fig. 2. The density-induced suppression of the α-particle condensate has also been studied by Funaki *et al.* (2008).

The THSR calculation also showed that the inelastic form factor from the ground state to the Hoyle state is very sensitive to the size of the Hoyle state (Funaki *et al.*, 2006). For example, artificially extending the size of the Hoyle state by 20% increases the inelastic form factor globally by a factor of 2. The THSR approach yields for the rms radius 3.83 fm (ground state 2.4 fm), so that the volume (density) of the Hoyle state is approximately 3–4 times larger (lower) than the one of the ground state. Those numbers are rather similar to what one finds for ^{8}Be reinforcing the picture of the Hoyle state of a low density three α system where the α particles are individually well borne out (see Fig. 4). Since the THSR wave function has 98%–100% squared overlap (depending on more or less elaborate versions of THSR) with the wave function for the Hoyle state of Kamimura (1981) which, together with Uegaki *et al.* (1977, 1978) can be considered as the most general *Ansatz* used so far with practically perfect precision even far out in the tail, one can deduce that implicitly the Kamimura approach also gives a $\sim$70% bosonic occupancy for the α particles in the Hoyle state. As a side remark, one

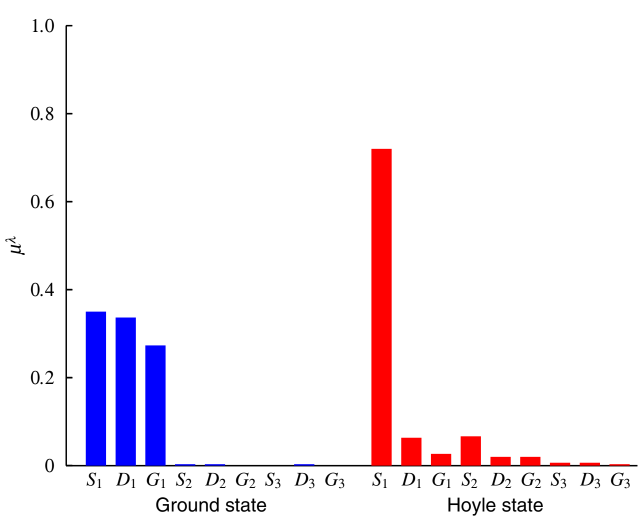

FIG. 2. α-particle occupation numbers in the ground state (left) and in the Hoyle state (right). From Funaki *et al.*, 2009.

may notice that for single proton or neutron states in nuclei one also obtains occupancies of 70%–80% (Pandharipande, Sick, and de Witt Huberts, 1997). One may, therefore, say that the bosonic quartets in nuclei excited to energies around the α decay threshold are about as far from (or as close to) the ideal gas case as are the fermions in the ground state. On the other hand, in cold atom devices, the bosonic atoms are at so low densities that their electronic clouds do not overlap at all and, thus, ideal BEC can develop (Pitaevskii and Stringari, 2003).

At this point, let us stress again that terms such as "α-particle condensation" or "Bose-Einstein condensation" strictu sensu apply only for macroscopic systems as homogeneous nuclear matter which we will treat later. In finite nuclei such terms can only be used in the sense that "condensate states" are to be considered as precursors to what happens potentially in the infinite matter case.

III. FURTHER APPROACHES TO THE HOYLE STATE

A. OCM of Horiuchi *et al.*

The precursor of all more or less realistic tentatives to explain the Hoyle state is the semimicroscopic description by Horiuchi (1974, 1975) using the OCM approach as mentioned in the Introduction. In the latter, the α particles are replaced by ideal bosons interacting with phenomenological two- and three-body forces. However, in an important aspect the Pauli principle is incorporated into the OCM approach. It is related to the fact that the physical states should be orthogonal to the so-called Pauli forbidden states. So, in the OCM the two-, three-, ... body bosonic equations are solved under the condition $\langle u_F | \Phi_{OCM} \rangle = 0$ where $u_F(r)$ are the Pauli forbidden states. For example, in the case of ^{8}Be those are given by harmonic oscillator $0S$, $1S$, and $0D$ wave functions (up to four $\hbar\omega$ quanta) because four neutrons plus four protons in a harmonic oscillator also occupy four $\hbar\omega$ quanta. Horiuchi (1974, 1975) stated for the first time that the Hoyle state is not a linear chain state but rather a state of "3α's weakly coupled to each other in relative S states." It also was concluded that the Hoyle state has quite an enlarged spatial structure compared to the ground state of ^{12}C. They did not investigate the Bose condensate character of the Hoyle state but it is clear that from a state of "loosely bound α particles" to a condensate of α's, there is only a short step.

As seen later in Sec. IV, concerning a study of the 0^+ spectrum in ^{16}O, OCM remains an efficient method for α cluster states. We show in Fig. 3 the radial part of the Hoyle wave function calculated with OCM (Yamada and Schuck, 2005) (solid line). We see no nodal behavior of the Hoyle orbit, only small oscillations in the inner region and a long tail up to $r \sim 10$ fm. The radial behavior of the Hoyle orbit is similar to a Gaussian (dotted line). On the contrary, in the ground state of ^{12}C where the α's strongly overlap, due to the active Pauli principle, strong oscillations develop with the number of nodes 2, 1, and 0 for S, D, and G waves, respectively. This reflects very well the SU(3) character of the ^{12}C ground state.

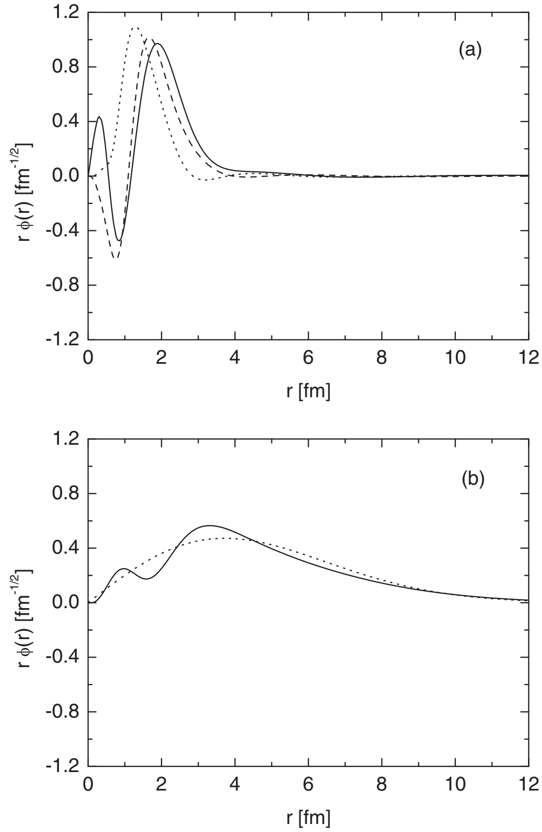

FIG. 3. (a) The ground state wave function is compared to the one of the (b) Hoyle state. We see the strong difference in spatial extensions. The strong overlap of α's in the ground state is responsible for the pronounced oscillations (a), whereas in the Hoyle state the S-wave function resembles a broad Gaussian. In (a), the solid line corresponds to the S wave, the dashed line corresponds to the D wave, and the dotted line corresponds to the G wave. From Yamada and Schuck, 2005.

B. The approaches by Kamimura *et al.* and Uegaki *et al.*

Kamimura (1981) made the following RGM *Ansatz* for the Hoyle state:

$$\Psi_{\mathrm{RGM}} \propto \mathcal{A}\chi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\phi_{\alpha_1}\phi_{\alpha_2}\phi_{\alpha_3}, \tag{5}$$

whereas Uegaki *et al.* (1977, 1978) considered the Brink-generator Ccoordinate method (GCM) wave function

$$\Psi_{\text{B-GCM}} \propto P_0 \int d^3\mathbf{S}_1 \int d^3\mathbf{S}_2 \int d^3\mathbf{S}_3 f(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3)\Phi_{\mathrm{B}}$$
$$\Phi_{\mathrm{B}} = \mathcal{A} e^{-(\mathbf{R}_1-\mathbf{S}_1)^2/b^2} e^{-(\mathbf{R}_2-\mathbf{S}_2)^2/b^2} e^{-(\mathbf{R}_3-\mathbf{S}_3)^2/b^2} \phi_{\alpha_1}\phi_{\alpha_2}\phi_{\alpha_3}, \tag{6}$$

where ϕ_{α_i} is again the intrinsic α-particle wave function of Eq. (3), $\boldsymbol{\xi}_i$ are Jacobi coordinates, and $\chi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ is a completely general translational invariant three boson wave function [note that in Eqs. (5) and (6) the scalar spin-isospin part of the wave function is not written out]. In Eq. (6) $f(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3)$ is the generator coordinate weight function and P_0 is a projector onto the zero total momentum. The Brink wave function Φ_{B} (Brink, 1966) places the small sized α particles with width parameter b at definite spatial points $\mathbf{S}_i$. These three-body wave functions have been determined variationally with RGM (GCM) by Kamimura (1981) (Uegaki *et al.*, 1977, 1978). Note that the angle of the third α with respect to the axis of the other two is completely free, as well as the distance with respect to the other two α particles. Therefore, all kinds of 3α arrangements from a linear chain over an open triangle to an equilateral triangle, etc., are in principle possible. On the other hand, since the THSR wave function is equivalent to Kamimura's wave function, this tells us that implicitly Kamimura's wave function also contains about 70% an α-particle condensate component. We show in Fig. 8 that the inelastic form factors of Kamimura and THSR are on top of one another explaining accurately the experimental data. In addition to all known properties of the Hoyle state, Kamimura *et al.* and Uegaki *et al.* explained a variety of other α gas states in ^{12}C with different quantum numbers such as, e.g., the second 2^+ state whose position was experimentally confirmed only recently (Itoh *et al.*, 2004, 2011; Freer *et al.*, 2009; Zimmerman *et al.*, 2011, 2013). As Horiuchi, also Kamimura and Uegaki concluded that the Hoyle state is a "weakly coupled system or gas of α particles" in relative S waves. We cite Uegaki *et al.* (1977, 1978): "In a number of excited states which belong to the new 'phase,' ^{12}C nucleus should be considered to dissociate into 3α-clusters which interact weakly with each other and move almost freely over a wide region." And further "The 0_2^+ state is the lowest state which belongs to the new phase, and could be considered to be a finite system of α-boson gas." These words are very similar to what we currently use in the context of the THSR approach. For instance, the "container" picture (Zhou *et al.*, 2014), of which we discuss in Sec. III.E, is already alluded to. The major difference between THSR and those earlier works consists of the fact that THSR predicted that α-particle condensation may not exist only in ^{12}C but also in heavier $n\alpha$ nuclei, as for instance in ^{16}O (Tohsaki *et al.*, 2001) and, thus, may be a more general phenomenon. Also the bosonic occupation numbers were not calculated at that time.

C. Antisymmetrized molecular dynamics and fermion molecular dynamics approaches by Kanada-En'yo *et al.* and Chernykh *et al.*

In 2007 the Hoyle state was also newly calculated by the practitioneers of antisymmetrized molecular dynamics (AMD) (Kanada-En'yo, 1998, 2007) and fermion molecular dynamics (FMD) (Chernykh *et al.*, 2007) approaches. In AMD one uses a Slater determinant of single-particle Gaussian wave packets where the centers of the packets $\mathbf{S}_i$ are replaced by complex numbers. This allows one to give the center of the Gaussians a velocity as easily realized. In FMD, in addition, the width parameters of the Gaussians are also complex numbers and, in principle, different for each nucleon. AMD and FMD do not contain any preconceived information of clustering. Both approaches found from a variational determination of the parameters of the wave function and a prior projection on good total linear and angular momenta that the Hoyle state has dominantly a 3-α cluster structure with no definite geometrical configurations. In this way the α cluster *Ansätze* of the earlier approaches were justified. As an achievement, Chernykh *et al.* (2007) successfully reproduced the inelastic form factor from the ground to the Hoyle state by employing an effective nucleon-nucleon interaction V_{UCOM} derived from the realistic bare Argonne V18 potential (plus a small phenomenological correction).

Kanada-En'yo (1998, 2007) pointed out that with AMD some breaking of the α clusters can and is taken into account. The Volkov force (Volkov, 1965) was employed by Kanada-En'yo (1998, 2007). Again all properties of the Hoyle state were explained with these approaches. As in the other works (Uegaki *et al.*, 1977, 1978; Kamimura, 1981; Tohsaki *et al.*, 2001; Chernykh *et al.*, 2007), the $E0$ transition probability came out ~20% too high. No bosonic occupation numbers were calculated. It seems technically difficult to do this with these types of wave functions. However, one can suspect that if occupation numbers were calculated, the results would not be very different from the THSR results. This stems from the high sensitivity of the inelastic form factor to the employed wave function. Nonetheless, it is important to produce the occupation numbers also with AMD and FMD.

Kanada-En'yo (1998, 2007) and Chernykh *et al.* (2007) showed some geometrical configurations of α particles in the Hoyle state. No special configuration out of several is dominant. This reflects the fact that the Hoyle state is not in a crystal-like α configuration but rather forms to a large extent a Bose condensate.

D. Pure bosonic approaches

In some works the Hoyle state is approached in treating the α particles completely as ideal bosons. Even the fact that the physical states should be orthogonal to the Pauli forbidden states, as is done in OCM, is not taken care of. The effect of antisymmetrization is entirely simulated by effective forces. The two most recent approaches of this sort are the ones of Lazauskas and Dufour (2011) [who used the nonlocal Papp-Moszkowski force (Papp and Moszkowski, 2008)] and of Ishikawa (2014b) using a modified Ali-Bodmer interaction (Ali and Bodmer, 1966) plus a three-body term. Lazauskas and Dufour (2011) and Ishikawa (2014b) reproduced the position of the Hoyle state and the α threshold energies. Lazauskas and Dufour (2011) also analyzed the relative angular momenta between the α particles in the Hoyle states. It turns out that there is S-wave dominance to about 80%. This implies that also the S-wave occupation number is of the order of 70%–80%. This is shown by Ishikawa (2014b) where such an analysis was also performed. It was found that the proportion of partial waves is practically the same as in Lazauskas and Dufour (2011). In addition the bosonic occupation numbers were calculated and the $0S$ occupancy turned out to be ~80%, thus confirming the Bose condensate picture (Ishikawa, 2014a). The strong link between relative S-wave dominance and high S-wave occupation numbers is likely a general feature. On the other hand, in Ishikawa (2014b) the simultaneous and democratic three α decay probability was given. This can be considered as a great achievement. The probability with respect to sequential $^8\mathrm{Be}+\alpha$ decay resulted to be negligible (branching ratio 10^{-4}). However, this does not speak against α-particle condensation. It simply means that three-body decay (tunneling under the Coulomb barrier) is strongly hindered.

E. Brink-type versus THSR wave function: Dumbbell versus container picture

A convenient way to compare Brink and THSR wave functions is the following hybrid *Ansatz* for ^{8}Be:

$$\Psi_{\text{THSR-hyb}} \propto P_0 \mathcal{A} e^{-(\mathbf{R}_1-\mathbf{S}_1)^2/B^2} e^{-(\mathbf{R}_2-\mathbf{S}_2)^2/B^2} \phi_{\alpha_1}\phi_{\alpha_2}. \quad (7)$$

In this way the THSR and Brink wave functions are encapsulated in one formula. For $B=b$, we have the Brink wave function Φ_{B}, Eq. (6), and for $S_i=0$, we have the THSR wave function (1). It turned out in a number of examples where the two variational parameters B and S have been put into competition that always $B>b$ and $S=0$ was the outcome of a variational calculation; see, e.g., Zhou *et al.* (2013). Therefore, the THSR picture where the large B parameter indicates free mean field motion of the cluster, the so-called container picture (Zhou *et al.*, 2014), prevails over the Brink *Ansatz* where the clusters are nailed down to definite positions via the $\mathbf{S}_i$ parameters. This latter evokes the "dumbbell" or "molecular" picture which was used almost exclusively in the past, for example, in the description of ^{8}Be. It is true that most of the time not a single Brink wave function was considered but a superposition smearing out the position of the clusters. This was believed to be a correction and the underlying picture was thought to remain the dumbbell or molecular one. However, as the studies with the hybrid wave function (7) show, the basic property of cluster motion is just the contrary: free motion in a cluster mean field, the container. Of course, the clusters in their motion cannot penetrate each other, due to the Pauli principle.

The α clusters can be considered as ideal bosons moving in their own bosonic mean field freely over the whole nuclear volume except for mutual overlaps. This can also be seen in Fig. 4 of Matsumara and Suzuki (2004), where the two α correlation in $^{12}\mathrm{C}(0_2^+)$ as a function of their mutual distance is displayed. It practically corresponds to the "excluded volume" idea often employed phenomenologically in cluster physics. This repulsive "force" between two α's also is the reason why they cannot be 100% in a condensate state but to a certain percentage the α's are scattered out of the condensate. A comparison between the Brink and THSR approaches is shown in Fig. 4 where we show the intrinsic density distribution of ^{8}Be calculated with the single THSR wave function (lower panel) with the one of a single Brink wave function (upper panel). The strong difference in localization of both distributions should be appreciated. Here the "intrinsic density" means that the system is in a symmetry broken deformed state which is close to a classical picture. Of course, the ground state (remember that ^{8}Be is slightly unstable with a width of only some eV whereas nuclear energy scales are MeV) of ^{8}Be has quantum number 0^+ and in the laboratory frame this state is spherical. This is obtained from the deformed intrinsic state in averaging it over the whole angular range in space.

In more recent works, similar results to the ones with a single Brink wave function have been obtained with a more

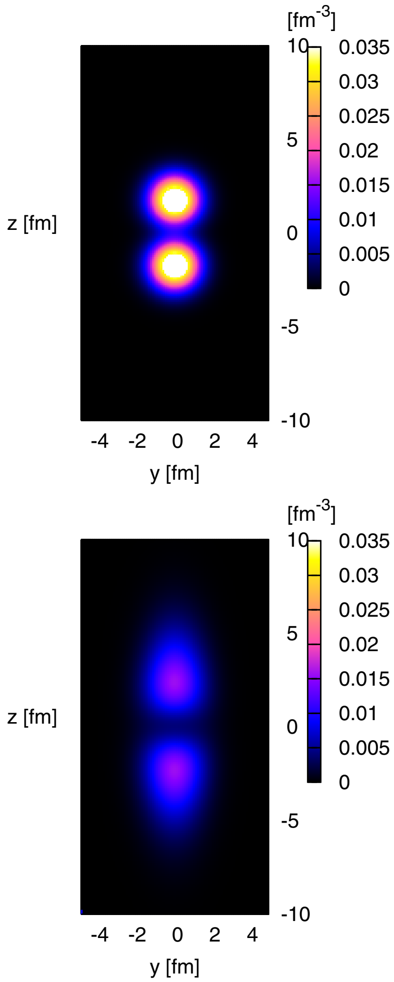

FIG. 4. Comparison of single Brink and THSR intrinsic densities for ^{8}Be, from top to bottom. We thank Y. Funaki for the preparation of this figure.

general mean field approach of the Gogny or relativistic mean field type (Girod and Schuck, 2013). As an example it was found that expanding a nucleus such as ^{16}O employing a constrained Hartree-Fock-Bogoliubov approach, at some critical low density, the nucleons spontaneously cluster into a tetrahedron of four α particles. These α particles have fixed positions so they can be formed with a single Brink wave function Φ_B in Eq. (6); see Fig. 4 upper panel. The general mean field approach has, however, the advantage that realistic density functionals can be used. Whether a GCM calculation can be applied on top of these configurations as with a Brink-GCM wave function remains to be seen.

F. Rotating triangle versus extended THSR approach for the "Hoyle band"

Another approach of α clustering in ^{12}C was put forward recently. In Marin-Lambarri *et al.* (2014) an algebraic model suggested by Bijker and Iachello (2000), originally due to Hafstadt and Teller (1938), was used on the hypothesis that the ground state of ^{12}C has an equilateral triangle structure. The model then allows one to calculate the rotational-vibrational (rot-vib) spectrum of three α particles. Notably a newly

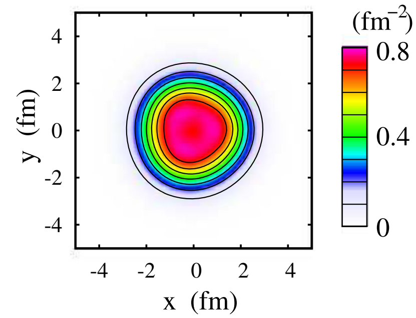

FIG. 5. Intrinsic density distribution of the ^{12}C ground state from a mean field calculation. We thank Y. Kanada-En'yo for providing this figure.

measured 5^- state fits into the rotational band of a spinning triangle. This interpretation is also reinforced by the fact that for such a situation the 4^+ and 4^- states should be degenerate which is effectively the case experimentally. In Fig. 5, we show the triangular density distribution of the ^{12}C ground state obtained from a pure mean field calculation. This means a calculation without any projection on parity nor angular momentum. Therefore, symmetry is spontaneously broken into a triangular shape. The calculation is obtained under the same conditions as in Suhara and Kanada-En'yo (2010), that is, in the AMD model space. However, in that work only figures with variation after the projection are shown (Suhara and Kanada-En'yo, 2010). This enhances the triangular shape. It must be stated, however, that the broken symmetry to a triangular shape is very subtle and depends on the force used (Kanada-En'yo, 2016). Anyway, such a triangular shape seems a possibility.

Marin-Lambarri *et al.* (2014) then tried to repeat their reasoning tentatively for the "rotational" band with the Hoyle state as the band head. However, in this case, the situation is much less clear. In Fig. 6 we see the experimental positions of the 0^+ states together with the 2_2^+ and

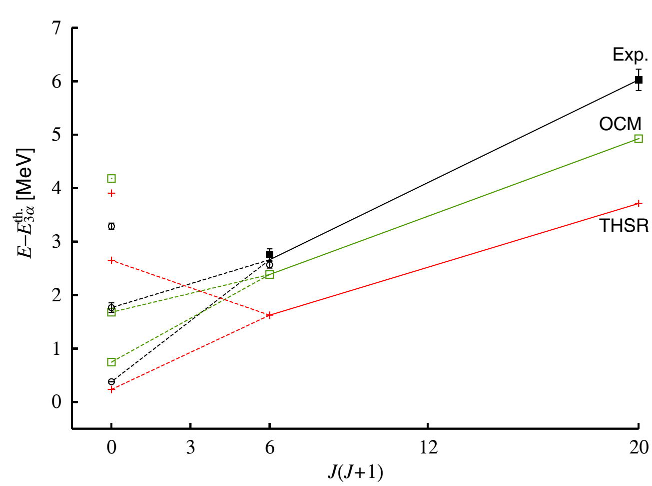

FIG. 6. Positions of 0^+ states together with 2^+ and 4^+ states of the Hoyle band as a function of $J(J+1)$. The origin at the vertical axis is the 3α disintegration threshold. OCM and extended THSR results are compared with experiment. We thank Y. Funaki for the preparation of this figure.

4_2^+ states plotted as a function of $J(J+1)$ and compared with the results of an OCM approach (Ohtsubo *et al.*, 2013) and a calculation by Funaki (2015) where a generalized THSR wave function was used involving a different B parameter for each Jacobi coordinate

$$\Psi_{3\alpha} \propto \mathcal{A}\left[\exp\left(-\frac{4}{3B_1^2}\xi_1^2 - \frac{1}{B_2^2}\xi_2^2\right)\right]\phi_{\alpha_1}\phi_{\alpha_2}\phi_{\alpha_3}. \qquad (8)$$

This physically transparent 12 nucleon wave function allows one to describe pairs of α's to have different relative distances. That is, this generalization of THSR includes α pair correlations. This is an important new feature of the THSR approach. With $B_1 = B_2$, one recovers Eq. (1). With Eq. (8), a variety of excited α gas states above the Hoyle state have been obtained by Funaki (2015). The result is that the Hoyle state (0_2^+) and the third zero plus state (0_3^+) have with $B(E2; 2_2^+ \to 0_2^+) = 295\ e^2\,\mathrm{fm}^4$ and $B(E2; 2_2^+ \to 0_3^+) = 104\ e^2\,\mathrm{fm}^4$, respectively, both a strong transition probability to the second two plus state (2_2^+). So no clear band head can be identified. It was also concluded by Chernykh *et al.* (2007) that the states $0_2^+, 2_2^+, 4_2^+$ do not form a rotational band. The line which connects the two other hypothetical members of the rotational band, see Fig. 6, has a slope which points to somewhere in between the 0_2^+ and 0_3^+ states. To conclude from there that this gives raise to a rotational band may be premature. One should also realize that the 0_3^+ state is strongly excited from the Hoyle state by a monopole transition whose strength is obtained from the extended THSR calculation to be $M(E0; 0_3^+ \to 0_2^+) = 35\ \mathrm{fm}^2$. So the 0_3^+ state seems to be a state where one α particle has been lifted out of the condensate to the next higher S level with a node [see also Kurokawa and Kato (2007), Ohtsubo *et al.* (2013), and Yamada and Funaki (2015)], where the 0_3^+ and 0_4^+ states have been identified as well. This is confirmed in Fig. 7 where the probabilities $S^2_{[I,l]}$ of the third α orbiting in an l wave around a ^{8}Be-like, two α correlated pair with relative angular momentum I, are displayed. One sees that except for the 0_4^+ state, the states have the largest contribution from the $[0, l]$ channel. So the picture which arises is as follows: in the Hoyle state, the three α's are all in relative $0S$ states with some α-pair correlations [even with $I \neq 0$, see, e.g., Lazauskas and Dufour (2011) and Ishikawa (2014b)], responsible for emptying the α condensate by 20%–30%. This S-wave dominance, so far found by about half a dozen different theoretical works [see, e.g., Horiuchi (1974, 1975), Uegaki *et al.* (1977, 1978), Kamimura (1981), Matsumara and Suzuki (2004), Funaki *et al.* (2009), Lazauskas and Dufour (2011), and Ishikawa (2014b)], is incompatible with the picture of a rotating triangle. As mentioned, the 0_3^+ state is one where an α particle is in a higher nodal S state and the 0_4^+ state is built out of an α particle orbiting in a D wave around a (correlated) two α pair, also in a relative $0D$ state; see Fig. 7. The 2_2^+ and 4_2^+ states are a mixture of various relative angular momentum states (Fig. 7). Whether they can be qualified as members of a rotational band or maybe rather of a vibrational band or a mixture of both is an open question. In any case, indeed, they are very strongly connected by $B(E2)$ transitions $B(E2; 4_2^+ \to 2_2^+) = 560\ e^2\,\mathrm{fm}^4$. Let us also mention that the excited α cluster states discussed previously have a width much larger ($\sim$1 MeV) than the Hoyle state ($\sim$1 eV). Nevertheless they are sufficiently sharp, so that they could be treated in the bound state approximation.

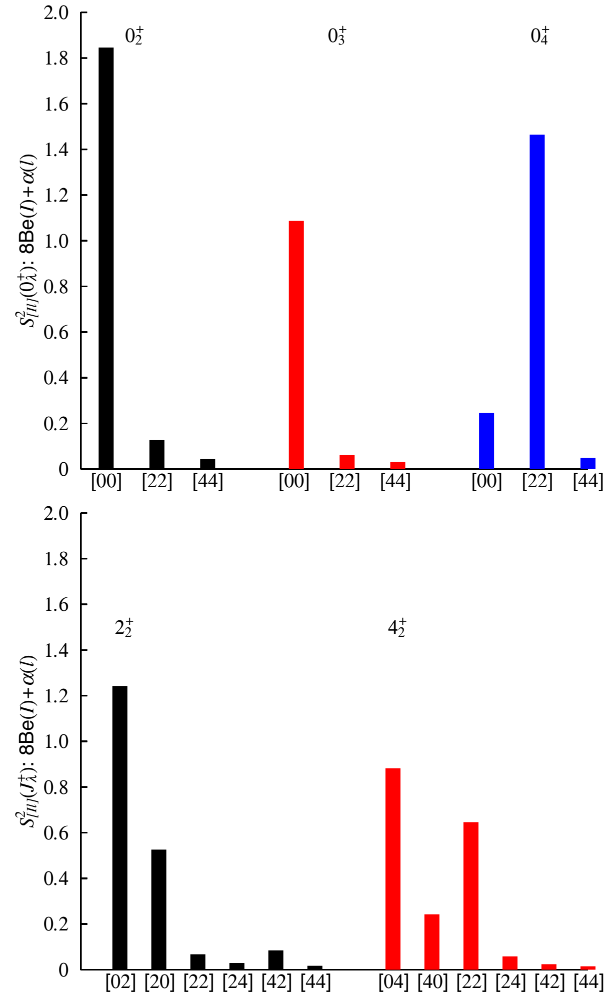

FIG. 7. Probability distributions for various components in the Hoyle and excitations of the Hoyle state. From Funaki, 2015.

One may also wonder why, with the extended THSR approach, there is a relatively strong difference between the calculated and experimental so-called Hoyle band? This may have to do with a deficiency inherent to the THSR wave function which so far has not been cured (there may be ways to do it in the future). It concerns the fact that with THSR (as, by the way, with the Brink wave function), it is difficult to include the spin-orbit potential. This has as a consequence that the first 2^+ and first 4^+ states are quite wrong in energy because the strong energy splitting between $p_{3/2}$ and $p_{1/2}$ states is missing. This probably has a repercussion on the position of the second 2^+ and 4^+ states. This can be deduced from the OCM calculation by Ohtsubo *et al.* (2013), also shown in Fig. 6, where the 2^+ and 4^+ states of the ground state rotational band have been adjusted to experiment with a phenomenological force and, thus, the position of the 2^+ and 4^+ states of the so-called Hoyle band is much improved. Additionally, this may also come from the fact that with this extended THSR wave function a different force has to be adopted. Such investigations are under way.

G. Quantum Monte Carlo and *ab initio* approaches

Recently a breakthrough in the description of the Hoyle state was achieved by two groups (Epelbaum *et al.*, 2012; Carlson *et al.*, 2015) using Monte Carlo techniques. In Epelbaum *et al.* (2012) Dean Lee *et al.* reproduced the low lying spectrum of ^{12}C, including the Hoyle state, very accurately with a so-called *ab initio* lattice QMC approach starting from effective chiral field theory. The sign problem was circumvented exploiting the fact that SU(4) symmetry is very well fulfilled, at least for the lighter nuclei. This parameter free first principle calculation is an important step forward in the explanation of the structure of ^{12}C. On the other hand, all quantities which are more sensitive to details of the wave function have so far either not been calculated (e.g., inelastic form factor to the Hoyle state) or the results are in quite poor agreement with the results of practically all other theoretical approaches. This, for instance, is the case for the rms radius of the Hoyle state which in Epelbaum *et al.* (2012) is barely larger than the one of the ground state, whereas it is usually believed that the Hoyle state is quite extended.

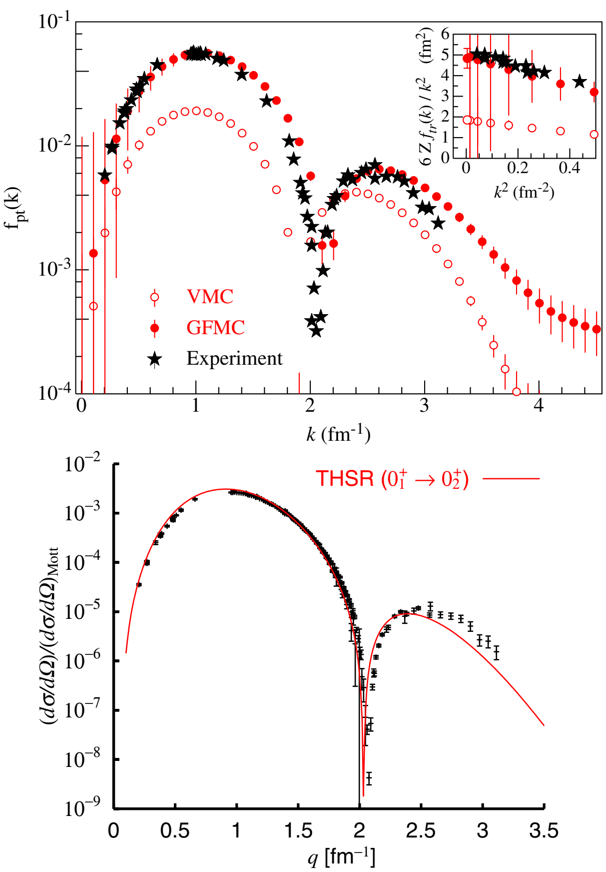

FIG. 8. Inelastic form factors from GFMC (Carlson *et al.*, 2015), upper panel, and THSR (Funaki *et al.*, 2006), lower panel. The THSR result cannot be distinguished from the one of Kamimura (1981) on the scale of the figure. We thank Y. Funaki for the preparation of the figure in the lower panel. In the upper panel, the black stars correspond to the experimental values and the solid red circles correspond to the GFMC results. The open circles correspond to some approximate QMC calculation; see Carlson *et al.* (2015).

Epelbaum *et al.* (2012) remarked that higher order contributions to the chiral expansion have to be included to account for the size of the Hoyle state.

On the other hand, there exist new Green's function Monte Carlo (GFMC) results with constrained path approximation using the Argonne v18 two-body and Illinois-7 three-body forces, where the inelastic form factor for most of the experimental points is reproduced very accurately (Carlson *et al.*, 2015). In Fig. 8, we compare this result with the one obtained from THSR. The results of Kamimura (1981) are on top of the THSR ones. They cannot be distinguished from the THSR ones on the scale of the graph demonstrating again the equivalence of both approaches. We see excellent agreement between the three calculations and with experiment. In the inset of the upper panel of Fig. 8, we see nevertheless that the rather precise experimental transition radius of 5.29 ± 0.14 fm^2 given by Chernykh *et al.* (2007) is much better reproduced than in α cluster models which all yield an about 20% too large value. This may also be the reason for the too slow dropoff of the THSR density in the surface region; see the lower panel of Fig. 9. The energy of the Hoyle state is around 10 MeV in Carlson *et al.* (2015), slightly worse than the one in Epelbaum *et al.* (2012). In Fig. 9, we compare the density of the Hoyle state (weighted with r^2) obtained with the THSR wave function and in Carlson *et al.* (2015). We again see quite good agreement between both figures up to about 4 fm. For instance the kind of plateau between 1.5 and 4 fm seems to be very characteristic. It is, however, more pronounced in the GFMC calculation than

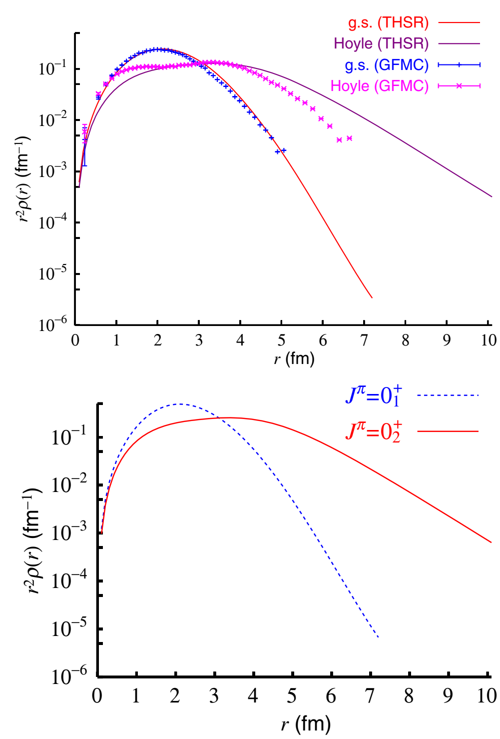

FIG. 9. Density of the Hoyle state with GFMC (Carlson *et al.*, 2015), upper panel, and with THSR, lower panel. In the upper panel, the pluses correspond to the ground state density and the crosses to the one of the Hoyle state. We thank Y. Funaki for the preparation of this figure.

from THSR. For a better appreciation, we repeat the results of THSR separately in the lower panel of Fig. 9. Beyond 4 fm, the density in Carlson *et al.* (2015) falls off more rapidly. As mentioned, this may be due to the fact that the GFMC results are more accurate for small q values. At any rate, the outcome of the three calculations in Kamimura (1981), Tohsaki *et al.* (2001), and Carlson *et al.* (2015) is so close that it is difficult to believe that results for other quantities should be qualitatively different when calculated with the GFMC technique. This should, for instance, hold for the strong proportion of relative S waves between the α's found with the other approaches discussed previously.

There has recently been substantial progress with the so-called symplectic no-core shell model (NCSM) (Dreyfuss *et al.*, 2013; Dytrych *et al.*, 2013; Tobin *et al.*, 2014). This is an extension of the usual NCSM which was not able to reproduce any of the α gas states. With the symplectic version, the positions of the Hoyle and the second 2^+ states in ^{12}C are well reproduced. Again, what is missing is the inelastic form factor. As pointed out several times, the very well-measured inelastic form factor is highly sensitive to the ingredients of the wave function of the Hoyle state and it is mandatory that a theory reproduces this decisive experimental quantity correctly.

H. Nuclear matter

Last but certainly not least, we consider α clustering and α condensation in nuclear matter. As a matter of fact, it was for nuclear matter where the possibility of α-particle condensation had been considered first (Röpke *et al.*, 1998), where the critical temperature from an in-medium four nucleon (two protons and two neutrons) equation was established. Sogo *et al.* (2009) presented an improved calculation; see Fig. 10.

This is in complete analogy to what is known as the Thouless criterion for the onset of pairing as a function of temperature. It was found that despite its strong binding, the α condensate, as a function of increasing density, rapidly breaks down as soon as the chemical potential passes substantially from negative values (binding) to positive ones. This is contrary to what happens for pairing where the strong coupling limit passes continuously to the weak, BCS-type of limit with positive values of the chemical potential and a long coherence length (size) of the Cooper pairs (Baldo, Lombardo, and Schuck, 1995). The density where α condensation as a function of temperature breaks down is about one-fifth of the saturation density.

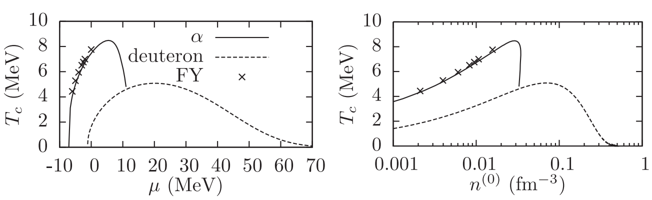

FIG. 10. Critical temperature for α condensation as a function of chemical potential (left panel) and as a function of uncorrelated density (right panel) compared to the one of neutron-proton (deuteron) pairing (dashed lines). The crosses correspond to a full solution of the in-medium Faddeev-Yakubovsky equations with the Malfliet-Tjohn potential. From Sogo *et al.*, 2009.

The reason for this very different behavior between pairing and quartetting has to do with the fact that the in-medium two particle level density

$$g_2(E) = \int \frac{d^3k}{(2\pi\hbar)^3}\delta(E - 2\epsilon_k),$$

for the two particles at rest, has a finite value at the Fermi level whereas this is not the case with the four-body level density

$$g_4(E) = \frac{1}{(2\pi\hbar)^{12}}\int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \times \theta_{1234}\delta(E - \epsilon_{k_1} - \epsilon_{k_2} - \epsilon_{k_3} - \epsilon_{k_4}), \tag{9}$$

which goes through zero at the Fermi energy, just at the point where quartet correlations should build up. Here ϵ_k are kinetic energies and $\theta_{1234} = \theta_1\theta_2\theta_3\theta_4 + \bar{\theta}_1\bar{\theta}_2\bar{\theta}_3\bar{\theta}_4$ and $\theta_i = \theta(\mu - \epsilon_{k_i}), \bar{\theta}_{k_i} = \theta(\epsilon_{k_i} - \mu)$ with μ the chemical potential. We leave it to the reader to verify this but it is also explained by Sogo *et al.* (2009). As a matter of fact all many body level densities of this kind go through zero at the Fermi level, besides, precisely in the one-body and two-body cases when the two particles are at rest. Another well-known example of this kind is the two particle–one hole (2p-1h) level density which enters the perturbative calculation of the mean free path of a fermion in a fermionic medium. At the Fermi energy the mean free path becomes infinite because the 2p-1h level density goes through zero there.

In conclusion, it is legitimate to see in the Hoyle state (and other similar states in heavier self-conjugate nuclei) the precursor of the infinite matter situation; see also Takemoto *et al.* (2004), where α matter was investigated with a crystal structure. The situation is then quite analogous to pairing in nuclei which can be considered as a precursor of pairing in neutron matter, i.e., neutron stars.

IV. A GLIMPSE OF ^{16}O

The situation in ^{16}O is again quite a bit more complicated than in ^{12}C. The fact is that between the 4α threshold and the ground state, there are a couple of 0^+ states which can be interpreted as $\alpha + {}^{12}$C cluster configurations. In Fig. 11 we show the result of an OCM calculation with a very large dimension (Funaki *et al.*, 2008).

We see that there is a one to one correspondence between the first six calculated 0^+ states and experiment. With regard to the complexity of the situation the agreement between both can be considered as very satisfactory. Only the highest state was identified with the 4α condensate state. The four other excited 0^+ states are $\alpha + {}^{12}$C configurations. For example, the fifth 0^+ state is interpreted as an α orbiting in a higher nodal S wave around the ground state of ^{12}C. The fourth 0^+ state contains an α orbiting in a P wave around the first 1^- state in ^{12}C. In the third 0^+ state the α is in a D wave coupled to the 2_1^+ state of ^{12}C, and in the second 0^+ state the α is in a $0S$ wave and the ^{12}C in its ground state. The single parameter THSR

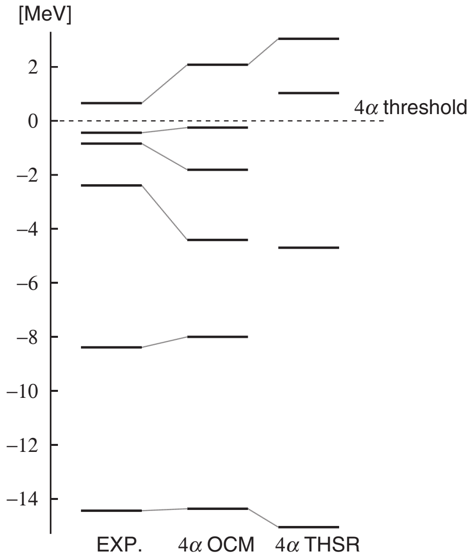

FIG. 11. Spectrum of 0^+ states in ^{16}O from the OCM and THSR approaches. From Funaki *et al.*, 2008.

calculation can correctly reproduce only the ground state and the α condensate state (0_6^+). By construction it cannot describe $\alpha + {}^{12}$C configurations. So the two intermediate states give some sort of average picture of the four $\alpha + {}^{12}$C configurations. One would have to employ a more general *Ansatz* as in Eq. (8) to cope with the situation. Work in this direction is in progress. The 0_6^+ state is theoretically identified as the α-condensate state from the overlap squared $|\langle 0_6^+|\alpha + {}^{12}\mathrm{C}(0_i^+)\rangle|^2$ (Yamada *et al.*, 2012). In Fig. 12 we see that the 0_6^+ state has an overwhelming contribution from the Hoyle

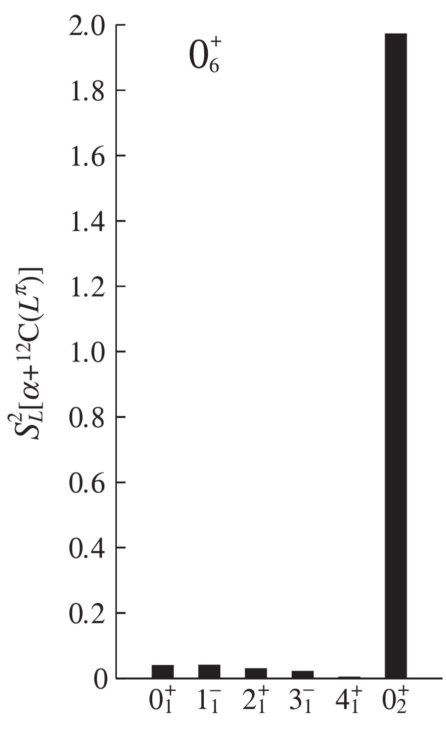

FIG. 12. Spectroscopic factor of the 0_6^+ state in ^{16}O with respect to α plus the Hoyle state. From Yamada *et al.*, 2012.

state plus α particle. Let us also mention that recently the results of Funaki *et al.* (2008) have essentially been reproduced with an AMD-like approach in Kanada-En'yo (2014).

It is important to measure, as is the case of ^{12}C for the Hoyle state, the inelastic form factor from the ground state to the 0_6^+ state to have at least an indirect confirmation of a Hoyle-analog state in ^{16}O.

V. EXPERIMENTAL EVIDENCES

Unfortunately, contrary to pairing, the experimental evidences for α condensation are rare and so far only indirect. We, nevertheless, elaborate here on this issue, even though the experimental situation concerning α-particle condensation is far from being clear. However, this may incite experimentors to perform more extensive and accurate measurements. The most prominent feature is the inelastic form factor which, as stated, is very sensitive to the extension of the Hoyle state and shows that the Hoyle state has a volume of 3–4 times larger than the ground state of ^{12}C. A state at low density is, of course, favorable to α condensation as seen from the infinite matter study. Nevertheless, this does not establish direct evidence. Also the analysis of hadronic reactions indicate an increased radius of the Hoyle state (Ohkubo and Hirabayashi, 2007; Tomita *et al.*, 2014). Other attempts to search for signatures of α condensate structures are heavy ion collisions around the Fermi energy where a condensate structure may be formed as an intermediate state and correlations between the final α particles may reveal this structure.

For example, von Oertzen *et al.* reanalyzed old data (Kokalova *et al.*, 2006) of the $^{28}\mathrm{Si}+{}^{24}\mathrm{Mg}\to{}^{52}\mathrm{Fe}\to{}^{40}\mathrm{Ca}+3\alpha$ reaction at 130 MeV which could not be explained with a Hauser-Feshbach approach for the supposedly statistical decay of the compound nucleus ^{52}Fe. Analyzing the spectrum of the decaying particles via γ decay, obtained in combination with a multiparticle detector, it was found that the spectrum is dramatically different for events where the three α's are emitted randomly hitting various detectors under different angles from the ones where the three α's were impinging on the same detector. This is shown in Fig. 13, where the upper panel corresponds to the case of the three α's in different detectors and lower panel, and three α's in the same detector. A spectacular enhancement of the ^{36}Ar line is seen in the lower panel. This is then explained by a strong lowering of the emission barrier, due to the presence of an α gas state, for the emission of $^{12}\mathrm{C}(0_2^+)$. This fact explains that the energies of the $^{12}\mathrm{C}(0_2^+)$ are concentrated at much lower energies as compared to the summed energy of three α particles under the same kinematical conditions (von Oertzen, 2010, 2011). In this way, the residual nucleus (^{40}Ca) attains a much higher excitation energy which leads to a subsequent α decay and to a pileup of ^{36}Ar in the γ spectrum. One could also ask the question whether four α's have been seen in the same detector. However, this will happen only at somewhat higher energies, an important experiment to be done in the future.

The interpretation of the experiment is, thus, the following [we cite von Oertzen (2010, 2011)]: "due to the coherent properties of the threshold states consisting of α particles with a large de Broglie wavelength, the decay of the compound nucleus ^{52}Fe did not follow the Hauser-Feshbach assumption

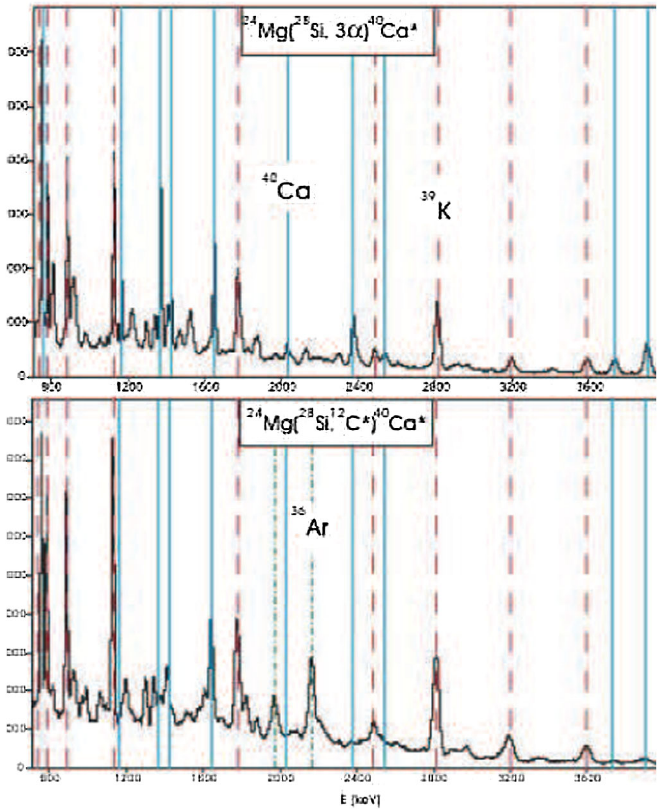

FIG. 13. Coincident γ spectra gated with the α particles hitting randomly three different detectors (upper panel) in comparison with the case where three α's hit the same detector (lower panel). Note the additional lines for ^{36}Ar in the lower panel.

of the statistical model: a sequential decay and that all decay steps are statistically independent. On the contrary, after emission of the first α particle, the residual α particles in the nucleus contain the phase of the first emission process. The subsequent decays will follow with very short time delays related to the nuclear reaction times. Actually, a simultaneous decay can be considered. Very relevant for this scenario is, as mentioned, the large spatial extension of the Bose condensate states, as discussed in von Oertzen (2010, 2011)."

However, as the saying goes: "one swallow does not make a summer" and, anyway, although suggestive, this may not be considered as a hard proof of α condensation. It, however, may become a rewarding research field to analyze heavy ion reactions more systematically for nonstatistical, coherent α decays.

A promising route may also be Coulomb excitation. In Fig. 14, we show emulsion images of coherent α decay of ^{20}Ne into three α's and one ^{8}Be, or into five α's with remarkable intensity from relativistic Coulomb excitation at the Dubna Nucletron accelerator (von Oertzen, 2010, 2011); see also Andreeva et al. (2004). The Coulomb breakup is induced by heavy target nuclei silver (Ag). The breakup of ^{16}O into four α's, or into two α's and one ^{8}Be is shown in Fig. 15. The presence of ^{8}Be in the two reactions shows that the α's travel coherently, otherwise the ^{8}Be resonance could not be formed. Of course, this is only a vague indication for some α-particle coherence and much more dedicated experiments should be performed for firmer conclusions.

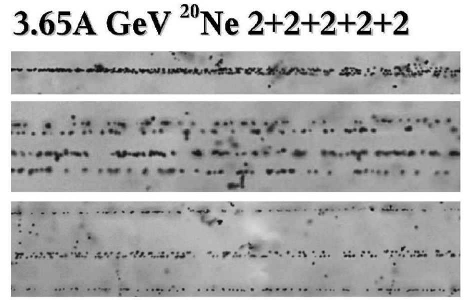

FIG. 14. Breakup of ^{20}Ne into five α's, partially containing a ^{8}Be. From von Oertzen, 2010, 2011.

Imagine once could Coulomb excite ^{40}Ca to over 60 MeV and observe a slow coherent α-particle Coulomb explosion. Coulomb explosions have been observed in highly charged atomic van der Waals clusters; see Last and Jortner (2000). Coulomb excitation is insofar an ideal excitation mechanism as it transfers very little angular momentum and the projectile essentially gets into a radial density expansion mode.

Next we argue that the ^{8}Be decay of the sixth 0^+ state at 15.1 MeV in ^{16}O can eventually show Bose enhancement, if the 15.1 MeV state is an α condensate.

We know that a pickup of a Cooper pair out of a superfluid nucleus is enhanced (Potel et al., 2013), if the remaining nucleus is also superfluid. As an example, one could think of the reaction $^{120}\mathrm{Sn} \to {}^{118}\mathrm{Sn} + \mathrm{Cooper\ pair}$. Of course the same is true for the pickup of two Cooper pairs simultaneously. We make an analogy between this and ^{8}Be decay of the 15.1 MeV state. In the decay probability of coincident two ^{8}Be, the following spectroscopic factor should enter:

$$S = \langle {}^{8}\mathrm{Be}{}^{8}\mathrm{Be}|15.1\ \mathrm{MeV}\rangle. \tag{10}$$

The reduced width amplitude y is roughly related to the spectroscopic factor as $y = 2^{-1/2}(4!/2!2!)^{1/2}S$. Adopting the condensation approximation of ^{8}Be and 15.1 MeV states, this yields

$$S = \langle B^2B^2|(B^+)^4\rangle/(2!2!4!)^{1/2} = (4!/2!2!)^{1/2} = 6^{1/2}$$

entailing $y = 6/(2^{1/2})(y^2 = 18)$. In the above expression for S, $B^+(B)$ stands for an ideal boson creator (destructor), representing the α particle.

When we say that S is large, we need to compare this S with some standard value. So we consider the case in which the 15.1 MeV state is a molecular state of ^{8}Be-^{8}Be. We have

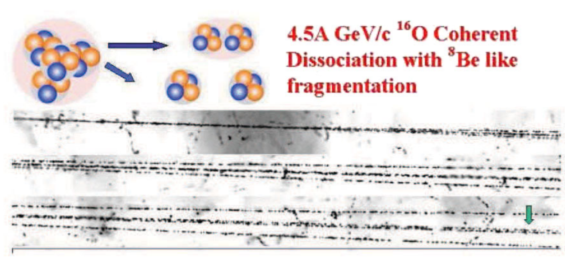

FIG. 15. Breakup of ^{16}O into four α's. Two α's are correlated into ^{8}Be. From von Oertzen, 2010, 2011.

$$S = \langle {}^8\mathrm{Be(I)}{}^8\mathrm{Be(II)}|{}^8\mathrm{Be(I)}{}^8\mathrm{Be(II)}\rangle = 1$$

and, therefore, $y = 3^{1/2}(y^2 = 3)$. This result shows that the condensation character of the 15.1 MeV state gives a ^{8}Be decay width which is 6 times larger than the molecular resonance character.

We should be aware that this estimate is extremely crude and one rather should rely on a microscopic calculation of the reduced width amplitude y that seems possible to do in the future. Nevertheless, this example shows that the decay of the 15.1 MeV state into two ^{8}Be's may be a rewarding subject experimentally as well as theoretically in order to elucidate further its α cluster structure.

A further indirect indication of an extended α gas state and, thus, of the eventual existence of an α condensate state, is the measurement of the momentum distribution of the α and/or ^{8}Be particles from a decaying α-particle condensed state. Yamada and Schuck (2005) showed that those decay products should have a very narrow momentum distribution, close to zero momentum. Again such experiments seem to be very difficult.

In conclusion, the experimental situation needs to be improved. However, new experimental results will soon be published (Natowitz, 2016) or are planned (Freer, 2015), so that there is hope that we will have a clearer picture of α-particle gas states in self-conjugate nuclei in the near future also from the experimental side. In this context, we also mention two other experimental works. First, there are the results of Raduta *et al.* (2011). An enhanced simultaneous 3α decay of the Hoyle state has been found involving a heavy ion reaction. However, this finding is in contradiction with three other experiments (Freer *et al.*, 1994; Kirsebom *et al.*, 2012; Itoh *et al.*, 2014) and one theoretical work (Ishikawa, 2014b) on the decay of an isolated $^{12}\mathrm{C}^*$ in the Hoyle state where a triple α decay is found to be below the threshold of detectability. It is important to investigate the reason for this enhanced 3α decay of the Hoyle state in a heavy ion reaction. Second, there is the recent publication of Marini *et al.* (2016), where it is claimed to have detected "signals of Bose-Einstein condensation and Fermi quenching in the decay of hot nuclear systems." In short, in complete vaporization events, the bosonlike particles (deuterons, α particles) are much denser packed than the corresponding fermionic particles (protons, helions, tritons). This then is in analogy to what has been seen in cold atom systems with fermion-boson mixures (Ebner and Edwards, 1971; Schreck *et al.*, 2001). We think, however, that much more precise measurements and investigations have to be performed before definite conclusions can be drawn.

Concerning future experiments, we repeat that an important quantity still to be measured is the inelastic form factor from ground to the sixth 0^+ state in ^{16}O. As mentioned, this form factor has long been known for the Hoyle state that allowed for strong theoretical conclusions. However, for ^{16}O this is so far not possible.

VI. DISCUSSION, CONCLUSION, AND OUTLOOK

In this Colloquium, we tried to assess the present situation with respect to a possible interpretation of the Hoyle state as an α-particle condensate. We pointed to the fact that so far three calculations exist which determine the bosonic occupation numbers of the α's in the Hoyle state (Ishikawa, 2014a; Matsumara and Suzuki, 2004); Funaki *et al.*, 2009). All those works concluded that the three α's of the Hoyle state occupy to $\sim$70%–80% a 0S state with their c.m. motion. However, about half a dozen works exist which predict a 80% relative 0S-wave dominance between the α's in the Hoyle state (Horiuchi, 1974, 1975; Uegaki *et al.*, 1977, 1978; Kamimura, 1981; Lazauskas and Dufour, 2011; Ishikawa, 2014b). Since Ishikawa found $\sim$80% relative 0S-wave dominance in his three boson (α) calculation and calculated the mean field boson (α) occupation numbers with also an 80% S wave, one logically can conclude a strong correlation between dominance of the relative S wave and dominance of the 0S-wave bosonic occupation number. According to this finding, one can say that the Hoyle state is to a large extent an α-particle condensate of low density (1/3–1/4 of saturation). Quite naturally, this can be considered as a precursor to α-particle condensation in low density nuclear matter; see Sec. III.H. This should be seen in analogy to the pairing case where only a handful of Cooper pairs are present and nuclear superfluidity can be considered as the precursor of superfluidity in neutron matter, i.e., neutron stars. The THSR wave function is a single variational wave function which fully respects the Pauli principle among all nucleons and which allows one to interpolate between a pure Slater determinant and a pure Bose condensate according to a single variational parameter B. We also considered a hybrid THSR wave function where in the single Brink wave function a variable width parameter B has been introduced. For the positions of the α's all going to zero, one recovers the THSR wave function and for finite positions but $B \to b$, with b the free space width of the α, one recovers the Brink wave function reflecting a crystal structure of the α arrangement. The two variational parameters B and positions have been put into competition with a variational calculation for the energy. The variation largely yields an answer close to the Bose condensate picture, i.e., a large B value, covering the whole nuclear volume, and with positions of the α's all centered at the origin. Such competition has equally been analyzed schematically by Zinner and Jensen (2013) who also concluded that a large extension of the α wave functions covering the whole nuclear volume is akin to Bose condensation. The parameter free reproduction of all experimentally known properties of the Hoyle state with THSR gives a further strong argument for the condensate picture. Since the THSR wave function has a $\sim$98% squared overlap with Kamimura's Hoyle state, the former is not just an approximation to the latter but is equivalent. One can, thus, argue that implicitly the work of Kamimura (and Uegaki) also describes the Hoyle state as a Bose condensate of α particles, a new insight to the otherwise very successful approaches of those about 40 years ago. We also pointed out that their work can still today be considered as the most advanced approach to the α cluster structure of ^{12}C. Their wave function does not contain any preconceived ingredients for α-particle condensation because, in principle, with RGM or Brink GCM the α's can take any arrangement they like. We surmise that all approaches which so far reproduced the measured properties of the Hoyle state, for instance, the inelastic form factor, implicitly describe the same α-particle condensate as does the THSR approach. This should notably be the case for AMD and FMD theories and, in particular, also with the recent GFMC approach. It is nevertheless very desirable that the bosonic occupation numbers will be calculated with those approaches as well.

Further indications of the validity of the condensation picture, also discussed by Zinner and Jensen (2013), are the fact that the de Broglie wavelength of the α particles in the Hoyle state is larger by factors than the extension of the Hoyle state, that is, larger than the inter-α distance (Yamada and Schuck, 2004; Yamada *et al.*, 2011a, 2011b; von Oertzen, 2010, 2011). Also, the calculated shape of the α-particle wave function in the condensate practically does not change, besides a trivial norm factor, from ^{12}C to ^{16}O (Yamada and Schuck, 2004; Yamada *et al.*, 2011a, 2011b), this being another criterium of α condensation; see Fig. 16 (Funaki *et al.*, 2010).

In this Colloquium, we only quickly discussed the situation in ^{16}O, where the 0_6^+ state at 15.1 MeV is identified as an α condensate state (Funaki *et al.*, 2008). Since the α disintegration threshold rises rather sharply with the number of α particles, one may wonder whether states at such high energies do not acquire a very large width, i.e., decay in very short times. The 15.1 MeV state in ^{16}O has a width of only 160 keV which is very small considering that excitation energy. This stems from the fact that all the states underneath have a strongly different structure. Nevertheless, the ground states have a certain percentage of α gas components and vice versa the condensate states have some shell model components. This gives rise to the decay probability which, of course, increases with more α particles but will stay unusually small.

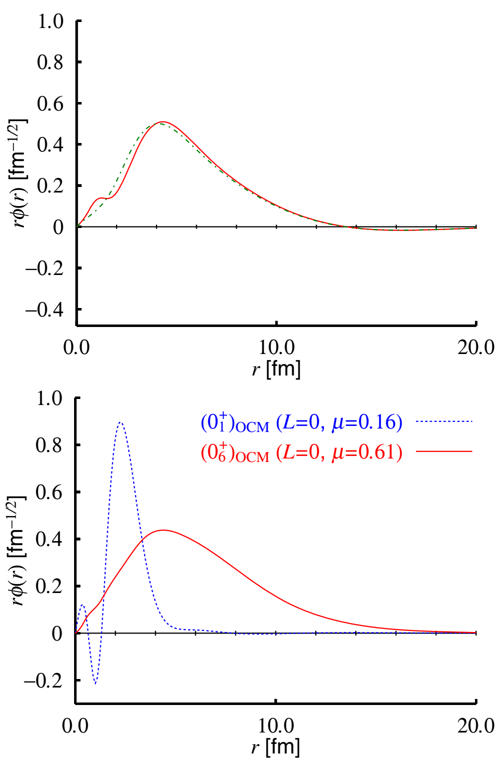

FIG. 16. Comparison of single α-particle wave functions in the condensate states of ^{12}C (upper panel) and ^{16}O (lower panel). Note the similarity of both wave functions (up to a scale factor). The dotted line in the upper panel is a best fit of a Gaussian to the calculated curve (solid line). In the lower panel, the dotted line represents the single α-partical wave function in the ground state. The strong distortion of an α particle in the compact ground state should be noted. The upper panel shows the same quantity as in Fig. 3 but from a slightly different calculation. We show it here again for a direct comparison with the ^{16}O case. From Funaki *et al.*, 2010.

Promising approaches to the Hoyle state are two recent attempts using QMC techniques. Epelbaum *et al.* (2012) used the so-called lattice QMC based on chiral perturbation theory with energy functional theory (EFT). The only open input parameters are the current quark masses. The low lying spectrum of ^{12}C is very well reproduced. However, no inelastic form factor is calculated as yet. Carlson *et al.* (2015) make use of the GFMC with the fixed node approximation. The inelastic form factor of the Hoyle state is very well reproduced; see Fig. 8 where we also show the inelastic form factor obtained with both the THSR approach and the one of Kamimura (1981). All three theories reproduce the inelastic form factor well. If at all possible to evaluate, it would be interesting to see what the GFMC approach yields for the bosonic occupation numbers. On the experimental side, the Bose condensate character is difficult to verify. However, we discussed heavy ion reactions and ^{8}Be + ^{8}Be decay out of the 0_6^+ state at 15.1 MeV in ^{16}O as possible future indicators of α-particle condensation. Also unusually low momenta of the decay products may give a hint.

Recently an interesting work appeared (Nakamura *et al.*, 2016), where some α gas states located just above the Hoyle state reproduced on grounds that the Hoyle state is an α condensate state. Only one adjustable parameter is involved. However, the approach used is novel and must be tested further before any firm conclusions can be drawn.

All in all, there exist many calculations, see, e.g., Horiuchi (1974, 1975), Uegaki *et al.* (1977, 1978), Kamimura (1981), Matsumara and Suzuki (2004), Funaki *et al.* (2009), Lazauskas and Dufour (2011), and Ishikawa (2014a, 2014b), which all point to the Hoyle state as being dominated by $0S$ waves among the three α's. We see no counter argument which would invalidate the hypothesis that the Hoyle state is to a large extent composed of an α-particle condensate with 70%–80% occupancy. These results are obtained from sophisticated but natural and transparent wave functions through a Raleigh-Ritz variational principle and the conclusions drawn from these investigations seem to us reliable. Additionally, there are clear theoretical indications that the sixth 0^+ at 15.1 MeV in ^{16}O is a Hoyle-analog state.

In this Colloquium, we concentrated on the case of the Hoyle state with only a small glimpse of the situation in ^{16}O. However, it seems clear that in heavier self-conjugate nuclei, such as ^{20}Ne, ^{24}Mg, up to ^{40}Ca close to the α disintegration threshold, analogous Hoyle-like α condensates may exist and that a whole series of excited states of which the Bose condensate can be considered as the ground state (Uegaki *et al.*, 1977, 1978) still is to be discovered and their precise nature to be clarified in the future. Studies in this direction were performed by Yamada and Schuck (2004) using the Gross-Pitaevskii equation for bosons. It seems that around ^{40}Ca the Coulomb barrier fades away and no long-lived α

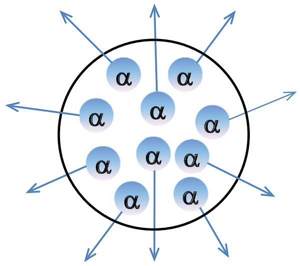

FIG. 17. Artist's view of a Coulomb explosion of ^{40}Ca into ten α's. We thank T. Yamada for the preparation of this figure.

condensate can exist any more. The α-like correlations and α formation is also of importance for nuclei with α decay such as ^{212}Po (Röpke *et al.*, 2014) and superheavy nuclei. Even the decay of heavier clusters has been observed such as ^{223}Ra into ^{209}Pb + ^{14}C and discussed theoretically (Barranco *et al.*, 1990). ^{20}Ne is similar to ^{212}Po with an α particle sitting on top of a doubly magic nucleus (^{16}O). In this respect, it is worth pointing out that mean field approaches (the independent particle model) can show sizable α cluster correlations (Girod and Schuck, 2013; Ebran *et al.*, 2014). We argued that heavy ion reactions with detection of coherent α-particle motion have been seen in one or two works in the past [see, e.g., Kokalova *et al.* (2006), and references therein]. However, these reactions seem to be a largely unexplored territory concerning α-particle coherence and condensation. We also pointed out that Coulomb excitation could be an ideal way of inducing important radial extension of a nucleus provoking (α) clustering. An artist's view of a hypothetical Coulomb explosion of ^{40}Ca into ten α's is shown in Fig. 17. It seems a truly exciting aspect that in the lighter $n\alpha$ nuclei there is a coexistence of two almost ideal quantum gases: fermions (nucleons) and bosons (α particles). Still many things have to be discovered in this context in future research where nuclear physics plays a prominent role. On the other hand, cluster physics is also developed concerning atomic clusters (Guet *et al.*, 2000). However, so far no bosonic condensation phenomena are discussed in this field to the best of our knowledge.

ACKNOWLEDGMENTS

We thank our collaborators Y. Funaki and T. Yamada who joined the α-particle cluster collaboration later but whose contributions have in several aspects been decisive. Also recent collaborations with Mengjiao Lyu, Z. Ren, Chang Xu, and Bo Zhou are much appreciated. P. S. is greatful to P. Nozières for his continuous interest in quartet condensation. We are grateful to Y. Kanada-En'yo for discussions and for preparing Fig. 5 for this article.

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