Neutrino electromagnetic interactions: A window to new physics

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(published 16 June 2015)

A review is given of the theory and phenomenology of neutrino electromagnetic interactions, which provide powerful tools to probe the physics beyond the standard model. After a derivation of the general structure of the electromagnetic interactions of Dirac and Majorana neutrinos in the one-photon approximation, the effects of neutrino electromagnetic interactions in terrestrial experiments and in astrophysical environments are discussed. The experimental bounds on neutrino electromagnetic properties are presented and the predictions of theories beyond the standard model are confronted.

DOI: 10.1103/RevModPhys.87.531 PACS numbers: 14.60.St, 13.15.+g, 13.35.Hb, 14.60.Lm

CONTENTS

I. INTRODUCTION

The theoretical and experimental investigation of neutrino properties and interactions is one of the most active fields of research in current high-energy physics. It brings us important information on the physics of the standard model and provides a powerful window on the physics beyond the standard model.

The possibility that a neutrino has a magnetic moment was considered by Pauli in his famous 1930 letter addressed to "Dear Radioactive Ladies and Gentlemen" (Pauli, 1991), in which he proposed the existence of the neutrino and he supposed that its mass could be of the same order of magnitude as the electron mass. Neutrinos remained elusive until the detection of reactor neutrinos by Reines and Cowan around 1956 (Reines *et al.*, 1960). However, there was no sign of a neutrino mass. After the discovery of parity violation in 1957, Landau (1957), Lee and Yang (1957), and Salam (1957) proposed the two-component theory of massless neutrinos, in which a neutrino is described by a Weyl spinor and there are only left-handed neutrinos and right-handed antineutrinos. It was, however, clear (Case, 1957; Mclennan, 1957; Radicati and Touschek, 1957) that two-component neutrinos could be massive Majorana fermions and that the two-component theory of a massless neutrino is equivalent to the Majorana theory in the limit of zero neutrino mass.

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The two-component theory of massless neutrinos was later incorporated in the standard model of Glashow (1961), Weinberg (1967), and Salam (1969), in which neutrinos are massless and have only weak interactions. In the standard model Majorana neutrino masses are forbidden by the $SU(2)_L \times U(1)_Y$ symmetry. Although in the standard model neutrinos are electrically neutral and do not possess electric or magnetic dipole moments, they have a charge radius which is generated by radiative corrections.

We now know that neutrinos are massive, because many experiments observed neutrino oscillations (Giunti and Kim, 2007; Bilenky, 2010; Xing and Zhou, 2011; Beringer *et al.*, 2012; Gonzalez-Garcia *et al.*, 2012; Bellini *et al.*, 2014), which are generated by neutrino masses and mixing (Pontecorvo, 1957, 1958, 1968; Maki, Nakagawa, and Sakata, 1962). Therefore, the standard model must be extended to account for the neutrino masses. There are many possible extensions of the standard model that predict different properties for neutrinos (Ramond, 1999; Mohapatra and Pal, 2004; Xing and Zhou, 2011). Among them, most important is their fundamental Dirac or Majorana character. In many extensions of the standard model neutrinos also acquire electromagnetic properties through quantum loop effects which allow direct interactions of neutrinos with electromagnetic fields and electromagnetic interactions of neutrinos with charged particles.

Hence, the theoretical and experimental study of neutrino electromagnetic interactions is a powerful tool in the search for the fundamental theory beyond the standard model. Moreover, the electromagnetic interactions of neutrinos can generate important effects, especially in astrophysical environments, where neutrinos propagate over long distances in magnetic fields in vacuum and in matter.

Unfortunately, in spite of many efforts in the search of neutrino electromagnetic interactions, up to now there is no positive experimental indication in favor of their existence. However, it is expected that the standard model neutrino charge radii should be measured in the near future. This will be a test of the standard model and of the physics beyond the standard model which contributes to the neutrino charge radii. Moreover, the existence of neutrino masses and mixing implies that neutrinos have magnetic moments. Since their values depend on the specific theory which extends the standard model in order to accommodate neutrino masses and mixing, experimentalists and theorists are eagerly looking for them.

The structure of this review is as follows. In Sec. II we summarize the basic theory of neutrino masses and mixing and the phenomenology of neutrino oscillations, which are important for the following discussion of theoretical models and for understanding the connection between neutrino masses and mixing and neutrino electromagnetic properties. In Sec. III we derive the general form of the electromagnetic interactions of Dirac and Majorana neutrinos in the one-photon approximation, which are expressed in terms of electromagnetic form factors. In Sec. IV we discuss the phenomenology of the neutrino magnetic and electric dipole moments in laboratory experiments. These are the most studied electromagnetic properties of neutrinos, both experimentally and theoretically. In Sec. V we discuss neutrino radiative decay in vacuum and in matter and related processes which are induced by the neutrino magnetic and electric dipole moments. These processes could have observable effects in astrophysical environments and could be detected on Earth by astronomical photon detectors. In Sec. VI we discuss some important effects due to the interaction of neutrino magnetic moments with classical electromagnetic fields. In particular, we derive the effective potential in a magnetic field and we discuss the corresponding spin and spin-flavor transitions in astrophysical environments. In Sec. VII we review the theory and experimental constraints on the neutrino electric charge (millicharge), the charge radius, and the anapole moment. In conclusion, in Sec. VIII we summarize the status of our knowledge of neutrino electromagnetic properties and we discuss the prospects for future research. This review also has several appendixes listed in Table I which are contained in an online Supplemental Material document (Giunti and Studenikin, 2015).

TABLE I. Appendixes contained in the online Supplemental Material document (Giunti and Studenikin, 2015). Page numbers in the table refer to the Supplemental Material document. References to equations which start with a capital alphabetic letter refer to the corresponding appendix in the table.

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We also remind one that neutrino electromagnetic properties and interactions are discussed by Bahcall (1989), Boehm and Vogel (1992), Kim and Pevsner (1993), Raffelt (1996), Fukugita and Yanagida (2003), Zuber (2003), Mohapatra and Pal (2004), Xing and Zhou (2011), Barger, Marfatia, and Whisnant (2012), and Lesgourgues *et al.* (2013), and in the previous reviews by Dolgov and Zeldovich (1981), Bilenky and Petcov (1987), Raffelt (1990a, 1999a, 1999b, 2000), Pulido (1992), Salati (1994), Dolgov (2002), Nowakowski, Paschos, and Rodriguez (2005), Wong and Li (2005), Giunti and Studenikin (2009), Studenikin (2009), Broggini, Giunti, and Studenikin (2012), and Akhmedov (2014). In this review we improved and extended the discussion presented in our previous reviews in order to cover in detail the most important aspects of neutrino electromagnetic interactions.

II. NEUTRINO MASSES AND MIXING

In the standard model of electroweak interactions (Glashow, 1961; Weinberg, 1967; Salam, 1969), neutrinos are described by two-component massless left-handed Weyl spinors (Giunti and Kim, 2007). The masslessness of neutrinos is due to the absence of right-handed neutrino fields, without which it is not possible to have Dirac mass terms, and to the absence of Higgs

TABLE II. Eigenvalues of the weak isospin I of its third component I_3 of the hypercharge Y, and of the charge $Q = I_3 + Y/2$ of the lepton and Higgs doublets and singlets in the extension of the standard model with the introduction of right-handed neutrinos.

$(\ell = e, \mu, \tau)$		I	I_3	Y	Q
Left-handed lepton doublets	$L_{\ell L} \equiv \begin{pmatrix} \nu_{\ell L} \\ \ell_L \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	-1	0 -1
Right-handed charged-lepton singlets	ℓ_R	0	0	-2	-1
Right-handed neutrino singlets	$\nu_{\ell R}$	0	0	0	0
Higgs doublet	$\Phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	$+1$	1 0

triplets, without which it is not possible to have Majorana mass terms. In the following we consider the extension of the standard model with the introduction of three right-handed neutrinos. We see that this seemingly innocent addition has the very powerful effect of introducing not only Dirac mass terms, but also Majorana mass terms for the right-handed neutrinos, which can induce Majorana masses for the observable light neutrinos through the seesaw mechanism.

Table II shows the values of the weak isospin, hypercharge, and electric charge of the lepton and Higgs doublets and singlets in the extended standard model under consideration. We work in the flavor basis in which the mass matrix of the charged leptons is diagonal. Hence, e, μ, and τ are the physical charged leptons with definite masses.

In the following subsections we briefly review the theory of masses and mixing of Dirac (Sec. II.A) and Majorana (Sec. II.B) neutrinos, the standard framework of three-neutrino mixing (Sec. II.C), neutrino oscillations in vacuum and in matter (Sec. II.D), the current phenomenological status of three-neutrino mixing (Sec. II.E), and the possibility of additional sterile neutrinos (Sec. II.F).

A. Dirac neutrinos

The fields in Table II allow us to construct the Yukawa Lagrangian term

$$\mathcal{L}_{\mathrm{Y}} = -\sum_{\ell,\ell'=e,\mu,\tau} Y_{\ell\ell'} \overline{L_{\ell L}}\, \tilde{\Phi}\, \nu_{\ell' R} + \mathrm{H.c.}, \tag{2.1}$$

where Y is a matrix of Yukawa couplings and $\tilde{\Phi} \equiv i\sigma_2 \Phi^*$. In the standard model, a nonzero vacuum expectation value of the Higgs doublet,

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{2.2}$$

induces the spontaneous symmetry breaking of the standard model symmetries $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$. From the Yukawa Lagrangian term in Eq. (2.1), we obtain the neutrino Dirac mass term

$$\mathcal{L}_{\mathrm{D}} = -\sum_{\ell,\ell'=e,\mu,\tau} \overline{\nu_{\ell L}}\, M^{\mathrm{D}}_{\ell\ell'} \nu_{\ell' R} + \mathrm{H.c.}, \tag{2.3}$$

with the complex 3×3 Dirac mass matrix

$$M^{\mathrm{D}} = \frac{v}{\sqrt{2}} Y. \tag{2.4}$$

If the total lepton number is conserved, $\mathcal{L}_{\mathrm{D}}$ is the only neutrino mass term and the three massive neutrinos obtained through the diagonalization of $\mathcal{L}_{\mathrm{D}}$ are Dirac particles. The diagonalization of $\mathcal{L}_{\mathrm{D}}$ is achieved through the transformations

$$\nu_{\ell L} = \sum_{k=1}^{3} U_{\ell k} \nu_{kL}, \tag{2.5}$$

$$\nu_{\ell' R} = \sum_{k=1}^{3} V_{\ell' k} \nu_{kR}, \tag{2.6}$$

with unitary 3×3 matrices U and V such that

$$(U^{\dagger} M^{\mathrm{D}} V)_{kj} = m_k \delta_{kj}, \tag{2.7}$$

with real and positive masses m_k (Bilenky and Petcov, 1987; Giunti and Kim, 2007). The resulting diagonal Dirac mass term is

$$\mathcal{L}_{\mathrm{D}} = -\sum_{k=1}^{3} m_k \overline{\nu_{kL}} \nu_{kR} + \mathrm{H.c.} = -\sum_{k=1}^{3} m_k \overline{\nu_k} \nu_k, \tag{2.8}$$

with the Dirac fields of massive neutrinos

$$\nu_k = \nu_{kL} + \nu_{kR}. \tag{2.9}$$

B. Majorana neutrinos

In the above derivation of Dirac neutrino masses we assumed that the total lepton number is conserved. However, since there is not any compelling argument which imposes the conservation of the total lepton number, it is plausible that the right-handed singlet neutrinos have the Majorana mass term

$$\mathcal{L}_R = \frac{1}{2} \sum_{\ell,\ell'=e,\mu,\tau} \nu_{\ell R}^T \mathcal{C}^{\dagger} M^R_{\ell\ell'} \nu_{\ell' R} + \mathrm{H.c.}, \tag{2.10}$$

which violates the total lepton number by two units. In Eq. (2.10), $\mathcal{C}$ is the charge-conjugation matrix defined by Eqs. (A34)–(A36) and the mass matrix M^R is complex and symmetric.

The Majorana mass term in Eq. (2.10) is allowed by the symmetries of the standard model, since right-handed neutrino fields are invariant. On the other hand, an analogous Majorana mass term of the left-handed neutrinos,

$$\mathcal{L}_L = \frac{1}{2} \sum_{\ell,\ell'=e,\mu,\tau} \nu_{\ell L}^T \mathcal{C}^{\dagger} M^L_{\ell\ell'} \nu_{\ell' L} + \mathrm{H.c.}, \tag{2.11}$$

is forbidden, since it has $I_3 = 1$ and $Y = -2$, as one can find easily using Table II. There is no Higgs triplet in the standard model to compensate these quantum numbers.

In the extension of the standard model with the introduction of right-handed neutrinos, the neutrino masses and mixing are given by the Dirac-Majorana mass term

$$\mathcal{L}_{\mathrm{D+M}} = \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{R}. \tag{2.12}$$

The neutrino fields with definite masses are obtained through the diagonalization of $\mathcal{L}_{\mathrm{D+M}}$. It is convenient to define the vector N_L of six left-handed fields

$$N_L^T \equiv (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}, \nu_{eR}^c, \nu_{\mu R}^c, \nu_{\tau R}^c), \tag{2.13}$$

with the charge-conjugated fields

$$\nu_{\ell R}^c = \mathcal{C}\overline{\nu_{\ell R}}^T. \tag{2.14}$$

The Dirac-Majorana mass term in Eq. (2.12) can be written in the compact form

$$\mathcal{L}_{\mathrm{D+M}} = \tfrac{1}{2} N_L^T \mathcal{C}^\dagger M^{\mathrm{D+M}} N_L + \mathrm{H.c.}, \tag{2.15}$$

with the 6×6 symmetric mass matrix

$$M^{\mathrm{D+M}} \equiv \begin{pmatrix} 0 & M^{\mathrm{D}T} \\ M^{\mathrm{D}} & M^R \end{pmatrix}. \tag{2.16}$$

The order of magnitude of the elements of the Dirac mass matrix M^{D} in Eq. (2.4) is smaller than $v \sim 10^2$ GeV, since the Dirac mass term (2.3) is forbidden by the symmetries of the standard model and can be generated only as a consequence of symmetry breaking below the electroweak scale v. On the other hand, since the Majorana mass term in Eq. (2.10) is a standard model singlet, the elements of the Majorana mass matrix M_R are not related to the electroweak scale. It is plausible that the Majorana mass term $\mathcal{L}_R$ is generated by new physics beyond the standard model and the right-handed chiral neutrino fields $\nu_{\ell R}$ belong to nontrivial multiplets of the symmetries of the high-energy theory. The corresponding order of magnitude of the elements of the mass matrix M_R is given by the symmetry-breaking scale of the high-energy physics beyond the standard model, which may be as large as the grand unification scale, of the order of 10^{14}–10^{16} GeV. In this case, the mass matrix can be diagonalized by blocks, up to corrections of the order $\epsilon = (M^R)^{-1} M^{\mathrm{D}}$:

$$W^T M^{\mathrm{D+M}} W \simeq \begin{pmatrix} M_{\mathrm{l}}^{\mathrm{M}} & 0 \\ 0 & M_{\mathrm{h}}^{\mathrm{M}} \end{pmatrix}, \tag{2.17}$$

with

$$W \simeq 1 - \frac{1}{2} \begin{pmatrix} \epsilon^\dagger \epsilon & 2\epsilon^\dagger \\ -2\epsilon & \epsilon\epsilon^\dagger \end{pmatrix}. \tag{2.18}$$

The light symmetric 3×3 Majorana mass matrix $M_{\mathrm{l}}^{\mathrm{M}}$ and the heavy symmetric 3×3 Majorana mass matrix $M_{\mathrm{h}}^{\mathrm{M}}$ are given by

$$M_{\mathrm{l}}^{\mathrm{M}} \simeq -M^{\mathrm{D}T} (M^R)^{-1} M^{\mathrm{D}}, \qquad M_{\mathrm{h}}^{\mathrm{M}} \simeq M^R. \tag{2.19}$$

There are three heavy masses given by the eigenvalues of $M_{\mathrm{h}}^{\mathrm{M}}$ and three light masses given by the eigenvalues of $M_{\mathrm{l}}^{\mathrm{M}}$, whose elements are suppressed with respect to the elements of the Dirac mass matrix M^{D} by the very small matrix factor $M^{\mathrm{D}T}(M^R)^{-1}$. This is the well-known *seesaw mechanism* (Minkowski, 1977; Gell-Mann, Ramond, and Slansky, 1979; Ramond, 1979; Yanagida, 1979; Mohapatra and Senjanovic, 1980), which naturally explains the smallness of light neutrino masses. Notice, however, that the values of the light neutrino masses and their relative sizes can vary over wide ranges, depending on the specific values of the elements of M^{D} and M^R.

Since the off-diagonal block elements of W are very small, the three flavor neutrinos are mainly composed by the three light neutrinos. Therefore, the seesaw mechanism implies the effective low-energy Majorana mass term

$$\mathcal{L}_{\mathrm{M}}^{\mathrm{eff}} = \frac{1}{2} \sum_{\ell,\ell'=e,\mu,\tau} \nu_{\ell L}^T \mathcal{C}^\dagger (M_{\mathrm{l}}^{\mathrm{M}})_{\ell\ell'} \nu_{\ell' L} + \mathrm{H.c.}, \tag{2.20}$$

which involves only the three active left-handed flavor neutrino fields. The symmetric 3×3 Majorana mass matrix $M_{\mathrm{l}}^{\mathrm{M}}$ is diagonalized by the transformation in Eq. (2.5) with a 3×3 unitary mixing matrix U such that

$$(U^T M_{\mathrm{l}}^{\mathrm{M}} U)_{kj} = m_k \delta_{kj}, \tag{2.21}$$

with real and positive masses m_k (Bilenky and Petcov, 1987; Giunti and Kim, 2007). In this way, the effective Majorana mass term in Eq. (2.20) can be written in terms of the massive fields as

$$\begin{aligned} \mathcal{L}_{\mathrm{M}}^{\mathrm{eff}} &= \frac{1}{2} \sum_{k=1}^{3} m_k \nu_{kL}^T \mathcal{C}^\dagger \nu_{kL} + \mathrm{H.c.} \\ &= \frac{1}{2} \sum_{k=1}^{3} m_k \nu_k^T \mathcal{C}^\dagger \nu_k, \end{aligned} \tag{2.22}$$

with the massive Majorana fields

$$\nu_k = \nu_{kL} + \nu_{kL}^c = \nu_{kL} + \mathcal{C}\overline{\nu_{kL}}^T, \tag{2.23}$$

which satisfy the Majorana constraint

$$\nu_k = \nu_k^c = \mathcal{C}\overline{\nu_k}^T. \tag{2.24}$$

Hence, a general result of the seesaw mechanism is an effective low-energy mixing of three massive Majorana neutrinos.

C. Three-neutrino mixing

In the previous two sections we have seen that an effective mixing of three light neutrinos is obtained in the Dirac case assuming the conservation of the total lepton number and in the Majorana case through the seesaw mechanism. In both cases the mixing relation between the three left-handed flavor neutrino fields ν_{eL}, $\nu_{\mu L}$, and $\nu_{\tau L}$ which partake in weak interactions and the three left-handed massive neutrino fields

ν_{1L}, ν_{2L}, and ν_{3L} is given by Eq. (2.5), which depends on a unitary 3×3 mixing matrix U.

The mixing matrix U is observable through its effects in charged-current weak-interaction processes in which leptons are described by the current

$$j^{\rho}_{\rm CC} = 2\sum_{\ell=e,\mu,\tau} \overline{\nu_{\ell L}}\gamma^{\rho}\ell_L = 2\sum_{\ell=e,\mu,\tau}\sum_{k=1}^{3} U^{*}_{\ell k}\overline{\nu_{kL}}\gamma^{\rho}\ell_L. \tag{2.25}$$

A unitary 3×3 matrix can be parametrized in terms of three mixing angles and six phases. However, in the mixing matrix three phases are unphysical, because they can be eliminated by rephasing the three charged-lepton fields in $j^{\rho}_{\rm CC}$. In the case of Majorana massive neutrinos, no additional phase can be eliminated, because the Majorana mass term in Eq. (2.22) is not invariant under rephasing of ν_{kL}. On the other hand, in the case of Dirac massive neutrinos, two additional phases can be eliminated by rephasing the massive neutrino fields. Hence, the mixing matrix has three physical phases in the case of Majorana massive neutrinos or one physical phase in the case of Dirac massive neutrinos. In general, in the case of Majorana massive neutrinos U can be written as

$$U = U^{\rm D}D^{\rm M}, \tag{2.26}$$

where $U^{\rm D}$ is a Dirac unitary mixing matrix which can be parametrized in terms of three mixing angles and one physical phase, called the *Dirac phase*, and $D^{\rm M}$ is a diagonal unitary matrix with two physical phases, usually called *Majorana phases*. In the case of Dirac neutrinos $U = U^{\rm D}$.

The standard parametrization of $U^{\rm D}$ is

$$U^{\rm D} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{2.27}$$

where $c_{ab} \equiv \cos\vartheta_{ab}$ and $s_{ab} \equiv \sin\vartheta_{ab}$. ϑ_{12}, ϑ_{13}, and ϑ_{23} are the three mixing angles ($0 \le \vartheta_{ab} \le \pi/2$) and δ_{13} is the Dirac phase ($0 \le \delta_{13} < 2\pi$). The diagonal unitary matrix $D^{\rm M}$ can be written as

$$D^{\rm M} = {\rm diag}(1, e^{i\lambda_{21}}, e^{i\lambda_{31}}), \tag{2.28}$$

in terms of the two Majorana phases λ_{21} and λ_{31},

All the phases in the mixing matrix violate the *CP* symmetry (Giunti and Kim, 2007; Branco, Felipe, and Joaquim, 2012).

We also note that in the leptonic weak neutral current,

$$j^{\rho}_{\rm NC} = \sum_{\ell=e,\mu,\tau} \overline{\nu_{\ell L}}\gamma^{\rho}\nu_{\ell L} = \sum_{k=1}^{3}\overline{\nu_{kL}}\gamma^{\rho}\nu_{kL}, \tag{2.29}$$

the unitarity of U implies the absence of neutral-current transitions among different massive neutrinos (GIM mechanism, Glashow, Iliopoulos, and Maiani, 1970).

D. Neutrino oscillations

Flavor neutrinos are produced and detected in charged-current weak-interaction processes described by the leptonic current in Eq. (2.25). Hence, a neutrino with flavor $\ell = e,\mu,\tau$ created in a charged-current weak-interaction process from a charged lepton ℓ^- or together with a charged antilepton ℓ^+ is described by the state

$$|\nu_\ell\rangle = \sum_k U^{*}_{\ell k}|\nu_k\rangle. \tag{2.30}$$

Since the mixing matrix is unitary, we have the inverted relation

$$|\nu_k\rangle = \sum_\ell U_{\ell k}|\nu_\ell\rangle. \tag{2.31}$$

The massive neutrino states $|\nu_k\rangle$ are eigenstates of the free Hamiltonian with energy eigenvalues

$$E_k = \sqrt{|\vec{p}_k|^2 + m_k^2}, \tag{2.32}$$

where $\vec{p}_k$ are the respective momenta. In the plane-wave approximation (Giunti and Kim, 2007), the space-time evolution of a massive neutrino is given by

$$|\nu_k(\vec{L},T)\rangle = e^{-iE_kT + i\vec{p}_k\cdot\vec{L}}|\nu_k\rangle, \tag{2.33}$$

where $(\vec{L},T)$ is the space-time distance from the production point. Inserting this equation into Eq. (2.30) and using Eq. (2.31), we obtain

$$\begin{aligned} |\nu_\ell(\vec{L},T)\rangle &= \sum_k U^{*}_{\ell k}e^{-iE_kT + i\vec{p}_k\cdot\vec{L}}|\nu_k\rangle \\ &= \sum_{\ell'=e,\mu,\tau}\left(\sum_k U^{*}_{\ell k}e^{-iE_kT + i\vec{p}_k\cdot\vec{L}}U_{\ell' k}\right)|\nu_{\ell'}\rangle. \end{aligned} \tag{2.34}$$

Then the phase differences of different massive neutrinos generate flavor transitions with probability

$$\begin{aligned} P_{\nu_\ell\to\nu_{\ell'}}(\vec{L},T) &= |\langle\nu_{\ell'}|\nu_\ell(\vec{L},T)\rangle|^2 \\ &= \left|\sum_k U^{*}_{\ell k}e^{-iE_kT + i\vec{p}_k\cdot\vec{L}}U_{\ell' k}\right|^2. \end{aligned} \tag{2.35}$$

Since the source-detector distance $L \equiv |\vec{L}|$ is macroscopic, we can consider all massive neutrino momenta $\vec{p}_k$ aligned along $\vec{L}$. Moreover, taking into account the smallness of neutrino masses, in oscillation experiments in which the neutrino propagation time T is not measured it is possible to approximate $T = L$ (Giunti and Kim, 2007). With these approximations, the phases in Eq. (2.35) reduce to

$$-E_k T + p_k L = -(E_k - p_k)L = -\frac{E_k^2 - p_k^2}{E_k + p_k} L$$
$$= -\frac{m_k^2}{E_k + p_k} L \simeq -\frac{m_k^2}{2E_\nu} L, \qquad (2.36)$$

at lowest order in the neutrino masses. Here $p_k \equiv |\vec{p}_k|$ and E_ν is the neutrino energy neglecting mass contributions. Equation (2.36) shows that the phases of massive neutrinos relevant for oscillations are independent of the values of the energies and momenta of different massive neutrinos, because of the relativistic dispersion relation in Eq. (2.32). The flavor transition probabilities are

$$P_{\nu_\ell \to \nu_{\ell'}}(L, E_\nu)$$
$$= \delta_{\ell\ell'} - 4\sum_{k>j} \mathrm{Re}(U_{\ell k}^* U_{\ell' k} U_{\ell j} U_{\ell' j}^*) \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E_\nu}\right)$$
$$- 2\sum_{k>j} \mathrm{Im}(U_{\ell k} U_{\ell j}^* U_{\ell' k}^* U_{\ell' j}) \sin\left(\frac{\Delta m_{kj}^2 L}{2E_\nu}\right), \qquad (2.37)$$

where $\Delta m_{kj}^2 = m_k^2 - m_j^2$.

In the approximation of two-neutrino mixing, in which one of the three massive neutrino components of two flavor neutrinos is neglected, the mixing matrix reduces to

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, \qquad (2.38)$$

where ϑ is the mixing angle ($0 \le \vartheta \le \pi/2$). In this approximation, there is only one squared-mass difference Δm^2 and the transition probability is given by

$$P_{\nu_\ell \to \nu_{\ell'}}^{2\nu}(L, E_\nu) = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E_\nu}\right) \quad (\ell \neq \ell'). \qquad (2.39)$$

The corresponding survival probabilities are given by

$$P_{\nu_\ell \to \nu_\ell}^{2\nu}(L, E_\nu) = 1 - \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E_\nu}\right). \qquad (2.40)$$

These simple expressions are often used in the analysis of experimental data.

When neutrinos propagate in matter, the potential generated by the coherent forward elastic scattering with the particles in the medium (electrons and nucleons) modifies mixing and oscillations (Wolfenstein, 1978). In a medium with varying density it is possible to have resonant flavor transitions (Mikheev and Smirnov, 1985). This is the famous MSW effect.

The effective potentials for ν_ℓ and $\bar{\nu}_\ell$ are, respectively,

$$V_\ell = V_{\mathrm{CC}} \delta_{\ell e} + V_{\mathrm{NC}}, \qquad \bar{V}_\ell = -V_\ell, \qquad (2.41)$$

with the charged-current and neutral-current potentials

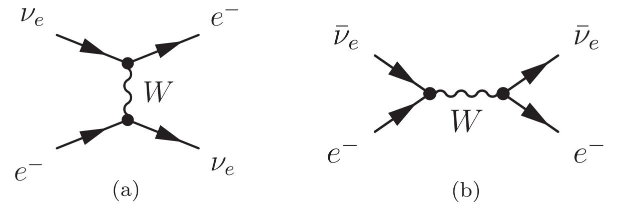

FIG. 1. Feynman diagrams of the coherent charged-current forward elastic scattering processes that generate the potentials (a) V_{CC} and (b) $\bar{V}_{\mathrm{CC}}$.

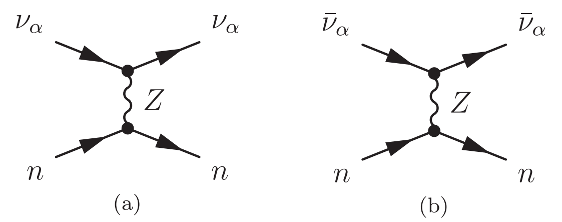

FIG. 2. Feynman diagram of the coherent neutral-current forward elastic scattering processes that generate the potentials (a) V_{NC} and (b) $\bar{V}_{\mathrm{NC}}$.

$$V_{\mathrm{CC}} = \sqrt{2} G_{\mathrm{F}} N_e, \qquad V_{\mathrm{NC}} = -\tfrac{1}{2}\sqrt{2} G_{\mathrm{F}} N_n, \qquad (2.42)$$

generated, respectively, by the Feynman diagrams in Figs. 1(a) and 2(a). Here N_e and N_n are the electron and neutron number densities in the medium (in an electrically neutral medium the neutral-current potentials of protons and electrons cancel each other). In normal matter, these potentials are very small, because

$$\sqrt{2} G_{\mathrm{F}} \simeq 7.63 \times 10^{-14} \frac{\mathrm{eV\ cm^3}}{N_{\mathrm{A}}}, \qquad (2.43)$$

where N_{A} is Avogadro's number given in Eq. (A1).

We consider, for simplicity, two-neutrino $\nu_e - \nu_a$ mixing, where ν_a is a linear combination of ν_μ and ν_τ (which can be pure ν_μ or ν_τ as special cases). This is a good approximation for solar neutrinos. In general, a neutrino produced at $x = 0$ is described at a distance x by a state

$$|\nu(x)\rangle = \varphi_e(x)|\nu_e\rangle + \varphi_a(x)|\nu_a\rangle. \qquad (2.44)$$

Taking into account the fact that for ultrarelativistic neutrinos the distance x is approximately equal to the propagation time t, the evolution of the flavor amplitudes $\varphi_e(x)$ and $\varphi_a(x)$ with the distance x is given by the Schrödinger equation (Wolfenstein, 1978)

$$i\frac{d}{dx}\begin{pmatrix} \varphi_e(x) \\ \varphi_a(x) \end{pmatrix} = \mathrm{H}\begin{pmatrix} \varphi_e(x) \\ \varphi_a(x) \end{pmatrix}, \qquad (2.45)$$

with the effective Hamiltonian matrix

$$\mathrm{H} = \frac{1}{4E_\nu}\begin{pmatrix} -\Delta m^2 \cos 2\vartheta + A_{\mathrm{CC}} & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta - A_{\mathrm{CC}} \end{pmatrix}, \qquad (2.46)$$

where $A_{CC} = 2E_\nu V_{CC}$. In Eq. (2.46) we took into account only the difference V_{CC} of the potentials of ν_e and ν_a, which affects neutrino oscillations. In the framework of three-neutrino mixing the neutral-current potential V_{NC}, which is common to the three neutrino flavors, does not have any effect. However, one must be aware that the neutral-current potential V_{NC} must be taken into account in extensions of three-neutrino mixing involving sterile states (see Sec. II.F) and/or spin-flavor transitions (see Sec. VI.B).

For an initial ν_e, as in the case of solar neutrinos, the boundary condition for the solution of the differential equation is

$$\begin{pmatrix} \varphi_e(0) \\ \varphi_a(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.47}$$

and the probabilities of $\nu_e \to \nu_a$ transitions and ν_e survival are, respectively,

$$P_{\nu_e\to\nu_a}(x) = |\varphi_a(x)|^2, \tag{2.48}$$

$$P_{\nu_e\to\nu_e}(x) = |\varphi_e(x)|^2 = 1 - P_{\nu_e\to\nu_a}(x). \tag{2.49}$$

The effective Hamiltonian matrix in Eq. (2.46) can be diagonalized with the transformation

$$\begin{pmatrix} \varphi_e(x) \\ \varphi_a(x) \end{pmatrix} = U_{\rm M} \begin{pmatrix} \varphi_1^{\rm M}(x) \\ \varphi_2^{\rm M}(x) \end{pmatrix}, \tag{2.50}$$

with the effective orthogonal ($U_{\rm M}^T = U_{\rm M}^{-1}$) mixing matrix in matter

$$U_{\rm M} = \begin{pmatrix} \cos\vartheta_{\rm M} & \sin\vartheta_{\rm M} \\ -\sin\vartheta_{\rm M} & \cos\vartheta_{\rm M} \end{pmatrix}, \tag{2.51}$$

such that

$$U_{\rm M}^T \mathrm{H} U_{\rm M} = \frac{\mathrm{diag}(-\Delta m_{\rm M}^2, \Delta m_{\rm M}^2)}{4E_\nu}. \tag{2.52}$$

The amplitudes $\varphi_1^{\rm M}(x)$ and $\varphi_2^{\rm M}(x)$ correspond to the effective massive neutrinos in matter $\nu_1^{\rm M}(x)$ and $\nu_2^{\rm M}(x)$, which have the effective squared-mass difference

$$\Delta m_{\rm M}^2 = \sqrt{(\Delta m^2\cos 2\vartheta - 2E_\nu V_{CC})^2 + (\Delta m^2 \sin 2\vartheta)^2}. \tag{2.53}$$

The effective mixing angle in matter $\vartheta_{\rm M}$ is given by

$$\tan 2\vartheta_{\rm M} = \frac{\tan 2\vartheta}{1 - 2E_\nu V_{CC}/\Delta m^2 \cos 2\vartheta}. \tag{2.54}$$

The most interesting characteristic of this expression is that there is a resonance (Mikheev and Smirnov, 1985) when

$$V_{CC} = \frac{\Delta m^2}{2E_\nu}\cos 2\vartheta, \tag{2.55}$$

which corresponds to the electron number density

$$N_e^{\rm R} = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2}E_\nu G_{\rm F}}. \tag{2.56}$$

At the resonance the effective mixing angle is equal to $\pi/4$, i.e., the mixing is maximal, leading to the possibility of total transitions between the two flavors if the resonance region is wide enough.

In general, the evolution equation (2.45) must be solved numerically or with appropriate approximations. In a constant matter density, it is easy to derive an analytic solution, leading to the transition probability

$$P_{\nu_e\to\nu_a}^{2\nu}(x) = \sin^2 2\vartheta_{\rm M} \sin^2\left(\frac{\Delta m_{\rm M}^2 x}{4E_\nu}\right). \tag{2.57}$$

This expression has the same structure as the two-neutrino transition probability in vacuum in Eq. (2.39), with the mixing angle and the squared-mass difference replaced by their effective values in matter.

The matter effect is especially important for solar neutrinos, which are created as electron neutrinos by thermonuclear reactions in the center of the Sun, where the electron number density N_e is of the order of $10^2 N_{\rm A}$ cm^{-3}, and propagate out of the Sun through an electron density which decreases approximately in an exponential way (Giunti and Kim, 2007). In a first approximation which neglects the small effects due to ϑ_{13}, ν_e is mixed only with ν_1 and ν_2, which are almost equally mixed with ν_μ and ν_τ (see Sec. II.E). In this approximation, the oscillations of solar neutrinos are well described by the two-neutrino $\nu_e - \nu_a$ mixing formalism with $\vartheta = \vartheta_{12}$. The oscillations are generated by the solar squared-mass difference

$$\Delta m_{\rm S}^2 \approx 8 \times 10^{-5}\ \mathrm{eV}^2, \tag{2.58}$$

and

$$|\nu_a\rangle \simeq \cos\vartheta_{23}|\nu_\mu\rangle - \sin\vartheta_{23}|\nu_\tau\rangle \approx (|\nu_\mu\rangle - |\nu_\tau\rangle)/\sqrt{2}. \tag{2.59}$$

An electron neutrino created in the center of the Sun is the linear combination of effective massive neutrinos

$$|\nu_e^0\rangle = \cos\vartheta_{\rm M}^0|\nu_1^0\rangle + \sin\vartheta_{\rm M}^0|\nu_2^0\rangle, \tag{2.60}$$

where ν_1^0 and ν_2^0 are the effective massive neutrinos at the point of neutrino production near the center of the Sun and $\vartheta_{\rm M}^0$ is the corresponding effective mixing angle. Since the resonance is crossed adiabatically, there are no transitions between the effective massive neutrinos during propagation and the state which emerges from the Sun is

$$|\nu_{\rm S}\rangle = \cos\vartheta_{\rm M}^0|\nu_1\rangle + \sin\vartheta_{\rm M}^0|\nu_2\rangle, \tag{2.61}$$

where ν_1 and ν_2 are the massive neutrinos in vacuum. Since the two massive neutrinos lose coherence during the long propagation from the Sun to the Earth (Dighe, Liu, and Smirnov, 1999), experiments on Earth measure the average electron neutrino survival probability (Parke, 1986)

$$\bar{P}^{S,2\nu}_{\nu_e\to\nu_e} = \cos^2\vartheta^0_M |\langle\nu_e|\nu_1\rangle|^2 + \sin^2\vartheta^0_M |\langle\nu_e|\nu_2\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\vartheta^0_M \cos 2\vartheta_{12}. \quad (2.62)$$

This is a surprisingly simple expression, which depends only on the mixing angle in vacuum ϑ_{12} and on the effective mixing angle in the center of the Sun ϑ^0_M, which can be easily calculated using Eq. (2.54). Notice that ϑ^0_M depends on the neutrino energy. With the value of Δm^2_S in Eq. (2.58), $\vartheta^0_M \simeq \vartheta_{12}$ for $E_\nu \lesssim 1$ MeV and $\vartheta^0_M \simeq \pi/2$ for $E_\nu \gtrsim 5$ MeV (Giunti and Kim, 2007). Therefore,

$$\bar{P}^{S,2\nu}_{\nu_e\to\nu_e} \simeq \begin{cases} 1 - 0.5\sin^2 2\vartheta_{12} & \text{for } E_\nu \lesssim 1 \text{ MeV}, \\ \sin^2\vartheta_{12} & \text{for } E_\nu \gtrsim 5 \text{ MeV}. \end{cases} \quad (2.63)$$

E. Status of three-neutrino mixing

The results of several solar, atmospheric, and long-baseline neutrino oscillation experiments have proved that neutrinos are massive and mixed particles (Giunti and Kim, 2007; Bilenky, 2010; Xing and Zhou, 2011; Gonzalez-Garcia *et al.*, 2012; Bellini *et al.*, 2014; Capozzi *et al.*, 2014; Gonzalez-Garcia, Maltoni, and Schwetz, 2014). There are two groups of experiments which measured two types of flavor transition generated by two independent squared-mass differences (Δm^2): the solar squared-mass difference in Eq. (2.58) and the atmospheric squared-mass difference

$$\Delta m^2_A \approx 2 \times 10^{-3} \text{ eV}^2. \quad (2.64)$$

Since in the framework of three-neutrino mixing described in Sec. II.C there are just two independent squared-mass differences, solar, atmospheric, and long-baseline data have led us to the current three-neutrino mixing paradigm, with the standard assignments

$$\Delta m^2_S = \Delta m^2_{21} \ll \Delta m^2_A = \tfrac{1}{2}|\Delta m^2_{31} + \Delta m^2_{32}|. \quad (2.65)$$

The absolute value in the definition of Δm^2_A is necessary, because there are the two possible orderings of the neutrino masses illustrated schematically in the insets of the two corresponding panels in Fig. 3: the normal ordering (NO) with $m_1 < m_2 < m_3$ and $\Delta m^2_{13}, \Delta m^2_{23} > 0$, and the inverted ordering (IO) with $m_3 < m_1 < m_2$ and $\Delta m^2_{13}, \Delta m^2_{23} < 0$.

The three-neutrino mixing parameters can be determined with good precision with a global fit of neutrino oscillation data. In Table III we report the results of the latest global fit presented by Capozzi *et al.* (2014), which agree, within the uncertainties, with the NuFIT-v1.2 (Gonzalez-Garcia, Maltoni, and Schwetz, 2014) update of the global analysis presented by Gonzalez-Garcia *et al.* (2012). One can see that all the oscillation parameters are determined with precision between about 3% and 10%. The largest uncertainty is that of ϑ_{23}, which is known to be close to maximal ($\pi/4$), but it is not known if it is smaller or larger than $\pi/4$. For the Dirac *CP*-violating phase δ, there is an indication in favor of $\delta \approx 3\pi/2$, which would give maximal *CP* violation, but at 3σ all the values of δ are allowed, including the *CP*-conserving values $\delta = 0, \pi$.

An open problem in the framework of three-neutrino mixing is the determination of the absolute scale of neutrino masses, which cannot be determined with neutrino oscillation experiments, because oscillations depend only on the differences of neutrino masses. However, the measurement in neutrino oscillation experiments of the neutrino squared-mass differences allows us to constrain the allowed patterns of neutrino masses. A convenient way to see the allowed patterns of neutrino masses is to plot the values of the masses as functions of the unknown lightest mass, which is m_1 in the normal ordering and m_3 in the inverted ordering, as shown in

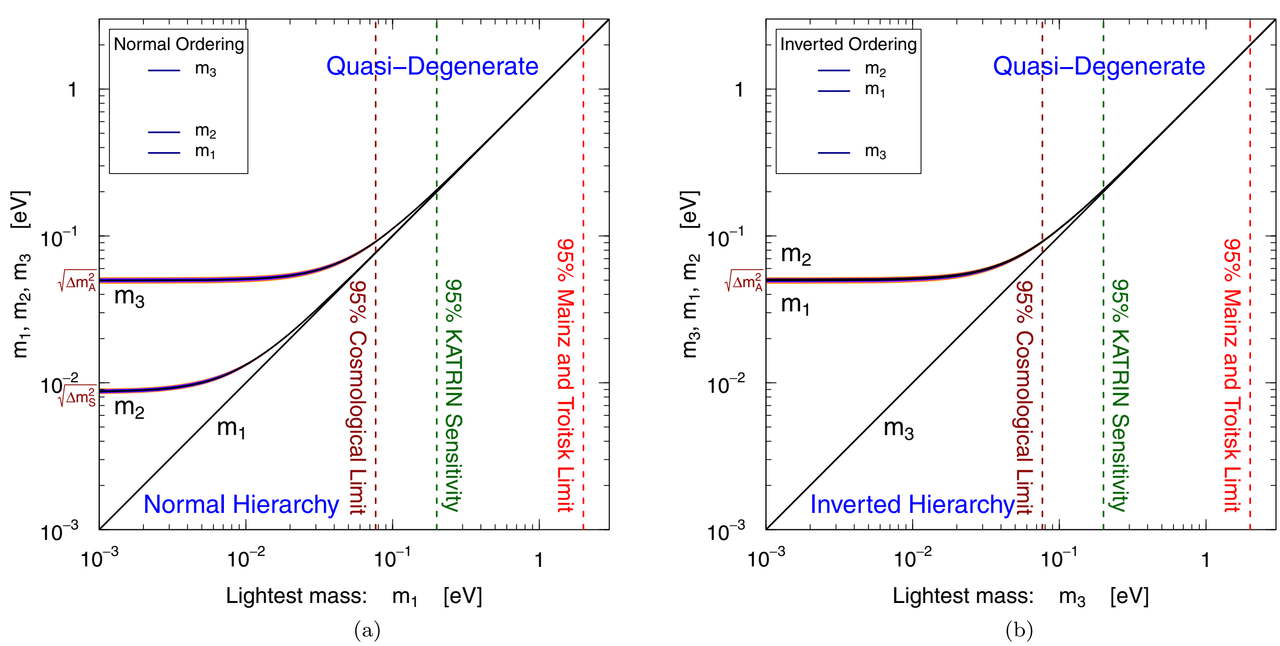

FIG. 3 (color online). Values of the neutrino masses as functions of the lightest mass in the two possible cases of (a) normal ordering and (b) inverted ordering. They have been obtained using the squared-mass differences in Table III.

TABLE III. Values of the neutrino mixing parameters obtained with a global analysis of neutrino oscillation data presented by Capozzi *et al.* (2014) in the framework of three-neutrino mixing with the normal ordering (NO) and the inverted ordering (IO). The relative uncertainty has been obtained from the 3σ range divided by 6.

Parameter	Ordering	Best fit	1σ range	2σ range	3σ range	Relative uncertainty
$\Delta m^2_S/10^{-5}$ eV^2		7.54	7.32–7.80	7.15–8.00	6.99–8.18	3%
$\sin^2\vartheta_{12}/10^{-1}$		3.08	2.91–3.25	2.75–3.42	2.59–3.59	5%
$\Delta m^2_A/10^{-3}$ eV^2	NO	2.43	2.37–2.49	2.30–2.55	2.23–2.61	3%
	IO	2.38	2.32–2.44	2.25–2.50	2.19–2.56	3%
$\sin^2\vartheta_{23}/10^{-1}$	NO	4.37	4.14–4.70	3.93–5.52	3.74–6.26	10%
	IO	4.55	4.24–5.94	4.00–6.20	3.80–6.41	10%
$\sin^2\vartheta_{13}/10^{-2}$	NO	2.34	2.15–2.54	1.95–2.74	1.76–2.95	8%
	IO	2.40	2.18–2.59	1.98–2.79	1.78–2.98	8%

Fig. 3. We used the squared-mass differences in Table III. Figure 3 shows that there are three extreme possibilities:

A normal hierarchy: $m_1 \ll m_2 \ll m_3$. In this case

$$m_2 \simeq \sqrt{\Delta m^2_S} \approx 9\times 10^{-3}\ \text{eV}, \tag{2.66}$$

$$m_3 \simeq \sqrt{\Delta m^2_A} \approx 5\times 10^{-2}\ \text{eV}. \tag{2.67}$$

An inverted hierarchy: $m_3 \ll m_1 \lesssim m_2$. In this case

$$m_1 \lesssim m_2 \simeq \sqrt{\Delta m^2_A} \approx 5\times 10^{-2}\ \text{eV}. \tag{2.68}$$

Quasidegenerate masses: $m_1 \lesssim m_2 \lesssim m_3 \simeq m_\nu$ in the normal scheme and $m_3 \lesssim m_1 \lesssim m_2 \simeq m_\nu$ in the inverted scheme, with

$$m_\nu \gg \sqrt{\Delta m^2_A} \approx 5\times 10^{-2}\ \text{eV}. \tag{2.69}$$

There are three main sources of information on the absolute scale of neutrino masses:

Beta decay: The most robust information on neutrino masses can be obtained in β-decay experiments which measure the kinematical effect of neutrino masses on the energy spectrum of the emitted electron. Tritium β-decay experiments obtained the most stringent bounds on the neutrino masses by limiting the effective electron neutrino mass m_β given by (Giunti and Kim, 2007; Bilenky, 2010; Xing and Zhou, 2011)

$$m_\beta^2 = \sum_{k=1}^{3} |U_{ek}|^2 m_k^2. \tag{2.70}$$

The most stringent 95% C.L. limits obtained in the Mainz (Kraus *et al.*, 2005) and Troitsk (Aseev *et al.*, 2011) experiments,

$$m_\beta \leq 2.3\ \text{eV} \quad (\text{Mainz}), \tag{2.71}$$

$$m_\beta \leq 2.1\ \text{eV} \quad (\text{Troitsk}), \tag{2.72}$$

are shown in Fig. 3. The KATRIN experiment (Fraenkle, 2011), which was scheduled to start data taking in 2014, is expected to have a sensitivity to m_β of about 0.2 eV (also shown in Fig. 3).

Neutrinoless double-beta decay: This process occurs only if massive neutrinos are Majorana fermions and depends on the effective Majorana mass (Giunti and Kim, 2007; Bilenky, 2010; Xing and Zhou, 2011; Bilenky and Giunti, 2014)

$$m_{\beta\beta} = \left| \sum_{k=1}^{3} U_{ek}^2 m_k \right|. \tag{2.73}$$

The most stringent 90% C.L. limits have been obtained combining the results of EXO (Auger *et al.*, 2012) and KamLAND-Zen (Gando *et al.*, 2013) experiments with ^{136}Xe,

$$m_{\beta\beta} \lesssim 0.12\text{–}0.25\ \text{eV}, \tag{2.74}$$

and combining the results of Heidelberg-Moscow (Klapdor-Kleingrothaus *et al.*, 2001), IGEX (Aalseth *et al.*, 2002), and GERDA (Agostini *et al.*, 2013) with ^{76}Ge,[1]

$$m_{\beta\beta} \lesssim 0.2\text{–}0.4\ \text{eV}. \tag{2.75}$$

The intervals are caused by nuclear physics uncertainties (Vergados, Ejiri, and Simkovic, 2012).

Cosmology: Since light massive neutrinos are hot dark matter, cosmological data give information on the sum of neutrino masses (Giunti and Kim, 2007;

[1]The claim of observation of neutrinoless double-beta decay of ^{76}Ge presented by Klapdor-Kleingrothaus *et al.* (2004) is strongly disfavored by the recent results of the GERDA experiment (Agostini *et al.*, 2013) and by the combined bound in Eq. (2.75); see also Aalseth *et al.* (2004), Elliott and Engel (2004), Strumia and Vissani (2006), Schwingenheuer (2013), and Bilenky and Giunti (2014).

Bilenky, 2010; Xing and Zhou, 2011; Lesgourgues *et al.*, 2013). The analysis of cosmological data in the framework of the standard cold dark matter model with a cosmological constant (ΛCDM) disfavors neutrino masses larger than some fraction of eV, but the value of the upper bound on the sum of neutrino masses depends on model assumptions and on the considered data set (Wong, 2011). Figure 3 shows the 95% limit

$$\sum_{k=1}^{3} m_k < 0.32 \text{ eV}, \tag{2.76}$$

obtained recently by the Planck Collaboration (Ade *et al.*, 2014). See Archidiacono *et al.* (2013), Lesgourgues and Pastor (2014), and Abazajian *et al.* (2015) for recent reviews of the implications of cosmological data for neutrino physics.

F. Sterile neutrinos

In the previous sections we considered the standard framework of three-neutrino mixing which can explain the numerous existing measurements of neutrino oscillations as explained in Sec. II.E. However, it is possible that there are additional massive neutrinos, such as those at the eV scale suggested by anomalies found in short-baseline oscillation experiments (Aguilar *et al.*, 2001; Abdurashitov *et al.*, 2006; Giunti and Laveder, 2011; Kopp, Maltoni, and Schwetz, 2011; Mention *et al.*, 2011; Giunti *et al.*, 2012, 2013; Conrad *et al.*, 2013; Kopp *et al.*, 2013) or those at the keV scale, which could constitute warm dark matter according to the neutrino minimal standard model (νMSM) (Asaka, Blanchet, and Shaposhnikov, 2005; Asaka and Shaposhnikov, 2005; Asaka, Kusenko, and Shaposhnikov, 2006; Asaka, Laine, and Shaposhnikov, 2006, 2007) [see also the reviews by Boyarsky, Ruchayskiy, and Shaposhnikov (2009), Kusenko (2009), Boyarsky, Iakubovskyi, and Ruchayskiy (2012), and Drewes (2013)]. In the flavor basis, which describes the interacting neutrino states, the additional neutrinos are sterile, because we know from the measurement of the invisible width of the Z boson in the LEP experiments that the number of light active neutrinos is three (Schael *et al.*, 2006), and the existence of a heavy fourth generation of active fermions with an active neutrino heavier than $m_Z/2$ is disfavored by the experimental data (Lenz, 2013; Vysotsky, 2013). From a theoretical point of view, it is likely that if there are sterile neutrinos, all neutrinos are Majorana particles, but the Dirac case is not excluded.

We consider the general case of N_s sterile neutrinos $\nu_{s_1}, \ldots, \nu_{s_{N_s}}$. In the mass basis there are $N = 3 + N_s$ massive neutrino fields $\nu_1, \ldots, \nu_N$ and the mixing of the left-handed neutrino fields is given by

$$\nu_{\ell L} = \sum_{k=1}^{N} U_{\ell k} \nu_{kL} \quad (\ell = e, \mu, \tau, s_1, \ldots, s_{N_s}), \tag{2.77}$$

where U is a $N \times N$ unitary mixing matrix. The three massive neutrinos ν_1, ν_2, and ν_3 coincide with those in the standard three-neutrino mixing framework discussed in Sec. II.C, and $\nu_4, \ldots, \nu_N$ are the additional nonstandard N_s massive neutrinos. In order to preserve approximately the three-neutrino mixing explanation of oscillation data described in Sec. II.E, the mixing of the three active neutrinos ν_e, ν_μ, and ν_τ with the nonstandard massive neutrinos $\nu_4, \ldots, \nu_N$ must be very small:

$$|U_{\ell k}| \ll 1 \quad \text{for } \ell = e, \mu, \tau \quad \text{and} \quad k \geq 4, \tag{2.78}$$

which implies that

$$|U_{s_n k}| \ll 1 \quad \text{for } n = 1, \ldots, N_s \quad \text{and} \quad k \leq 3. \tag{2.79}$$

Since the mixing in the sterile sector is arbitrary, it is convenient to choose

$$U_{s_n k} = 0 \quad \text{for } n \neq k - 3 \quad \text{and} \quad k \geq 4. \tag{2.80}$$

Then, from Eq. (2.79) we have

$$1 - |U_{s_{k-3} k}|^2 \ll 1, \quad \text{for } k \geq 4. \tag{2.81}$$

The numerical values of the inequalities (2.78)–(2.81) depend on the model and on the experimental data under consideration. In this review we consider only these generic inequalities in order to present general results on the neutrino dipole moments in Secs. IV.A and IV.B and on neutrino radiative decay in Sec. V.A.

III. ELECTROMAGNETIC FORM FACTORS

The importance of neutrino electromagnetic properties was first mentioned by Pauli in 1930, when he postulated the existence of this particle and discussed the possibility that the neutrino might have a magnetic moment (Pauli, 1991). Systematic theoretical studies of neutrino electromagnetic properties started after it was shown that in the extended standard model with right-handed neutrinos the magnetic moment of a massive neutrino is, in general, nonvanishing and that its value is determined by the neutrino mass (Lee and Shrock, 1977; Marciano and Sanda, 1977; Petcov, 1977; Fujikawa and Shrock, 1980; Pal and Wolfenstein, 1982; Shrock, 1982; Bilenky and Petcov, 1987).

Neutrino electromagnetic properties are important because they are directly connected to fundamentals of particle physics. For example, neutrino electromagnetic properties can be used to distinguish Dirac and Majorana neutrinos, because Dirac neutrinos can have both diagonal and off-diagonal magnetic and electric dipole moments, whereas only the off-diagonal ones are allowed for Majorana neutrinos (Schechter and Valle, 1981; Kayser, 1982, 1984; Nieves, 1982; Pal and Wolfenstein, 1982; Shrock, 1982). This is shown in detail in Secs. III.A and III.B. Another important case in which Dirac and Majorana neutrinos have quite different observable effects is the spin-flavor precession in an external magnetic field discussed in Sec. VI.B. Neutrino electromagnetic properties are also probes of new physics beyond the standard model, because in the standard model neutrinos can have only a charge radius (see Secs. III.C and VII.B). The discovery of other neutrino electromagnetic properties would be a signal of new physics beyond the standard model (Bell *et al.*, 2005, 2006; Bell, 2007; Novales-Sanchez *et al.*, 2008).

In this section we discuss the general form of the electromagnetic interactions of Dirac and Majorana neutrinos in the one-photon approximation. In Sec. III.A we derive the general expression of the effective electromagnetic coupling of Dirac neutrinos in terms of electromagnetic form factors and we discuss the properties of the form factors under CP and CPT transformations. In Sec. III.B we consider Majorana neutrinos and in Sec. III.C we consider the standard model case of massless Weyl neutrinos.

A. Dirac neutrinos

In the standard model, the interaction of a fermionic field $\mathfrak{f}(x)$ with the electromagnetic field $A^\mu(x)$ is given by the interaction Hamiltonian

$$\mathcal{H}_{\text{em}}^{(\mathfrak{f})}(x) = j_\mu^{(\mathfrak{f})}(x)A^\mu(x) = \mathbb{q}_\mathfrak{f}\bar{\mathfrak{f}}(x)\gamma_\mu\mathfrak{f}(x)A^\mu(x), \quad (3.1)$$

where $\mathbb{q}_\mathfrak{f}$ is the charge of the fermion $\mathfrak{f}$. Figure 4(a) shows the corresponding tree-level Feynman diagram [the photon γ is the quantum of the electromagnetic field $A^\mu(x)$].

For neutrinos the electric charge is zero and there are no electromagnetic interactions at tree level.[2] However, such interactions can arise at the quantum level from loop diagrams at higher order of the perturbative expansion of the interaction. In the one-photon approximation,[3] the electromagnetic interactions of a neutrino field $\nu(x)$ can be described by the effective interaction Hamiltonian

$$\mathcal{H}_{\text{em}}^{(\nu)}(x) = j_\mu^{(\nu)}(x)A^\mu(x) = \bar{\nu}(x)\Lambda_\mu\nu(x)A^\mu(x), \quad (3.2)$$

where $j_\mu^{(\nu)}(x)$ is the neutrino effective electromagnetic current four-vector and Λ_μ is a 4×4 matrix in spinor space which can contain space-time derivatives, such that $j_\mu^{(\nu)}(x)$ transforms as a four-vector. Since radiative corrections are generated by weak interactions which are not invariant under a parity transformation, $j_\mu^{(\nu)}(x)$ can be a sum of polar and axial parts. The corresponding diagram for the interaction of a neutrino with a photon is shown in Fig. 4(b), where the blob represents the quantum loop contributions.

As we see in the following, the neutrino electromagnetic properties corresponding to the diagram in Fig. 4(b) include charge and magnetic form factors. We emphasize that these neutrino electromagnetic properties can exist even if neutrinos are elementary particles, without an internal structure, because they are generated by quantum loop effects. Thus, the neutrino charge and magnetic form factors have a different origin from the neutron charge and magnetic form factors (also called Dirac and Pauli form factors), which are mainly due to its internal quark structure. For example, the neutrino magnetic moment [which is the magnetic form factor for interactions with real photons, i.e., $q^2 = 0$ in Fig. 4(b)] have the same

[2]However, in some theories beyond the standard model neutrinos can be millicharged particles (see Sec. VII.A).

[3]Some cases in which the one-photon approximation breaks down are discussed in Sec. VII.A.

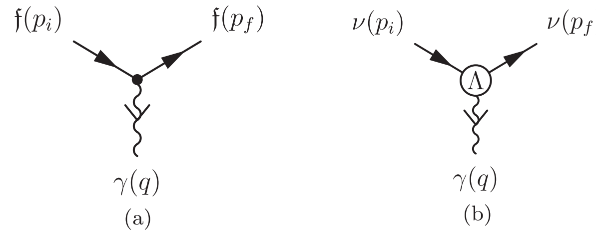

FIG. 4. Tree-level coupling of a charged fermion $\mathfrak{f}$ with (a) a photon γ and (b) effective one-photon coupling of a neutrino with a photon.

quantum origin as the anomalous magnetic moment of the electron (Greiner and Reinhardt, 2009).

We are interested in the neutrino part of the amplitude corresponding to the diagram in Fig. 4(b), which is given by the matrix element

$$\langle\nu(p_f,h_f)|j_\mu^{(\nu)}(x)|\nu(p_i,h_i)\rangle, \quad (3.3)$$

where p_i (p_f) and h_i (h_f) are the four-momentum and helicity of the initial (final) neutrino. Taking into account the fact that

$$\partial^\mu j_\mu^{(\nu)}(x) = i[\mathcal{P}^\mu, j_\mu^{(\nu)}(x)], \quad (3.4)$$

where $\mathcal{P}^\mu$ is the four-momentum operator which generate translations, the effective current can be written as

$$j_\mu^{(\nu)}(x) = e^{i\mathcal{P}\cdot x}j_\mu^{(\nu)}(0)e^{-i\mathcal{P}\cdot x}. \quad (3.5)$$

Since $\mathcal{P}^\mu|\nu(p)\rangle = p^\mu|\nu(p)\rangle$, we have

$$\langle\nu(p_f)|j_\mu^{(\nu)}(x)|\nu(p_i)\rangle = e^{i(p_f-p_i)\cdot x}\langle\nu(p_f)|j_\mu^{(\nu)}(0)|\nu(p_i)\rangle, \quad (3.6)$$

where we suppressed for simplicity the helicity labels which are not of immediate relevance. Here we see that the unknown quantity which determines the neutrino-photon interaction is $\langle\nu(p_f)|j_\mu^{(\nu)}(0)|\nu(p_i)\rangle$. Considering that the incoming and outgoing neutrinos are free particles which are described by free Dirac fields with the Fourier expansion in Eq. (A55), we have

$$\langle\nu(p_f)|j_\mu^{(\nu)}(0)|\nu(p_i)\rangle = \bar{u}(p_f)\Lambda_\mu(p_f,p_i)u(p_i). \quad (3.7)$$

The electromagnetic properties of neutrinos are embodied by the vertex function $\Lambda_\mu(p_f,p_i)$, which is a matrix in spinor space and can be decomposed in terms of linearly independent products of Dirac γ matrices and the available kinematical four-vectors p_i and p_f. As shown in Appendix B, the most general decomposition can be written as

$$\begin{aligned}\Lambda_\mu(p_f,p_i) = {} & \mathbb{f}_1(q^2)q_\mu + \mathbb{f}_2(q^2)q_\mu\gamma_5 + \mathbb{f}_3(q^2)\gamma_\mu \\ & + \mathbb{f}_4(q^2)\gamma_\mu\gamma_5 + \mathbb{f}_5(q^2)\sigma_{\mu\nu}q^\nu \\ & + \mathbb{f}_6(q^2)\epsilon_{\mu\nu\rho\gamma}q^\nu\sigma^{\rho\gamma}, \end{aligned} \quad (3.8)$$

where $\mathbb{f}_k(q^2)$ are six Lorentz-invariant form factors ($k = 1, \ldots, 6$) and q is the four-momentum of the photon, which is given by

$$q = p_i - p_f, \tag{3.9}$$

from energy-momentum conservation. Notice that the form factors depend only on q^2, which is the only available Lorentz-invariant kinematical quantity, since $(p_i + p_f)^2 = 4m^2 - q^2$. Therefore, $\Lambda_\mu(p_f, p_i)$ depends only on q and from now on we denote it as $\Lambda_\mu(q)$.

Since the Hamiltonian and the electromagnetic field are Hermitian ($\mathcal{H}_{\rm em}^{(\nu)\dagger} = \mathcal{H}_{\rm em}^{(\nu)}$ and $A^{\mu\dagger} = A^\mu$), the effective current must be Hermitian, $j_\mu^{(\nu)\dagger} = j_\mu^{(\nu)}$. Hence, we have

$$\langle \nu(p_f)| j_\mu^{(\nu)}(0) |\nu(p_i)\rangle = \langle \nu(p_i)| j_\mu^{(\nu)}(0) |\nu(p_f)\rangle^*, \tag{3.10}$$

which leads to

$$\Lambda_\mu(q) = \gamma^0 \Lambda_\mu^\dagger(-q)\gamma^0. \tag{3.11}$$

Using the properties of the Dirac matrices (see Appendix A), one can find that this constraint implies that

$$\mathbb{f}_2, \mathbb{f}_3, \mathbb{f}_4 \text{ are real}, \tag{3.12}$$

and

$$\mathbb{f}_1, \mathbb{f}_5, \mathbb{f}_6 \text{ are imaginary}. \tag{3.13}$$

The number of independent form factors can be reduced by imposing current conservation $\partial^\mu j_\mu^{(\nu)}(x) = 0$, which is required by gauge invariance [i.e., invariance of $\mathcal{H}_{\rm em}^{(\nu)}(x)$ under the transformation $A^\mu(x) \to A^\mu(x) + \partial^\mu \varphi(x)$ for any $\varphi(x)$, which leaves invariant the electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$]. Using Eq. (3.4), current conservation implies that

$$\langle \nu(p_f)| [\mathcal{P}^\mu, j_\mu^{(\nu)}(0)] |\nu(p_i)\rangle = 0. \tag{3.14}$$

Hence, in momentum space we have the constraint

$$q^\mu \bar{u}(p_f)\Lambda_\mu(q)u(p_i) = 0, \tag{3.15}$$

which implies that

$$\mathbb{f}_1(q^2)q^2 + \mathbb{f}_2(q^2)q^2\gamma_5 + 2m\mathbb{f}_4(q^2)\gamma_5 = 0. \tag{3.16}$$

Since γ_5 and the unity matrix are linearly independent, we obtain the constraints

$$\mathbb{f}_1(q^2) = 0, \qquad \mathbb{f}_4(q^2) = -\mathbb{f}_2(q^2)q^2/2m. \tag{3.17}$$

Therefore, in the most general case consistent with Lorentz and electromagnetic gauge invariance, the vertex function $\Lambda_\mu(q)$ is defined in terms of four form factors (Kayser, 1982, 1984; Nieves, 1982),

$$\Lambda_\mu(q) = \mathbb{f}_Q(q^2)\gamma_\mu - \mathbb{f}_M(q^2) i\sigma_{\mu\nu}q^\nu + \mathbb{f}_E(q^2)\sigma_{\mu\nu}q^\nu\gamma_5 + \mathbb{f}_A(q^2)(q^2\gamma_\mu - q_\mu \not{q})\gamma_5, \tag{3.18}$$

where $\mathbb{f}_Q = \mathbb{f}_3$, $\mathbb{f}_M = i\mathbb{f}_5$, $\mathbb{f}_E = -2i\mathbb{f}_6$, and $\mathbb{f}_A = -\mathbb{f}_2/2m$ are the real charge, dipole magnetic and electric, and anapole neutrino form factors. The term involving the electric form factor corresponds to the last term in Eq. (3.8), in which we took into account the identity in Eq. (A26). In the term involving the anapole form factor we used the identity $\bar{u}(p_f)\not{q}\gamma^5 u(p_i) = 2m\bar{u}(p_f)\gamma^5 u(p_i)$, which is easily obtained from Eqs. (A17) and (A42).

The physical meaning of the dipole magnetic and electric neutrino form factors is discussed in Sec. IV and that of the charge and anapole in Sec. VII. Here we remark only that for the coupling with a real photon ($q^2 = 0$)

$$\mathbb{f}_Q(0) = \mathbb{q}, \qquad \mathbb{f}_M(0) = \mu, \qquad \mathbb{f}_E(0) = \epsilon, \qquad \mathbb{f}_A(0) = \mathbb{a}, \tag{3.19}$$

where $\mathbb{q}$, μ, ϵ, and $\mathbb{a}$ are, respectively, the neutrino charge, magnetic moment, electric moment, and anapole moment. Although previously we stated that $\mathbb{q} = 0$, here we did not enforce this equality because in some theories beyond the standard model neutrinos can be millicharged particles, as explained in Sec. VII.A.

Now it is interesting to study the properties of $\mathcal{H}_{\rm em}^{(\nu)}(x)$ under a CP transformation, in order to find which of the terms in Eq. (3.18) violate CP. The reason is that, whereas it is well known that weak interactions violate maximally C and P, the violation of CP is a more exotic phenomenon, which has been observed so far only in the hadron sector (Bilenky, 2008).

Using the transformation (A66) of a fermion field under an active CP transformation one can find that for the standard model electric current $j_\mu(x)$ in Eq. (3.1) we have

$$j_\mu(x) \xrightarrow{CP} \mathsf{U}_{CP} j_\mu(x) \mathsf{U}_{CP}^\dagger = -j^\mu(x_P). \tag{3.20}$$

Hence, the standard model electromagnetic interaction Hamiltonian $\mathcal{H}_{\rm em}^{(\nu)}(x)$ is left invariant by[4]

$$A_\mu(x) \xrightarrow{CP} -A^\mu(x_P). \tag{3.21}$$

CP is conserved in neutrino electromagnetic interactions (in the one-photon approximation) if $j_\mu^{(\nu)}(x)$ transforms as $j_\mu(x)$:

$$CP \Leftrightarrow \mathsf{U}_{CP} j_\mu^{(\nu)}(x) \mathsf{U}_{CP}^\dagger = -j_{(\nu)}^\mu(x_P). \tag{3.22}$$

For the matrix element (3.7) we obtain

$$CP \Leftrightarrow \Lambda_\mu(q) \xrightarrow{CP} -\Lambda^\mu(q). \tag{3.23}$$

Using the formulas in Appendix A, one can find that under a CP transformation we have[5]

[4]The transformation $x \to x_P$ is irrelevant since all amplitudes are obtained by integrating over d^4x, as in Eq. (5.2).

[5]The operators in $j_\mu^{(\nu)}(x)$ are implicitly assumed to be normally ordered (Giunti and Kim, 2007).

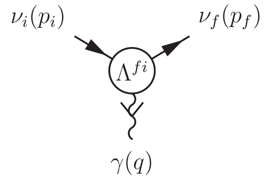

FIG. 5. Effective one-photon coupling of neutrinos with the electromagnetic field, taking into account possible transitions between two different initial and final massive neutrinos ν_i and ν_f.

$$\Lambda_\mu(q) \xrightarrow{CP} \gamma^0 \mathcal{C} \Lambda_\mu^T(q_P) \mathcal{C}^\dagger \gamma^0, \tag{3.24}$$

with $q_P^\mu = q_\mu$. Using the form-factor expansion in Eq. (3.18), we obtain

$$\Lambda_\mu(q) \xrightarrow{CP} -[\mathbb{f}_Q(q^2)\gamma^\mu - \mathbb{f}_M(q^2) i\sigma^{\mu\nu} q_\nu - \mathbb{f}_E(q^2)\sigma^{\mu\nu} q_\nu \gamma_5 + \mathbb{f}_A(q^2)(q^2\gamma^\mu - q^\mu \not{q})\gamma_5]. \tag{3.25}$$

Therefore, only the electric dipole form factor violates CP:

$$CP \Leftrightarrow \mathbb{f}_E(q^2) = 0. \tag{3.26}$$

So far, in this section we considered only one massive neutrino field $\nu(x)$, but from the discussion of neutrino mixing in Sec. II we know that there are at least three massive neutrino fields in nature. Therefore, we must generalize the discussion to the case of N massive neutrino fields $\nu_k(x)$ with respective masses m_k $(k = 1, \ldots, N)$. The effective electromagnetic interaction Hamiltonian in Eq. (3.2) is generalized to

$$\mathcal{H}_{\text{em}}^{(\nu)}(x) = j_\mu^{(\nu)}(x) A^\mu(x) = \sum_{k,j=1}^{N} \overline{\nu_k}(x) \Lambda_\mu^{kj} \nu_j(x) A^\mu(x), \tag{3.27}$$

where we take into account possible transitions between different massive neutrinos. The physical effect of $\mathcal{H}_{\text{em}}^{(\nu)}$ is described by the effective electromagnetic vertex in Fig. 5, with the neutrino matrix element

$$\langle \nu_f(p_f) | j_\mu^{(\nu)}(0) | \nu_i(p_i) \rangle = \overline{u_f}(p_f) \Lambda_\mu^{fi}(p_f, p_i) u_i(p_i). \tag{3.28}$$

As in the case of one massive neutrino field (see Appendix B), $\Lambda_\mu^{fi}(p_f, p_i)$ depends only on the four-momentum q transferred to the photon and can be expressed in terms of six Lorentz-invariant form factors:

$$\Lambda_\mu^{fi}(q) = \mathbb{f}_1^{fi}(q^2) q_\mu + \mathbb{f}_2^{fi}(q^2) q_\mu \gamma_5 + \mathbb{f}_3^{fi}(q^2)\gamma_\mu + \mathbb{f}_4^{fi}(q^2)\gamma_\mu\gamma_5 + \mathbb{f}_5^{fi}(q^2)\sigma_{\mu\nu} q^\nu + \mathbb{f}_6^{fi}(q^2)\epsilon_{\mu\nu\rho\gamma} q^\nu \sigma^{\rho\gamma}. \tag{3.29}$$

The Hermitian nature of $j_\mu^{(\nu)}$ implies that $\langle \nu_f(p_f) | j_\mu^{(\nu)}(0) | \nu_i(p_i) \rangle = \langle \nu_i(p_i) | j_\mu^{(\nu)}(0) | \nu_f(p_f) \rangle^*$, leading to the constraint

$$\Lambda_\mu^{fi}(q) = \gamma^0 [\Lambda_\mu^{if}(-q)]^\dagger \gamma^0. \tag{3.30}$$

Considering the $N \times N$ form-factor matrices $\mathbb{f}_k$ in the space of massive neutrinos with components $\mathbb{f}_k^{fi}$ for $k = 1, \ldots, 6$, we find that

$$\mathbb{f}_2, \mathbb{f}_3, \mathbb{f}_4 \text{ are Hermitian}, \tag{3.31}$$

and

$$\mathbb{f}_1, \mathbb{f}_5, \mathbb{f}_6 \text{ are anti-Hermitian}. \tag{3.32}$$

Following the same method used in Eqs. (3.4)–(3.16), one can find that current conservation implies the constraints

$$\mathbb{f}_1^{fi}(q^2) q^2 + \mathbb{f}_3^{fi}(q^2)(m_f - m_i) = 0, \tag{3.33}$$

$$\mathbb{f}_2^{fi}(q^2) q^2 + \mathbb{f}_4^{fi}(q^2)(m_f + m_i) = 0. \tag{3.34}$$

Therefore, we obtain

$$\Lambda_\mu^{fi}(q) = (\gamma_\mu - q_\mu \not{q}/q^2)[\mathbb{f}_Q^{fi}(q^2) + \mathbb{f}_A^{fi}(q^2) q^2 \gamma_5] - i\sigma_{\mu\nu} q^\nu [\mathbb{f}_M^{fi}(q^2) + i\mathbb{f}_E^{fi}(q^2)\gamma_5], \tag{3.35}$$

where $\mathbb{f}_Q^{fi} = \mathbb{f}_3^{fi}$, $\mathbb{f}_M^{fi} = i\mathbb{f}_5^{fi}$, $\mathbb{f}_E^{fi} = -2i\mathbb{f}_6^{fi}$, and $\mathbb{f}_A^{fi} = -\mathbb{f}_2^{fi}/(m_f + m_i)$, with

$$\mathbb{f}_\Omega^{fi} = (\mathbb{f}_\Omega^{if})^* \qquad (\Omega = Q, M, E, A). \tag{3.36}$$

Note that since $\overline{u_f}(p_f)\not{q} u_i(p_i) = (m_f - m_i)\overline{u_f}(p_f) u_i(p_i)$, if $f = i$ Eq. (3.35) correctly reduces to Eq. (3.18).

The form factors with $f = i$ are called "diagonal," whereas those with $f \neq i$ are called "off-diagonal" or "transition form factors." This terminology follows from

$$\Lambda_\mu(q) = (\gamma_\mu - q_\mu \not{q}/q^2)[\mathbb{f}_Q(q^2) + \mathbb{f}_A(q^2) q^2 \gamma_5] - i\sigma_{\mu\nu} q^\nu [\mathbb{f}_M(q^2) + i\mathbb{f}_E(q^2)\gamma_5], \tag{3.37}$$

in which $\Lambda_\mu(q)$ is a $N \times N$ matrix in the space of massive neutrinos expressed in terms of the four Hermitian $N \times N$ matrices of form factors

$$\mathbb{f}_\Omega = \mathbb{f}_\Omega^\dagger \quad (\Omega = Q, M, E, A). \tag{3.38}$$

For the coupling with a real photon ($q^2 = 0$) we have

$$\mathbb{f}_Q^{fi}(0) = \mathbb{q}_{fi}, \qquad \mathbb{f}_M^{fi}(0) = \mu_{fi}, \qquad \mathbb{f}_E^{fi}(0) = \epsilon_{fi}, \qquad \mathbb{f}_A^{fi}(0) = \mathbb{a}_{fi}, \tag{3.39}$$

where $\mathbb{q}_{fi}$, μ_{fi}, ϵ_{fi}, and $\mathbb{a}_{fi}$ are, respectively, the neutrino charge, magnetic moment, electric moment, and anapole moment of diagonal ($f = i$) and transition ($f \neq i$) types.

Considering now CP invariance, the transformation (3.22) of $j_\mu^{(\nu)}(x)$ implies the constraint in Eq. (3.23) for the $N \times N$ matrix $\Lambda_\mu(q)$ in the space of massive neutrinos. Using the formulas in Appendix A, we obtain

$$\Lambda_\mu^{fi}(q) \xrightarrow{CP} \xi_f^{CP} \xi_i^{CP*} \gamma^0 \mathcal{C} [\Lambda_\mu^{if}(q_P)]^T \mathcal{C}^\dagger \gamma^0, \tag{3.40}$$

where ξ_k^{CP} is the CP phase of ν_k. Since the massive neutrinos take part to standard charged-current weak interactions,[6] their CP phases are equal if CP is conserved (Giunti and Kim, 2007). Hence, we have

$$\Lambda_\mu^{fi}(q) \xrightarrow{CP} \gamma^0 \mathcal{C} [\Lambda_\mu^{if}(q_P)]^T \mathcal{C}^\dagger \gamma^0. \tag{3.41}$$

Using the form-factor expansion in Eq. (3.35), we obtain

$$\begin{aligned}\Lambda_\mu^{fi}(q) \xrightarrow{CP} & -\{(\gamma^\mu - q^\mu \not{q}/q^2)[\mathbb{f}_Q^{if}(q^2) + \mathbb{f}_A^{if}(q^2) q^2 \gamma_5] \\ & - i\sigma^{\mu\nu} q_\nu [\mathbb{f}_M^{if}(q^2) - i\mathbb{f}_E^{if}(q^2)\gamma_5]\}.\end{aligned} \tag{3.42}$$

Imposing the constraint in Eq. (3.23), for the form factors we obtain

$$CP \Leftrightarrow \begin{cases} \mathbb{f}_\Omega^{fi} = \mathbb{f}_\Omega^{if} = (\mathbb{f}_\Omega^{fi})^* & (\Omega = Q, M, A), \\ \mathbb{f}_E^{fi} = -\mathbb{f}_E^{if} = -(\mathbb{f}_E^{fi})^*, \end{cases} \tag{3.43}$$

where, in the last equalities, we took into account the constraints (3.36). Therefore, diagonal electric form factors violate CP, in agreement with the one-generation constraint in Eq. (3.26). For the Hermitian $N \times N$ form-factor matrices, we obtain that if CP is conserved $\mathbb{f}_Q$, $\mathbb{f}_M$, and $\mathbb{f}_A$ are real and symmetric and $\mathbb{f}_E$ is imaginary and antisymmetric:

$$CP \Leftrightarrow \begin{cases} \mathbb{f}_\Omega = \mathbb{f}_\Omega^{\mathrm{T}} = \mathbb{f}_\Omega^* & (\Omega = Q, M, A), \\ \mathbb{f}_E = -\mathbb{f}_E^{\mathrm{T}} = -\mathbb{f}_E^*. \end{cases} \tag{3.44}$$

We now consider antineutrinos. Using for the massive neutrino fields the Fourier expansion in Eq. (A55), the effective antineutrino matrix element for $\bar\nu_i(p_i) \to \bar\nu_f(p_f)$ transitions is given by

$$\langle \bar\nu_f(p_f) | j_\mu^{(\nu)}(0) | \bar\nu_i(p_i) \rangle = -\bar v_i(p_i) \Lambda_\mu^{if}(q) v_f(p_f). \tag{3.45}$$

Using the relation (A47) we can write it as

$$\langle \bar\nu_f(p_f) | j_\mu^{(\nu)}(0) | \bar\nu_i(p_i) \rangle = \overline{u_f}(p_f) \mathcal{C} [\Lambda_\mu^{if}(q)]^T \mathcal{C}^\dagger u_i(p_i), \tag{3.46}$$

where transposition operates in spinor space. Therefore, the effective form-factor matrix in spinor space for antineutrinos is given by

$$\bar\Lambda_\mu^{fi}(q) = \mathcal{C} [\Lambda_\mu^{if}(q)]^T \mathcal{C}^\dagger. \tag{3.47}$$

Using the properties of the charge-conjugation matrix, Eq. (3.35) for $\Lambda_\mu^{if}(q)$, and the Hermiticity in Eq. (3.36), we obtain the antineutrino form factors

$$\bar{\mathbb{f}}_\Omega^{fi} = -\mathbb{f}_\Omega^{if} = -(\mathbb{f}_\Omega^{fi})^* \qquad (\Omega = Q, M, E), \tag{3.48}$$

$$\bar{\mathbb{f}}_A^{fi} = \mathbb{f}_A^{if} = (\mathbb{f}_A^{fi})^*. \tag{3.49}$$

Therefore, in particular, the diagonal magnetic and electric moments of neutrinos and antineutrinos, which are real, have the same size with opposite signs, as the charge, if it exists. On the other hand, the real diagonal neutrino and antineutrino anapole moments are equal.

It is interesting to note that the relations in Eqs. (3.48) and (3.49) between neutrino and antineutrino form factors are a consequence of CPT symmetry, which is a fundamental symmetry of local relativistic quantum field theory (Greenberg, 2006). In order to prove this statement, we first consider the CPT transformation of the standard model electric current $j_\mu(x)$ in Eq. (3.1): using Eq. (A68) we have

$$j_\mu(x) \xrightarrow{CPT} \mathsf{U}_{CPT} j_\mu(x) \mathsf{U}_{CPT}^\dagger = -j_\mu(-x). \tag{3.50}$$

Therefore, the standard model electromagnetic interaction Hamiltonian $\mathcal{H}_{\mathrm{em}}^{(\nu)}(x)$ is left invariant by

$$A_\mu(x) \xrightarrow{CPT} -A_\mu(-x). \tag{3.51}$$

CPT is conserved by the neutrino effective electromagnetic interaction Hamiltonian in Eq. (3.27) if $j_\mu^{(\nu)}(x)$ transforms as $j_\mu(x)$:

$$CPT \Leftrightarrow \mathsf{U}_{CPT} j_\mu^{(\nu)}(x) \mathsf{U}_{CPT}^\dagger = -j_\mu^{(\nu)}(-x). \tag{3.52}$$

In order to find the implications of this relation for the antineutrino matrix element in Eq. (3.45), we need to consider it taking into account the helicities of the initial and final neutrinos, because CPT reverses helicities. Thus, assuming CPT and inserting $\mathsf{U}_{CPT}^\dagger \mathsf{U}_{CPT} = 1$ on both sides of $j_\mu^{(\nu)}(0)$, we obtain

$$\begin{aligned}\bar M_{fi} &= \langle \bar\nu_f(p_f, h_f) | j_\mu^{(\nu)}(0) | \bar\nu_i(p_i, h_i) \rangle \\ &= -\langle \bar\nu_f(p_f, h_f) | \mathsf{U}_{CPT}^\dagger j_\mu^{(\nu)}(0) \mathsf{U}_{CPT} | \bar\nu_i(p_i, h_i) \rangle.\end{aligned} \tag{3.53}$$

Now we take into account that the application of U_{CPT} to a neutrino state transforms it into an antineutrino state. Using the notation and conventions of Giunti and Kim (2007) we have

$$\mathsf{U}_{CPT} |\bar\nu_k(p_k, h_k)\rangle = -\zeta(h) \xi_k^{CPT*} |\nu_k(p_k, -h_k)\rangle, \tag{3.54}$$

where $\zeta(h)$ is a phase coming from

$$\gamma^5 v^{(-h)}(p) = \zeta(h) u^{(h)}(p), \tag{3.55}$$

and $\zeta(-h) = -\zeta(h)$. For the CPT phases ξ_k^{CPT}, we assume that they are all equal, as we have done for the CP phases in

[6]Here we consider massive neutrinos which are mixed with the three active flavor neutrinos ν_e, ν_μ, and ν_τ. This is the case in standard three-neutrino mixing (see Sec. II) and in its extensions with Dirac sterile neutrinos which mix with the active ones. If there are Dirac sterile neutrinos which are not mixed with the active ones and have nonstandard interactions, the CP phases of the corresponding massive neutrinos could be different from that of the standard massive neutrinos. However, since the production and detection of such sterile neutrinos would be very problematic, this case is not interesting in practice.

Eq. (3.40). Then, using Eq. (3.54) and taking into account the antiunitarity of U_{CPT}, Eq. (3.53) becomes

$$\bar{M}_{fi} = -\zeta(h_f)\zeta^*(h_i)\langle\nu_i(p_i,-h_i)|j_\mu^{(\nu)}(0)|\nu_f(p_f,-h_f)\rangle. \quad (3.56)$$

This is the crucial relation between the neutrino and antineutrino matrix elements which follows from CPT invariance. Using for the neutrino matrix element Eq. (3.28) and Eq. (3.55), we obtain

$$\bar{M}_{fi} = \overline{v_i^{(h_i)}}(p_i)\gamma^5\Lambda_\mu^{if}(-q)\gamma^5 v_f^{(h_f)}(p_f). \quad (3.57)$$

Taking into account the form-factor expression of $\Lambda_\mu^{fi}(q)$ in Eq. (3.35), we have $\gamma^5\Lambda_\mu^{if}(-q)\gamma^5 = -\Lambda_\mu^{if}(q)$, which leads to

$$\bar{M}_{fi} = -\overline{v_i^{(h_i)}}(p_i)\Lambda_\mu^{if}(q)v_f^{(h_f)}(p_f). \quad (3.58)$$

This expression for the antineutrino matrix element coincides with Eq. (3.45) and implies Eqs. (3.48) and (3.49) for the form factors.

Thus, we obtained the expression (3.45) for the antineutrino matrix element in a complicated way, assuming only CPT invariance and Eq. (3.28) for the neutrino matrix element. This result is a tautology in the theoretical framework in which we are working, because CPT is a fundamental symmetry of any local relativistic quantum field theory (Greenberg, 2006). However, in some theories beyond the standard model small CPT violations can exist (Tsukerman, 2010), which may be revealed by finding violations of the equalities in Eqs. (3.48) and (3.49).

B. Majorana neutrinos

A Majorana neutrino is a neutral spin $1/2$ particle which coincides with its antiparticle. The 4 degrees of freedom of a Dirac field (two helicities and two particle and antiparticle) are reduced to two (two helicities) by the Majorana constraint in Eq. (2.24). Since a Majorana field has half the degrees of freedom of a Dirac field, it is possible that its electromagnetic properties are reduced. From Eqs. (3.48) and (3.49) between neutrino and antineutrino form factors in the Dirac case, we can infer that in the Majorana case the charge, magnetic, and electric form-factor matrices are antisymmetric and the anapole form-factor matrix is symmetric. In order to confirm this deduction, we calculate the neutrino matrix element corresponding to the effective electromagnetic vertex in Fig. 5, with the effective interaction Hamiltonian in Eq. (3.27), which takes into account possible transitions between two different initial and final massive Majorana neutrinos ν_i and ν_f. Using the Fourier expansion (A59) for the neutrino Majorana fields we obtain

$$\langle\nu_f(p_f)|j_\mu^{(\nu)}(0)|\nu_i(p_i)\rangle = \overline{u_f}(p_f)\Lambda_\mu^{fi}(p_f,p_i)u_i(p_i) - \overline{v_i}(p_i)\Lambda_\mu^{if}(p_f,p_i)v_f(p_f). \quad (3.59)$$

Using Eq. (A47), we can write it as

$$\overline{u_f}(p_f)\{\Lambda_\mu^{fi}(p_f,p_i) + \mathcal{C}[\Lambda_\mu^{if}(p_f,p_i)]^T\mathcal{C}^\dagger\}u_i(p_i), \quad (3.60)$$

where transposition operates in spinor space. Therefore the effective form-factor matrix in spinor space for Majorana neutrinos is given by

$$\Lambda_\mu^{\mathrm{M}fi}(p_f,p_i) = \Lambda_\mu^{fi}(p_f,p_i) + \mathcal{C}[\Lambda_\mu^{if}(p_f,p_i)]^T\mathcal{C}^\dagger. \quad (3.61)$$

As in the case of Dirac neutrinos, $\Lambda_\mu^{fi}(p_f,p_i)$ depends only on $q = p_f - p_i$ and can be expressed in terms of six Lorentz-invariant form factors according to Eq. (3.29). Hence, we can write the $N\times N$ matrix $\Lambda_\mu^{\mathrm{M}}(p_f,p_i)$ in the space of massive Majorana neutrinos as

$$\Lambda_\mu^{\mathrm{M}}(q) = \mathbb{f}_1^{\mathrm{M}}(q^2)q_\mu + \mathbb{f}_2^{\mathrm{M}}(q^2)q_\mu\gamma_5 + \mathbb{f}_3^{\mathrm{M}}(q^2)\gamma_\mu + \mathbb{f}_4^{\mathrm{M}}(q^2)\gamma_\mu\gamma_5 + \mathbb{f}_5^{\mathrm{M}}(q^2)\sigma_{\mu\nu}q^\nu + \mathbb{f}_6^{\mathrm{M}}(q^2)\epsilon_{\mu\nu\rho\gamma}q^\nu\sigma^{\rho\gamma}, \quad (3.62)$$

with

$$\mathbb{f}_k^{\mathrm{M}} = \mathbb{f}_k + \mathbb{f}_k^T \Rightarrow \mathbb{f}_k^{\mathrm{M}} = (\mathbb{f}_k^{\mathrm{M}})^T \quad \text{for } k = 1,2,4, \quad (3.63)$$

$$\mathbb{f}_k^{\mathrm{M}} = \mathbb{f}_k - \mathbb{f}_k^T \Rightarrow \mathbb{f}_k^{\mathrm{M}} = -(\mathbb{f}_k^{\mathrm{M}})^T \quad \text{for } k = 3,5,6. \quad (3.64)$$

Now we can follow the discussion in Sec. III.A for Dirac neutrinos taking into account the additional constraints (3.63) and (3.64) for Majorana neutrinos. The Hermiticity of $j_\mu^{(\nu)}$ and current conservation lead to an expression similar to that in Eq. (3.37):

$$\Lambda_\mu^{\mathrm{M}}(q) = (\gamma_\mu - q_\mu \not{q}/q^2)[\mathbb{f}_Q^{\mathrm{M}}(q^2) + \mathbb{f}_A^{\mathrm{M}}(q^2)q^2\gamma_5] - i\sigma_{\mu\nu}q^\nu[\mathbb{f}_M^{\mathrm{M}}(q^2) + i\mathbb{f}_E^{\mathrm{M}}(q^2)\gamma_5], \quad (3.65)$$

with $\mathbb{f}_Q^{\mathrm{M}} = \mathbb{f}_3^{\mathrm{M}}$, $\mathbb{f}_M^{\mathrm{M}} = i\mathbb{f}_5^{\mathrm{M}}$, $\mathbb{f}_E^{\mathrm{M}} = -2i\mathbb{f}_6^{\mathrm{M}}$, and $\mathbb{f}_A^{\mathrm{M}} = -\mathbb{f}_2^{\mathrm{M}}/(m_f + m_i)$. For the Hermitian $N\times N$ form-factor matrices in the space of massive neutrinos,

$$\mathbb{f}_\Omega^{\mathrm{M}} = (\mathbb{f}_\Omega^{\mathrm{M}})^\dagger \quad (\Omega = Q,M,E,A), \quad (3.66)$$

the Majorana constraints (3.63) and (3.64) imply that

$$\mathbb{f}_\Omega^{\mathrm{M}} = -(\mathbb{f}_\Omega^{\mathrm{M}})^T \quad (\Omega = Q,M,E), \quad (3.67)$$

$$\mathbb{f}_A^{\mathrm{M}} = (\mathbb{f}_A^{\mathrm{M}})^T. \quad (3.68)$$

These relations confirm the expectation discussed earlier that for Majorana neutrinos the charge, magnetic, and electric form-factor matrices are antisymmetric and the anapole form-factor matrix is symmetric.

Since $\mathbb{f}_Q^{\mathrm{M}}$, $\mathbb{f}_M^{\mathrm{M}}$, and $\mathbb{f}_E^{\mathrm{M}}$ are antisymmetric, a Majorana neutrino does not have diagonal charge and dipole magnetic and electric form factors (Case, 1957; Radicati and Touschek, 1957). It can have only a diagonal anapole form factor. On the other hand, Majorana neutrinos can have as many off-diagonal (transition) form factors as Dirac neutrinos.

Since the form-factor matrices are Hermitian as in the Dirac case, $\mathbb{f}_Q^{\mathrm{M}}$, $\mathbb{f}_M^{\mathrm{M}}$, and $\mathbb{f}_E^{\mathrm{M}}$ are imaginary, whereas $\mathbb{f}_A^{\mathrm{M}}$ is real:

$$\mathbb{f}_\Omega^{\mathrm{M}} = -(\mathbb{f}_\Omega^{\mathrm{M}})^* \quad (\Omega = Q,M,E), \quad (3.69)$$

$$\mathbb{f}_A^{\rm M} = (\mathbb{f}_A^{\rm M})^*. \tag{3.70}$$

Taking into account these properties, in the standard case of three-neutrino mixing the charge, magnetic, and electric Majorana form factors can be written as

$$\mathbb{f}_\Omega^{{\rm M}fi}(q^2) = i\sum_{j=1}^{3} \epsilon^{fij}\tilde{\mathbb{f}}_\Omega^{{\rm M}j}(q^2), \tag{3.71}$$

for $\Omega = Q, M, E$, in terms of three vectors of real form factors

$$(\tilde{\mathbb{f}}_\Omega^{\rm M1}, \tilde{\mathbb{f}}_\Omega^{\rm M2}, \tilde{\mathbb{f}}_\Omega^{\rm M3}) = -i(\tilde{\mathbb{f}}_\Omega^{\rm M23}, \tilde{\mathbb{f}}_\Omega^{\rm M31}, \tilde{\mathbb{f}}_\Omega^{\rm M12}). \tag{3.72}$$

Considering now CP invariance, the case of Majorana neutrinos is rather different from that of Dirac neutrinos, because the CP phases of Majorana neutrinos are constrained by the CP invariance of the Majorana mass term. In order to prove this statement, we first notice that since a massive Majorana neutrino field ν_k is constrained by the Majorana relation in Eq. (2.24), only the parity transformation part is effective in a CP transformation. Indeed, from Eqs. (2.24) and (A66) we obtain

$$\mathsf{U}_{CP}\nu_k(x)\mathsf{U}_{CP}^\dagger = \xi_k^{CP}\gamma^0\nu_k(x_P). \tag{3.73}$$

Considering the Majorana mass term in Eq. (2.22), we have

$$\mathsf{U}_{CP}\nu_k^T\mathcal{C}^\dagger\nu_k\mathsf{U}_{CP}^\dagger = -\xi_k^{CP2}\nu_k^T\mathcal{C}^\dagger\nu_k. \tag{3.74}$$

Therefore,

$$CP \Leftrightarrow \xi_k^{CP} = \eta_k i, \tag{3.75}$$

with $\eta_k = \pm 1$. These CP signs can be different for the different massive neutrinos, even if they all take part in the standard charged-current weak interactions through neutrino mixing, because they can be compensated by the Majorana CP phases in the mixing matrix (Giunti and Kim, 2007). Therefore, from Eq. (3.40) we have

$$\Lambda_\mu^{{\rm M}fi}(q) \xrightarrow{CP} \eta_f\eta_i\gamma^0\mathcal{C}[\Lambda_\mu^{{\rm M}if}(q_P)]^T\mathcal{C}^\dagger\gamma^0. \tag{3.76}$$

Imposing a CP constraint analogous to that in Eq. (3.23), we obtain

$$CP \Leftrightarrow \begin{cases} \mathbb{f}_\Omega^{{\rm M}fi} = \eta_f\eta_i\mathbb{f}_\Omega^{{\rm M}fi} = \eta_f\eta_i(\mathbb{f}_\Omega^{{\rm M}fi})^*, \\ \mathbb{f}_E^{{\rm M}fi} = -\eta_f\eta_i\mathbb{f}_E^{{\rm M}fi} = -\eta_f\eta_i(\mathbb{f}_E^{{\rm M}fi})^*, \end{cases} \tag{3.77}$$

with $\Omega = Q, M, A$. Taking into account the constraints (3.69) and (3.70), we have two cases:

$$CP \quad \text{and} \quad \eta_f = \eta_i \Leftrightarrow \mathbb{f}_Q^{{\rm M}fi} = \mathbb{f}_M^{{\rm M}fi} = 0, \tag{3.78}$$

and

$$CP \quad \text{and} \quad \eta_f = -\eta_i \Leftrightarrow \mathbb{f}_E^{{\rm M}fi} = \mathbb{f}_A^{{\rm M}fi} = 0. \tag{3.79}$$

Therefore, if CP is conserved two massive Majorana neutrinos can have either a transition electric form factor or a transition magnetic form factor, but not both, and the transition electric form factor can exist together only with a transition anapole form factor, whereas the transition magnetic form factor can exist together only with a transition charge form factor. In the diagonal case $f = i$, Eq. (3.78) does not give any constraint, because only diagonal anapole form factors are allowed for Majorana neutrinos.

We consider now the CPT symmetry. Following the method used at the end of Sec. III.A for Dirac neutrinos and taking into account the particle-antiparticle equality of Majorana neutrinos, one can show that Eqs. (3.67) and (3.68) are a consequence of CPT symmetry (Kayser, 1982, 1984; Nieves, 1982). Therefore, in particular, the existence of diagonal magnetic or electric moments of Majorana neutrinos would be a signal of CPT violation.

We finally note that the determination of which are the allowed form factors for Majorana neutrinos can also be performed at the field level considering the neutrino electromagnetic current $j_\mu^{(\nu)}$ in Eq. (3.27) and taking into account the chiral decomposition (2.23) of a Majorana field. For example, the magnetic dipole moment $\mu_{kj}^{\rm M}$ is generated by

$$\overline{\nu_k}\sigma^{\mu\nu}\nu_j = \overline{\nu_{kL}}\sigma^{\mu\nu}\nu_{jL}^c + \overline{\nu_{kL}^c}\sigma^{\mu\nu}\nu_{jL}. \tag{3.80}$$

Taking into account the antisymmetry of fermion fields and the properties of the charge-conjugation matrix, one can find that

$$\overline{\nu_k}\sigma^{\mu\nu}\nu_j = -\overline{\nu_j}\sigma^{\mu\nu}\nu_k. \tag{3.81}$$

Therefore, Majorana neutrinos can have only off-diagonal (transition) magnetic dipole moments.

C. Massless Weyl neutrinos

In Sec. II we saw that neutrinos are known to be massive and mixed. However, it is interesting to study the electromagnetic properties of neutrinos in the standard model, where they are described by the two-component massless left-handed Weyl spinors $\nu_{\ell L}(x)$, with $\ell = e, \mu, \tau$. In this case, taking into account that there is no mixing, the neutrino effective electromagnetic current is

$$j_\mu^{(\nu)}(x) = \sum_{\ell,\ell'=e,\mu,\tau} \overline{\nu_{\ell L}}(x)\Lambda_\mu^{\ell\ell'}\nu_{\ell' L}(x). \tag{3.82}$$

Since neutrinos are strictly left handed, the effective electromagnetic vertex in Fig. 5 is given by the matrix element

$$\langle\nu_\ell(p_\ell,-)|j_\mu^{(\nu)}(0)|\nu_{\ell'}(p_{\ell'},-)\rangle = \overline{u_\ell^{(-)}}(p_\ell)\Lambda_\mu^{\ell\ell'}(q)u_{\ell'}^{(-)}(p_{\ell'}), \tag{3.83}$$

with $q = p_{\ell'} - p_\ell$. Since for massless neutrinos Eq. (C6) leads to the equality

$$\gamma^5 u^{(-)}(p) = -u^{(-)}(p), \tag{3.84}$$

we can reduce the general expression of Λ_μ in Eq. (3.37) to (Bernstein, Ruderman, and Feinberg, 1963)

$$\Lambda_\mu(q) = (\gamma_\mu - q_\mu \not{q}/q^2)\mathbb{f}(q^2), \tag{3.85}$$

with

$$\mathbb{f}(q^2) = \mathbb{f}_Q(q^2) - \mathbb{f}_A(q^2)q^2. \tag{3.86}$$

Therefore, massless left-handed Weyl neutrinos have only one type of form factor given by the difference of the charge form factor and the anapole form factor multiplied by q^2.

It is important that massless left-handed Weyl neutrinos cannot have diagonal or off-diagonal electric or magnetic dipole moments, because

$$\overline{\nu_{\ell L}}\sigma^{\mu\nu}\nu_{\ell' L} = \overline{\nu_{\ell L}}\sigma^{\mu\nu}\gamma_5\nu_{\ell' L} = 0. \tag{3.87}$$

The physical reason is that in the case of massless neutrinos the interactions generated by electric and magnetic dipole moments flip helicity, as explained in Appendix C, but the helicity flip of a massless left-handed Weyl neutrino is not possible if the corresponding right-handed state does not exist.

In the standard model neutrinos are electrically neutral and $\mathbb{f}(0) = \mathbb{f}_Q(0) = 0$. However, radiative corrections generate a finite $\mathbb{f}(q^2)$ for $q^2 \neq 0$, as explained in Sec. VII.B, where $d\mathbb{f}_Q(q^2)/dq^2|_{q^2=0}$ is interpreted as the neutrino charge radius. The equivalence between the charge radius and anapole moment interpretations of $\mathbb{f}(q^2)$ is explained in Sec. VII.C.

We also note that the Lorentz symmetry allows one to write an effective current of the type

$$\tilde{j}_\mu^{(\nu)}(x) = \sum_{\ell,\ell'=e,\mu,\tau} \overline{\nu_{\ell L}}(x)\tilde{\Lambda}_\mu^{\ell\ell'}\nu^c_{\ell' L}(x) + \text{H.c.} \tag{3.88}$$

However, this current violates the total lepton number by two units and cannot be generated in the framework of the standard model where the total lepton number is conserved. In theories beyond the standard model in which the total lepton number is violated, neutrinos are Majorana particles and the discussion in Sec. III.B applies. For example, the magnetic moment terms in Eq. (3.88) are of the form in Eq. (3.80).

IV. MAGNETIC AND ELECTRIC DIPOLE MOMENTS

The magnetic and electric dipole moments are theoretically the most well-studied electromagnetic properties of neutrinos. They also attract the interest of experimentalists, although the magnetic moments of Dirac neutrinos in the simplest extension of the standard model with the addition of right-handed neutrinos are proportional to the corresponding neutrino mass and therefore they are many orders of magnitude smaller than the present experimental limits. However, if there is new physics beyond the minimally extended standard model with right-handed neutrinos, the magnetic and electric dipole moments of neutrinos can be much larger and observable by future experiments.

In Sec. IV.A we discuss this prediction for Dirac neutrinos and in Sec. IV.B we present the predictions for the transition magnetic moments of Majorana neutrinos in minimal extensions of the standard model. In Sec. IV.C we discuss the observable effects of electric and magnetic dipole moments in neutrino-electron elastic scattering and in Sec. IV.D we review the derivation of the effective dipole moments in scattering experiments. In Sec. IV.E we present the most relevant experimental limits on the values of the effective dipole moments and in Sec. IV.F we conclude with some considerations on the theoretical possibilities to have large magnetic moments.

A. Theoretical predictions for Dirac neutrinos

The first calculations of the one-loop electromagnetic vertex of an initial fermion $\mathfrak{f}$, a final fermion $\mathfrak{f}'$ (with $\mathfrak{f}' = \mathfrak{f}$ or $\mathfrak{f}' \neq \mathfrak{f}$), and a photon γ in the minimal extension of the standard model with right-handed neutrinos were presented by Lee and Shrock (1977), Marciano and Sanda (1977), and Petcov (1977), with applications to $\mu \to e\gamma$ and $\mu \to ee\bar{e}$ decays and to the radiative neutrino decay process discussed in Sec. V.A, which depends on the transition electric and magnetic moments of the corresponding neutrinos. The electric and magnetic moments of neutrinos have been explicitly calculated by Fujikawa and Shrock (1980), Pal and Wolfenstein (1982), Shrock (1982), and Dvornikov and Studenikin (2004a, 2004b) by evaluating the one-loop radiative diagrams shown in Fig. 6. The result is (Shrock, 1982)

$$\left.\begin{matrix}\mu^{\text{D}}_{kj}\\ i\epsilon^{\text{D}}_{kj}\end{matrix}\right\} = \frac{eG_{\text{F}}}{8\sqrt{2}\pi^2}(m_k \pm m_j)\sum_{\ell=e,\mu,\tau} f(a_\ell)U^*_{\ell k}U_{\ell j}, \tag{4.1}$$

where the superscript "D" indicates Dirac neutrinos,

$$f(a_\ell) = \frac{3}{4}\left[1 + \frac{1}{1-a_\ell} - \frac{2a_\ell}{(1-a_\ell)^2} - \frac{2a_\ell^2 \ln a_\ell}{(1-a_\ell)^3}\right], \tag{4.2}$$

and

$$a_\ell = \frac{m_\ell^2}{m_W^2} \leq \frac{m_\tau^2}{m_W^2} \simeq 5 \times 10^{-4}, \tag{4.3}$$

for $\ell = e, \mu, \tau$. Since all the a_ℓ's are very small, we can approximate

$$f(a_\ell) \simeq \frac{3}{2}\left(1 - \frac{a_\ell}{2}\right), \tag{4.4}$$

and obtain

$$\left.\begin{matrix}\mu^{\text{D}}_{kj}\\ i\epsilon^{\text{D}}_{kj}\end{matrix}\right\} \simeq \frac{3eG_{\text{F}}}{16\sqrt{2}\pi^2}(m_k \pm m_j) \times \left(\delta_{kj} - \frac{1}{2}\sum_{\ell=e,\mu,\tau} U^*_{\ell k}U_{\ell j}\frac{m_\ell^2}{m_W^2}\right). \tag{4.5}$$

It is clear that in this model there are no diagonal electric dipole moments ($\epsilon^{\text{D}}_{kk} = 0$). The diagonal magnetic moments are given by

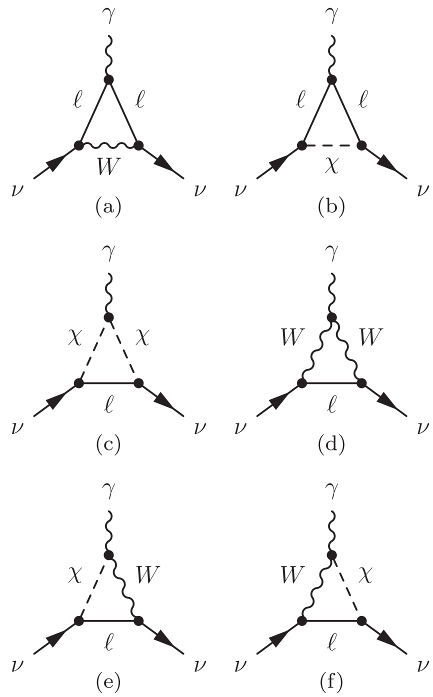

FIG. 6. Feynman diagrams of proper vertices contributing to the neutrino vertex function at one loop in the extended standard model with right-handed neutrinos. χ is the unphysical would-be charged scalar boson. From Dvornikov and Studenikin, 2004a, 2004b.

$$\mu^{\rm D}_{kk} \simeq \frac{3eG_{\rm F}m_k}{8\sqrt{2}\pi^2}. \tag{4.6}$$

Here we neglected the corrections due to the very small a_ℓ's in Eq. (4.3). Note also that higher-order electromagnetic corrections, which have been neglected in Eq. (4.1), can be of the same order of magnitude or larger (for example, the ratio of the contributions of two-loop and one-loop diagrams can be of the order of $\alpha/\pi \simeq 2\times 10^{-3}$).

Equation (4.6) exhibits the following important features. Each diagonal magnetic moment is proportional to the corresponding neutrino mass and vanishes in the massless limit, even if in the extension of the standard model under consideration there are right-handed neutrinos. This case is different from that of massless Weyl neutrinos discussed in Sec. III.C, in which all electric and magnetic, diagonal and off-diagonal dipole moments are forbidden by the absence of right-handed states. In this case we have both spinors $u^{(-)}(p)$ and $u^{(+)}(p)$. As shown in Appendix C, in the massless limit helicity equals chirality, because $\gamma^5 u^{(\pm)}(p) = \pm u^{(\pm)}(p)$. Since $\overline{u^{(\pm)}}(p)\sigma^{\mu\nu}u^{(\pm)}(p) = 0$ and $\overline{u^{(\pm)}}(p)\sigma^{\mu\nu}u^{(\mp)}(p) \neq 0$, the existence of a magnetic moment corresponds to the existence of an helicity and chirality flipping interaction with the electromagnetic field. However, in the minimal extension of the standard model with right-handed neutrinos a magnetic moment is generated by the radiative diagrams in Fig. 6, which cannot flip chirality, because the weak-interaction vertices in the diagrams in Fig. 6 involve only left-handed neutrinos.

At the leading order in the small ratios m_ℓ^2/m_W^2, the diagonal magnetic moments are independent of the neutrino mixing matrix and of the values of the charged-lepton masses. Their numerical values are given by

$$\mu^{\rm D}_{kk} \simeq 3.2\times 10^{-19}\left(\frac{m_k}{\rm eV}\right)\mu_{\rm B}. \tag{4.7}$$

Taking into account the existing constraint of the order of 1 eV on the neutrino masses (see Sec. II.E), these values are several orders of magnitude smaller than the present experimental limits, which are discussed in Sec. IV.E.

We consider now the neutrino transition dipole moments, which are given by Eqs. (4.1) and (4.5) for $k \neq j$. Considering only the leading term $f(a_\ell) \simeq 3/2$ in the expansion (4.4), one gets vanishing transition dipole moments, because of the unitarity relation

$$\sum_{\ell=e,\mu,\tau} U^*_{\ell k}U_{\ell j} = \delta_{kj}. \tag{4.8}$$

Therefore, the first nonvanishing contribution comes from the second term in the expansion (4.4) of $f(a_\ell)$, which contains the additional small factor $a_\ell = m_\ell^2/m_W^2$:

$$\left.\begin{matrix}\mu^{\rm D}_{kj}\\ i\epsilon^{\rm D}_{kj}\end{matrix}\right\} \simeq -\frac{3eG_{\rm F}}{32\sqrt{2}\pi^2}(m_k \pm m_j)\sum_{\ell=e,\mu,\tau} U^*_{\ell k}U_{\ell j}\frac{m_\ell^2}{m_W^2}, \tag{4.9}$$

for $k \neq j$. Thus, the transition magnetic moment $\mu^{\rm D}_{kj}$ is suppressed with respect to the largest of the diagonal magnetic moments of ν_k and ν_j, which are given by Eq. (4.6). This suppression is called the "GIM mechanism," in analogy with the suppression of flavor-changing neutral currents in hadronic processes discovered by Glashow, Iliopoulos, and Maiani (1970). Numerically, the transition dipole moments are given by

$$\left.\begin{matrix}\mu^{\rm D}_{kj}\\ i\epsilon^{\rm D}_{kj}\end{matrix}\right\} \simeq -3.9\times 10^{-23}\mu_{\rm B}\left(\frac{m_k \pm m_j}{\rm eV}\right) \times \sum_{\ell=e,\mu,\tau} U^*_{\ell k}U_{\ell j}\left(\frac{m_\ell}{m_\tau}\right)^2. \tag{4.10}$$

Hence, the suppression of $\mu^{\rm D}_{kj}$ with respect to the numerical values of the largest of the diagonal magnetic moments of ν_k and ν_j, which are given by Eq. (4.7), is at least a factor of the order of 10^{-4}. The transition electric moments are even smaller than the transition magnetic moment because of the mass difference, and they are the only electric moments in the extension of the standard model under consideration.

So far in this section we considered the standard framework of three-neutrino mixing in which the unitarity relation (4.8) applies. However, it is possible that there are additional nonstandard sterile neutrinos, as discussed in Sec. II.F. In this case, the unitarity relation (4.8) becomes

$$\sum_{\ell=e,\mu,\tau} U^*_{\ell k}U_{\ell j} = \delta_{kj} - \sum_{n=1}^{N_s} U^*_{s_n k}U_{s_n j}, \tag{4.11}$$

where N_s is the number of sterile neutrinos, which correspond in the mass basis to N_s nonstandard massive neutrinos. From Eqs. (4.1) and (4.4), the diagonal magnetic moments are given by

$$\mu^{\mathrm{D}}_{kk} \simeq \frac{3eG_{\mathrm{F}}m_k}{8\sqrt{2}\pi^2}\left(1-\sum_{n=1}^{N_s}|U_{s_n k}|^2\right). \tag{4.12}$$

From the inequality (2.79) it follows that the diagonal magnetic moments of the three standard massive neutrinos ($k = 1, 2, 3$) are practically the same as those in Eq. (4.6). On the other hand, for the nonstandard massive neutrinos Eq. (2.80) implies that

$$\mu^{\mathrm{D}}_{kk} \simeq \frac{3eG_{\mathrm{F}}m_k}{8\sqrt{2}\pi^2}(1-|U_{s_{k-3}k}|^2) \quad \text{for } k \geq 4. \tag{4.13}$$

Hence, the diagonal magnetic moments of the nonstandard massive neutrinos are suppressed by the inequality (2.81).

The GIM mechanism does not operate for the transition dipole moments, which are given by

$$\left.\begin{matrix}\mu^{\mathrm{D}}_{kj}\\ i\epsilon^{\mathrm{D}}_{kj}\end{matrix}\right\} \simeq -\frac{3eG_{\mathrm{F}}}{16\sqrt{2}\pi^2}(m_k \pm m_j) \times\left(\sum_{n=1}^{N_s}U^*_{s_n k}U_{s_n j}+\frac{1}{2}\sum_{\ell=e,\mu,\tau}U^*_{\ell k}U_{\ell j}\frac{m_\ell^2}{m_W^2}\right), \tag{4.14}$$

for $k \neq j$. However, the inequality (2.79) quadratically suppresses the additional contribution $\sum_n U^*_{s_n k}U_{s_n j}$ to the transition dipole moments between two standard massive neutrinos ($k, j \leq 3$). From Eqs. (2.78) and (2.80), the transition dipole moments between two nonstandard massive neutrinos ($k, j \geq 4$) are strongly suppressed. On the other hand, the transition dipole moments between a standard massive neutrino and a nonstandard massive neutrino ($k \leq 3$ and $j \geq 4$ or vice versa) are suppressed only linearly by the inequality (2.79).

B. Theoretical predictions for Majorana neutrinos

Majorana neutrinos can have only transition magnetic and electric moments, as discussed in Sec. III.B. The simplest models with Majorana neutrinos can be obtained by extending the standard model with the addition of a $\mathrm{SU}(2)_L$ Higgs triplet (Gelmini and Roncadelli, 1981) or with the addition of right-handed neutrinos and a $\mathrm{SU}(2)_L$ Higgs singlet (Chikashige, Mohapatra, and Peccei, 1980; Mohapatra and Pal, 2004). Neglecting the model-dependent Feynman diagrams which depend on the details of the scalar sector, the Majorana magnetic and electric transition moments are given by (Shrock, 1982)

$$\mu^{\mathrm{M}}_{kj} \simeq -\frac{3ieG_{\mathrm{F}}}{16\sqrt{2}\pi^2}(m_k+m_j)\sum_{\ell=e,\mu,\tau}\mathrm{Im}[U^*_{\ell k}U_{\ell j}]\frac{m_\ell^2}{m_W^2}, \tag{4.15}$$

$$\epsilon^{\mathrm{M}}_{kj} \simeq \frac{3ieG_{\mathrm{F}}}{16\sqrt{2}\pi^2}(m_k-m_j)\sum_{\ell=e,\mu,\tau}\mathrm{Re}[U^*_{\ell k}U_{\ell j}]\frac{m_\ell^2}{m_W^2}. \tag{4.16}$$

Apart from the increase by a factor of 2 of the first coefficient with respect to the Dirac case in Eq. (4.9), it is difficult to compare the expressions of the Dirac and Majorana dipole moments, because the mixing matrices are different in the two cases, due to the possible presence of additional phases in the Majorana case [see Eq. (2.26)]. In any case, it is clear that also the Majorana transition dipole moments are suppressed by the GIM mechanism and they are expected to have the same order of magnitude [see Eq. (4.10)] of the Dirac transition dipole moments. However, the model-dependent contributions of the scalar sector can enhance the Majorana transition dipole moments (Pal and Wolfenstein, 1982; Barr, Freire, and Zee, 1990; Pal, 1991).

If CP is conserved, we must distinguish the two cases in which ν_k and ν_j have the same or opposite CP phases, as explained in Sec. III.B. It can be shown (Giunti and Kim, 2007) that if CP is conserved, the elements of the mixing matrix can be written as

$$U_{\ell k} = \mathcal{O}_{\ell k}e^{i\lambda_k}, \tag{4.17}$$

where $\mathcal{O}$ is a real orthogonal matrix [e.g., U^{D} in Eq. (2.27) with $\delta_{13} = 0, \pi$] and the Majorana CP phases λ_k such that

$$e^{-2i(\lambda_k-\lambda_j)} = \eta_k/\eta_j. \tag{4.18}$$

Here $\eta_k = \pm 1$ is the sign of the CP phase in Eq. (3.75) of the massive Majorana neutrino ν_k. Then we have

$$U^*_{\ell k}U_{\ell j} = \mathcal{O}_{\ell k}\mathcal{O}_{\ell j}e^{-i(\lambda_k-\lambda_j)} = \mathcal{O}_{\ell k}\mathcal{O}_{\ell j}\sqrt{\eta_k/\eta_j}. \tag{4.19}$$

Then if ν_k and ν_j have the same CP phase ($\eta_k = \eta_j$), the products $U^*_{\ell k}U_{\ell j} = \mathcal{O}_{\ell k}\mathcal{O}_{\ell j}$ are real and the dipole moments are given by (Schechter and Valle, 1981; Pal and Wolfenstein, 1982)

$$\mu^{\mathrm{M}}_{kj} = 0 \quad \text{and} \quad \epsilon^{\mathrm{M}}_{kj} = 2\epsilon^{\mathrm{D}}_{kj}, \tag{4.20}$$

with $\epsilon^{\mathrm{D}}_{kj}$ and μ^{D}_{kj} given by Eq. (4.1). On the other hand, if ν_k and ν_j have opposite CP phases ($\eta_k = -\eta_j$), the products $U^*_{\ell k}U_{\ell j} = i\mathcal{O}_{\ell k}\mathcal{O}_{\ell j}$ are imaginary and the dipole moments are given by (Schechter and Valle, 1981; Pal and Wolfenstein, 1982)

$$\mu^{\mathrm{M}}_{kj} = 2\mu^{\mathrm{D}}_{kj} \quad \text{and} \quad \epsilon^{\mathrm{D}}_{kj} = 0. \tag{4.21}$$

The vanishing of μ^{M}_{kj} in the first case and the vanishing of $\epsilon^{\mathrm{D}}_{kj}$ in the second case are consistent with the general results in Eqs. (3.78) and (3.79).

We consider now the case of additional sterile neutrinos discussed in Sec. II.F. Taking into account the unitarity relation (4.11), in the Majorana case one can infer from Shrock (1982) that the transition dipole moments are given by

$$\mu_{kj}^{\rm M} \simeq -\frac{3ieG_{\rm F}}{16\sqrt{2}\pi^2}(m_k + m_j) \times {\rm Im}\left[\sum_{n=1}^{N_s} U_{s_n k}^* U_{s_n j} + \sum_{\ell=e,\mu,\tau} U_{\ell k}^* U_{\ell j}\frac{m_\ell^2}{m_W^2}\right], \quad (4.22)$$

$$\epsilon_{kj}^{\rm M} \simeq \frac{3ieG_{\rm F}}{16\sqrt{2}\pi^2}(m_k - m_j) \times {\rm Re}\left[\sum_{n=1}^{N_s} U_{s_n k}^* U_{s_n j} + \sum_{\ell=e,\mu,\tau} U_{\ell k}^* U_{\ell j}\frac{m_\ell^2}{m_W^2}\right]. \quad (4.23)$$

Here the situation is similar to the case of Dirac neutrinos discussed at the end of Sec. IV.A: the additional contribution $\sum_n U_{s_n k}^* U_{s_n j}$ to the transition dipole moments between two standard massive neutrinos ($k, j \leq 3$) is suppressed quadratically by the inequality (2.79). The transition dipole moments between two nonstandard massive neutrinos ($k, j \geq 4$) are strongly suppressed by Eqs. (2.78) and (2.80). The transition dipole moments between a standard massive neutrino and a nonstandard massive neutrino ($k \leq 3$ and $j \geq 4$ or vice versa) are suppressed only linearly by the inequality (2.79).

C. Neutrino-electron elastic scattering

The most sensitive and widely used method for the experimental investigation of the neutrino magnetic moment is provided by direct laboratory measurements of low-energy elastic scattering of neutrinos and antineutrinos with electrons in reactor, accelerator, and solar experiments.[7] Detailed descriptions of several experiments can be found in Wong and Li (2005) and Beda *et al.* (2007).

Extensive experimental studies of the neutrino magnetic moment, performed during many years, are stimulated by the hope to observe a value much larger than the prediction in Eq. (4.7) of the minimally extended standard model with right-handed neutrinos. It would be a clear indication of new physics beyond the extended standard model. For example, the effective magnetic moment in $\bar{\nu}_e$-e elastic scattering in a class of extra-dimension models can be as large as about $10^{-10}\mu_{\rm B}$ (Mohapatra, Ng, and Yu, 2004). Future higher precision reactor experiments can therefore be used to provide new constraints on large extra dimensions.

The possibility for neutrino-electron elastic scattering due to neutrino magnetic moment was first considered by Carlson and Oppenheimer (1932) and the cross section of this process was calculated by Bethe (1935) [for related short historical notes, see Kyuldjiev (1984)]. Here we recall the paper by Domogatsky and Nadezhin (1970), where the cross section of Bethe (1935) was corrected and the antineutrino-electron cross section was considered in the context of the earlier experiments with reactor antineutrinos of Cowan, Reines, and Harrison (1954) and Cowan and Reines (1957), which were aimed to reveal the effects of the neutrino magnetic moment. Discussions on the derivation of the cross section and the optimal conditions for bounding the neutrino magnetic moment, as well as a collection of cross section formulas for elastic scattering of neutrinos (antineutrinos) on electrons, nucleons, and nuclei can be found in Kyuldjiev (1984) and Vogel and Engel (1989).

We consider the elastic scattering

$$\overset{(-)}{\nu}_\ell + e^- \to \overset{(-)}{\nu}_\ell + e^- \quad (4.24)$$

of a neutrino or antineutrino with flavor $\ell = e, \mu, \tau$ and energy E_ν with an electron at rest in the laboratory frame. There are two observables: the kinetic energy T_e of the recoil electron and the recoil angle χ with respect to the neutrino beam, which are related by

$$\cos\chi = \frac{E_\nu + m_e}{E_\nu}\left[\frac{T_e}{T_e + 2m_e}\right]^{1/2}. \quad (4.25)$$

The electron kinetic energy is constrained from the energy-momentum conservation by

$$T_e \leq \frac{2E_\nu^2}{2E_\nu + m_e}. \quad (4.26)$$

Since, in the ultrarelativistic limit, the neutrino magnetic moment interaction changes the neutrino helicity and the standard model weak interaction conserves the neutrino helicity (see Appendix C), the two contributions add incoherently in the cross section[8] which can be written as (Vogel and Engel, 1989)

$$\frac{d\sigma_{\nu_\ell e^-}}{dT_e} = \left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e}\right)_{\rm SM} + \left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e}\right)_{\rm mag}. \quad (4.27)$$

The weak-interaction cross section is given by

$$\left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e}\right)_{\rm SM} = \frac{G_{\rm F}^2 m_e}{2\pi}\left\{(g_V^{\nu_\ell} + g_A^{\nu_\ell})^2 + (g_V^{\nu_\ell} - g_A^{\nu_\ell})^2\left(1 - \frac{T_e}{E_\nu}\right)^2 + [(g_A^{\nu_\ell})^2 - (g_V^{\nu_\ell})^2]\frac{m_e T_e}{E_\nu^2}\right\}, \quad (4.28)$$

with the standard coupling constants g_V and g_A given by

$$g_V^{\nu_e} = 2\sin^2\theta_W + 1/2, \qquad g_A^{\nu_e} = 1/2, \quad (4.29)$$

$$g_V^{\nu_{\mu,\tau}} = 2\sin^2\theta_W - 1/2, \qquad g_A^{\nu_{\mu,\tau}} = -1/2. \quad (4.30)$$

For antineutrinos one must substitute $g_A \to -g_A$.

The neutrino magnetic moment contribution to the cross section is given by (Vogel and Engel, 1989)

$$\left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e}\right)_{\rm mag} = \frac{\pi\alpha^2}{m_e^2}\left(\frac{1}{T_e} - \frac{1}{E_\nu}\right)\left(\frac{\mu_{\nu_\ell}}{\mu_{\rm B}}\right)^2, \quad (4.31)$$

[7] The effects of a neutrino magnetic moment in other processes which can be observed in laboratory experiments have been discussed by Kim, Mathur, and Okubo (1974), Kim (1978), Dicus *et al.* (1979), and Rosado and Zepeda (1982).

[8] The small interference term due to neutrino masses was derived by Grimus and Stockinger (1998).

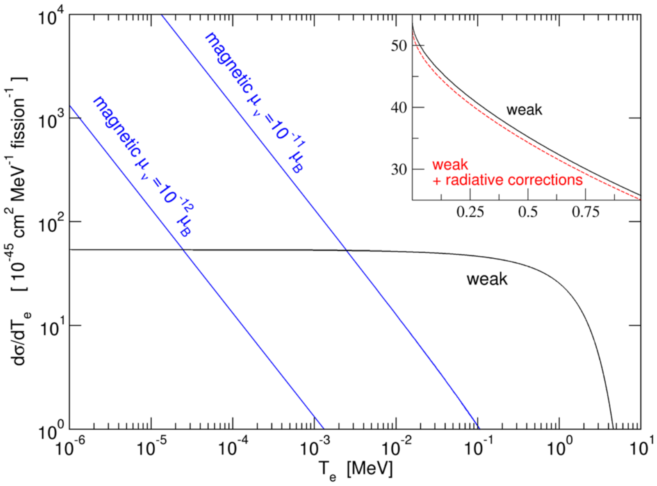

FIG. 7 (color online). Standard model weak and magnetic moment electromagnetic contributions to the differential cross section of neutrino-electron scattering averaged over the antineutrino spectrum of fissioning ^{235}U. The inset plot is the weak correction on the linear scale both with (dashed line) and without (solid line) radiative corrections (Sarantakos, Sirlin, and Marciano, 1983). From Balantekin and Vassh, 2014.

where μ_{ν_ℓ} is the effective magnetic moment discussed in Sec. IV.D. It is traditionally called the "magnetic moment," but it receives equal contributions from both the electric and magnetic dipole moments.

The two terms $(d\sigma_{\nu_\ell e^-}/dT_e)_{\rm SM}$ and $(d\sigma_{\nu_\ell e^-}/dT_e)_{\rm mag}$ exhibit quite different dependencies on the experimentally observable electron kinetic energy T_e, as illustrated in Fig. 7 from Balantekin and Vassh (2014) [see also Vogel and Engel (1989) and Beda *et al.* (2007)]. One can see that small values of the neutrino magnetic moment can be probed by lowering the electron recoil energy threshold. In fact, considering $T_e \ll E_\nu$ in Eq. (4.31) and neglecting the coefficients due to $g_V^{\nu_\ell}$ and $g_A^{\nu_\ell}$ in Eq. (4.28), one can find that $(d\sigma/dT_e)_{\rm mag}$ exceeds $(d\sigma/dT_e)_{\rm SM}$ for

$$T_e \lesssim \frac{\pi^2\alpha^2}{G_F^2 m_e^3}\left(\frac{\mu_\nu}{\mu_B}\right)^2. \tag{4.32}$$

D. Effective magnetic moment

In scattering experiments the neutrino is created at some distance from the detector as a flavor neutrino, which is a superposition of massive neutrinos. Therefore, the magnetic moment that is measured in these experiment is not that of a massive neutrino, but it is an effective magnetic moment which takes into account neutrino mixing and the oscillations during the propagation between source and detector (Grimus and Stockinger, 1998; Beacom and Vogel, 1999).

We consider an initial neutrino with flavor $\ell = e, \mu, \tau$, which is described by the flavor state in Eq. (2.30). The state of the neutrino which is detected through a scattering process at a space-time distance $(\vec{L}, T)$ from the source is given by the superposition of massive neutrinos in the first line of Eq. (2.34). Considering an incoming left-handed neutrino, the amplitude of ν_j production in low-q^2 electromagnetic scattering of a neutrino which has traveled a space-time distance $(\vec{L}, T)$ from a source of ν_ℓ is

$$A_{\ell j}(\vec{L}, T) \propto \sum_k U_{\ell k}^* e^{-iE_kT + i\vec{p}_k\cdot\vec{L}} \times \sum_{h_j} u_j^{(\bar{h}_j)}\,\sigma_{\mu\nu}q^\nu(\mu_{jk} + i\epsilon_{jk}\gamma_5)u_k^{(-)}. \tag{4.33}$$

Since for an incoming ultrarelativistic left-handed neutrino the additional γ^5 in the electric dipole term has only the effect of changing a sign [see Eq. (C6)], the amplitude of $\nu_k \to \nu_j$ transitions is proportional to $\mu_{jk} - i\epsilon_{jk}$, leading to

$$A_{\ell j}(\vec{L}, T) \propto \sum_k U_{\ell k}^* e^{-iE_kT + i\vec{p}_k\cdot\vec{L}}(\mu_{jk} - i\epsilon_{jk}). \tag{4.34}$$

The total cross section of electromagnetic scattering with an electron or a nucleon is given by

$$\sigma_{\nu_\ell e^-}(\vec{L}, T) \propto \sum_j |A_{\ell j}(\vec{L}, T)|^2. \tag{4.35}$$

Taking into account that for ultrarelativistic neutrinos $T = L$, from the approximation in Eq. (2.36) we obtain the fact that the cross section is proportional to the squared effective magnetic moment

$$\mu_{\nu_\ell}^2(L, E_\nu) = \sum_j \left|\sum_k U_{\ell k}^* e^{-i\Delta m_{kj}^2 L/2E_\nu}(\mu_{jk} - i\epsilon_{jk})\right|^2. \tag{4.36}$$

In this expression of the effective μ_{ν_ℓ} one can see that in general both the magnetic and electric dipole moments contribute to the elastic scattering. Note also that, as neutrino oscillations discussed in Sec. II, the effective magnetic moment $\mu_{\nu_\ell}(L, E_\nu)$ depends on the neutrino squared-mass differences, not on the absolute values of neutrino masses.

Considering antineutrinos, the mixing of antineutrinos is obtained from that of neutrinos in Eq. (2.30) with the substitution $U \to U^*$. From Eq. (3.48) it follows that the electric and magnetic moments of antineutrinos are obtained with the substitutions $\mu_{jk} \to -\mu_{jk}^*$ and $\epsilon_{jk} \to -\epsilon_{jk}^*$. Moreover, we must take into account that incoming antineutrinos are right handed. Hence, for antineutrinos we have

$$\bar{A}_{\ell j}(\vec{L}, T) \propto \sum_k U_{\ell k} e^{-iE_kT + i\vec{p}_k\cdot\vec{L}} \times \sum_{h_j} u_j^{(\bar{h}_j)}\,\sigma_{\mu\nu}q^\nu(\mu_{jk}^* + i\epsilon_{jk}^*\gamma_5)u_k^{(+)}. \tag{4.37}$$

For an incoming ultrarelativistic right-handed neutrino the additional γ^5 in the electric dipole term has no effect [see Eq. (C6)] and we obtain

$$\mu^2_{\bar{\nu}_\ell}(L, E_\nu) = \sum_j \left| \sum_k U_{\ell k} e^{-i\Delta m^2_{kj} L/2E_\nu} (\mu^*_{jk} + i\epsilon^*_{jk}) \right|^2$$
$$= \sum_j \left| \sum_k U^*_{\ell k} e^{i\Delta m^2_{kj} L/2E_\nu} (\mu_{jk} - i\epsilon_{jk}) \right|^2. \quad (4.38)$$

Therefore, there can be only a phase difference between the terms contributing to $\mu^2_{\nu_\ell}(L, E_\nu)$ and $\mu^2_{\bar{\nu}_\ell}(L, E_\nu)$, which is induced by neutrino oscillations.

As discussed in Sec. IV.E, the laboratory experiments which are most sensitive to small values of the effective magnetic moment are reactor and accelerator experiments which detect the elastic scattering of flavor neutrinos on electrons at a short distance from the neutrino source. In this case, the value in Eq. (2.64) of the largest squared-mass difference Δm^2_{A} in the standard case of three-neutrino mixing is such that $\Delta m^2_{\mathrm{A}} L/2E_\nu \ll 1$. Therefore, it is possible to approximate all the exponentials in Eqs. (4.36) and (4.38) with unity and obtain the effective short-baseline magnetic moment of flavor neutrinos and antineutrinos

$$\mu^2_{\nu_\ell} \simeq \mu^2_{\bar{\nu}_\ell} \simeq \sum_j \left| \sum_k U^*_{\ell k} (\mu_{jk} - i\epsilon_{jk}) \right|^2$$
$$= [U(\mu^2 + \epsilon^2)U^\dagger + 2\mathrm{Im}(U\mu\epsilon U^\dagger)]_{\ell\ell}, \quad (4.39)$$

where we took into account the fact that $\mu = \mu^\dagger$ and $\epsilon = \epsilon^\dagger$. In this approximation the effective magnetic moment is independent of the neutrino energy and from the source-detector distance.

In the following, when we refer to an effective magnetic moment of a flavor neutrino without indication of a source-detector distance L it is implicitly understood that L is small and the effective magnetic moment is given by Eq. (4.39).

It is interesting to note that flavor neutrinos can have effective magnetic moments even if massive neutrinos are Majorana particles. In this case, since massive Majorana neutrinos do not have diagonal magnetic and electric dipole moments, the effective magnetic moments of flavor neutrinos receive contributions only from the transition dipole moments. For example, in the three-generation case, following Eq. (3.71), we can write μ_{jk} and ϵ_{jk} as

$$\mu_{jk} = i \sum_{m=1}^{3} \epsilon_{jkm} \tilde{\mu}_m, \qquad \epsilon_{jk} = i \sum_{m=1}^{3} \epsilon_{jkm} \tilde{\epsilon}_m, \quad (4.40)$$

with real $\tilde{\mu}_m$ and $\tilde{\epsilon}_m$. Thus, we obtain

$$\mu^2_{\nu_\ell} \simeq \sum_{k=1}^{3} (\tilde{\mu}^2_k + \tilde{\epsilon}^2_k) - \left| \sum_{k=1}^{3} U_{\ell k} (\tilde{\mu}_k - i\tilde{\epsilon}_k) \right|^2. \quad (4.41)$$

Another case in which the effective magnetic moment does not depend on the neutrino energy and on the source-detector distance is when the source-detector distance is much larger than all the oscillation lengths $L_{kj} = 4\pi E_\nu / |\Delta m^2_{kj}|$. In this case the interference terms in Eqs. (4.36) and (4.38) are washed out by the finite energy resolution of the detector, leading to

$$\mu^2_{\nu_\ell}(\infty) \simeq \mu^2_{\bar{\nu}_\ell}(\infty) \simeq \sum_k |U_{\ell k}|^2 \sum_j |\mu_{jk} - i\epsilon_{jk}|^2$$
$$= \sum_k |U_{\ell k}|^2 [(\mu^2)_{kk} + (\epsilon^2)_{kk} + 2\mathrm{Im}(\mu\epsilon)_{kk}]. \quad (4.42)$$

For three generations of Majorana neutrinos, from Eq. (4.40) we obtain

$$\mu^2_{\nu_\ell}(\infty) \simeq \mu^2_{\bar{\nu}_\ell}(\infty) \simeq \sum_{k=1}^{3} (1 - |U_{\ell k}|^2)(\tilde{\mu}^2_k + \tilde{\epsilon}^2_k). \quad (4.43)$$

So far, in this section we considered the effects of neutrino mixing and oscillations on the effective magnetic moment for neutrinos propagating in vacuum. In the case of solar neutrinos, which have been used by the Super-Kamiokande (Liu *et al.*, 2004) and Borexino (Arpesella *et al.*, 2008) experiments to search for neutrino magnetic moment effects, one must take into account the matter effects discussed in Sec. II.D. The state which describes the neutrinos emerging from the Sun is the following generalization of the state in Eq. (2.61) which takes into account three-neutrino mixing and the squared-mass hierarchy in Eq. (2.65):

$$|\nu_{\mathrm{S}}\rangle = \sum_{k=1}^{3} (U^{\mathrm{M}}_{ek})^* |\nu_k\rangle, \quad (4.44)$$

with

$$U^{\mathrm{M}}_{e1} = \cos\vartheta_{13} \cos\vartheta^0_{\mathrm{M}}, \quad (4.45)$$

$$U^{\mathrm{M}}_{e2} = \cos\vartheta_{13} \sin\vartheta^0_{\mathrm{M}}, \quad (4.46)$$

$$U^{\mathrm{M}}_{e3} = U_{e3} = \sin\vartheta_{13} e^{-i\delta_{13}}, \quad (4.47)$$

where ϑ^0_{M} is the effective mixing angle at the point of neutrino production inside the Sun. Following the same reasoning that led to Eq. (4.36), we obtain that the effective magnetic moment measured by an experiment on Earth is

$$\mu^2_{\mathrm{S}}(L, E_\nu) = \sum_j \left| \sum_k (U^{\mathrm{M}}_{ek})^* e^{-i\Delta m^2_{kj} L/2E_\nu} (\mu_{jk} - i\epsilon_{jk}) \right|^2, \quad (4.48)$$

where L is the Sun-Earth distance. Since the Sun-Earth distance is much larger than the oscillation lengths, the interference terms in Eq. (4.48) are washed out by the finite energy resolution of the detector and we obtain the effective magnetic moment

$$\mu^2_{\mathrm{S}}(E_\nu) = \sum_k |U^{\mathrm{M}}_{ek}|^2 \sum_j |\mu_{jk} - i\epsilon_{jk}|^2. \quad (4.49)$$

This expression is similar to that in Eq. (4.42), but takes into account the effective mixing at the point of neutrino production inside the Sun. Note that μ_{S} depends on the neutrino energy through the dependence of ϑ^0_{M} on E_ν [see Eq. (2.54)]. As remarked before Eq. (2.63), in practice we have $\vartheta^0_{\mathrm{M}} \simeq \vartheta_{12}$ for $E_\nu \lesssim 1$ MeV and $\vartheta^0_{\mathrm{M}} \simeq \pi/2$ for $E_\nu \gtrsim 5$ MeV. Therefore,

TABLE IV. Experimental limits for different neutrino effective magnetic moments.

Method	Experiment	Limit	C.L.	Reference
Reactor $\bar{\nu}_e$-e^-	Krasnoyarsk	$\mu_{\nu_e} < 2.4 \times 10^{-10} \mu_B$	90%	Vidyakin *et al.* (1992)
	Rovno	$\mu_{\nu_e} < 1.9 \times 10^{-10} \mu_B$	95%	Derbin *et al.* (1993)
	MUNU	$\mu_{\nu_e} < 9 \times 10^{-11} \mu_B$	90%	Daraktchieva *et al.* (2005)
	TEXONO	$\mu_{\nu_e} < 7.4 \times 10^{-11} \mu_B$	90%	Wong *et al.* (2007)
	GEMMA	$\mu_{\nu_e} < 2.9 \times 10^{-11} \mu_B$	90%	Beda *et al.* (2012)
Accelerator ν_e-e^-	LAMPF	$\mu_{\nu_e} < 1.1 \times 10^{-9} \mu_B$	90%	Allen *et al.* (1993)
Accelerator $(\nu_\mu, \bar{\nu}_\mu)$-e^-	BNL-E734	$\mu_{\nu_\mu} < 8.5 \times 10^{-10} \mu_B$	90%	Ahrens *et al.* (1990)
	LAMPF	$\mu_{\nu_\mu} < 7.4 \times 10^{-10} \mu_B$	90%	Allen *et al.* (1993)
	LSND	$\mu_{\nu_\mu} < 6.8 \times 10^{-10} \mu_B$	90%	Auerbach *et al.* (2001)
Accelerator $(\nu_\tau, \bar{\nu}_\tau)$-$e^-$	DONUT	$\mu_{\nu_\tau} < 3.9 \times 10^{-7} \mu_B$	90%	Schwienhorst *et al.* (2001)
Solar ν_e-e^-	Super-Kamiokande	$\mu_S(E_\nu \gtrsim 5\ \text{MeV}) < 1.1 \times 10^{-10} \mu_B$	90%	Liu *et al.* (2004)
	Borexino	$\mu_S(E_\nu \lesssim 1\ \text{MeV}) < 5.4 \times 10^{-11} \mu_B$	90%	Arpesella *et al.* (2008)

$$\mu_S(E_\nu \lesssim 1\ \text{MeV}) \simeq \mu_{\nu_e}(\infty), \tag{4.50}$$

and

$$\mu_S^2(E_\nu \gtrsim 5\ \text{MeV}) \simeq \cos^2\vartheta_{13} \sum_j |\mu_{j2} - i\epsilon_{j2}|^2 + \sin^2\vartheta_{13} \sum_j |\mu_{j3} - i\epsilon_{j3}|^2. \tag{4.51}$$

E. Experimental limits

The constraints on the neutrino magnetic moment in direct laboratory experiments have been obtained so far from the lack of any observable distortion of the recoil electron energy spectrum. Experiments of this type started in the 1950s at the Savannah River Laboratory where the $\bar{\nu}_e$-e^- elastic scattering process was studied (Cowan, Reines, and Harrison, 1954; Cowan and Reines, 1957; Reines, Gurr, and Sobel, 1976) with somewhat controversial results, as discussed by Vogel and Engel (1989). The most significant experimental limits on the effective magnetic moment μ_{ν_e} which have been obtained in reactor $\bar{\nu}_e$-e^- experiments after about 1990 are listed in Table IV [some details of the different experimental setups are reviewed by Broggini, Giunti, and Studenikin (2012)].[9]

The current best limit on μ_{ν_e} was obtained in 2012 in the GEMMA experiment at the Kalinin Nuclear Power Plant (Russia) with a 1.5 kg highly pure germanium detector exposed at a $\bar{\nu}_e$ flux of $2.7 \times 10^{13}\ \text{cm}^{-2}\,\text{s}^{-1}$ at a distance of 13.9 m from the core of a 3 GW_{th} commercial water-moderated reactor (Beda *et al.*, 2012). The competitive TEXONO experiment is based at the Kuo-Sheng Reactor Neutrino Laboratory (Taiwan), where a 1.06 kg highly pure germanium detector was exposed to the flux of $\bar{\nu}_e$ at a distance of 28 m from the core of a 2.9 GW_{th} commercial reactor (Wong *et al.*, 2007).[10]

Searches for effects of neutrino magnetic moments have also been performed in accelerator experiments. The LAMPF bounds on μ_{ν_e} in Table IV have been obtained with ν_e from μ^+ decay (Allen *et al.*, 1993). The LAMPF and LSND bounds on μ_{ν_μ} in Table IV have been obtained with ν_μ and $\bar{\nu}_\mu$ from π^+ and μ^+ decay (Allen *et al.*, 1993; Auerbach *et al.*, 2001). The DONUT Collaboration (Schwienhorst *et al.*, 2001) investigated ν_τ-e^- and $\bar{\nu}_\tau$-e^- elastic scattering, finding the limit on μ_{ν_τ} in Table IV.

Solar neutrino experiments can also search for a neutrino magnetic moment signal by studying the shape of the electron spectrum (Beacom and Vogel, 1999). The effective magnetic moment μ_S in solar ν_e-e^- scattering experiments is given in Eq. (4.49). Table IV gives the limits obtained in the Super-Kamiokande experiment (Liu *et al.*, 2004) for $\mu_S(E_\nu \gtrsim 5\ \text{MeV})$ and those obtained in the Borexino experiment (Arpesella *et al.*, 2008) for $\mu_S(E_\nu \lesssim 1\ \text{MeV})$ [see Eqs. (4.50) and (4.51)].

Information on neutrino magnetic moments was obtained also with global fits of solar neutrino data (Joshipura and Mohanty, 2002; Grimus *et al.*, 2003; Tortola, 2003). Considering Majorana three-neutrino mixing, Tortola (2003) obtained, at 90% C.L.,

$$\sqrt{|\mu_{12}|^2 + |\mu_{23}|^2 + |\mu_{31}|^2} < 4.0 \times 10^{-10} \mu_B, \tag{4.52}$$

[9]An attempt to improve the experimental bound on μ_{ν_e} in reactor experiments was undertaken by Wong, Li, and Lin (2010), where it was suggested that in $\bar{\nu}_e$ interactions on an atomic target the atomic electron binding ("atomic-ionization effect") can significantly increase the electromagnetic contribution to the differential cross section with respect to the free-electron approximation. However, as explained in Appendix D, the dipole approximation used to derive the atomic-ionization effect is not valid for the electron antineutrino cross section in reactor neutrino magnetic moment experiments. Instead, the free-electron approximation is appropriate for the interpretation of the data of reactor neutrino experiments and the current constraints in Table IV cannot be improved by considering the atomic electron binding (Voloshin, 2010; Kouzakov and Studenikin, 2011a, 2011b; Kouzakov, Studenikin, and Voloshin, 2011a, 2011b, 2012; Chen *et al.*, 2014). The history and present status of the theory of neutrino-atom collisions is reviewed by Kouzakov and Studenikin (2014).

[10]The TEXONO and GEMMA data have also been used by Barranco *et al.* (2012) and Healey, Petrov, and Zhuridov (2013) to constrain neutrino nonstandard interactions.

from the analysis of solar and KamLAND, and

$$\sqrt{|\mu_{12}|^2 + |\mu_{23}|^2 + |\mu_{31}|^2} < 1.8 \times 10^{-10} \mu_B, \tag{4.53}$$

adding the Rovno (Derbin *et al.*, 1993), TEXONO (Li, 2003), and MUNU (Daraktchieva *et al.*, 2003) constraints.

As seen in Sec. IV.C the neutrino magnetic moment contribution to the $\overset{(-)}{\nu}_{\ell}$-$e^-$ elastic scattering process flips the neutrino helicity. If neutrinos are Dirac particles, this process transforms active left-handed neutrinos into sterile right-handed neutrinos, leading to dramatic effects on the explosion of a core-collapse supernova (Dar, 1987; Nussinov and Rephaeli, 1987; Barbieri and Mohapatra, 1988; Goldman *et al.*, 1988; Lattimer and Cooperstein, 1988; Notzold, 1988; Voloshin, 1988b; Ayala, D'Olivo, and Torres, 1999, 2000; Balantekin, Volpe, and Welzel, 2007), where there are also contributions from $\overset{(-)}{\nu}_{\ell}$-$p$ and $\overset{(-)}{\nu}_{\ell}$-$n$ elastic scattering. Requiring that the entire energy in a supernova collapse is not carried away by the escaping sterile right-handed neutrinos created in the supernova core, Ayala, D'Olivo, and Torres (1999, 2000) obtained the following upper limit on a generic neutrino magnetic moment:

$$\mu_\nu \lesssim (0.1\text{–}0.4) \times 10^{-11} \mu_B, \tag{4.54}$$

which is slightly more stringent than the bound $\mu_\nu \lesssim (0.2\text{–}0.8) \times 10^{-11} \mu_B$ obtained by Barbieri and Mohapatra (1988).

F. Theoretical considerations

There is a gap of many orders of magnitude between the present experimental limits on neutrino magnetic moments of the order of $10^{-11} \mu_B$ (discussed in Sec. IV.E) and the prediction smaller than about $10^{-19} \mu_B$ in Eq. (4.7) of the minimal extension of the standard model with right-handed neutrinos. The hope to reach in the near future an experimental sensitivity of this order of magnitude is very weak, taking into account that the experimental sensitivity of reactor $\bar{\nu}_e$-e elastic scattering experiments has improved by only 1 order of magnitude during a period of about 20 years [see Vogel and Engel (1989), where a sensitivity of the order of $10^{-10} \mu_B$ is discussed]. However, the experimental studies of neutrino magnetic moments are stimulated by the hope that new physics beyond the minimally extended standard model with right-handed neutrinos might give much stronger contributions.

One of the examples in which it is possible to avoid the neutrino magnetic moment being proportional to a (small) neutrino mass, that would in principle make a neutrino magnetic moment accessible for experimental observations, is realized in the left-right symmetric model with direct right-handed neutrino interactions (Shrock, 1974, 1982; Kim, 1976; Marciano and Sanda, 1977; Beg, Marciano, and Ruderman, 1978; Duncan *et al.*, 1987; Liu, 1987; Rajpoot, 1990; Czakon, Gluza, and Zralek, 1998; Nemevsek, Senjanovic, and Tello, 2013; Boyarkin and Boyarkina, 2014a). In this model there is a new charged boson W_R which mediates right-handed charged-current weak interactions and mixes with the standard model W_L boson which mediates left-handed charged-current weak interactions. The massive gauge boson states W_1 and W_2 are given by

$$W_1 = W_L \cos\xi - W_R e^{i\varphi} \sin\xi, \tag{4.55}$$

$$W_2 = W_L e^{-i\varphi} \sin\xi + W_R \cos\xi, \tag{4.56}$$

where ξ is a small mixing angle and φ is a possible CP-violating phase. Neglecting the contributions of neutrino masses and the terms suppressed by the small ratio m_{W_1}/m_{W_2}, the magnetic moments of Dirac neutrinos are given by (Shrock, 1982; Fukugita and Yanagida, 2003)

$$\mu_{kj} = \frac{e G_F}{4\sqrt{2}\pi^2} \sin 2\xi \times \sum_{\ell=e,\mu,\tau} m_\ell [e^{i\varphi} U_{\ell k}^* V_{\ell j} + e^{-i\varphi} V_{\ell k}^* U_{\ell j}], \tag{4.57}$$

where U is the standard mixing matrix of left-handed neutrinos and V is the mixing matrix of right-handed neutrinos. Hence, in this case the neutrino magnetic moments depend on the values of the charged-lepton masses. However, one must take into account the coefficient $\sin 2\xi$, which must be very small in order to have small Dirac neutrino masses (Czakon, Gluza, and Zralek, 1998). For example, in the model of Chang and Mohapatra (1987) $\sin\xi \lesssim 10^{-7}$ for $m_{W_2} \gtrsim 2.5$ TeV (Beall, Bander, and Soni, 1982; Ecker and Grimus, 1985; Maiezza *et al.*, 2010), which implies that $\mu_{kj} \lesssim 10^{-16} \mu_B$. However, larger values of the magnetic moments have been obtained by Rajpoot (1990) by adding to the left-right symmetric model a charged scalar singlet, following the idea of Fukugita and Yanagida (1987).

Other interesting possibilities of obtaining neutrino magnetic moments larger than the prediction in Eq. (4.7) of the minimal extension of the standard model with right-handed neutrinos have been considered. For example, the analysis performed by Aboubrahim *et al.* (2014) of the Dirac neutrino magnetic moment in the framework of a minimal supersymmetric standard model[11] extension with a vectorlike lepton generation showed that a neutrino magnetic moment as large as $(10^{-12}\text{–}10^{-14}) \mu_B$ can be obtained. These values lie within reach of improved laboratory experiments in the future.

Gozdz *et al.* (2006) obtained Majorana transition magnetic moments as large as about $10^{-17} \mu_B$, significantly larger than those in Eq. (4.10), in the framework of the minimal supersymmetric standard model with R-parity violating interactions, constrained by grand unification.

It is possible to estimate a generic relation between the size of a neutrino magnetic moment μ_ν and the corresponding neutrino mass m_ν (Voloshin, 1988a; Barr, Freire, and Zee, 1990; Pal, 1992; Davidson, Gorbahn, and Santamaria, 2005; Bell *et al.*, 2006; Bell, 2007). Suppose that a large neutrino magnetic moment is generated by physics beyond a minimal

[11]Other supersymmetric models have been considered by Biswas, Goyal, and Passi (1983), Frank (1999), Fukuyama, Kikuchi, and Okada (2004), and Gozdz and Kaminski (2009).

extension of the standard model at an energy scale characterized by Λ. This contribution to μ_ν is described by the Feynman diagram in Fig. 4(b), with the blob representing the effects of new physics beyond the standard model. The contribution of this diagram to the magnetic moment is

$$\mu_\nu \sim \frac{eG}{\Lambda}, \tag{4.58}$$

where e is the electric charge and G is a combination of coupling constants and loop factors (Bell *et al.*, 2006; Bell, 2007). The diagram of Fig. 4(b) without the photon line gives a new physics contribution to the neutrino mass of the order

$$\delta m_\nu \sim G\Lambda. \tag{4.59}$$

Combining the estimates (4.58) and (4.59), one can obtain

$$\delta m_\nu \sim \frac{\Lambda^2}{2m_e}\frac{\mu_\nu}{\mu_B} = \frac{\mu_\nu}{10^{-18}\mu_B}\left(\frac{\Lambda}{\text{TeV}}\right)^2 \text{eV} \tag{4.60}$$

between the new physics contribution to the neutrino mass and the neutrino magnetic moment.

It follows that, generally, in theoretical models that predict large values for the neutrino magnetic moment, simultaneously large contributions to the neutrino mass arise. Therefore, a particular fine-tuning is needed to get a large value for the neutrino magnetic moment while keeping the neutrino mass within experimental bounds, unless the ratio m_ν/μ_ν is suppressed by a symmetry. Voloshin (1988a) proposed a $SU(2)_\nu$ under which the neutrino and antineutrino fields, ν and ν^c, transform as a doublet. Taking into account that fermion fields anticommute, a Dirac mass term can be written as

$$\bar{\nu}\nu = -\nu^T\bar{\nu}^T = -\nu^T\mathcal{C}^\dagger\mathcal{C}\bar{\nu}^T = -\nu^T\mathcal{C}^\dagger\nu^c, \tag{4.61}$$

and a magnetic moment term can be written as

$$\bar{\nu}\sigma_{\alpha\beta}\nu = -\nu^T\sigma_{\alpha\beta}^T\bar{\nu}^T = \nu^T\mathcal{C}^\dagger\sigma_{\alpha\beta}\nu^c. \tag{4.62}$$

One can see that the mass term is invariant under the change $\nu \leftrightarrows \nu^c$, whereas the magnetic moment term changes sign. Therefore, the magnetic moment term is a singlet under the $SU(2)_\nu$ symmetry, whereas the mass term transforms as a triplet and is forbidden.[12] If, as it happens in a realistic model, the $SU(2)_\nu$ symmetry is broken and if this breaking is small, the ratio m_ν/μ_ν is also small, giving a natural way to obtain a magnetic moment of the order of $\sim 10^{-11}\mu_B$ without contradictions with the neutrino mass experimental constraints. Several possibilities based on the general idea of Voloshin (1988a) were considered by Barbieri and Mohapatra (1989), Ecker, Grimus, and Neufeld (1989), Babu and Mohapatra (1990a), Georgi and Randall (1990), Leurer and Marcus (1990), and Chang *et al.* (1991).

Another idea of neutrino mass suppression without suppression of the neutrino magnetic moment was discussed by Barr, Freire, and Zee (1990) within the Zee model (Zee, 1980), which is based on the standard model gauge group $SU(2)_L \times U(1)_Y$ and contains at least three Higgs doublets and a charged field which is a singlet of $SU(2)_L$. For this kind of model there is a suppression of the neutrino mass diagram, while the magnetic moment diagram is not suppressed.

Bell *et al.* (2005, 2006), Davidson, Gorbahn, and Santamaria (2005), and Bell (2007) derived "natural" upper bounds for the magnetic moments of Dirac and Majorana neutrinos generated by new physics above the electroweak scale. They considered an effective low-energy theory in which the effects of the new physics above the electroweak scale are described by high-dimension nonrenormalizable operators whose coefficients are not fine-tuned. The low-energy effective Lagrangian must respect the standard model $SU(2)_L \times U(1)_Y$ symmetry and is constructed with standard model fields plus right-handed neutrino fields ν_R (with implicit flavor indices), in order to have Dirac neutrino masses. This low-energy effective Lagrangian can be written as

$$\mathcal{L}_{\text{eff}} = \sum_{n,j}\frac{\mathcal{C}_j^n(\mu)}{\Lambda^{n-4}}\mathcal{O}_j^{(n)}(\mu) + \text{H.c.}, \tag{4.63}$$

where μ is the renormalization scale, $n \geq 4$ denotes the operator dimension, and j runs over independent operators of a given dimension. For $n = 4$, a Dirac neutrino mass arises from the operator $\mathcal{O}_1^{(4)} = \bar{L}\,\tilde{\Phi}\,\nu_R$, where $\tilde{\Phi} = i\sigma_2\Phi^*$. For $n = 6$ there are two operators which generate, after electroweak symmetry breaking, the magnetic moment operator $\bar{\nu}\sigma_{\alpha\beta}\nu F^{\alpha\beta}$. These operators can generate a contribution to the neutrino mass operator $\mathcal{O}_1^{(4)}$ through loop diagrams. Using dimensional analysis, Bell *et al.* (2005) estimated that the corresponding contribution $\delta m_\nu^{(4)}$ to the Dirac neutrino mass is given by

$$\delta m_\nu^{(4)} \sim \frac{\alpha}{16\pi}\frac{\Lambda^2}{m_e}\frac{\mu_\nu^D}{\mu_B}. \tag{4.64}$$

Apart from the different coefficient, the dependence on Λ and μ_B is the same as in Eq. (4.60). The Λ^2 dependence is due to the quadratic divergence in the renormalization of the dimension-four neutrino mass operator. Imposing that δm_ν is smaller than the neutrino mass m_ν, we obtain

$$\mu_\nu^D \lesssim 3 \times 10^{-15}\mu_B\left(\frac{m_\nu}{\text{eV}}\right)\left(\frac{\text{TeV}}{\Lambda}\right)^2. \tag{4.65}$$

For $\Lambda \sim 1$ TeV and $m_\nu \lesssim 1$ eV, one obtains $\mu_\nu^D \lesssim 3 \times 10^{-15}\mu_B$, which is some orders of magnitude stronger than the experimental constraints in Table IV.

Bell *et al.* (2005) noted that if the scale Λ is close to the electroweak scale, an important contribution to the neutrino mass can arise also from an $n = 6$ operator. In order to obtain a natural upper bound on μ_ν^D they assumed that at the scale Λ the coefficient of the $n = 6$ mass operator is zero, so that the

[12] Denoting the doublet as $\psi^T = (\nu \quad \nu^c)^T$ and the Pauli matrices acting in the $SU(2)_\nu$ space as $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$, we have $\bar{\nu}\nu = -(1/2)\psi^T\mathcal{C}^\dagger\tau^1\psi$ and $\bar{\nu}\sigma_{\alpha\beta}\nu = (1/2)\psi^T\mathcal{C}^\dagger\sigma_{\alpha\beta}i\tau^2\psi$. One can verify that the magnetic moment is invariant under a $SU(2)_\nu$ transformation $\psi \to \psi' = e^{i\vec{\lambda}\cdot\vec{\tau}}\psi$, whereas the mass term is not invariant.

contribution to the neutrino mass is generated entirely by radiative corrections involving insertions of the $n = 6$ magnetic moment operators. Solving the renormalization group equations from the scale Λ to the electroweak scale, they found the following relation between the contribution $\delta m_\nu^{(6)}$ neutrino mass and the neutrino magnetic moment:

$$\frac{\mu_\nu^{\rm D}}{\mu_{\rm B}} = \frac{16\sqrt{2}G_{\rm F} m_e \sin^4\theta_W}{9\alpha^2 |f| \ln(\Lambda/v)} \delta m_\nu^{(6)}, \tag{4.66}$$

where α is the fine structure constant, v is the vacuum expectation value of the Higgs doublet,

$$f = 1 - r - \tfrac{2}{3}\tan^2\theta_W - \tfrac{1}{3}(1+r)\tan^4\theta_W, \tag{4.67}$$

and r is a ratio of effective operator coefficients defined at the scale Λ which is of the order of unity without fine-tuning. If the neutrino magnetic moment is generated by new physics at a scale $\Lambda \sim 1$ TeV and the corresponding contribution to the neutrino mass is $\delta m_\nu^{(6)} \lesssim 1$ eV, then $\mu_\nu \lesssim 8 \times 10^{-15} \mu_{\rm B}$. Also this bound is some orders of magnitude stronger than the experimental constraints in Table IV.

Following a similar method, Bell *et al.* (2006) calculated natural upper bounds for the transition magnetic moments of Majorana neutrinos [see also Davidson, Gorbahn, and Santamaria (2005)]. They found that the most general naturalness upper bounds for the Majorana transition magnetic moments in the flavor basis are given by

$$\mu_{\ell\ell'}^{\rm M} \lesssim 4 \times 10^{-9} \mu_{\rm B} \left(\frac{M_{\ell\ell'}^{\rm M}}{\rm eV}\right)\left(\frac{\rm TeV}{\Lambda}\right)^2 \left|\frac{m_\tau^2}{m_\ell^2 - m_{\ell'}^2}\right|, \tag{4.68}$$

where $M_{\ell\ell'}^{\rm M}$ is the Majorana neutrino mass matrix in the flavor basis. For Majorana neutrinos the flavor and mass bases are related by a transformation similar to that in Eq. (2.21): $(U^T M^{\rm M} U)_{kj} = m_k \delta_{kj}$, where U is the neutrino mixing matrix. For the magnetic moments we have

$$\mu_{kj}^{\rm M} = \sum_{\ell,\ell'} U_{\ell k} \mu_{\ell\ell'}^{\rm M} U_{\ell' j}. \tag{4.69}$$

The limits (4.68) are much weaker than those in the Dirac case, because for a Majorana neutrino the magnetic moment contribution to the mass is Yukawa suppressed.[13] Hence, if a neutrino transition magnetic moment larger than about $10^{-14}\mu_{\rm B}$ is observed in an experiment, it would indicate that it is plausible that neutrinos are Majorana rather than Dirac particles.

We emphasize that the natural upper bounds on neutrino magnetic moments derived by Bell *et al.* (2005, 2006), Davidson, Gorbahn, and Santamaria (2005), and Bell (2007) apply in models with new physics well above the electroweak scale, for which only the first terms of the effective Lagrangian expansion in Eq. (4.63) are not negligible. This is not the case, for example, in the model discussed by Aboubrahim *et al.* (2014), in which there is new physics at the electroweak scale.

An unusual case of a large observable effect of the small magnetic moments in Eq. (4.7) is that of $\bar{\nu}_e$-e^- elastic scattering in large extra-dimension brane-bulk models with three bulk neutrinos discussed by Mohapatra, Ng, and Yu (2004). They showed that the magnetic moment contribution to $\bar{\nu}_e$-e^- elastic scattering due to the tower of Kaluza-Klein right-handed neutrino states, each contribution with a magnetic moment given by Eq. (4.7), can be comparable with that of a single neutrino in four-dimensional space-time with a magnetic moment of the order of $10^{-10}\mu_{\rm B}$ and the different shapes of the spectra can distinguish the two cases. Hence, $\bar{\nu}_e$-e^- elastic scattering experiments searching for the effects of neutrino magnetic moments can probe the existence of large extra dimensions.

V. RADIATIVE DECAY AND RELATED PROCESSES

The magnetic and electric (transition) dipole moments of neutrinos, as well as possible very small electric charges (millicharges), describe direct couplings of neutrinos with photons which induce several observable decay processes. In this section we discuss the decay processes generated by the diagrams in Fig. 8: Fig. 8(a) generates neutrino radiative decay $\nu_i \to \nu_f + \gamma$ and the processes of neutrino Cherenkov radiation and spin light ($SL\nu$) of a neutrino propagating in a medium; Fig. 8(b) generates photon (plasmon) decay to an neutrino-antineutrino pair in a plasma ($\gamma^* \to \nu\bar{\nu}$).

In Secs. V.A and V.B we review neutrino radiative decay in vacuum and in matter, respectively. In Sec. V.C we discuss neutrino Cherenkov radiation. In Sec. V.D we consider the process of plasmon decay into a neutrino-antineutrino pair, which can be important in dense astrophysical environments as the interior of stars. In Sec. V.E we review the spin light process of a neutrino propagating in a medium.

A. Radiative decay

If the masses of neutrinos are nondegenerate, the radiative decay of a heavier neutrino ν_i into a lighter neutrino ν_f (with $m_i > m_f$) with emission of a photon,

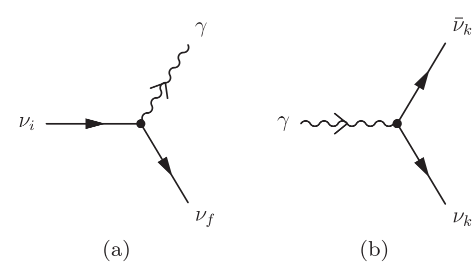

FIG. 8. Feynman diagrams for neutrino radiative decay and (a) Cherenkov radiation and (b) plasmon decay.

[13]Since in the Majorana case the magnetic moment matrix is antisymmetric, it is generated by an antisymmetric magnetic moment operator. On the other hand, the mass matrix and the corresponding mass operator of Majorana neutrinos are diagonal in the mass basis and symmetric in the flavor basis. Therefore, with respect to the Dirac case in which there are no such constraints, additional Yukawa couplings are needed to convert an antisymmetric magnetic moment operator into a symmetric mass operator (Davidson, Gorbahn, and Santamaria, 2005; Bell *et al.*, 2006; Bell, 2007).

$$\nu_i \to \nu_f + \gamma, \tag{5.1}$$

may proceed in vacuum (Shrock, 1974, 1982; Goldman and Stephenson, 1977; Lee and Shrock, 1977; Marciano and Sanda, 1977; Petcov, 1977; Zatsepin and Smirnov, 1978; Pal and Wolfenstein, 1982). Early discussions of the possible role of neutrino radiative decay in different astrophysical and cosmological settings can be found in Dicus, Kolb, and Teplitz (1977), Sato and Kobayashi (1977), De Rujula and Glashow (1980a), Stecker (1980), Kimble, Bowyer, and Jakobsen (1981), and Melott and Sciama (1981).

The neutrino radiative decay process is generated by the interaction in Fig. 5 with a real photon. The decay amplitude is given by

$$\begin{aligned}&\langle \nu_f(p_f, h_f), \gamma(q, \varepsilon)| \int d^4x \mathcal{H}_{\rm em}^{(\nu)}(x) |\nu_i(p_i, h_i)\rangle \\ &\quad = (2\pi)^4 \delta^4(q - p_i + p_f) \overline{u^{(h_f)}}(p_f) \Lambda_\mu^{fi}(q) u^{(h_i)}(p_i) \varepsilon^\mu,\end{aligned} \tag{5.2}$$

where p_i (p_f) and h_i (h_f) are the four-momentum and helicity of the initial (final) neutrino and q and ε are the four-momentum and polarization four-vectors of the photon. The Dirac δ function implements energy-momentum conservation.

Taking into account that for a real photon

$$q^2 = 0 \quad \text{and} \quad \varepsilon^\mu q_\mu = 0, \tag{5.3}$$

from the general expression of $\Lambda_\mu(q)$ for Dirac neutrinos in Eq. (3.35) and from Eq. (3.39), we obtain

$$\Lambda_\mu^{fi}(q) \varepsilon^\mu = \mathbb{q}_{fi} \not{\varepsilon} - i \sigma_{\mu\nu} \varepsilon^\mu q^\nu (\mu_{fi} + i \epsilon_{fi} \gamma_5), \tag{5.4}$$

where $\mathbb{q}_{fi} \neq 0$ only if neutrinos are millicharged particles (see Sec. VII.A). Therefore, the radiative decay of a neutrino ν_i into a lighter neutrino ν_f depends on the corresponding transition charge, magnetic moment, and electric moment. Assuming $\mathbb{q}_{fi} = 0$, the decay rate in the rest frame (rf) of the decaying neutrino ν_i is given by (Raffelt, 1996, 1999a, 1999b)

$$\Gamma^{\rm rf}_{\nu_i \to \nu_f + \gamma} = \frac{1}{8\pi} \left(\frac{m_i^2 - m_f^2}{m_i} \right)^3 (|\mu_{fi}|^2 + |\epsilon_{fi}|^2). \tag{5.5}$$

This expression is valid for both Dirac and Majorana neutrinos, because both can have transition magnetic and electric moments and the corresponding expression (3.62) for $\Lambda_\mu(q)$ in the Majorana case is equivalent to that in Eq. (3.35) for Dirac neutrinos.

The transition magnetic and electric dipole moments of Dirac neutrinos in the minimal extension of the standard model with right-handed neutrinos are given approximately by Eq. (4.9). In this case, the radiative decay rate is given by (Shrock, 1982)

$$\begin{aligned}\Gamma^{\rm rf}_{\nu_i^{\rm D} \to \nu_f^{\rm D} + \gamma} \simeq& \frac{\alpha}{2} \left(\frac{3 G_{\rm F}}{32 \pi^2} \right)^2 \left(\frac{m_i^2 - m_f^2}{m_i} \right)^3 (m_i^2 + m_f^2) \\ &\times \left| \sum_{\ell = e, \mu, \tau} U_{\ell i}^* U_{\ell f} \frac{m_\ell^2}{m_W^2} \right|^2.\end{aligned} \tag{5.6}$$

The radiative decay rate is suppressed by the small phase space due to the smallness of neutrino masses, by the proportionality of the magnetic (electric) transition moment to the sum (difference) of the masses of the two neutrinos involved in the decay, and by a coefficient which is smaller than $(m_\tau / m_W)^4 \simeq 2 \times 10^{-7}$. Note, however, that there are models [see, for instance, Petcov (1982), Aboubrahim, Ibrahim, and Nath (2013), and Aboubrahim *et al.* (2014)] in which the neutrino radiative decay rates (as well as the magnetic moment discussed in Sec. IV) of a Dirac neutrino are much larger than those predicted in the minimally extended standard model.

The expression of the decay rate for Majorana neutrinos in the simplest extensions of the standard model (without taking into account model-dependent contributions of the scalar sector) can be derived from the expressions in Eqs. (4.15) and (4.16) of the Majorana magnetic and electric transition moments (Shrock, 1982):

$$\begin{aligned}\Gamma^{\rm rf}_{\nu_i^{\rm M} \to \nu_f^{\rm M} + \gamma} \simeq& \alpha \left(\frac{3 G_{\rm F}}{32 \pi^2} \right)^2 \left(\frac{m_i^2 - m_f^2}{m_i} \right)^3 \\ &\times \left\{ (m_i + m_f)^2 \left| \sum_{\ell = e, \mu, \tau} U_{\ell i}^* U_{\ell f} \frac{m_\ell^2}{m_W^2} \right|^2 \right. \\ &\left. - 4 m_i m_f \left({\rm Re} \left[\sum_{\ell = e, \mu, \tau} U_{\ell i}^* U_{\ell f} \frac{m_\ell^2}{m_W^2} \right] \right)^2 \right\}.\end{aligned} \tag{5.7}$$

In the case of CP conservation, from Eqs. (4.20) and (4.21), it follows that the decay process is induced purely by the neutrino electric or magnetic transition dipole moment if the CP phases of ν_i and ν_f are, respectively, equal or opposite.

For numerical estimations it is convenient to express the lifetime $\tau_{\nu_i \to \nu_f + \gamma} = \Gamma^{-1}_{\nu_i \to \nu_f + \gamma}$ in the following form:

$$\tau^{\rm rf}_{\nu_i \to \nu_f + \gamma} \simeq 0.19 \left(\frac{m_i^2}{m_i^2 - m_f^2} \right)^3 \left(\frac{\rm eV}{m_i} \right)^3 \left(\frac{\mu_{\rm B}}{\mu_{fi}^{\rm eff}} \right)^2 \ {\rm s}, \tag{5.8}$$

with the neutrino effective magnetic moment

$$\mu_{fi}^{\rm eff} = \sqrt{|\mu_{fi}|^2 + |\epsilon_{fi}|^2}. \tag{5.9}$$

Since $\mu_{fi}^{\rm eff}$ is very small, the lifetime in Eq. (5.8) is very long. Indeed, in the case of Dirac neutrinos in the minimal extension of the standard model with right-handed neutrinos, considering only the dominant τ contribution in Eq. (5.6) and neglecting m_f, we obtain

$$\tau^{\rm rf}_{\nu_i^{\rm D} \to \nu_f^{\rm D} + \gamma} \simeq \frac{6.2 \times 10^{43} \ {\rm s}}{|U_{\tau i}|^2 |U_{\tau f}|^2} \left(\frac{\rm eV}{m_i} \right)^5. \tag{5.10}$$

For $m_i \lesssim 1$ eV, this lifetime is much larger than the age of the Universe, which is about 4.3×10^{17} s (Beringer *et al.*, 2012).

The neutrino radiative decay can be constrained by the absence of decay photons in reactor $\bar{\nu}_e$ and solar ν_e fluxes. The limits on μ_{fi}^{eff} that are obtained from these considerations are much weaker than those obtained from neutrino scattering terrestrial experiments. Stronger constraints on μ_{fi}^{eff} (although still weaker than those obtained in terrestrial experiments) are obtained from the neutrino decay limit set by SN 1987A (Kolb and Turner, 1989; Jaffe and Turner, 1997) and from the measurements of the diffuse cosmic infrared background and those of the cosmic microwave background (Cowsik, 1977; Sato and Kobayashi, 1977; Dicus *et al.*, 1978; De Rujula and Glashow, 1980b; Stecker, 1980; Dolgov and Zeldovich, 1981; Kimble, Bowyer, and Jakobsen, 1981; Ressell and Turner, 1990; Biller *et al.*, 1998; Raffelt, 1998; Masso and Toldra, 1999; Mirizzi, Montanino, and Serpico, 2007). These limits, shown in Fig. 9, can be expressed as (Raffelt, 1996, 1999a, 1999b)

$$\frac{\mu_{fi}^{\text{eff}}}{\mu_{\text{B}}} < \begin{cases} 0.9 \times 10^{-1}\ (\text{eV}/m_\nu)^2 & \text{reactor } (\bar{\nu}_e) \\ 0.5 \times 10^{-5}\ (\text{eV}/m_\nu)^2 & \text{Sun } (\nu_e), \\ 1.5 \times 10^{-8}\ (\text{eV}/m_\nu)^2 & \text{SN 1987A (all flavors)}, \\ 1.0 \times 10^{-11}\ (\text{eV}/m_\nu)^{9/4} & \text{cosmic background (all flavors)}. \end{cases} \tag{5.11}$$

We also recall the studies of the effect of neutrino radiative decay on primordial big-bang nucleosynthesis in Sato and Kobayashi (1977), Dicus *et al.* (1978), Miyama and Sato (1978), Audouze, Lindley, and Silk (1985), and Terasawa, Kawasaki, and Sato (1988); see also the review by Dolgov (2002).

Until now in this section we considered the standard framework of three-neutrino mixing in which there are three massive neutrinos, but it is possible that additional massive neutrinos which are mainly sterile exist, as explained in Sec. II.F. The radiative decay of heavy massive neutrinos is a topic of current interest in view of the recent indication[14] of an astrophysical monochromatic x-ray line at and energy of about 3.5 keV (Boyarsky *et al.*, 2014; Bulbul *et al.*, 2014), which could be due to the radiative decay of a heavy neutrino with a mass of about 7 keV (Abazajian, 2014; Harada, Kamada, and Yoshida, 2014; Vincent *et al.*, 2014) in agreement with the prediction of the νMSM (Boyarsky, Ruchayskiy, and Shaposhnikov, 2009; Kusenko, 2009; Boyarsky, Iakubovskyi, and Ruchayskiy, 2012; Drewes, 2013). In fact, from energy-momentum conservation in the two-body decay (5.1) the energy of the emitted photon in the rest frame of the decaying neutrino ν_i is given by

$$E_\gamma = \frac{m_i^2 - m_f^2}{2m_i} \simeq \frac{m_i}{2} \quad \text{for } m_i \gg m_f. \tag{5.12}$$

We first consider the radiative decay of Dirac neutrinos. Using Eq. (5.5), the transition dipole moments in Eq. (4.14) imply the decay rates

$$\Gamma^{\text{rf}}_{\nu_i^{\text{D}} \to \nu_f^{\text{D}} + \gamma} \simeq \frac{\alpha}{2}\left(\frac{3G_{\text{F}}}{16\pi^2}\right)^2 \left(\frac{m_i^2 - m_f^2}{m_i}\right)^3 (m_i^2 + m_f^2) \times \left| \sum_{n=1}^{N_s} U^*_{s_n i} U_{s_n f} + \frac{1}{2} \sum_{\ell=e,\mu,\tau} U^*_{\ell i} U_{\ell f} \frac{m_\ell^2}{m_W^2} \right|^2. \tag{5.13}$$

The inequality (2.79) suppresses quadratically the sterile contribution to the decays between two standard massive neutrinos ($k, j \leq 3$) and the decays between two nonstandard massive neutrinos are strongly suppressed by Eqs. (2.78) and (2.80). On the other hand, the decay of a nonstandard heavy massive neutrino ν_h with $h \geq 4$ into a lighter standard massive neutrino ν_l with $l \leq 3$ can be significant if $|U_{s_{h-3} l}|$ is not too small. Neglecting the small contributions due to the charged-lepton masses and considering $m_h \gg m_l$, we have

$$\Gamma^{\text{rf}}_{\nu_h^{\text{D}} \to \nu_l^{\text{D}} + \gamma} \simeq \frac{\alpha}{2}\left(\frac{3G_{\text{F}}}{16\pi^2}\right)^2 m_h^5 |U_{s_{h-3} h}|^2 |U_{s_{h-3} l}|^2. \tag{5.14}$$

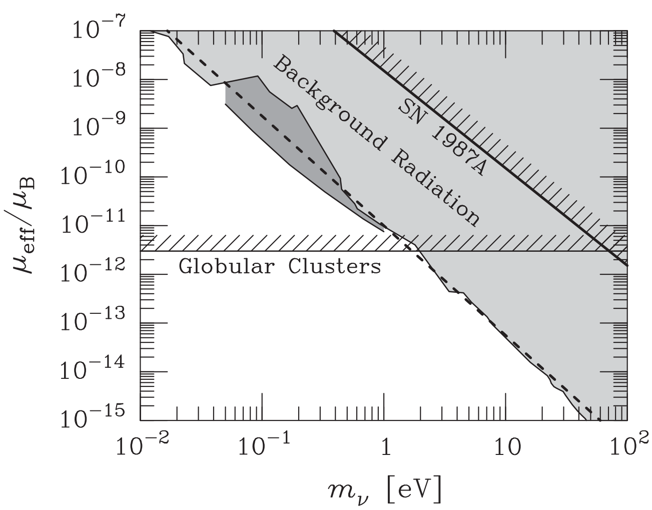

FIG. 9. Astrophysical limits on neutrino transition moments. From Raffelt, 1999b.

[14]See, however, the negative result of the searches in Anderson, Churazov, and Bregman (2014), Carlson, Jeltema, and Profumo (2014), Jeltema and Profumo (2014), Malyshev, Neronov, and Eckert (2014), and Tamura *et al.* (2014).

If the mixing of $\nu_{s_{h-3}}$ with the three light neutrinos is dominated by $|U_{s_{h-3}l}|^2$, we can define an approximate effective mixing angle ϑ_{hl} such that

$$\cos^2\vartheta_{hl} \simeq |U_{s_{h-3}h}|^2, \qquad \sin^2\vartheta_{hl} \simeq |U_{s_{h-3}l}|^2, \tag{5.15}$$

and we can write the decay rate as

$$\Gamma^{\rm rf}_{\nu_h^{\rm D}\to\nu_l^{\rm D}+\gamma} \simeq \frac{\alpha}{2}\left(\frac{3G_{\rm F}}{32\pi^2}\right)^2 m_h^5 \sin^2 2\vartheta_{hl}. \tag{5.16}$$

This approximation is convenient for the analysis of experimental data, because the decay rate depends on only two unknown parameters, the heavy neutrino mass m_h and the effective mixing angle ϑ_{hl}.

We consider now the decay of heavy nonstandard massive neutrinos in the Majorana framework, which applies to the νMSM explanation of the astrophysical 3.5 keV x-ray line mentioned previously (Boyarsky, Ruchayskiy, and Shaposhnikov, 2009; Kusenko, 2009; Boyarsky, Iakubovskyi, and Ruchayskiy, 2012; Drewes, 2013). The decay rates are generalizations of those in Eq. (5.7) taking into account the transition magnetic and electric moments in Eqs. (4.22) and (4.23). For simplicity, we consider only the decay of a heavy neutrino ν_h with $h \geq 4$ into a light neutrino ν_l with $l \leq 3$. Neglecting the small contributions due to the charged-lepton masses and considering $m_h \gg m_l$ we obtain

$$\Gamma^{\rm rf}_{\nu_h^{\rm M}\to\nu_l^{\rm M}+\gamma} \simeq \alpha\left(\frac{3G_{\rm F}}{16\pi^2}\right)^2 m_h^5 |U_{s_{h-3}h}|^2 |U_{s_{h-3}l}|^2. \tag{5.17}$$

This expression is twice that in Eq. (5.14) in the Dirac case. Under the approximation (5.15) we obtain

$$\Gamma^{\rm rf}_{\nu_h^{\rm M}\to\nu_l^{\rm M}+\gamma} \simeq \alpha\left(\frac{3G_{\rm F}}{32\pi^2}\right)^2 m_h^5 \sin^2 2\vartheta_{hl}. \tag{5.18}$$

This expression is typically used in the phenomenological studies of heavy neutrino radiative decay in the νMSM model (Boyarsky, Ruchayskiy, and Shaposhnikov, 2009; Kusenko, 2009; Boyarsky, Iakubovskyi, and Ruchayskiy, 2012; Drewes, 2013).

We finally mention that the radiative decay of heavy neutrinos may be observable also in hadron collider experiments (Boyarkin and Boyarkina, 2014b).

B. Radiative decay in matter

As explained in Sec. II.D, the evolution of neutrinos propagating in matter is affected by the potential generated by the coherent forward elastic scattering with the particles in the medium. It turns out that the coherent interaction with an electron background induces the radiative decay in Eq. (5.1) with a rate that is not suppressed by the GIM mechanism as the decay rate in vacuum in Eq. (5.6) (D'Olivo, Nieves, and Pal, 1990). Following the approach of Giunti *et al.* (1992), the process of radiative decay in an electron background can be represented by the two Feynman diagrams in Fig. 10 which are obtained from the CC potential diagram in Fig. 1(a)

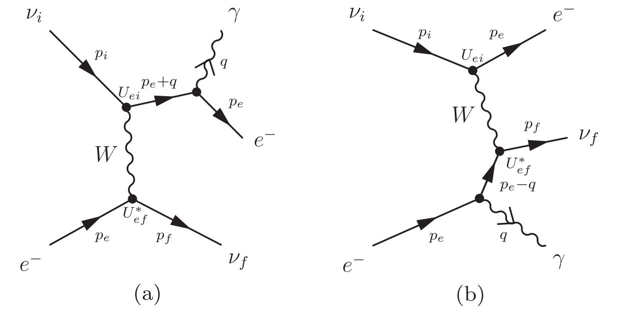

FIG. 10. Feynman diagrams of the coherent contribution of background electrons to the radiative decay $\nu_i(p_i) \to \nu_f(p_f) + \gamma(q)$ in matter.

attaching a final photon line at the initial or final electron line. As in the case of the calculation of the potential (Giunti and Kim, 2007), the coherent contribution of the electron background is obtained by considering equal initial and final four-momenta of the electron. The resulting decay rate in the rest frame of the electron background is

$$\Gamma^{\rm mat}_{\nu_i\to\nu_f+\gamma} = \frac{\alpha G_{\rm F}^2 N_e^2}{2m_e^2}\left(\frac{m_i^2 - m_f^2}{m_i}\right)|U_{ei}|^2|U_{ef}|^2 F(v_i), \tag{5.19}$$

where N_e is the electron number density, $v_i = |\vec{p}_i|/E_i$ is the velocity of the initial neutrino, and

$$F(v_i) = \sqrt{1-v_i^2}\left[\frac{2}{v_i}\ln\left(\frac{1+v_i}{1-v_i}\right) - 3 + \frac{m_f^2}{m_i^2}\right]. \tag{5.20}$$

In the realistic case of ultrarelativistic initial neutrinos, we have

$$F(v_i) \underset{v_i\to 1}{\longrightarrow} 4m_i/E_i. \tag{5.21}$$

Note that the matter-induced radiative decay is independent of the Dirac or Majorana nature of neutrinos, because it is generated by the coherent weak interactions with matter, which are the same for left-handed neutrinos.

Neglecting the final neutrino mass in Eq. (5.19), the numerical value of the lifetime $\tau^{\rm mat}_{\nu_i\to\nu_f+\gamma} = (\Gamma^{\rm mat}_{\nu_i\to\nu_f+\gamma})^{-1}$ for ultrarelativistic initial neutrinos is given by

$$\tau^{\rm mat}_{\nu_i\to\nu_f+\gamma} \simeq \frac{4.0\times 10^{30}\ {\rm s}}{|U_{ei}|^2|U_{ef}|^2}\left(\frac{\rm eV}{m_i}\right)^2\left(\frac{E_i}{\rm MeV}\right)\left(\frac{N_{\rm A}{\rm cm}^{-3}}{N_e}\right)^2. \tag{5.22}$$

In order to compare the radiative lifetime in matter in Eq. (5.22) with the radiative lifetime in vacuum in Eq. (5.10), obtained in the minimal extension of the standard model with right-handed neutrinos, we must take into account the Lorentz boost factor $\gamma_i = E_i/m_i$ from the rest frame of the decaying neutrino to the rest frame of the electron background:

$$\frac{\tau^{\text{mat}}_{\nu_i\to\nu_f+\gamma}}{\gamma_i\tau^{\text{rf}}_{\nu_i^{\text{D}}\to\nu_f^{\text{D}}+\gamma}} \simeq 1.1\times10^{-19}\left(\frac{|U_{\tau i}|^2|U_{\tau f}|^2}{|U_{ei}|^2|U_{ef}|^2}\right)\times\left(\frac{m_i}{\text{eV}}\right)^4\left(\frac{N_{\text{A}}\text{cm}^{-3}}{N_e}\right)^2. \tag{5.23}$$

Therefore, the radiative decay rate in an electron background is many orders of magnitude larger than the radiative decay rate in vacuum in the minimal extension of the standard model with right-handed neutrinos. However, the large value of the lifetime in Eq. (5.22) indicates that it is very difficult, if not impossible, to find a realistic application of this effect.

So far we have considered the radiative decay rate in a background of electrons, assuming that the temperature is not very high. For a temperature $T \gg m_e$ both electrons and positrons are present in the background and the radiative decay rate is given by (D'Olivo, Nieves, and Pal, 1990)

$$\Gamma^{(T\gg m_e)}_{\nu_i\to\nu_f+\gamma} = \frac{\alpha G_{\text{F}}^2 T^4}{72}\left(\frac{m_i^2-m_f^2}{m_i}\right)|U_{ei}|^2|U_{ef}|^2F(v_i). \tag{5.24}$$

Neglecting the final neutrino mass for ultrarelativistic initial neutrinos we have

$$\tau^{(T\gg m_e)}_{\nu_i\to\nu_f+\gamma} \simeq \frac{1.2\times10^{16}\ \text{s}}{|U_{ei}|^2|U_{ef}|^2}\left(\frac{\text{eV}}{m_i}\right)^2\left(\frac{E_i}{\text{MeV}}\right)\left(\frac{\text{MeV}}{T}\right)^4. \tag{5.25}$$

Therefore, in this case the radiative decay in matter is enormously faster than that in vacuum in the minimal extension of the standard model with right-handed neutrinos:

$$\frac{\tau^{(T\gg m_e)}_{\nu_i\to\nu_f+\gamma}}{\gamma_i\tau^{\text{rf}}_{\nu_i^{\text{D}}\to\nu_f^{\text{D}}+\gamma}} \simeq 3.3\times10^{-34}\left(\frac{|U_{\tau i}|^2|U_{\tau f}|^2}{|U_{ei}|^2|U_{ef}|^2}\right)\times\left(\frac{m_i}{\text{eV}}\right)^4\left(\frac{\text{MeV}}{T}\right)^4. \tag{5.26}$$

We finally mention that Nieves and Pal (1997) calculated the radiative decay rate of neutrinos propagating in a thermal background of electrons and photons, taking into account the effect of the stimulated emission of photons in the thermal bath; Grasso and Semikoz (1999) calculated the decay rate of a neutrino induced by the emission or absorption of a photon in a plasma taking into account the effective mass of the photons (plasmons); Skobelev (1995), Zhukovsky, Eminov, and Grigoruk (1996), and Kachelriess and Wunner (1997) calculated the radiative decay rate of neutrinos propagating in magnetic fields; Ternov and Eminov (2003, 2013) calculated the radiative decay rate of neutrinos propagating in a magnetized plasma.

C. Cherenkov radiation

It is well known that a charged particle moving through a medium at a velocity greater than the speed of light in the medium $v > c/n$ (n is the medium refractive index) can emit Cherenkov radiation. In the same way, neutrinos with a magnetic moment (and/or an electric dipole moment) propagating in a medium with a velocity larger than the velocity of light in the medium can emit Cherenkov radiation. This possibility was first discussed by Radomski (1975), who studied a solution of the solar neutrino problem in which the rate of solar neutrino detection is lowered by the loss of energy of the neutrinos due to the emission of Cherenkov radiation in the solar matter. However, the effect was found to be too small to significantly reduce the solar neutrino flux.

The Cerenkov radiation is the helicity flip process

$$\nu_L(p) \to \nu_R(p') + \gamma(k), \tag{5.27}$$

where $\nu_L(p)$ and $\nu_R(p')$ denote the initial and final states of the same neutrino with negative and positive helicities, respectively. The amplitude of the transition due to a neutrino magnetic moment μ is given by

$$M = \frac{\mu}{n}\overline{u^{(+)}}(p')\sigma_{\mu\nu}k^\mu u^{(-)}(p)\varepsilon^\nu(k,\lambda), \tag{5.28}$$

where $p = (E, \vec{p})$ and $p' = (E', \vec{p}\,')$ are the four-momenta of the initial and final neutrinos and $k = (\omega, \vec{k})$ and $\varepsilon^\nu(k,\lambda)$ are the four-momenta and polarization four-vector of the emitted photon (λ denotes the photon helicity), with $|\vec{k}| = n\omega$ and $n > 1$ in matter. The rate of the Cherenkov process is given by (Grimus and Neufeld, 1993; Mohanty and Samal, 1996)

$$\Gamma = \frac{1}{2(2\pi)^2E}\int\frac{d^3p'}{2E'}\frac{d^3k}{2\omega}|M|^2\delta^4(p-p'-k). \tag{5.29}$$

After integration with the use of the δ function, we obtain

$$\Gamma = \frac{1}{16\pi E^2 v}\int n^2 d\omega d(\cos\theta)|M|^2 \times\delta\left(\cos\theta - \frac{2\omega E + (n^2-1)\omega^2}{2n\omega E v}\right), \tag{5.30}$$

where $v = |\vec{p}|/E$ is the initial neutrino velocity and θ is the angle between the emitted photon and the direction of propagation of the initial neutrino. The remaining δ function constrains the photon emission angle to have the value

$$\cos\theta = \frac{1}{nv}\left(1 + (n^2-1)\frac{\omega}{2E}\right). \tag{5.31}$$

After performing the analytic integrals and taking into account Eq. (5.31), we obtain

$$\Gamma = \frac{\mu^2}{4\pi E^2 v}\int_{\omega_{\text{min}}}^{\omega_{\text{max}}}\left\{\left[\frac{(n^2-1)^2}{n^2}E^2 + (n^2-1)m_\nu^2\right]\omega^2 - \frac{(n^2-1)^2}{n^2}E\omega^3 - \frac{(n^2-1)^3}{4n^2}\omega^4\right\}d\omega. \tag{5.32}$$

The range of integration from ω_{min} to ω_{max} corresponds to the range of frequencies of the emitted photon which is allowed by the kinematical condition $|\cos\theta| \le 1$. The determination of this range is nontrivial, because in general the refractive index n depends on ω.

The general expression (5.32) of the rate of the Cherenkov process can be used for analyses of possible phenomenological consequences of the neutrino magnetic moment Cherenkov radiation in different environments. For example, Grimus and Neufeld (1993) estimated that if solar neutrinos have an effective magnetic moment of about $3 \times 10^{-11} \mu_B$, they emit about five Cherenkov photons per day in a terrestrial 1 km^3 water detector.

The Cherenkov mechanism is of interest for astrophysical applications also because it flips the neutrino helicity. If neutrinos are Dirac particles, this helicity flip transforms active left-handed neutrinos into sterile right-handed neutrinos. This mechanism can have important consequences, for instance, for the evolution of a supernova core. Imposing the fact that the luminosity of the sterile neutrinos is less than the total energy 10^{53} ergs s^{-1} emitted by a typical core-collapse supernova, Mohanty and Samal (1996) found an upper bound on the neutrino effective magnetic moment of about $3 \times 10^{-14} \mu_B$.

Neutrinos can emit Cherenkov radiation also when they propagate in vacuum in the presence of a magnetic field. This can occur even if neutrinos are massless with only standard model couplings, because the magnetic field induces an effective neutrino-photon vertex and modifies the photon dispersion relation in such a way that the Cherenkov condition is fulfilled (Ioannisian and Raffelt, 1997). This mechanism was discussed by Galtsov and Nikitina (1972), Skobelev (1976, 1995), Gvozdev, Mikheev, and Vasilevskaya (1992, 1996), Ioannisian and Raffelt (1997), and Kachelriess and Wunner (1997). However, in order to produce appreciable effects this mechanism requires an extremely strong magnetic field. The strongest magnetic fields known in nature are those near pulsars. Even considering a magnetic field as strong as the critical field $B_{cr} = m_e^2/e = 4.41 \times 10^{13}$ G, since its spatial extension near a pulsar is only of some tens of kilometers, the Cherenkov radiation emitted by the neutrinos escaping from the pulsar is too small to be of practical importance for pulsar physics (Ioannisian and Raffelt, 1997).

There is also another possible mechanism of electromagnetic radiation of neutrinos in a medium, also called "Cherenkov radiation" (Sawyer, 1992; D'Olivo, Nieves, and Pal, 1996). This mechanism is based on the expectation that neutrinos moving in a medium acquire an electric charge as a consequence of their weak interaction with the particles of the background (Oraevsky, Semikoz, and Smorodinsky, 1986). Note that this effect exists even for massless neutrinos and no physics beyond the standard model is needed. The magnetic moment Cherenkov radiation estimated by Grimus and Neufeld (1993) in the optical range is much larger than the Cherenkov radiation due to the induced charge. However, the Cherenkov radiation due to the induced neutrino charge becomes important for photons with higher energies and might be of interest for applications in astrophysics.

We finally mention the studies of photon emission of a massive neutrino with a magnetic moment in magnetic fields and in electromagnetic waves (Skobelev, 1976, 1991; Borisov, Zhukovsky, and Ternov, 1988, 1989; Chistyakov and Mikheev, 1999) and that of a neutrino with a magnetic moment which crosses the interface of two media with different refractive indices (Sakuda, 1994; Grimus and Neufeld, 1995; Sakuda and Kurihara, 1995; Ioannisian, Ioannisian, and Kazarian, 2011; D'Olivo and Loza, 2012).

D. Plasmon decay into a neutrino-antineutrino pair

The most interesting process, for the purpose of constraining neutrino electromagnetic properties, is the photon (plasmon) decay into a neutrino-antineutrino pair $\gamma^* \to \nu + \bar{\nu}$ (Bernstein, Ruderman, and Feinberg, 1963; Sutherland *et al.*, 1976). This plasmon process becomes kinematically allowed in media, because a photon with the dispersion relation $\omega_\gamma^2 - \vec{k}_\gamma^2 > 0$ roughly behaves as a particle with an effective mass. For example, photons in a nonrelativistic plasma have the dispersion relation $\omega_\gamma^2 - \vec{k}_\gamma^2 = \omega_P^2$, where $\omega_P = 4\pi\alpha N_e/m_e$ is the plasma frequency (Raffelt, 1996). For $\omega_P > 2m_\nu$ the plasmon decay $\gamma^* \to \nu + \bar{\nu}$ is kinematically possible.

The plasmon-decay rate is (Sutherland *et al.*, 1976; Raffelt, 1996)

$$\Gamma_{\gamma^* \to \nu\bar{\nu}} = \frac{\mu_\nu^2}{24\pi} Z \frac{(\omega_\gamma^2 - k_\gamma^2)^2}{\omega_\gamma}, \tag{5.33}$$

where μ_ν is the effective magnetic moment

$$\mu_\nu = \sum_{k,j} (|\mu_{kj}|^2 + |\epsilon_{kj}|^2). \tag{5.34}$$

The quantity Z is a renormalization factor which depends on the polarization of the plasmon. For transverse plasmons $\omega_\gamma^2 - k_\gamma^2 = \omega_P^2$ and $Z = 1$, whereas for longitudinal plasmons $\omega_\gamma \simeq \omega_P$ and $Z \simeq (1 - k_\gamma^2/\omega_P^2)^{-1}$ (Raffelt, 1996).

The process of plasmon decay into a neutrino-antineutrino pair was first considered by Bernstein, Ruderman, and Feinberg (1963) as a new energy-loss channel for the Sun. In general, a plasmon decay in a star liberates the energy ω_γ in the form of neutrinos that freely escape the stellar environment. The corresponding energy-loss rate per unit volume is

$$Q_{\gamma^* \to \nu\bar{\nu}} = \frac{g}{(2\pi)^3} \int \omega_\gamma f_{k_\gamma} \Gamma_{\gamma \to \nu\bar{\nu}} d^3 k_\gamma, \tag{5.35}$$

where f_{k_γ} is the photon Bose-Einstein distribution function and $g = 2$ is the number of polarization states.

The requirement that the plasmon-decay energy-loss channel does not exceed the standard solar model (SSM) luminosity leads to the constraint (Raffelt, 1996, 1999a, 1999b)

$$\mu_\nu \lesssim 4 \times 10^{-10} \mu_B. \tag{5.36}$$

However, the tightest astrophysical bound on μ_ν comes from the constraints on the possible delay of helium ignition of a red giant star in globular clusters due to the cooling induced by the plasmon-decay energy loss. From the lack of observational evidence of this effect, the following limit has been found (Raffelt, 1990b, 1990c; Raffelt and Weiss, 1992):

$$\mu_\nu \lesssim 3 \times 10^{-12} \mu_B. \tag{5.37}$$

See also Castellani and Degl'Innocenti (1993) and Catelan, Pacheco, and Horvath (1996). Recently the limit was updated by Viaux *et al.* (2013) using state-of-the-art astronomical observations and stellar evolution codes, with the results

$$\mu_\nu < \begin{cases} 2.6\times 10^{-12}\mu_{\rm B} & (68\%\ \text{C.L.}), \\ 4.5\times 10^{-12}\mu_{\rm B} & (95\%\ \text{C.L.}). \end{cases} \quad (5.38)$$

This astrophysical constraint on a neutrino magnetic moment is applicable to both Dirac and Majorana neutrinos and constrains all diagonal and transition dipole moments according to Eq. (5.34).

It was also shown by Heger *et al.* (2009) that the additional cooling due to processes induced by neutrino magnetic moments [plasmon decay $\gamma^* \to \nu\bar{\nu}$, photo processes $\gamma e^- \to e^-\nu\bar{\nu}$, pair processes $e^+e^- \to \nu\bar{\nu}$, bremsstrahlung $e^-(Ze) \to (Ze)e^-\nu\bar{\nu}$] generate qualitative changes to the structure and evolution of stars with masses between 7 and 18 solar masses, rather than simply changing the time scales of their burning. The resulting sensitivity to the neutrino magnetic moment was estimated by Heger *et al.* (2009) to be at the level of $(2\text{–}4)\times 10^{-11}\mu_{\rm B}$.

E. Spin light

It is known from classical electrodynamics that a system with zero electric charge but nonzero magnetic (or electric) moment can produce electromagnetic radiation which is called "magnetic (or electric) dipole radiation." It is due to the rotation of the magnetic (or electric) moment.

A similar mechanism of radiation exists in the case of a neutrino with a magnetic (or electric) moment propagating in matter (Lobanov and Studenikin, 2003). This phenomenon, called "spin light of neutrino" ($SL\nu$), is different from the neutrino Cherenkov radiation in matter discussed in Sec. V.C, because it can exist even when the emitted photon refractive index is equal to unity. The $SL\nu$ is a radiation produced by the neutrino on its own, rather than a radiation of the background particles. Since the $SL\nu$ process is a transition between neutrino states with equal masses, it can become possible only because of an external environment influence on the neutrino states.

The $SL\nu$ was first studied by Lobanov and Studenikin (2003, 2004), and Grigoriev, Dvornikov, and Studenikin (2005a) with a quasiclassical treatment based on a Lorentz-invariant approach to the neutrino spin evolution that implies the use of the generalized Bargmann-Michel-Telegdi equation (Egorov, Lobanov, and Studenikin, 2000; Lobanov and Studenikin, 2001) (for further details see Appendix F). The full quantum theory of the $SL\nu$ has been elaborated on by Grigoriev, Studenikin, and Ternov (2005a, 2005b, 2006), Studenikin and Ternov (2005), Studenikin (2007), and Grigoriev *et al.* (2012b) [see also Lobanov (2005a, 2005b). The method is based on the exact solution of the modified Dirac equation for the neutrino wave function in matter (Studenikin, 2006a, 2006b, 2008; Grigoriev, Studenikin, and Ternov, 2009).

The Feynman diagram of the $SL\nu$ process is shown in Fig. 11, where the neutrino initial (ψ_i) and final (ψ_f) states (indicated by broad lines) are exact solutions of the corresponding Dirac equations accounting for the interactions with matter. The neutrino wave functions and energy spectrum are given by Eqs. (H16) and (H17) of Appendix F. Here we consider a generic flavor neutrino with an effective magnetic moment μ_ν and effective mass m_ν. The $SL\nu$ process for a relativistic neutrino is a transition from an initial neutrino state to a less energetic final neutrino state with the emission of a photon and a neutrino helicity flip (Grigoriev, Studenikin, and Ternov, 2005a; Studenikin and Ternov, 2005).

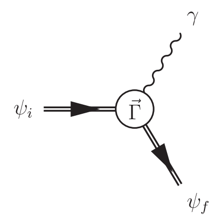

FIG. 11. The spin light of a neutrino ($SL\nu$) radiation diagram.

The amplitude of the $SL\nu$ process is given by (Studenikin and Ternov, 2005)

$$S_{fi} = -\mu_\nu\sqrt{4\pi}\int d^4x\bar{\psi}_f(x)(\vec{\Gamma}\cdot\vec{\varepsilon}^*)\frac{e^{ikx}}{\sqrt{2\omega L^3}}\psi_i(x), \quad (5.39)$$

where L^3 is the normalization volume and

$$\vec{\Gamma} = i\omega\{[\vec{\Sigma}\times\vec{\varkappa}] + i\gamma^5\vec{\Sigma}\}, \quad \text{with} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \quad (5.40)$$

Here $k^\mu = (\omega, \vec{k})$ and $\vec{\varepsilon}$ are the photon momentum and polarization vectors, and $\vec{\varkappa} = \vec{k}/\omega$ is the unit vector pointing in the direction of propagation of the emitted photon. $\psi_i(x)$ and $\psi_f(x)$ are the initial and final neutrino wave functions in the presence of matter obtained as the exact solutions of the effective Dirac equation

$$\{i\gamma_\mu\partial^\mu - \tfrac{1}{2}\gamma_\mu(1+\gamma_5)\tilde{f}^\mu - m_\nu\}\psi_{i,f}(x) = 0 \quad (5.41)$$

[see Eqs. (H16) and (H17) of Appendix H]. From the energy-momentum conservation relations

$$p_0 = p_0' + \omega, \qquad \vec{p} = \vec{p}' + \vec{\varkappa}, \quad (5.42)$$

where $(p_0, \vec{p})$ and $(p_0', \vec{p}')$ are the initial and final neutrino energy and momenta, it follows that the photon energy is given by

$$\omega = \frac{2\tilde{\alpha}m_\nu p[(p_0 - \tilde{\alpha}m_\nu) - (p + \tilde{\alpha}m_\nu)\cos\theta]}{(p_0 - \tilde{\alpha}m_\nu - p\cos\theta)^2 - (\tilde{\alpha}m_\nu)^2}, \quad (5.43)$$

where $p = |\vec{p}|$ and θ is the angle between $\vec{\varkappa}$ and the initial neutrino propagation. For an electron neutrino propagating in a medium composed of electrons, protons, and neutrons, the matter density parameter $\tilde{\alpha}$ is given by

$$\tilde{\alpha} = \frac{G_{\rm F}}{2\sqrt{2}m_\nu}[n_e(1 + 4\sin^2\theta_W) + n_p(1 - 4\sin^2\theta_W) - n_n], \quad (5.44)$$

where n_e, n_p, and n_n are the number densities of the background electrons, protons, and neutrons, respectively. From the amplitude (5.39) and the photon energy (5.43) the $SL\nu$ transition rate and total radiation power can be obtained

$$\Gamma = \mu_\nu^2 \int_0^\pi \frac{\omega^3}{1+\tilde{\beta}' y} S \sin\theta d\theta, \qquad (5.45)$$

$$I = \mu_\nu^2 \int_0^\pi \frac{\omega^4}{1+\tilde{\beta}' y} S \sin\theta d\theta, \qquad (5.46)$$

where

$$S = (\tilde{\beta}\tilde{\beta}' + 1)(1 - y\cos\theta) - (\tilde{\beta} + \tilde{\beta}')(\cos\theta - y), \qquad (5.47)$$

and

$$\tilde{\beta} = \frac{p + \tilde{\alpha} m_\nu}{p_0 - \tilde{\alpha} m_\nu}, \qquad \tilde{\beta}' = \frac{p' - \tilde{\alpha} m_\nu}{p_0' - \tilde{\alpha} m_\nu}, \qquad y = \frac{\omega - p\cos\theta}{p'}, \qquad (5.48)$$

where $p' = |\vec{p}'|$. For the case of a relativistic neutrino with $p \gg m_\nu$, the total rate and power are given by

$$\Gamma = \begin{cases} \frac{64}{3}\mu_\nu^2 \tilde{\alpha}^3 p^2 m_\nu & \text{for } \tilde{\alpha} \ll \frac{m_\nu}{p}, \\ 4\mu_\nu^2 \tilde{\alpha}^2 m_\nu^2 p & \text{for } \frac{m_\nu}{p} \ll \tilde{\alpha} \ll \frac{p}{m_\nu}, \\ 4\mu_\nu^2 \tilde{\alpha}^3 m_\nu^3 & \text{for } \tilde{\alpha} \gg \frac{p}{m_\nu}, \end{cases} \qquad (5.49)$$

$$I = \begin{cases} \frac{128}{3}\mu_\nu^2 \tilde{\alpha}^4 p^4 & \text{for } \tilde{\alpha} \ll \frac{m_\nu}{p}, \\ \frac{4}{3}\mu_\nu^2 \tilde{\alpha}^2 m_\nu^2 p^2 & \text{for } \frac{m_\nu}{p} \ll \tilde{\alpha} \ll \frac{p}{m_\nu}, \\ 4\mu_\nu^2 \tilde{\alpha}^4 m_\nu^4 & \text{for } \tilde{\alpha} \gg \frac{p}{m_\nu}. \end{cases} \qquad (5.50)$$

Since the rate and power of $SL\nu$ are proportional to μ_ν^2, they are in general very small. However, some specific features of the $SL\nu$ might be phenomenologically interesting for astrophysics (Lobanov and Studenikin, 2003, 2004; Studenikin, 2004b, 2006a, 2006b, 2007, 2007, 2008; Grigoriev, Studenikin, and Ternov, 2005a, 2005b, 2006, 2009; Grigoriev, Dvornikov, and Studenikin, 2005a; Lobanov, 2005a, 2005b; Studenikin and Ternov, 2005; Kuznetsov and Mikheev, 2006; Grigoriev *et al.*, 2008, 2012b).

As can be seen from Eqs. (5.49) and (5.50), for a wide range of matter densities the $SL\nu$ rate and power increase with the neutrino momentum. For ultrahigh energy neutrinos ($p_0 \sim 10^{18}$ eV) propagating through a dense matter characterized by the value of the density parameter $\tilde{\alpha} m_\nu \sim 10$ eV (this value is typical for a neutron star with $n_{e,p,n}$ of the order of 10^{38} cm^{-3}), the rate of the $SL\nu$ process is about 0.7 s^{-1}.

For the average emitted photon energy

$$\langle\omega\rangle = I/\Gamma, \qquad (5.51)$$

we obtain

$$\langle\omega\rangle \simeq \begin{cases} 2\tilde{\alpha} p^2/m_\nu & \text{for } \tilde{\alpha} \ll \frac{m_\nu}{p}, \\ \frac{1}{3} p & \text{for } \frac{m_\nu}{p} \ll \tilde{\alpha} \ll \frac{p}{m_\nu}, \\ \tilde{\alpha} m_\nu & \text{for } \tilde{\alpha} \gg \frac{p}{m_\nu}. \end{cases} \qquad (5.52)$$

Therefore, in the most interesting case of astrophysical ultrahigh-energy neutrinos, the average energy of the $SL\nu$ photons is one-third of the neutrino momentum and the $SL\nu$ spectrum spans the energy range of gamma rays.

Another interesting property of the $SL\nu$ is its spatial distribution. As it follows from Eqs. (5.45) and (5.46) the radiation is collimated along the direction of neutrino propagation. In the case of relativistic neutrinos ($p \gg m_\nu$) we have $1 \ll \tilde{\alpha} \ll p/m_\nu$ for a wide range of matter densities and the radiation power is emitted in a narrow cone with thickness $\delta\theta \simeq m_\nu/p$ around a very small angle $\theta_{\max}$ given by $\cos\theta_{\max} \simeq 1 - (2/3)\tilde{\alpha} m_\nu/p$. The image drawn by the $SL\nu$ radiation in the plane perpendicular to the neutrino direction of motion in dense matter is a narrow ring with a very small radius centered on the neutrino path.

When neutrinos propagate in a plasma, the $SL\nu$ radiation is affected by the influence of the background plasma on the propagation of the emitted photons. This effect was first discussed by Grigoriev, Studenikin, and Ternov (2005a, 2005b, 2006), and Studenikin and Ternov (2005), and was further studied by Kuznetsov and Mikheev (2006, 2007), where the role of the $SL\nu$ plasmon mass was taken into account. In the case of the ultra-high-energy neutrino, the $SL\nu$ rate of Kuznetsov and Mikheev (2006, 2007) exactly reproduces the results obtained by Grigoriev, Studenikin, and Ternov (2005a), 2005b, 2006), and Studenikin and Ternov (2005). For a more detailed discussion on the historical aspects of this issue, see Grigoriev *et al.* (2006, 2008), Kuznetsov and Mikheev (2006), Studenikin (2008), and Grigoriev, Studenikin, and Ternov (2009). The most complete and consistent study of the $SL\nu$ accounting for the plasma effects can be found in Grigoriev *et al.* (2012b).

The $SL\nu$ process with transitions between neutrinos with different masses was considered by Grigoriev *et al.* (2012a) and the $SL\nu$ mechanism taking into account possible effects of Lorentz invariance violation was discussed by Kruglov (2014).

VI. INTERACTIONS WITH ELECTROMAGNETIC FIELDS

If neutrinos have nontrivial electromagnetic properties, they can interact with classical electromagnetic fields. Significant effects can occur, in particular, in neutrino astrophysics, since neutrinos can propagate over very long distances in astrophysical environments with magnetic fields. In this case even a very weak interaction can have large cumulative effects.

In Sec. VI.A we derive the effective potential of a neutrino propagating in a classical electromagnetic field. This potential can generate spin and spin-flavor transitions, which are discussed in Sec. VI.B. We also review the limits on the neutrino effective magnetic moment obtained from analyses of solar neutrino data. In Sec. VI.C we discuss the modifications of neutrino magnetic moments in very strong magnetic fields. In Sec. VI.D we review the effects of a strong magnetic field on neutron decay. In Sec. VI.E we review neutrino-antineutrino pair production in a magnetic field and in Sec. VI.F we discuss neutrino-antineutrino pair production due to vacuum instability in a very strong magnetic field. In Sec. VI.G we review the energy quantization of neutrinos propagating in rotating media.

A. Effective potential

The coherent interactions of neutrinos with classical electromagnetic fields generate potentials which are similar to the matter potentials in Eq. (2.42) and must be taken into account in the study of flavor and spin evolution with an equation analogous to the MSW equation (2.45). This evolution in a magnetic field is discussed in detail in Sec. VI.B. Here we discuss the derivation of the neutrino effective potential in a classical electromagnetic field, which corresponds to the amplitude of coherent forward elastic scattering:

$$V_{h_i\to h_f} = \lim_{q\to 0}\frac{\langle \nu(p_f,h_f)|\int d^3x \mathcal{H}_{\rm em}^{(\nu)}(x)|\nu(p_i,h_i)\rangle}{\langle \nu(p,h)|\nu(p,h)\rangle}, \qquad (6.1)$$

where $q = p_i - p_f$ as above and the denominator enforces the correct normalization ($p = p_i = p_f$ in the limit $q \to 0$ and h is arbitrary). The interaction Hamiltonian $\mathcal{H}_{\rm em}^{(\nu)}(x)$ is that in Eq. (3.2). Here we consider for simplicity only one neutrino (the generalization to more than one neutrino, with the possibility of coherent transitions between different massive neutrinos generated by transition form factors, is discussed later), allowing for possible helicity transitions ($h_f \neq h_i$), which are important in magnetic fields (see Sec. VI.B). Note that the Hermiticity of $\mathcal{H}_{\rm em}^{(\nu)}(x)$ implies that

$$V_{h_f\to h_i} = V^*_{h_i\to h_f}. \qquad (6.2)$$

From the normalization of states in Eq. (A58) and Eqs. (3.2)–(3.8), we obtain

$$V_{h_i\to h_f} = \frac{1}{2E_\nu VT}\lim_{q\to 0}\overline{u^{(h_f)}}(p_f)\Lambda_\mu(q)u^{(h_i)}(p_i)\tilde{A}^\mu(q), \qquad (6.3)$$

where T is the normalization time, $E_\nu = E_i = E_f$ in the limit $q \to 0$, and

$$\tilde{A}^\mu(q) = \int d^4x e^{-iq\cdot x}A^\mu(x) \qquad (6.4)$$

is the Fourier transform of $A^\mu(x)$. Integrating by parts and neglecting an irrelevant surface term (which vanishes for well-behaved physical fields which vanish at infinity), we have

$$q_\alpha \tilde{A}^\mu(q) = -i\int d^4x e^{-iq\cdot x}\partial_\alpha A^\mu(x). \qquad (6.5)$$

Using Eq. (3.18) for $\Lambda_\mu(q)$, and the Gordon identity (A60) for the γ^μ term, we obtain, in the limit $q \to 0$,

$$V_{h_i\to h_f} = \frac{1}{VT}\int d^4x\Big[\mathbb{q}\frac{p_\mu}{E_\nu}A^\mu(x)\delta_{h_f h_i} + \frac{1}{4E_\nu}\overline{u^{(h_f)}}(p)\sigma_{\mu\nu}F^{\mu\nu}(x)\left(\frac{\mathbb{q}}{2m}+\mu+i\epsilon\gamma_5\right)u^{(h_i)}(p) - \frac{\mathbb{a}}{2E_\nu}\overline{u^{(h_f)}}(p)j^\mu(x)\gamma_\mu\gamma_5 u^{(h_i)}(p)\Big], \qquad (6.6)$$

where $p^\mu = p_i^\mu = p_f^\mu$. The electromagnetic tensor $F^{\mu\nu}(x)$ defined in Eq. (A72) contains the physical electric field $\vec{E}(x)$ and magnetic field $\vec{B}(x)$ [see Eq. (A74)].

Now we take into account the fact that propagating neutrinos are described by wave packets whose size is limited (Giunti and Kim, 2007). Considering fields which are approximately constant over the extension of the neutrino wave packet, we can extract them from the integral in Eq. (6.6). Then the integral simplifies with VT, leading to

$$V_{h_i\to h_f} = \mathbb{q}\frac{p_\mu}{E_\nu}A^\mu\delta_{h_f h_i} + \frac{1}{4E_\nu}\overline{u^{(h_f)}}(p)\sigma_{\mu\nu}F^{\mu\nu}\left(\frac{\mathbb{q}}{2m}+\mu+i\epsilon\gamma_5\right)u^{(h_i)}(p) - \frac{\mathbb{a}}{2E_\nu}\overline{u^{(h_f)}}(p)j^\mu\gamma_\mu\gamma_5 u^{(h_i)}(p). \qquad (6.7)$$

From Eq. (6.7) one can see that $V_{h_i\to h_f}$ depends on the four neutrino electromagnetic form factors at $q^2 = 0$, but the anapole moment contributes only in very special environments in which the medium is charged. Since we discuss this special case in Sec. VII.C devoted to the anapole moment, in the following part of this section we do not consider the anapole moment, assuming $j^\mu(x) = 0$.

We consider the first term in Eq. (6.7). In an electrostatic field $A^\mu = (A^0, 0, 0, 0)$, we have $V^{(1)}_{h_i\to h_f} = \mathbb{q}A^0\delta_{h_f h_i}$. This is the expected result, taking into account that A^0 is the electric potential. Of course this term can contribute to the neutrino potential only if neutrinos are millicharged particles (see Sec. VII.A).

We now consider the more interesting contribution of the second term in Eq. (6.7), which depends on the dipole magnetic and electric moments. Note that the charge generates a magnetic moment

$$\mu_{\mathbb{q}} = \frac{\mathbb{q}}{2m} = g\mu_{\rm cl}^{(1/2)}, \quad \text{with} \quad g = 2, \qquad (6.8)$$

where $\mu_{\rm cl}^{(s)} = \mathbb{q}s/2m$ is the classical magnetic moment of a spin-s particle (Jackson, 1999). This is the same magnetic moment obtained from the Dirac equation of a charged particle, with the well-known gyromagnetic ratio $g = 2$. For a normally charged particle the additional contribution μ in Eq. (6.7) to the magnetic moment would be called an "anomalous magnetic moment," which is generated by an internal structure in the case of nucleons or by quantum loop corrections in the case of leptons (measured in the famous $g - 2$ experiments). Since neutrinos are at most millicharged particles, the μ in Eq. (6.7) is traditionally called the magnetic moment, and the possible contribution of $\mu_{\mathbb{q}}$ is neglected. Moreover, $\mu_{\mathbb{q}}$ does not contribute to helicity transitions, because it generates a spin precession which has the same frequency as the precession of the angular momentum generated by $\mathbb{q}$ (Sakurai, 1967).

In the following, we study the effects of μ and ϵ assuming $\mathbb{q} = 0$. We also establish the connection of the neutrino potential with the classical potential for a nonrelativistic particle (Jackson, 1999),

$$V_{\rm cl} = -\vec{\mu}\cdot\vec{B} - \vec{\epsilon}\cdot\vec{E}, \tag{6.9}$$

and the torque

$$\vec{T}_{\rm cl} = \vec{\mu}\times\vec{B} + \vec{\epsilon}\times\vec{E}, \tag{6.10}$$

which generates the precession of the spin $\vec{S}$ through $d\vec{S}/dt = \vec{T}_{\rm cl}$.

We first consider the helicity-conserving potential $V_{h\to h}$. Using the method described in Appendix E, we obtain

$$V_{h\to h} = -\frac{m}{E_\nu}(\vec{\mu}\cdot\vec{B} + \vec{\epsilon}\cdot\vec{E}), \tag{6.11}$$

with

$$\vec{\mu} = h\frac{\vec{p}}{|\vec{p}|}\mu, \qquad \vec{\epsilon} = h\frac{\vec{p}}{|\vec{p}|}\epsilon. \tag{6.12}$$

Hence, the helicity-conserving potential is proportional to the longitudinal components of the magnetic and electric fields. In the nonrelativistic limit ($E_\nu \simeq m$) we obtain a potential which corresponds to the classical one in Eq. (6.9). Note, however, that this potential is strongly suppressed by the small fraction m/E_ν for ultrarelativistic neutrinos in realistic experiments.

Considering now the helicity-flipping potential $V_{-h\to h}$, using the method described in Appendix E, if there is only an electric field $\vec{E}$, we obtain

$$V_{-h\to h}(\vec{E}) = \left(\epsilon + ih\frac{|\vec{p}|}{E_\nu}\mu\right)E_\perp, \tag{6.13}$$

where $E_\perp$ is the transverse component of the electric field, i.e., orthogonal to $\vec{p}$. In the case of a pure magnetic field $\vec{B}$, we have, with a similar notation,

$$V_{-h\to h}(\vec{B}) = \left(\mu - ih\frac{|\vec{p}|}{E_\nu}\epsilon\right)B_\perp, \tag{6.14}$$

where $B_\perp$ is the transverse component of the magnetic field. The expression of $V_{-h\to h}$ in the general case of an electromagnetic field is given in Eq. (E11), from which one can see that in any case the helicity-flipping potential depends only on the transverse components of the electric and magnetic fields.

Note that for nonrelativistic neutrinos ($|\vec{p}| \ll E_\nu$) in practice $V_{-h\to h}(\vec{E})$ depends only on ϵ and $V_{-h\to h}(\vec{B})$ depends only on μ, as one may have expected:

$$V^{\rm nr}_{-h\to h}(\vec{E}) \simeq \epsilon E_\perp = |\vec{\epsilon}\times\vec{E}|, \tag{6.15}$$

$$V^{\rm nr}_{-h\to h}(\vec{B}) \simeq \mu B_\perp = |\vec{\mu}\times\vec{B}|. \tag{6.16}$$

Hence, in the nonrelativistic limit the helicity-flipping potential corresponds to the classical torque in Eq. (6.10), which rotates the spin of the particle, causing periodic changes of the helicity.

The additional dependences of $V_{-h\to h}(\vec{E})$ on μ and that of $V_{-h\to h}(\vec{B})$ on ϵ for relativistic neutrinos are explained in Appendix E as a consequence of the relativistic transformations of the electric and magnetic fields and the correspondence of the electric and magnetic dipole moments with their classical counterparts only in the nonrelativistic limit.

We finally consider the potential between different massive neutrinos, which is generated by transition electric and magnetic dipole moments,

$$V_{\nu_i^{(h_i)}\to\nu_f^{(h_f)}} = \lim_{q\to 0}\frac{\langle\nu_f(p_f,h_f)|\int d^3x\mathcal{H}^{(\nu)}_{\rm em}(x)|\nu_i(p_i,h_i)\rangle}{\langle\nu(p,h)|\nu(p,h)\rangle}, \tag{6.17}$$

which is especially interesting for Majorana neutrinos which do not have diagonal electric and magnetic dipole moments. Here one can notice that it is impossible to have $p_i = p_f$ if $m_i \neq m_f$. However, we must remember that observable neutrinos are ultrarelativistic and their energy-momentum uncertainty is much larger than their mass differences (Giunti and Kim, 2007). In this case, $\nu_i \to \nu_f$ transitions are possible in an electromagnetic field, as well as the coherent production of different massive neutrinos which is necessary for the oscillations discussed in Sec. II.D. In practice this means that in the calculation of V_{fi} we can approximate the neutrinos as massless. Under this approximation, the helicity-flipping potential in a transverse magnetic field in Eq. (6.14) can be generalized to

$$V_{\nu_i^{(-h)}\to\nu_f^{(h)}}(\vec{B}) = \left(\mu_{fi} - ih\frac{|\vec{p}|}{E_\nu}\epsilon_{fi}\right)B_\perp. \tag{6.18}$$

This potential is interesting because it determines the neutrino spin-flavor precession in a transverse magnetic field discussed in Sec. VI.B.

B. Spin-flavor precession

If neutrinos have magnetic moments, the spin can precess in a transverse magnetic field (Cisneros, 1971; Fujikawa and Shrock, 1980; Okun, Voloshin, and Vysotsky, 1986; Voloshin and Vysotsky, 1986).

We first derive the spin precession of an ultrarelativistic Dirac neutrino generated by its diagonal magnetic moment μ. We consider a neutrino with four-momentum p which at the initial time $t = 0$ has a definite helicity h_i and is described by the state $|\nu(p,h_i)\rangle$. After propagation in a magnetic field $\vec{B}$, at time t the neutrino is described by a superposition of both helicities:

$$|\nu(t)\rangle = \sum_{h=\pm 1}\psi_h(t)|\nu(p,h)\rangle. \tag{6.19}$$

The temporal evolution of $|\nu(t)\rangle$ is given by the Schrödinger equation

$$i\frac{d}{dt}|\nu(t)\rangle = \mathbb{H}_{\text{em}}(t)|\nu(t)\rangle, \tag{6.20}$$

where $|\nu(0)\rangle = |\nu(p,h_i)\rangle$ and $\mathbb{H}_{\text{em}}(t) = \int d^3x\mathcal{H}_{\text{em}}^{(\nu)}(x)$ is the effective interaction Hamiltonian, which can depend on time if the magnetic field is not constant. Here we neglect the irrelevant contribution of the vacuum Hamiltonian, which does not cause any change in helicity because the two helicity states have the same mass.

Multiplying Eq. (6.20) on the left by $\langle\nu(p,h)|$, we obtain the evolution equation for the helicity amplitudes

$$i\frac{d\psi_h(t)}{dt} = \sum_{h'=\pm1}\psi_{h'}(t)V_{h'\to h}(t), \tag{6.21}$$

with the potential $V_{h'\to h}(t)$ given in Eq. (6.1) and $\psi_h(0) = \delta_{hh_i}$.

In Eq. (6.11) we have seen that the helicity-conserving potential, which depends on the longitudinal component of the magnetic field, is strongly suppressed for ultrarelativistic neutrinos. Hence, in practice only the transverse component of the magnetic field contributes through the helicity-flipping potential in Eq. (6.14). Considering for simplicity only the contribution of the magnetic moment μ, we have

$$V_{h'\to h}(t) = \mu B_\perp(t)\delta_{-hh'}. \tag{6.22}$$

Then the evolution equation (6.21) can be written in the standard matrix form

$$i\frac{d}{dx}\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} = \begin{pmatrix}0 & \mu B_\perp(x)\\ \mu B_\perp(x) & 0\end{pmatrix}\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix}, \tag{6.23}$$

where we approximated the distance x along the neutrino trajectory with time t for ultrarelativistic neutrinos, and we adopted the standard notation $\psi_L \equiv \psi_{-1}$ and $\psi_R \equiv \psi_{+1}$ for the negative and positive helicity amplitudes of the left-handed and right-handed neutrinos, which are described, respectively, by the states $|\nu_L\rangle = |\nu(p,-1)\rangle$ and $|\nu_R\rangle = |\nu(p,+1)\rangle$. The differential equation (6.23) can be solved through the transformation

$$\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\varphi_-(x)\\ \varphi_+(x)\end{pmatrix}. \tag{6.24}$$

The amplitudes $\varphi_-(x)$ and $\varphi_+(x)$ satisfy decoupled differential equations, whose solutions are

$$\varphi_\mp(x) = \exp\left[\pm i\int_0^x dx'\mu B_\perp(x')\right]\varphi_\mp(0). \tag{6.25}$$

If we consider an initial left-handed neutrino, we have

$$\begin{pmatrix}\psi_L(0)\\ \psi_R(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}\varphi_-(0)\\ \varphi_+(0)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{6.26}$$

Then the probability of $\nu_L \to \nu_R$ transitions is given by

$$P_{\nu_L\to\nu_R}(x) = |\psi_R(x)|^2 = \sin^2\left(\int_0^x dx'\mu B_\perp(x')\right). \tag{6.27}$$

Note that the transition probability is independent of the neutrino energy (contrary to the case of flavor oscillations) and the amplitude of the oscillating probability is unity. Hence, when the argument of the sine is equal to $\pi/2$ there is complete $\nu_L \to \nu_R$ conversion.

The precession $\nu_{eL} \to \nu_{eR}$ in the magnetic field of the Sun was considered in 1971 (Cisneros, 1971) as a possible solution of the solar neutrino problem. If neutrinos are Dirac particles, right-handed neutrinos are sterile and a $\nu_{eL} \to \nu_{eR}$ conversion could explain the disappearance of active solar ν_{eL}'s.

In 1986 Okun, Voloshin, and Vysotsky (1986) and Voloshin and Vysotsky (1986) realized that the matter effect during neutrino propagation inside of the Sun suppresses $\nu_{eL} \to \nu_{eR}$ transition by lifting the degeneracy of ν_{eL} and ν_{eR} [see also Barbieri and Fiorentini (1988)]. Indeed, taking into account matter effects, the evolution equation (6.23) becomes

$$i\frac{d}{dx}\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} = \begin{pmatrix}V(x) & \mu B_\perp(x)\\ \mu B_\perp(x) & 0\end{pmatrix}\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix}, \tag{6.28}$$

with the appropriate potential V which depends on the neutrino flavor, according to Eq. (2.41). In the case of a constant matter density, this differential equation can be solved analytically with the orthogonal transformation

$$\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} = \begin{pmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{pmatrix}\begin{pmatrix}\varphi_-(x)\\ \varphi_+(x)\end{pmatrix}. \tag{6.29}$$

The angle ξ is chosen in order to diagonalize the matrix operator in Eq. (6.28):

$$\sin 2\xi = \frac{2\mu B_\perp}{\Delta E_{\text{M}}}, \tag{6.30}$$

with the effective energy splitting in matter

$$\Delta E_{\text{M}} = \sqrt{V^2 + (2\mu B_\perp)^2}. \tag{6.31}$$

The decoupled evolution of $\varphi_\mp(x)$ is given by

$$\varphi_\mp(x) = \exp\left[-\frac{i}{2}(V \mp \Delta E_{\text{M}})\right]\varphi_\mp(0). \tag{6.32}$$

Considering an initial left-handed neutrino,

$$\begin{pmatrix}\varphi_-(0)\\ \varphi_+(0)\end{pmatrix} = \begin{pmatrix}\cos\xi\\ \sin\xi\end{pmatrix}, \tag{6.33}$$

we obtain the oscillatory transition probability

$$P_{\nu_L\to\nu_R}(x) = |\psi_R(x)|^2 = \sin^2 2\xi \sin^2(\tfrac{1}{2}\Delta E_{\text{M}}x). \tag{6.34}$$

Since in matter $\Delta E_{\text{M}} > 2\mu B_\perp$, the matter effect suppresses the amplitude of $\nu_L \to \nu_R$ transitions. However, these transitions are still independent of the neutrino energy, which does not enter in the evolution equation (6.28).

When it was known in 1986 (Okun, Voloshin, and Vysotsky, 1986; Voloshin and Vysotsky, 1986) that the matter potential has the effect of suppressing $\nu_L \to \nu_R$ transitions because it breaks the degeneracy of left-handed and right-handed states, it did not take long to realize in 1988 (Akhmedov, 1988; Lim and Marciano, 1988) that the matter potentials can cause resonant spin-flavor precession if different flavor neutrinos have transition magnetic moments [spin-flavor precession in vacuum was previously discussed by Schechter and Valle (1981)]. The application of this mechanism to solar neutrinos was discussed in the following years (Akhmedov and Bychuk, 1989; Minakata and Nunokawa, 1989; Balantekin, Hatchell, and Loreti, 1990; Akhmedov, 1991a, 1991b; Raghavan *et al.*, 1991; Balantekin and Loreti, 1992; Pulido, 1992; Akhmedov, Lanza, and Petcov, 1993a, 1995; Akhmedov, Petcov, and Smirnov, 1993b; Shi *et al.*, 1993; Akhmedov and Pulido, 2000, 2003; Dev and Sharma, 2000; Pulido and Akhmedov, 2000; Barranco *et al.*, 2002; Chauhan and Pulido, 2002, 2004; Chauhan, Pulido, and Torrente-Lujan, 2003; Miranda *et al.*, 2004a, 2004b; Balantekin and Volpe, 2005; Chauhan, Pulido, and Raghavan, 2005; Friedland, 2005; Guzzo, de Holanda, and Peres, 2005, 2012; Pulido, Chauhan, and Raghavan, 2005; Picariello *et al.*, 2007; Chauhan, Pulido, and Picariello, 2007; Yilmaz, 2008; Das, Pulido, and Picariello, 2009; Raffelt and Rashba, 2010).

We consider a neutrino state which is a superposition of different massive neutrinos with both helicities:

$$|\nu(t)\rangle = \sum_k \sum_{h=\pm 1} \psi_{k,h}(t) |\nu_k(p,h)\rangle, \tag{6.35}$$

where $\psi_{kh}(t)$ is the amplitude of the neutrino with mass m_k and helicity h. The temporal evolution of $|\nu(t)\rangle$ is given by the Schrödinger equation

$$i\frac{d}{dt}|\nu(t)\rangle = \mathbb{H}(t)|\nu(t)\rangle, \tag{6.36}$$

with the initial condition $|\nu(0)\rangle = |\nu_i(p,h_i)\rangle$. Multiplying the evolution equation on the left by $\langle\nu_k(p,h)|$, we obtain the evolution equation for the helicity amplitudes of the different massive neutrinos

$$i\frac{d\psi_{kh}(t)}{dt} = \sum_j \sum_{h'=\pm 1} \frac{\langle\nu_k(p,h)|\mathbb{H}(t)|\nu_j(p,h')\rangle}{\langle\nu(p,h)|\nu(p,h)\rangle} \psi_{j,h'}(t), \tag{6.37}$$

with $\psi_{kh}(0) = \delta_{hh_i}\delta_{ki}$. The effective Hamiltonian $\mathbb{H}(t)$ is the sum of a vacuum Hamiltonian $\mathbb{H}_0$, a weak-interaction Hamiltonian $\mathbb{H}_w(t)$ which generates the effective potentials (2.41) of flavor neutrinos in matter, and the electromagnetic Hamiltonian $\mathbb{H}_{\rm em}(t)$ already considered in Eq. (6.20). For ultrarelativistic neutrinos,

$$\frac{\langle\nu_k(p,h)|\mathbb{H}_0|\nu_j(p,h')\rangle}{\langle\nu(p,h)|\nu(p,h)\rangle} = \left(E_\nu + \frac{m_k^2}{2E_\nu}\right)\delta_{kj}\delta_{hh'}, \tag{6.38}$$

where E_ν is the neutrino energy neglecting mass contributions.

In order to calculate the matrix element of $\mathbb{H}_w(t)$, we must take into account the mixing of neutrino states in Eq. (2.31), which applies to left-handed neutrinos:

$$|\nu_k(p,-)\rangle = \sum_\ell U_{\ell k}|\nu_\ell(p,-)\rangle. \tag{6.39}$$

For right-handed Dirac neutrinos the mixing is arbitrary, because right-handed Dirac neutrinos are sterile to weak interactions. On the other hand, since right-handed Majorana neutrinos interact as right-handed Dirac antineutrinos, their mixing is given by

$$|\nu_k^{\rm M}(p,+)\rangle = \sum_\ell U^*_{\ell k}|\nu_\ell(p,+)\rangle. \tag{6.40}$$

Therefore, we define the generalized mixing relation

$$|\nu_k(p,h)\rangle = \sum_\ell U^{(h)}_{\ell k}|\nu_\ell(p,h)\rangle, \tag{6.41}$$

with $U^{(-)} = U$ and

$$\text{Dirac}:\ U^{(+)} = U; \tag{6.42}$$

$$\text{Majorana}:\ U^{(+)} = U^*. \tag{6.43}$$

The arbitrary choice for Dirac neutrinos has been made for simple convenience. Then for the matrix element of $\mathbb{H}_w(t)$ we obtain

$$\frac{\langle\nu_k(p,h)|\mathbb{H}_w(t)|\nu_j(p,h')\rangle}{\langle\nu(p,h)|\nu(p,h)\rangle} = \sum_\ell U^{(h)*}_{\ell k} U^{(h)}_{\ell j} V^{(h)}_\ell(t)\delta_{hh'}, \tag{6.44}$$

where $V^{(-)}_\ell = V_\ell$, with the potential V_ℓ in Eq. (2.41), and

$$\text{Dirac}:\ V^{(+)}_\ell = 0; \tag{6.45}$$

$$\text{Majorana}:\ V^{(+)}_\ell = -V_\ell. \tag{6.46}$$

As remarked before Eq. (6.22), the helicity-conserving potential generated by $\mathbb{H}^{(\nu)}_{\rm em}(t)$, which depends on the longitudinal component of the magnetic field, is strongly suppressed for ultrarelativistic neutrinos. Then, from Eq. (6.18), considering for simplicity only the contribution of the magnetic moments, we have

$$\frac{\langle\nu_k(p,h)|\mathbb{H}^{(\nu)}_{\rm em}(t)|\nu_j(p,h')\rangle}{\langle\nu(p,h)|\nu(p,h)\rangle} = \mu_{kj}B_\perp(t)\delta_{-hh'}. \tag{6.47}$$

Plugging Eqs. (6.38), (6.44), and (6.47) into Eq. (6.37), neglecting the irrelevant common energy contribution in Eq. (6.38) and approximating the distance x along the neutrino trajectory with time t for ultrarelativistic neutrinos, one obtains the evolution equations of the helicity amplitudes of the different massive neutrinos:

$$i\frac{d\psi_{k,h}(x)}{dx} = \sum_j \sum_{h'=\pm 1} \Bigg[\Bigg(\frac{m_k^2}{2E_\nu} \delta_{kj} + \sum_\ell U_{\ell k}^{(h)*} V_\ell^{(h)}(x) U_{\ell j}^{(h)} \Bigg) \delta_{hh'} + \mu_{kj} B_\perp(x) \delta_{-hh'} \Bigg] \psi_{j,h'}(x). \tag{6.48}$$

In order to study flavor and helicity transitions, it is more convenient to work in the flavor basis. Using the mixing of neutrino states in Eq. (2.30), the state (6.35) with $t = x$ can be written as

$$|\nu(x)\rangle = \sum_\ell \sum_{h=\pm 1} \psi_{\ell,h}(x) |\nu_\ell(p,h)\rangle, \tag{6.49}$$

with the flavor and helicity amplitudes

$$\psi_{\ell,h}(x) = \sum_k U_{\ell k}^{(h)} \psi_{k,h}(x), \tag{6.50}$$

which obey the evolution equation

$$i\frac{d\psi_{\ell,h}(x)}{dx} = \sum_{\ell'} \sum_{h'=\pm 1} \Bigg[\Bigg(\sum_k U_{\ell k}^{(h)} \frac{m_k^2}{2E_\nu} U_{\ell' k}^{(h)*} + V_\ell^{(h)}(x) \delta_{\ell\ell'} \Bigg) \delta_{hh'} + \mu_{\ell\ell'}^{(h,h')} B_\perp(x) \delta_{-hh'} \Bigg] \psi_{\ell',h'}(x), \tag{6.51}$$

with the effective magnetic moments in the flavor basis

$$\mu_{\ell\ell'}^{(h,h')} = \sum_{k,j} U_{\ell k}^{(h)} \mu_{kj} U_{\ell' j}^{(h')*}. \tag{6.52}$$

For Dirac neutrinos, from Eq. (6.42) we have

$$\mu_{\ell\ell'}^{(-,+)} = \mu_{\ell\ell'}^{(+,-)} = \sum_{k,j} U_{\ell k} \mu_{kj} U_{\ell' j}^* \equiv \mu_{\ell\ell'}. \tag{6.53}$$

Then, from Eq. (3.36) we obtain

$$\mu_{jk} = \mu_{kj}{}^* \Rightarrow \mu_{\ell'\ell} = \mu_{\ell\ell'}^*. \tag{6.54}$$

For Majorana neutrinos, from Eq. (6.43) we have

$$\mu_{\ell\ell'}^{(-,+)} = \sum_{k,j} U_{\ell k} \mu_{kj} U_{\ell' j}, \tag{6.55}$$

$$\mu_{\ell\ell'}^{(+,-)} = \sum_{k,j} U_{\ell k}^* \mu_{kj} U_{\ell' j}^*. \tag{6.56}$$

From Eqs. (3.67) and (3.69) it follows that for Majorana neutrinos the matrix of magnetic moments is antisymmetric and the transition magnetic moments are imaginary:

$$\mu_{jk} = -\mu_{kj} = \mu_{kj}^*. \tag{6.57}$$

The antisymmetric property is preserved in the flavor basis:

$$\mu_{\ell\ell'}^{(-,+)} = -\mu_{\ell'\ell}^{(-,+)}, \qquad \mu_{\ell\ell'}^{(+,-)} = -\mu_{\ell'\ell}^{(+,-)}. \tag{6.58}$$

Hence, there are no diagonal magnetic moments in the flavor basis as in the mass basis. Moreover, we have

$$\mu_{\ell\ell'}^{(+,-)} = -\mu_{\ell\ell'}^{(-,+)*}. \tag{6.59}$$

In the following we discuss the spin-flavor evolution equation in the two-neutrino mixing approximation, which is interesting for understanding the relevant features of neutrino spin-flavor precession. Keeping in mind the application to solar neutrinos, we consider the ν_e-ν_a mixing discussed in Sec. II.D, where ν_a is the linear combination of ν_μ and ν_τ in Eq. (2.59). Neglecting the small effects due to ϑ_{13}, we have

$$\begin{pmatrix} \psi_{e,h}(x) \\ \psi_{a,h}(x) \end{pmatrix} = R_{12} \begin{pmatrix} \psi_{1,h}(x) \\ \psi_{2,h}(x) \end{pmatrix}, \tag{6.60}$$

with

$$R_{12} = \begin{pmatrix} \cos\vartheta_{12} & \sin\vartheta_{12} \\ -\sin\vartheta_{12} & \cos\vartheta_{12} \end{pmatrix}. \tag{6.61}$$

Considering Dirac neutrinos, from Eq. (6.51) it follows that the generalization of Eq. (6.23) to two-neutrino ν_e-ν_a mixing is, using the analogous notation $\psi_{\ell L} \equiv \psi_{\ell,-1}$ and $\psi_{\ell R} \equiv \psi_{\ell,+1}$,

$$i\frac{d}{dx} \begin{pmatrix} \psi_{eL}(x) \\ \psi_{aL}(x) \\ \psi_{eR}(x) \\ \psi_{aR}(x) \end{pmatrix} = \mathrm{H} \begin{pmatrix} \psi_{eL}(x) \\ \psi_{aL}(x) \\ \psi_{eR}(x) \\ \psi_{aR}(x) \end{pmatrix}, \tag{6.62}$$

with the effective Hamiltonian matrix

$$\mathrm{H} = \begin{pmatrix} -\frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} + V_e & \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} & \mu_{ee} B_\perp(x) & \mu_{ea} B_\perp(x) \\ \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} + V_a & \mu_{ea}^* B_\perp(x) & \mu_{aa} B_\perp(x) \\ \mu_{ee} B_\perp(x) & \mu_{ea} B_\perp(x) & -\frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} \\ \mu_{ea}^* B_\perp(x) & \mu_{aa} B_\perp(x) & \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} \end{pmatrix}, \tag{6.63}$$

with the effective magnetic moments in the flavor basis given by

$$\begin{pmatrix} \mu_{ee} & \mu_{ea} \\ \mu_{ea}^* & \mu_{aa} \end{pmatrix} = R_{12} \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12}^* & \mu_{22} \end{pmatrix} R_{12}^T. \quad (6.64)$$

The matter potential can generate resonances, which occur when two diagonal elements of H become equal. Besides the standard MSW resonance in the $\nu_{eL} \leftrightarrows \nu_{aL}$ channel discussed in Sec. II.D, there are two possibilities:

(1) There is a resonance in the $\nu_{eL} \leftrightarrows \nu_{aR}$ channel for

$$V_e = \frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.65)$$

The density at which this resonance occurs is not the same as that of the MSW resonance, given by Eq. (2.55), because of the neutral-current contribution to $V_e = V_{\rm CC} + V_{\rm NC}$. The location of this resonance depends on both N_e and N_n.

(2) There is a resonance in the $\nu_{aL} \leftrightarrows \nu_{eR}$ channel for

$$V_a = -\frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.66)$$

If $\cos 2\vartheta_{12} > 0$, this resonance is possible in normal matter, since the sign of $V_a = V_{\rm NC}$ is negative, as one can see from Eq. (2.42).

In practice the effect of these resonances could be the disappearance of active ν_{eL} or ν_{aL} into sterile right-handed states.

We consider now the case of Majorana neutrinos. The evolution equation of the amplitudes is given by Eq. (6.62) with the effective Hamiltonian matrix

$$\mathrm{H} = \begin{pmatrix} -\frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} + V_e & \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} & 0 & \mu_{ea} B_\perp(x) \\ \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} + V_a & -\mu_{ea} B_\perp(x) & 0 \\ 0 & -\mu_{ea}^* B_\perp(x) & -\frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} - V_e & \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} \\ \mu_{ea}^* B_\perp(x) & 0 & \frac{\Delta m^2}{4E_\nu}\sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu}\cos 2\vartheta_{12} - V_a \end{pmatrix}, \quad (6.67)$$

with

$$\mu_{ea} \equiv \mu_{ea}^{(-,+)} = \mu_{12} e^{i\lambda_{12}}, \quad (6.68)$$

where λ_{12} is the Majorana phase in Eq. (2.28).

As in the Dirac case, there are two possible resonances besides the standard MSW resonance in the $\nu_{eL} \leftrightarrows \nu_{aL}$ channel:

(1) There is a resonance in the $\nu_{eL} \leftrightarrows \nu_{aR}$ channel for

$$V_{\rm CC} + 2V_{\rm NC} = \frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.69)$$

(2) There is a resonance in the $\nu_{aL} \leftrightarrows \nu_{eR}$ channel for

$$V_{\rm CC} + 2V_{\rm NC} = -\frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.70)$$

The locations of both resonances depend on both N_e and N_n. If $\cos 2\vartheta_{12} > 0$, only the first resonance can occur in normal matter, where $N_n \simeq N_e/6$. A realization of the second resonance requires a large neutron number density, as that in a neutron star.

The neutrino spin oscillations in a transverse magnetic field with a possible rotation of the field-strength vector in a plane orthogonal to the neutrino propagation direction (such rotating fields may exist in the convective zone of the Sun) have been considered by Vidal and Wudka (1990), Smirnov (1991), Akhmedov, Petcov, and Smirnov (1993a), and Likhachev and Studenikin (1995). The effect of the magnetic-field rotation may substantially shift the resonance point of neutrino oscillations. Neutrino spin oscillations in electromagnetic fields with other different configurations, including a longitudinal magnetic field and the field of an electromagnetic wave, were examined by Akhmedov and Khlopov (1988a, 1988b), Egorov, Lobanov, and Studenikin (2000), Dvornikov and Studenikin (2001, 2004c), Lobanov and Studenikin (2001), and Studenikin (2004a, 2004b) (see also Appendix F).

It is possible to formulate a criterion (Likhachev and Studenikin, 1995) for finding out if the neutrino spin and spin-flavor precession is significant for given neutrino and background medium properties. The probability of oscillatory transitions between two neutrino states $\nu_{\ell L} \leftrightarrows \nu_{\ell' R}$ can be expressed in terms of the elements of the effective Hamiltonian matrices (6.63) and (6.67) as

$$P_{\nu_{\ell L} \leftrightarrows \nu_{\ell' R}} = \sin^2 \vartheta_{\rm eff} \sin^2 \frac{x\pi}{L_{\rm eff}}, \quad (6.71)$$

where

$$\sin^2 \vartheta_{\rm eff} = \frac{4\mathrm{H}_{\ell\ell'}^2}{4\mathrm{H}_{\ell\ell'}^2 + (\mathrm{H}_{\ell'\ell'} - \mathrm{H}_{\ell\ell})^2}, \quad (6.72)$$

$$L_{\rm eff} = \frac{2\pi}{\sqrt{4\mathrm{H}_{\ell\ell'}^2 + (\mathrm{H}_{\ell'\ell'} - \mathrm{H}_{\ell\ell})^2}}. \quad (6.73)$$

The transition probability can be of the order of unity if the following two conditions hold simultaneously: (1) the amplitude of the transition probability must be sizable (at least $\sin^2 \vartheta_{\rm eff} \gtrsim 1/2$), and (2) the neutrino path length in a medium with a magnetic field should be longer than half the effective length of oscillations $L_{\rm eff}$. In accordance with this criterion, it is possible to introduce the critical strength of a magnetic field $B_{\rm cr}$ which determines the region of field values $B_\perp > B_{\rm cr}$ at which the probability amplitude is not small ($\sin^2\vartheta_{\rm eff} > 1/2$):

$$B_{\rm cr} = \frac{1}{2\mu_{\ell'\ell}}\sqrt{(\mathrm{H}_{\ell'\ell'} - \mathrm{H}_{\ell\ell})^2}. \tag{6.74}$$

Consider, for instance, the case of $\nu_{eL} \leftrightarrows \nu_{aR}$ transitions of Majorana neutrinos. From Eqs. (6.67) and (6.74), it follows (Likhachev and Studenikin, 1995) that

$$B_{\rm cr} = \left|\frac{1}{2\mu_{ae}}\left(\frac{\Delta m^2\cos 2\vartheta_{12}}{2E_\nu} - \sqrt{2}G_{\rm F}N_{\rm eff}\right)\right|, \tag{6.75}$$

where $N_{\rm eff} = N_e - N_n$. For getting numerical estimates of $B_{\rm cr}$ it is convenient to rewrite Eq. (6.75) in the following form:

$$B_{\rm cr} \approx 43\frac{\mu_{\rm B}}{\mu_{ae}}\left|A\left(\frac{\Delta m^2}{\mathrm{eV}^2}\right)\left(\frac{\mathrm{MeV}}{E_\nu}\right) - 2.5\times10^{-31}\left(\frac{N_{\rm eff}}{\mathrm{cm}^{-3}}\right)\right|\mathrm{G}. \tag{6.76}$$

An interesting feature of the evolution equation (6.62) in the case of Majorana neutrinos is that the interplay of spin precession and flavor oscillations can generate $\nu_{eL} \to \nu_{eR}$ transitions (Akhmedov, 1991a). Since ν_{eR} interacts as right-handed Dirac antineutrinos, it is often denoted by $\bar{\nu}_{eR}$, or only $\bar{\nu}_e$, and called an "electron antineutrino." This state can be detected through the inverse β-decay reaction

$$\bar{\nu}_e + p \to n + e^+, \tag{6.77}$$

having a threshold $E_{\rm th} = 1.8$ MeV.

The possibility of $\nu_{eL} \to \bar{\nu}_{eR}$ transitions generated by a spin-flavor precession of Majorana neutrinos is particularly interesting for solar neutrinos, which experience matter effects in the interior of the Sun in the presence of the solar magnetic field (Pulido, 1992; Shi *et al.*, 1993). Taking into account the dominant $\nu_e \to \nu_a$ transitions due to neutrino oscillations, with ν_a given by Eq. (2.59), the probability of solar $\nu_{eL} \to \bar{\nu}_{eR}$ transitions is given by Akhmedov and Pulido (2003)

$$P_{\nu_{eL}\to\bar{\nu}_{eR}} \simeq 1.8\times10^{-10}\sin^2 2\vartheta_{12} \times\left(\frac{\mu_{ea}}{10^{-12}\mu_{\rm B}}\frac{B_\perp(0.05R_\odot)}{10\ \mathrm{kG}}\right)^2, \tag{6.78}$$

where μ_{ea} is the transition magnetic moment in Eq. (6.68), $R_\odot$ is the radius of the Sun, and the values of ϑ_{12} and ϑ_{23} are given in Table III.

It is also possible that spin-flavor precession occurs in the convective zone of the Sun, where there can be random turbulent magnetic fields (Miranda *et al.*, 2004a, 2004b; Friedland, 2005). In this case (Raffelt and Rashba, 2010),

$$P_{\nu_{eL}\to\bar{\nu}_{eR}} \approx 10^{-7}S^2\left(\frac{\mu_{ea}}{10^{-12}\mu_{\rm B}}\right)^2\left(\frac{B}{20\ \mathrm{kG}}\right)^2 \times\left(\frac{3\times10^4\ \mathrm{km}}{L_{\rm max}}\right)^{p-1}\left(\frac{8\times10^{-5}\ \mathrm{eV}^2}{\Delta m^2_{\rm S}}\right)^p \times\left(\frac{E_\nu}{10\ \mathrm{MeV}}\right)^p\left(\frac{\cos^2\vartheta_{12}}{0.7}\right)^p, \tag{6.79}$$

where S is a factor of the order of unity depending on the spatial configuration of the magnetic field, B is the average strength of the magnetic field at the spatial scale $L_{\rm max}$, which is the largest scale of the turbulence, p is the power of the turbulence scaling, $\Delta m^2_{\rm S}$ is the solar neutrino squared-mass difference in Table III, and E_ν is the neutrino energy. A possible value of p is $5/3$ (Miranda *et al.*, 2004a, 2004b; Friedland, 2005), corresponding to Kolmogorov turbulence. Conservative values for the other parameters are $B = 20$ kG and $L_{\rm max} = 3\times10^4$ km.

In 2002, the Super-Kamiokande Collaboration established for the flux of solar $\bar{\nu}_e$'s a 90% C.L., an upper limit of 0.8% of the SSM neutrino flux in the range of energy from 8 to 20 MeV (Gando *et al.*, 2003) by taking as a reference the BP00 SSM prediction $\phi^{\rm BP00}_{^8\rm B} = 5.05\times10^6\ \mathrm{cm^{-2}\,s^{-1}}$ for the solar ^{8}B flux (Bahcall, Pinsonneault, and Basu, 2001) and assuming an undistorted ^{8}B spectrum for the $\bar{\nu}_e$'s. This limit was improved in 2003 by the KamLAND Collaboration (Eguchi *et al.*, 2004) to 2.8×10^{-4} of the BP00 SSM prediction at 90% C.L. by measuring $\phi_{\bar{\nu}_e} < 370\ \mathrm{cm^{-2}\,s^{-1}}$ (90% C.L.) in the energy range 8.3–14.8 MeV, which corresponds to $\phi_{\bar{\nu}_e} < 1250\ \mathrm{cm^{-2}\,s^{-1}}$ (90% C.L.) in the entire ^{8}B energy range assuming an undistorted spectrum.

Recently, the Borexino Collaboration established the best limit on the probability of solar $\nu_{eL} \to \bar{\nu}_{eR}$ transitions (Bellini, 2011),

$$P_{\nu_{eL}\to\bar{\nu}_{eR}} < 1.3\times10^{-4} \qquad (90\%\ \mathrm{C.L.}), \tag{6.80}$$

by taking as a reference $\phi^{\rm SSM}_{^8\rm B} = 5.88\times10^6\ \mathrm{cm^{-2}\,s^{-1}}$ (Serenelli *et al.*, 2009) and assuming an undistorted ^{8}B spectrum for the $\bar{\nu}_e$'s. They measured $\phi_{\bar{\nu}_e} < 320\ \mathrm{cm^{-2}\,s^{-1}}$ (90% C.L.) for $E_{\bar{\nu}_e} > 7.3$ MeV, which corresponds to $\phi_{\bar{\nu}_e} < 760\ \mathrm{cm^{-2}\,s^{-1}}$ (90% C.L.) in the entire ^{8}B energy range assuming an undistorted spectrum

The implications of the limits on the flux of solar $\bar{\nu}_e$'s on Earth for the spin-flavor precession of solar neutrinos have been studied (Akhmedov and Pulido, 2003; Chauhan, Pulido, and Torrente-Lujan, 2003; Miranda *et al.*, 2004a, 2004b; Balantekin and Volpe, 2005; Friedland, 2005; Guzzo, de Holanda, and Peres, 2005; Yilmaz, 2008), taking into account the dominant $\nu_e \to \nu_\mu, \nu_\tau$ transitions due to neutrino oscillations (see Sec. II.D). Using Eqs. (6.78) and (6.80), we obtain

$$\mu_{ea} \lesssim 1.3\times10^{-12}\frac{7\ \mathrm{MG}}{B_\perp(0.05R_\odot)}\mu_{\rm B}, \tag{6.81}$$

with $600\ \mathrm{G} \lesssim B_\perp(0.05R_\odot) \lesssim 7\ \mathrm{MG}$ (Bellini, 2011). In the case of spin-flavor precession in the convective zone of the Sun with random turbulent magnetic fields, Eqs. (6.79) and (6.80) give, assuming $p = 5/3$,

$$\mu_{ea} \lesssim 4 \times 10^{-11} S^{-1} \frac{20 \text{ kG}}{B} \left(\frac{L_{\max}}{3 \times 10^4 \text{ km}} \right)^{1/3} \mu_{\text{B}}. \qquad (6.82)$$

The spin-flavor precession mechanism was also considered (Pulido, Chauhan, and Raghavan, 2005) in order to describe time variations of solar neutrino fluxes in gallium experiments. The effect of a nonzero neutrino magnetic moment is also of interest in connection with the analysis of helioseismological observations (Couvidat, Turck-Chieze, and Kosovichev, 2003).

The idea that the neutrino magnetic moment may solve the problem of the explosion of core-collapse supernovae, i.e., that the neutrino spin-flip transitions in a magnetic field can provide an efficient mechanism of energy transfer from a protoneutron star, was discussed by Fujikawa and Shrock (1980), Dar (1987), Nussinov and Rephaeli (1987), Barbieri and Mohapatra (1988), Goldman *et al.* (1988), Lattimer and Cooperstein (1988), and Voloshin (1988b). The possibility of a loss of up to one-half of the active left-handed neutrinos because of their transition to sterile right-handed neutrinos in strong magnetic fields at the boundary of the neutron star (the so-called boundary effect) was considered by Likhachev and Studenikin (1995).

The possibility to observe the effects of neutrino spin-flip transitions in terrestrial measurements of the neutrino flux of a core-collapse supernova was studied by Akhmedov and Fukuyama (2003), Ando and Sato (2003), Cuesta and Lambiase (2008), and Yoshida *et al.* (2009, 2011).

Recently de Gouvea and Shalgar (2012, 2013) studied the effects of spin-flavor precession on the evolution of neutrinos with Majorana transition magnetic moments inside the core of a supernova, where the magnetic field can be as large as 10^{12} G at a radius of about 50 km. The high neutrino density in the protoneutron star induces neutrino-neutrino interactions (Notzold and Raffelt, 1988) that lead to collective neutrino flavor oscillations (Duan and Kneller, 2009; Duan, Fuller, and Qian, 2010; Volpe, 2013). This effect can swap the spectrum of different flavor neutrinos and antineutrinos emerging from the supernova core above a "split" energy. de Gouvea and Shalgar (2012, 2013) studied the additional effects of spin-flavor precession by considering a Hamiltonian of the type in Eq. (6.67) with the addition of neutrino-neutrino interactions. They found that there can be collective spin-flavor oscillations in addition to the usual mass-generated collective neutrino oscillations, which can lead to spectral swaps between neutrinos and antineutrinos[15] for Majorana transition magnetic moments of the order of $10^{-21}\mu_{\text{B}}$. These are extremely small values for the Majorana transition magnetic moments, which are only 2 orders of magnitude larger than those predicted by the simplest extensions of the standard model [see Sec. IV.B, where it is explained that the Majorana transition magnetic moments are expected to have the same order of magnitude (4.10) of the Dirac transition magnetic moments]. This may be the only potentially observable phenomenon sensitive to such small values of the Majorana transition magnetic moments.

The neutrino spin (and spin-flavor) procession can be stimulated in the presence of moving matter when the matter speed transverse to the neutrino propagation is not zero or when matter is polarized. A detailed discussion of this phenomena can be found in Studenikin (2004a, 2004b) [see also Lobanov and Studenikin (2001)]. Note that these types of spin procession and the corresponding oscillations in matter occur without the presence of any electromagnetic field.

C. Magnetic moment in a strong magnetic field

The discussion of the neutrino electromagnetic properties in Sec. III is based on the one-photon approximation of the response of a neutrino to the presence of an electromagnetic field. This approximation is appropriate when the strength of the electromagnetic field is not too high. In the case of a very strong electromagnetic field one must take into account multiphoton contributions, which can be effectively incorporated in the neutrino form factors derived in Sec. III by allowing the form factors to depend on the strength of the external electromagnetic field. In this section we discuss the dependence of the effective neutrino magnetic moments on the strength of an external magnetic field, which was investigated by Borisov *et al.* (1985), Borisov, Zhukovsky, and Ternov (1987, 1988, 1989), and Masood *et al.* (2002) through the calculation of the self-energy of a neutrino in the presence of an arbitrary electromagnetic field. In the following we generalize the results of Borisov *et al.* (1985) in order to take into account neutrino mixing.

The evaluation of the dependence of the neutrino magnetic moments on the magnetic field is based on the Dirac-Schwinger equation for the wave function $\Psi_k(x)$ of a neutrino with mass m_k:

$$(i\partial_\mu \gamma^\mu - m_k)\Psi_k(x) = \int M_k(x, x'; \vec{B})\Psi_k(x')dx', \qquad (6.83)$$

where $M_k(x', x; \vec{B})$ is the neutrino mass operator in the presence of a magnetic field $\vec{B}$. The diagonal matrix element calculated on the mass shell ($p_k^2 = m_k^2$) between the neutrino vacuum states gives the radiative correction to the mass of the neutrino in the external field,

$$\Delta m_k = \frac{E_k}{m_k} \Delta E_k. \qquad (6.84)$$

The shift of the neutrino energy due to the presence of the external field is given by

$$\Delta E_k(\vec{B}) = \int dx dx' \overline{\psi_k}(x) M_k(x, x'; \vec{B}) \psi_k(x'), \qquad (6.85)$$

where $\psi_k(x) = (2E_k)^{-1/2} u(p_k) e^{-ip_k \cdot x}$ is the neutrino wave function in vacuum with four-momentum $p_k^\mu = (E_k, \vec{p})$ and energy $E_k = \sqrt{\vec{p}_k^2 + m_k^2}$. The radiative correction Δm_k to the

[15]In the traditional terminology, although strictly speaking in the Majorana case there is no difference between a neutrino and an antineutrino, it is common to call neutrinos the left-handed helicity states and antineutrinos the right-handed helicity states, which have the same weak interactions of the right-handed helicity states of Dirac antineutrinos.

neutrino mass in a constant electromagnetic field described by the tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ includes the Lorentz invariant $s_k^\mu \tilde{F}_{\mu\nu} p_k^\nu$ that depends on $\tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ and on the neutrino polarization vector [see, for instance, Akhiezer and Berestetskii (1965)]

$$s_k^\mu = \left(\frac{\vec{S} \cdot \vec{p}}{m_k}, \vec{S} + \frac{\vec{p}(\vec{S} \cdot \vec{p})}{m_k(E_k + m_k)} \right), \tag{6.86}$$

where $\vec{S}$ is the normalized neutrino spin vector in the rest frame.

The contribution to Δm_k proportional to the Lorentz invariant $s_k^\mu \tilde{F}_{\mu\nu} p_k^\nu$ is due to the interaction of the neutrino magnetic moment with the external field. Following Ritus (1972), for the real part of Δm_k one gets

$$\mathrm{Re}\Delta m_k = \frac{\mu_k}{m_k} s_k^\mu \tilde{F}_{\mu\nu} p_k^\nu, \tag{6.87}$$

where $\mu_k = \mu_{kk}$ are the diagonal magnetic moments of the massive neutrinos. In the neutrino rest frame we obtain

$$\mathrm{Re}\Delta m_k = -\mu_k(\vec{B} \cdot \vec{S}). \tag{6.88}$$

Using this equation one can extract from Δm_k the dependence of the effective magnetic moment on the field strength $B = |\vec{B}|$.

In the framework of the minimal extension of the standard model with right-handed neutrinos, the virtual one-loop processes $\nu_k \to e^- W^+ \to \nu_k$, $\nu_k \to \mu^- W^+ \to \nu_k$, and $\nu_k \to \tau^- W^+ \to \nu_k$ contribute to the mass operator

$$M_k(x, x'; \vec{B}) = -i \frac{g^2}{8} \sum_{\ell = e,\mu,\tau} |U_{\ell k}|^2 (1 - \gamma^5) \times \gamma_\mu S_\ell(x, x'; \vec{B}) \gamma_\nu (1 + \gamma^5) D_W^{\mu\nu}(x, x'; \vec{B}), \tag{6.89}$$

where $U_{\ell k}$ are the elements of the neutrino mixing matrix, $S_\ell(x, x'; \vec{B})$ and $D_W^{\mu\nu}(x, x'; \vec{B})$ are the charged leptons and W boson propagators in the presence of the external magnetic field $\vec{B}$, and g is the $\mathrm{SU}(2)_L$ weak-interaction coupling constant, which is related to the Fermi coupling constant by $G_\mathrm{F} = \sqrt{2} g^2 / 8 m_W^2$. Neglecting terms proportional to $m_k^2/m_\ell^2 \ll 1$ and considering a magnetic field $B \ll B_0^e = m_e^2/e \simeq 4.41 \times 10^{13}$ G, from a generalization of the results of Borisov *et al.* (1985) to the case of neutrino mixing we obtain

$$\mu_k(B) = \mu_k(0) \left[1 + \frac{4}{9} \left(\frac{B}{B_0^W} \right)^2 \sum_{\ell = e,\mu,\tau} |U_{\ell k}|^2 \ln \frac{m_W^2}{m_\ell^2} \right], \tag{6.90}$$

where $B_0^W = m_W^2/e \simeq 1.1 \times 10^{24}$ G. In this case the one-loop correction to the magnetic moment given by the external magnetic field is very small, because $(B/B_0^W)^2 \ll 10^{-22}$.

A significant difference of $\mu_k(B)$ from $\mu_k(0)$ is obtained when the strength of the magnetic field approaches B_0^W. For $B_0^W - B_0^e \ll B \lesssim B_0^W$ we have (Borisov *et al.*, 1985)

$$\mu_k(B) = \frac{2}{3} \mu_k(0) \ln \left(\frac{B_0^W}{B_0^W - B} \right) \sum_{\ell = e,\mu,\tau} |U_{\ell k}|^2 \frac{m_W^2}{m_\ell^2}. \tag{6.91}$$

The divergence of this expression for $B \to B_0^W$ must be treated with caution, because when the magnetic field B is close to the critical value B_0^W the vacuum becomes unstable with respect to $W^+ W^-$ pair production, giving rise to W boson condensation (Nielsen and Olesen, 1978; Skalozub, 1985, 1987; Ambjorn and Olesen, 1989).

We recall that very strong fields are supposed to exist in some astrophysical domains. For instance, magnetic fields of the order of 10^{16} G or even up to 10^{18} G can be produced in a supernova explosion or in the vicinity of magnetars, as discussed by Lai (2001), Akiyama *et al.* (2003), and Mereghetti (2008). For magnetar cores made of quark matter the interior magnetic field can reach values up to about 10^{20} G (Paulucci *et al.*, 2011). A more exotic possibility of super-strong magnetic fields is discussed by Ostriker, Thompson, and Witten (1986), where it is shown that magnetic fields stronger than 10^{30} G can be generated in the vicinity of superconducting cosmic strings.

Borisov *et al.* (1985) calculated also the dependence of the effective neutrino magnetic moment on the energy of a neutrino. In the case of a magnetic field which is not extremely strong ($B \ll B_0^W$), a neutrino with transverse momentum $p_\perp \gg m_W$ with respect to the magnetic field direction and

$$\frac{B}{B_0^\ell} \frac{p_\perp}{m_\ell} \gg \left(\frac{m_W}{m_\ell} \right)^3, \tag{6.92}$$

we have

$$\mu_k(B) = \frac{3^{5/6} \Gamma^4(1/3)}{20\pi} \mu_k(0) \times \sum_{\ell = e,\mu,\tau} |U_{\ell k}|^2 \left(\frac{B p_\perp}{B_0^\ell m_\ell} \right)^{-2/3} \left(\frac{m_W}{m_\ell} \right)^2, \tag{6.93}$$

where $B_0^\ell = m_\ell^2/e$. In this case, the magnetic moment of a neutrino with very high energy decreases to zero with the increase of the neutrino energy.

We finally recall the studies of the neutrino self-energy and electromagnetic vertex in matter without and with a magnetic field. The neutrino self-energy and the electromagnetic vertex function in matter without a magnetic field have been studied by Notzold and Raffelt (1988), D'Olivo, Nieves, and Pal (1989), and Nieves and Pal (1989). The vacuum dispersion relation in the presence of a constant magnetic field has been studied by Erdas and Feldman (1990). Finite-temperature corrections to the neutrino self-energy in a background medium without magnetic field have been calculated by D'Olivo, Nieves, and Torres (1992). Those in the presence of an electromagnetic field have been calculated by Zhukovsky, Shoniya, and Eminov (1993), Esposito and Capone (1996), and Nieves (2003). The general expressions for the neutrino dispersion relation in a magnetized plasma with wide ranges of temperature, chemical potential, and magnetic field strengths have been derived by Elmfors, Grasso, and Raffelt (1996), and Elizalde, Ferrer, and de la

Incera (2002, 2004). The one-loop thermal self-energy of a neutrino in an arbitrary strong magnetic field has been calculated by Erdas, Kim, and Lee (1998) and Erdas and Isola (2000). These calculations of the effective neutrino properties in a magnetized plasma are useful for the study of the behavior of neutrinos in the early Universe.

D. Beta decay of the neutron in a magnetic field

The first studies of neutrino interactions in the presence of external electromagnetic fields were performed by Korovina (1964) and Ternov, Lysov, and Korovina (1965), who considered the β decay $n \to p + e^- + \bar{\nu}_e$ of a polarized neutron in a magnetic field.[16] It was shown that the differential rate of the process exhibits resonance spikes which appear when the final electron energy is equal to one of the allowed Landau energies in the magnetic field. It was also shown that the total rate depends on the initial neutron polarization and that the neutrino emission is asymmetric. The range of magnetic field strengths considered in these papers spans up to subcritical fields $B \lesssim B_0^e = m_e^2/e \simeq 4.41 \times 10^{13}$ G. It is worth noting that these studies were performed before the discovery by Hewish *et al.* (1968) of pulsars, where such strong magnetic fields are believed to exist.

In two papers by Fassio-Canuto (1969) and Matese and O'Connell (1969), published a few years later, the results of Korovina (1964) and Ternov, Lysov, and Korovina (1965) for the neutron decay rate in a magnetic field were rederived, but there was no discussion of the asymmetry in the neutrino emission.

Very strong magnetic fields are also supposed to exist in the early Universe (Grasso and Rubinstein, 2001). As first discussed by Greenstein (1969) and Matese and O'Connell (1970), the weak reaction rates of the URCA processes

$$\begin{aligned} &n \to p + e^- + \bar{\nu}_e, \qquad \nu_e + n \leftrightarrows e^- + p, \\ &p + \bar{\nu}_e \leftrightarrows n + e^+, \end{aligned} \tag{6.94}$$

which determine the conversions between neutrons and protons and set the n/p ratio in various environments, can be significantly modified under the influence of magnetic fields. This can be important for big-bang nucleosynthesis and neutron star cooling (Cheng, Schramm, and Truran, 1993).

The aforementioned studies of neutrino interactions in the presence of magnetic fields performed by Korovina (1964), Ternov, Lysov, and Korovina (1965), Fassio-Canuto (1969), Greenstein (1969), and Matese and O'Connell (1969, 1970) created neutrino astrophysics in magnetic fields.

The β-decay process can be described by the well-known four-fermion Lagrangian

$$\mathcal{L} = \frac{\tilde{G}}{\sqrt{2}}[\bar{\psi}_p\gamma_\mu(1 + g_A\gamma_5)\psi_n][\bar{\psi}_e\gamma^\mu(1 + \gamma_5)\psi_\nu], \tag{6.95}$$

where $\tilde{G} = G_{\rm F}\cos\theta_{\rm C}$, $\theta_{\rm C}$ is the Cabibbo angle, and $g_A \simeq 1.27$ (Beringer *et al.*, 2012) is the axial coupling constants. After standard calculations one can obtain the neutron decay rate

$$\Gamma = \sum_{\substack{\text{phase}\\\text{space}}} |M|^2 \delta(E_n - E_p - E_e - E_\nu), \tag{6.96}$$

where the matrix element

$$M = \frac{\tilde{G}}{\sqrt{2}} \int d^4x [\bar{\psi}_p\gamma_\mu(1 + g_A\gamma_5)\psi_n][\bar{\psi}_e\gamma^\mu(1 + \gamma_5)\psi_\nu] \tag{6.97}$$

accounts for the influence of the magnetic field through the wave functions of the electron and proton. For the electron wave function one has to use the exact solutions of the Dirac equation in the magnetic field given in Appendix H by Eqs. (H8), (H11), (H12), and (H13). The wave function for a proton has similar form and is given by Studenikin (1989). The initial neutron and neutrino are supposed to not be directly affected by the magnetic field and the plane waves are used for these particle wave functions.

The argument of the δ function in Eq. (6.96), being equated with zero, gives the law of energy conservation for the particles in the process that for the case of the neutron decay at rest is

$$m_n = \sqrt{m_e^2 + 2eBN_e + p_e^{3\,2}} + \sqrt{m_p^2 + 2eBN_p + p_p^{3\,2}} + E_\nu, \tag{6.98}$$

where N_e and N_p are the numbers of Landau levels in the magnetic field for the electron and proton. The summation in Eq. (6.96) is performed over the phase space of the final particles: $\vec{p}_\nu, p_p^2, p_p^3, N_p, s_p, p_e^2, p_e^3, N_e, s_e$, where values $s_e, s_p = \pm 1$ denote the two possible spin states of the electron and proton. For not very strong magnetic fields $B < B_{\rm cr} = (\Delta^2 - m_e^2)/2e \simeq 1.8 \times 10^{14}$ G, where $\Delta = m_p - m_n$, the decay rate is

$$\begin{aligned} \Gamma(B) = \frac{\Gamma(0)}{2} \int \sin\theta_\nu d\theta_\nu \bigg\{ 1 + \frac{2(g_A^2 + g_A)}{1 + 3g_A^2} s_n \cos\theta_\nu \\ - 4.9 \frac{eB}{\Delta^2} \left(\frac{g_A^2 - 1}{1 + 3g_A^2} \cos\theta_\nu + \frac{2(g_A^2 - g_A)}{1 + 3g_A^2} s_n \right) \bigg\}, \end{aligned} \tag{6.99}$$

where θ_ν is the angle between the neutrino propagation and the magnetic field vector and $\Gamma(0)$ is the decay rate of the neutron in the absence of the magnetic field, given by

$$\Gamma(0) = 0.47 \frac{\tilde{G}^2 \Delta^5}{120\pi^3}(1 + 3g_A^2), \tag{6.100}$$

where $s_n = \pm 1$ correspond to the neutron spin polarization parallel or antiparallel to the magnetic field vector.

From Eq. (6.99) it follows that there is an asymmetry in the spatial distribution of neutrinos. This asymmetry is due to the parity violation in weak interactions and it is modified by the magnetic field presence. In addition, as it is also clear from Eq. (6.99) the average momentum of antineutrinos on the magnetic field strength and the direction of propagation with

[16] This process and the other URCA processes in Eq. (6.94) are important for the energy loss of stars (Gamow and Schoenberg, 1941). The name URCA was inspired to George Gamow and Mario Schoenberg by that of the casino in Rio de Janeiro, joking on the rapidity of energy disappearance in the nucleus of a supernova, which is as quick as the disappearance of money at a roulette table.

respect to the magnetic field vector. That is why we consider the total effect of the antineutrino spatial distribution asymmetry as the neutrino electromagnetic properties manifestation.

Note that the same asymmetry appears in the case of much stronger magnetic fields $B > B_{\rm cr}$ as well as for other similar processes (6.94). Recently the relativistic approach to the inverse β decay of a polarized neutron $\nu_e + n \to p + e^-$ in a magnetic field was developed by Shinkevich and Studenikin (2005).[17] It was shown that in strong magnetic fields the cross section can be highly anisotropic with respect to the neutrino angle. In the particular case of polarized neutrons, matter becomes even transparent for neutrinos if neutrinos propagate against the direction of neutrons polarization.

It was first claimed by Chugai (1984), Dorofeev, Rodionov, and Ternov (1984, 1985), and Zakhartsov and Loskutov (1985) that asymmetric neutrino (antineutrino) emission in the direct URCA processes (6.94) during the first seconds after a magnetized massive star collapse could provide explanations for the observed pulsar velocities. As shown by Studenikin (1988), in order to get a correct prediction for the direction and value of the kick velocity of a pulsar one has to account not only for the amount of neutrinos radiated in the processes (6.94) but also for the fact that the values of the average momentum of neutrinos propagating in the opposite directions are not the same. More detailed studies of the neutrino asymmetry in relation to magnetized stars have been performed by Leinson and Perez (1998), Lai and Qian (1998), Arras and Lai (1999), Goyal (1999), Gvozdev and Ognev (1999), Roulet (1998), Duan and Qian (2004), and Kauts, Savochkin, and Studenikin (2006).

We recall also different other mechanisms for the asymmetry in the neutrino emission from a magnetized pulsar studied by Bisnovatyi-Kogan (1993), Kusenko and Segre (1996), Akhmedov, Lanza, and Sciama (1997), and Lai and Qian (1998). For more complete references to the performed studies on the neutrino mechanisms of the pulsar kicks see the introductions presented by Bhattacharya and Pal (2004) and Shinkevich and Studenikin (2005). Presently there is no solid explanation for the observed pulsars kick velocities. Thus, the origin of pulsar kicks is still an unsolved problem (Tamborra *et al.*,2014). The phenomenon seems to be very complicated and is probably the result of different mechanisms which are acting simultaneously. One of these mechanisms can be the neutrino asymmetry considered in this section.

E. Neutrino pair production by an electron

It is well known that in the presence of external electromagnetic fields, particle interaction processes, that are forbidden in vacuum, become possible. One may consider the corresponding processes of neutrinos interaction with real particles that could become possible only under the influence of external electromagnetic fields as manifestation of neutrino electromagnetic properties.

One of these processes is the production of a neutrino-antineutrino pair by an electron moving in a constant magnetic field

$$e \to e + \nu_e + \bar{\nu}_e. \tag{6.101}$$

Astrophysical significance of this process, termed the synchrotron radiation of neutrinos, was discussed by Landstreet (1967). Here it is worth noting that the possibility of $\nu\bar{\nu}$ emission by an electron through the bremsstrahlung process on a nuclei

$$e + A \to e + A + \nu_e + \bar{\nu}_e \tag{6.102}$$

was first discussed by Pontecorvo (1959) who also pointed out that for certain stages of a star evolution the proposed mechanism of $\nu\bar{\nu}$ emission might be important.

In vacuum, i.e., in the absence of the magnetic field, the process (6.101) is obviously forbidden. The dependence of the rate of the process (6.101) on the magnetic field was initially derived by Baier and Katkov (1966), Loskutov and Zakhartsov (1969), and Ritus (1969) within the local four-fermion weak-interaction model of Gell-Mann-Feynman. In the Weinberg-Salam model this process was considered by Ternov, Rodionov, and Studenikin (1982, 1983). In the low-energy approximation of the model for the amplitude of the process (6.101) we used

$$M = -\frac{G_{\rm F}}{\sqrt{2}}\overline{\psi'_e}\gamma_\mu(g_V + g_A\gamma_5)\psi_e\bar{\psi}_{\nu_1}\gamma^\mu(1+\gamma_5)\psi_{\nu_2}, \tag{6.103}$$

where ψ_e and ψ'_e are the initial and final electron wave functions and ψ_{ν_1} and ψ_{ν_2} are the two neutrino wave functions. In the case of the electron $\nu\bar{\nu}$ pair emission in Eq. (6.101), $g_V = \sin^2\theta_W + 1/2$ and $g_A = 1/2$. The effect of a constant magnetic field presence is accounted for by the wave functions of the initial and final electron that are the exact solutions of the Dirac equation in magnetic field given in Appendix H. Performing standard calculations accounting for the rotational symmetry of the problem with respect to the magnetic field $\vec{B}$ oriented along the z axis one arrives at the rate given by Ternov, Rodionov, and Studenikin (1982, 1983)

$$\begin{aligned}\Gamma = \frac{G_{\rm F}^2}{3(2\pi)^2}\sum_N\int_{|\vec{f}|\le f_0} d^3f[f_0^2H_{00} \\ - (f_0^2 - |\vec{f}|^2)(H_{00} - H_{11} - H_{22} - H_{33}) \\ + |\vec{f}|^2(H_{22}\sin^2\theta + H_{33}\cos^2\theta) \\ - 2f_0|\vec{f}|(H_{20}\sin\theta + H_{30}\cos\theta) + 2|\vec{f}|^2H_{32}\cos\theta\sin\theta],\end{aligned} \tag{6.104}$$

where the sum is performed over the Landau quantum number of the final electron $f^\mu = (f_0, \vec{f}) = p_\nu^\mu + p_{\bar{\nu}}^\mu = p_e^\mu - p_e'^\mu$, and θ is the angle between $\vec{f}$ and $\vec{B}$. The matrix elements $H_{\alpha\beta} = j_\alpha j_\beta^*$ are determined by the electron currents

[17] This process is also important for the neutrino transport inside the magnetized pulsar and contributes to the kick velocities, as shown by Roulet (1998), Bhattacharya and Pal (2004), and Duan and Qian (2004).

$$j_\alpha = \int dxdy\overline{\psi'_e}\gamma_\alpha(g_V + g_A\gamma_5)\psi_e \times \exp\{-i[(\varkappa_1+\eta_1)x + (\varkappa_2+\eta_2)y]\}, \quad (6.105)$$

where $\varkappa_i$ and η_i are the corresponding neutrino and antineutrino momenta components. The functions $H_{\alpha\beta}$ are expressed in terms of quadratic combinations of Laguerre functions which depend on the argument $\rho = |\vec{f}|^2\sin^2\theta/(2eB)$. In the case of the ultrarelativistic electron energies the resulting expressions for the rate depend on the electromagnetic field dynamical parameter

$$\chi = \frac{e\sqrt{(F_{\mu\nu}p^\nu)^2}}{m_e^2} = \frac{B}{B_0^e}\frac{p_0}{m_e}. \quad (6.106)$$

Integration in Eq. (6.104) can be performed analytically. The final expressions for the rate Γ were obtained by Ternov, Rodionov, and Studenikin (1982, 1983):

$$\Gamma = \frac{G_F^2 m_e^6 \chi^5}{1152\sqrt{3}\pi^3 p_0}[49g^2 + 437g_A^2], \quad (6.107)$$

for $\chi \ll 1$ and

$$\Gamma = \frac{G_F^2 m_e^6 \chi^2}{216\pi^3 p_0}(g_V^2 + g_A^2)\left[\ln\chi - C - \frac{\ln 3}{2} - \frac{5}{6}\right], \quad (6.108)$$

for $\chi \gg 1$, where $C = 0.577$ is the Euler constant.

From Eqs. (6.107) and (6.108) one can see that the rate is governed by the value of the parameter χ. It follows that the rate is significantly dependent on the magnetic field strength and the initial electron energy. Therefore, for ultrarelativistic energies and strong enough magnetic fields the $\nu\bar{\nu}$ synchrotron radiation by an electron can be important for astrophysics.

As demonstrated, for instance, by Kaminker *et al.* (1992), more consistent consideration of the process $e \to e + \nu_e + \bar{\nu}_e$ appropriate for astrophysical applications implies account for the presence of background matter in addition to an external magnetic field.

F. Neutrino pair production by a strong magnetic field

Over the years, starting from the observation of Klein (1929), it has been known that the vacuum is not stable under the influence of an external electric field. Schwinger (1951) showed that electron-positron pairs can be produced from the vacuum in the presence of a strong electric field, with a strength that exceeds the critical value $E_{\rm cr} = m_e^2/e$. It is also known that under the influence of a homogeneous magnetic field the vacuum is stable, because such a field does not produce work. On the contrary, the presence of a strong inhomogeneous magnetic field can produce an instability of the vacuum with respect to neutral fermion-antifermion pair creation if the fermion has a magnetic moment.

The interest in neutral particle-antiparticle pair creation from the vacuum through the Pauli interaction of a magnetic moment with external electromagnetic fields was raised by Lee and Yoon (2006, 2007), and Lin (1999). Recently Lee and Yoon (2008) and Lee (2011) discussed the vacuum instability in a strong magnetic field due to neutrino-antineutrino pair production through the Pauli interaction. However, their results are questionable, because they admit the creation of neutrino-antineutrino pairs from the vacuum in a homogeneous magnetic field.

Gavrilov and Gitman (2013) recently presented a nonperturbative calculation of neutrino-antineutrino pair creation in a strong inhomogeneous magnetic field in the framework of quantum field theory. In particular, they showed that in specific cases (appropriate to typical astrophysical applications) the problem can be technically reduced to the problem of charged-particle creation by an electric field.

Considering a generic neutrino ν with mass m_ν and magnetic moment μ_ν, the neutrino states in a magnetic field are described by the Dirac-Pauli equation

$$\left\{i\partial_\mu\gamma^\mu - m_\nu + \frac{\mu_\nu}{2}\sigma_{\alpha\beta}F^{\alpha\beta}\right\}\Psi_\nu(x) = 0, \quad (6.109)$$

where $\Psi_\nu(x)$ is the neutrino wave function and $F^{\alpha\beta}$ is the electromagnetic field tensor. Gavrilov and Gitman (2013) showed that the energy spectrum of a neutrino that interacts with an inhomogeneous magnetic field through a magnetic moment consists of two branches separated by a gap. Considering a magnetic field which is linearly growing on a given spatial interval L, they demonstrated that the rate of pair creation is determined by the gradient of the magnetic field.

A first condition for neutrino-antineutrino pair production in a magnetic field B is that the magnetic energy must be enough to create a neutrino-antineutrino pair, i.e., $\mu_\nu B > 2m_\nu$. Therefore, the minimum value of the magnetic field for which neutrino-antineutrino pairs are created is

$$B_{\rm cr} = 2\frac{m_\nu}{\mu_\nu} \simeq 3.4\times 10^8\left(\frac{m_\nu}{\rm eV}\right)\left(\frac{\mu_B}{\mu_\nu}\right)\ {\rm G}. \quad (6.110)$$

Magnetic fields generated during a supernova explosion or in the vicinity of magnetars can be of the order of $10^{15} - 10^{16}$ G or even stronger, up to about 10^{18} G. In this extreme case, neutrino-antineutrino pair production can occur for $\mu_\nu \sim 10^{-12}\mu_B$ and $m_\nu \lesssim 10^{-2}$ eV. However, it is also necessary to have a large gradient B' of the magnetic field. Considering a magnetic field which is linearly growing in a spatial interval L, Gavrilov and Gitman (2013) obtained the condition $|\mu_\nu B'| \gtrsim m_\nu^2$, which can be written as $|B'| \gtrsim m_\nu B_{\rm cr}$. Then for the maximum value $B_{\rm max}$ of the magnetic field in the spatial interval L we have the condition

$$|B_{\rm max}| \gtrsim L m_\nu B_{\rm cr}. \quad (6.111)$$

Hence, if the magnetic field is larger than $B_{\rm cr}$ as required by the first condition above, neutrino-antineutrino pair production can occur if the size L over which the magnetic field raises to such large values is small enough:

$$L \lesssim 10^{-10}\left(\frac{|B_{\rm max}|}{B_{\rm cr}}\right)\left(\frac{\rm eV}{m_\nu}\right)\ {\rm km} \sim 10^{-18}\left(\frac{|B_{\rm max}|}{\rm G}\right)\left(\frac{\rm eV}{m_\nu}\right)^2\left(\frac{\mu_\nu}{\mu_{\rm B}}\right)\ {\rm km}. \quad (6.112)$$

Even considering the large values $|B_{\rm max}| \sim 10^{18}$ G and $\mu_\nu \sim 10^{-12}\mu_{\rm B}$, we need $m_\nu \lesssim 10^{-6}$ eV in order to obtain a distance of the order of a kilometer, which may be appropriate for the spatial size of the magnetic field variations in a supernova explosion or in the vicinity of magnetars. Figure 3 shows that neutrino oscillation data allow one of the massive neutrinos to be very light and even massless. Hence, there can be pair production of the lightest neutrino in extreme astrophysical environments if its mass is very small and its magnetic moment is very large. This is a condition which is contrary to the usual proportionality between the neutrino mass and the neutrino magnetic moment and requires the intervention of powerful new physics beyond the standard model, as explained in Sec. IV.F.

G. Energy quantization in rotating media

In Sec. VII.A we discuss the possibility of nonzero neutrino electric charge that is predicted in a set of standard model extensions. If a neutrino is really a millicharged particle, in the presence of a constant magnetic field it behaves in a way similar to an electron. In particular, the energy of a millicharged neutrino is quantized in a magnetic field (see Appendix H)

$$p_0^\nu = \sqrt{m_\nu^2 + p_3^2 + 2\mathbb{q}_\nu B N_\nu}, \quad (6.113)$$

where $\mathbb{q}_\nu$ is a millicharge of the neutrino and $N_\nu = 0, 1, 2, \ldots$ is the Landau number of the millicharged neutrino energy levels. The corresponding radius of the neutrino classical orbits in the magnetic field is given by Balantsev, Popov, and Studenikin (2011)

$$\langle R_B^\nu \rangle = \sqrt{\frac{2N_\nu}{\mathbb{q}_\nu B}}. \quad (6.114)$$

It is interesting to compare the radius of classical orbits in a magnetic field of the millicharged neutrino $\langle R_B^\nu \rangle$ with that of the electron $\langle R_B^e \rangle$. If the relativistic electron and millicharged neutrino are moving with the same energy in a constant magnetic field then the ratio of orbits radiuses is equal to the inverse ratio of electric charges

$$\frac{\langle R_B^\nu \rangle}{\langle R_B^e \rangle} = \frac{e}{\mathbb{q}_\nu}, \quad (6.115)$$

if for both particles the momentum components along the magnetic field vector are zero. From the obtained estimation for the ratio of orbits radiuses, taking into account existing experimental constraints on neutrino millicharge, we conclude that for the same strength of the external magnetic field the motion of a charged neutrino is much less localized as compared with an electron motion.

The same method of wave equation exact solutions that is used in studies of charged particles under the influence of external electromagnetic fields (including millicharged neutrinos and neutrinos with nonzero magnetic moment, see previous discussions of this section and Appendix H), as has been explicitly demonstrated by Studenikin and Ternov (2005) and Studenikin (2008), can also be used for investigations of neutrinos moving in the background matter. In particular, using the method of exact solutions for a neutrino wave function in the presence of matter it has been shown by Grigoriev, Savochkin, and Studenikin (2007) and Studenikin (2008) that the energy spectrum of a neutrino moving in a rotating media is quantized. This effect is very similar to charged particles energy quantization in a magnetic field.

The neutrino wave function exactly accounting for the neutrino interaction with matter can be obtained by solving the modified Dirac equation given by Studenikin and Ternov (2005) (see Appendix H),

$$\{i\gamma_\mu\partial^\mu - \tfrac{1}{2}\gamma_\mu(1+\gamma_5)f^\mu - m_\nu\}\Psi(x) = 0. \quad (6.116)$$

In case an electron neutrino is propagating through a rotating matter composed of neutrons then the matter potential, according to Balantsev, Popov, and Studenikin (2009, 2011), is

$$f^\mu = -G(n, n\vec{v}), \qquad \vec{v} = (-\omega y, \omega x, 0), \quad (6.117)$$

where ω is the angular frequency of matter rotation around the z axis and $G = G_{\rm F}/\sqrt{2}$. The neutrino energy spectrum obtained by Balantsev, Popov, and Studenikin (2009, 2011)

$$p_0 = \sqrt{m_\nu^2 + p_3^2 + p_\perp^2} - Gn \quad (6.118)$$

contains the transverse momentum

$$p_\perp = 2\sqrt{NGn\omega}, \qquad N = 0, 1, 2, \ldots \quad (6.119)$$

that is quantized (Grigoriev, Savochkin, and Studenikin, 2007). The quantum number N also determines the radius of classical orbits of neutrino in rotating matter (it is supposed that $N \gg 1$ and $p_3 = 0$),

$$R = \sqrt{\frac{N}{Gn\omega}}. \quad (6.120)$$

It was shown by Studenikin (2008) that for low-energy neutrinos it can be $R \sim R_{NS} = 10$ km that might be thought to be of interest in applications for neutron stars.

It is interesting to note that within the quasiclassical approach the neutrino binding on circular orbits is due to an effective force that is orthogonal to the particle speed. And an analogy between a charged-particle motion in a magnetic field and a neutrino motion in a rotating matter can be established (Studenikin, 2008). It is possible to explain the neutrino quasiclassical circular orbits as a result of action of the attractive central force,

$$\vec{F}_m^{(\nu)} = q_m^{(\nu)}\vec{\beta}\times\vec{B}_m, \quad \vec{B}_m = \vec{\nabla}\times\vec{A}_m, \quad \vec{A}_m = n\vec{v}, \qquad (6.121)$$

where the neutrino effective "charge" in matter (composed of neutrons in the discussed case) is $q_m^{(\nu)} = -G$, whereas $\vec{B}_m$ and $\vec{A}_m$ play the roles of effective "magnetic" field and the correspondent "vector potential." Like the magnetic part of the Lorentz force, $\vec{F}_m^{(\nu)}$ is orthogonal to the speed $\vec{\beta}$ of the neutrino.

For the most general case the "matter-induced Lorentz force" is given by

$$\vec{F}_m^{(\nu)} = q_m^{(\nu)}\vec{E}_m + q_m^{(\nu)}\vec{\beta}\times\vec{B}_m, \qquad (6.122)$$

where the effective electric and magnetic fields are, respectively,

$$\vec{E}_m = -\vec{\nabla}n - \vec{v}\frac{\partial n}{\partial t} - n\frac{\partial \vec{v}}{\partial t}, \qquad (6.123)$$

and

$$\vec{B}_m = n\vec{\nabla}\times\vec{v} - \vec{v}\times\vec{\nabla}n. \qquad (6.124)$$

The force acting on a neutrino, produced by the first term of the effective electric field in the neutron matter, was considered also by Loeb (1990) and the quasiclassical treatment of a neutrino motion in the electron plasma was considered by Mendonca *et al.* (1998).

Note that while considering a neutrino effective electromagnetic interaction with media an effective electric charge of the neutrino was introduced by Oraevsky, Semikoz, and Smorodinsky (1986, 1994), Oraevsky and Semikoz (1987), Nieves and Pal (1994), Mendonca *et al.* (1998), Bhattacharya, Ganguly, and Konar (2001), Nieves (2003), and Studenikin (2008).

In the most general case the description of the millicharged neutrino with anomalous magnetic moment motion in the presence of matter and external electromagnetic fields can be obtained by solving the modified Dirac equation

$$\left\{\gamma_\mu(p^\mu + \mathbb{q}_\nu A^\mu) - \frac{1}{2}\gamma_\mu(1+\gamma_5)f^\mu - \frac{i}{2}\mu_\nu\sigma_{\mu\nu}F^{\mu\nu} - m_\nu\right\}\Psi(x) = 0, \qquad (6.125)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, A^μ is the electromagnetic field potential, and μ_ν is the neutrino anomalous magnetic moment. For several particular cases this equation can be solved exactly and the neutrino wave functions and the corresponding energy spectra can be found (Grigoriev, Savochkin, and Studenikin, 2007; Balantsev, Studenikin, and Tokarev, 2012, 2013; Studenikin and Tokarev, 2014). In particular, for a neutrino moving in a rotating matter with the potential

$$f^\mu = -GN_n(1, -\epsilon y\omega, \epsilon x\omega, 0) \qquad (6.126)$$

and superimposed constant electric $\vec{E}$ and magnetic field $\vec{B}$, $\epsilon = \pm 1$ corresponds to parallel and antiparallel directions of vectors $\vec{\omega}$ and $\vec{B}$, and for the neutrino energy spectrum we obtain

$$p_0 = \sqrt{p_3^2 + 2N|2GN_n\omega - \epsilon\mathbb{q}_\nu B| + m_\nu^2} - GN_n - \mathbb{q}_\nu\phi, \qquad (6.127)$$

where ϕ is the scalar potential of the electric field. In this case the generalized effective Lorentz force introduced by Studenikin (2008) is

$$\vec{F}_{\text{eff}} = \mathbb{q}_{\text{eff}}\vec{E}_{\text{eff}} + \mathbb{q}_{\text{eff}}[\vec{\beta}\times\vec{B}_{\text{eff}}]. \qquad (6.128)$$

Here $\vec{\beta}$ is the neutrino speed and

$$\begin{aligned} \mathbb{q}_{\text{eff}}\vec{E}_{\text{eff}} &= \mathbb{q}_m\vec{E}_m + \mathbb{q}_\nu\vec{E}, \\ \mathbb{q}_{\text{eff}}\vec{B}_{\text{eff}} &= |\mathbb{q}_m B_m + \epsilon\mathbb{q}_\nu B|\vec{e}_z, \end{aligned} \qquad (6.129)$$

where $\mathbb{q}_m, \vec{B}_m$, and $\vec{E}_m$ are the matter-induced charge, electric, and magnetic fields correspondingly,

$$\mathbb{q}_m = -G, \qquad \vec{E}_m = -\vec{\nabla}N_n, \qquad \vec{B}_m = -2N_n\vec{\omega}. \qquad (6.130)$$

Note that the effective Lorentz force (6.128), that directly follows from the exact form of the obtained energy spectrum (6.127), is generated by both weak and electromagnetic interactions. The effect of the millicharged neutrino energy quantization in a rotating magnetized matter was discussed by Grigoriev, Savochkin, and Studenikin (2007) and Studenikin (2008), where it is shown that the neutrino trapping in circular orbits exists due to the neutrino millicharge interaction with the magnetic field and also due to neutrino weak interaction with the rotating matter.

Under the influence of the effective Lorentz force (6.129) the neutrino will move with acceleration given by (Studenikin, 2008)

$$\vec{a} = \frac{1}{m_\nu}(G\vec{\nabla}N_n + \mathbb{q}_\nu\vec{\nabla}\phi + |2GN_n\omega - \epsilon\mathbb{q}_\nu B|\vec{\beta}\times\vec{e}_z), \qquad (6.131)$$

where $\vec{e}_z$ is a unit vector in the direction of the magnetic field and matter rotation. The accelerated neutrino should produce the electromagnetic radiation. In the quasiclassical treatment the radiation power of induced electromagnetic radiation is given by

$$I_{LC\nu} = \frac{2\mathbb{q}_\nu^2}{3}\left(\frac{\vec{a}^2}{(1-|\vec{\beta}|^2)^2} + \frac{(\vec{a}\cdot\vec{\beta})^2}{(1-|\vec{\beta}|^2)^3}\right). \qquad (6.132)$$

Such a mechanism of the neutrino electromagnetic radiation due to the neutrino millicharge, that can be emitted in the presence of the nonuniform rotating matter and electromagnetic fields, is termed in Studenikin and Tokarev (2014) the "light of (milli)charged neutrino" ($LC\nu$). It should be stressed that the phenomenon exists even in the absence of the electromagnetic fields, when the acceleration (6.131) is produced due only to the weak interactions of neutrinos with the background particles, so that the discussed mechanism is of a different nature than that of the cyclotron radiation of a charged particle in magnetic fields.

The $LC\nu$ mechanism manifests itself during the neutrino propagation from the central part of a rotating neutron star outward through the crust. The gradient of the matter density (the density variation along the neutrino path) gives the following contribution to the $LC\nu$ radiation power [see Eq. (6.131)]:

$$I_{LC\nu} = \frac{2\mathbb{q}_\nu^2}{3m_\nu^2}(G\vec{\nabla}N_n)^2, \tag{6.133}$$

and the effect of the matter rotation yields

$$I_{LC\nu} = \frac{2\mathbb{q}_\nu^2\gamma^2}{3m_\nu^2}(-\epsilon\mathbb{q}_\nu B + 2GN_n\omega)^2, \tag{6.134}$$

where $\gamma = (1-|\vec{\beta}|^2)^{-1/2}$. The numerical estimations, that account for the $LC\nu$ power for the present limits on the neutrino millicharge and for a realistic gradient of a neutron star matter density $|G\vec{\nabla}N_n| \sim 1$ eV/1 km and the rotation frequency $\omega \sim 2\pi \times 10^3$ s^{-1}, show that the role of the $LC\nu$ in the explosion energetics is negligible with respect to the total energy of the collapse. However, as discussed in Sec. VII (Oraevsky, Semikoz, and Smorodinsky, 1994; Nieves, 2003; Duan and Qian, 2004), in the presence of a dense plasma the induced neutrino effective electric charge can be reasonably large. In addition, the phenomenon is of interest for astrophysics in light of the recently reported measurement of ultra-high-energy PeV neutrinos in the IceCube experiment (Aartsen *et al.*, 2013a, 2013b, 2014).

VII. CHARGE AND ANAPOLE FORM FACTORS

The magnetic and electric dipole moments are the most studied electromagnetic properties in theoretical and experimental works, but some attention has also been devoted to the possibility that neutrinos have very small electric charges, usually called “millicharges.” Moreover, even if neutrinos are exactly neutral, they can have nonzero charge radii, which can be probed in scattering experiments. In Secs. VII.A and VII.B we review the theory of electric charge and charge radius, respectively, and we present the corresponding experimental limits. In Sec. VII.C we discuss the neutrino anapole moment, which is the less known neutrino electromagnetic property.

A. Neutrino electric charge

It is usually believed (Bernstein, Ruderman, and Feinberg, 1963) that the neutrino electric charge is exactly zero. This is true in the standard model, but in extensions of the standard model neutrinos may be millicharged particles.

In the standard model of $SU(2)_L \times U(1)_Y$ electroweak interactions the neutrality of neutrinos is a consequence of the quantization of electric charge (Babu and Mohapatra, 1989, 1990b; Geng and Marshak, 1989; Foot *et al.*, 1990b; Minahan, Ramond, and Warner, 1990) [see also the earlier discussions by Bardeen, Gastmans, and Lautrup (1972), Gross and Jackiw (1972), and Lee and Shrock (1977) and the reviews by Foot *et al.* (1990a) and Foot, Lew, and Volkas (1993)]. In the standard model the electric charges of the particles are related to the eigenvalue of the third component I_3 of the weak isospin I and to the eigenvalue Y of the hypercharge by

$$Q = I_3 + \frac{Y}{2}. \tag{7.1}$$

The hypercharges of the fermion multiplets are fixed by the requirement of cancellation of the triangle anomalies, which is necessary for renormalizability. For each generation, we denote with Y_Φ, Y_L, Y_Q, Y_e, Y_u, and Y_d the hypercharges of the Higgs doublet, the left-handed lepton doublet, the left-handed quark doublet, the right-handed electron singlet, the right-handed up-quark singlet, and the right-handed down-quark singlet, respectively. The electric charge can be defined in units of the charge of the Higgs field ϕ^+ (see Table II) by choosing $Y_\Phi = +1$. Then the $U(1)_Y$ gauge invariance of the Yukawa couplings that generate the charged leptons and quarks masses requires that

$$Y_e = Y_L - 1, \tag{7.2}$$

$$Y_u = Y_Q + 1, \tag{7.3}$$

$$Y_d = Y_Q - 1. \tag{7.4}$$

Taking into account the fact that quarks have three colors, the values of Y_L and Y_Q are constrained by the cancellation of the $SU(2)_L$ triangle anomaly by

$$Y_Q = -Y_L/3. \tag{7.5}$$

Finally, the cancellation of the $U(1)_Y$ triangle anomaly requires that

$$0 = \mathrm{Tr}[Y^3] = 2Y_L^3 + 6Y_Q^3 - Y_e^3 - 3(Y_u^3 + Y_d^3), \tag{7.6}$$

where the right-handed fields enter with a minus sign. Using Eqs. (7.2)–(7.5) in Eq. (7.6), we obtain

$$0 = \mathrm{Tr}[Y^3] = (Y_L + 1)^3 \Rightarrow Y_L = -1. \tag{7.7}$$

Therefore, the charge is quantized and from Eq. (7.1) neutrinos are exactly neutral [see also the explicit calculations by Bardeen, Gastmans, and Lautrup (1972), Beg, Marciano, and Ruderman (1978), Marciano and Sirlin (1980), Sakakibara (1981), Lucio, Rosado, and Zepeda (1984, 1985), and Cabral-Rosetti *et al.* (2000).

This proof of charge quantization is spoiled by the introduction of a right-handed $SU(2)_L$ singlet neutrino ν_R in order to have a Dirac neutrino mass. Denoting with Y_ν the hypercharge of ν_R, the $U(1)_Y$ gauge invariance of the Yukawa coupling that generates a Dirac neutrino mass requires that

$$Y_\nu = Y_L + 1. \tag{7.8}$$

Then, Eq. (7.7) becomes

$$\mathrm{Tr}[Y^3] = (Y_L + 1)^3 - (Y_L + 1)^3 = 0. \tag{7.9}$$

Therefore, there is no $\mathrm{U}(1)_Y$ triangle anomaly for any value of Y_L, a right-handed $\mathrm{SU}(2)_L$ singlet neutrino ν_R which remains unconstrained. With the definition $Y_\nu = 2\varepsilon$, using the relations in Eqs. (7.1)–(7.5) and (7.8), we obtain

$$Q_\nu = \varepsilon, \tag{7.10}$$

$$Q_e = -1 + \varepsilon, \tag{7.11}$$

$$Q_u = 2/3 - \varepsilon/3, \tag{7.12}$$

$$Q_d = -1/3 - \varepsilon/3. \tag{7.13}$$

For the proton and the neutron we have

$$Q_p = 1 - \varepsilon, \qquad Q_n = -\varepsilon. \tag{7.14}$$

Hence, the hydrogen atom is neutral, but all the atoms with neutrons are not. Obviously, the limits on the non-neutrality of matter (Marinelli and Morpurgo, 1984; Bressi *et al.*, 2011) imply that the value of ε must be very small. In this case, neutrinos may be electrically millicharged particles (Babu and Mohapatra, 1989, 1990b; Foot *et al.*, 1990b; Minahan, Ramond, and Warner, 1990); see also the discussions by Okun, Voloshin, and Zakharov (1984) and Shrock (1996).

From Eqs. (7.10)–(7.13) one can see that the nonstandard hypercharge proportional to ε is proportional to $B - L$, where B and L are the baryon and lepton numbers. With the introduction of the right-handed neutrino ν_R the $\mathrm{U}(1)_{B-L}$ symmetry of the standard model becomes anomaly free. Adopting a notation similar to that used for the hypercharges, in the standard model the $\mathrm{U}(1)_{B-L}$ triangle anomaly is proportional to

$$\begin{aligned}\mathrm{Tr}[(B-L)^3] &= 2(B-L)^3_L + 6(B-L)^3_Q - (B-L)^3_e \\ &\quad - 3[(B-L)^3_u + (B-L)^3_d] \\ &= -1. \end{aligned} \tag{7.15}$$

Hence, in the standard model the $\mathrm{U}(1)_{B-L}$ symmetry is not anomaly free and cannot be gauged. On the other hand, with the introduction of ν_R which has $(B-L)_\nu = -1$ we obtain $\mathrm{Tr}[(B-L)^3] = 0$. In this case the $\mathrm{U}(1)_{B-L}$ symmetry is anomaly free and can be gauged. Then there can be a mixing of the standard model hypercharge Y_{SM} and $B - L$, which leads to the hypercharge

$$Y = Y_{\mathrm{SM}} - 2\varepsilon(B - L), \tag{7.16}$$

and the dequantized electric charges in Eqs. (7.10)–(7.13). Hence, the dequantization of the electric charge is due to the appearance of an anomaly-free U(1) symmetry which can be gauged and can mix with the standard hypercharge (Babu and Mohapatra, 1989, 1990b; Foot *et al.*, 1990b). The addition of an anomaly-free U(1) symmetry to the symmetries of the Lagrangian is a general way to obtain charge dequantization (Holdom, 1986).

A well-known way to recover electric charge quantization in theories with right-handed $\mathrm{SU}(2)_L$ singlet neutrinos is to consider grand unified theories (GUT) in which there is no U(1) symmetry (Georgi and Glashow, 1974; Pati and Salam, 1974). However, there is also the natural possibility to allow the right-handed neutrino to have a Majorana mass (Babu and Mohapatra, 1989,1990b). In this case the gauge invariance of the Majorana mass term $\nu_R^T \mathcal{C}^\dagger \nu_R$ requires that $Y_\nu = 0$ and, from Eq. (7.8), $Y_L = -1$, which gives the same charge quantization as in the standard model. This is consistent with the violation of the $\mathrm{U}(1)_{B-L}$ symmetry by the Majorana mass term, which forbids the addition of the $B - L$ term to Y_{SM} in Eq. (7.16).

Until now in this section we considered only one generation, but we know that there are three generations and the standard model Lagrangian has four global U(1) symmetries: $\mathrm{U}(1)_B$, $\mathrm{U}(1)_{L_e}$, $\mathrm{U}(1)_{L_\mu}$, and $\mathrm{U}(1)_{L_\tau}$, associated with the conservation of the baryon number B, the electron lepton number L_e, the muon lepton number L_μ, and the tau lepton number L_τ. It turns out that there is an infinite number of linear combinations of these U(1) symmetries which are anomaly free and lead to electric charge dequantization in the standard model with three generations (Foot *et al.*, 1990a; Foot, 1991; Foot, Lew, and Volkas, 1993). Charge quantization can be recovered by introducing right-handed neutrinos with Majorana mass terms which violate the conservation of all lepton numbers (Foot *et al.*, 1990a; Foot, 1991; Sladkowski and Zralek, 1992; Foot, Lew, and Volkas, 1993).

Some approximate constraints obtained with various assumptions from reactor, accelerator, and astrophysical data are listed in Table V (Babu and Volkas, 1992; Davidson, Hannestad, and Raffelt, 2000; Raffelt, 1996; Beringer *et al.*, 2012).

The most severe experimental constraint on neutrino electric charges is that on the effective electron neutrino charge $\mathbb{q}_{\nu_e}$, which can be obtained from electric charge conservation in neutron beta decay $n \to p + e^- + \bar{\nu}_e$, from the experimental limits on the non-neutrality of matter which

TABLE V. Approximate limits for different neutrino effective charges. The limits on $\mathbb{q}_\nu$ apply to all flavors.

Limit	Method	Reference
$\lvert\mathbb{q}_{\nu_\tau}\rvert \lesssim 3\times10^{-4}e$	SLAC e^- beam dump	Davidson, Campbell, and Bailey (1991)
$\lvert\mathbb{q}_{\nu_\tau}\rvert \lesssim 4\times10^{-4}e$	BEBC beam dump	Babu, Gould, and Rothstein (1994)
$\lvert\mathbb{q}_{\nu}\rvert \lesssim 6\times10^{-14}e$	Solar cooling (plasmon decay)	Raffelt (1999a)
$\lvert\mathbb{q}_{\nu}\rvert \lesssim 2\times10^{-14}e$	Red giant cooling (plasmon decay)	Raffelt (1999a)
$\lvert\mathbb{q}_{\nu_e}\rvert \lesssim 3\times10^{-21}e$	Neutrality of matter	Raffelt (1999a)
$\lvert\mathbb{q}_{\nu_e}\rvert \lesssim 3.7\times10^{-12}e$	Nuclear reactor	Gninenko, Krasnikov, and Rubbia (2007)
$\lvert\mathbb{q}_{\nu_e}\rvert \lesssim 1.5\times10^{-12}e$	Nuclear reactor	Studenikin (2014)

constrain the sum of the proton and electron charges $\mathbb{q}_p + \mathbb{q}_e$ and from the experimental limits on the neutron charge $\mathbb{q}_n$ (Raffelt, 1996, 1999a). Several experiments which measured the neutrality of matter give their results in terms of

$$\mathbb{q}_{\text{mat}} = \frac{Z(\mathbb{q}_p + \mathbb{q}_e) + N\mathbb{q}_n}{A}, \tag{7.17}$$

where $A = Z + N$ is the atomic mass of the substance under study, Z is its atomic number, and N is its neutron number. From electric charge conservation in neutron beta decay, we have

$$\mathbb{q}_{\nu_e} = \mathbb{q}_n - (\mathbb{q}_p + \mathbb{q}_e) = \frac{A}{Z}(\mathbb{q}_n - \mathbb{q}_{\text{mat}}). \tag{7.18}$$

The best recent bound on the non-neutrality of matter (Bressi *et al.*, 2011),

$$\mathbb{q}_{\text{mat}} = (-0.1 \pm 1.1) \times 10^{-21} e, \tag{7.19}$$

has been obtained with SF_6, which has $A = 146.06$ and $Z = 70$. Using the independent measurement of the charge of the free neutron (Baumann *et al.*, 1988)

$$\mathbb{q}_n = (-0.4 \pm 1.1) \times 10^{-21} e, \tag{7.20}$$

we obtain

$$\mathbb{q}_{\nu_e} = (-0.6 \pm 3.2) \times 10^{-21} e. \tag{7.21}$$

This value is compatible with the neutrality of the matter limit in Table V, which has been derived (Raffelt, 1996, 1999a) from the value of $\mathbb{q}_n$ in Eq. (7.20) and $\mathbb{q}_{\text{mat}} = (0.8 \pm 0.8) \times 10^{-21} e$ (Marinelli and Morpurgo, 1984).

It is also interesting that the effective charge of $\bar{\nu}_e$ can be constrained by the SN 1987A neutrino measurements taking into account that galactic and extragalactic magnetic fields can lengthen the path of millicharged neutrinos and requiring that neutrinos with different energies arrive on Earth within the observed time interval of a few seconds (Barbiellini and Cocconi, 1987):

$$|\mathbb{q}_{\nu_e}| \lesssim 3.8 \times 10^{-12} \frac{(E_\nu/10\ \text{MeV})}{(d/10\ \text{kpc})(B/1\ \mu\text{G})} \sqrt{\frac{\Delta t/t}{\Delta E_\nu/E_\nu}}, \tag{7.22}$$

considering a magnetic field B acting over a distance d and the corresponding time $t = d/c$. $E_\nu \approx 15$ MeV is the average neutrino energy, $\Delta E_\nu \approx E_\nu/2$ is the energy spread, and $\Delta t \approx 5$ s is the arrival time interval. Barbiellini and Cocconi (1987) considered two cases:

(1) An intergalactic field $B \approx 10^{-3}\ \mu$G acting over the whole path $d \simeq 50$ kpc, which corresponds to $t \simeq 5 \times 10^{12}$ s, gives

$$|\mathbb{q}_{\nu_e}| \lesssim 2 \times 10^{-15} e. \tag{7.23}$$

(2) A galactic field $B \approx 1\ \mu$G acting over a distance $d \simeq 10$ kpc, which corresponds to $t \simeq 1 \times 10^{12}$ s, gives

$$|\mathbb{q}_{\nu_e}| \lesssim 2 \times 10^{-17} e. \tag{7.24}$$

The last two limits in Table V have been obtained (Gninenko, Krasnikov, and Rubbia, 2007; Studenikin, 2014) considering the results of reactor neutrino magnetic moment experiments (see Secs. IV.C and IV.E). The differential cross section of the $\bar{\nu}_e$-e^- elastic scattering process due to an neutrino effective charge $\mathbb{q}_{\nu_e}$ is given by (Berestetskii, Lifshitz, and Pitaevskii, 1979)

$$\left(\frac{d\sigma}{dT_e}\right)_{\text{charge}} \simeq 2\pi\alpha \frac{1}{m_e T_e^2} \mathbb{q}_{\nu_e}^2. \tag{7.25}$$

In reactor experiments the neutrino magnetic moment is searched by considering data with $T_e \ll E_\nu$, for which the ratio of the charge cross section (7.25) and the magnetic moment cross section in Eq. (4.31), for which we consider only the dominant part proportional to $1/T_e$, is given by

$$R = \frac{(d\sigma/dT_e)_{\text{charge}}}{(d\sigma/dT_e)_{\text{mag}}} \simeq \frac{2m_e}{T_e} \frac{(\mathbb{q}_{\nu_e}/e)^2}{(\mu_{\nu_e}/\mu_{\text{B}})^2}. \tag{7.26}$$

Considering an experiment which does not observe any effect of μ_{ν_e} and obtains a limit on μ_{ν_e}, it is possible to obtain, following Studenikin (2014), a bound on $\mathbb{q}_{\nu_e}$ by demanding that the effect of $\mathbb{q}_{\nu_e}$ is smaller than that of μ_{ν_e}, i.e., that $R \lesssim 1$:

$$\mathbb{q}_{\nu_e}^2 \lesssim \frac{T_e}{2m_e} \left(\frac{\mu_{\nu_e}}{\mu_{\text{B}}}\right)^2 e^2. \tag{7.27}$$

The last limit in Table V has been obtained from the 2012 results (Beda *et al.*, 2012) of the GEMMA experiment, considering T_e at the experimental threshold of 2.8 keV.

We finally note that a strong limit on a generic neutrino electric charge $\mathbb{q}_\nu$ can be obtained by considering the influence of millicharged neutrinos on the rotation of a magnetized star which is undergoing a core-collapse supernova explosion (the neutrino star turning mechanism νST) (Studenikin and Tokarev, 2014). During the supernova explosion, the escaping millicharged neutrinos move along curved orbits inside the rotating magnetized star and slow down the rotation of the star. This mechanism could prevent the generation of a rapidly rotating pulsar in the supernova explosion. Imposing that the frequency shift of a forming pulsar due to the neutrino star turning mechanism is less than a typical observed frequency of 0.1 s^{-1} and assuming a magnetic field of the order of 10^{14} G, Studenikin and Tokarev (2014) obtained

$$|\mathbb{q}_\nu| \lesssim 1.3 \times 10^{-19} e. \tag{7.28}$$

Note that this limit is much stronger than the astrophysical limits in Table V.

B. Neutrino charge radius

Even if the electric charge of a neutrino is zero, the electric form factor $\mathbb{f}_Q(q^2)$ can contain nontrivial information about the neutrino electric properties. In fact, a neutral particle can be characterized by a (real or virtual) superposition of two different charge distributions of opposite signs, which is described by a form factor $\mathbb{f}_Q(q^2)$ which is nonzero for $q^2 \neq 0$.

The neutrino charge radius is determined by the second term in the expansion of the neutrino charge form factor $\mathbb{f}_Q(q^2)$ in a series of powers of q^2:

$$\mathbb{f}_Q(q^2) = \mathbb{f}_Q(0) + q^2 \frac{d\mathbb{f}_Q(q^2)}{dq^2}\bigg|_{q^2=0} + \cdots. \quad (7.29)$$

In the so-called "Breit frame," in which $q_0 = 0$, the charge form factor $\mathbb{f}_Q(q^2)$ depends only on $|\vec{q}| = \sqrt{-q^2}$ and can be interpreted as the Fourier transform of a spherically symmetric charge distribution $\rho(r)$, with $r = |\vec{x}|$:

$$\mathbb{f}_Q(q^2) = \int \rho(r) e^{-i\vec{q}\cdot\vec{x}} d^3x = \int \rho(r) \frac{\sin(|\vec{q}|r)}{|\vec{q}|r} d^3x. \quad (7.30)$$

Deriving with respect to $q^2 = -|\vec{q}|^2$, we obtain

$$\frac{d\mathbb{f}_Q(q^2)}{dq^2} = \int \rho(r) \frac{\sin(|\vec{q}|r) - |\vec{q}|r\cos(|\vec{q}|r)}{2q^3 r} d^3x, \quad (7.31)$$

and

$$\lim_{q^2\to 0} \frac{d\mathbb{f}_Q(q^2)}{dq^2} = \int \rho(r) \frac{r^2}{6} d^3x = \frac{\langle r^2 \rangle}{6}. \quad (7.32)$$

Therefore, the squared neutrino charge radius is given by

$$\langle r^2 \rangle = 6 \frac{d\mathbb{f}_Q(q^2)}{dq^2}\bigg|_{q^2=0}. \quad (7.33)$$

Note that $\langle r^2 \rangle$ can be negative, because the charge density $\rho(r)$ is not a positively defined quantity.

As seen in Sec. III.C, massless left-handed Weyl neutrinos have the electromagnetic form factor in Eq. (3.86). This is the case of the standard model, in which in addition neutrinos have zero electric charge $\mathbb{f}_Q(0) = 0$ as explained in Sec. VII.A. Taking into account Eqs. (7.29) and (7.33), in the standard model the neutrino electromagnetic form factor for small values of q^2 is given by

$$\mathbb{f}(q^2) \simeq \left(\frac{\langle r^2 \rangle}{6} - \mathbb{a}\right) q^2, \quad (7.34)$$

where $\mathbb{a}$ is the anapole moment. Hence, in the standard model the form factor $\mathbb{f}(q^2)$ can be interpreted as a neutrino charge radius or as an anapole moment (or as a combination of both). In this section we consider the charge radius interpretation. The equivalence between the charge radius and anapole moment interpretations of $\mathbb{f}(q^2)$ is discussed further in Sec. VII.C.

The standard model theory of the neutrino charge radius has a long history, with some controversies which are shortly summarized in the following.

In one of the first studies, Bardeen, Gastmans, and Lautrup (1972) claimed that in the standard model and in the unitary gauge the neutrino charge radius is ultraviolet divergent and so it is not a physical quantity. A direct one-loop calculation (Dvornikov and Studenikin, 2004a; Dvornikov and Studenikin, 2004b) of proper vertices (Fig. 6) and γ-Z self-energy (Figs. 12 and 13) contributions to the neutrino charge radius performed in a general R_ξ gauge for a massive Dirac neutrino also gave a divergent result. However, it was shown (Lee, 1972), using the unitary gauge, that by including in addition to the usual terms also contributions from diagrams of the neutrino-lepton neutral-current scattering (Z boson diagrams) it is possible to obtain for the neutrino charge radius a gauge-dependent but finite quantity. Later on, it was also shown (Lee and Shrock, 1977) that in order to define the neutrino charge radius as a physical quantity one has to consider additional box diagrams and that in combination with contributions from the proper diagrams it is possible to obtain a finite and gauge-independent value for the neutrino charge

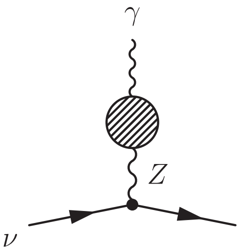

FIG. 12. Contribution to the neutrino vertex function of γ-Z self-energy. Figure 13 shows the diagrams contributing to the blob at one loop in the extended standard model with right-handed neutrinos.

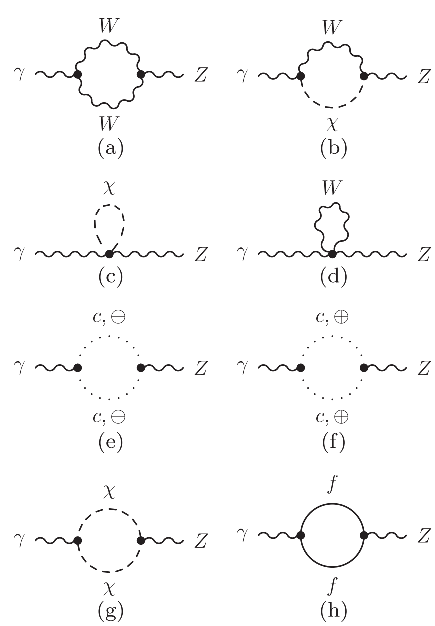

FIG. 13. γ-Z self-energy diagrams contributing to the neutrino vertex function at one loop in the extended standard model with right-handed neutrinos (Dvornikov and Studenikin, 2004a, 2004b). f denotes a generic charged lepton (e, μ, τ) or a quark (u, c, t, d, s, b). χ is the unphysical would-be charged scalar boson. The charge of ghosts c is indicated by the symbols $\oplus$ and $\ominus$.

radius. In this way, the neutrino electroweak radius was defined by Lucio, Rosado, and Zepeda (1984, 1985) and an additional set of diagrams that give contribution to its value was discussed by Degrassi, Sirlin, and Marciano (1989). Finally, Bernabeu *et al.* (2000), and Bernabeu, Papavassiliou, and Vidal (2002, 2004) introduced the neutrino electroweak radius as a physical observable. In the corresponding calculations, performed in the one-loop approximation including additional terms from the γ-Z boson mixing and the box diagrams involving W and Z bosons, the following gauge-invariant result for the neutrino charge radius was obtained:

$$\langle r^2_{\nu_\ell} \rangle_{\rm SM} = \frac{G_{\rm F}}{4\sqrt{2}\pi^2}\left[3 - 2\log\left(\frac{m^2_\ell}{m^2_W}\right)\right], \tag{7.35}$$

where m_W and m_ℓ are the W boson and lepton masses ($\ell = e, \mu, \tau$). This result, however, revived the discussion (Fujikawa and Shrock, 2003, 2004; Papavassiliou *et al.*, 2004; Bernabeu, Binosi, and Papavassiliou, 2005) on the definition of the neutrino charge radius. Numerically, Eq. (7.35) gives (Bernabeu *et al.*, 2000; Bernabeu, Papavassiliou, and Vidal, 2004)

$$\langle r^2_{\nu_e} \rangle_{\rm SM} = 4.1 \times 10^{-33}\ {\rm cm}^2, \tag{7.36}$$

$$\langle r^2_{\nu_\mu} \rangle_{\rm SM} = 2.4 \times 10^{-33}\ {\rm cm}^2, \tag{7.37}$$

$$\langle r^2_{\nu_\tau} \rangle_{\rm SM} = 1.5 \times 10^{-33}\ {\rm cm}^2. \tag{7.38}$$

These values are of the same order of magnitude of the numerical estimation $\langle r^2_{\nu_\ell} \rangle \approx 10^{-33}$ cm^2 obtained by Lucio, Rosado, and Zepeda (1985).

The effects of new physics beyond the standard model can contribute to the neutrino charge radius. However, Novales-Sanchez *et al.* (2008) showed that in the context of an effective electroweak Yang-Mills theory the anomalous $WW\gamma$ vertex contribution to the neutrino effective charge radius is smaller than about 10^{-34} cm^2, which is 1 order of magnitude smaller than the standard model values in Eqs. (7.36)–(7.38).

The neutrino charge radius has an effect in the scattering of neutrinos with charged particles. The most useful process is the elastic scattering with electrons, which was discussed in Sec. IV.C in connection with the searches of neutrino magnetic moments. Since in the ultrarelativistic limit the charge form factor conserves the neutrino helicity (see Appendix C), a neutrino charge radius contributes to the weak-interaction cross section $(d\sigma/dT_e)_{\rm SM}$ of ν_ℓ-e^- elastic scattering through the following shift of the vector coupling constant $g_V^{\nu_\ell}$ (Grau and Grifols, 1986; Degrassi, Sirlin, and Marciano, 1989; Vogel and Engel, 1989; Hagiwara *et al.*, 1994):

$$g_V^{\nu_\ell} \to g_V^{\nu_\ell} + \tfrac{2}{3} m_W^2 \langle r^2_{\nu_\ell} \rangle \sin^2\theta_W. \tag{7.39}$$

Using this method, experiments which measure neutrino-electron elastic scattering can probe the neutrino charge radius. Some experimental results are listed in Table VI. In addition, Hirsch, Nardi, and Restrepo (2003) obtained the following 90% C.L. bounds on $\langle r^2_{\nu_\mu} \rangle$ from a reanalysis of CHARM-II (Vilain *et al.*, 1995) and CCFR (McFarland *et al.*, 1998) data:

$$-0.52 \times 10^{-32} < \langle r^2_{\nu_\mu} \rangle < 0.68 \times 10^{-32}\ {\rm cm}^2. \tag{7.40}$$

Recently, Barranco, Miranda, and Rashba (2008) obtained the following 90% C.L. bounds on $\langle r^2_{\nu_e} \rangle$ from a combined fit of all available ν_e-e^- and $\bar{\nu}_e$-e^- data:

$$-0.26 \times 10^{-32} < \langle r^2_{\nu_e} \rangle < 6.64 \times 10^{-32}\ {\rm cm}^2. \tag{7.41}$$

The single photon production process $e^+ + e^- \to \nu + \bar{\nu} + \gamma$ has been used to get bounds on the effective ν_τ charge radius, assuming a negligible contribution of the ν_e and ν_μ charge radii (Altherr and Salati, 1994; Tanimoto, Nakano, and Sakuda, 2000; Hirsch, Nardi, and Restrepo, 2003). For Dirac neutrinos, Hirsch, Nardi, and Restrepo (2003) obtained

$$-5.6 \times 10^{-32} < \langle r^2_{\nu_\tau} \rangle < 6.2 \times 10^{-32}\ {\rm cm}^2. \tag{7.42}$$

Comparing the theoretical standard model values in Eqs. (7.36)–(7.38) with the experimental limits in Table VI and those in Eqs. (7.40)–(7.42), one can see that they differ at most by 1 order of magnitude. Therefore, one may expect that the experimental accuracy will soon reach the value needed to probe the standard model predictions for the neutrino charge radii. This will be an important test of the standard model calculation of the neutrino charge radii. If the experimental value of a neutrino charge radius is found to be different from the standard model prediction in Eqs. (7.36)–(7.38) it will be necessary to clarify the precision of the theoretical calculation

TABLE VI. Experimental limits for the electron neutrino charge radius.

Method	Experiment	Limit (cm^2)	C.L.	Reference
Reactor $\bar{\nu}_e$-e^-	Krasnoyarsk	$\lvert\langle r^2_{\nu_e} \rangle\rvert < 7.3 \times 10^{-32}$	90%	Vidyakin *et al.* (1992)
	TEXONO	$-4.2 \times 10^{-32} < \langle r^2_{\nu_e} \rangle < 6.6 \times 10^{-32}$	90%	Deniz *et al.* (2010)[a]
Accelerator ν_e-e^-	LAMPF	$-7.12 \times 10^{-32} < \langle r^2_{\nu_e} \rangle < 10.88 \times 10^{-32}$	90%	Allen *et al.* (1993)[a]
	LSND	$-5.94 \times 10^{-32} < \langle r^2_{\nu_e} \rangle < 8.28 \times 10^{-32}$	90%	Auerbach *et al.* (2001)[a]
Accelerator ν_μ-e^-	BNL-E734	$-4.22 \times 10^{-32} < \langle r^2_{\nu_\mu} \rangle < 0.48 \times 10^{-32}$	90%	Ahrens *et al.* (1990)[a]
	CHARM-II	$\lvert\langle r^2_{\nu_\mu} \rangle\rvert < 1.2 \times 10^{-32}$	90%	Vilain *et al.* (1995)[a]

[a]The published limits are half, because they use a convention which differs by a factor of 2 [see also Hirsch, Nardi, and Restrepo (2003)].

in order to understand if the difference is due to new physics beyond the standard model.

The neutrino charge radius has also some impact on astrophysical phenomena and on cosmology. The limits on the cooling of the Sun and white dwarfs due to the plasmon-decay process discussed in Sec. V.D induced by a neutrino charge radius led Dolgov and Zeldovich (1981) to estimate the respective limits $|\langle r_\nu^2\rangle| \lesssim 10^{-28}$ cm^2 and $|\langle r_\nu^2\rangle| \lesssim 10^{-30}$ cm^2 for all neutrino flavors. From the cooling of red giants Altherr and Salati (1994) inferred the limit $|\langle r_\nu^2\rangle| \lesssim 4\times 10^{-31}$ cm^2.

If neutrinos are Dirac particles, e^+-e^- annihilations can produce right-handed neutrino-antineutrino pairs through the coupling induced by a neutrino charge radius. This process would affect primordial big-bang nucleosynthesis and the energy release of a core-collapse supernova. From the measured ^{4}He yield in primordial big-bang nucleosynthesis Grifols and Masso (1987) obtained

$$|\langle r_\nu^2\rangle| \lesssim 7\times 10^{-33}\ \text{cm}^2, \tag{7.43}$$

and from SN 1987A data Grifols and Masso (1989) obtained

$$\langle r_\nu^2\rangle \lesssim 2\times 10^{-33}\ \text{cm}^2, \tag{7.44}$$

for all neutrino flavors.

C. Neutrino anapole moment

The notion of an anapole moment for a Dirac particle was introduced by Zel'dovich (1958) after the discovery of parity violation. The anapole form factor was not known before because it violates P. Indeed, taking into account that

$$A_\mu(x) \xrightarrow{P} A^\mu(x_P), \tag{7.45}$$

P is conserved if

$$\Lambda_\mu(q) \xrightarrow{P} \Lambda^\mu(q). \tag{7.46}$$

Using the formulas in Appendix A, one can find that

$$\Lambda_\mu(q) \xrightarrow{P} \gamma^0 \Lambda_\mu(q_P)\gamma^0. \tag{7.47}$$

Using the form-factor expansion in Eq. (3.18), we obtain

$$\begin{aligned}\Lambda_\mu(q) \xrightarrow{P} & \mathbb{f}_Q(q^2)\gamma^\mu - \mathbb{f}_M(q^2) i\sigma^{\mu\nu}q_\nu \\ & - \mathbb{f}_E(q^2)\sigma^{\mu\nu}q_\nu\gamma_5 - \mathbb{f}_A(q^2)(q^2\gamma^\mu - q^\mu \not{q})\gamma_5.\end{aligned} \tag{7.48}$$

Hence, parity is violated by the electric and anapole moments. Since the anapole moment conserves CP (and T, as a consequence of CPT symmetry), as shown in Sec. III.A, it follows that the anapole moment also violates C.

In order to understand the physical characteristics of the anapole moment, we consider its effect in the interactions with external electromagnetic fields. From the last term in Eq. (6.7) one can see that the anapole moment describes an interaction with the current which generates the external electromagnetic fields.

Using the method described in Appendix E, we obtain the helicity-conserving potential

$$V_{h\to h} = -\mathbb{a} h \frac{m}{E} s^\mu j_\mu, \tag{7.49}$$

which is strongly suppressed for ultrarelativistic neutrinos. In the nonrelativistic limit, we obtain

$$V_{h\to h}^{\text{nr}} \simeq \vec{\mathbb{a}}\cdot\vec{j}, \quad \text{with} \quad \vec{\mathbb{a}} = h\frac{\vec{p}}{|\vec{p}|}\mathbb{a}. \tag{7.50}$$

This is the anapole moment potential that was introduced by Zel'dovich (1958). It is proportional to the longitudinal component of the current.

Considering now the helicity-flipping potential, as shown in Appendix E, we obtain

$$V_{-h\to h} = \mathbb{a}\frac{m}{E} j_\perp, \tag{7.51}$$

where $j_\perp$ is the component of $\vec{j}$ orthogonal to $\vec{p}$. For ultrarelativistic neutrinos, the helicity-flipping potential is strongly suppressed, but in the nonrelativistic limit we have

$$V_{-h\to h}^{\text{nr}} \simeq \mathbb{a} j_\perp = |\vec{\mathbb{a}}\times\vec{j}|. \tag{7.52}$$

This potential corresponds to a classical torque (Zel'dovich, 1958) which rotates the spin of the particle, causing periodic changes of the helicity.

The anapole moment is a quantity which is difficult to understand, because it does not generate interactions with a free electromagnetic field, but only contact interactions with the charge and current density which generates an electromagnetic field. A classical model which can help to visualize the behavior of the anapole moment has been given by Zel'dovich (1958) [see also Bukina, Dubovik, and Kuznetsov (1998a)]. In this model the anapole is represented by a current-carrying rigid toroidal solenoid. The current generates a magnetic field only inside the toroidal solenoid. Since the solenoid is rigid, there is no external magnetic field which can act on the toroidal solenoid as a whole. The only action on the toroidal solenoid can be generated by a current which passes through the solenoid and interacts with the magnetic field inside. For example, the toroidal solenoid can be immersed in an electrolytic solution which also fills the space inside the solenoid. If a current flows through the electrolytic solution, it interacts with the magnetic field inside the solenoid and generates a torque proportional to the sine of the angle between the direction of the current and the axis of the toroid. In this model the axis of the toroid corresponds to the direction of $\vec{\mathbb{a}}$ in Eqs. (7.50) and the torque corresponds to the helicity-flipping potential in Eq. (7.52).

The neutrino anapole moment contributes to the scattering of neutrinos with charged particles. In order to discuss its effects, it is convenient to consider strictly neutral neutrinos with $\mathbb{f}_Q(0) = 0$ and define a reduced charge form factor $\tilde{\mathbb{f}}_Q(q^2)$ such that

$$\mathbb{f}_Q(q^2) = q^2\tilde{\mathbb{f}}_Q(q^2). \tag{7.53}$$

Then, from Eq. (7.33), apart from a factor 1/6, the reduced charge form factor at $q^2 = 0$ is just the squared neutrino charge radius:

$$\tilde{\mathbb{f}}_Q(0) = \langle r^2 \rangle / 6. \qquad (7.54)$$

We now consider the charge and anapole parts of the neutrino electromagnetic vertex function in Eq. (3.37), which can be written as

$$\Lambda_\mu^{Q,A}(q) = (\gamma_\mu q^2 - q_\mu \not{q})[\tilde{\mathbb{f}}_Q(q^2) + \mathbb{f}_A(q^2)\gamma_5]. \qquad (7.55)$$

Since for ultrarelativistic neutrinos the effect of γ_5 is only a sign which depends on the helicity of the neutrino [see Eq. (C6)], the phenomenology of neutrino anapole moments is similar to that of neutrino charge radii. Hence, the limits on the neutrino charge radii discussed in Sec. VII.B also apply to the neutrino anapole moments multiplied by 6.

As discussed in Sec. VII.A, in the standard model the neutrino electric charges are exactly zero. Hence, Eq. (7.55) applies to the standard model and can be further simplified taking into account that in the standard model neutrinos are described by two-component massless left-handed Weyl spinors. As discussed in Sec. III.C, the γ^5 in Eq. (7.55) becomes a minus sign, leading to

$$\Lambda_{\mathrm{SM}\mu}^{Q,A}(q) = (\gamma_\mu q^2 - q_\mu \not{q})\mathbb{f}^{\mathrm{SM}}(q^2), \qquad (7.56)$$

with

$$\mathbb{f}^{\mathrm{SM}}(q^2) = \tilde{\mathbb{f}}_Q(q^2) - \mathbb{f}_A(q^2) \underset{q^2 \to 0}{\longrightarrow} \frac{\langle r^2 \rangle}{6} - \mathbb{a}. \qquad (7.57)$$

These equations correspond to Eqs. (3.85) and (3.86) for $\mathbb{f}_Q(0) = 0$. Hence, in the standard model the neutrino charge radius and the anapole moment are not defined separately and one can interpret arbitrarily $\mathbb{f}^{\mathrm{SM}}(0)$ as a charge radius or as an anapole moment. This is the correct interpretation of the statement often found in the literature that in the standard model $\mathbb{a} = -\langle r^2 \rangle/6$. Therefore, the standard model values for the neutrino charge radii in Eqs. (7.35)–(7.38) can be interpreted also as values of the corresponding neutrino anapole moments.

Some deep insight into an interpretation of the decompositions of the vertex function (3.18) and the neutrino form factors can be obtained in the framework of a multipole expansion of the corresponding classical electromagnetic currents (Dubovik and Cheshkov, 1974; Dubovik and Tosunian, 1983; Dubovik and Kuznetsov, 1998). Since in this limit the anapole form factor does not correspond to a certain multipole distribution [that is why the term "anapole" was introduced by Zel'dovich (1958)], the anapole moment has a quite intricate classical analog. Therefore, Bukina, Dubovik, and Kuznetsov (1998a, 1998b), and Dubovik and Kuznetsov (1998) proposed to consider the toroidal dipole moment as a characteristic of the neutrino which is more convenient and transparent than the anapole moment for the description of T-invariant interactions with nonconservation of the P and C symmetries. In this case, the electromagnetic vertex of a neutrino can be rewritten in the alternative multipole (toroidal) parametrization

$$\Lambda_\mu(q) = \mathbb{f}_Q(q^2)\gamma_\mu - \mathbb{f}_M(q^2) i\sigma_{\mu\nu}q^\nu + \mathbb{f}_E(q^2)\sigma_{\mu\nu}q^\nu\gamma_5 + i\mathbb{f}_T(q^2)\epsilon_{\mu\nu\lambda\rho}P^\nu q^\lambda \gamma^\rho, \qquad (7.58)$$

where $\mathbb{f}_T$ is the toroidal dipole form factor and $P = p_i + p_f$. From the following identity:

$$\bar{u}_f(p_f)\{(m_i - m_f)\sigma_{\mu\nu}q^\nu + (q^2\gamma_\mu - \not{q}q_\mu) - i\epsilon_{\mu\nu\lambda\rho}P^\nu q^\lambda \gamma^\rho \gamma_5\}\gamma_5 u_i(p_i) = 0, \qquad (7.59)$$

it can be seen that the toroidal and anapole moments coincide in the static limit when the masses of the initial and final neutrino states are equal to each other $m_i = m_f$ (Bukina, Dubovik, and Kuznetsov, 1998b), i.e., the toroidal and anapole parametrizations coincide in this case.

In some sense the toroidal parametrization has a more transparent and clear physical interpretation, because it provides a one-to-one correspondence between the multipole moments and the corresponding form factors. From the properties of each term in Eq. (7.58) for the vertex function under C, P, and T transformations, it follows that in the Majorana case only the toroidal form factor survives (Zel'dovich, 1958; Kobzarev and Okun, 1972) and the toroidal moment of the Dirac neutrino is half of that in the Majorana case.

In one-loop calculations (Dubovik and Kuznetsov, 1998) of the toroidal (and anapole) moment of a massive and a massless Majorana neutrino (the diagrams in Figs. 6, 12, and 13 contribute) it was shown that its value does not significantly depend on the neutrino mass (through the ratios $m_{\nu_i}^2/m_W^2$) and is of the order of

$$\mathbb{f}_T(q^2 = 0) \sim e \times (10^{-33} - 10^{-34})\ \mathrm{cm}^2, \qquad (7.60)$$

depending on the values of the quark masses that propagate in the loop diagrams in Fig. 13.

Note that the toroidal form factors can contribute to the neutrino vertex function in both the diagonal and off-diagonal cases.

The toroidal (anapole) interactions of a Majorana as well as a Dirac neutrino are expected to contribute to the total cross section of neutrino elastic scattering off electrons, quarks, and nuclei. Because of the fact that the toroidal (anapole) interactions contribute to the helicity preserving part of the scattering of neutrinos on electrons, quarks, and nuclei, its contributions to cross sections are similar to those of the neutrino charge radius. In principle, these contributions can be probed and information about toroidal moments can be extracted in low-energy scattering experiments in the future.

Different effects of the neutrino toroidal moment are discussed by Ginzburg and Tsytovich (1985), Bukina, Dubovik, and Kuznetsov (1998a, 1998b), and Dubovik and Kuznetsov (1998). In particular, it has been shown that the neutrino toroidal electromagnetic interactions can produce Cherenkov radiation of neutrinos propagating in a medium.

VIII. SUMMARY AND PERSPECTIVES

In this review we discussed the theory and phenomenology of neutrino electromagnetic properties and interactions. We have seen that most of the theoretical and experimental research has been devoted to the study of magnetic and electric dipole moments, but there has also been some interest in the investigation of neutrino millicharges and of the charge radii and anapole moments of neutrinos.

Unfortunately, so far there is not any experimental indication in favor of neutrino electromagnetic interactions and all neutrino electromagnetic properties are known to be small, with rather stringent upper bounds obtained in laboratory experiments or from astrophysical observations.

The most accessible neutrino electromagnetic property may be the charge radius, discussed in Sec. VII.B, for which the standard model gives a value which is only about 1 order of magnitude smaller than the experimental upper bounds. A measurement of a neutrino charge radius at the level predicted by the standard model would be another confirmation of the standard model, after the recent discovery of the Higgs boson (Ellis, 2013). However, such a measurement would not give information on new physics beyond the standard model unless the measured value is shown to be incompatible with the standard model value in a high-precision experiment.

The strongest current efforts to probe the physics beyond the standard model by measuring neutrino electromagnetic properties is the search for a neutrino magnetic moment effect in reactor $\bar{\nu}_e$-e^- scattering experiments. The current upper bounds reviewed in Sec. IV.E are more than 8 orders of magnitude larger than the prediction discussed in Sec. IV.A of the Dirac neutrino magnetic moments in the minimal extension of the standard model with right-handed neutrinos. Hence, a discovery of a neutrino magnetic moment effect in reactor $\bar{\nu}_e$-e^- scattering experiments would be an exciting discovery of nonminimal new physics beyond the standard model.

In particular, the GEMMA-II Collaboration expects to reach around the year 2017 a sensitivity to $\mu_{\nu_e} \approx 1 \times 10^{-11}\mu_B$ in a new series of measurements at the Kalinin Nuclear Power Plant with a doubled neutrino flux obtained by reducing the distance between the reactor and the detector from 13.9 to 10 m and by reducing the energy threshold from 2.8 to 1.5 keV (Beda *et al.*, 2012, 2013). The corresponding sensitivity to the neutrino electric millicharge discussed in Sec. VII.A will reach the level of $|q_{\nu_e}| \approx 3.7 \times 10^{-13} e$ (Studenikin, 2014).

There is also a GEMMA-III project[18] to further lower the energy threshold to about 350 eV, which may allow the experimental collaboration to reach a sensitivity of $\mu_{\nu_e} \approx 9 \times 10^{-12}\mu_B$. The corresponding sensitivity to neutrino millicharge will be $|q_{\nu_e}| \approx 1.8 \times 10^{-13} e$ (Studenikin, 2014).

An interesting possibility for exploring very small values of μ_{ν_e} in $\bar{\nu}_e$-e^- scattering experiments was proposed by Bernabeu, Papavassiliou, and Passera (2005) on the basis of the observation (Segura *et al.*, 1994) that "dynamical zeros" induced by a destructive interference between the left-handed and right-handed chiral couplings of the electron in the charged and neutral-current amplitudes appear in the standard model contribution to the scattering cross section. It may be possible to enhance the sensitivity of an experiment to μ_{ν_e} by selecting recoil electrons contained in a forward narrow cone corresponding to a dynamical zero [see Eq. (4.25)].

In the future experimental searches of neutrino electromagnetic properties may be performed also with new neutrino sources, as a tritium source (McLaughlin and Volpe, 2004), a low-energy beta beam (McLaughlin and Volpe, 2004; de Gouvea and Jenkins, 2006), a stopped-pion neutrino source (Scholberg, 2006), or a neutrino factory (de Gouvea and Jenkins, 2006). Recently Coloma, Huber, and Link (2014) proposed to improve the existing limit on the electron neutrino magnetic moment with a megacurie ^{51}Cr neutrino source and a large liquid xenon detector.

Neutrino electromagnetic interactions could have important effects in astrophysical environments and in the evolution of the Universe and the current rapid advances of astrophysical and cosmological observations may lead soon to the exciting discovery of nonstandard neutrino electromagnetic properties. In particular, future high-precision observations of supernova neutrino fluxes may reveal the effects of collective spin-flavor oscillations due to Majorana transition magnetic moments as small as $10^{-21}\mu_B$ (de Gouvea and Shalgar, 2012, 2013).

We finally emphasize the importance of pursuing the experimental and theoretical studies of electromagnetic neutrino interactions, which could open a powerful window to new physics beyond the standard model.

[18]Victor Brudanin and Vyacheslav Egorov, private communication.

ACKNOWLEDGMENTS

We thank Vadim Bednyakov, Samoil Bilenky, Victor Brudanin, Vyacheslav Egorov, Konstantin Kouzakov, Alexander Starostin, and Alexey Ternov for fruitful discussions. The work of A. S. was partially supported by the Russian Basic Research Foundation Grants No. 14-22-03043_ofi_m and No. 15-52-53112. The work of C. G. was partially supported by the research grant Theoretical Astroparticle Physics No. 2012CPPYP7 under the program PRIN 2012 funded by the Ministero dell'Istruzione, Università e della Ricerca (MIUR).

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