

# Lévy walks

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Random walk is a fundamental concept with applications ranging from quantum physics to econometrics. Remarkably, one specific model of random walks appears to be ubiquitous across many fields as a tool to analyze transport phenomena in which the dispersal process is faster than dictated by Brownian diffusion. The Lévy-walk model combines two key features, the ability to generate anomalously fast diffusion and a finite velocity of a random walker. Recent results in optics, Hamiltonian chaos, cold atom dynamics, biophysics, and behavioral science demonstrate that this particular type of random walk provides significant insight into complex transport phenomena. This review gives a self-consistent introduction to Lévy walks, surveys their existing applications, including latest advances, and outlines further perspectives.

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## I. INTRODUCTION

In this review we demonstrate how a simple idea of giving a finite velocity to a diffusing particle increases flexibility and diversity of diffusion models in describing complex transport phenomena. We consider processes resulting from a motion of many identical noninteracting particles. There are two key complimentary approaches to the statistical description of such motion. The first approach is based on the concept of random

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walks (Weiss, 1994), while the second is based on stochastic differential equations. Among the latter are the Langevin equation (Coffey and Kalmykov, 2012) and the concept of Brownian motion (Mörters and Peres, 2010). Although having different terminologies and mathematical apparatus, the two approaches are closely related and their exact equivalence can be demonstrated in some cases. The framework of our review is random walks and we start with a historical overview of the development of the concept, with a special focus on how the idea of the finite velocity of walking particles was born and matured over the years.

As if to predict its interdisciplinary future, the theory of random walks was developed independently in the context of biology (Brown, 1828), probability theory (Bernoulli, 1713), finance (Bachelier, 1900), and physics (Rayleigh, 1880; Pearson, 1905). The seminal works of Einstein (1905) and von Smoluchowski (1906) marked the start of rigorous and quantitative approach connecting microscopic dynamics of particles to the macroscopic process of diffusion [see, e.g., Nelson (1967) for more historical background on Brownian motion]. The diffusion equation was already known for nearly a century as it was derived to describe the heat conduction by Fourier (1822). Despite the success of this equation in various applications, it had one particular drawback that did not escape the attention of contemporary scientists. According to the diffusion equation, when starting with a localized initial condition, even after an infinitesimally short elapsed time there will be a nonzero density of diffusing particles at any arbitrary distance from the starting point. This implies an infinite propagation speed of some particles and thus contradicts our understanding of how physical objects move [see the discussion on this issue in the context of relativistic statistical physics in Dunkel and Hänggi (2009) and Dunkel, Hänggi, and Hilbert (2009)]. The infinite speed is also inconsistent with the original schematization of a random walk process by Pearson (1905): “A man starts from a point 0 and walks  $l$  yards in a straight line; he then turns through any angle whatever and walks another  $l$  yards in a second straight line. He repeats this process  $n$  times. I require the probability that after these  $n$  stretches he is at a distance between  $r$  and  $r + dr$  from his starting point, 0.” After this drawback was noted, two approaches to resolve the issue were proposed. In 1920, G. I. Taylor, concerned with the problem of turbulent transport, formulated a random walk model in which the motion of a particle between two turning events was characterized by a finite velocity (Taylor, 1922). The same year the finiteness of the velocity was mentioned by Fürth (1920) in the context of the so-called persistent Brownian motion. Both these models assume that there should be no particles outside the ballistic cone defined by the maximal velocity of the particles. In 1935 Davydov proposed to use the telegraph equation, which contains an additional second order time derivative, to address the existence of the ballistic cone (Davydov, 1934; Bakunin, 2003). As with the diffusion equation, the telegraph equation was discovered much earlier by Kirchhoff and Heaviside in the context of electric current transmission through a conducting line. Around 1950 it was demonstrated that the telegraph equation could be derived from the random walk model proposed by Taylor (Goldstein, 1951). The next milestone in the development of the modern random walk theory was due to

Montroll and Weiss who introduced the continuous-time random walk model (CTRW) (Montroll and Weiss, 1965; Scher and Montroll, 1975). The main innovation of that model is that a particle has to wait for a random time before moving to another point. This model provided the framework necessary for describing anomalous diffusion with the spreading of particles *slower* than in the Brownian diffusion, a process that was named “subdiffusion.”

Richardson (1926) pointed out the possibility of the anomalous diffusion in turbulent flows, where particles spread faster than in normal diffusion, and are referred to as “superdiffusion.” To accommodate for superdiffusive transport, the random walk model was modified to allow particles to perform very long excursions. To step beyond the premises of the central limit theorem (CLT), slow decaying functions with power-law tails and diverging second moment were used as the distributions of the excursion lengths. The scaling properties of the corresponding particle distributions were found to be different from those of the standard Brownian diffusion and thus required a new mathematical apparatus. At this point, a link between the superdiffusion and Lévy stable distributions (Lévy, 1937; Gnedenko and Kolmogorov, 1954) was established. The random walk model with walkers covering long distances instantaneously received the name of *Lévy flight* (Mandelbrot, 1982). In its simplest schematization, this stochastic process could drive a particle over very long distance in a single motion event that is called “flight” (although, in fact, it is a *jump*), so that the mean squared flight length is infinite (Shlesinger and Klafter, 1986). Similar to the concept of Brownian diffusion, Lévy flights served well to describe different transport phenomena. However, Lévy flights have the same trait of infinite propagation speed as the diffusion equation. In addition, the distribution of the particles performing Lévy flight has a divergent second and all higher moments. This poses a significant difficulty in relating Lévy-flight models to experimental data, especially when analyzing the scaling of the measured moments in time.<sup>1</sup> Akin to the Taylor model, the Lévy-flight model was then equipped with a finite velocity of moving particles and therefore produced distributions which are confined to ballistic cones and thus have finite moments. As a contrast to the flight process with instantaneous jumps, the name *Lévy walk* was coined by Shlesinger, Klafter, and Wong (1982). The aim of this review is to show how versatile and powerful the concept

<sup>1</sup>It is not correct, however, to think of the Lévy flight as an abstract mathematical formalism. The mechanisms leading to the dispersion of the observable of interest may not be related to a physical motion of an entity in Euclidean space. For example, it may be caused by long-range interactions (Barkai *et al.*, 2003), or by a nontrivial “crumpled” topology of a phase (or configuration) space of polymer systems (Sokolov, Mai, and Blumen, 1997; Brockmann and Geisel, 2003) and small-world networks (Kozma, Hastings, and Korniss, 2005), or by spectral characteristics of disordered media, amorphous materials, and glasses (Klauder and Anderson, 1962; Zumofen and Klafter, 1994b); see Bouchaud and Georges (1990) for more information and other examples where the Lévy flights are of relevance.

of Lévy walks is in describing a wide spectrum of physical and biological processes involving stochastic transport.

In order to orient the reader in the existing literature on random walks in the context of anomalous diffusion, we mention several monographs which can serve as good introductory material to continuous-time random walks (Klafter and Sokolov, 2011), anomalous diffusion, Lévy flights, and subdiffusion (Montroll and Shlesinger, 1984; Havlin and Ben-Avraham, 1987; Bouchaud and Georges, 1990; Isichenko, 1992; Metzler and Klafter, 2000, 2004), and some reviews on particular applications of these formalisms (Bardou *et al.*, 2001; Balescu, 2005).

### A. Lévy stable laws

One of the fundamental theorems in probability theory is the central limit theorem. It states that the sum of independent identically distributed random variables with a finite second moment is a random variable with the distribution tending to a normal distribution as the number of summands increases. It has a history of development spanning several hundred years, from initial considerations by Laplace and Poisson at the end of the 18th century to the stage of rigorous analysis by Markov, Chebyshev, Lyapunov, Feller, Lévy, and others in the beginning of the 20th century; see Fischer (2010) for a historical overview. Remarkably, the CLT can be cast into the dynamical problem of a particle hopping at random distances. The sum of all displacements (independent random variables) will then determine the final position of a particle (their sum). As a result, the distribution of a particle's position is normal (or Gaussian) if the second moment of the displacement length distribution is finite. Normal distributions are also known to be stable distributions meaning that the sum (or, more generally, a linear combination with positive weights) of two independent random variables has the same distribution (up to a scaling factor and shift). Around 1920, Paul Lévy showed that there are other stable distributions which now bear the name of Lévy alpha-stable distributions. In particular, they have power-law tails and diverging second moments. The generalized central limit theorem (gCLT) was then formulated to state that the sum of identically distributed random variables with distributions having power-law tails converges to one of the Lévy distributions. We now look at the total displacement of a particle whose individual hops are distributed as a power law. The position of the particle after many hops will be described, according to the gCLT, by a Lévy distribution; see, e.g., Uchaikin (2003). That is why such random walks are also known as Lévy flights. It has been found that a large variety of natural and man-made phenomena exhibit power-law statistics (Bouchaud, 1995; Uchaikin and Zolotarev, 1999; Clauset, Shalizi, and Newman, 2009). While relating the empirical data to the theoretical models with Lévy distributions, it became clear that the model solutions could not be characterized by the second moment: Like every individual jump in a sequence, the distribution of particle's final positions has an infinite variance.

One of the most straightforward ways to resolve this inconsistency is to regularize the power-law distributions by truncating them at large values (Mantegna and Stanley, 1994). That would make the moments of the distribution finite

while still retaining some properties of the power-law distributions for intermediate values. However, the truncation introduces a certain arbitrariness and, as a phenomenological procedure, it cannot always be justified in a particular physical (or economical, biological, etc.) context.

Importantly, there is an alternative way to remedy the problem of divergent moments. A fundamental property of having a finite velocity while moving couples the displacement of a walker and the time it takes to cover the corresponding distance and puts a larger time cost to a longer displacement. In the simple picture of a hopping particle that would mean that at any moment of time the position of the particle after many hops is bounded by the ballistic cone with the fronts matching the maximal particle velocity multiplied by the observation time. In between these fronts, the long displacements of the particle still exist, as necessary for the Lévy-like statistics, but all moments of the distribution of the particle's position will be finite for any given time (Shlesinger, Klafter, and West, 1986).

We hope that at this point we have already convinced the reader that random walks are an appropriate language to describe the stochastic transport phenomena. We now proceed to introduce the theoretical framework of continuous-time random walks (Klafter and Sokolov, 2011) and describe how it changes when the finite velocity of walkers is taken into account. Next we mainly focus on one-dimensional systems (some open problems concerning the generalization to higher dimensions are mentioned in Sec. VIII).

### B. Continuous-time random walks

Consider a random motion of passive particles in homogeneous media. We are interested in the macroscopic behavior of the density of particles  $P(x, t)$  as a function of space and time. Each particle can make instantaneous jumps to the left or to the right with equal probabilities. The probability density function (PDF) of the jump lengths  $x$  and  $g(x)$  is chosen to be symmetric  $g(x) = g(-x)$  and independent of the starting point. Before making a jump, a particle waits for a time  $\tau$  defined by another PDF  $\psi(\tau)$ ; see Fig. 1(a). Both distributions are normalized:  $\int_{-\infty}^{\infty} g(x)dx = 1$  and  $\int_0^{\infty} \psi(\tau)d\tau = 1$ . These two functions determine the macroscopic properties of the transport process. In the standard continuous-time random walk model, random variables  $x$  and  $\tau$  are independent from each other. We can define the survival probability  $\Psi(t)$ , that is, the probability for a particle not to jump away until time  $t$ , as

$$\Psi(t) = 1 - \int_0^t \psi(\tau)d\tau. \quad (1)$$

The first transport equation governs the outgoing flow of particles  $Q(x, t)$ , which defines how many particles leave the point  $x$  per unit of time. The equation connects the flux at the current point in space and time to the flux from all neighboring points in the past (Klafter and Silbey, 1980):

$$Q(x, t) = \int_{-\infty}^{+\infty} g(y) \int_0^t \psi(\tau)Q(x-y, t-\tau)dyd\tau + P_0(x)\psi(t). \quad (2)$$



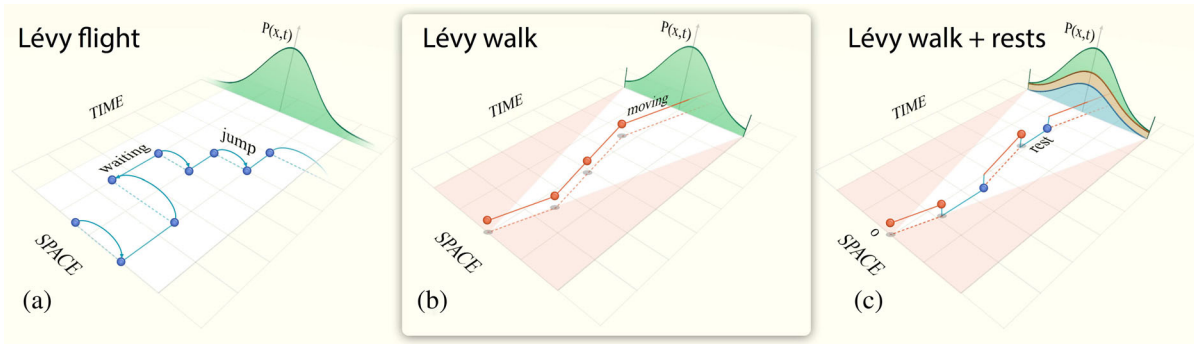


FIG. 1 (color online). Random walk models of superdiffusion. (a) Lévy flight: A particle performs instantaneous jumps alternated with waiting pauses. The length of jumps and durations of waiting events are independent random variables. The resulting PDF  $P(x, t)$  is not local in space at any moment of time. (b) Lévy walk: A particle moves with a constant velocity for a random time and then, at a turning point, instantaneously chooses a new direction and moves again. In this basic model the particle's velocity can assume two values  $\pm v$  only. As a result the length and duration of each event of ballistic motion are coupled. There is a ballistic cone  $x_f = \pm vt$  beyond which no particle can go (shaded area). As a result the PDF is bounded in space and its fronts are marked by two delta peaks. (c) Lévy walk with rests: Ballistic flights of a particle are alternated with pauses during which the particle does not move. At any instant of time the statistical ensemble consists of two fractions of particles, flying and resting. The total PDF is the sum of the two.

It is time for a particle to leave from point  $x$  if its waiting time  $\tau$  has elapsed, which is taken care of via the multiplication by  $\psi(\tau)d\tau$ . The particle could arrive to the point  $x$  time  $\tau$  ago from some other point  $x - y$  by making a jump of length  $y$  with probability density  $g(y)$ . We next integrate over all possible waiting times and jump distances. The last term on the right-hand side assumes that at the moment of time  $t = 0$  particles had an initial distribution  $P_0(x) = P(x, t = 0)$ . Particles gradually leave their initial spots according to the waiting time distribution. We also assume that all particles were introduced to the system at  $t = 0$  and the probability density of making the first jump is given by  $\psi(\tau)$ . The situation is different if the particles have some prehistory. In that case, the probability distribution of the first waiting time is in general different from  $\psi(\tau)$  (Haus and Kehr, 1987); see Sec. IV.D for a more detailed discussion.

The next step is to connect the outgoing flux in the past to the current density of particles at a given point in space and time,

$$P(x, t) = \int_{-\infty}^{+\infty} g(y) \int_0^t \Psi(\tau) Q(x - y, t - \tau) dy d\tau + P_0(x) \Psi(t). \quad (3)$$

The density is a sum of outgoing particles from all other points at different times, weighted by the jump length probability, provided the particles survived for a time  $\tau$  after their arrival to  $x$  at  $t - \tau$ . The last term on the right-hand side of Eq. (3) accounts for the particles that stay in their starting points until the observation time  $t$ .

The above set of equations specifies the standard CTRW process with an arbitrary initial condition. These integral equations can be solved by using the combination of Fourier (with respect to space) and Laplace (with respect to time) integral transforms (Erdélyi, 1954). We exploit the fundamental property of these transforms which turns convolution integrals into products in the Fourier-Laplace space. We use  $k$  and  $s$  to denote coordinates in Fourier and Laplace space,

respectively. By explicitly providing the argument of a function, we distinguish between the normal or transformed space, for example,  $\psi(\tau) \rightarrow \psi(s)$  and  $g(x) \rightarrow g(k)$ . As a result, the solution for the density of particles is given by the Montroll-Weiss equation (Montroll and Weiss, 1965; Klafter, Blumen, and Shlesinger, 1987):

$$P(k, s) = \frac{\Psi(s)P_0(k)}{1 - \psi(s)g(k)}, \quad (4)$$

where  $\Psi(s) = [1 - \psi(s)]/s$ . This solution allows us to reduce the pair of original equations (2) and (3) to a single equation for the density  $P(x, t)$ :

$$P(x, t) = \int_{-\infty}^{+\infty} g(y) \int_0^t \psi(\tau) P(x - y, t - \tau) dy d\tau + \Psi(t) P_0(x). \quad (5)$$

In our derivation, we intentionally used an intermediate step of introducing the flow of particles  $Q(x, t)$ . For a more general initial condition with a nontrivial distribution of particles over their lifetimes, it provides the proper way to obtain the corresponding transport equation for the density of particles (Zaburdaev, 2008). In addition, a very similar set of equations will be used for the models that incorporate the velocity of particles.

Below we frequently use the notion of the *propagator* (and sometimes, depending on the context, Green's function). It is the solution of the transport equation for the deltalike initial distribution<sup>2</sup>  $P_0(x) = \delta(x)$ . From Eq. (4) the propagator can be identified as

<sup>2</sup>A biology-oriented definition of the propagator was made by Ronald Ross, the Nobel laureate for medicine in 1902 (Ross, 1905): "... suppose a box containing a million gnats were to be opened in the center of a large plain, and that the insects were allowed to wander freely in all directions, how many of them would be found after death at a given distance from the place where the box was opened?"

$$G(k, s) = \frac{\Psi(s)}{1 - \psi(s)g(k)}. \tag{6}$$

Then, for any arbitrary initial distribution, the solution will be given by the convolution of an initial profile with the propagator,

$$P(x, t) = \int_{-\infty}^{\infty} G(x - y)P_0(y)dy. \tag{7}$$

The above Eqs. (4) and (6) give a formal solution of the transport equation. When given functions  $g(y)$ ,  $\psi(\tau)$ , and  $P_0(x)$ , one should find their Fourier and Laplace transforms, insert them into Eq. (4), and calculate their inverse transform. Unfortunately, in general it is almost impossible to find this inverse transform analytically. Instead an asymptotic analysis for large time and space scales can be employed. To proceed, we must specify the probability densities  $g(y)$  and  $\psi(\tau)$ . Motivated by applications and also mathematical convenience, we choose a power-law form of these PDFs. By varying the exponent of their power-law tails, different regimes of diffusion can be accessed. Assume the following particular forms:

$$\psi(\tau) = \frac{1}{\tau_0} \frac{\gamma}{(1 + \tau/\tau_0)^{1+\gamma}}, \quad \gamma > 0, \tag{8}$$

$$g(x) = \frac{\Gamma[\beta + 1/2]}{x_0 \sqrt{\pi} \Gamma[\beta] [1 + (x/x_0)^2]^{\beta+1/2}}, \quad \beta > 0. \tag{9}$$

The exact details of these distributions are not qualitatively important in the asymptotic limit. Crucial are their power-law tails which determine the behavior of their moments. All other details will be absorbed into constant prefactors; yet we will keep track of those for the sake of completeness.

There are two important moments of these PDFs, the mean squared jump length

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 g(x) dx, \tag{10}$$

and the mean waiting time

$$\langle \tau \rangle = \int_0^{\infty} \tau \psi(\tau) d\tau. \tag{11}$$

If these moments exist, for the chosen functions in Eqs. (8) and (9), they are given by simple expressions:

$$\langle \tau \rangle = \frac{\tau_0}{\gamma - 1}, \quad \gamma > 1, \quad \langle x^2 \rangle = \frac{x_0^2}{2(\beta - 1)}, \quad \beta > 1. \tag{12}$$

When both quantities are finite, the resulting transport equation reduces to the standard diffusion equation with a diffusion coefficient  $D = \langle x^2 \rangle / (2\langle \tau \rangle)$ ,

$$\frac{\partial P}{\partial t} = D \Delta P(x, t). \tag{13}$$

It is easy to demonstrate by assuming that the typical spatial and temporal scales of interest are significantly larger than

$\langle x^2 \rangle$  and  $\langle \tau \rangle$ , respectively. It is then possible to expand the expression under the integral in Eq. (5) in a Taylor series with respect to  $y$  and  $\tau$  yielding the diffusion equation above; its propagator is the well-known Gaussian distribution:

$$P(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}. \tag{14}$$

The second moment of this distribution is the mean squared displacement (MSD) which scales linearly with time:

$$\langle x^2(t) \rangle = \int_{-\infty}^{\infty} x^2 P(x, t) dx = 2Dt. \tag{15}$$

Another important property of the diffusion process is the scaling of the density profile. As we will see for different models and regimes of random walks, in the limit of large times, the propagator may be represented as

$$G(x, t) = t^{-\alpha} F\left(\frac{x}{t^\alpha}\right), \tag{16}$$

where  $F(\xi)$  is a scaling function (for example, Gaussian, in the case of normal diffusion) and  $\alpha$  is a model-specific scaling exponent (with  $\alpha = 1/2$  in the case of normal diffusion). Such a functional form suggests a scaling variable  $\xi = x/t^\alpha$ , meaning that a characteristic spatial scale on which the density changes  $\bar{x}$  scales with time as  $\bar{x} \propto t^\alpha$ . From the solution given by Eq. (14), we see that the width of the cloud of particles grows as  $\bar{x} \propto t^{1/2}$ .

The asymptotic limit  $x, t \rightarrow \infty$  corresponds to the dual transition  $k, s \rightarrow 0$  in the Fourier-Laplace domain. That is why, instead of the full Fourier and Laplace transforms of  $g$  and  $\psi$  in Eq. (4), their expansion in a Taylor series with respect to small  $k$  and  $s$  can be used.<sup>3</sup> In the Fourier-Laplace coordinates, the leading terms of the expansion for the chosen power-law functions are (Prudnikov, Brychkov, and Marichev, 1986)

$$g(k) = 1 - \frac{x_0^2}{\beta - 1} \frac{k^2}{4} - \frac{x_0^{2\beta} \Gamma[1 - \beta]}{2^{2\beta} \Gamma[1 + \beta]} |k|^{2\beta} + O(k^{2+2\beta}), \tag{17}$$

$$\psi(s) = 1 - \frac{\tau_0}{\gamma - 1} s - \tau_0^\gamma \Gamma[1 - \gamma] s^\gamma + O(s^{1+\gamma}). \tag{18}$$

In the marginal cases, when  $\gamma$  or  $\beta$  have values where one of the moments starts to diverge (for example,  $\gamma, \beta = 1$ ), there are logarithmic correction terms appearing in this expansion. We do not consider these cases here [see, e.g., Zumofen and Klafter (1993) and Chukbar (1995) for more information]. The finite waiting time and the mean squared jump distance correspond to  $\gamma, \beta > 1$ . In this case, the first two terms in the above expansions are dominant for small  $k$  and  $s$ . The prefactors in front of  $k^2$  and  $s$  can be recognized as the mean squared jump distance and the mean waiting time from

<sup>3</sup>Taylor series are normally understood as an expansion in integer powers of the argument. In our case the leading terms of expansion with respect to the small  $k$  and  $s$  may involve fractional powers. We use the notion of Taylor series in this extended sense.

Eq. (12), respectively. By substituting them into the formal solution, Eq. (4), we can rewrite it as the diffusion equation in the Fourier-Laplace space, or simply compute the inverse transforms and obtain the Gaussian distribution Eq. (14). For  $\gamma < 1$  and  $\beta < 1$ , terms with fractional powers dominate over linear and quadratic terms in Eqs. (18) and (17), respectively. By substituting them into Eq. (4) we obtain the following equation in the Fourier-Laplace space:

$$s^\gamma \Gamma[1 - \gamma] P(k, s) = -|k|^{2\beta} K' P(k, s) + s^{\gamma-1} \Gamma[1 - \gamma] P_0(k), \quad (19)$$

where  $K' = (x_0/2)^{2\beta} \Gamma[1 - \beta] / \tau_0^\gamma \Gamma[1 + \beta]$ . After returning to the original space-time domain, we obtain an equation with integral nonlocal operators:

$$\frac{\partial}{\partial t} \int_0^t \frac{P(x, \tau)}{(t - \tau)^\gamma} d\tau = K \int_{-\infty}^{\infty} \frac{P(y, t)}{|x - y|^{2\beta+1}} dy + \frac{P_0(x)}{t^\gamma}, \quad (20)$$

where  $K = (x_0)^{2\beta} \Gamma[\beta + 1/2] / \tau_0^\gamma \sqrt{\pi} \Gamma[\beta]$ . The integral nonlocal operators in the above equation are the *fractional derivatives* (the integral on the right-hand side is understood as the principle value) (Samko, Kilbas, and Marichev, 1993; Podlubny, 1999; Kilbas, Srivastava, and Trujillo, 2006; West, 2014). The notion of a fractional derivative allows us to rewrite the asymptotic balance equation in a compact form of the fractional diffusion equation (Saichev and Zaslavsky, 1997; Metzler and Klafter, 2000; Barkai, 2002):

$$\frac{\partial^\gamma P}{\partial t^\gamma} = K_{\beta, \gamma} \Delta^\beta P + \frac{P_0(x)}{t^\gamma}. \quad (21)$$

This equation describes the stochastic spreading of a cloud of particles and besides the case of normal diffusion has several interesting regimes. If the mean squared jump length is finite but the waiting times are anomalously long,  $\langle \tau \rangle = \infty$ , the resulting dispersal is anomalously slow or subdiffusive. The typical width of the cloud scales as  $\bar{x} \propto t^{\gamma/2}$ , with  $\gamma < 1$ . In the opposite case, when the average waiting time is finite but jumps are very long,  $\langle x^2 \rangle = \infty$ , the equation describes superdiffusion. The typical width of the distribution of particles scales as  $\bar{x} \propto t^{1/2\beta}$ . Finally, when long waiting times compete with long jumps, the scaling is defined by both distributions of waiting times and jump lengths,  $\bar{x} \propto t^{\gamma/2\beta}$ . The stochasticity of the transport process reveals itself in the “forgetting” of the initial distribution and the tendency of the particles’ density to the universal self-similar profile of the corresponding propagator, Eq. (16), with  $\alpha = \gamma/2\beta$ .

Before closing this section, we take a closer look at the superdiffusion regime. The jump length distribution has a diverging second moment ( $\beta < 1$ ), whereas the mean waiting time is finite. Therefore the leading terms in the expansion of  $\psi(s)$  in Eq. (18) could be written as  $\psi(s) \approx 1 + s\langle \tau \rangle$ . The propagator in the Fourier-Laplace space is then given by

$$G(k, s) = \frac{1}{s + K_\beta |k|^{2\beta}} \quad (22)$$

with  $K_\beta = (x_0/2)^{2\beta} \Gamma[1 - \beta] / \langle \tau \rangle \Gamma[1 + \beta]$ . By performing the inverse Laplace transform we obtain

$$G(k, t) = \exp(-K_\beta |k|^{2\beta} t). \quad (23)$$

This expression is the Fourier transform (a characteristic function) of the symmetric Lévy distribution  $L_\kappa[x, \sigma(t)]$ , which describes the distribution of the sum of independent and identically distributed variables with power-law PDFs (Uchaikin, 2003). Here  $\kappa = 2\beta$  is the Lévy index and  $\sigma^\kappa = K_\beta t$  is the scaling parameter. For some particular values of  $\kappa$ , the Lévy distribution has an analytical expression in coordinate space, such as a Cauchy distribution ( $\kappa = 1$ ), Lévy-Smirnov ( $\kappa = 1/2$ ), or a Holtsmark distribution ( $\kappa = 3/2$ ) (Uchaikin, 2003; Klafter and Sokolov, 2011). The key feature of all of these distributions is the asymptotic power-law tail  $G(x, t) \sim t/|x|^{\kappa+1}$  (Chukbar, 1995; Klafter and Sokolov, 2011). In the scaling relation given by Eq. (16), the scaling function  $F$  is also given by the Lévy distribution with  $\alpha = 1/\kappa$ . Because of this intimate relation of the particles density to the Lévy distribution, the model of random walks with instantaneous jumps received the name of Lévy flight<sup>4</sup> (Mandelbrot, 1982). We now ask about the behavior of the second moment of the density profile given by the Lévy distribution. This and higher moments in the Fourier (or the Fourier-Laplace) space are

$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x, t) dx = (-i)^n \frac{d^n}{dk^n} P(k, t) |_{k=0}. \quad (24)$$

By substituting Eq. (23) into this formula, we immediately see that all moments with  $n \geq 2$  diverge. It is thus impossible to compute the MSD as a function of time for a particle performing Lévy flights for a simple reason: it does not exist. Already after the first jump all particles are distributed as  $g(x)$  and this distribution has an infinite second moment. This feature of Lévy flights is often referred to as a shortcoming of the model when applied to physical processes, in which one expects regular behavior of moments (Mantegna and Stanley, 1994). However, as already mentioned, physical intuition points to the possibility to modify the Lévy-flight model assuming that longer jumps must have a higher cost; there has to be a certain coupling between the length of the flight and its duration. The simplest coupling is the finite velocity of particles, when the time to accomplish a flight is linearly proportional to its length. As we show next, the introduction of the finite velocity of particles into the Lévy-flight model retains the anomalous character of the transport process while regularizing the behavior of moments of the particle density. We believe that the compliance of the new model with physical intuition and the ability to account for the velocity of particles explains its success in applications.

<sup>4</sup>Shlesinger and Klafter (1986) proposed an alternative schematization of Lévy flights: There are no waiting pauses and the duration of each step is constant so that the velocity of a flight is proportional to the step length drawn from a Lévy distribution.



## II. LÉVY WALKS

We discuss several ways to introduce a coupling between the jump length and time in the upcoming sections. Here we start with the conventional dynamical coupling of the particle position and current time via a constant velocity of the particle.

There are two closely related models which incorporate a finite velocity of random walkers. The first one is a direct modification of the CTRW model: After spending its waiting time, a random walker does not jump instantaneously but instead moves with a constant speed to its destination (Klafter and Zumofen, 1994; Zaburdaev and Chukbar, 2002); see Fig. 1(c). Long excursions are still responsible for the anomalously fast diffusion, but now there are well-defined ballistic fronts and behavior of all moments is regularized. In the second model, waiting periods are eliminated and the particle is always on the move (Shlesinger, Klafter, and West, 1986; Zumofen and Klafter, 1993; Klafter and Zumofen, 1994); see Fig. 1(b). The distance of ballistic flights is distributed randomly and each flight is performed with a finite speed. At the end of the flight, the particle randomly changes direction. The latter model is what historically received the name of Lévy walks. Note that the Lévy-walk model has only one distribution function to parametrize the motion of the particles as it discards waiting. This minimalistic setup remains the most popular model in modern applications.

### A. Lévy-walk model

The formulation of the microscopic model is very similar to that by Pearson cited in the Introduction. A particle moves on a straight line with a fixed speed for some random time. At the end of the excursion, the particle randomly chooses a new direction of motion and moves for another random time with the same speed; see Fig. 1(b). There are only two characteristics of this model, that are the PDF of the duration of movements, which we denote by  $\psi(\tau)$ , and the speed  $v$  of the particles. Despite its simplicity, this model is able to describe various regimes of stochastic transport, from classical to ballistic superdiffusion.

We now derive the transport equations of the Lévy-walk model. First we introduce the frequency of velocity changes at a given point (analog of the flux of particles from a given point in the CTRW model),

$$\nu(x, t) = \int_{-\infty}^{\infty} dy \int_0^t \phi(y, \tau) \nu(x - y, t - \tau) d\tau + \delta(t) P_0(x). \quad (25)$$

Here we incorporated a *coupled* transition probability density  $\phi(y, t)$  which takes care of the fact that only particles flying from  $x - v\tau$  and  $x + v\tau$  can reach  $x$  in time  $\tau$  and change the direction of their velocities after the flight time of  $\tau$ :

$$\phi(y, \tau) = \frac{1}{2} \delta(|y| - v\tau) \psi(\tau). \quad (26)$$

Therefore, a particle changes its velocity if it is at the end of the flight of duration  $\tau$  which originated from  $x \pm v\tau$ . We assume that at  $t = 0$  all particles at once choose new velocities

and hence the second term on the right-hand side of Eq. (25) contains a delta function [note a difference to a gradual leaving of particles from their waiting positions for the CTRW model, Eq. (2)].

To calculate the actual amount of particles at a given point, we write

$$P(x, t) = \int_{-\infty}^{\infty} dy \int_0^t \Phi(y, \tau) \nu(x - y, t - \tau) d\tau, \quad (27)$$

where

$$\Phi(y, \tau) = \frac{1}{2} \delta(|y| - v\tau) \Psi(\tau) \quad (28)$$

is the probability density to travel a distance  $y$  and remain in the state of flight [note that with respect to  $\tau$  Eq. (26) has the meaning of the probability density whereas Eq. (28) is the probability]. Therefore, a particle is at the point  $(x, t)$  if it has started some time  $\tau$  ago at  $x \pm v\tau$  and is still in the state of the flight, taken care of by multiplication with  $\Psi(\tau)$ , Eq. (1). Note that in Eq. (27) the influence of the initial condition appears only indirectly, through the frequency of velocity changes  $\nu(x, t)$  [cf. Eq. (3) for the CTRW model with an extra term for immobile particles survived from the start]. By taking the limit  $t \rightarrow 0+$  we can substitute  $\nu(x, t)$  by  $P_0(x)\delta(t)$  and recover  $P(x, t) \rightarrow P_0(x)$ .

The equations can be solved by using the combined Fourier-Laplace transform, but an additional technical complexity due to the coupling of space and time variables occurs (Klafter, Blumen, and Shlesinger, 1987; Zumofen and Klafter, 1993). We resolve it by using the shift property of the Laplace and Fourier transforms; as a result the corresponding Laplace transformed functions hold a linear combination of Fourier-Laplace coordinates  $s \pm ikv$  as its argument:

$$P(k, s) = \frac{[\Psi(s + ikv) + \Psi(s - ikv)]P_0(k)}{2 - [\psi(s + ikv) + \psi(s - ikv)]}. \quad (29)$$

This is a formal solution of the problem and, as in the case of Lévy flights, the next step is to perform the asymptotic analysis. Because of the simple ballistic coupling  $x = v\tau$ , the possible scaling regimes of diffusion are governed by the power-law tail of the flight-time distribution, which we again take in the form given by Eq. (8).

### 1. Telegraph equation

If the mean squared flight distance is finite,  $\gamma > 2$ , the classical diffusion takes place in the asymptotic limit. However, the effects of finite velocity can be seen in this regime too. Consider for a moment an exponentially distributed flight time  $\psi(\tau) = (1/\tau_0) \exp(-\tau/\tau_0)$ . By taking its Laplace transform and substituting it into Eq. (29), we can invert the Fourier and Laplace transforms to obtain the telegraph equation (Goldstein, 1951):

$$\frac{\partial P}{\partial t} + \tau_0 \frac{\partial^2 P}{\partial t^2} = D \Delta P(x, t), \quad (30)$$

where  $D = v^2 \tau_0$ . On very short times, it describes almost ballistic spreading of particles. As time goes, ballistic fronts

run away much faster than spreading of the diffusive evolution ( $\bar{x} \propto t$  vs  $\bar{x} \propto t^{1/2}$ , for  $t \gg \tau_0$ ) which starts to dominate the central part of the density profile. Finite velocity ensures, however, that there are no particles beyond the ballistic fronts.

## 2. Superdiffusion

As the flights get longer,  $1 < \gamma < 2$ , the mean squared flight length diverges (but the average flight time is still finite) and we turn to the regime of superdiffusion. By using the expansion from Eq. (18) for small  $s$  and substituting its leading terms into Eq. (29) we arrive at a similar answer as Eq. (22) for the propagator

$$G(k, s) \simeq \frac{1}{s + K_v |k|^\gamma} \quad (31)$$

with  $K_v = \tau_0^{\gamma-1} v_0^\gamma (\gamma - 1) \Gamma[1 - \gamma] \sin(\pi\gamma/2)$ . There are several things to be noted here. After the inverse Fourier-Laplace transform we get the stable time-parametrized Lévy distribution with a scaling given by Eq. (16) and  $F(\xi) = L_\gamma(\xi)$ ,  $\alpha = 1/\gamma$ . For  $1 < \gamma < 2$  the cloud of particles spreads faster than classical diffusion but still slower than the running ballistic fronts. Therefore, ballistic fronts do not appear in this analysis and affect (as in Sec. II.A.1) only the far tails of the particle density distribution. Nevertheless, the existence of fronts is crucial for the calculation of moments, as we show next. Now we take a closer look at what is happening at the ballistic fronts. Assume that at the moment of time  $t = 0$  we start with all particles initiating their flights at  $x = 0$ . Ballistic fronts are formed mostly by the particles which are still in their very first flights. The probability to remain in the flight is  $\Psi(t)$  and therefore we can write down the density of particles in the ballistic peaks or “chubchiks” (Klafter and Sokolov, 2011)

$$G_{\text{front}}(x, t) = \frac{1}{2} \Psi(t) [\delta(x - vt) + \delta(x + vt)]. \quad (32)$$

This gives the first approximation of the whole density of particles as a Lévy distribution sandwiched between two running ballistic deltalike peaks; see Fig. 2. A more detailed understanding of the density can be achieved by using the so-called infinite density measure (Rebenshtok *et al.*, 2014a, 2014b), as discussed in Sec. IV.C.

## 3. Ballistic diffusion

In case of even longer flights  $0 < \gamma < 1$ , when the mean flight time diverges, the diffusion process changes dramatically. In the case of a Lévy flight, the scaling  $\bar{x} \propto t^{1/\gamma}$  for small  $\gamma$  results in spreading which is faster than ballistic, leading to a conflict with the light front limitation in the Lévy-walk setup. Clearly, the ballistic front is playing a crucial role here. An asymptotic expansion of the propagator [obtained from Eq. (29) with  $P_0(x) = \delta(x)$ ] has the following form:

$$G(k, s) \simeq \frac{(s + ikv)^{\gamma-1} + (s - ikv)^{\gamma-1}}{(s + ikv)^\gamma + (s - ikv)^\gamma}. \quad (33)$$

Now Fourier and Laplace variables appear in the same scaling  $s \sim k$  and indicate the ballistic behavior. It was suggested to

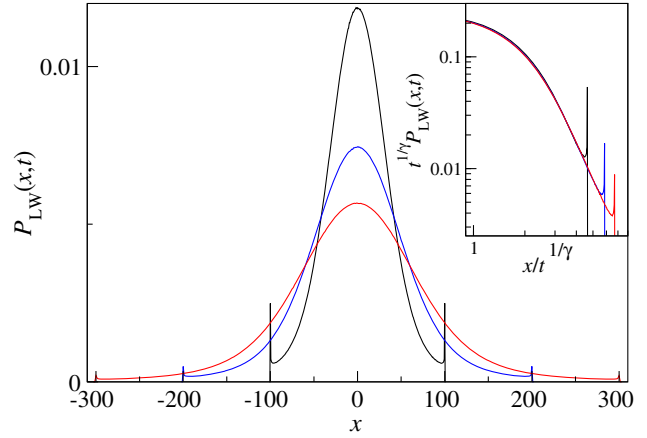


FIG. 2 (color online). Propagators of the superdiffusive Lévy walk for different times. The propagators have a central part of the profile approximated by the Lévy distribution sandwiched between two ballistic peaks. In the inset, the same curves are shown in double logarithmic scale after the rescaling as given on the axis labels. A characteristic linear slope on the log-log plot illustrates the power-law tails of the density. The propagators were obtained by numerical simulations with  $\gamma = 3/2$ ,  $\tau_0 = 1$ , and all particles starting their flights at  $t = 0$ ,  $x = 0$  with velocities  $v = \pm 1$ . The PDFs were sampled at  $t = 100, 200$ , and  $300$  (the width increases with time).

call the inverse Fourier-Laplace transform of  $(s + ikv)^\gamma + (s - ikv)^\gamma$  as a fractional generalization of the substantial or material derivative operator,  $(v^{-1} \partial / \partial t \pm \partial / \partial x)^{1/\gamma}$  (Sokolov and Metzler, 2003). In the ballistic regime there is again a technical difficulty to find the inverse Fourier-Laplace transform. The ballistic case is special in that its analytical solution can be found by the method discussed in Sec. IV.B. Here we just illustrate the shape of the propagator on a particular example  $\gamma = 1/2$  [for arbitrary  $\gamma$  the answer is given by the Lamperti distribution (Lamperti, 1958; Bel and Barkai, 2005); see also Eqs. (61) and (83)]:

$$G(x, t) = \frac{1}{\pi(v_0^2 t^2 - x^2)^{1/2}}. \quad (34)$$

Figure 3 shows a U-shaped profile [for  $\gamma \gtrsim 0.6$  the shape is W like; see Fig. 6(b)] with a divergent density at the ballistic fronts (Zumofen and Klafter, 1993). This divergence is, however, integrable and the total number of particles is conserved. Although the density profile is very different from the Gaussian profile of the classical diffusion, the ballistic diffusion remains a stochastic transport phenomena where the initial condition is gradually forgotten with time and the solution approaches the universal self-similar profile of the corresponding Green’s function.

## 4. Mean squared displacement and other moments

The PDFs of the Lévy walk in general do not possess a global scaling where the whole propagator can be represented in the form of Eq. (16), unlike the case of the Gaussian profile and normal diffusion. A clear example is the superdiffusive regime ( $1 < \gamma < 2$ ), where the middle



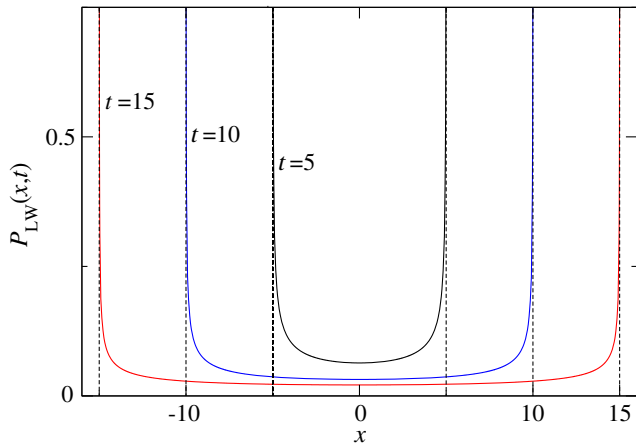


FIG. 3 (color online). PDF of the Lévy-walk model in the ballistic regime. The density of particles, Eq. (34), at different moments of time is shown. The parameters are  $\gamma = 1/2$ ,  $v = \pm 1$ ,  $\tau_0 = 1$ , and  $P_0(x) = \delta(x)$ . The densities have integrable divergences at the ballistic fronts due to the conservation of the total amount of particles.

part of the profile scales as  $\bar{x} \propto t^{1/\gamma}$ , but, at the same time, the fronts exhibit the ballistic scaling. As a consequence, in Lévy walks the scaling exponents of the propagator and MSD are not the same; for a summary of these issues see Schmiedeberg, Zaburdaev, and Stark (2009), Rebenshtok *et al.* (2014a, 2014b).

As discussed, the asymptotic dynamics of propagators can be analyzed by considering the limits  $k \rightarrow 0$  and  $s \rightarrow 0$ . Remarkably, in coupled models these two limits do not commute (Schmiedeberg, Zaburdaev, and Stark, 2009). In general, by changing the order of these limits we imply which effect is dominating: larger distance or longer time, or maybe the interplay of both. The MSD can be calculated by using Eq. (24) via the second derivative with respect to  $k$  and taking the limit  $k \rightarrow 0$ . To compute the asymptotic time dependence of the MSD we can take the second limit  $s \rightarrow 0$  and then calculate the inverse Laplace transform. By inverting the order of limits and first taking  $s \rightarrow 0$ , we can follow the behavior of the density of particles closer to the origin, from where the scaling exponent of the propagator  $\alpha$  [see Eq. (16)] could be obtained. Finally, in order to find the shape of the propagator, both limits have to be taken simultaneously. We first provide the results for the scaling of the MSD (Zumofen and Klafter, 1993). To compute the MSD for a superdiffusive sub-ballistic regime (the Lévy-walk processes with the finite mean flight time  $\langle \tau \rangle$ , corresponding to  $1 < \gamma < 2$ ), one more term in the expansion of the nominator has to be included, in order to capture the effect of the ballistic fronts. This leads to

$$\langle x^2(t) \rangle \propto \begin{cases} t^2 & 0 < \gamma < 1, \\ t^2 / \ln t & \gamma = 1, \\ t^{3-\gamma} & 1 < \gamma < 2, \\ t \ln t & \gamma = 2, \\ t & \gamma > 2. \end{cases} \quad (35)$$

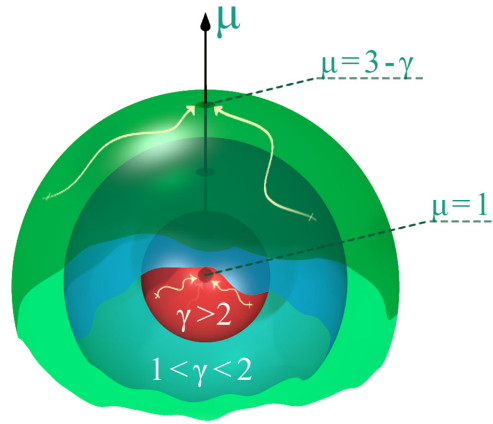


FIG. 4 (color online). Scaling exponent of the mean squared displacement for Lévy walks, Eq. (35). When the second moment of the PDF  $\psi(\tau) \sim \tau^{-\gamma-1}$  exists,  $\gamma > 2$ , the scaling  $\langle x^2(t) \rangle \propto t^\mu$  is universal with the exponent  $\mu = 1$ . When  $1 < \gamma < 2$  and the variance  $\langle \tau^2 \rangle$  diverges, the mean squared displacement scales with the exponent  $\mu = 3 - \gamma$ . Finally, for very heavy tails  $0 < \gamma < 1$ , the scaling is ballistic  $\mu = 2$ . Inspired by a sketch in Bouchaud and Georges, 1990.

Figure 4 gives a pictorial view of the MSD scaling regimes. The scaling exponent  $\alpha$  is given by Zumofen and Klafter (1993):

$$\alpha = \begin{cases} 1 & 0 < \gamma < 1, \\ 1/\gamma & 1 < \gamma < 2, \\ 1/2 & \gamma > 2. \end{cases} \quad (36)$$

It is important to note that in the sub-ballistic regimes the scaling exponent  $\alpha$  refers to the central part of the density profile.

There is an interesting concept to characterize the stochastic transport phenomena by using a spectrum of fractional moments (Castiglione *et al.*, 1999; Metzler and Klafter, 2000; Artuso and Cristadoro, 2003; Sanders and Larralde, 2006; de Anna *et al.*, 2013; Rebenshtok *et al.*, 2014b; Seuront and Stanley, 2014):

$$\langle |x|^q \rangle = \int_{-\infty}^{\infty} |x|^q P(x, t) dx \approx M_q \cdot t^{q\nu(q)}. \quad (37)$$

For normal diffusion, because of its self-similar shape and the unique scaling  $\bar{x} \propto t^{1/2}$ , Eq. (37) leads to a constant value of  $\nu(q) = 1/2$ . If  $\nu(q)$  is not constant, this kind of diffusion process is referred to as strongly anomalous (Castiglione *et al.*, 1999). Because of its multiscaling property, sub-ballistic Lévy walks belong to this class. Figure 5 shows a characteristic bilinear shape of  $q\nu(q)$  as a function of the moment order  $q$ . The linear dependences  $q/\gamma$  for small  $q$  is replaced by the dependence  $q - \gamma + 1$  for higher moments. For small values of  $q$ , the dominating contribution to the averaging integral comes from the central part of the propagator, where it can be approximated by the self-similar Lévy distribution. For  $1 < \gamma < 2$  the fractional moments of Lévy distribution exist for  $q < 2$  and  $q\nu(q) = q/\gamma$ . For higher moments, the far tails

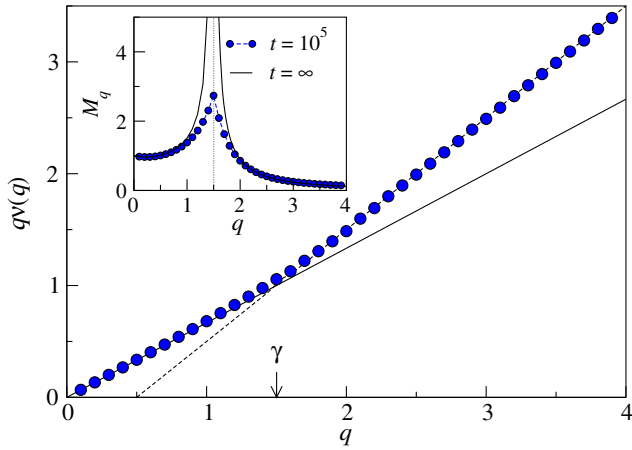


FIG. 5 (color online). Scaling of the fractional moments of the Lévy-walk process. Scaling exponent  $q\nu(q)$  for the  $q$ th moment of the PDF, see Eq. (37), as a function of  $q$  for the sub-ballistic Lévy-walk model,  $\gamma = 3/2$ . It has a characteristic bilinear behavior  $q\nu(q) = q/\gamma$  for  $q < \gamma$  (solid line) and  $q\nu(q) = q + 1 - \gamma$  (dashed line) otherwise. The inset shows the prefactors of fractional moments  $M_q$ . Dots correspond to the numerical data sampled for  $t = 10^5$ , whereas lines diverging at  $q = \gamma$  are analytical predictions in the limit  $t \rightarrow \infty$ . Adapted from Rebenshtok *et al.*, 2014a.

of the propagator are important and this is where the ballistic cutoff by the running peaks plays a crucial role. The scaling of the fractional moments for large  $q$  can be obtained by assuming that the PDF has an asymptotic shape  $P(x, t) \sim t/|x|^{1+\gamma}$  and has to be integrated with  $|x|^q$  until the cutoff distance  $|x| = v_0 t$ . That would lead to the  $q - \gamma + 1$  result or  $\nu(q) \sim 1$ . Exact results on the behavior of fractional moments could be obtained by using the concept of infinite densities (Rebenshtok *et al.*, 2014b), which is discussed in Sec. IV.C.

For  $q \rightarrow 0$ , the limit  $\nu(q) \rightarrow \alpha$  gives a possible way of estimating the scaling from the experimental data. Recently Gal and Weihs (2010) measured the spectrum of exponents  $q\nu(q)$  for the dispersion of polystyrene bead particles internalized by live human metastatic breast cancer epithelial cells and found for large  $q$  a linear behavior  $q\nu(q) \sim cq$  with  $c \approx 0.8 - 0.6$ . That means that the observed spreading is of the sub-ballistic superdiffusion type. This is probably related to the active transport of the beads within a cell.

## B. Lévy walks with rests

When performing Lévy walks, a particle always moves [see Fig. 1(b)], whereas during CTRW evolution it makes instantaneous jumps alternated with waiting events [see Fig. 1(a)]. By combining both of them, we arrive at the model where waiting periods alternate with periods of ballistic motion [see Fig. 1(c)]. One can describe this model as a Lévy walk interrupted by rests (Klafter and Zumofen, 1994; Zaburdaev and Chukbar, 2002; Klafter and Sokolov, 2011). As in the standard Lévy-walk model, there cannot be particles beyond the fronts  $|x| > vt$ . Interestingly, there is a natural separation of particles into two groups: sitting at a given point and moving somewhere else. The total density of particles at a

given point  $x$  is the sum of two fractions (Zaburdaev and Chukbar, 2002; Uchaikin, 2003). The PDF of resting times we denote as  $\psi_r(\tau)$  (Klafter and Sokolov, 2011), and the PDF  $\psi(\tau)$ , as before for Lévy walks, is used to describe the durations of ballistic phases. Both functions are of the same power-law form but may have different exponents. By  $\tilde{\nu}(x, t)$  we denote the flux of particles which finished their rest and started moving out of a given point  $x$  (analogy to the velocity reorientation points in the standard Lévy-walk model). It satisfies the following balance equation:

$$\begin{aligned} \tilde{\nu}(x, t) = & \int_0^t \psi_r(\tau) \int_0^{t-\tau} \phi(y, \tau_1) \tilde{\nu}(x-y, t-\tau-\tau_1) d\tau_1 d\tau \\ & + \psi_r(t) P_0(x), \end{aligned} \quad (38)$$

where  $\phi(y, \tau)$  is the coupled transition probability of the Lévy-walk model. The densities of sitting and flying particles are then given by

$$\begin{aligned} P_r(x, t) = & \int_0^t \Psi_r(\tau) \int_0^{t-\tau} \phi(y, \tau_1) \tilde{\nu}(x-y, t-\tau-\tau_1) d\tau_1 d\tau \\ & + \Psi_r(t) P_0(x), \end{aligned} \quad (39)$$

$$P_{\text{fly}}(x, t) = \int_0^t \Phi(y, \tau) \tilde{\nu}(x-y, t-\tau) d\tau, \quad (40)$$

where  $\Phi(x, t)$  is the coupled survival probability of Lévy walks (28). The total density of particles is the sum of flying and sitting PDFs  $P_\Sigma = P_{\text{fly}} + P_r$ . In the Fourier-Laplace space it can be expressed as (Klafter and Sokolov, 2011)

$$P_\Sigma(k, s) = \frac{[\Phi(k, s)\psi_r(s) + \Psi_r(s)]P_0(k)}{1 - \psi_r(s)\phi(k, s)}. \quad (41)$$

The first and second terms in the brackets of the nominator correspond to the contributions from the flying and sitting particles, respectively. As in the case of the CTRW, the long trapping times can compete with long excursions. If, however, the mean trapping time is finite, the scaling of the propagator and the corresponding MSD is the same as in the Lévy-walk model. It is also easy to see from Eq. (41) that if both mean resting time and the mean moving time are finite, the density of the flying particles is locally proportional to that of the resting particles. The coefficient of proportionality is the ratio of times a particle spends on average in each phase (Zaburdaev and Chukbar, 2002),  $P_{\text{fly}}(x, t) = \langle \tau \rangle_{\text{fly}} / \langle \tau \rangle_r P_r(x, t)$ . In the regime when the mean flight time diverges, there is an irreversible transition of resting particles into flying ones (see Fig. 6) and therefore a convergence to the standard Lévy-walk process. If at  $t = 0$  all particles are resting, their total population will decrease in time as  $\int_{-\infty}^{\infty} P_r(x, t) dx \propto t^{\nu-1}$ .

With that we close the discussion of the relatives of the standard Lévy-walk model. We meet them again in Secs. V and VI, when discussing their applications in both physics and biology. We now proceed to the generalizations of the Lévy-walk model.

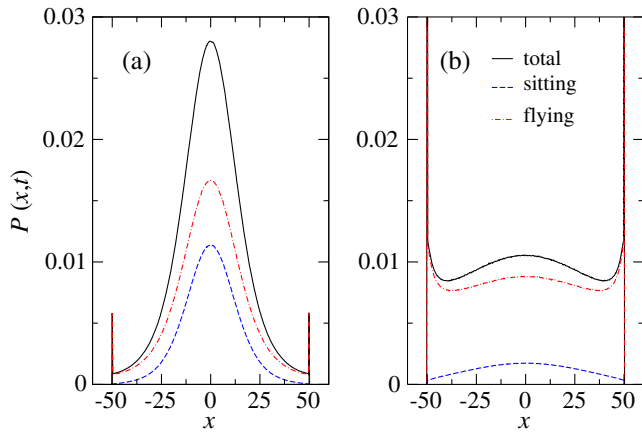


FIG. 6 (color online). Lévy walks with rests. (a) The PDF of the Lévy-walk model with exponentially distributed resting times and power-law distribution of flight times with  $\gamma = 3/2$ . Both average resting ( $\tau_r$ ) and flying times are finite. The total density of particles (solid line) is the sum of sitting (dashed line) and flying (dash-dotted line) particles. (b) The PDF for the ballistic regime with  $\gamma = 0.8$ . Here the number of sitting particles is greatly reduced. In the asymptotic limit  $t \rightarrow \infty$  the total PDF and the PDF of flying particles will coincide. The parameters are  $v = \tau_0 = \langle \tau_r \rangle = 1$  and  $t = 50$ .

### III. GENERALIZATIONS OF THE LÉVY-WALK MODEL

#### A. Random walks with random velocities

A natural generalization of the Lévy-walk model is the process in which the velocity of a particle is not fixed but is a random variable itself (Barkai and Klafter, 1998; Zaburdaev, Schmiedeberg, and Stark, 2008). A number of examples where a random walker has a changing velocity is discussed by Zaburdaev, Schmiedeberg, and Stark (2008). When the velocity of particles is characterized by a heavy tailed distribution, the palette of possible diffusion regimes is defined by the interplay of flight time and velocity distributions. We denote the velocity PDF by  $h(v)$  and write down the corresponding transport equations of random walks with random velocities (RWRV) (Zaburdaev, Schmiedeberg, and Stark, 2008):

$$\nu(x, t) = \int_{-\infty}^{\infty} dv \int_0^t \nu(x - v\tau, t - \tau) h(v) \psi(\tau) d\tau + \delta(t) P_0(x), \quad (42)$$

$$P(x, t) = \int_{-\infty}^{\infty} dv \int_0^t \nu(x - v\tau, t - \tau) \Psi(\tau) h(v) d\tau. \quad (43)$$

Despite the fact that the equations now are more complicated they can still be resolved by using the integral transforms

$$P(k, s) = \frac{\int_{-\infty}^{\infty} \Psi(s + ikv) h(v) dv}{1 - \int_{-\infty}^{\infty} \psi(s + ikv) h(v) dv}. \quad (44)$$

It is easy to see that for  $h(v) = [\delta(v - u_0) + \delta(v + u_0)]/2$  we recover the standard Lévy-walk model result of Eq. (29). Because of the additional complexity added through velocity

distribution it is even harder to find an example where an exact analytical solution can be obtained. However, one remarkable example is the case of the Lorentzian or Cauchy velocity distribution:

$$h(v) = \frac{1}{u_0 \pi} \frac{1}{1 + v^2/u_0^2}. \quad (45)$$

This velocity distribution appears in physical problems of two-dimensional turbulence (Tong and Goldburg, 1988; Min, Mezić, and Leonard, 1996; Chukbar, 1999), as a model distribution of kinetic theory (Ben-Naim, Machta, and Machta, 2005; Trizac, Barrat, and Ernst, 2007), and also as a particular case of the generalized kappa distributions of plasma physics applications (Hasegawa, Mima, and Duong-van, 1985; Meng, Thorne, and Summers, 1992) and statistics (Tsallis, 1988, 1999). It was also reported for the distribution of velocities of starving amoeba cells (Takagi *et al.*, 2008). In this case the density of particles does not depend on the flight-time distribution at all and also has a Lorentzian shape:

$$P(x, t) = \frac{u_0 t}{\pi(x^2 + u_0^2 t^2)}. \quad (46)$$

To understand the scaling behavior of the RWRV model, we use the scaling regimes of the CTRW model as a guideline. For that we calculate the effective jump length distribution, which, due to a simple coupling  $x = vt$ , can be obtained by the following integration:

$$g_{\text{eff}}(x) = \int_{-\infty}^{+\infty} dv \int_0^{+\infty} \delta(x - v\tau) h(v) \psi(\tau) d\tau. \quad (47)$$

For the velocity distribution we assume a generic power-law form  $h(v) \propto |v|^{-1-2\delta}$ . We can now integrate Eq. (47) and find the exponent of the tail of the effective jump length distribution  $\beta(\gamma, \delta)$ . The waiting time distribution of the CTRW model represents the time cost of the flight; therefore we can use the flight-time distribution exponent. By substituting  $\gamma$  and  $\beta(\gamma, \delta)$  into the scaling relation for the CTRW, we find the scaling exponent  $\alpha = \gamma/2\beta$  of the RWRV model; see Fig. 7.

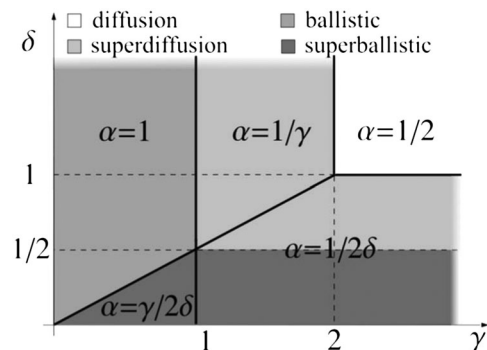


FIG. 7. Scaling regimes of the random walks with random velocities. By varying the exponents of the power-law tails of the velocity distribution  $\delta$  and the flight-time distribution  $\gamma$ , the model can be tuned into different diffusion regimes, from classical diffusion to superballistic superdiffusion. From Zaburdaev, Schmiedeberg, and Stark, 2008.



Besides the classical diffusive, superdiffusive, and ballistic transport, superballistic scaling is possible. In the latter case, the mean absolute velocity has to be infinite ( $\delta < 1/2$ ). As in the Lévy-walk model, the regime of subdiffusion is inaccessible; with nonzero velocities there is no possibility to trap a particle for a long time. As we see, the introduction of the velocity distribution significantly increases the flexibility of the model while still keeping it amenable to the analytical approach.

### B. Random walks with velocity fluctuations

In all previous models we neglected interactions of the walker with its environment or assumed that it had no effect on the particle as it moved. In this section we discuss a model of random walks in active media. We assume that a particle can interact with its surroundings which results in the weak fluctuations of the particle's velocity. The term "active" emphasizes the fact that particles not simply lose velocity as a result of passive friction but can gain positive and negative velocity increments such that on average their velocity remains constant during a single flight event; see Fig. 8. This model was applied to reproduce the perturbation spreading in Hamiltonian many-particle systems by Zaburdaev, Denisov, and Hänggi (2011) and Denisov, Zaburdaev, and Hänggi (2012); see Sec. V.B. To set up the model accounting for velocity fluctuations, we modify the Lévy-walk model (Zaburdaev, Denisov, and Hänggi, 2011).

During each flight of a particle, its position is described by a simple Langevin equation  $\dot{x} = v_0 + \zeta(t)$  (Van Kampen, 2011), where  $\zeta(t)$  is a delta-correlated Gaussian noise of zero mean and finite intensity  $D_v$ , i.e.,  $\langle \zeta(t)\zeta(s) \rangle = D_v \delta(t-s)$ . This equation describes the well-known biased Wiener process with drift  $v_0$  (Karatsas and Shreve, 1997). After an integration over a time interval  $\tau$ , we obtain

$$x(t + \tau) = x(t) + v_0\tau + w(\tau), \quad (48)$$

where  $w(\tau) = \int_t^{t+\tau} \zeta(s) ds$  is characterized by a Gaussian PDF  $p(w, \tau)$  with the dispersion  $\sigma_w^2 = \langle [x(t) - v_0\tau]^2 \rangle = D_v\tau$ . Transport equations for this model can also be written and solved in the Fourier-Laplace space. When velocity fluctuations are small,  $(D_v\langle\tau\rangle)^{1/2} \ll v_0\langle\tau\rangle$ , the central part of the density profile of particles is given by the same Lévy distribution as in the case of the standard model. New

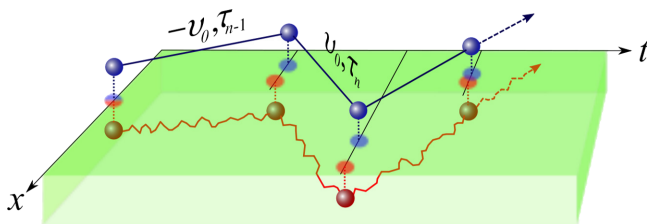


FIG. 8 (color online). The random walk in active media. Velocity during flights fluctuates around a fixed averaged value. As a result the fluctuations accumulate with time and the final position of the particle, passing through the active medium, will differ from that produced by an ideal Lévy-walk process. From Zaburdaev, Denisov, and Hänggi, 2011.

phenomena appear in the ballistic regions, where fronts, due to fluctuations, now look like humps (see Fig. 9):

$$P_{\text{hump}}(x, t) = \Psi(t)[p(x + v_0t, t) + p(x - v_0t, t)]/2. \quad (49)$$

As before,  $\Psi(t)$  is the probability of not changing the direction of flight during time  $t$  [Eq. (1)] and has a power-law asymptotic  $\Psi(t) \propto (t/\tau_0)^{-\gamma}$ . Consequently, the area under the ballistic humps, Eq. (49), also scales as  $t^{-\gamma}$ . During ballistic flights, the particles undergo random fluctuations caused by velocity variations. All particles in the hump are in the state of their first flight of duration  $t$ , thus the dispersion of the Gaussian-like humps grows as  $t^{1/2}$ , and we arrive at the following scaling for the particles density in the humps:

$$P_{\text{hump}}(\bar{x}, t') \approx u^{-\gamma-1/2} P_{\text{hump}}(\bar{x}/u^{-1/2}, t), \quad (50)$$

where  $u = t'/t$  and  $\bar{x} = x - v_0t$ ; see the inset in Fig. 9. From this result we learn that ballistic humps may carry some additional information about the interactions between the random walking particles and their environment.

The two models we discussed are the most frequently used modifications of the original Lévy-walk setup. In the next section, we mention two more models of coupled random walks which introduce higher time cost for longer jumps, but still have instantaneous jumps as the standard CTRW model.

### C. Other coupled models

There are two modifications of the CTRW model which are very similar to the Lévy-walk model, but still do not have a well-defined velocity of particles (Kotulski, 1995; Becker-Kern, Meerschaert, and Scheffler, 2004; Meerschaert and Scalas, 2006; Straka and Henry, 2011; Jurlewicz *et al.*, 2012). In both models the waiting time and jump distance are

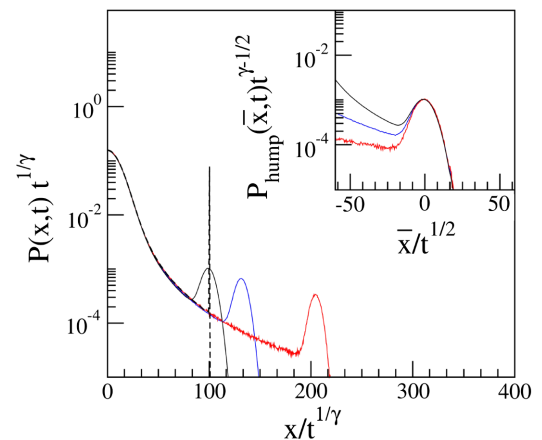


FIG. 9 (color online). Profile of the Lévy-walk model with velocity fluctuations. The inset shows the scaling of humps. Please note that the figure is adapted from Zaburdaev, Denisov, and Hänggi (2011) [see also Zaburdaev, Denisov, and Hänggi (2012)], where a so-called equilibrated initial condition was used (see Sec. IV.D for details). As a result the height of the hump decays slower than discussed in the text, Eq. (50). Independent of the type of the initial condition, the shape of humps is Gaussian, with their width growing as  $t^{1/2}$ .

coupled but jumps are instantaneous. There are two possibilities: In the “jump first” model, a random walker first jumps to a random distance  $y$  and then waits at the arrival point for a time  $\tau = a|y|$ , where  $a$  is a positive coupling constant (Zaburdaev, 2006). In the “wait first” model, a particle first waits for a random time  $\tau$  and then makes a jump of length  $|y| = \tau a^{-1}$  (Shlesinger, Klafter, and Wong, 1982; Barkai, 2002). Figure 10 shows trajectories of these two models compared to the one of the standard Lévy-walk model. We see that in the  $(x, t)$  plane the turning points of all three random walks are identical, and it is the paths which are different. At time  $t$  the models differ only by their last step. However, this difference is crucial. The transport equations for jump first and wait first models can be written down and solved in the Fourier-Laplace space; see Schmiedeberg, Zaburdaev, and Stark (2009). The MSD of the jump first model for the anomalous diffusion regime (long jumps) is diverging, which is clear if one looks at the distribution after the first jump. The wait first model resembles a Lévy-walk model in that it also has a defined light front and therefore finite moments (Schmiedeberg, Zaburdaev, and Stark, 2009). All three models have the same scaling properties of their propagators, but the shapes of propagators are model specific. Figure 11 shows the density profiles for the three models with similar (leading to the same scaling) jump, waiting, and flight-time distributions, and with all remaining proportionality constants set to 1,  $a = v = 1$ .

These two simple coupled models can serve as a starting point for further generalizations. Here we looked only at linear couplings but it can be extended to the case when the flight

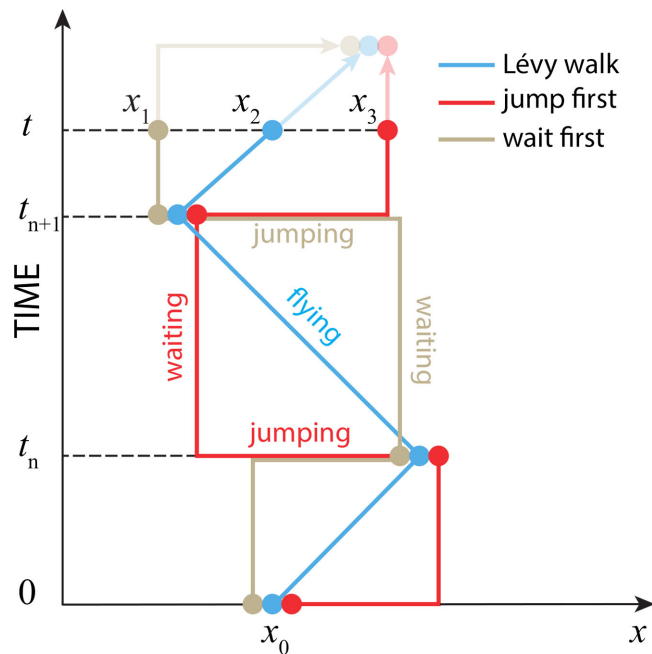


FIG. 10 (color online). Comparison of trajectories of the Lévy-walk model and two coupled models, “wait first” and “jump first.” Trajectories of all three models are passing through the same points on the  $(x, t)$  diagram, but taking different paths. The positions of particles at time  $t$  are determined by their last steps. From Froemberg *et al.*, 2015.

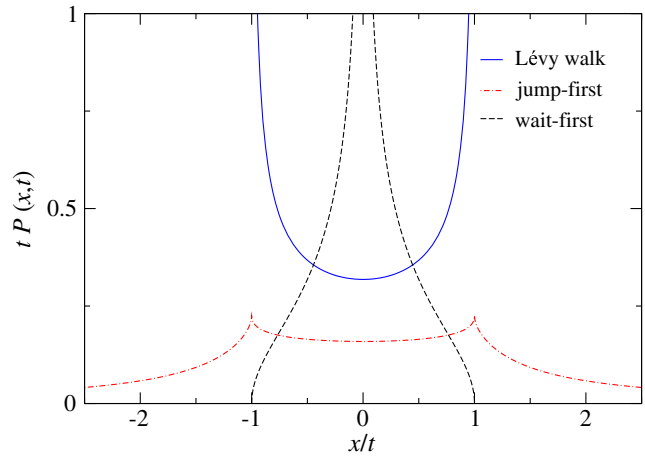


FIG. 11 (color online). Propagators of coupled models. Analytical solutions for the propagators of the wait first (dashed line) and jump first (dash-dotted line) models are compared with a Lévy-walk propagator (solid line) in the ballistic regime with  $\gamma = 1/2$ . The results are obtained by the method discussed in Sec. IV.B; see Froemberg *et al.*, (2015).

distance is some power-law function of the flight time (Metzler and Klafter, 2004); it will then affect the scaling of the propagator and its moments. One interesting example of the wait first model assumes that the jump length is distributed as a Gaussian function with a variance which linearly depends on the waiting time. In this case, for the power-law distributed waiting times, the resulting equation contains the diffusion operator but in a fractional power (Shlesinger, Klafter, and Wong, 1982; Becker-Kern, Meerschaert, and Scheffler, 2004). In the spirit of the Lévy-walk model with rests, one can also add velocity to the above two coupled models. It is interesting to see how a new effective velocity arises as a combination of the coupling  $a^{-1}$  and actual velocity of the particles  $v$ ,  $v_{\text{eff}} = v/(1 + av)$  (Zaburdaev, 2006). In this direction, one could consider more general models. For example, variation of the Lévy walk, in which the waiting times are power law distributed but jump lengths scale nonlinearly with waiting times  $\psi(x|\tau) = \psi(\tau)\delta(|x| - u\tau^\beta)$ , with  $\beta > 0$  and  $\beta \neq 1$ . A nonlinear coupling provides a way to step beyond a ballistic scaling regime of Lévy walks. It appears in the description of anomalous Richardson diffusion (Shlesinger, Klafter, and West, 1986; Klafter, Blumen, and Shlesinger, 1987; Shlesinger, West, and Klafter, 1987) with  $\beta = 3/2$  or, in a slightly more involved form, in the context of cold atom dynamics with the same  $\beta = 3/2$ , which we discuss in Sec. V.E.

With this section we finalize the review of the existing random walk models which embrace the concept of finite velocity of particles and the coupled nature of the spatio-temporal transport process. The following Sec. IV is dedicated to more sophisticated properties of Lévy walks and more advanced tools for their analysis.

#### IV. PROPERTIES OF LÉVY WALKS

While the results of the previous sections can be rated as basic tools needed for applications of Lévy walks in practice,

the following material goes into more detail and as a result is more involved. However, it touches upon fundamental concepts of physics, such as aging, ergodicity, and space-time correlations. Most of these results are very recent thus indicating that the properties of Lévy walks are still being explored.

### A. Space-time velocity autocorrelation function

Finite velocity of walking particles brings a random walk model closer to the basic physical principles and makes it more suitable for the description of real-life phenomena. However, the presence of the well-defined velocity in random walks brings additional properties to these stochastic process. In the realm of the continuous mechanics, the space-time velocity autocorrelation function is a fundamental quantity characterizing the dynamics of a fluid or other media. It reveals the relation between the velocities at two distant points and two different instants of time. Remarkably, the notion of continuous theory can be adapted to the single-particle process of random walks. We can ask how the velocity of a random walker is correlated to its own velocity at some later moment of time but also at a certain distance from the starting point. A naive expectation for a random walk, where each next step is independent from the previous one, is that the correlations will be zero at a distances larger than a single flight. It was shown, however, that even for the regime of classical diffusion, but with finite velocity, the space-time velocity correlation function is different from zero and has a nontrivial space-time dependence (Zaburdaev, Denisov, and Hänggi, 2013).

The space-time velocity autocorrelation function for a single-particle process can be redefined from the conventional expression (Monin, Yaglom, and Lumley, 2007)

$$C_{vv}(x, t) = \langle v(0, 0)v(x, t) \rangle. \quad (51)$$

We assume that the particle starts its walk with an initial velocity  $v(x = 0, t = 0) = v_0$ . After a time  $t$  the particle is found at the point  $x$  with some velocity  $v(x, t)$ . To estimate  $C_{vv}(x, t)$ , an observer at time  $t$  averages the product of the actual and the initial velocities of all particles that are located within a bin  $[x, x + dx]$ . It can be formalized in the following way:

$$C_{vv}(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v v_0 \frac{P(v, x, v_0, t)}{P(x, t)} dv_0 dv, \quad (52)$$

where  $P(v, x, v_0, t)$  is the joint PDF for a particle to start with velocity  $v_0$  and to be in the point  $x$  at time  $t$  with velocity  $v$ . The particle has first to arrive at the point  $x$  for the measurement to occur; therefore we use Bayes' rule (Grinstead and Snell, 1997) for the conditional probability to obtain the integral above. The spatial density  $P(x, t)$  is usually a known quantity for a given random walk model. In contrast, a challenging quantity to tackle is the joint probability of particles' positions and velocities. To focus on its role, the spatial density of the velocity correlation function can be introduced:

$$C(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v v_0 P(v, x, t|v_0) h(v_0) dv_0 dv. \quad (53)$$

Here  $h(v_0)$  is the distribution of the initial velocities, which also signals that we are using the formulation of the random walk with random velocities. The integration over  $x$ , Eq. (53) yields the standard temporal velocity autocorrelation function  $C(t) = \langle v(0)v(t) \rangle$ . Normalizing  $C(x, t)$  by the particle density  $P(x, t)$ , we return to the original velocity-autocorrelation function:

$$C_{vv}(x, t) = C(x, t)/P(x, t). \quad (54)$$

As for every model we considered so far, the integral transport equations can be derived for  $P(v, x, t|v_0)$  and solved by using the Fourier-Laplace transforms. The definition of the velocity autocorrelation function contains two additional integrals with respect to the final and initial velocities, which makes the final answer more involved (Zaburdaev, Denisov, and Hänggi, 2013).

However, the asymptotic analysis of the general answer in the limit of large time and space scales retrieves some surprisingly simple results. Consider first the Lévy-walk regime of the velocity model,  $h(v) = [\delta(v - v_0) + \delta(v + v_0)]/2$ , with a power-law distributed flight time, Eq. (8). The density of particles is sandwiched between the two ballistic peaks. For the peaks we get  $C(x = \pm v_0 t, t) = v_0^2 \Psi(t) \delta(x \pm v_0 t)/2$  and  $C_{v,v} = v_0^2$ . For the central part of the propagator, in the regime of classical diffusion,  $\gamma > 2$ , the density of the correlation function is proportional to the first time derivative of the particle's density:

$$C_{\text{centr}}(x, t) = v_0^2 D \langle \tau \rangle \Delta P(x, t) = v_0^2 \langle \tau \rangle \frac{\partial P(x, t)}{\partial t}. \quad (55)$$

The above asymptotic result is valid for any flight-time distribution with a finite second moment; see Fig. 12(a). We consider this result more closely. As mentioned, by integrating the density of the space-time velocity-autocorrelation function over the coordinate we should obtain the standard temporal correlation function  $C(t)$ . In the long time limit, the integral of the central part is approaching zero [because of the Laplacian operator in Eq. (55)]. The only nonzero contribution comes from the ballistic peaks which lead to  $C(t) = u_0^2 \Psi(t)$ : velocities of particles remain correlated only during the flight time. If, for example, the flight time is exponentially distributed, the standard temporal velocity autocorrelation function decays very fast,  $C(t) = v_0^2 e^{-t/\tau_0}$ . If we look at the density of velocity-velocity autocorrelation at zero  $x = 0$ , from Eq. (55) it follows that  $C(x = 0, t) = -v_0^2 \langle \tau \rangle (\pi D)^{-1/2} t^{-3/2}$ . First, correlations at  $x = 0$  are negative, and second, they decay in time algebraically like  $t^{-3/2}$ . This decay to zero is faster than that of the density of particles,  $P(x = 0, t) \propto t^{-1/2}$ , but still much slower than the exponential decay of the temporal correlation function. It highlights the fact that the space-time velocity correlation function provides access to long-lived correlations and therefore increases the chance of their detection in experiments.



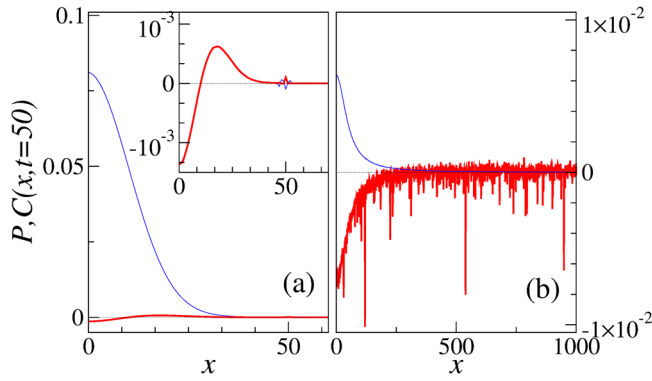


FIG. 12 (color online). Propagators (thin line) and space-time velocity autocorrelation function (thick line) at time  $t = 50$  for two extreme cases of the random walk with random velocities (see Sec. III.A), with the flight-time PDF  $\psi(\tau) = \delta(\tau - 1)$ , for (a)  $h(v) = [\delta(v - u_0) + \delta(v - u_0)]/2$  and (b)  $h(v)$  in the form of Cauchy distribution, Eq. (45). In the first case, the autocorrelation function is proportional to the second spatial derivative of the propagator  $C(x, t) \propto \Delta P(x, t)$  [see the inset where both functions are plotted together, with  $\Delta P(x, t)$  weighted as in Eq. (55)], while in the second case  $C(x, t) \sim -P(x, t)$ . The functions were sampled with  $N = 10^7$  realizations. The parameter  $u_0 = 1$ .

For the regime of the superdiffusive Lévy walk  $1 < \gamma < 2$ , the formula with the first time derivative remains valid. By further exploiting the properties of the time derivative we see that the velocity autocorrelation function is negative near the point  $x = 0$ ; see Fig. 13. Upon the departure from the origin the correlation density becomes positive and produces two local maxima. These maxima are traveling with the power-law scaling  $x_m^\pm \propto \pm t^{1/\gamma}$ , while the height of the maxima decays as  $t^{-1-1/\gamma}$ .

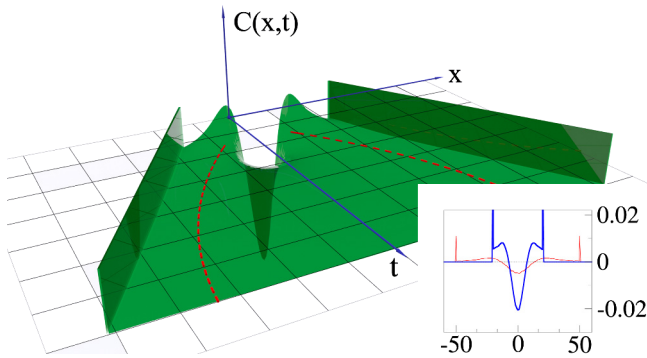


FIG. 13 (color online). Space-time velocity autocorrelation function of Lévy walks in the superdiffusive regime. The space-time evolution of correlations shows a negative dip near  $x = 0$  and two spreading maxima. The integral of the central part with respect to the coordinate is equal to zero. The ballistic peaks carry the correlations of the particles which are still in their first flight. The dashed lines indicate the positions of local maxima  $x_m^\pm$  on the  $x-t$  plane which follow the power-law scaling  $x_m^\pm \propto \pm t^{1/\gamma}$ , while the height of the maxima decays as  $t^{-1-1/\gamma}$ . The inset depicts spatial profiles of  $C(x, t)$  for two different instants of time,  $t = 20$  (thick line) and  $50$  (thin line),  $\gamma = 3/2$ . From Zaburdaev, Denisov, and Hänggi, 2013.

As we continue to move toward more anomalous behavior, for example, to the ballistic regime of Lévy walks, the correlations decay in time even slower (Zaburdaev, Denisov, and Hänggi, 2013). Finally, an example was given, when the velocity distribution of the particles was Lorentzian; see Sec. III.A. In that case the density of the space-time velocity autocorrelation function was proportional to the particle density  $C(x, t) \sim -P(x, t)$ ; see Fig. 12(b).

In all considered regimes there is a region of negative correlations at the vicinity of the starting point. This means that the majority of particles found there are moving in the direction opposite to that of their initial motion. The shape of the “echo” region and the time scaling of its width are model-specific characteristics. Interestingly, simulations of a stochastic process described by a system of Langevin equations in the regime of classical Brownian diffusion show analogous results [see Zaburdaev, Denisov, and Hänggi (2013) and its Supplementary Material for additional plots], thus suggesting that these findings are applicable to a broad class of stochastic transport processes characterized by a finite velocity of moving particles.

Here we considered only a simple initial condition, when all particles instantly change their velocity at  $t = 0$ . As a result, the temporal correlation function  $C(t)$  obtained by the integration of the density of the space-time velocity correlations describes velocity correlations in a specific setting: the initial velocity is always taken right after the reorientation event and the second velocity is measured after the lag time  $t$ . In general, the temporal velocity autocorrelation function depends on two arbitrary times  $t_1$  and  $t_2$ , when the corresponding velocities of particles are measured. Such a two-point correlation function was considered before in the context of CTRW (Barkai and Sokolov, 2007; Baule and Friedrich, 2007; Zaburdaev, 2008; Dechant *et al.*, 2014), and also for the case of Lévy walks [see Sec. VI.C and Froemberg and Barkai (2013a) and Taktikos, Stark, and Zaburdaev (2013)]. These results call for the generalization of the space-time velocity correlation function to a broader class of initial conditions.

## B. Exact solutions for ballistic random walks

The asymptotic analysis is useful but in many cases it is still impossible to analytically obtain the expression for the propagators in real time and space. Interestingly, for random walk models which have the ballistic scaling, there is a particular method to calculate the inverse Fourier-Laplace transform without performing it directly. It immediately gives the shape of the scaling function  $F(\xi)$ ; see Eq. (16). The method is similar to the one proposed by Godrèche and Luck (2001) and used for the analysis of the renewal process and the inversion of the double Laplace transform. The problem of finding the PDF in ballistic regimes is intimately related to the problem of time averages (Rebenshtok and Barkai, 2008), as the scaled position of the particle after time  $T$  is given by

$$x/T = \frac{1}{T} \int_0^T v(t) dt, \quad (56)$$

which is a time average of the particle’s velocity. In application to the random walk concept, the method of Godrèche

and Luck has the following formulation. Assume that the propagator of the random walk model has the following scaling form:

$$G(x, t) = \frac{1}{t} F\left(\frac{x}{t}\right). \quad (57)$$

In the Fourier-Laplace space it can be represented as

$$G(k, s) = \frac{1}{s} f\left(\frac{ik}{s}\right) = \frac{1}{s} f(\zeta), \quad \zeta = \frac{ik}{s}. \quad (58)$$

Finally, by using the Sokhotsky-Weierstrass theorem [see [Froemberg \*et al.\* \(2015\)](#) for more details]

$$F(\xi) = -\frac{1}{\pi\xi} \lim_{\epsilon \rightarrow 0} \text{Im} f\left(-\frac{1}{\xi + i\epsilon}\right), \quad (59)$$

where  $\xi = x/t$  is the scaling variable. This formula allows us to compute the shape of the propagator without calculating the inverse Fourier-Laplace transforms.

We first give an example corresponding to the standard Lévy-walk model with a constant speed  $v$  and set it to unity for simplicity  $v = 1$ . The asymptotic profile of the Green's function in the Fourier-Laplace space is given by Eq. (33) from which we can easily identify

$$f(\zeta) = \frac{(1 - \zeta)^{\gamma-1} + (1 + \zeta)^{\gamma-1}}{(1 - \zeta)^\gamma + (1 + \zeta)^\gamma}. \quad (60)$$

Now by using Eq. (59) we can find the shape of the scaling function to be

$$F(\xi) = \frac{\sin \pi\gamma}{\pi} \times \frac{|\xi - 1|^\gamma |\xi + 1|^{\gamma-1} + |\xi + 1|^\gamma |\xi - 1|^{\gamma-1}}{|\xi - 1|^{2\gamma} + |\xi + 1|^{2\gamma} + 2|\xi - 1|^\gamma |\xi + 1|^\gamma \cos \pi\gamma}, \quad (61)$$

which is the Lamperti distribution ([Lamperti, 1958](#)). For another illustration we ask how the shape of the velocity distribution of a random walking particle affects the shape of the corresponding propagator. For that consider the model with random velocities, Sec. III.A, in the ballistic regime ( $\gamma = 1/2$ ) with four different velocity distributions  $h(v)$ : (a) two delta peaks (Lévy-walk regime), (b) Gaussian, (c) uniform on a symmetric bounded interval, and (d) Lorentzian (Cauchy). Equation (59) gives analytical answers for the scaling function  $F(x/t)$  for all four cases. Figure 14 shows that the velocity distribution has a pronounced effect on the shape of the particles' density. Namely, we see the familiar U-like profile for the standard Lévy-walk model, more of a bell-shaped profile but still bounded by fronts for the uniform velocity distribution, and unbounded bell-shaped profile for the Gaussian velocity distribution. The Lorentzian case is special as it does not depend on the flight-time distribution and has diverging moments.

We note here that this approach does not require the finite velocity of particles, it only relies on the existence of the

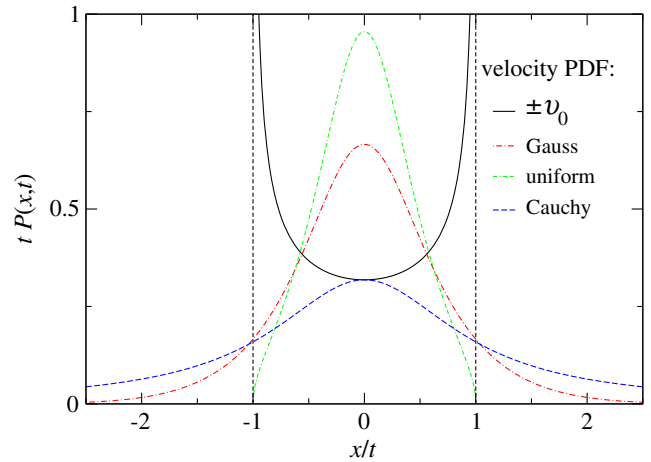


FIG. 14 (color online). Exact solutions of ballistic Lévy walks. Analytical results obtained from Eq. (59) for the scaling functions of the random walks with random velocity model in the ballistic regime ( $\gamma = 1/2$ ), for four different velocity distributions: two delta peaks corresponding to the Lévy walk (solid line), Gaussian (dash-dotted), uniform on an interval (double dash-dotted), and Cauchy (dashed). See [Froemberg \*et al.\* \(2015\)](#) for details.

ballistic scaling. Therefore it can also be applied to the coupled setups as wait or jump first models (this is how the plots in Fig. 11 were obtained). We present these analytical results to emphasize that even a model of random walks with random velocities can be thoroughly analyzed with the help of the combination of asymptotic analysis and elegant mathematical machinery. It would be challenging to try to extend or find similar approaches to other scaling regimes of random walks.

### C. Infinite densities of Lévy walks

Many statistical properties of a Lévy-walk process can be evaluated from the corresponding propagator, Eqs. (6) and (7). For superdiffusive sub-ballistic regimes the central part of the propagator is subjected to the generalized central limit theorem and thus it is given by the symmetric Lévy distribution  $L_\gamma[x, \sigma(t)]$ , with  $\sigma(t) = (K_\gamma t)^{1/\gamma}$ . However, this fundamental fact does not allow for calculations of the moments, starting from the second one, simply because they do not exist for Lévy distributions. The confinement of the process to the ballistic cone should be taken into account and in order to calculate higher-order moments one needs to know the behavior of the propagator at the vicinity of the ballistic fronts. These regions are out of the validity domain of the gCLT and a complementary theory is needed. The concept of infinite measure ([Aaronson, 1997](#); [Thaler and Zweimüller, 2006](#)) provides us with such theoretical framework ([Rebenshtok \*et al.\*, 2014a, 2014b](#)).

In the intermediate region  $t^{1/\gamma} < |x| < t$  the Lévy-walk propagator scales as  $P(x, t) \sim t/|x|^{1+\gamma}$ . Since a power law is a scale-free function, there are infinitely many different scaling transformations which match power-law tails of the propagators for different times  $t^{\xi_{\text{PDF}}} P(x/t^{\xi_x}, t)$ , with linearly related scaling exponents  $\xi_{\text{PDF}} = \xi_x(1 + \gamma) - 1$ . When  $\xi_x = 1/\gamma$  we have the familiar Lévy scaling, Eq. (16), with  $\alpha = 1/\gamma$ , which matches the central parts of the propagators. For  $\xi_x = 1$  we have a “ballistic” scaling  $t^\gamma P(x/t, t)$ , which matches the outmost

fronts of the propagators. This scaling is suitable for the analysis of the asymptotic evolutions of the high-order moments determined by the propagator tails. We next introduce a scaled ballistic variable  $\xi = x/t$  and define a density over this variable as (Rebenshtok *et al.*, 2014b)

$$\mathcal{I}(\xi) = \lim_{t \rightarrow \infty} t^\gamma P(x/t, t). \quad (62)$$

This function scales differently from the Lévy distribution  $L_\gamma[x, \sigma(t)]$  and it is non-normalizable,<sup>5</sup>  $\int_{-\infty}^{\infty} \mathcal{I}(\xi) d\xi = \infty$ , because of the power-law singularity in the limit  $\xi \rightarrow 0$ ,  $\mathcal{I}(\xi) \approx c_\gamma K_\gamma |\xi|^{-1-\gamma}$ . For the general case of the Lévy walks with random velocities, the infinite density is given by (Rebenshtok *et al.*, 2014a)

$$\mathcal{I}(\xi) = B \left[ \frac{\gamma \mathcal{F}_\gamma(|\xi|)}{|\xi|^{1+\gamma}} - \frac{(\gamma - 1) \mathcal{F}_{\gamma-1}(|\xi|)}{|\xi|^\gamma} \right], \quad (63)$$

where

$$\mathcal{F}_\gamma(\xi) = \int_{|\xi|}^{\infty} dv v^\gamma h(v). \quad (64)$$

It is not surprising that, in contrast to the universal Lévy-like central part of the Green's function, the infinite density, which describes function's tails, is specific to the velocity distribution  $h(v)$  [it is assumed that this PDF is not heavy tailed so the integral (64) is finite]. It accounts for the particles with highly correlated flying histories so that the velocity PDF is imprinted into the tails of the PDF  $P(x, t)$ . Two ends meet in the intermediate region where both functions scale similarly  $L_\gamma[x/\sigma(t)] \sim \mathcal{I}(x/t) \sim x^{-1-\gamma}$ . For large  $t$  two densities match nearly perfectly at the point

$$x_c(t) = \left[ \frac{c_\gamma}{L_\gamma(0)} \right]^{1/1+\gamma} (K_\gamma t)^{1/\gamma}$$

so that the propagator  $P(x, t)$  can be approximated with high accuracy by gluing two functions (properly scaled beforehand) at the point  $x_c(t)$ ; see Fig. 15.

Now the fractional moments can be calculated as

$$\begin{aligned} \langle |x|^q \rangle \approx & 2 \underbrace{\int_0^{x_c(t)} L_\gamma \left[ \frac{x}{(K_\gamma t)^{1/\gamma}} \right] (K_\gamma t)^{-1/\gamma} x^q dx}_{\text{inner region}} \\ & + 2 \underbrace{\int_{x_c(t)}^{\infty} \mathcal{I} \left( \frac{x}{t} \right) t^{-\gamma} x^q dx}_{\text{outer tails}}. \end{aligned} \quad (65)$$

<sup>5</sup>The non-normalizable density function  $\mathcal{I}(\xi)$  should not be thought of as a *probability* density function (the latter is always normalizable). The infinite density relates to a mathematical concept of a spatially varying function, defined on a smooth manifold that is locally integrable almost everywhere. Non-normalizable densities are no strangers to physics; check, for example, for non-normalizable energy densities in black holes and non-normalizable densities of states in relativistic quantum dynamics.

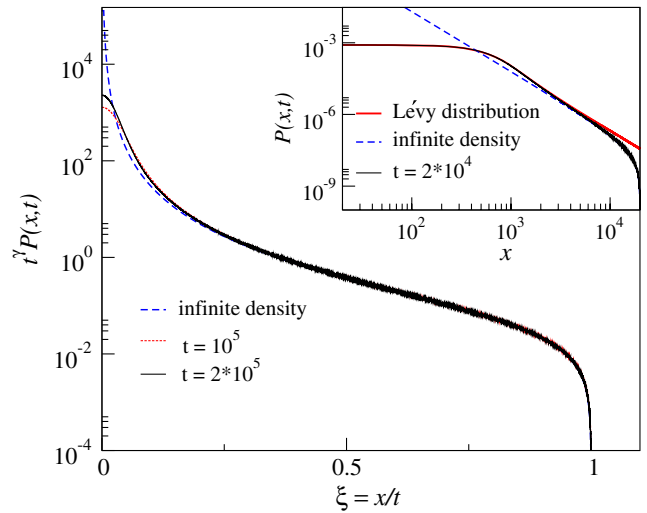


FIG. 15 (color online). Ballistic scaling of the propagators of the Lévy-walk model with random velocities (see Sec. III.B) for the velocity distribution  $h(v)$  uniform on the interval  $\in [-1, 1]$  and  $\alpha = 3/2$ . The corresponding infinite density (dashed line), Eq. (62), matches the tails of the rescaled propagators. The inset shows the propagator for a shorter time plotted together with a rescaled Lévy distribution (solid line) and infinite density (dashed line). The propagator is barely visible due to the perfect matching of the theoretical curves at the central region near  $x = 5000$ . The propagators were sampled with  $N = 10^{10}$  realizations. Adapted from Rebenshtok *et al.*, 2014a.

In the long-time limit, the lower limit of the second integral  $x_c(t)/t \rightarrow 0$  while the upper limit of the first integral is a constant. For  $q > \gamma$  the second integral is by far larger than the first, hence we may neglect the inner region and get

$$\langle |x|^q \rangle \sim 2t^{q+1-\gamma} \int_0^{\infty} \mathcal{I}(\xi) \xi^q d\xi. \quad (66)$$

When  $q < \gamma$  the contribution from the second integral is negligible in the limit  $t \rightarrow \infty$  and the upper limit of the first integral is taken to infinity; hence, after a change of variables  $y = x/\sigma(t)$ , and using the symmetry  $L_\gamma(y) = L_\gamma(-y)$ , we are left with

$$\langle |x|^q \rangle \sim (K_\gamma t)^{q/\gamma} \int_{-\infty}^{\infty} L_\gamma(y) |y|^q dy. \quad (67)$$

These results prove that sub-ballistic Lévy walks belong to the class of strongly anomalous diffusion processes whose asymptotic moments satisfy Eq. (37) with  $q\nu(q) = q/\gamma$  for  $q < \gamma$  and  $q\nu(q) = q + 1 - \gamma$  for  $q > \gamma$ ; see Fig. 5. There is also a phase-transition-like crossover at the point  $q_c = \gamma$  in terms of the moment prefactors  $M_q$ ; see the inset in Fig. 5. Finally, the knowledge of both the Lévy PDF and infinite density allows one to calculate observables that are not integrable with respect to either of the two densities, for example,  $f(x) = 1 + x^2$  (Rebenshtok *et al.*, 2014a).



#### D. Memory effects and ergodicity breaking in Lévy walks

In general, a continuous-time random walk model is a non-Markovian process, meaning that the future of a particle depends not only on the particle's current state, namely, its position and velocity, but also on its prehistory, like how long it was waiting, or how long it was flying already.

As a result, a CTRW process cannot be fully characterized by its PDF, but requires the knowledge of all higher-order correlation functions (Hänggi and Thomas, 1982). However, CTRWs and Lévy-walk models we considered so far, where each next step is independent of the previous, represent the so-called semi-Markov processes. In a semi-Markov process, the points where jumps or velocity changes occur form a Markov chain; the renewal events at those points erase all previous memory. In between the renewal points, to predict the future of the particle, we need to know how long it was in its current state. In some cases, the limiting transport equations are consistent with Markovian dynamics, as in the case of normal diffusion Eq. (13) or superdiffusion Eqs. (22) and (23), and preserve the continuity of evolution for the times exceeding the average waiting times. In other cases, when the mean time diverges, the asymptotic transport equations are obviously of a non-Markovian nature, as in the case of CTRW in the subdiffusive regime, when the corresponding transport equation has a fractional time derivative. A large body of work addressing the semi-Markov property and its consequences for the CTRW models exists and next we will look only at those which are pertinent to Lévy walks.

In the context of Lévy walks, there are two important interrelated issues which relate to the power-law distributed flight times. The first issue concerns the effects of the initial distribution of particles with respect to their flight times on future evolution of the PDF  $P(x, t)$  (Sokolov, Blumen, and Klafter, 2001; Barkai, 2003; Barkai and Cheng, 2003; Zaburdaev and Chukbar, 2003; Aquino *et al.*, 2004; Zaburdaev and Sokolov, 2009). The second issue is the weak ergodicity breaking (WEB) and it points to the fact that time and ensemble averaged quantities can be different from each other (Bel and Barkai, 2005; Rebenshtok and Barkai, 2007, 2008).

The problem of the initial preparation of the system of particles is important for all random walk models. So far we always assumed that all particles were introduced to the system at  $t = 0$ , that is they had no history. In this case, the probabilities to make the first jump after a certain waiting time, or to make the first turn after a flight time, are governed by the same waiting time or flight time PDFs  $\psi(\tau)$ . However, if at  $t = 0$  a particle has already collected some “history,” for example, it was sitting at a given point or it was in the state of flight for some time  $\tau_1$ , then the PDF for it to make the first jump (first turn) at time  $\tau$  is in general different from  $\psi(\tau)$ . This case is handled by the so-called renewal theory (in this simple case, it is just the implementation of the conditional probability formula) (Tunaley, 1974; Haus and Kehr, 1987):

$$\psi_{\text{first}}(\tau|\tau_1) = \frac{\psi(\tau + \tau_1)}{\Psi(\tau_1)}. \quad (68)$$

The only distribution function which is not affected by the prehistory is the exponential distribution and because of that it is often called memoryless. In general, the initial distribution of particles over the flight or waiting times may affect the following evolution. An approach to incorporate this distribution was developed for the CTRW model and can be extended to the Lévy-walk case (Zaburdaev and Chukbar, 2003; Zaburdaev, 2008). Here we mention one important example of the memory effects. Assume that before starting observation we let the system evolve for time  $t_1$  and then require that  $t_1 \rightarrow \infty$ . This is a so-called *equilibrated* [or stationary (Klafter and Zumofen, 1993)] setup where we assume that the system reaches a certain equilibrium before we start measuring it. In contrast, the setup where all particles are introduced to the system at  $t = 0$  and do not have prehistories is called a *nonequibrated* [nonstationary (Klafter and Zumofen, 1993)] setup. For the equilibrated case, one has to imagine a system with an infinite number of particles in unbounded domain but with a fixed uniform density. Particles evolve according to their random walk model for an infinite time. At some time point which we denote as  $t = 0$  we mark all particles located at  $x = 0$  and then follow the evolution of marked particles only. It can be shown that, in the equilibrated setup, the probability of making the first reorientation event after the observation started is given by the following PDF [see, e.g., Denisov, Zaburdaev, and Hänggi (2012) for a simple derivation]:

$$\bar{\psi}(t) = \frac{1}{\langle \tau \rangle} \int_0^\infty \psi(t + \tau) d\tau. \quad (69)$$

If the flight times are too long, such that the mean flight time diverges, there is no reason to talk about the equilibrated setup as it simply does not exist. The prehistory affects only the probability of the very first reorientation effect to occur; in terms of the transport equations, it will lead to the new terms on the right-hand sides of Eqs. (25)–(27):

$$\nu(x, t) = \dots + \bar{\psi}(t)\delta(|x| - vt)P_0, \quad (70)$$

$$P(x, t) = \dots + \bar{\Psi}(t)\delta(|x| - vt)P_0, \quad (71)$$

and consequently to different propagators. Here the corresponding probability of not changing the direction till time  $t$ ,  $\bar{\Psi}(t)$ , is given by the similar integration as in Eq. (69):  $\bar{\Psi}(t) = (1/\langle \tau \rangle) \int_0^\infty \Psi(t + \tau) d\tau$ . Finally, the exponent in the power-law tail of  $\bar{\psi}(\tau) \propto t^{-\gamma}$  is smaller in the case of the equilibrated setup than in the nonequibrated case  $\psi(\tau) \propto t^{-1-\gamma}$ , meaning a longer lasting influence of the initial distribution on the consequent evolution. Understanding of these memory effects is important for the analysis of experimental data or comparison of theory and simulations, as we will exemplify when discussing applications.

The problem of WEB (Bouchaud, 1992) is of great interest in both theoretical and experimental communities (Brokmann *et al.*, 2003; Margolin and Barkai, 2005; He *et al.*, 2008; Lubelski, Sokolov, and Klafter, 2008; Jeon *et al.*, 2011; Weigel *et al.*, 2011). WEB, similarly to memory effects discussed previously, is found in systems with temporal

dynamics governed by power-law distributed time variables with diverging means (Barkai, 2008). Ergodicity breaking is called weak if the whole phase space of the system can be explored, but the ergodicity is never reached because the characteristic times involved in the corresponding process are always of the order or longer than the total measurement time. In practice it means that the time average of a certain quantity itself is random and can be characterized by a nontrivial distribution. In the case of the fully ergodic system the distribution of time averages has a shape of the delta function at the value of the corresponding ensemble average. The most pronounced effects of WEB can be observed for subdiffusive systems (Bel and Barkai, 2005; He *et al.*, 2008; Lubelski, Sokolov, and Klafter, 2008); however, Lévy walks, as they may involve flight-time distributions with infinite flight times, also exhibit WEB. In several recent studies (Akimoto, 2012; Froemberg and Barkai, 2013b; Godec and Metzler, 2013a), the effects of WEB in Lévy walks were investigated on the example of the mean squared displacement calculated as time and ensemble average. The time averaged MSD is defined as

$$\overline{\delta x^2(\tau)} = \frac{1}{(T-\tau)} \int_0^{T-\tau} [x(t+\tau) - x(t)]^2 dt, \quad (72)$$

where  $T$  is the measurement time and  $\tau$  is the lag time. Several observations were made. In the superdiffusive sub-ballistic regime, the time averaged MSD for finite measurement time  $T$  shows different apparent scaling for large  $\tau$  (but  $\tau$  is still much smaller than  $T$ ). Some of the individual trajectories can even demonstrate the subdiffusive behavior. Godec and Metzler (2013a) attributed this effect to the finiteness of the trajectories. Another quantity which can be constructed is the ensemble average of the time averaged MSD  $\overline{\langle \delta x^2(\tau) \rangle}$ . This quantity now can be compared with the ensemble averaged MSD  $\langle x^2(\tau) \rangle$ , namely, by calculating the ratio of the former to the latter giving the so-called ergodicity breaking parameter  $\mathcal{EB} = \overline{\langle \delta x^2(\tau) \rangle} / \langle x^2(\tau) \rangle$ . For the superdiffusive Lévy walk in the limit  $\tau \rightarrow \infty$  it tends to a constant value  $\mathcal{EB} = 1/(\gamma - 1)$ . To draw the connection to the previously discussed memory effects we note that the ensemble-time averaged MSD  $\overline{\langle \delta x^2(\tau) \rangle}$  corresponds to the ensemble averaged MSD of the equilibrated setup, whereas simple ensemble average MSD is calculated for the nonequilibrated setup (Klafter and Zumofen, 1993). Froemberg and Barkai (2013b) considered the ballistic regime of Lévy walks as well. In that case the WEB effect is also present, but, surprisingly, is not as pronounced as in the superdiffusive case. The ensemble-time average can also be calculated analytically. Its leading term (assuming  $\tau/T \ll 1$ ) is given by  $\overline{\langle \delta x^2(\tau) \rangle} \sim (v_0\tau)^2$ . Note that the ensemble averaged MSD has a different prefactor  $\langle x^2(\tau) \rangle = (1-\gamma)(v_0\tau)^2$ . Furthermore the fluctuations of the shifted quantity  $\overline{\langle \delta x^2(\tau) \rangle} - (v_0\tau)^2$  can also be quantified; see Froemberg and Barkai (2013b) for details.

Ergodicity breaking effects discussed here are essential for the analysis of the experimental data, particularly in biology, where due to the limited number of measurements one has to resort to the time averaging and always deals with trajectories of finite length. In addition, understanding of the fluctuations

in time averaged observables can help to obtain more information about the underlying stochastic process (Dechant *et al.*, 2014; Schulz, Barkai, and Metzler, 2014). There is also a very recent work on Einstein relation, fluctuation dissipation, and linear response (Froemberg and Barkai, 2013b; Godec and Metzler, 2013b). We refer the interested reader to a recent review by Metzler *et al.* (2014).

### E. Langevin approach and fractional Kramers equation

In the Introduction we mentioned the Langevin equation as an approach to stochastic transport phenomena complementary to the random walk paradigm. Random walks have their strength in the flexibility of the model construction and amenability of the corresponding transport equations to the analytical treatment. Langevin equations utilize the machinery of stochastic differential equations and provide a link to the Fokker-Planck equation (Risken, 1996). The Langevin equation was originally proposed in 1908 to describe the Brownian motion (Lemons and Gythiel, 1997). Since then it has grown into a powerful tool of modern physics (Coffey and Kalmykov, 2012). This success was certainly supported by rigorous mathematical foundations laid by mathematicians, such as N. Wiener, Itô, and Stratonovich. In many cases, the equivalence of random walks and the corresponding Langevin equations can be explicitly demonstrated in a proper limit. That includes also regimes of normal and anomalous diffusion, with both subdiffusion and superdiffusion. This line of research led to the formulation of the fractional Fokker-Planck (Metzler, Barkai, and Klafter, 1999; Barkai, Metzler, and Klafter, 2000; Barkai, 2001; Chechkin, Klafter, and Sokolov, 2003; Heinsalu *et al.*, 2007) and Klein-Kramers equations (Barkai and Silbey, 2000; Metzler and Sokolov, 2002; Friedrich *et al.*, 2006a; Eule *et al.*, 2007; Dieterich *et al.*, 2008), and it remains an active field of research up to now.

One of the Langevin pathways to Lévy walks was recently proposed by Kessler and Barkai (2012) (discussed in more detail in Sec. V.E). In brief, the dynamics of the particle is governed by the Langevin equation with a standard white noise term but with a nonlinear friction, Eq. (88). On a mesoscopic scale the trajectory of the particle  $\{x(t), p(t)\}$  can be divided into flights that are events of unidirectional motion. Time duration of the  $i$ th event  $\tau_i$  is given by the time lag between two consecutive “turns” marked by sign alternations of the momentum  $p(t)$ . Marksteiner, Ellinger, and Zoller (1996) showed that, within a certain parameter range, the PDF of the flight time scales as  $\psi(\tau) \propto \tau^{-1-\gamma}$ , with a parameter-dependent exponent  $\gamma$ . The flight time and flight distance are coupled in a nontrivial way, such that the corresponding random walk description of the particle dynamics does not reduce to the simple Lévy-walk model with linear coupling between  $x$  and  $t$  (Kessler and Barkai, 2012; Barkai, Aghion, and Kessler, 2014).

In an attempt to model real-life continuous trajectories similar to those of Brownian motion but exhibiting anomalous diffusion, a Langevin equation with a special form of multiplicative noise term was suggested (Lubashevsky, Friedrich, and Heuer, 2009a, 2009b). A trajectory generated by this Langevin equation, when sampled at fixed time intervals, will reproduce the behavior of the Lévy-flight model.

Finally, to achieve a one-to-one correspondence of the Langevin picture and the Lévy-walk model one can use the method of subordination (Fogedby, 1994). In this case an additional variable is introduced, which is called an operational time. The dynamics of velocity occurs in this operational time and can be tuned to produce the desired velocity distributions  $h(v)$ . The real time is connected to the operational time via its own stochastic equation with a noise term which generates long traps in real time space. Those traps correspond to the long flight intervals as required for the Lévy-walk model; see Eule *et al.* (2012) for more detail. This phenomenological approach allows one to connect the world of Langevin equations to Lévy walks where the constant speed of a particle during a long time interval is crucial, remaining, at the same time, very different from a standard Brownian trajectory where the velocity is constantly changing.

A complementary approach to study anomalous stochastic transport is to generalize the Kramers-Fokker-Planck equation. Several versions of generalized Kramers-Fokker-Planck equations were suggested in the literature [summarized in Eule *et al.* (2007)]. Here we follow a scheme by Friedrich *et al.* (2006a) with a ballistically moving particle subjected to random kicks which alter its velocity. Provided the times between consecutive collisions are distributed as a power law, the fractional Kramers-Fokker-Planck equation for the joint position-velocity distribution  $f(\mathbf{r}, \mathbf{v}, t)$  can be obtained

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla_{\mathbf{r}} + \mathbf{F}(\mathbf{r})\nabla_{\mathbf{v}}\right)f(\mathbf{r}, \mathbf{v}, t) = L_{\text{FP}}\mathcal{D}_t^{1-\gamma}f(\mathbf{r}, \mathbf{v}, t). \quad (73)$$

Here  $L_{\text{FP}}$  is the Fokker-Planck operator  $L_{\text{FP}}f = \tilde{\gamma}\nabla_{\mathbf{v}} \cdot (\mathbf{v}f) + \kappa\Delta_{\mathbf{v}}f$ , and  $\mathcal{D}_t^{1-\gamma}$  is the fractional substantial derivative defined through its Laplace transform as  $\mathcal{L}[\mathcal{D}_t^{1-\gamma}f(t)] = [s + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \mathbf{F}(\mathbf{r}) \cdot \nabla_{\mathbf{v}}]^{1-\gamma}f(s)$ . Further,  $\mathbf{F}(\mathbf{r})$  is the external force,  $\tilde{\gamma}$  is the generalized friction coefficient, and  $\kappa$  is related to the amplitude of the noise in the corresponding Langevin equation for the velocity of the particle. For rigorous derivation and many technical details see Friedrich *et al.* (2006a, 2006b), and Carmi and Barkai (2011). By appropriate modifications, the above equation can be simplified to give the equations of the random walk with random velocity model (Zaburdaev, Schmiedeberg, and Stark, 2008) and of the Lévy-walk model (Eule, Friedrich, and Jenko, 2008). The genetic link between Lévy walks, Langevin equations, and fractional Fokker-Planck equations certainly needs to be investigated further (Lubashevsky, Friedrich, and Heuer, 2009a, 2009b; Turgeman, Carmi, and Barkai, 2009; Magdziarz, Szczotka, and Zebrowski, 2012).

At this point it would be timely to mention two relevant approaches (which, however, fall beyond the scope of this review). That is the fractional Brownian motion, which is characterized by a Gaussian but time correlated noise (Mandelbrot and Ness, 1968), and the generalized Langevin equation, which contains an integral operator with a memory kernel on the right-hand side of the equation governing the velocity increments (Zwanzig, 2001). Both approaches are useful in describing various transport processes across disciplines. They possess, however, very distinct

features that are different from those of the random walk concept; in relation to the questions already discussed in this review, we direct the interested reader to Magdziarz *et al.* (2009), Eliazar and Shlesinger (2013), and Meroz, Sokolov, and Klafter (2013).

## V. LÉVY WALKS IN PHYSICS

Physics is a natural habitat of random walk models (de Gennes, 1979; Fernandez, Fröhlich, and Sokal, 1992; Weiss, 1994). During the last 25 years, the Lévy-walk model has found a number of applications, mostly in classical chaos and nonlinear hydrodynamics (Klafter, Shlesinger, and Zumofen, 1996; Shlesinger, Klafter, and Zumofen, 1999; Klafter and Sokolov, 2011). Geisel and Thomae (1984) were the first to consider an intermittent ballistic motion with power-law flight-time PDFs in the context of deterministic chaos. Later on, Geisel, Nierwetberg, and Zacherl (1985) studied a model of the rotational phase dynamics in a Josephson junction, by using a one-dimensional map

$$x_{n+1} = g(x_n), \quad (74)$$

assuming discrete translational and reflection symmetries,

$$g(x+N) = g(x) + N, \quad g(-x) = -g(x), \quad (75)$$

where  $N$  denotes the number of the unit box  $[N-1, N]$ . With this setup, the definition of the map is required only for the reduced range  $0 \leq x \leq 1$ . It can be extended then over  $x \in [-\infty, \infty]$  by using symmetries in Eq. (75). Geisel, Nierwetberg, and Zacherl (1985) used a nonlinear map,

$$\bar{g}(x) = \begin{cases} (1+\epsilon)x + ax^z - 1, & 0 \leq x \leq 1/2, \\ 2 - (1+\epsilon)(1-x) - a(1-x)^z, & 1/2 \leq x \leq 1, \end{cases} \quad (76)$$

where  $\epsilon$  is a small constant and  $a = 2^z(1-\epsilon/2)$ . The profile of the corresponding extended map  $g(x)$  for  $z = 5/3$  is shown in Fig. 16(a). Because of the power-law form of the second term on the right-hand side of Eq. (76), the reduced variable  $x \bmod 1$  tends to cluster near the semistable points  $x_N = N$ . Once entered into the vicinity of one of these points, a trajectory performs a unirotational motion with a near constant rate  $|x_{n+1} - x_n| = 1$ . The rotation direction depends on the point to which the trajectory stuck,  $v = 1$  when it stuck to  $x_N^+ = N - 0$  [ $x = 1$  in the reduced map  $\bar{g}(x)$ ] and  $v = -1$  when to  $x_N^- = N + 0$  [ $x = 0$  in the reduced map  $\bar{g}(x)$ ]. It was found that a histogram of the numbers of iterations, or the time, if we set  $t = n$ , spent by the system in a rotation state, yields a long-tailed distribution  $\psi(t) \propto t^{-z/(z-1)}$ . The results of a numerical sampling reveal that the MSD  $\langle x^2(t) \rangle$  scales as in Eq. (35), with  $\gamma = 1/(z-1)$ . It was then shown to be a clear-cut case of a Lévy walk with the constant speed  $v = 1$  and the exponent  $\gamma$  (Shlesinger and Klafter, 1985); see Fig. 16(b). A complete evaluation of the diffusion in the intermittent maps within the Lévy-walk framework was presented by Zumofen and Klafter (1993). The next ‘‘dynamical’’ realization of the Lévy walk was found in Hamiltonian chaotic systems



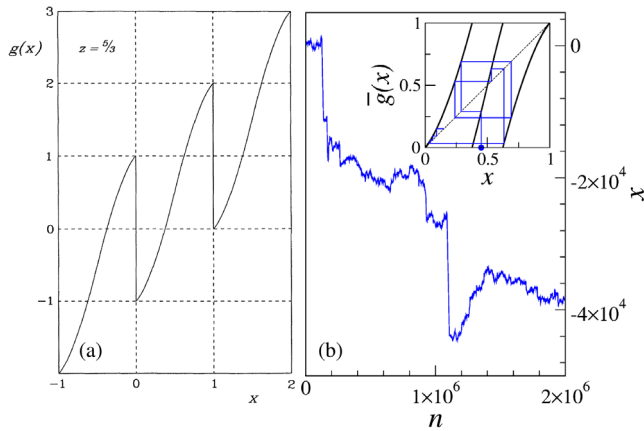


FIG. 16 (color online). (a) Map  $g(x)$ , Eqs. (74)–(76), for  $z = 5/3$ . Adapted from Zumofen and Klafter, 1993. (b) A trajectory  $x(n)$  obtained by iterating the map from the initial point  $x(0) = 0.45$ . The inset shows the reduced map  $\tilde{g}(x)$ , Eq. (76), with the first ten iterations of the initial point  $x(0)$  ( $\bullet$ ). The branch on the left (right) part is responsible for a decrement (an increment) of the cell number  $N = \text{int}(x)$  produced by the extended map  $g(x)$ . The parameter  $\epsilon = 10^{-4}$ .

(Shlesinger, Zaslavsky, and Klafter, 1993; Klafter and Zumofen, 1994; Zumofen and Klafter, 1994a). This was a case when the Lévy-walk concept perfectly matched a peculiar dynamical effect, in appearance similar to the intermittency in the dissipative maps with power-law singularities (MacKay, Meiss, and Percival, 1984a; Geisel, Zacherl, and Radons, 1987). The machinery behind the Hamiltonian stickiness is related to specific fractal structures living in the phase space of chaotic Hamiltonian systems (MacKay, Meiss, and Percival, 1984b; Meiss, 1992). We discuss this issue in more detail in Sec. V.A.

With this section we are not up to a comprehensive historical review. We want to present Lévy walks in physics as something (re)emergent and promising rather than something residual and completed. In Secs. V.A–V.E we concentrate on the most recent advances and results, both theoretical and experimental, which underline the potential and universality of the concept.

### A. Lévy walks in single-particle Hamiltonian systems

The subject of Lévy walks in low-dimensional Hamiltonian chaos is already 20 years old (Klafter and Zumofen, 1994; Zumofen and Klafter, 1994a). We start with a brief outline of it not because of the historical reason but because it will help to better understand the recent developments that will be discussed next.

The phase space of a nonintegrable single-particle Hamiltonian system is mixed and consists of different invariant manifolds that are chaotic layers, regular islands, tori, etc. (Sagdeev, Usikov, and Zaslavsky, 1992). The  $i$ th manifold can be characterized by an averaged value of any observable, for example, velocity  $v_i = \langle v_i(t) \rangle_{t \rightarrow \infty}$ . The average velocity of a manifold might be nonzero and for a regular island it is determined by the winding number of the elliptic

orbit at the island center. A chaotic layer is well separated from regular manifolds by Kolmogorov-Arnold-Moser (KAM) tori (Sagdeev, Usikov, and Zaslavsky, 1992) so that a trajectory initiated inside the layer cannot enter a regular island even when the latter is embedded into a chaotic sea. A “coastal area” near the island is structured by *cantori* (MacKay, Meiss, and Percival, 1984b), which form partial barriers for the trajectories. Once entered into the region enclosed by a cantorus, a trajectory will be trapped in the vicinity of the corresponding island for a long time. During this sticking event (Meiss and Ott, 1986), the trajectory reproduces the dynamics of the orbits located inside the island. If the corresponding island is transporting,  $v_i \neq 0$ , the sticking event produces a long ballistic flight with velocity  $v_i$ . It has been found that power-law tails of sticking time PDFs,  $\psi(\tau) \propto \tau^{-1-\gamma}$ , are general features of Hamiltonian chaos which is related to the self-similar hierarchical structure of cantori (Meiss and Ott, 1986; Geisel, Zacherl, and Radons, 1987; Meiss, 1992). With this finding all needed ingredients were collected and a link between “strange kinetics” of Hamiltonian chaos and Lévy walks was established (Shlesinger, Zaslavsky, and Klafter, 1993).

In a Hamiltonian system possessing the time-reversal symmetry, ballistic islands always exist in pairs and have identical sticking time PDFs. If, in addition, the long-time dynamics of the system is governed by only two symmetry-related sticky ballistic islands, with a sticking-time exponent  $\gamma$ , then the system dynamics will realize the standard Lévy walk; see Fig. 1(b). Zumofen and Klafter (1994a) considered the kicked-rotor map, an archetypical Hamiltonian model (Sagdeev, Usikov, and Zaslavsky, 1992),

$$x_{n+1} = x_n + K \sin(2\pi\theta), \quad \theta_{n+1} = \theta_n + x_{n+1}, \quad (77)$$

where  $K$  is the stochasticity parameter, as an example. For  $K = 1.03$  the system phase space represents a chaotic sea which extends over the whole  $x$  region. The long-time dynamics of the system is governed by two symmetry-related islands enclosing period-five elliptic orbits with velocities  $v = \pm 1$ . The islands are sticky [see Fig. 17(a)], and the locations of the cantori are marked by the sudden increase of the sticking times. The corresponding sticking-time PDF [see the inset in Fig. 17(b)] for  $t \gtrsim 10^2$  approximately follows a power law with an exponent  $\gamma = 1.2$ . Figure 17(b) shows propagators obtained for different times after they were scaled as in Eq. (16), with exponent  $\alpha = 1/\gamma$ . The curves fall on top of each other thus indicating the scaling expected for the propagators of a Lévy walk; see the inset in Fig. 2.

A Lévy-walk kinetics has also been found in continuous ac-driven one-dimensional (Glück, Kolovsky, and Korsch, 1998; Denisov, Klafter, and Urbakh, 2002) and stationary two-dimensional (Klafter and Zumofen, 1994) Hamiltonian systems. There the key mechanism responsible for the appearance of anomalous transport was the cantori-induced stickiness. There are still ongoing debates on the universality of sticking-time exponent(s) at the asymptotic limit  $t \rightarrow \infty$ , with a number of pros and cons for different “universal” values (Chirikov and Shepelyansky, 1999; Weiss, Hufnagel, and Ketzmerick, 2002; Cristadoro and Ketzmerick, 2008; Venegeroles, 2009; Shepelyansky, 2010). The Lévy walk



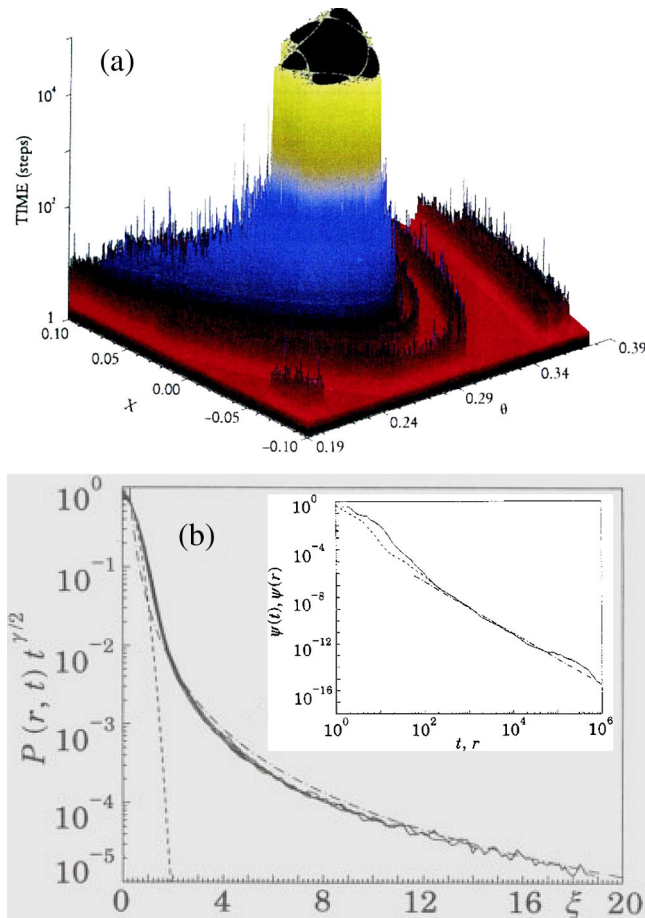


FIG. 17 (color online). Lévy walks in the standard map. (a) The time  $t_{\text{exit}}(x, \theta)$  it takes for a trajectory initiated at the point  $(x, \theta)$  to the exit from a vicinity of a regular island around the period-five orbit (black area on the top of the distribution). Adapted from [Klafter, Shlesinger, and Zumofen, 1996](#). (b) The rescaled propagators for different times  $t = 100, 200, 400, 800, 1600$ , and  $\xi = x/t^{1/\gamma}$ . The dashed line is a Gaussian fit and the dash-dotted line is a power-law fit  $f(\xi) \propto \xi^{-1-\gamma}$ . The inset shows the sticking time (solid line),  $\psi(t)$ , and the flight length (dashed line),  $\psi(x)$ , and PDFs at the vicinity of the period-five island. The dash-dotted line is a power law with exponent  $\gamma = 1.2$ . The spatial variable  $x$  is denoted  $r$  in the original work. The y-axis label in (b) should read as  $P(r, t)t^{1/\gamma}$ . Adapted from [Zumofen and Klafter, 1994a](#).

can stand this uncertainty: If a PDF of sticking times is well approximated by a power law with a particular exponent  $\gamma$  over some substantial time interval (for example, over several decades in  $t$ ) then the corresponding propagator for these times will scale as in Eq. (16), with the scaling exponent  $\alpha = 1/\gamma$ .

It is noteworthy that in Hamiltonian systems only sub-ballistic superdiffusion (see Sec. II.A.2) is possible in the asymptotic limit. This follows from the Kac theorem on the finiteness of recurrence time in Hamiltonian systems ([Kac, 1959](#); [Zaslavsky, 2002](#)). Therefore all sticky ballistic manifolds should have finite mean sticking times, so that the corresponding sticking-time PDFs (which are the flight-time PDFs of the corresponding Lévy walks),  $\psi(\tau) \propto \tau^{-1-\gamma}$ , are

characterized by exponents in the range  $1 < \gamma \leq 2$  ([Denisov, Klafter, and Urbakh, 2002](#)).

Two-dimensional chaotic advection is another field where the chaotic Hamiltonian phase space, with all its trademarks, including cantori and the stickiness phenomenon, appears ([Aref et al., 2014](#)). On the theory level, the dynamics of a chaotic flow can be modeled with symplectic equations and the flow stream function as a Hamiltonian. The path of a passive tracer in the flow can be seen as a trajectory of the Hamiltonian system. Periodic flow modulations lead to the appearance of mixed phase space and regular islands. That idea was behind the first experimental observation of the Lévy walk in a real physical system. In their experiment, [Solomon, Weeks, and Swinney \(1993\)](#) used a rotating annulus tank filled up with fluid. The flow was generated by pumping the fluid into and out of the annulus through the holes in its bottom. This resulted in the appearance of the stable two-dimensional flow pattern on the surface of the fluid. Rotational motion of the tracer, monitored by using a tracer's azimuthal angle, consisted of ballistic episodes interrupted by trappings of the tracer by a periodic chain of vortices; see Fig. 18.

It followed from the measurements that ballistic flights had near constant velocities and the flight-time PDFs followed power-law asymptotics ([Solomon, Weeks, and Swinney, 1994](#)); see Fig. 19. At the same time, the sticking-time PDFs revealed either an exponential decay or power-law tails with exponent  $\gamma_{\text{st}} > 1$  so that the mean sticking time is finite. Therefore, following our classification (see Fig. 1), the process can be taken as a Lévy walk with rests. However, there was a feature: Because of the annulus rotation, ballistic motion happened predominantly in one direction, clockwise or counterclockwise, depending on the rotation direction. This modification of the walk process can be absorbed into the theory by introducing bias in the standard Lévy-walk model,

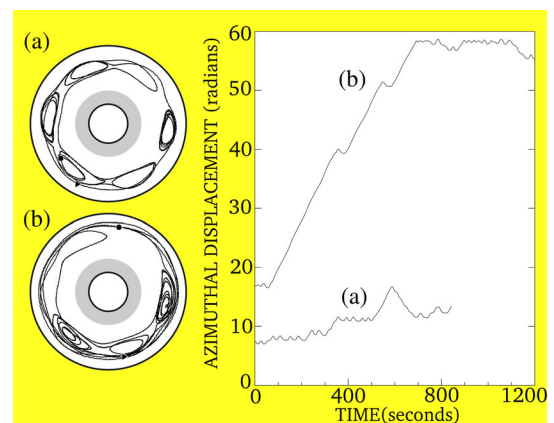


FIG. 18 (color online). Tracer dynamics in a chaotic flow. The flow produces a regular chain of six stable vortices which can trap the tracer as it circles ballistically around the annulus. Two short-time trajectories are shown on the left. The corresponding azimuthal dynamics is presented on the right: while trajectory (a) (lowest on the graph) was trapped by the chain during all the observation time, trajectory (b) (upper on the graph) shows that the tracer produced a long ballistic flight before being trapped. Figure courtesy of E. R. Weeks.

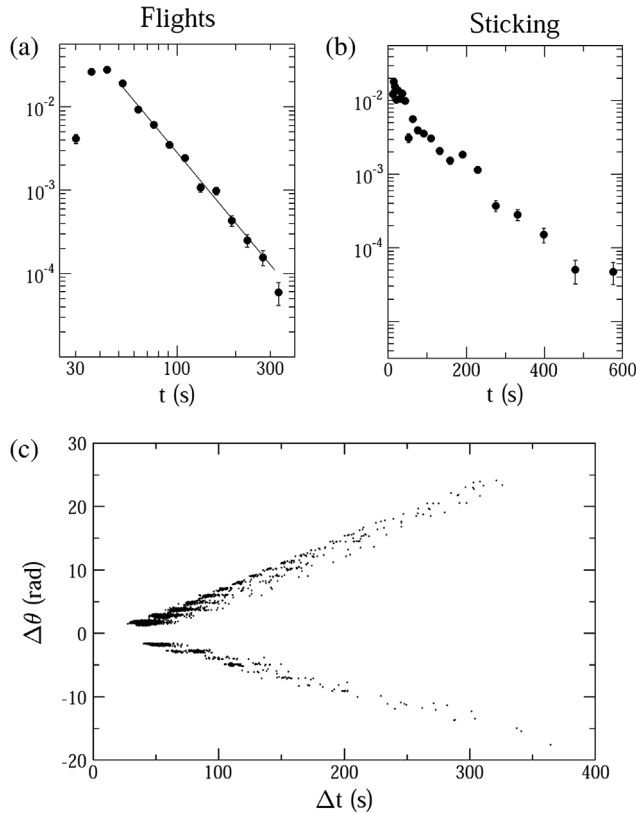


FIG. 19. Statistical characteristics of the tracer diffusion in an ac-driven chaotic flow. (a) Flight- and (b) sticking-time PDFs obtained for a flow with the six-vortex lattice; see Fig. 18. The line corresponds to a power law  $t^{-1-\gamma}$ , with  $\gamma = 1.5$ . (c) Flight length  $\Delta\theta$  vs flight duration  $\Delta t$ . The forklike structure reveals that all flights have near constant velocity. Adapted from Weeks, 1997.

e.g., by making ballistic flights in the positive direction less probable than in the negative one. The corresponding update was made by Weeks and Swinney (1998) and the model outcomes were found to be in good agreement with the experimental measurements. See del Castillo Negrete (1998) and Isichenko (1992) for a theoretical overview of “anomalous advection” and dynamical mechanisms behind it.

A potential of the Lévy-walk model for generalizations can be illustrated with *Hamiltonian ratchets*, ac-driven Hamiltonian systems which are able to generate a constant current in the absence of a bias (Denisov and Flach, 2001; Schanz *et al.*, 2001; Denisov *et al.*, 2002; Schanz, Dittrich, and Ketzmerick, 2005). A directed chaotic transport appears due to violation of the time reversal symmetry with a zero-mean drive (Denisov, Flach, and Hänggi, 2014). The set of regular islands, submerged into the chaotic layer, becomes asymmetric, so that there are islands with nonzero velocities which do not have symmetry-related twins. This leads to the violation of the balance between ballistic flights in opposite directions and the appearance of a strong current (Denisov and Flach, 2001). The asymmetric generalization of the Lévy-walk process by Weeks and Swinney (1998) is able to capture many features of the Hamiltonian ratchet dynamics (Denisov *et al.*, 2002; Denisov, Klafter, and Urbakh, 2004).

## B. Lévy walks in many-particle Hamiltonian systems

In many-particle systems with unbounded interaction potentials, such as nonlinear chains and lattices, it is no longer reasonable to talk about diffusion of particles. The individual particle dynamics has an oscillatory character due to the confinement induced by the interaction with its neighbors. The collective system dynamics creates a “tissue” which can react to small perturbations locally affecting its dynamics. The perturbation transport defines overall energy, correlation, and information transport through a lattice (Helfand, 1960; Torcini, Grassberger, and Politi, 1995; Torcini and Lepri, 1997; Giacomelli *et al.*, 2000; Primo *et al.*, 2007).

Consider a many-particle system at microcanonical equilibrium with a Hamiltonian

$$H_{\text{total}}(\{x_i, p_i\}) = \sum_{i=1}^N H_i, \quad (78)$$

where  $H_i = H(x_i, x_{i-1}, x_{i+1}, p_i)$  is the energy of the  $i$ th particle with position  $x_i$  and momentum  $p_i$ . It is also assumed that the Hamiltonian guarantees the preservation of the zero total momentum of the system  $P = \sum_{i=1}^N p_i = 0$ . At the initial time  $t = 0$  one of the bulk particles receives an external perturbation. The system gains a small amount of extra energy  $E_p$  which is conserved due to the Hamiltonian character of the system evolution. However, the perturbation does spread as the perturbation energy is shared by a constantly growing number of particles. Remarkably, the spreading is universally limited by a finite velocity,  $v_0 < \infty$ , that at a given time  $t$  the perturbation is almost completely confined to the interval  $[-v_0 t, v_0 t]$  (“almost” means that outside the cone the perturbation is exponentially diminished). The fundamental fact of the cone’s existence, the so-called “Lieb-Robinson bound” for classical systems, has a status of a mathematical existence theorem (Marchioro *et al.*, 1978; Nachtergaele *et al.*, 2009). Altogether, that was a strong hint to consider the perturbation spreading as a diffusion process, treat the normalized local excess of energy  $\Delta E(i, t)$ ,  $\sum_{i=1}^N \Delta E(i, t) = E_p$ , as a PDF,  $q(i, t) = \overline{\Delta E(i, t)}/E_p$  (denotes a microcanonical average), and estimate its second moment.

For a one-dimensional hard-point gas with alternating masses, a protozoan Hamiltonian many-particle model (Casati and Ford, 1976), it was found that the mean squared displacement  $\sigma^2(t) = \sum_{i=1}^N i^2 q(i, t)$  scaled as  $\sigma^2(t) \propto t^\mu$  with the exponent  $\mu$  very close to  $4/3$  (Cipriani, Denisov, and Politi, 2005). Moreover, a quasi-PDF  $q(i, t)$  appeared to be the exact propagator of a Lévy walk with velocity fluctuations (Sec. III.B), and exponent  $\gamma = 3 - \mu = 5/3$ , if we set  $i \equiv x$ ; see Fig. 20. Zaburdaev, Denisov, and Hänggi (2011, 2012) further strengthened this finding by showing that the scaling of the ballistic peaks is identical to that predicted by the Lévy-walk model, Eq. (50). Perturbation profiles for different values of microcanonical “temperature,” energy per particle  $\varepsilon$ , perfectly matched each other by assuming that the averaged velocity of the walk and the fluctuations variance both scale as

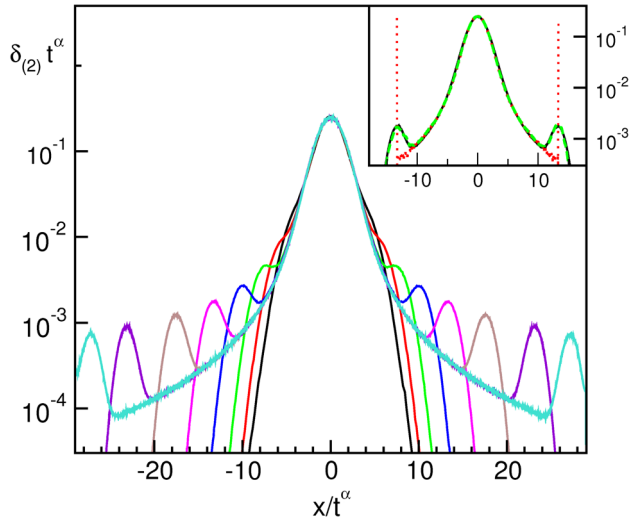


FIG. 20 (color online). Lévy walks in a hard-point gas. Perturbation profiles  $\varrho(i, t)$  [denoted by  $\delta_{(2)}(i, t)$  in the original publication] at  $t = 40, 80, 160, 320, 640, 1280, 2560,$  and  $3840$  (the width increases with time) for the energy per particle  $\varepsilon = 1$ , rescaled as in Eq. (16) with the exponent  $\alpha = 1/\gamma = 3/5$ . In the inset, the profile at  $t = 640$  (solid line) is compared with the propagators of the standard Lévy with the velocity  $v = 1$  (dotted line) and fluctuating velocities with  $D_v = 0.036$  (dashed line). Adapted from Cipriani, Denisov, and Politi, 2005.

$v_0, D_v \propto \sqrt{\varepsilon}$ . Similar results were obtained for a Fermi-Pasta-Ulam (FPU)  $\beta$  chain by using the local energy-energy correlation function  $e(i, t)$  (Zhao, 2006) instead of a finite perturbation. This switch was induced by the fact that it is not feasible to sample the evolution of perturbations in FPU-type systems due to emerging statistical fluctuations. Although less sharp than in the case of hard-point gas, the results obtained for the times  $t < 10^4$  revealed the correspondence between the correlation function profiles and the propagators of a Lévy walk with fluctuating velocity and exponent  $\gamma = 5/3$  (Zaburdaev, Denisov, and Hänggi, 2011, 2012).

There is a genetic link between the problem of energy diffusion and the issue of deterministic heat conduction (Helfand, 1960; Liu *et al.*, 2012, 2014). The latter is typically anomalous in most of the nonlinear chains, meaning that the thermal conductivity  $\kappa_T$  scales with the length of a chain  $L$  as  $\kappa_T \propto L^\eta$ , with  $\eta$  between 0 (normal heat conduction) and 1 (ballistic heat conduction) (Lepri, Livi, and Politi, 2003). Denisov, Klafter, and Urbakh (2003) built up a model of a dynamical heat channel in which energy is carried by an ensemble of noninteracting Lévy walkers. A relatively simple evaluation led to the linear relation between the exponents,

$$\eta = 3 - \gamma. \quad (79)$$

There are still ongoing debates on both the (non)universality of Fourier exponent  $\eta$  and the validity of the single-particle Lévy-walk approach to the heat conduction by many-particle chains (Lepri, Livi, and Politi, 2008; Liu *et al.*, 2012). Meanwhile the Lévy-walk model has been used to reproduce the temperature profiles of finite chains (Lepri and Politi, 2011; Dhar, Saito, and Derrida, 2013) and analyze heat

fluctuations in conducting rings (Dhar, Saito, and Derrida, 2013). Recently, Vermeersch *et al.* (2014) proposed an interpretation of the interfacial thermal transport through metal-semiconductor interfaces in terms of exponentially truncated Lévy flights. Similar to the problem of the light transmission (see Sec. V.C), the setup of the performed experiments does not allow one to differentiate in a clear-cut manner between Lévy flights and Lévy walks, yet experimentalists could think about new experiments that can.

The findings presented by Cipriani, Denisov, and Politi (2005) and Zaburdaev, Denisov, and Hänggi (2011) are phenomenological. To understand the mechanisms which sculpt Lévy kinetics out of many-particle dynamics, the problem should be considered in a broader context. van Beijeren (2012) used a hydrodynamic approach to build a mode-coupling theory for the Fourier components of the densities of conserved quantities, which are the number of particles, total momentum, and energy. A linear transformation splits the transport over three channels facilitated by the three different modes, a heat mode and two sound modes propagating in opposite directions. Thus, instead of a single energy-energy correlator curve for a given time, as in Fig. 20, the hydrodynamic approach produces three profiles (two of them, for the sound modes, are related by the inversion  $x \rightarrow -x$ ). The key result by van Beijeren (2012) is that the scaling of the ballistic sound peaks is of the Kardar-Parisi-Zhang (KPZ) universality class (Kardar, Parisi, and Zhang, 1986). The anomalous scaling of the central heat peak with the exponent  $\gamma = 5/3$  was predicted, which corresponds to the anomalous scaling of the conductivity  $\kappa_T \propto L^{1/3}$  (Lepri, Livi, and Politi, 2003). Recently Spohn (2014) presented a complete version of the hydrodynamic formalism which addresses also the dynamics of the heat peak in more detail. Das *et al.* (2014) and Mendl and Spohn (2014) found for FPU chains and the hard-point gas the Lévy scaling for the correlator of their heat modes (see Fig. 21) and confirmed the KPZ-type scaling for the correlator of their sound modes. The Lévy-like profiles for the heat mode exhibit cutoffs at the points  $x = \pm ct$ , where  $c$  is the speed of sound. It is tempting to think that further progress in this direction can provide one with a “hydrodynamic” foundation of Lévy walks.

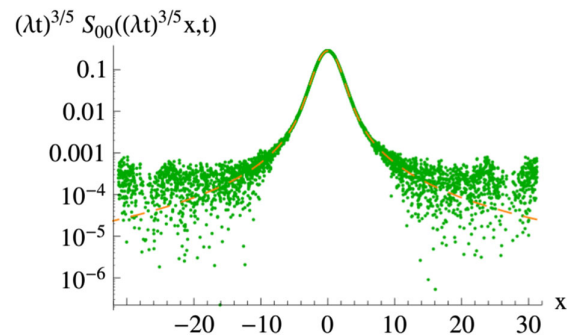


FIG. 21 (color online). Heat peak for a hard-point gas with alternating masses at time  $t = 1024$ . The dashed curve is the Lévy distribution  $L_\gamma(x)$  with  $\gamma = 5/3$ . Adapted from Mendl and Spohn, 2014.



### C. Lévy flights of light and Lévy walks of photons

When passing through a medium, light is subjected to multiple scattering by medium inhomogeneities. Physics of this process depends on the characteristic size of inhomogeneities and different scattering mechanisms can coexist. For example, scattering by molecules (Rayleigh scattering) and by water droplets (Mie scattering) work together in a cloudy sky (Kerker, 1969). In some cases one particular mechanism dominates and thus specifies characteristic scales of the path length between consecutive scattering events (for example, Mie scattering dominates inside a cloud). If a medium is a fractal (Mandelbrot, 1982), then the structure of its inhomogeneities is scale free [a stratocumulus cloud is a good example (Cahalan *et al.*, 1994)]. The path of a photon inside a fractal media can be represented as a random walk consisting of free-path segments connecting subsequent scattering points with the PDF of the segment length in a power-law form (Davis and Marshak, 1997). The question now is shall we use the Lévy flight or Lévy walk to correctly model the process? If we are interested in the stationary transmission through the medium only then is the answer “either” (Buldyrev *et al.*, 2001).

Consider a propagation of a light beam through a slab of thickness  $L$ , with a photon free-path PDF  $p(\ell) \sim \ell^{-1-\gamma}$ . A local stationary transmission on the output surface is defined by all the trajectories leading to the corresponding point from the illuminated spot on the entry side. The total transmission is given by the integral over the local transmission and equals the probability of the absorption of a photon that starts at the illuminated spot by the absorption boundary on the opposite side of the slab.<sup>6</sup> Within this setup two approaches are equivalent. The total path length of a Lévy flight corresponds to the total traveling time of a Lévy walk, and results obtained with both models are interchangeable (Buldyrev *et al.*, 2001). By using a one-dimensional Lévy-flight model<sup>7</sup> with  $1 < \gamma \leq 2$ , Davis and Marshak (1997) derived the transmission as a function of  $L$ ,

$$T(L) = \frac{1}{1 + (L/\bar{\ell})^{\gamma/2}}, \quad (80)$$

where  $\bar{\ell}$  is the mean free path and the unity in the denominator regularizes the expression at  $L = 0$ . In the continuous limit  $L \gg \bar{\ell}$ , the problem can be recast in terms of the fractional diffusion equation, Eq. (21), and, by treating the particle PDF as the light intensity, the scaling in Eq. (80) can be obtained. Identically, the scaling could be derived within the Lévy-walk framework by using the integral equations (25)–(28) for the PDF of walking photons (Drysdales

and Robinson, 1998). Photons move with finite velocity in any medium and therefore the Lévy walk is physically more adequate to model the photon dynamics than the Lévy flight. However, in the context of the transmission problem and from the mathematical point of view the difference between the two approaches is absent. The difference could become tangible when the problem setup is changed and, for example, autocorrelation (Sec. IV.A) and/or interference effects are taken into account. It remains for future work to set up the corresponding experiments. Next we review the up-to-date experimental results.

*Solar light transmission by cloudy skies.*—The first attempt to gain insight into the morphology of a scattering media by utilizing the Lévy-flight concept was made by Pfeilsticker (1999). He used statistical data obtained by measuring the mean geometrical paths of photons coming from a cloudy sky. By assuming the fractal cloud morphology and resorting to the scaling Eq. (80), the flight-length exponent was estimated as  $\gamma \simeq 1.74 - 1.78$ . Pfeilsticker found that the exponent value depends on the cloud type: it tends to 1.5 for convective clouds and to 2 for stratiform clouds.

*Photon transmission through a Lévy glass.*—Modern technologies provide the possibility to synthesize scattering materials with a tunable fractal structure (Tsuji, 2008). One of the recent advances is the creation of *Lévy glass*, a polymer matrix with embedded high-refractive-index scattering nanoparticles (Barthelemy, Bertolotti, and Wiersma, 2008). The matrix also contains a set of glass microspheres with a power-law diameter distribution,  $p(\emptyset) \sim \emptyset^{-\eta-1}$ ; see Fig. 22(a). The microspheres do not scatter because their refractive index is the same as that of the host polymer and therefore scattering happens on nanoparticles only. The photon transport inside a Lévy glass is dominated by long “jumps” performed by the photon when it propagates through the glass spheres; see Fig. 22(b). When the diameter distribution of the spheres is sampled exponentially in  $\emptyset$  space, the jump length PDF scales as  $p(\ell) \sim \ell^{-1-\gamma}$ , with  $\gamma = \eta - 1$  (Bertolotti *et al.*, 2010). This is a clear-cut case of the Lévy walk (flight) of photons. Measurements performed with a Cauchy glass  $\gamma = 1$  by illuminating the slab with a narrow collimated laser beam corroborated the scaling given by Eq. (80); see Fig. 22(c).

Recently, Savo *et al.* (2014) reported on an experimental retrieval of the scaling exponent  $\alpha$ , Eqs. (16) and (36), by analyzing the scaling of the *time-resolved* transmission with  $L$ . The performed measurements verified the universal relation between the three exponents  $\alpha = 1/\gamma = 1/(\eta - 1)$ , thus strengthening the position of Lévy-walk (flight) formalism as an adequate theoretical approach to the process of photon (light) diffusion through fractal media.

An interesting aspect of the photon diffusion inside a Lévy glass is the role of quenched disorder. The distribution of glass spheres in a matrix does not evolve in time and so there is room for correlations between flight directions and angles. By using a one-dimensional chain of barriers with a power-law spacing distribution, Beenakker, Groth, and Akhmerov (2009) found that a walk along the chain is not the standard uncorrelated Lévy walk because of the strong correlations of subsequent step sizes. Similar results have been obtained by

<sup>6</sup>The absorption effects are neglected within the framework of the approach. A photon that entered the slab will appear on the opposite side almost surely in the asymptotic limit  $t \rightarrow \infty$ .

<sup>7</sup>The problem setup considered by Davis and Marshak (1997) assumed the scattering probability peak in the forward  $x$  direction. It was shown that the directional correlations in scattering angles can be absorbed into a rescaling of the free path within the one-dimensional framework; see the appendix in the cited work.



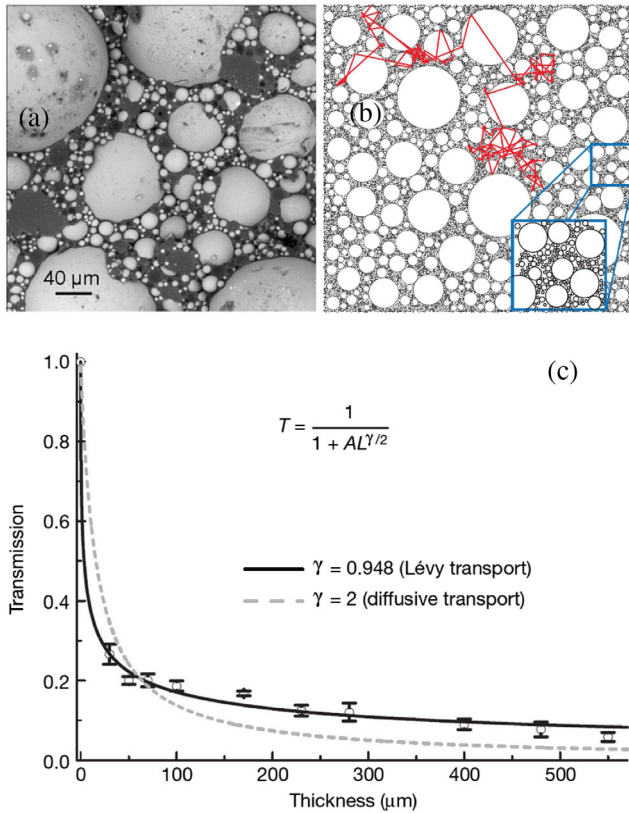


FIG. 22 (color online). Lévy flights of photons in a Lévy glass. (a) Electron micrograph of a Lévy glass. The gray zones are interiors of the glass spheres, whereas the darker area corresponds to the polymer matrix. Scattering nanoparticles are too small to be resolved. (b) A photon walk inside a two-dimensional version of a Lévy glass. The inset shows the scale invariance of the glass. (c) Measured transmission through a Lévy glass slab as a function of the slab thickness. The gray dashed curve is obtained with Eq. (80) for  $\gamma = 2$  (normal diffusion) while the black line is obtained for  $\gamma = 0.948$ . Note that the exponent  $\gamma$  is denoted  $\alpha$  in the original publications. Adapted from Barthelemy, Bertolotti, and Wiersma, 2008, and Burrelli *et al.*, 2012.

Burioni, Caniparoli, and Vezzani (2010) and Vezzani *et al.* (2011). However, by using a two-dimensional model of a Lévy glass, Barthelemy *et al.* (2010) demonstrated that the influence of the quenched disorder can be neglected (in the sense that it can be accounted for by a simple parameter tuning) when stepping into higher dimensions. Therefore the transport of photons in two- and three-dimensional Lévy glasses is close to an uncorrelated Lévy walk [although this could also change when a scattering media is perfectly self-similar and represents a regular fractal, as shown by Buonsante, Burioni, and Vezzani (2011)].

*Lévy flights of photons in atomic vapors.*—The two considered realizations of Lévy walks of light assumed the elastic scattering of photons. The needed power-law distributions are produced by the fractal spatial inhomogeneity of scattering media and the corresponding flight-time exponents are linearly related to characteristic fractal exponents. In the case of inelastic scattering, the distance traveled by a photon depends on its frequency, which changes after every scattering event

(which is in fact an absorption or emission). The spatial inhomogeneity is no longer needed and a power-law distribution of flight length can be obtained from the spectral inhomogeneity of the medium (Molisch and Oehry, 1998). In an atomic vapor (Baudouin, Guerin, and Kaiser, 2014) the absorption probability of a photon with a frequency  $\omega$  at a distance  $r$  from its emission point is  $p(\omega|r) = \Phi(\omega) \times \exp[-\Phi(\omega)r]$ , where  $\Phi(\omega)$  is the absorption spectrum of the atoms. The average absorption probability can be obtained as a frequency average of  $p(\omega|r)$  weighted with an emission spectrum  $\Theta(\omega)$  (Holstein, 1947; Molisch and Oehry, 1998),

$$p(r) = \int_0^\infty \Theta(\omega)\Phi(\omega)e^{-\Phi(\omega)r}d\omega. \quad (81)$$

When emission and absorption spectra are identical, the Doppler spectrum  $\Phi_D(\omega) = \exp(-\omega^2)/\sqrt{\pi}$  leads to  $p(r) \sim r^{-2}[\ln(r)]^{-1/2}$ , while the Cauchy spectrum  $\Phi_C(\omega) = 1/[\pi(1 + \omega^2)]$  yields  $p(r) \sim r^{-3/2}$ . Pereira, Martinho, and Berberan-Santos (2004) proposed this as a means to realize a three-dimensional Lévy flight of photons in a hot atomic vapor where the spectra-equality condition may hold. They have also raised two important points. First, in a high opacity atomic vapor many elastic scattering events happen before an inelastic scattering event occurs. This is a natural call for an extended intermittent model in which Lévy walks are alternated with periods of Brownian diffusion (see also Sec. VII.B, where such processes appear in the context of animal search). Second, they pointed out that in laboratory vapors the time of flight is negligible compared to the waiting time between absorption and emission events. Therefore the use of a Lévy-flight model [Fig. 1(a)] is well justified. However, as noted by Pereira, Martinho, and Berberan-Santos (2004), in interstellar gases the flight time can be larger than the characteristic absorption or emission time and the Lévy walk will be more appropriate in the astrophysical context.

By using a specially designed experimental setup, Mercadier *et al.* (2009) measured the first step-length distribution of Doppler-broadened photons which enter a hot rubidium vapor. The obtained PDF follows a power law with  $\gamma = 1.41$ . The step-length PDF changes after each scattering event while remaining a near perfect power law. The dependence of the exponent  $\alpha(n) = \gamma(n) + 1$  on the number  $n$  of scattering events saturates to a value close to 2, as expected from the theory (Pereira, Martinho, and Berberan-Santos, 2004). Recently, Baudouin *et al.* (2014) measured transmission through a hot rubidium vapor by changing the opacity of the media  $O = L/\bar{\ell}$  over two decades. This was realized by controlling the density of atoms (and thus  $\bar{\ell}$ ) by adjusting the temperature inside the vapor chamber. The results fit the dependence predicted by Eq. (80) with the exponent  $\gamma \approx 1.01$ ; see Fig. 23(d). The radial profile of the transmitted light has a power-law tail  $I(r) \propto r^{-3-\mu}$  as expected from the Lévy-based theory they developed; see Figs. 23(c) and 23(d). By comparing these results with the single step-length PDFs obtained before, Baudouin *et al.* (2014) stated an excellent agreement with a Lévy-walk approach.

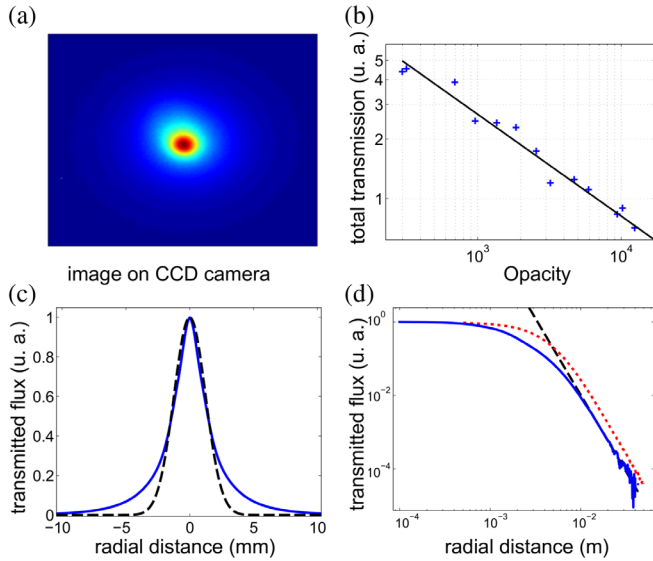


FIG. 23 (color online). Photon transmission through a hot rubidium vapor. (a) The radial profile of the outgoing light on the charge-coupled device camera. (b) Crosses show experimentally measured transmission as a function of the opacity. The power-law fit  $T \propto O^{-0.516}$  is shown by the solid curve. (c) Experimental radial profile of the light transmitted through the vapor chamber (solid line) is compared with a Gaussian distribution with the same width at half maximum (dashed line). (d) Radial profile of the transmitted light. The dashed line is a power-law fit  $I(r) \propto r^{-4.03}$ . Adapted from Baudouin *et al.*, 2014.

#### D. Blinking quantum dots

Blinking quantum dots serve another realization of the ballistic Lévy walk. Similar to the Lévy flights of light, photons are again involved but in a different way.

A quantum dot (QD) is a nanocrystal made out of semi-conducting material and is several nanometers in size

(Alivisatos, 1996). The size is crucial for determining the specific properties of QDs which are governed by quantum effects and are on the border between bulk and molecular behavior. One of the important features of QDs is the so-called quantum confinement, when the exciton Bohr radius is of the order of the object size, leading to the discrete energy levels and a band gap which depends on the size of the object. When under the laser light with energy above the band gap, QDs can adsorb light by creating an exciton pair and then reemit a photon when the exciton decays. The frequency of the emitted light is increasing with decreasing QD size and can be accurately tuned in applications. One of the important QD applications is bioimaging; in addition to their small size, QDs have higher brightness as compared to organic fluorescent dyes and show minimal photobleaching. However, there is one interesting effect: QDs blink (Nirmal *et al.*, 1996). Experimentally it was found that quantum dots alternate periods of fluorescence with no emission of photons, and the durations of these periods are not exponentially distributed but instead have a fat tailed power-law distribution with diverging average time. A quantum dot can fluoresce or be completely dark during the whole measurement time, which can be on the order of hours. Current experimental results provide the *on* and *off* times statistics spanning 4 orders of magnitude. In Fig. 24 one realization of the QD fluorescence intensity track is shown. By defining a certain threshold in the intensity, a sequence of on and off times can be identified and characterized. Many experiments with different QD materials and at different temperatures show power-law distributions of those times, and in many cases the exponent is nearly the same for both of them:  $\gamma \approx 0.5$  (Margolin *et al.*, 2005). Therefore the blinking dynamics of a QD can be described as a two state model, where the durations of phases in each state [on,  $I(t) = 1$ , and off,  $I(t) = 0$ ] are distributed as power laws with diverging means. As discussed in Sec. IV.D, systems with such distributions exhibit memory effects, aging, and weak

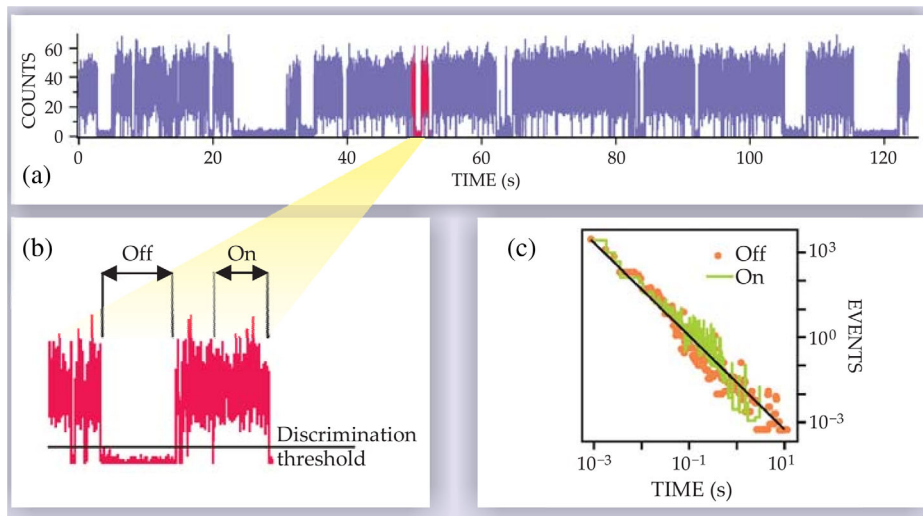


FIG. 24 (color online). Blinking quantum dots. (a) A sample trace of a quantum dot fluorescence. A zoom in (b) shows how a threshold is defined, which allows one to identify the periods of on and off times. The PDFs of those times are shown in (c) and have a clear power-law dependence with the same tail exponent for on and off times. The solid black line indicates a slope of 1.65, i.e.,  $\gamma = 0.65$ . Adapted from Stefani, Hoogenboom, and Barkai, 2009.

ergodicity breaking. Interestingly, the problem of a blinking nanodot can be mapped onto the Lévy-walk model. Consider the fluorescence intensity  $I(t)$  and define its time average as

$$\bar{I} = \frac{\int_0^T I(t) dt}{T}. \quad (82)$$

As learned from Sec. IV.D, for the weak ergodicity breaking problems the time average is itself a random variable with a certain distribution. It can be shown that the PDF of the time averaged intensity  $P(\bar{I})$  is given by the Lamperti distribution (Lamperti, 1958; Margolin *et al.*, 2005):

$$P(\bar{I}) = \frac{\pi^{-1} \sin(\pi\gamma) \bar{I}^{\gamma-1} (1-\bar{I})^{\gamma-1}}{\bar{I}^{2\gamma} + (1-\bar{I})^{2\gamma} + 2\bar{I}^\gamma (1-\bar{I})^\gamma \cos(\pi\gamma)}. \quad (83)$$

We know this distribution from the analysis of the ballistic Lévy-walk model, and it is easy to draw an analogy. In the ballistic regime of the Lévy walk, we can define a time averaged position of the particle  $x/T$  as an integral from 0 to  $T$  of the particles velocity  $v(t)$ ; see Eq. (56). As in quantum dots, the time spent in each velocity state has a power-law distribution with infinite mean. The only difference to the QD blinking problem is that the velocity of particles can have values of  $v(t) = \pm v_0$ , while the intensity switches between 0 and 1. As a result, in the case of Lévy walks, the PDF  $P(x/T)$  is symmetric around zero, whereas for the time average intensity it is shifted and has a support from 0 to 1. One particular example of  $\gamma = 1/2$  gives a simple particular case of the Lamperti distribution; see Eq. (34) and Fig. 3. An intuitive expectation that the QD will be half time on and half time off appears to be least probable: the corresponding PDF has a minimum at  $\bar{I} = 1/2$ . Instead the  $P(\bar{I})$  has a divergent behavior at  $\bar{I} = 0$  and  $\bar{I} = 1$  (similarly to the divergence of the PDFs at ballistic fronts in the case of Lévy walks). Therefore a quantum dot is either on or off for most of the observation time.

The particular mechanism responsible for the appearance of the power-law distributed blinking times in quantum dots remains unknown. There are several working models which relate the statistics of on and off times to the dynamics of the exciton pair including its transport, diffusion, and trapping [see Stefani, Hoogenboom, and Barkai (2009) for an overview], but none of the models are able to describe all available experimental observations. For the context of this review it is important that the experimental data on a blinking QD can be directly mapped to the model of Lévy walks in the ballistic regime (Margolin *et al.*, 2006). We speculate that the analytical results available for the Lévy-walk model with random velocities could be useful for the interpretation of experiments with QDs with a whole distribution of intensities and not just two levels. Reciprocally, a possible correlation between consequent long on (off) times, when a long on (off) time is followed with higher probability by another long on (off) time (Stefani, Hoogenboom, and Barkai, 2009) calls for further generalizations of the Lévy-walk model.

### E. Lévy walks of cold atoms

Lévy distributions have been known in the field of cold atom optics since the 1990s, when Bardou *et al.* (1994) and

Reichel *et al.* (1995) discussed the relation between the process of the so-called subrecoil laser cooling (Aspect *et al.*, 1988) and anomalous diffusion in terms of Lévy flights. Lévy walks appeared in cold atom optics in the context of Sisyphus cooling of atoms loaded into an optical bipotential created by two counterpropagating linearly polarized laser beams (Dalibard and Cohen-Tannoudji, 1989). There are two internal atomic states and atoms with different internal states feel potentials of different polarizations. The laser-induced transitions of an atom between its internal states influence translational motion of the atom along the bipotential. Elaborated within the Monte Carlo wave function framework, this connection was shown to be responsible for Lévy-walk-like dynamics of atoms (Marksteiner, Ellinger, and Zoller, 1996). Katori, Schlipf, and Walther (1997) measured the mean squared displacement of Mg ions in a bipotential optical lattice and found the scaling  $\sigma^2(t) \propto t^\mu$  with the exponent  $\mu > 1$  for potential depths below the critical value. Sagi *et al.* (2012) performed more sophisticated experiments and measured the spreading of a packet of cold Rb atoms in optical lattices of different depths. The obtained atomic distributions scaled with the characteristic scaling, Eq. (16), Fig. 25(a), and their shapes could be nicely fit by Lévy distributions, Fig. 25(b). However, a Lévy-walk description did not work well in this case, because in both, experiments and Monte Carlo wave function simulations, strong correlations between velocities and durations of atom flights were found.

Marksteiner, Ellinger, and Zoller (1996) derived that on the semiclassical level the distribution  $W(x, p, t)$  of atoms can be described by the Kramers equation

$$\frac{\partial W}{\partial t} + p \frac{\partial W}{\partial x} = \left[ D \frac{\partial^2}{\partial p^2} - \frac{\partial}{\partial p} F(p) \right] W, \quad (84)$$

with the cooling force (Castin, Dalibard, and Cohen-Tannoudji, 1991)

$$F(p) = -\frac{p}{1+p^2}, \quad (85)$$

where momentum is expressed in dimensionless units  $p/p_c$ , with the capture momentum  $p_c$  set to unity. For small momenta the force is of the conventional linear form  $F(p) \sim -p$ , while  $F(p) \sim -1/p$  for large  $p$  so that the atom becomes frictionless at the high-momentum limit. The diffusion constant  $D$  combines all relevant parameters such as the depth of the optical potential, recoil energy, see Eq. (87). The equilibrium momentum-momentum correlation function corresponding to Eq. (84) scales as  $t^{-\lambda}$ , with the exponent

$$\lambda = (1/2D) - 3/2. \quad (86)$$

The control parameter

$$D = cE_R/U_0 \quad (87)$$

depends on the recoil energy  $E_R$  and the depth of the optical lattice potential  $U_0$ . The constant  $c$  is specific to the type of atom or ion cooled, with the typical value around 10. Therefore, one could, by tuning the potential depth  $U_0$  while



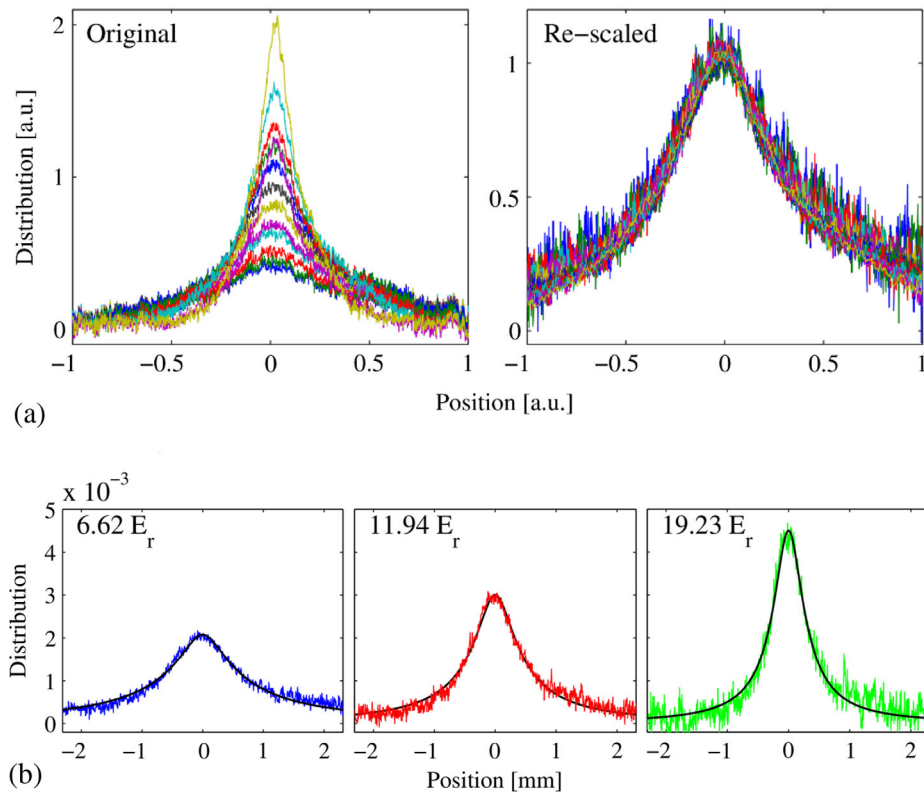


FIG. 25 (color online). Lévy walks of atoms in optical lattices. (a) Atomic distributions obtained for different times,  $t \in [10 - 40]$  ms, before (left) and after the scaling transformation (16) with exponent  $\alpha = 0.8$  (right). (b) Atomic distributions after 30 ms of spreading for three different optical potential depths. Lines correspond to Lévy distributions with depth-specific exponents. Adapted from Sagi *et al.*, 2012.

keeping all other parameters fixed, control the exponent  $\lambda$  and switch between the regimes of normal and anomalous atom diffusion.

When  $\lambda < 1$ , the integral of the correlation function over time diverges and an anomalously fast diffusion appears. The equilibrium velocity distribution for Eq. (84) has a form of the Tsallis distribution (Lutz, 2003; Douglas, Bergamini, and Renzoni, 2006). Although interesting as an indication of a strong deviation from the Boltzmann-Gibbs thermodynamics (Lutz and Renzoni, 2013), these distributions themselves do not provide sufficient insight into the diffusion of atoms in real space.

By unraveling the Kramers equation (84) into a Langevin equation with a white Gaussian noise as a drive,

$$\dot{p} = F(p) + \sqrt{2D}\zeta(t), \quad (88)$$

$$\dot{x} = p, \quad (89)$$

Kessler and Barkai (2012) analyzed the atom diffusion from the microscopic point of view. The theory developed by Barkai, Aghion, and Kessler (2014) predicts the existence of three phases in the dynamics, generated by Eqs. (88) and (89), depending on the value of  $D$ . Namely, it can exhibit normal diffusion, Lévy-walk superdiffusion, and Richardson's diffusion (Richardson, 1926), when the MSD scales as  $\sigma(t) \propto t^3$ .

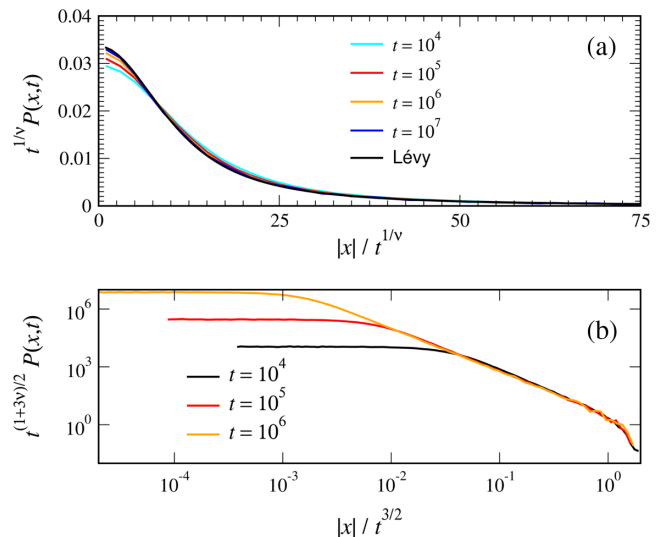


FIG. 26 (color online). Scaling of the propagators  $P(x, t)$  of the stochastic process, Eqs. (88) and (89), modeling the diffusion of cold atoms. The scaling exponent  $\gamma$  (denoted by  $\nu$  in the original publications) is  $(1 + D)/3D$ . (a) With the increase of time the rescaled profiles start to fall onto the Lévy distribution  $L_\gamma(\xi)$ . (b) Another scaling reveals the cutoff induced by a nonlinear space-time coupling, Eq. (90). The parameter  $D = 2/5$ . Adapted from Kessler and Barkai, 2012.



The existence of the Lévy-walk regime was proved analytically for the range  $1/5 < D < 1$ . In the asymptotic limit of large  $t$ , the central part of the propagator  $P(x, t)$  scales with the distinctive scaling, Eq. (16), where  $\Phi(\xi) = L_\gamma(\xi)$ , and  $\gamma = (1 + D)/3D$ . The coupled transition probably is different from that for the standard Lévy walk, Eq. (26), and has the form (Kessler and Barkai, 2012; Barkai, Aghion, and Kessler, 2014)

$$\phi(y, \tau) = \psi(\tau)p(y|\tau), \quad (90)$$

with the conditional PDF  $p(y|\tau) \sim \tau^{-3/2}B(y/\tau^{3/2})$ . The normalized nonlinear function  $B(x)$  is responsible for the cutoff of the propagator tails, so that all moments of the process are finite; see Fig. 26. Together with the mode-coupling theory of the energy transport in classical nonlinear chains (Sec. V.B), these results pave the way toward physical foundations of Lévy walks.

## VI. LÉVY WALKS IN BIOLOGY

From rather complex but objective systems of physics, we are shifting to the field of biology and biophysics where effects and phenomena are much harder to quantify because of their intrinsic diversity and variability. In recent years, the topic of Lévy walks resonated in research communities working on motility of living organisms, their foraging, and search strategies. By the level of debates, the topic may even be called controversial. Fortunately our review is preceded by two very recent monographs devoted to these subjects (Viswanathan *et al.*, 2011; Méndez, Campos, and Bartumeus, 2014). Here we present our point of view through the prism of the Lévy-walk framework and point to the examples that are directly relevant to this model. Before moving on to particular examples we outline the general complexity of the problem in question.

### A. Motility is a complex issue across many scales

Motility spans many scales, ranging from swimming micron-sized bacteria to albatrosses which can travel hundreds of kilometers at a time. Motility involves interactions of moving animals with their environment and habitats, which in most cases is hard to quantify or predict. In ecology, the interest in motility usually does not arise *per se* but in relation to some greater issues, for example, questions of how animals search for food, how they navigate home, how they find each other to mate or to agglomerate into colonies, and others.

In an interesting twist, Lévy walks are involved in a particular topic of effectiveness of search and foraging strategies. Lévy walks are argued to be the most efficient search strategy under certain conditions imposed on the distribution and properties of targets. There is a constantly growing number of accounts where Lévy statistics is reported for the trajectories of animals. Quite often these results get criticized or disputed, based on insufficient data, an inconsistent analysis, or just out of different beliefs. As a side effect of these still ongoing discussions, new papers constantly appear where researchers report the analysis of the motion

patterns of yet another living species and claim that the patterns do or do not look as Lévy-flight or Lévy-walk trajectories. There is even a philosophical flavor in this discussion (Baron, 2014). The possible reasons of this controversy are manifold. Next we summarize them from rather evident to more complex levels.

- (i) Difference in sizes, forms of locomotion, habitats etc.: All these differences dictate different experimental techniques and also call for different statistical techniques. As pointed out by Méndez, Campos, and Bartumeus (2014), on the micron scales of single cells, positional data can be acquired with high space and time resolution leading to almost continuous recorded trajectories. Such observations are common in a laboratory since the 1970s. Tracking of big animals in their habitats is a much harder task due to complex interactions of the animals with the environment and large spatial scales they travel over. This field advanced only recently, to a greater extent due to the miniaturization and growing accuracy of the portable global positioning system (GPS) devices. Therefore there is much less and sparser statistical data for big animals. Still, while it is possible to follow 1500 individual sperm cells at a time (Su, Xue, and Ozcan, 2012), this number remains unrealistic for sharks or deer. There is a data-driven gap in the applied methodology. Some researchers are trying to use Langevin-type equations for continuous tracks while others prefer more coarse-grained random walk models for the trajectories recorded with limited resolution. Currently the gap is narrowing, as there are examples of random walks used to model the motility of bacteria and attempts to apply the Langevin machinery to analyze the trajectories of bumblebees and beetles.
- (ii) Complex trajectories: Some trajectories resemble neither Lévy flights nor Lévy walks but are still modeled as such. These are usually almost smooth continuous tracks of cells or other organisms (Levandowsky, White, and Schuster, 1997; Dieterich *et al.*, 2008; de Jager *et al.*, 2011). There is often a problem of how to define a flight or a step of a random walk for such tracks, to resolve which several methods were suggested (Turchin, 1998; Rhee *et al.*, 2011; Humphries, Weimerskirch, and Sims, 2013; Raichlen *et al.*, 2014). The proposed random walk models often are of academic interest only, since most of the information encoded in continuous trajectories is lost or disregarded. These approaches provide, however, some statistical characteristics of the foraging patterns that can be compared with those for the known search strategies. An alternative approach is to look into the microscopic details of motility patterns by using, for example, Langevin dynamics (Selmecki *et al.*, 2008; Zaburdaev *et al.*, 2011; Lenz, Chechkin, and Klages, 2013), and then pose a question of

how it can lead to the appearance of the Lévy-like behavior on larger spatial scales (Lubashevsky, Friedrich, and Heuer, 2009b). In a few cases, the information provided by trajectories was sufficient to suggest biological mechanisms of the motility as was demonstrated for some cells and bacteria (Li, Norrelykke, and Cox, 2008; Gibiansky *et al.*, 2010; Jin *et al.*, 2011; Marathe *et al.*, 2014; Zaburdaev *et al.*, 2014).

- (iii) Lévy flight versus Lévy walk: Although it is evident that living organisms can move only with a finite speed, there is a big subset of studies where the Lévy flight is used to model the observed trajectories. Some papers mention both approaches, walks and flights, interchangeably, but then they mostly consider the statistics of displacements at fixed time intervals or the MSD. The distribution of displacements at fixed time intervals in fact yields a velocity distribution (López-López *et al.*, 2013) and therefore suggests a very different model of random walks with random velocities; as we have seen already, its properties are different from both the Lévy-flight and Lévy-walk models. We also know that the MSD of a Lévy flight diverges and therefore the corresponding model is not suitable for the analysis of the MSDs obtained from the experimental data. The ignorance to the difference between the Lévy-flight and Lévy-walk concepts does not add positively to the clarity of the issue.
- (iv) Other biological reasons: As mentioned, whether a bacterium or a deer, both interact with the environment. The more complex the organism, the more rich and unpredictable are the effects of this interaction. While laboratory conditions for bacteria or cell experiments can be controlled to a high degree, the question of how much of a deer's motion is influenced by the type of a forest the deer moves in is much harder to disentangle. Individuals may have different responses to the same stimuli, because, for example, they can be at different developmental stages. Therefore it should not be forgotten that some effects which look like anomalous behavior for the ensemble of organisms, may come about only as a result of variability between the individuals, where each individual behaves quite normally but on its own scale; see Méndez, Campos, and Bartumeus (2014) and original works by Hapca, Crawford, and Young (2009) and Petrovskii, Mashanova, and Jansen (2011).

It is certainly beyond our goals and abilities to resolve all these challenging issues in this review. We can only welcome attempts to summarize and critically address these points by Selmezi *et al.* (2008) and Méndez, Campos, and Bartumeus (2014). We hope that the theoretical background provided in this review will help to introduce the Lévy-walk model (and its appropriate modifications) to the community of biologists and biophysicists with more rigor so that it can be applied to the collected data in a proper way.

## B. Soil amoeba

One of the first mentions of the Lévy-walk model in biological context was made in the work on crawling amoeba by Levandowsky, White, and Schuster (1997). Amoeba are unicellular organisms that can move on surfaces and three-dimensional media by growing cell protrusions called pseudopodia. Levandowsky, White, and Schuster (1997) tracked 17 amoeba isolates with the help of a microscope and a video recorder. Different traces of the overall duration 15–60 min were recorded with a time step of 1 or 2 min. Considered species represented a range of sizes 10–100  $\mu\text{m}$  and average speeds of 0.16–1.3  $\mu\text{m/s}$ . This means that cells roughly moved about one cell size per one step (one minute); see Fig. 27. After each step, they measured turning angles, velocity distribution, and the MSD. For all observed cells the MSD scaled as  $\langle x^2 \rangle \propto t^\mu$  with  $\mu \sim 1.5 - 1.9$ , which led them to the conclusion that the Lévy walk could be a good candidate for a model. The obtained histograms of turning angles indicated little directional change. They also stated that their tracking was not long enough to check whether cells switch to the normal diffusion at longer times. Although not a clear-cut example of the Lévy walk, this was the first and balanced assessment of the experimental observations. After two decades of similar research one could suggest that the Ornstein-Uhlenbeck process, i.e., a Langevin equation for velocity increments containing friction and random force (Risken, 1996), could be a reasonable alternative approach. The corresponding stochastic process is characterized by an exponentially decaying velocity autocorrelation function, and if the observation time is less or of the order of the correlation time the MSD behaves almost ballistically and only at later times switches to diffusive behavior; see similar results for

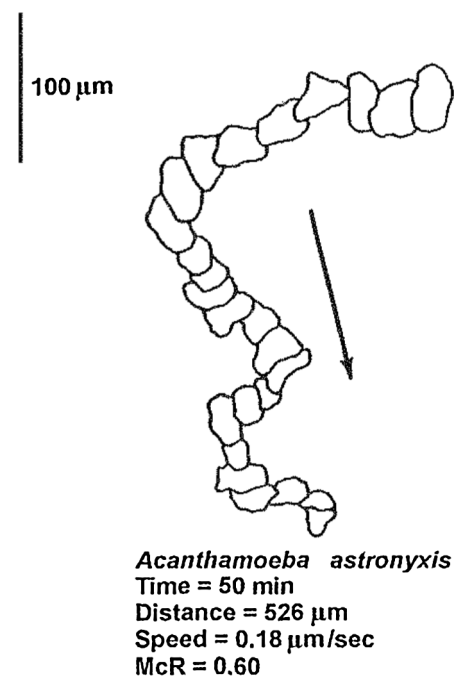


FIG. 27. A sample trajectory of soil amoeba showing the outline of the cell at 1 min intervals. Adapted from Levandowsky, White, and Schuster, 1997.

beetles (Reynolds, Leprêtre, and Bohan, 2013). A recent comprehensive study of *Dictyostelium discoideum* amoeba motility considered several possible mechanisms, including the generalized Langevin equation with a memory kernel, nontrivial fluctuations, and a more microscopic, zigzag motion strategy (Li, Norrelykke, and Cox, 2008).

### C. Run and tumble of bacteria

Until recently, motion of a swimming *E. coli* bacteria was considered as a clear example of the standard diffusion. The diffusive dynamics naturally follows from the mesoscopic picture of random walks describing the run-and-tumble motion (Berg, 1993, 2004). *E. coli* have multiple flagella, helical filaments which rotate and thus propel the cell in the fluid. Because of the microscopic size of the cell, the swimming occurs at low Reynolds numbers, which has its implications on the physics of the process (Purcell, 1977; Lauga and Goldstein, 2012). Molecular motors can rotate flagella in two opposite directions, clockwise (CW) and counterclockwise (CCW). In CCW mode multiple flagella form a bundle and the cell swims following almost a straight path, which is called a “run.” When one or several motors switch the direction to CW, the bundle dissolves and the cell rotates almost on the same spot, the so-called “tumble” phase. When the bundle forms again in the CCW mode, the cell begins its next run. The angle between the directions of the two consequent runs is not completely random but has a nonuniform distribution with a mean around  $70^\circ$ . *E. coli* are rod shaped bacteria of about  $2\ \mu\text{m}$  long, the average run time is 1 s, and the corresponding almost constant speed is  $\sim 20\ \mu\text{m/s}$ . Therefore the length of a run is roughly 10 times the cell body length. Tumbles are approximately 10 times shorter than the runs and usually neglected in theoretical models. Since the motion occurs in a fluid, runs are not entirely straight but are subjected to the effects of the rotational diffusion. If, for a moment, we neglect the rotational diffusion, the swimming cell can be seen as a biological realization of the Lévy-walk model: it moves with an almost constant velocity, then tumbles, and chooses a new swimming direction. The experimentally measured run time distribution of *E. coli* was usually described by an exponential distribution (Berg, 2004). However, in a recent experiment with individual tethered cells by Korobkova *et al.* (2004), it was shown that the PDFs of durations of CCW rotation of flagella (corresponds to run of the cell) fit the power-law distribution with an exponent  $\gamma = 1.2$ ; see Fig. 28.

It was also shown theoretically that the genetic circuit responsible for the duration of motor rotation in the CCW direction can generate power-law distributed times in the presence of chemical signal fluctuations (Matthäus *et al.*, 2011). Experiments with tethered cells suggest that power-law distributed run times could also be observed in individual swimming cells, but there is no experimental confirmation of this yet.

To encompass the possibility of the power-law distributed run times, the Lévy-walk model is the natural choice to describe the dispersal of idealized *E. coli* bacteria (by neglecting the effects of rotational diffusion during the runs) in two or three dimensions.

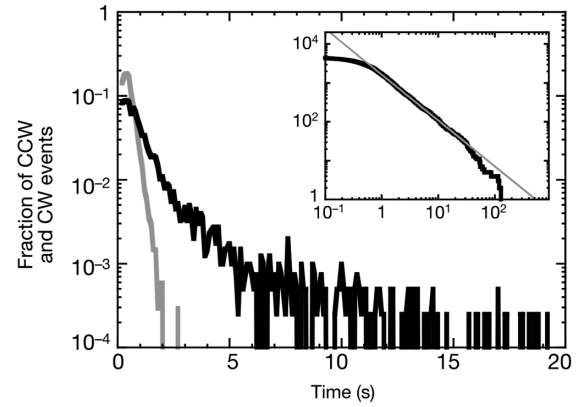


FIG. 28. Distributions of run-and-tumble times of *E. coli* bacteria. Counterclockwise (CCW) rotation (black line) corresponds to the run of bacteria, whereas clockwise (CW) rotation (gray line) corresponds to tumbling. Measurements are presented for a single bacterium. While tumbling times are shorter and well described by the exponential distribution, the durations of CCW rotations exhibit a long nonexponential tail which can be fitted by a power law. The inset shows the cumulative distribution function for CCW rotation times and the gray line is the power law with an exponent  $\sim 2.2$ . From Korobkova *et al.*, 2004.

Interestingly, many bacteria (and some eukaryotic cells) swimming in fluid or moving by other means on surfaces produce similar patterns, reminiscent of the run-and-tumble motion. Those include, for example, a run-and-reverse pattern, where the direction of the next run is opposite to the previous one, or a run-reverse-flick motion, where reversals are alternated with random turns (Xie *et al.*, 2011; Taktikos, Stark, and Zaburdaev, 2013); see Fig. 29(c). Experimentally such trajectories are often observed within the single focal plane of the microscope or in a confined planar geometry. These biologically relevant motility patterns suggest an alternative description of a Lévy-walk process in two dimensions, namely, via the angle determining the orientation of the cell velocity,  $\phi(t): \mathbf{v}(t) = v(\cos \phi(t), \sin \phi(t))$ . The time evolution of the angle can be split into two components, abrupt angular changes during the reorientation events and an independent component of the noise leading to the rotational

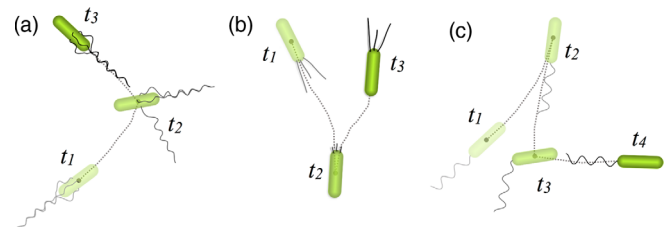


FIG. 29 (color online). Different motility patterns of bacteria: (a) run-and-tumble motion, where straight runs are alternated by tumbling events; the angle between the consecutive runs can have a certain preferred value; (b) run reverse, where the next always has an opposite direction; (c) run reverse and flick, where reversals strictly alternate with completely randomizing turn, similar to *E. coli*, in other words it is an alternation of (a) and (b) that makes up a more complex pattern of (c). From Taktikos, Stark, and Zaburdaev, 2013.



diffusion  $\phi(t) = \phi_{\text{rw}}(t) + \phi_{\text{rot}}(t)$ .  $\phi_{\text{rw}}(t)$  characterizes a one-dimensional CTRW in the angle space. The run-and-tumble motion is characterized by waiting times when the angle does not change (run) and random jumps according to the turning angle distribution (tumble). For example, in the case of run and reverse, the jumps are of the size  $\pm\pi$ . Therefore the problem of the Lévy walk in two dimensions can be mapped onto a one-dimensional CTRW process for the angle. It has interesting consequences for the calculation of the standard quantities as, for example, the velocity autocorrelation function or the MSD. The velocity correlation function of bacteria's velocities at times  $t_1$  and  $t_2$ ,  $C(t_1, t_2)$  is given by

$$C(t_1, t_2) = \langle \mathbf{v}(t_1) \mathbf{v}(t_2) \rangle = v^2 \langle e^{-i[\phi(t_2) - \phi(t_1)]} \rangle. \quad (91)$$

The contribution to correlations coming from the rotational diffusion is well known and appears as an exponential prefactor, a more nontrivial part is the random walking component of the angle. It can be shown that

$$C_{\text{rw}}(t_1, t_2) = v^2 \int_{-\infty}^{+\infty} d\phi_1 \int_{-\infty}^{+\infty} d\delta\phi e^{-i\delta\phi} P(\phi_1, t_1; \delta\phi, t_2). \quad (92)$$

Here  $P(\phi_1, t_1; \delta\phi, t_2)$  is the joint probability density to find a cell moving in direction  $\phi_1$  at time  $t_1$  and direction  $\phi_1 + \Delta\phi$  at time  $t_2$ . It is easy to see that Eq. (92) is the double Fourier transform with respect to  $\phi_1$  and  $\delta\phi$ , where the corresponding coordinates in Fourier space are set to  $k_1 = 0$  and  $k_2 = 1$ , respectively. Therefore, to find the velocity autocorrelation function one needs to find the two-point PDF for the random walk of the angle. It is a nontrivial, especially for the case of power-law distributed waiting times, but exactly solvable problem (Barkai and Sokolov, 2007; Baule and Friedrich, 2007; Zaburdaev, 2008; Dechant *et al.*, 2014). The MSD can now be calculated by using the Kubo relation:

$$\langle [\mathbf{r}(t) - \mathbf{r}(0)]^2 \rangle = \int_0^t dt_1 \int_0^t dt_2 \langle \mathbf{v}(t_1) \cdot \mathbf{v}(t_2) \rangle. \quad (93)$$

There are two important particular cases of the above general formulas. For the exponentially distributed run times,  $\psi_{\text{run}}(\tau) = \tau_{\text{run}}^{-1} \exp(-\tau/\tau_{\text{run}})$ , many things simplify dramatically and yield the following answer for the MSD:

$$\langle [\mathbf{r}(t) - \mathbf{r}(0)]^2 \rangle_{\text{rw}} = 2v^2 \tilde{\tau}^2 \left( \frac{t}{\tilde{\tau}} - 1 + e^{-t/\tilde{\tau}} \right), \quad (94)$$

where the effective decorrelation time  $\tilde{\tau}$  depends on the average run time  $\tau_{\text{run}}$  and the average cosine of the turning angle  $\cos \phi_0$  (Lovely and Dahlquist, 1975):

$$\tilde{\tau} = \frac{\tau_{\text{run}}}{1 - \cos \phi_0}. \quad (95)$$

For *E. coli*,  $\cos \phi_0 \approx 0.33$ , whereas for reversing cells it is equal to  $-1$ . Equation (94) is a well-known result for the Ornstein-Uhlenbeck process (Risken, 1996) which was mentioned before, but here it was derived from the Lévy-walk

model and not from the Langevin equation. For short times  $t \lesssim \tilde{\tau}$  the MSD scales ballistically and then turns to the diffusive regime. In the case of the power-law distributed run times [as in Eq. (8)] with  $1 < \gamma < 2$ , the MSD scales as  $t^{3-\gamma}$ , a well-known result. An interesting observation is that in the superdiffusive regime the turning angle distribution plays no role (unless the turning angle is not zero) in the asymptotic regime.

To finalize this section we discuss two more modifications of the Lévy walk used to model motility of bacteria. Previously we mentioned the run-reverse and flick motility pattern which was reported for *V. alginolyticus* bacteria by Xie *et al.* (2011). In this case, the reversals are alternating with completely randomizing turns with  $\cos \phi_0 = 0$ . In *V. alginolyticus* this happens because its single flagellum is unstable, when switching from CW to CCW rotation. The durations of runs after flick and reversals may also be governed by two different distributions. When translated into a CTRW model for the angle, that means that jumps with two distributions for the jump amplitude and waiting times are alternating. As a remarkable difference to the model with a single turning angle distribution where the velocity correlation function is always positive, the run-reverse-flick model has an interval of negative velocity correlations (Taktikos, Stark, and Zaburdaev, 2013).

For another type of swimming bacteria, *P. putida*, it was found that cells predominantly adopted the run-and-reverse pattern, but, in addition, the speed of a single cell changed roughly by a factor of 2 between forward and backward swimming directions (Theves *et al.*, 2013). For the corresponding one-dimensional Lévy-walk model with two alternating speeds that would result in the back and forth motion, but with the ballistic scaling in the direction of the higher speed. The cells swimming in a fluid are subjected to fluctuations and therefore the rotational diffusion regularizes the ballistic scaling. As a result, bacteria undergoing run-and-reverse motion with alternating velocities diffuse faster than bacteria showing run-and-reverse behavior but with a constant intermediate velocity.

The above examples demonstrate that the class of Lévy-walk models provides a perspective tool for the mesoscopic description of the bacterial motility. Whether the involved times are anomalously long or exponentially distributed, Lévy-walk framework is flexible and can be adjusted to the needs of a particular problem—rotational diffusion during runs, different turning angles and speeds, pausing during tumbles—while remaining in the domain of analytically solvable models.

## D. Short note on chemotaxis

Bacteria, amoeba, sperms, and many other cells and microorganisms are known to be able to perform chemotaxis: they can actively alternate motility in response to the gradients of certain chemicals, signaling molecules, nutrients, or waste products. Different organisms adopt different chemotactic strategies (Eisenbach and Lengeler, 2004). Larger cells, such as amoeba, can detect the gradients across their own cell body length via multiple chemoreceptors. Reacting to the occupancy of those receptors, amoeba can preferentially grow the



pseudopodia in the corresponding direction and therefore continuously reorient during its motion. Bacteria are too small to do that and instead use a temporal integration of the chemical concentration which they experience along the trajectory. The chemical signal is passed on to the genetic pathway which regulates the flagella motor reversals [we are omitting many interesting biological details, which, at least for *E. coli*, are well understood (Berg, 2004)]. If a cell swims in the direction of increasing concentration of the favorable signal it extends its run phase. The response of the cell to the pulses of certain chemicals was measured experimentally by observing the frequency of motor reversals; it revealed a nontrivial two lobed response function, showing the properties of adaptation (Segall, Block, and Berg, 1986; Celani and Vergassola, 2010). By assuming that tumbles follow after an exponentially distributed run time, the rate of tumbling events in the presence of the signaling chemical with not too strong variations can be represented as

$$\lambda(t) = \lambda_0 \left( 1 - \int_{-\infty}^t dt' c(t') R(t-t') \right), \quad (96)$$

where  $\lambda_0 = 1/\tau_{\text{run}}$  is the cell's tumbling rate in a homogeneous environment,  $c(t)$  is the concentration of the chemical along the path, and  $R(t)$  is the memory or response function of bacteria obtained from the experiments. de Gennes (2004) used this formula and the random walk model of run and tumble to analytically calculate the resulting average drift velocity along the small gradients of  $c(x)$ . This approach can be generalized to all the above discussed motility patterns of bacteria and shows the importance of the theoretical modeling by means of simple random walks. One of the open questions in this field is how to generalize the de Gennes' approach to a general distribution of tumbling events, going beyond the exponential function and including the power laws.

## E. T cells

In a recent experimental study by Harris *et al.* (2012), the migration of CD8<sup>+</sup> T cells in the brain explant of mice was analyzed. CD8<sup>+</sup> T cells are a special type of white blood cells which are responsible for killing cancer cells, those infected by viruses, or otherwise damaged or abnormal cells. Direct contact of the T cell and the target cell is required for killing the abnormal cell. In this study, T cells were targeting the cells infected by a parasite *T. gondii* which invades the cells of the central nervous system and causes the toxoplasmosis infection. T cells which produce a fluorescent protein were imaged in 3D by using two-photon microscopy of the brain explant of mice with chronic toxoplasmic encephalitis in different experimental conditions.

As one of the important factors involved in the regulation of T-cells motility, a small signaling protein, chemokine CXCL10, was noted. By varying the concentration of this chemokine they showed that T cells were changing the average speed but not other statistical characteristics of their trajectories. Along with the standard MSD measurements, several additional properties of the acquired trajectories were analyzed: PDF of displacements at different times and its scaling properties, correlation function of cell displacements, and overall shape of the tracks. MSD showed a clear superdiffusive behavior with the exponent 1.4:  $\langle r^2(t) \rangle \propto t^{1.4}$ . Consistent with previous observations of runs and pauses in lymphocytes they suggested the model of Lévy walks with rests as the working hypothesis. Indeed by comparing this model with more than ten other possible random walk models, it was shown to give the best representation of the experimental data (see the extensive supplementary material to the original paper). In Fig. 30 we show the representation of 3D tracks and in Fig. 31 the PDF of cell' displacements at different times (corresponds to the control case in Fig. 30).

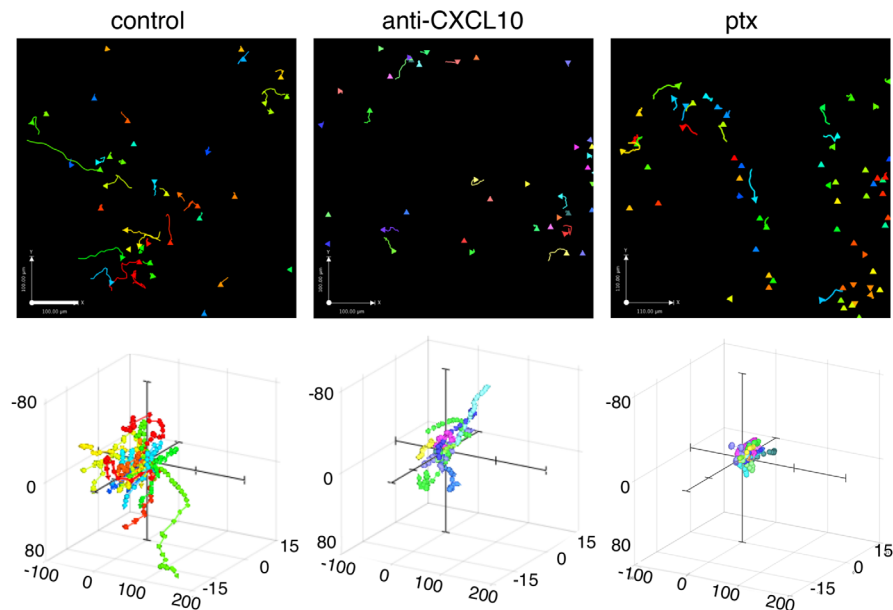


FIG. 30 (color online). (Top row) Tracks of T cells recorded by two-photon fluorescence microscopy in three different experimental conditions. (Bottom row) Reconstructed three-dimensional trajectories of individual cells. Adapted from Harris *et al.*, 2012.

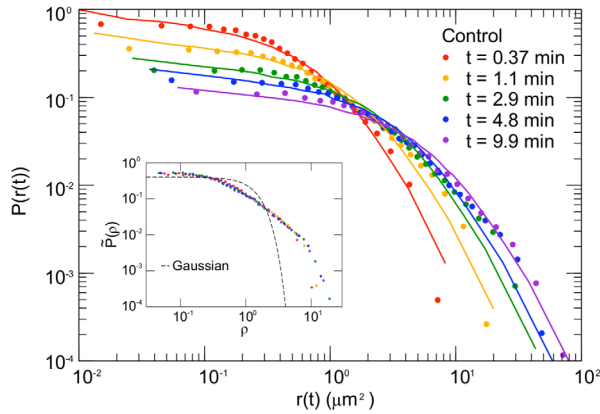


FIG. 31 (color online). PDF of T-cells' displacements for the control case at different time points. Symbols denote the experimental data, and lines are fits according to the generalized Lévy-walk model. The inset shows that the profiles, rescaled according to  $\tilde{P} = t^\alpha P$  and  $\rho = r/t^\alpha$  with  $\alpha = 0.63$ , collapse on top of each other. From [Harris et al., 2012](#).

Rescaled profiles convincingly fall onto a single master curve. In the concluding remarks of the paper, it is mentioned that the Lévy-walk model for the motility of T cells is consistent with the idea of more effective search, as compared to Brownian motion in the case of sparse targets. Overall it is one of the most thorough trajectory analyses to date which leads to the Lévy-walk model. Probably because ten other models were shown to fail to reproduce the experimental data, it effectively exhausted the arsenal of arguments from the opponents of the Lévy-walk foraging hypothesis [as a counterexample, see a trail of publications on the Lévy walk of mussels ([de Jager et al., 2011](#))].

It is instructive to look at the scales involved in this study. A typical duration of the recorded trajectories was 15–30 min with average moving speeds of 3 to 6  $\mu\text{m}/\text{min}$ , depending on the levels of the chemokine. With the size of the T cell about 10  $\mu\text{m}$ , similar to the case of amoeba, cells traveled a couple of tens of their sizes. For bacteria that would correspond to a distance of a single run. However, for the case of T cells that might be the relevant scale for finding the infected cells, and undoubtedly it is intriguing that Lévy walks can be evoked in this context.

## F. Humans

Humans are the most sophisticated organisms whose motility is governed by complex environmental, sociological, technological, and urban factors. The field of human mobility is an active domain of research because of its evident connection to real-life applications. Development of transportation systems, design of mobile networks, prevention of contagious disease spreading, all these issues are linked to the human mobility. Starting from dollar bill tracking by [Brockmann, Hufnagel, and Geisel \(2006\)](#) and to mobile phone tracking by the group of Barabási ([Gonzalez, Hidalgo, and Barabasi, 2008](#)), and to a recent study of influenza virus spreading by [Brockmann and Helbing \(2013\)](#), works on this topic gained a lot of attention, in the public domain and media also. Here we review three empirical

studies with very different settings, in which the Lévy-walk patterns were found.

An interesting experiment is described by [Méndez, Campos, and Bartumeus \(2014\)](#) on p. 275 of their book. Nineteen blindfolded volunteers were ordered to search for targets randomly distributed over a soccer field. Each searcher was followed by a person who was recording the moving times of the searcher between the reorientation events. As an observed process it is a good example of the standard random walk with constant velocity. Searchers were not priorly informed of the purpose of the study and about what was going to be measured. Each searcher was given ten minutes of time, and a prize was awarded to the person finding most of the targets. After a certain target was found on the field, it was returned to the field but displaced by a 1.5 m distance in a random direction. In total there were 200 targets with a characteristic size of 1 m, distributed on a field of the size 100  $\times$  50 m. Interestingly, after the data were analyzed and pooled according to the number of collected targets, (0 – 1, 2 – 4, 5 – 8), and the distribution of run times was plotted, it appeared that the first two groups had exponentially distributed run times, whereas the third group had a

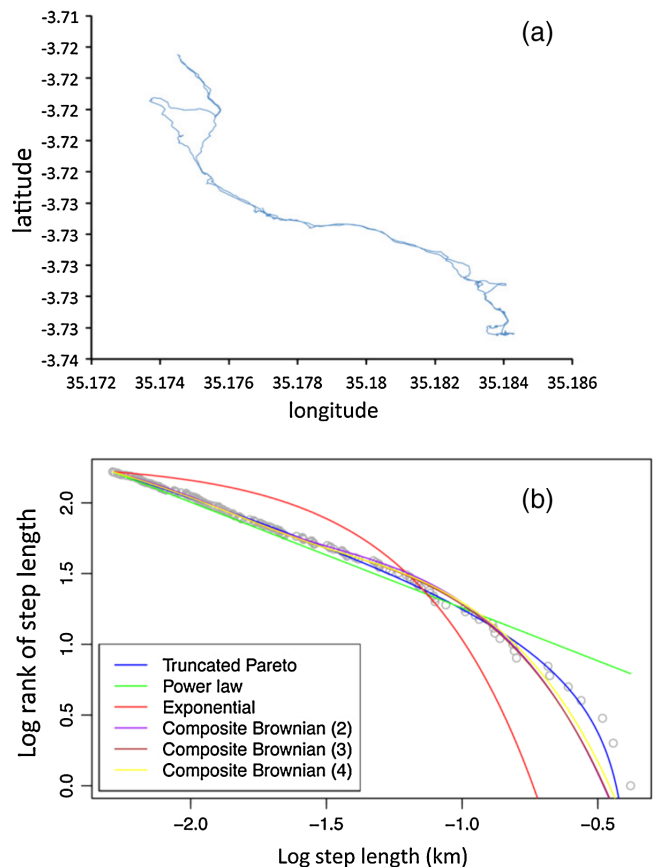


FIG. 32 (color online). Lévy walks of Hadza hunter-gatherers. (a) A representative trajectory of a Hadza hunter-gatherer bout obtained by GPS tracking. (b) The PDF of displacements during outbound parts of bouts, showing Lévy statistics in about one-half of all cases. Symbols represent the experimental data and lines correspond to different theoretical approximations to this distribution. Adapted from [Raichlen et al., 2014](#).

distribution reminiscent of the power law with an exponent of the tail  $\mu = 2.3$  [that corresponds to  $\gamma = 1.3$  in our notations for the flight-time distribution, Eq. (8)]. Certainly, the span of run times was only about 1 order of magnitude (there could be no runs longer than  $\sim 2$  min because of the size of the field) and statistical tests could not give a clear preference to the power-law fit. Nevertheless, it is still remarkable how the deviation from the exponential distribution arose and how this deviations correlated with the number of targets found.

While the previous example might look like a fun experiment, some people rely on search for their survival. In another recent study by Raichlen *et al.* (2014), human hunters-gatherers Hadza in northern Tanzania were shown to use Lévy walks in about one-half of their foraging bouts; see Fig. 32. The Hadza hunter-gatherers have no modern tools or developed agriculture, they hunt with bow and arrow, and collect wild plant food. Forty-four subjects were monitored with the help of GPS devices during their foraging bouts for several days and at different seasons. They analyzed the step-length distribution for outbound bouts (defined as travel between the camp and the farthest away from the starting point). The steps were defined either by pauses or by turning angles, which in turn were analyzed with different threshold values from  $0^\circ$  to  $180^\circ$  with a step of  $10^\circ$ . The obtained data were tested against Lévy walks, Brownian motion, or composite Brownian motion combining up to four exponential distributions. In around 50% of all bouts the distribution of step lengths was best described by either a power law or a truncated power law with tail exponents of 1.9 and 1.5, respectively. The Lévy-walk behavior appeared in both male and female subgroups despite the fact that they often had different goals of their bouts: hunting and searching for wild honey, or collecting berries and plant foods. The MSD of the corresponding tracks also showed an anomalous

superdiffusive behavior. Inclusion of round bounds did not change the results significantly. They argued that the human foragers, despite their higher cognitive complexity, still follow the same search pattern as used by other animals. Furthermore, the similar motion pattern of humans arises in much more complex urban environments, as discussed next.

In a comprehensive study by Rhee *et al.*, 2011, 226 daily GPS traces were collected from 101 volunteers in five different outdoor sites: two university campuses, a state fair, the theme park Disney World, and the New York metropolitan area (see sample tracks in Fig. 33).

They acquired data with high space resolution of 3 m and a time step of 10 s, one of the most precise tracking to date. The following quantities were extracted from the traces: flight length, pause time, direction, and velocity. They used three different methods to define the flights on the smoothed data: rectangular (when a piece of trajectory between the two end points does not leave the boundary of a certain width from the line connecting those two points), based on the turning angle, and marked by pausing events. For all locations it was found that a (truncated) power-law distribution fit the data better than other model distributions. In comparison to previously discussed examples here the span of flight lengths covers 4 orders of magnitude. The tail exponents of those distributions were found to be in the range 1.5–2.8 based on the turning angle definitions of flights; see Fig. 34.

Only in the case of the state fair, the exponential distribution was not so different from the power law. They explained this by the truncation of the step length, as the state fair was indeed the smallest location of all five. The pausing events were also power-law distributed with heavy tail exponents in the range of 2.3–3.5. The velocity of displacements was close to constant for short displacements, but increased steeply for larger travels. The reason behind this was that longer excursions could be made by using ground transportation, which was faster than walking. In terms of developing the appropriate Lévy-walk-type model that would require one to introduce an additional coupling between the distance and velocity, which can be read out from the experimental data. The MSD for all five locations

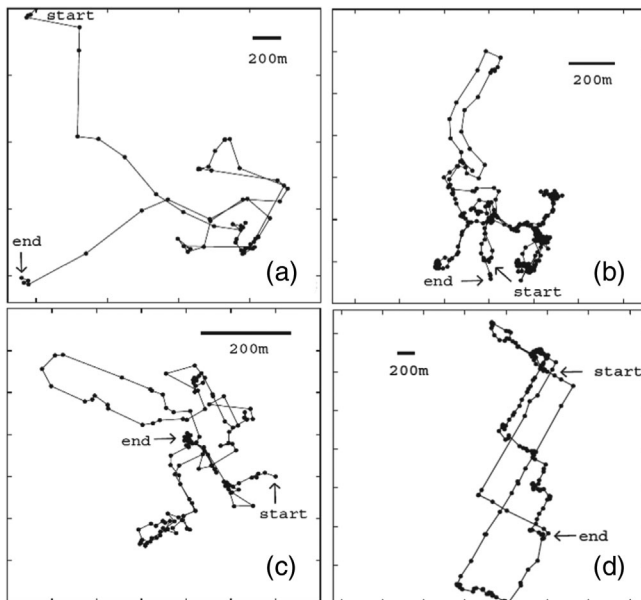


FIG. 33. GPS data on human movements during their daily activities in four different locations: (a) a university campus, (b) the Disney World theme park, (c) a state fair, and (d) New York City. Figure courtesy of I. Rhee.

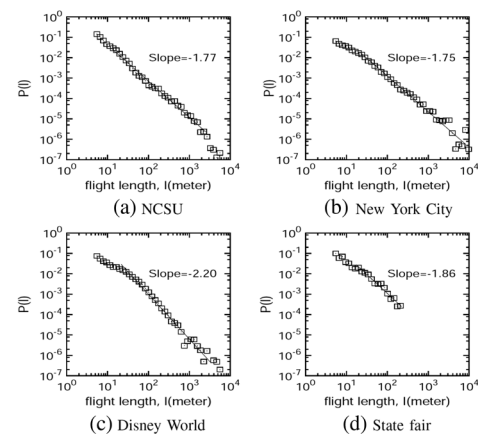


FIG. 34. Distribution of human displacements. Step-length distribution fitted with truncated Pareto distributions using the pause-based model to define the step lengths: (a) a university campus, (b) New York City, (c) Disney World, and (d) a state fair. Figure courtesy of I. Rhee.



always had two regimes: superdiffusive at short times, less than 30 min to 1 h, and subdiffusive afterward. The superdiffusion is explained by long excursions, whereas the subdiffusive scaling was caused by the bounded travel domain and also due to the fact that humans do not do a completely random walk, but rather travel to certain destinations and often return to the same points, like home, office, or class. Because of similar factors, mobility of humans is certainly more complex than just a Lévy walk, but still this model appears to be one of the best to describe human relocations as if they were a truly random process.

To finalize this section we return to one of the first and influential studies of human travel data approximated by the dispersal of dollar bank notes. Some fraction of dollar bills in the U.S. carry a stamp encouraging a person who gets a hold of it to visit a dedicated Web page, enter the bill number, current date, and location, and see its past trace. Brockmann, Hufnagel, and Geisel (2006) used the databank of bank note traces and proposed a Lévy-flight model combined with anomalously long traps, leading to a fractional diffusion equation, Eq. (21). Although one could argue that instantaneous jumps might not be the most adequate representation of human travel, which on the vast scales of North America could happen by car, bus, train, or air fair, each having its typical speed, for the data acquired it was practically impossible to take into account the finite velocity of travelers. Therefore, formally this study is outside our focus, but certainly deserves mentioning as one of the first works in this field. As the techniques of following individuals continue to progress, it is to be expected that in the near future we will learn more about the human mobility.

### G. Bumblebees, seabirds, monkeys, and others

As mentioned in the beginning of Sec. VI, the amount of data on animal motions has constantly been growing during the last decade. Not every new paper reports a Lévy-walk motion pattern as a result, but at least tries to relate the observed motion patterns to the Lévy-walk model. Current research trends in ecology were greatly influenced by the idea that under some circumstances a superdiffusive Lévy walk can be an advantageous search strategy when compared to a classical Brownian-like diffusion pattern. We review the search problem in Sec. VII, and here we briefly list very diverse and interesting examples of data on animal tracking.

*Insects:* Insects can be traced by using different methods, such as traps, video cameras, entomological radars, or scanning harmonic radars combined with miniature transponders attached to individual insects. To date, there is an impressive list of insects which were studied with respect to their motility patterns: ants (Schultheiss and Cheng, 2013), bumblebees (Lenz, Chechkin, and Klages, 2013), honeybees (Reynolds *et al.*, 2007), moths (Cardé, Cardé, and Girling, 2012), beetles (Reynolds, Leprêtre, and Bohan, 2013), stone flies (Knighton, Dapkey, and Cruz, 2014), and fruit flies (Cole, 1995; Reynolds and Frye, 2007).

*Sea animals:* Underwater creatures are much more difficult to follow and in general are traced by small, pressure-sensitive data-logging tags giving the depth information, or by high-frequency acoustic transmitters in combination with a

directional hydrophone, or with satellite relayed data loggers. The list of tracked species is also quite long: various sharks, penguins, tuna (Sims *et al.*, 2008), turtles (Hays *et al.*, 2006; Dodge *et al.*, 2014), dolphins (Bailey and Thompson, 2006), mussels (de Jager *et al.*, 2011), cuttlefish, octopus, various rays, sole, and angler fish (Wearmouth *et al.*, 2014), jelly fish (Hays *et al.*, 2012), gray seals (Austin, Bowen, and McMillan, 2004), and, finally, fishermen (Bertrand *et al.*, 2007).

*Birds:* Birds are usually tracked with the help of small GPS loggers attached to their bodies. One of the first studies in the field was done on wandering and black-browed albatrosses (Viswanathan *et al.*, 1996; Edwards *et al.*, 2007; Humphries *et al.*, 2012) with several more to follow on pelagic seabird Corys shearwaters (Focardi and Cecere, 2014), frigate birds (De Monte *et al.*, 2012), and Egyptian vultures (López-López *et al.*, 2013).

*Mammals:* Most observations of mammals foraging on terrain is done via visual contact and approximate GPS location determined by an observer using range finders. Several kinds of animals were tracked by this method: baboons (Schreier and Grove, 2014), spider monkeys (Ramos-Fernández *et al.*, 2004), fallow deer (Focardi, Montanaro, and Pecchioli, 2009), jackals (Atkinson *et al.*, 2002), reindeer (Marell, Ball, and Hofgaard, 2002), langurs (Vandercone *et al.*, 2013), and bearded sakis (Shaffer, 2014).

## VII. LÉVY WALKS AND SEARCH STRATEGIES

Searching and foraging is enormously important in the ecological context and it is not surprising that more and more physicists and mathematicians contribute to this field. A growing database allows one to propose and test various models with an increasing level of detail and complexity. The first mentioning of Lévy walks being advantageous in search as compared to classical random walks belongs to Shlesinger and Klafter (Shlesinger, Klafter, and West, 1986). Further on, a Lévy-walk hunting strategy in the context of feeding behavior in grazing microzooplankton was discussed by Levandowsky, Klafter, and White (1988). It is widely recognized now that two papers by the group of Stanley, first on the Lévy flights of albatrosses (Viswanathan *et al.*, 1996), and three years latter on optimality of the Lévy search (Viswanathan *et al.*, 1999), lead to the birth of the new interdisciplinary field dealing with the quantitative analysis of animal motility patterns and optimality of search. The maturity of the field is marked by several comprehensive monographs on the topic (Bénichou *et al.*, 2011; Viswanathan *et al.*, 2011; Méndez, Campos, and Bartumeus, 2014), and the field itself spreads beyond animals and humans to robotics; see Sec. VII.C.

The problem of animal search is complex, as well characterized by Shlesinger (2009): “Actual search patterns of animals will depend on many factors: amount of energy expended in different modes of travel; the probability of finding food during various locomotions (flying, running, walking, hopping, etc); whether a single animal or a group is executing the search; day or night conditions; topography; weather; fixed food sources (water and vegetation) or moving targets (prey); homogeneous or scarce food sources; whether the animal randomly searches for food or has knowledge of

food locations.” As an idealization of these features, when there is no prior information about the location of targets and complex interactions of a searcher with the environment and its prey, a so-called random search approach is used, which assumes that the searcher adopts a certain random motion pattern. The superdiffusive Lévy walk was proposed as an optimal search strategy (Viswanathan *et al.*, 1999) in case of sparse nondestructible targets. However, the validity and the straightforward use of the Lévy-walk concept for the analysis of animal search patterns was questioned both experimentally and theoretically (Bénichou *et al.*, 2006, 2007; Benhamou, 2007; Edwards *et al.*, 2007; Plank and Codling, 2009; Jansen, Mashanova, and Petrovskii, 2012; Reynolds, Schultheiss, and Cheng, 2014). For an opinion on “Should foraging animals really adopt Lévy strategies?”, see the recent review by Bénichou *et al.* (2011). The resolution of this issue is out of our scope. Yet we do believe that there is a balanced middle point between the two extremes, “Lévy” and “no Lévy,” which follows from the universal principle: mathematics and physics cannot take the place of nature but they certainly can help to understand the former. Indeed, wandering albatrosses “do not care about math” (Travis, 2007) and it is naive to think that a bird utilizes a Lévy walk when preying, by independently drawing a length of the next flight from a PDF with power-law tails. Lévy-walk-like motion patterns are not necessarily produced by a Lévy-walk process.<sup>8</sup> Moreover, patterns themselves—even when they look very similar to those obtained in theory—could not identify complex mechanisms of animal locomotion that produced them. This does not contradict the fact that the Lévy-walk concept represents a powerful tool for *quantification* and *analysis* of statistical data and provides us with more insight into animal foraging strategies than the conventional Brownian-based approach (Buchanan, 2008).

In Secs. VII.A, VII.B, and VII.C we overview the current state of the field. Special emphasis is put on the original paper by Viswanathan *et al.* (1999) which greatly promoted the Lévy-walk model as an advantageous search strategy.

### A. Lévy walk as an optimal search strategy

Viswanathan *et al.* (1999) considered a walker which performed a Lévy walk in two dimensions and searched for targets, randomly distributed in space with a density  $\rho$ . The searcher can detect targets at the sight radius  $r$ . If a walker sees a target it proceeds straight to it. If there is no target in sight it chooses a random direction and moves for a random time with a fixed speed. If no target is found during a flight a new flight starts in another random direction. The distribution of flight distances is chosen in the power-law form  $g(l) \propto l^{-\mu}$ . Because of a simple coupling  $l = v\tau$  we can identify

$$\mu = \gamma + 1, \quad (97)$$

<sup>8</sup>In a recent study of trace fossils, Sims *et al.* (2014) showed that the artificial trails produced by following three simple rules, (i) “do not cross your trail,” (ii) “stay close to it,” and (iii) “make  $U$  turns,” appeared to be Lévy-walk patterns when analyzed with the conventional methods used in the field.

where  $\gamma$  denotes the tail exponent of the flight-time distribution, Eq. (8). As discussed in Sec. II,  $\mu > 3$  will result in the finite mean squared length of the jump and normal diffusive dispersal. A regime of  $1 < \mu < 3$  corresponds to the superdiffusive Lévy walks. In this model, it is important that the searcher keeps looking for a target while moving and that the current flight is terminated if the target is found. One of the ways to define the efficiency of the search is by the ratio of the number of targets found to the time spent in search or, in the case of constant speed, to the total distance traveled:

$$\eta = \frac{1}{\langle l \rangle N}, \quad (98)$$

where  $\langle l \rangle$  is the mean flight distance and  $N$  is the average number of flights between the two successive targets. The only characteristic scale of the problem is given by an average distance between two detected targets  $\lambda = (2r\rho)^{-1}$ . With its help, the mean flight distance can be given as

$$\langle l \rangle = \frac{\int_r^\lambda l \cdot l^{-\mu} dl + \lambda \int_\lambda^\infty l^{-\mu} dl}{\int_r^\infty l^{-\mu} dl}. \quad (99)$$

The first term in the nominator arises from the usual definition of the average flight length, but it has an upper bound of the typical distance between the two found targets. These flights do not terminate at the target. The second term counts the flights which were chosen to be longer than  $\lambda$  but do terminate after the target encounter. The denominator is a normalizing factor. Next, the mean number of steps between the two successive targets needs to be found. At this point it is important to distinguish between two possible scenarios: targets can be either destroyed after being found (destructive case), or they become temporally depleted but can be revisited at later times (nondestructive). In these two cases, the average number of steps between two successive destructible and nondestructible targets can be estimated as [for a detailed explanation see Viswanathan *et al.* (1999)]

$$N_d \simeq (\lambda/r)^{\mu-1}, \quad N_n \simeq (\lambda/r)^{(\mu-1)/2}. \quad (100)$$

Now the question of optimality may be asked: Is there an optimal value of  $\mu$  which leads to a maximal number of found targets, but keeps the length of excursions sufficiently short? If targets are plentiful,  $\lambda \lesssim r$ , then  $N_d \approx N_n \approx 1$  and  $\langle l \rangle \approx \lambda$ . In that case the search efficiency does not depend on  $\mu$  at all. In the case of sparse resources  $\lambda \gg r$ , the situation is different. For destructive foraging the efficiency is maximal for smallest  $\mu$  meaning that moving along one straight line is the best strategy in that case. However, the situation is more interesting in the case of nondestructive search. By substituting the expressions for  $\langle l \rangle$ , Eq. (99), and for the number of steps  $N_n$ , Eq. (100), into Eq. (98), and equating its derivative with respect to  $\mu$  to zero, we obtain the optimal value of the power-law exponent:

$$\mu_{\text{opt}} = 2 - 1/[\ln(\lambda/r)]^2. \quad (101)$$

The second term is a small correction in the case of sparse targets, so roughly the exponent of  $\mu \approx 2$  ( $\gamma \approx 1$ ) arises as a solution. This value of a power-law tail of the traveled distances corresponds to the border regime between superdiffusive and ballistic Lévy walks. Qualitatively the advantage of Lévy walks with  $\mu \approx 2$  is explained by a compromise between diffusive trajectories returning to the same target zone ( $\mu > 3$ ) and ballistic motion ( $\mu \sim 1$ ) which is the best strategy to explore space. This result greatly promoted the notion of Lévy walks as an optimal search strategy in the case of randomly distributed, nondestructible, sparse targets.

One could question whether the assumptions made when formulating the above search model are realistic. An animal, even a protozoan, is a much more intellectual being than a pointlike particle driven by a finite-length algorithm. After all, why should a donkey leave a water pond in the oasis (a nondestructible target following the nomenclature) he has once found in a desert? Well, another could answer, the donkey has other needs also and he will turn to satisfy them once he has quenched his thirst and appeased his hunger; for example, he might like to find a mating partner. It is a perfectly correct argument but it goes far beyond the premises of the model. Animal search is a multilayered activity determined by a vast number of external and internal (instincts, etc.) factors and it is impossible to catch even the most essential of them with a simple stochastic model. The good point is that the model introduced by [Viswanathan et al. \(1999\)](#) allows for a gradual complexification and can absorb new assumptions and conditions. Since the paper was published, many modifications were proposed, which include, for example, moving and/or regenerating, patchy targets ([Bénichou et al., 2011](#); [Palyulin, Chechkin, and Metzler, 2014](#)). It was also found that, in some situations like searching for a single target in confinement ([Tejedor, Voituriez, and Bénichou, 2012](#)), or under the presence of a bias ([Palyulin, Chechkin, and Metzler, 2014](#)), persistent random walks or Brownian strategies perform better than Lévy walks.

## B. Intermittent search strategies

A simple assumption that animals or humans have lower search capabilities when they are moving fast leads to the idea of the so-called *intermittent search*, when periods of localized diffusivelike search activity are altered with ballistic relocation to a new spot (searching for a lost key in an apartment is a good example). The intermittence has been detected in motion patterns of biological species ranging from protists to primates ([Bartumeus, 2007](#); [Bénichou et al., 2011](#)). Different research fields contributed with different theories, as, for example, ecologists discussed phases of “tactical habitat utilization” (local search events) and “strategic displacements” (ballistic relocations) ([Gautestad and Mysterud, 2006](#)), while experts on random walks served a spectrum of phenomenological models ([Bénichou et al., 2011](#)). For us, further extensions of the standard Lévy-walk model which are motivated by these studies are of interest.

[Lomholt et al. \(2008\)](#) suggested that an intermittent search in which the relocation happens according to the Lévy walk could lead to a more efficient search than, for example, exponentially distributed displacements between the diffusive

search phases. From the point of view of modeling, such a process might be called a composite process. The Lévy walk is not simply diluted with resting events, when a walker is immobile, like the process shown in Fig. 1(c), but it is alternated with periods of different activity, for example, diffusion. Such processes are not new in the field of random walks but they experienced a revival of interest because of the new context. [Bartumeus et al. \(2003\)](#) claimed that precisely this type of search strategy is realized by *Oxyrrhis marina*, a dinoflagellate living in the sea depth, when it preys on a microzooplankton. Namely, when the prey decreases in abundance, a predator switches from a slow-rate Brownian motion, characterized by an exponential PDF of flight time, to a helical Lévy motion, characterized by an inverse square power-law PDF.

In addition to the analysis of search patterns of biological species, the formalism of composite random walks allows one to find analytic solutions for the density of particles and calculate the scaling of the corresponding MSD. Recently [Thiel, Schimansky-Geier, and Sokolov \(2012\)](#) used a composite random walk to describe the run-and-tumble dynamics where the durations of the tumbles were explicitly taken into account and the runs were assumed to have a heavy tailed flight-time PDF. It was also assumed that during tumbling events particles perform normal diffusion. Depending on the interplay between the tail of the flight times and durations of the tumbling phases (which also could, in principle, be characterized by a tunable power-law distribution) the MSD was shown to span the regimes from the normal diffusion to ballistic superdiffusion.

## C. Lévy walks for intelligent robotics: Following suit

Biological systems are a constant source of inspiration for the robot designers. It is not a surprise then that the wave of studies on Lévy-walk foraging and animal search strategies has attracted the attention of the researchers working in the field of robotics. The current aim of the “Lévy robotics” is twofold. First, it is a development of new nature-inspired search algorithms for autonomous mobile robots ([Nurzaman et al., 2009](#); [Pasternak, Bartumeus, and Grasso, 2009](#); [Lenagh and Dasgupta, 2010](#); [Sutantyo et al., 2010](#); [Keeter et al., 2012](#); [Fujisawa and Dobata, 2013](#); [Sutantyo, Moslinger, and Read, 2013](#)). A complementary research line aims at the understanding of how Lévy-walk motion patterns emerge from combinations of different external factors and theoretical assumptions on animal strategies and behavior ([Fricke et al., 2013](#)).

An idea to combine Lévy walks with chemotaxis in order to produce “Lévy taxis,” a search algorithm for an autonomous agent to find a source of chemical contamination in a turbulent aquatic environment, was proposed by [Pasternak, Bartumeus, and Grasso \(2009\)](#). It is not a typical search task because the searcher should scan a constantly changing chemical field and follow plumes in order to find their origin. In the computational studies, a virtual autonomous underwater vehicle (AUV), floating in a virtual two-dimensional riverlike turbulent flow, contaminated from a pointlike source, was used. Events of unidirectional motion, characterized by a power-law distribution of their lengths and a wrapped Cauchy



distribution of their direction angles, were intermingled with short reorientation events. During the latter the vehicle was randomly choosing a new movement direction along the local concentration upstream flow. This strategy somehow corresponds to a Lévy walk in a flow-oriented reference frame. When compared to other strategies, based on a Brownian walk, a simple Lévy walk, correlated Brownian walk, and a brute-force zigzag scanning, a Lévy taxis outperformed all of them, in terms of both detection success rate and detection speed.

Another searching strategy for a mobile robot, a sequence of Lévy walks alternated with taxis events, was proposed by [Nurzaman et al. \(2009\)](#). In computer simulations, the robot task was to locate a loudspeaker by using the information on the local sound intensity obtained from a robot-mounted microphone. The loudspeaker was stationary and the robot's speed  $v$  was constant. The robot orientation was defined by the angle  $\theta$ . The robot dynamics was governed by three stochastic equations,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix} = A(t) \begin{bmatrix} v \cos \theta(t) \\ v \sin \theta(t) \\ 0 \end{bmatrix} + [1 - A(t)] \begin{bmatrix} 0 \\ 0 \\ \varepsilon_{\theta}(t) \end{bmatrix}, \quad (102)$$

where the Cartesian coordinates  $x(t)$  and  $y(t)$  specify the position of the robot at time  $t$ . Activity  $A(t)$  is a dichotomous function switching between 1 and 0 so that the robot either is moving forward with velocity  $v$  (activity is “1”) or is randomly choosing a new direction of motion (activity is “0”). When the duration of a single 1-event is distributed according to a power law [see Fig. 35(a)], the robot performs a two-dimensional version of the Lévy walk with rests shown in Fig. 1(c). A stochastic sonotaxis strategy, when the robot tries to locate and move toward the loudspeaker, was also probed. However, neither of the two strategies was able to accomplish the task when used alone. The sonotaxis turned out to be effective in a

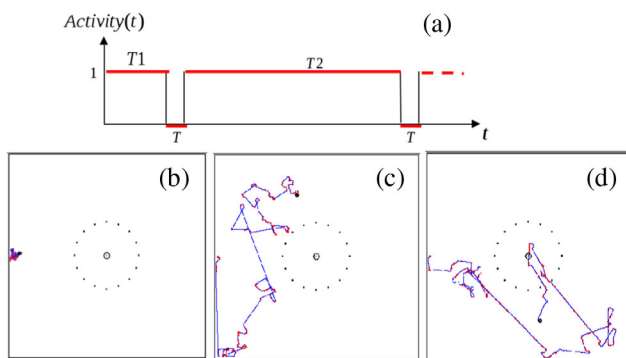


FIG. 35 (color online). Performance of a sonotactic robot. (a) Activity of the robot: Durations  $T_1, T_2, \dots$  follow a power-law tail distribution while the duration of reorientation events  $T$  is constant. (b)–(d) Trajectories of the robot using (b) the sonotaxis strategy, (c) the Lévy walk, and (d) the combination of the two. The speaker (small solid circle) is located at the center of the squared test area (box) and the dashed line encircles the area with sound gradient above a threshold. The starting point is located at the middle of the left box border. Figure courtesy of S. G. Nurzaman.

close vicinity of the speaker only and did not work when the sound gradient was small; see Fig. 35(a). The Lévy walk did not care about the sound intensity by default and produced unbiased wandering only; see Fig. 35(b). The combination of the two solved the problem: the Lévy walk first brought the robot to the area where the sound-intensity gradient was high enough and from there the sonotaxis strategy was able to lead the robot to the loudspeaker; see Fig. 35(c). A Lévy looped search algorithm to locate mobile targets with a swarm of noninteracting robots was proposed by [Lenagh and Dasgupta \(2010\)](#). The idea was to replace straight ballistic segments with loops so that each searcher returned to its initial position. The length of each loop was sampled from a power-law distribution, whereas the starting angle was sampled from the uniform distribution in the interval  $[0, 2\pi]$ . The reported results showed that the looped search outperformed the standard Lévy search in tracking mobile targets.

The idea that a search efficiency can be increased by using a number of autonomous agents is natural and relevant in many contexts. It is evident, for example, that the search time is inversely proportional to the number of independent searchers provided all other conditions remain the same. However, if an interaction or exchange of information between the searchers is allowed, the search time can be decreased even further. Swarm communication is widely used among animals and insects, and it is known among biologists and roboticists as “stigmergy” ([Beckers, Holland, and Deneubourg, 1994](#)). A multirobot searching algorithm based on a combination of a Lévy walk and an artificial potential field inducing repulsion among robots was proposed and tested by [Sutantyo et al. \(2010\)](#). The obtained results for up to 20 robots showed that the repulsion increases search efficiency in terms of the search time. It is noteworthy that the effect diminishes with an increase of the robot number, because crowding robots start to change their directions earlier than expected from the governing power-law distribution. Experimental results obtained for two Lévy-swimming AUVs in a 3D aquatic test bed ([Keeter et al., 2012](#)) show that in this case the best performance corresponds to a simple divide-and-conquer strategy, when the tank is divided into two equal volumes and each submarine scouts only its assigned region. However, this situation may change when the number of AUVs is larger than 2 so that communication between searchers could be beneficial. Group Lévy foraging with an artificial pheromone communication between robots was studied recently by [Fujisawa and Dobata \(2013\)](#). Each robot had a tank filled with a “pheromone” (alcohol) which was sprayed around by a micropump. Rovers also carried alcohol and touch sensors and their motion was controlled by a program which took into account the local pheromone concentration. The swarm foraging efficiency peaked when the robots were programmed beforehand to perform a Lévy walk in the absence of communication. Multirobot underwater exploration and target location were studied with a swarm of Lévy-swimming AUVs by [Sutantyo, Moslinger, and Read \(2013\)](#). Interaction between the robots was introduced by using a modification of the firefly optimization, an algorithm popular in the field of particle swarm optimization ([Kennedy and Eberhart, 2001](#)). The “attractiveness” of each AUV was defined by the time since the robot last found a target; it increased every time a

target was located and then slowly decayed. The task was for each searcher to find all the targets. The results of the experiments showed that the interaction substantially decreased the averaged search time.

Finally, an attempt to get insight into the machinery causing the emergence of Lévy-walk-like patterns in the motion of different biological species was made recently by [Fricke \*et al.\* \(2013\)](#). Inspired by the results obtained for T cells ([Harris \*et al.\*, 2012](#)) (see Sec. VI.E), researchers from the University of New Mexico and Santa Fe Institute used six small rovers, equipped with ultrasound sensors, compasses, and cameras. This navigation set enabled each robot to find patches of resources distributed over a 2D area. Tunable adaptive algorithms based on five different search strategies were tested. It turned out that the algorithm using correlated random walks, in which correlations between consequent step angles of a rover depend on the target last observed by the rover, produces Lévy-like motion patterns.

Lévy robotics is only one example that illustrates the practical value of the Lévy-walk concept. We do believe that there are more to come and discuss potential candidates in the final section of this review.

### VIII. OUTLOOK

The Lévy-walk concept is almost 30 years old and now possibly at the beginning of the most interesting phase of its life. The gradually developing theoretical framework was there in time to support the burst of applications across different fields. As can be seen from the previous sections, most of the empirical data obtained with cold atoms, nanostructured media, quantum dots, and ecology emerged only recently. Lévy walks remain in a stage of active development, and we now see them being used in robotics and mobile communication technologies ([Lee \*et al.\*, 2013](#)). In this concluding section we discuss some open problems in the field and sketch what we think are the next perspectives and challenges.

For the physicists, probably one of the central questions is to understand how the Lévy walk, which is a mathematical model, emerges in diverse physical phenomena. There is certain progress in this respect in the fields of classical

many-particle chaos ([Mendl and Spohn, 2014](#)) and cold atom dynamics ([Barkai, Aghion, and Kessler, 2014](#)). In the problem of light diffusion in hot atomic vapors, general principles of light emission and absorption were suggested to be relevant mechanisms ([Baudouin \*et al.\*, 2014](#)). In experimental plasma physics, the anomalous nonlocal transport is regularly reported in various works, but its origin remains a subject of ongoing debates. This can be partially explained by the high complexity of modern plasma experiments which are often performed in nonequilibrium regimes, involve nonlinear interactions, the formation of large coherent structures, etc. We can see that even simple approximations of plasma ion dynamics by using the Lévy walk immediately call for nonlinear space-time couplings ([Zimbaro, Greco, and Veltri, 2000](#); [Gustafson and Ricci, 2012](#)).

Most of the analytical results presented in this review are restricted to one dimension. This reflects the current situation on the theory front. Although the analysis can be formally generalized to higher dimensions by replacing the Fourier coordinate with a Fourier vector  $k \rightarrow \mathbf{k}$ , this technical step will immediately pose a number of questions. For example, a two-dimensional Lévy walk can be defined in two different intuitive ways, namely, as a process when (i) the length of the upcoming flight and its random *orientation* are both chosen from continuous PDFs (as in the case of run and tumble of bacteria, Sec. VI.C) or, alternatively, (ii) a random displacement is chosen independently but always along one of the two basis vectors. How do the propagators of these processes look alike? Evidently, because of the isotropy of the first process, the corresponding propagator will be circular symmetric,  $P(\mathbf{r}, t) = P(r, t)$ ,  $r = \sqrt{x^2 + y^2}$ ; see Fig. 36(a). It is tempting to say that in this case the problem can be reduced to the one-dimensional setup by substituting  $r$  instead of  $|x|$  in the propagator. However, it has yet to be clarified which equation governs the evolution of  $P(r, t)$ . A Lévy-walk process of type (ii) has been observed in numerical studies of the superdiffusion in two-dimensional chaotic Hamiltonian systems by [Klafter and Zumofen \(1994\)](#); see Fig. 36(b). It was shown that at the asymptotic limit and far from the center  $\mathbf{r} = 0$ , the corresponding propagator factorizes,  $P(\mathbf{r}, t) \approx P(x, t/2)P(y, t/2)$ , where  $\mathbf{r} = x \cdot \mathbf{e}_x + y \cdot \mathbf{e}_y$  and  $P(x, t)$  and  $P(y, t)$  are one-dimensional propagators. This

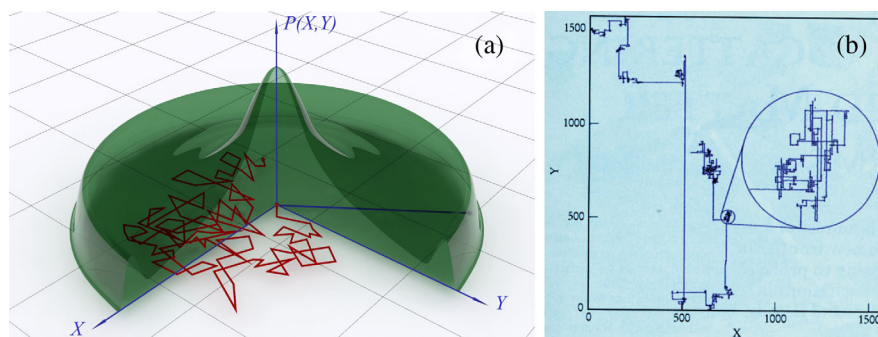


FIG. 36 (color online). Two intuitive generalizations of the Lévy walk to two dimensions. Both models are characterized by power-law PDFs of the flight time. (a) In an “isotropic” model the direction of a flight is given by a random angle uniformly distributed on the interval  $[0, 2\pi]$ . (b) In a “lattice” model ballistic flights happen along one of the two basis vectors of a square lattice. The latter process was observed when tracking trajectories of a Hamiltonian particle moving over an egg-grate potential; see [Klafter and Zumofen \(1994\)](#). Adapted from [Klafter, Shlesinger, and Zumofen, 1996](#).

type of two-dimensional Lévy walks with the exponent  $\gamma = 2$  is relevant for the description of the diffusion in Sinai billiards with infinite horizon (Bouchaud and Georges, 1990), as shown recently by Cristadoro *et al.* (2014).

For the next step one can consider Lévy walks on lattices of different geometries and try to elucidate the effects of the underlying geometry on the corresponding propagators. This question is particularly motivated by the experimental studies of light propagation in regular foams, where, due to the effect of total internal reflection, the light gets trapped in the liquid phase of the foam (Gittings, Bandyopadhyay, and Durian, 2004). Theoretically it was shown that, in the case of a honeycomb foam lattice, the light propagation can be superdiffusive (Schmiedeberg, Miri, and Stark, 2005; Schmiedeberg and Stark, 2006).

An issue of correlated Lévy walks not only constitutes a theoretical challenge but is of relevance in the context of several recent experiments. One example is the diffusion of light in Lévy glasses (see Sec. V.C), where the quenched disorder of scatterers may induce correlations between the flights. Independently, this question was posed by theoreticians some time ago (Kutner and Maass, 1997, 1998; Levitz, 1997; Barkai, Fleurov, and Klafter, 2000) and still requires further analytical investigation. A correlated ballistic Lévy walk could also serve as an advanced model to account for the correlations in blinking times of quantum dots (Stefani, Hoogenboom, and Barkai, 2009).

A fundamental characteristic of any random walk process is the so-called first passage time, which defines how soon a random walker would visit a point located at a certain distance from the origin; see Redner (2001). The first passage time problem for Lévy walks naturally occurs in the context of searching strategies, where it quantifies the time it takes to hit a target. Many of the results obtained for the first passage time and related problems for the standard random walks (Redner, 2001) were generalized to subdiffusion and Lévy flights. At the same time, the problem of the first passage time for Lévy walks remains largely unexplored (Korabel and Barkai, 2011).

As already mentioned, the origins of Lévy walks in biology and ecological context are an unsettled issue. Although some examples exist that show how the power-law distributed run times emerge from the underlying genetic circuits of bacteria (Matthäus *et al.*, 2011), it remains to be seen whether similar evidence can be found for more complex organisms that exhibit Lévy-walk-like behavior. In the mean time, Lévy-walk strategies are implemented to construct robots that can assist humans in finding sources of contamination and to develop efficient strategies to rescue people from disaster areas (Akpoiyibo, Lakshmi Narayanan, and Ibe, 2014). The concept of Lévy foraging has made its way into the field of criminology, potentially leading to implications in predictive policing. Johnson (2014) used criminal records of more than 1000 offenders who committed a series of crimes and found that the distribution of distances between the consequent events was consistent with Lévy-walk dynamics. There is also an interview with a burglar which corroborates the Lévy-walk behavior as an optimal *evading* strategy and relates it to the perception of risk to be caught. In a recent paper titled “Voles don’t take taxis,” Pease (2014) comments on the work of

Johnson (2014) and puts it in the context of modern quantitative criminology research.

To conclude, we provided an overview of the theoretical aspects of a simple but remarkably flexible model of Lévy walks. We illustrated theoretical considerations with a variety of examples, where the model and its offspring served to quantify the stochastic transport phenomena and help elucidate underlying mechanisms. We would like to think that this review will stimulate researchers from even more distant fields to use the model in their studies and thus will help to advance Lévy walks into new unexplored territories.

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