Colloquium: Theory of intertwined orders in high temperature superconductors

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The electronic phase diagrams of many highly correlated systems, and, in particular, the cuprate high temperature superconductors, are complex, with many different phases appearing with similar (sometimes identical) ordering temperatures even as material properties, such as dopant concentration, are varied over wide ranges. This complexity is sometimes referred to as “competing orders.” However, since the relation is intimate, and can even lead to the existence of new phases of matter such as the putative “pair-density wave,” the general relation is better thought of in terms of “intertwined orders.” Some of the experiments in the cuprates which suggest that essential aspects of the physics are reflected in the intertwining of multiple orders, not just in the nature of each order by itself, are selectively analyzed. Several theoretical ideas concerning the origin and implications of this complexity are also summarized and critiqued.

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CONTENTS

I. INTRODUCTION

Highly correlated electronic materials, and, in particular, those that exhibit unconventional superconductivity (SC), have phase diagrams that are intrinsically complex. Multiple distinct broken-symmetry phases occur as a function of parameters such as composition, pressure, and magnetic field. For example, there is a proximate antiferromagnetic (AF) state in the phase diagrams of superconductors such as cuprates, iron pnictides and chalcogenides, organics (both the quasi-1D TMTSF salts and the quasi-2D ET salts), and certain alkali-doped C_{60} compounds. It has become commonplace to describe ordering tendencies such as superconductivity and antiferromagnetism as “competing orders” since microscopic coexistence of the two broken symmetries is relatively rare, and where they do coexist, one order manifestly suppresses the other.

The primary purpose of this Colloquium is to emphasize a different perspective, taking the cuprates as a case study and focusing on the cooperative character of different orders. We note that the temperature and energy scales associated with antiferromagnetism and SC are comparable. Furthermore, while too much antiferromagnetism quenches superconductivity, experiments indicate that too little, in the form of residual (fluctuating) antiferromagnetism, is equally bad for superconductivity. Increasingly, it has become clear that this is just the tip of the iceberg and that various other orders—charge-density wave (CDW), long period spin-density wave (SDW), nematic, and possibly other forms of symmetry-breaking order—also occur with comparable onset temperatures in a wide range of material parameters. We present the case that the best way forward is to view these phenomena in terms of the "intertwining" of multiple orders.

A continuing conundrum in the cuprates concerns the nature of the normal state from which the superconductivity develops. For a large range of carrier concentration, suppression of the superconductivity with temperature or magnetic field leads to the pseudogap regime, a state with an ambiguous name that reflects an empirically well-defined set of electronic changes in the electronic structure whose underlying meaning is still much debated.[1] There is evidence for various fluctuating or static order parameters within the pseudogap regime. A consistent theoretical description of such broad fluctuational regimes with multiple orders is possible in 1D and quasi-1D, from which some insight into the pseudogap can be gleaned, but no similarly compelling theory exists in 2D or 3D. Nevertheless, some of the essential features of the pseudogap are addressed in our second theme, which is an exploration of a novel broken-symmetry phase, the pair-density wave (PDW), that intertwines CDW, SDW, and SC orders. There is increasingly compelling (although not yet definitive) computational evidence that this novel phase exists robustly in the phase diagrams of simple models of strongly interacting electrons, and experimental evidence that it occurs in at least one cuprate SC, $La_{2-x}Ba_xCuO_4$. More speculatively, we propose that the existence of such a "parent" phase which spontaneously breaks a large number of symmetries can be the key to understanding broad aspects of the phase diagram in the sense that a large number of "daughter" phases can be viewed as partially melted versions of the parent phase, in which "vestigial order" still exists in the form of a smaller subset of broken symmetries.

As the PDW is a new quantum phase of matter (Himeda, Kato, and Ogata, 2002; Berg *et al.*, 2007), we need to define what we mean by it. It is a state in which the superconducting order itself is spatially modulated in such a way that the uniform component is zero or nearly zero, but in which an oscillatory piece is strong.[2] This phase has unprecedented properties, of which the most readily experimentally identified are dynamical layer decoupling and anomalous sensitivity to disorder (Berg *et al.*, 2009). The PDW state can be viewed as a "self-organized" Larkin-Ovchinnikov state (Larkin and Ovchinnikov, 1964) but without the accompanying net magnetization.

The rest of the paper is organized as follows: Sec. II is a qualitative discussion of the sorts of broken-symmetry phases, especially somewhat less familiar electronic liquid crystalline phases, which can be expected in strongly correlated electron fluids. In particular, based on the analogy with the liquid crystalline phases that occur in classical complex fluids, we offer some intuitive theoretical reasons to expect intertwined orders to be an important generic feature of broad classes of highly correlated electron systems. From a somewhat different perspective, the "landscape" of possible ordered phases that appear at low temperatures in dynamical mean-field theoretic studies of strongly correlated systems (Kotliar, 2005) presumably reflects the same underlying physics.

In Sec. III, we discuss effective field theories of multiple interacting orders. To give focus to the discussion, we consider the case in which there are two fundamental orders—a uniform (d wave) SC and a PDW. From this starting point, other orders—notably CDW, nematic, and charge $4e$ SC order—appear as composite orders. To some degree, the choice of which orders are treated as fundamental and which are derivative is a matter of convenience; for instance, while it is possible to describe CDW order as a composite, in regions of the phase diagram where no PDW condensation occurs it is probably simpler to consider SC and CDW as the fundamental fields. In any case, SDW order, which is clearly an important part of the physics in portions of the cuprate phase diagram, involves additional order parameter fields that we have not included to simplify the discussion.

Section IV reviews the results of a variety of theoretical studies of simple models of correlated electronic systems, mostly one version or another of the Hubbard model. For the most part, we confine ourselves to a discussion of problems for which controlled analytical theory or arguably conclusive numerical solutions can be obtained. In Secs. IV.A–IV.C we study models that exhibit various general features of intertwined order. In Sec. IV.D, we focus on models which can be shown to have PDW ground states. We also review recent, illuminating variational results on the 2D t-J model which exhibit an astonishing near degeneracy of a variety of different broken-symmetry states, including a PDW phase, over a broad range of t/J and doping concentration x. In addition, we briefly summarize a related approach to the problem (Lee, 2014) which envisages a PDW state arising from "amperian pairing" of spinons in an underlying fractionalized phase.

Section V is a rather compressed summary of some of the most direct experimental evidence of the existence of a large variety of ordering tendencies in the cuprates. The discussion here is more descriptive than analytic.

In Sec. VI, we summarize some of the spectroscopic features that are associated with the pseudogap, and, in particular, highlight the conflicting evidence, some of which is highly suggestive that the pseudogap is a fluctuational descendant of the d-wave superconducting gap, and some of which suggests it arises from entirely distinct correlation effects, possibly associated with another form of order. While we certainly do not resolve this debate, we do suggest that the existence of two distinct forms of superconducting order and/or order parameter fluctuations may provide a useful framework for resolving the apparent "one-gap" versus "two-gap" dichotomy.

Section VII discusses several possible direct experimental tests which could unambiguously verify the existence of PDW order. A more extensive discussion of most of the same points has appeared previously in Berg *et al.* (2009).

[1]To view the range of ideas proposed to explain the pseudogap, see Norman, Pines, and Kallin (2005), Lee, Nagaosa, and Wen (2006), and Rice, Yang, and Zhang (2012).

[2]We note that Chen *et al.* (2004) used PDW to describe a different state, one of localized pairs. In terms of broken symmetries, we classify the latter state as a CDW.

In Sec. VIII, we consider some of the broader issues raised in the course of this Colloquium. The issue of whether it is reasonable to view the pseudogap scale T^* as a crossover associated with the development of a local "amplitude" of the order parameter (or parameters) is discussed, but not resolved, in Sec. VIII.A. Using as illustrative examples the results from somewhat artificial model problems that are susceptible to a controlled theoretical solution, a nontechnical physical discussion of known features of the complex phase diagrams with intertwined orders is contained in Sec. VIII.B. The role of dimensionality is explored, and special emphasis is placed on cases where partial melting of a highly ordered ground state can give rise to a variety of intermediate phases with vestigial order, and to cases (which probably, for technical reasons, are restricted to quasi-1D models) in which the ordered phases emerge on lowering T from non-Fermi-liquid "normal" states. As discussed in Sec. VIII.C, various versions of an alternative approach, in which the existence of intertwined orders is associated with an assumed emergent higher (approximate) symmetry [e.g., SU(2), SO(5), or SO(6)] that relates the various distinct forms of order, have been advocated by Zhang (1997), Efetov, Meier, and Pepin (2013), and Hayward *et al.* (2014). While this approach has many attractive features, we point out what we consider to be rather general theoretical and phenomenological shortcomings of these scenarios. We summarize our conclusions in Sec. VIII.D, especially concerning the role of PDW order as a potential origin of intertwined orders in the cuprates.

II. ELECTRONIC LIQUID CRYSTAL PHASES

The intertwined orders observed in the cuprates appear to emerge with relatively light doping of a strongly correlated (Mott) insulating antiferromagnet. A generic feature of short-range models of lightly doped Mott insulators, such as the the *t-J* model, extended Hubbard models, and others, is a strong tendency toward phase separation in which the doped holes are expelled from locally antiferromagnetic regions (Emery, Kivelson, and Lin, 1990; Grilli *et al.*, 1991; Emery and Kivelson, 1993; Vermeulen, Barford, and Gagliano, 1994; Poilblanc, 1995; Misawa and Imada, 2014).

Motivated by these observations and by the discovery in 1995 by one of us of stripe phases in the lanthanum family of cuprates (Tranquada *et al.*, 1995), two of us introduced the concept of electronic liquid crystal phases, which we argued are a general feature of the phase diagrams of strongly correlated systems (Kivelson, Fradkin, and Emery, 1998).[3] The recent spectacular experimental discoveries of diverse charge orders in the pseudogap phase of essentially all the cuprates [as well as in other materials, including evidence of electronic nematic order in iron based (Chu *et al.*, 2010) and heavy-fermion superconductors (Okazaki *et al.*, 2011; Riggs *et al.*, 2015)] have generally validated the applicability of this concept. (We discuss these experiments in Sec. V.)

[3]Electronic liquid crystal phases have also been studied in the context of two-dimensional electron fluids in large magnetic fields (Balents, 1996; Musaelian and Joynt, 1996; Fradkin and Kivelson, 1999; Fradkin *et al.*, 2010).

Also associated with local phase separation is a tendency for the spins in the hole-poor regions to form local spin singlets, and for the holes to pair as they aggregate (White and Scalapino, 1997). Hence, valence-bond crystals (Sachdev, 2003) and uniform resonating valence-bond (RVB) liquid states (Anderson, 1987; Kivelson, Rokhsar, and Sethna, 1987; Lee, Nagaosa, and Wen, 2006) are possible consequences of the same local physics. In particular, stripe (smectic) phases with a spin gap (which we discuss in Sec. IV.B) can be regarded as spatially nonuniform RVB states. At an intuitive level, the important point is that below a crossover temperature (which in the cuprates we associate with the crossover temperature to the pseudogap regime T^*), the electron fluid should be thought of in terms of a fluid of spin singlets and small charged clusters, rather than of electron quasiparticles. These clusters then behave in much the same way as the molecules in complex classical fluids, and consequently all sorts of ordering tendencies should appear in delicate balance below T^*, leading to "ineluctable complexity" (Fradkin and Kivelson, 2012) of the phase diagram.

Electronic liquid crystals are phases that spontaneously break translation and/or rotation symmetries; to make the analogy with classical liquid crystalline phases complete (de Gennes and Prost, 1993; Chaikin and Lubensky, 1995), one might restrict attention to phases that remain conducting (fluid) despite the broken symmetries, although this condition is sometimes overlooked in common usage (as for a spin nematic). Examples range from multicomponent CDW phases (which break translational symmetry in all directions), through stripe (or smectic) phases (which break translation symmetry along one direction and rotational symmetry), to nematic phases (which break spontaneously only rotational invariance) (Kivelson, Fradkin, and Emery, 1998). For the electrons in a crystal, the symmetries available to be broken are the discrete translation and point-group symmetries. Unlike their classical cousins, electronic liquid crystalline phases can be strongly quantum mechanical. From this comes the added richness of an interplay between the order parameters associated with spatial symmetry breaking and intrinsically quantum orders, including superconductivity and magnetism.

Much has already been written concerning the microscopic mechanisms that lead to electronic liquid crystalline phases in strongly correlated electron systems; for reviews, see Kivelson *et al.* (2003), Vojta (2009), Fradkin *et al.* (2010), Fradkin (2012), Hu and Xu (2012), and Fernandes, Chubukov, and Schmalian (2014). What constitutes the principal focus of the following is an analysis of the way in which the existence of liquid crystalline phases leads to complex phase diagrams for electronic systems, just as it does for classical complex fluids. In particular, we explore the important ways that superconductivity and liquid crystalline orders are naturally intertwined, so that the critical temperatures of various superconducting and charge ordered phases remain comparable to each other for a wide range of conditions.

III. FIELD THEORIES OF INTERTWINED ORDERS

We now consider the effective field theories that describe a particularly interesting set of intertwined orders. In this context it is useful to consider both the relevant classical

Landau-Ginzburg-Wilson (LGW) and nonlinear sigma models; the former is an expansion in powers of the order parameter fields and so gives a reasonable description as long as the ordering is weak, while the latter assumes a well-developed local amplitude of the order parameter field and focuses on the physics of the Goldstone modes (in the ordered phase) or the nearly Goldstone modes in the fluctuational regime above T_c. The number of order parameters that are potentially involved, from both theory and experiment in the cuprates, is dauntingly large, including as it does uniform SC, PDW, CDW, SDW, and nematic orders, at least.

To keep the analysis manageable, we ignore magnetism all together and treat both CDW and nematic orders as parasitic, deriving from a microscopic tendency to PDW order. Starting with these we can describe the uniform SC, the stripe-ordered (CDW + SC), and the PDW states directly at mean-field level, while a pure CDW and a nematic phase, as well as various more exotic phases including a charge $4e$ SC, can be obtained as states with vestigial order (Nie, Tarjus, and Kivelson, 2014) (involving composite order parameters) when fluctuation effects are treated carefully. For simplicity, we also neglect the effects of quenched disorder (which is always a relevant perturbation as far as at least the CDW component of the ordering is concerned) and quantum fluctuations. Of course, it is straightforward if tedious to include additional order parameter fields in the analysis, which is necessary for explicit application to some materials.

A. Landau-Ginzburg free energy

To begin with, we consider the LGW free energy density to quartic order in the fields where Δ_0 is the uniform d-wave order parameter, $\mathbf{\Delta}_a = (\Delta_{Q_a}, \Delta_{-Q_a})$ is the complex spinor representing the two components of the PDW with ordering vector in the a direction,

$$\begin{aligned} F = &\frac{r_0}{2}|\Delta_0|^2 + \frac{r_Q}{2}[|\mathbf{\Delta}_x|^2 + |\mathbf{\Delta}_y|^2] + \frac{\kappa_0}{2}|\mathbf{\nabla}\Delta_0|^2 \\ &+ \frac{\kappa_1}{2}[|\partial_x \mathbf{\Delta}_x|^2 + |\partial_y \mathbf{\Delta}_y|^2] + \frac{\kappa_2}{2}[|\partial_y \mathbf{\Delta}_x|^2 + |\partial_x \mathbf{\Delta}_y|^2] \\ &+ \frac{u}{4}[|\Delta_0|^2 + |\mathbf{\Delta}_x|^2 + |\mathbf{\Delta}_y|^2]^2 + \frac{\gamma_1}{2}|\mathbf{\Delta}_x|^2|\mathbf{\Delta}_y|^2 \\ &+ \frac{\gamma_2}{2}|\Delta_0|^2[|\mathbf{\Delta}_x|^2 + |\mathbf{\Delta}_y|^2] + \frac{\gamma_3}{2}[|\mathbf{\Delta}_x^\dagger \tau_3 \mathbf{\Delta}_x|^2 + |\mathbf{\Delta}_y^\dagger \tau_3 \mathbf{\Delta}_y|^2] \\ &+ \frac{\gamma_4}{2}[(\mathbf{\Delta}_x^\dagger \tau^+ \mathbf{\Delta}_y)(\mathbf{\Delta}_x^\dagger \tau^- \mathbf{\Delta}_y) + \text{c.c.}] \\ &+ \frac{\gamma_5}{2}[(\Delta_0^*)^2(\mathbf{\Delta}_x^T \tau^- \mathbf{\Delta}_x + \mathbf{\Delta}_y^T \tau^- \mathbf{\Delta}_y) + \text{c.c.}], \end{aligned} \tag{3.1}$$

where τ_3 and $\tau^\pm = (\tau_1 \pm i\tau_2)/2$ are the three 2×2 Pauli matrices. [This free energy was partly given by Agterberg and Tsunetsugu (2008) using a different notation, and it is not even quite the most general possible form to this order.] Because there are a total of five complex scalar fields involved, under fine-tuned conditions $\kappa_j = \kappa$, $\gamma_j = 0$, and $r_0 = r_Q$ (eight conditions), this model has a large O(10) symmetry.

Away from such fine-tuned points, the symmetries of F represent the microscopic symmetries of the system we consider—gauge invariance, translational symmetry in the x and y directions, time-reversal symmetry, and various mirror and discrete rotational symmetries that exchange the x and y axes (and, at the same time, interchange $\mathbf{\Delta}_x$ and $\mathbf{\Delta}_y$).

Even though we greatly limited the number of "primary" order parameter fields treated explicitly, it is possible to study a variety of other order parameters as composites of the primary fields. For instance, the unidirectional PDW ground state with $\langle \mathbf{\Delta}_x \rangle \neq 0$, but $\langle \Delta_0 \rangle = \langle \mathbf{\Delta}_y \rangle = 0$, breaks gauge symmetry (it is a superconductor), translational symmetry (it has a CDW component), and C_4 rotational symmetry (it has a nematic component). However, there are conditions in which, upon raising the temperature, such a PDW phase melts by a sequence of two transitions or more, at the lower of which gauge symmetry is restored but not translational symmetry, resulting in an intermediate unidirectional CDW phase which melts only at a second, higher transition temperature. From this perspective, the CDW phase is viewed as a state with vestigial order, in the sense that it breaks some but not all of the symmetries that are broken by the fully ordered PDW ground state. Naturally, if a set of primary fields have a nonzero expectation value, so do all products of those fields. However, in a state with vestigial order, the expectation values of the primary fields vanish, while certain composite order parameters still have nonzero expectation value (Berg *et al.*, 2009; Lee, 2014; Wang and Chubukov, 2014).

While many forms of vestigial spin singlet orders can be envisaged in terms of the primary fields we have introduced, for the present discussion, three forms of order are most relevant (Berg, Fradkin, and Kivelson, 2009a; Berg *et al.*, 2009)

- Charge $4e$ uniform SC, with order parameters

$$\Delta_{4e,a} = \Delta_{Q_a}\Delta_{-Q_a}, \tag{3.2}$$

  which can exist in a nematic form ($|\Delta_{4e,x}| \neq |\Delta_{4e,y}|$) or in various rotational symmetry preserving (s, d, etc.) forms.
- $2Q$ CDW (unidirectional or bidirectional), whose order parameter is

$$\rho_{2Q_a} = \Delta_{-Q_a}^*\Delta_{Q_a}. \tag{3.3}$$

- $1Q$ CDW (likewise, unidirectional or bidirectional) whose order parameter is

$$\rho_{Q_a} = \Delta_0^*\Delta_{Q_a} + \Delta_{-Q_a}^*\Delta_0. \tag{3.4}$$

- Nematic order, whose order parameter is

$$\mathcal{N} = |\mathbf{\Delta}_x|^2 - |\mathbf{\Delta}_y|^2. \tag{3.5}$$

So, for example, a striped CDW phase modulated along the x direction arises if $\langle \Delta_{Q_a} \rangle = \langle \Delta_0 \rangle = \langle \rho_{Q_y} \rangle = 0$ but $\langle \rho_{Q_x} \rangle \neq 0$, while a pure nematic phase arises if, in addition, $\langle \rho_{Q_x} \rangle = 0$ but $\langle \mathcal{N} \rangle \neq 0$.

B. Nonlinear sigma model: Near multicriticality

A nonlinear sigma model description assumes the existence of a local amplitude of the order parameter which is well established, but which does not strongly distinguish between which form of superconducting correlations, uniform or modulated, is ultimately favored at long wavelengths. This amounts to assuming that one is in the vicinity of a multicritical point at which the critical temperatures of the orders are equal.

Were we to assume that the system were fine-tuned to the point of maximal symmetry, this would be an O(10) nonlinear sigma model, with many interesting features, including (by selectively breaking subsets of this symmetry) all the interesting features of SO(5), SO(6), and SU(2). Instead, we assume only the actual symmetries of the problem, but treat the problem under the assumption that the temperature dependence of the amplitudes of the various order parameters can be neglected over a suitable range of temperatures; this is an innocuous assumption for the purposes of studying critical phenomena, but is more problematic if we are interested in the properties of the system over an extended range of T. It is analogous to treating spin antiferromagnetism in an approximation that assumes the existence of a fixed magnitude local moment (Chakravarty, Halperin, and Nelson, 1988, 1989) or the phase fluctuations of a superconductor with fixed magnitude local pairing strength (Carlson *et al.*, 1999; Eckl *et al.*, 2002). This is a highly nontrivial assumption; it implies that the "mean-field transition temperature" (itself an ill-defined concept) for both the uniform SC and PDW order is higher than the temperatures at which any of the ordering phenomena of interest occur, and that therefore (again in a somewhat ill-defined sense) the local magnitude of the corresponding order parameters is well defined over a broad range of temperatures which extends well above any observed T_c. We discuss the issue of whether this assumption is consistent with experiment in the cuprates in Sec. VIII.A.

Putting aside worries about the range of validity of such an approximation, there is still an issue concerning the role of various possible patterns of discrete symmetry breaking that can arise. To simplify the discussion, i.e., to avoid the many special considerations needed to be completely general, we explicitly assume that time reversal and inversion symmetry remain unbroken, which implies that the amplitudes at $\pm Q_a$ are the same, $|\Delta_{+Q_a}|^2 = |\Delta_{-Q_a}|^2$. We can then define five phase fields as $\Delta_0 \equiv |\Delta_0| e^{i\theta_0}$ and $\Delta_{\pm Q_a} \equiv |\Delta_{Q_a}| e^{i(\theta_a \pm \phi_a)}$, and a real scalar field (representing possible Ising-nematic order), $\mathcal{N} = |\Delta_{Q_x}|^2 - |\Delta_{Q_y}|^2$. In any isotropic phase (where $\mathcal{N} = 0$), the effective Hamiltonian in terms of the phase fields takes the form

$$\begin{aligned} H = {} & \frac{K_0}{2}(\boldsymbol{\nabla}\theta_0)^2 + \frac{K_1}{2}[(\partial_x\theta_x)^2 + (\partial_y\theta_y)^2] \\ & + \frac{K_2}{2}[(\partial_y\theta_x)^2 + (\partial_x\theta_y)^2] + \frac{K_3}{2}[(\partial_x\phi_x)^2 + (\partial_y\phi_y)^2] \\ & + \frac{K_4}{2}[(\partial_y\phi_x)^2 + (\partial_x\phi_y)^2] - \tilde{V}\cos(2\theta_x - 2\theta_y) \\ & - V[\cos(2\theta_0 - 2\theta_x) + \cos(2\theta_0 - 2\theta_y)]. \end{aligned} \tag{3.6}$$

From Eq. (3.3) it is apparent that $2\phi_a$ is the phase of the $2Q$ CDW component of the order. It is important to remember that while the CDW and SC phases seem to be totally uncoupled, they are related through the conditions that $e^{i(\theta_x \pm \phi_x)}$ and $e^{i(\theta_y \pm \phi_y)}$ are single valued. As we will see in Sec. VIII.B, this controls the nature of the topological excitations (vortices). On the other hand, the superconducting phases θ_0, θ_x, and θ_y are coupled to each other by higher order Josephson-like terms—the cosine terms in Eq. (3.6). This leaves us with three U(1)'s and two $\mathbb{Z}_2$ symmetries, the latter being the remnant of the superconducting relative phases.

In an Ising-nematic phase, the effective phase stiffnesses are anisotropic, $K_j \to K_{ja}(\mathcal{N})$, where $[K_{jx}(\mathcal{N}) - K_{jy}(\mathcal{N})] = -[K_{jx}(-\mathcal{N}) - K_{jy}(-\mathcal{N})]$, similarly $V \to V_{ja}(\mathcal{N})$ with $[V_x(\mathcal{N}) - V_y(\mathcal{N})] = -[V_x(-\mathcal{N}) - V_y(-\mathcal{N})]$, and where H must be augmented by an effective Hamiltonian for $\mathcal{N}$. Deep in a nematic phase, with $\mathcal{N} > 0$, we simply assume that the fluctuations of ϕ_y and θ_y are so violent that these fields can be ignored, leaving an effective model (which we study in Sec. VIII.B) with $U(1) \times U(1) \times \mathbb{Z}_2$ symmetry.

IV. INTERTWINED ORDER IN HUBBARD MODELS

A. Lightly doped antiferromagnetic insulators

The Hubbard model and the related t-J models are widely thought to capture the essential physics of a class of highly correlated systems. On a two-dimensional square lattice, when the band parameters (e.g., the ratio between the first and second neighbor hopping matrix elements t'/t) are adjusted to reproduce the salient features of the experimentally measured Fermi surface, these models may be sufficiently "realistic" that their properties can be compared with experiment in the cuprates (Emery, 1987; Varma *et al.*, 1989; Scalapino, 2012).

In the weak-coupling limit $U/t \ll 1$, an asymptotically exact solution of the Hubbard model (Raghu, Kivelson, and Scalapino, 2010) is possible: down to exponentially low temperatures of order $T_{SC} \sim 4t\exp[-\alpha(t/U)^2]$, the model exhibits Fermi liquid (FL) behavior, α is a number of the order of 1 which depends on details of the band structure and the value of electron concentration per site n. Below this temperature, the system exhibits $d_{x^2-y^2}$ superconductivity for a broad range of n in the neighborhood of $n = 1$ with $T_c \sim T_{SC}$. Although the nature of the superconducting state itself is reminiscent of the superconducting state in the cuprates, this is where the resemblance ends. In particular, there are no strong fluctuation effects to give rise to pseudogap phenomena, and no trace of competing orders of any sort. This is generic behavior for any weakly interacting Fermi fluid in more than one dimension.

In the strong coupling limit $U/t \gg 1$ or $t/J \gg 1$, recent numerical studies confirmed (Liu *et al.*, 2012) what was long believed (Nagaoka, 1966), that both models exhibit fully polarized ferromagnetic metallic phases for a broad range of n near $n = 1$. Again, there are no competing orders.

Thus, to the extent that intertwined or even conventional competing orders are features of the theoretically expected landscape, they must arise exclusively at intermediate coupling. For intermediate coupling, where U is of the order of the bandwidth $U \sim 8t$ or $J/t \sim 1/2$, there have been many approximate (mean-field) and numerically implemented approaches to the problem. Various different conjectured phases have been found in different studies—already evidence that no single pattern of broken symmetry is strongly favored.

For $n = 1$ ($x = 0$), U of the order of the bandwidth and t' not too large, the Hubbard model has an antiferromagnetically ordered, insulating ground state (Lin and Hirsch, 1987; Arovas and Auerbach, 1988; Chakravarty, Halperin, and Nelson, 1989). In the decade following the discovery of superconductivity in the cuprates, the issue of how the system evolves with weak doping $|x| \ll 1$ was one of the most studied problems in condensed matter physics. The problem is complex since there is inherent frustration between the tendency to maintain local antiferromagnetic correlations and the doped hole itineracy. Three possible scenarios have been considered.

One frequently occurring possibility (Emery, Kivelson, and Lin, 1990; Emery and Kivelson, 1993) is that this evolution is discontinuous, leading to macroscopic phase separation into regions of an undoped antiferromagnet where the antiferromagnetic exchange is unfrustrated, and critically doped regions with $x = x_c(U/t)$, where the zero point energy of the doped holes is dominant. Phase separation has been shown to occur in the limit of large spatial dimension $d \gg 1$ (Carlson *et al.*, 1998), large spin $S \gg 1$ (Auerbach and Larson, 1991b), in certain large N generalizations of the problem (Auerbach and Larson, 1991a), and for $U/t \gg 1$ (Emery, Kivelson, and Lin, 1990; Hellberg and Manousakis, 1997; Liu *et al.*, 2012; Misawa and Imada, 2014). Given that the ground state at $x = 0$ is a Néel state, if one assumes, consistent with an RVB scenario, that the lightly doped system corresponds to a doped spin liquid (Anderson, 1987; Kivelson, Rokhsar, and Sethna, 1987; Rokhsar and Kivelson, 1988; Fradkin and Kivelson, 1990; Anderson *et al.*, 2004; Lee, Nagaosa, and Wen, 2006), the transition between these two states must be first order, and thus there must be a two-phase region for small enough x.

Another possibility is some form of local phase separation, especially stripe formation, which is driven by more or less the same local energetic considerations. The earliest such proposals (Machida, 1989; Zaanen and Gunnarsson, 1989; Schulz, 1990) suggested insulating stripes, i.e., unidirectional charge-density waves with a doping dependent period such that x_s, the density of doped holes per site perpendicular to the charge ordering vector, is fixed at $x_s = 1$. Later proposals (Tsunetsugu, Troyer, and Rice, 1995; Nayak and Wilczek, 1997; White and Scalapino, 1998) based on various approaches to the 2D *t-J* and Hubbard models suggested conducting stripes, with $0 < x_s < 1$. Some of these studies found evidence that the doped holes in the conducting stripes are strongly paired and possibly superconducting. Since in the presence of long-range Coulomb interactions (not included in Hubbard-like models) macroscopic phase separation of charged particles is not possible, where the short-range interactions tend to produce phase separation, Coulomb-frustrated phase separation generically results in CDW order of a sort that is difficult to distinguish from direct forms of local phase separation, and similarly, where the short-range interactions favor stripe formation, long-range Coulomb forces will tend to shift the period, with the associated tendency to turn insulating to conducting stripe phases. Thus, in practice, the difference between cases 1 and 2 is not physically significant.

The third possibility is one form or another of uniform phase with coexisting antiferromagnetic order (Zhang, 1997; Demler, Hanke, and Zhang, 2004). Note that a discontinuous drop of the sublattice magnetization would necessarily imply a first order transition and hence phase separation (Misawa and Imada, 2014).

B. Intertwined orders in model quasi-1D systems

Powerful field theory methods (bosonization) permit an essentially complete understanding of the long-distance properties of interacting electrons in 1D, while efficient numerical methods [especially density-matrix renormalization group (DMRG)] permit the short-distance microscopic physics of specific models, even of rather complicated multileg ladders, to be treated reliably. [For a review of bosonization and Luttinger and Luther-Emery liquids see Fradkin (2013)]. By matching results at intermediate length scales it is possible to obtain an essentially complete theoretical understanding of strongly correlated systems in 1D. Moreover, so long as interchain couplings are sufficiently weak, various forms of interchain mean-field theory allow these results to be extended to quasi-1D systems. Taken literally, these models are relevant only to the properties of quasi-1D materials, but in many cases, results in this “solvable” limit give qualitative insight into the behavior of fully 2D or 3D highly correlated electron systems (Kivelson, Fradkin, and Emery, 1998; Carlson *et al.*, 2000; Emery *et al.*, 2000; Granath *et al.*, 2001; Mukhopadhyay, Kane, and Lubensky, 2001; Vishwanath and Carpentier, 2001; Carr and Tsvelik, 2002; Essler and Tsvelik, 2002; Jaefari, Lal, and Fradkin, 2010; Teo and Kane, 2014). In particular, this is the only well-understood class of problems in which various ordered phases emerge from non-Fermi liquids.

It is known from DMRG studies (Noack *et al.*, 1997) and bosonization theories (Balents and Fisher, 1996; Wu, Liu, and Fradkin, 2003; Controzzi and Tsvelik, 2005) that *t-J* and Hubbard-type models on a two-leg ladder have a broad regime in which they have a spin gap $\Delta_s \approx J/2$ and d-wave-like superconducting correlations. DMRG studies found (Noack *et al.*, 1997; Siller *et al.*, 2001; White, Affleck, and Scalapino, 2002) that the spin gap decreases monotonically with doping x and vanishes at a critical doping $x_c \approx 0.3$. The upshot is that the two-leg ladder is in a Luther-Emery (LE) liquid phase whose effective field theory consists of a field ϕ, describing the phase of the incommensurate CDW amplitude $\rho_Q(x) \sim \exp[i\sqrt{2\pi}\phi(x)]$ (with $Q = 2k_F$) and its dual field θ, describing the phase of the superconducting order parameter $\Delta_0(x) \sim \exp[i\sqrt{2\pi}\theta(x)]$. The field theory contains two important parameters—the charge Luttinger parameter K_c and the charge velocity v_c. Both are complicated functions of the microscopic parameters of the ladder. DMRG studies showed that $K_c \to 2$ as $x \to 0$ and decreases with increasing doping, reaching the value $K_c = 1/2$ at $x_c \approx 0.3$ (where $\Delta_s \to 0$). For $T \ll \Delta_s$, the SC and CDW susceptibilities obey the scaling laws $\chi_{\rm SC} \sim \Delta_s/T^{2-K_c^{-1}}$ and $\chi_{\rm CDW} \sim \Delta_s/T^{2-K_c}$.

Thus, an interesting quasi-1D model to analyze is an array of spacing a of two-leg Hubbard ladders (Emery *et al.*, 2000;

Arrigoni, Fradkin, and Kivelson, 2004). At energies low compared to the single ladder spin gap, the array is represented by a set of CDW phase fields $\{\phi_j\}$ and their conjugate SC phase fields $\{\theta_j\}$. Although microscopically there are many local couplings between neighboring two-leg ladders which are allowed, if the couplings are weak compared to Δ_s, most of these are irrelevant. In particular, electron tunneling, that normally drives the quasi-1D system into a 2D Fermi liquid state, is suppressed. The marginal or potentially relevant interladder interactions are (a) $\mathcal{J}_{\rm SC}$, the Josephson coupling between the SC order parameters of nearest neighbor ladders (i.e., Cooper pair tunneling) which locks $\theta_j(x)$ on neighboring ladders, (b) $\mathcal{J}_{\rm CDW}$, the coupling of the CDW order parameters of neighboring ladders which locks $\phi_j(x)$ on neighboring ladders, and (c) $\tilde{V}$, interladder forward-scattering interactions (which are strictly marginal operators).

Models of this type have been studied extensively using a variety of techniques (Carlson *et al.*, 2000; Emery *et al.*, 2000; Vishwanath and Carpentier, 2001; Arrigoni, Fradkin, and Kivelson, 2004; Jaefari, Lal, and Fradkin, 2010). In the range in which there is a spin gap, SC and CDW orders compete. The resulting state is roughly determined by which coupling is more relevant. The scaling dimensions of the Josephson coupling and the CDW coupling are $D_{\rm SC} = 1/K_c$ and $D_{\rm CDW} = K_c$, respectively. For small x, $D_{\rm SC} \approx 1/2$ and $D_{\rm CDW} \approx 2$, so the Josephson coupling is strongly relevant while the CDW coupling is barely relevant. Conversely, for larger values of x the CDW coupling is more relevant than the Josephson coupling. At an intermediate value of x (corresponding to $x_P \simeq 0.1$) $K_c = 1$ at which point $D_{\rm CDW} = D_{\rm SC} = 1$ so the two orders are equally relevant. This balance can be affected, as well, by the marginal interactions $\tilde{V}$ (Emery *et al.*, 2000; Vishwanath and Carpentier, 2001).

The generic phase diagram (Arrigoni, Fradkin, and Kivelson, 2004; Jaefari, Lal, and Fradkin, 2010) for weakly coupled two-leg Hubbard ladders is shown in Fig. 1 (see also Fig. 5). One thing to note about this phase diagram is that the ordered phases emerge from a non-Fermi-liquid normal state which for $T > \Delta_s$ consists of effectively decoupled LE liquids. For K_c not too close to $K_c = 1$, either SC or CDW order is dominant. The corresponding critical temperatures can be estimated from dimensional crossover mean-field theory and are found to have a power-law dependence on the interladder couplings with an overall scale set by the spin gap (Carlson *et al.*, 2000; Arrigoni, Fradkin, and Kivelson, 2004)

$$T_{\rm SC} \sim \Delta_s(\mathcal{J}_{\rm SC})^{\alpha_{\rm SC}}, \qquad T_{\rm CDW} \sim \Delta_s(\mathcal{J}_{\rm CDW})^{\alpha_{\rm CDW}}, \tag{4.1}$$

where the exponents are $\alpha_{\rm SC} = K_c/(2K_c - 1)$ and $\alpha_{\rm CDW} = 1/(2 - K_c)$, respectively. $T_{\rm CDW}$ vanishes as $x \to x_c$ due to the vanishing of Δ_s; for larger x, the most relevant interladder coupling is single-particle hopping which leads to a crossover to a higher-dimensional Fermi liquid regime that is stable to exponentially low temperatures. The vanishing of $T_{\rm SC}$ in proportion to x as $x \to 0$ involves additional considerations associated with the approach to the correlated (Mott) insulating state at $x = 0$, as discussed by Arrigoni, Fradkin, and Kivelson (2004).

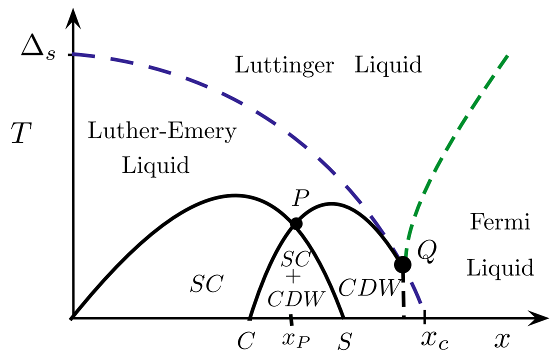

FIG. 1 (color online). Schematic phase diagram (temperature T vs doping x) for an array of weakly coupled two-leg Hubbard ladders. The long-dashed curve is the spin gap, which vanishes at $x_c \approx 0.3$, and which also indicates a crossover temperature to a LE liquid regime. The short-dashed line indicates a crossover scale to a higher-dimensional Fermi liquid (FL) regime. $x_P \approx 0.1$ is the point at which $K_c = 1$ and P is the corresponding tetracritical point discussed in the text. C and S are quantum critical points, between which SC and CDW orders coexist. We have shown Q as a trictitical point, below which the transition (dark short-dashed line) between the CDW and the FL becomes first order, although there are other possibilities here. For details, see text.

For $K_c \approx 1$, there is a multicritical point (shown as P in Fig. 1) at which $T_{\rm SC}$ and $T_{\rm CDW}$ meet. The dimensional crossover mean-field theory predicts that the 2D array has a SC phase and a CDW phase, but it does not tell us if these phases are separated by a first order transition (Emery *et al.*, 2000; Carr and Tsvelik, 2002) or if there is a phase in which SC and CDW orders coexist. It is a special (and likely nongeneric) feature of the two-leg ladder that the CDW coupling becomes marginal for the same value x_c where the spin gap vanishes. In the phase diagram of Fig. 1 we allowed for a tricritical point Q and a first order transition out of the CDW state into a FL state (which has no spin gap) to remove this accidental feature.

To address this problem, Jaefari, Lal, and Fradkin (2010) derived an effective field theory, a nonlinear sigma model (NLSM) in $2+1$ dimensions that describes the fluctuations of both the SC and the CDW order parameters of the ladder array under the special fine-tuned conditions that $K_c = 1$ and $\mathcal{J}_{\rm SC} = \mathcal{J}_{\rm CDW}$. Although the microscopic model has only a global $\mathrm{U}(1) \times \mathrm{U}(1) \simeq \mathrm{O}(2) \times \mathrm{O}(2)$ symmetry, they showed that under these special circumstances there exists an enlarged O(4) symmetry which unifies the CDW and SC orders. Thus, for K_c close to 1 and $|\mathcal{J}_{\rm SC} - \mathcal{J}_{\rm CDW}| \ll W$, where W is a high-energy cutoff for the charge sector of the ladder (Carlson *et al.*, 2000), the critical fluctuations near this fine-tuned multicritical point (P in Fig. 1) can be described by the effective action[4] for an O(4) NLSM (Jaefari, Lal, and Fradkin, 2010):

$$S[n] = \frac{1}{2g_0}(\partial_\mu n)^2 + w\partial_\mu n_a O_{ab}\partial^\mu n_b + h n_a O_{ab} n_b, \tag{4.2}$$

[4]Here we omitted the spatial anisotropy in the gradient terms of the 2D system since under the renormalization group (RG) the anisotropy is a redundant operator (Affleck and Halperin, 1996).

where n_a is a four-component order parameter subject to the constraint $n_a n_a = 1$, where $n_1/n_2 = \tan(\theta_{SC})$, $n_3/n_4 = \tan(\phi_{CDW})$, and $\mu = 0, 1, 2$ are the space-time indices. The first term of the action of Eq. (4.2) has an effective coupling constant $g_0 \simeq [8\pi(\Delta_s/v_c)W/(\mathcal{J}_{SC} + \mathcal{J}_{CDW})a]^{1/2}$. The last two terms represent the breaking of $O(4) \to O(2) \times O(2)$. Here $O_{ab} = \mathrm{diag}(1, 1, -1, -1)$ is a diagonal 4×4 matrix, $w \propto (K_c - 1)$ and $h \propto \Delta_s(\mathcal{J}_{SC} - \mathcal{J}_{CDW})/2W$.[5]

The nature of the phase diagram, and, in particular, the existence of the tetracritical point P was obtained by Jaefari, Lal, and Fradkin (2010) from a standard analysis of the (quantum) NLSM in 2D (and its relation with the LGW ϕ^4 field theory formulation), both at finite temperatures and at zero temperature. Of course, in 2D at finite temperature the designations SC and CDW refer to phases with quasi-long-range order, while they are long range ordered in 3D. Both of the symmetry-breaking terms w and h are perturbatively relevant, so the tetracritical point generically has only the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of the microscopic model. Consequently, even in 2D, P occurs at finite temperature. However, under the doubly fine-tuned circumstances $w = h = 0$, the resulting O(4) symmetry implies that in 2D the tetracritical point is suppressed to $T = 0$. The emergent O(4) symmetry at $w = h = 0$ is similar to such symmetries that arise in the SO(5) theory of antiferromagnetism and superconductivity of Zhang (1997) and in the O(6) of Hayward *et al.* (2014) and SU(2) of Efetov, Meier, and Pepin (2013) of CDW and SC order, which we discuss in Sec. VIII.C.

In summary, this analysis shows that the higher symmetries are, in general, *not emergent* symmetries but instead are fragile. In spite of that, systems of this type in general have intertwined orders with complex phase diagrams with several quantum critical points under their dome(s) (Kivelson, Fradkin, and Emery, 1998). On the other hand, this analysis shows that (in this case, at least) these quantum critical points cannot be regarded as the origin of the SC (or the CDW) phase, contrary to what is often assumed in the literature of high temperature superconductivity.

C. Variational results for the 2D *t*-*J* model

Recently Corboz, Rice, and Troyer (2014) found the "best," to date, variational solution of the 2D *t*-*J* model at intermediate coupling on the square lattice, in the technical sense that they obtained the lowest variational energy. While there is always the danger with variational studies that the true ground state could have properties that are incompatible with the assumed form of the states considered, the only obvious prejudice of the present study is that it favors states with relatively lower "quantum entanglement." Given this and the large number of variational parameters involved, it is quite plausible that the present results can be taken at face value.

Corboz *et al.* found that for the range of parameters studied (i.e., J/t between 0.2 and 0.8 and doped hole concentration $x \equiv 1 - n$ between 0 and 0.16) the states listed below all have energies that are, to a high degree of accuracy, equal to each other: (1) A uniform *d*-wave superconducting phase (SC) corresponding to $\langle\Delta_0\rangle \neq 0$, and $\langle\Delta_{Q_a}\rangle = 0$. For $x < x_c$ (where $x_c = 0.1$ for $J/t = 0.4$) this has coexisting antiferromagnetic Néel-type magnetic order. (2) A state with coexisting $2Q_x$ charge-density wave and uniform *d*-wave superconducting order (CDW + SC). This is a striped superconducting phase with $\langle\Delta_0\rangle \neq 0$, $\langle\rho_{2Q_x}\rangle \neq 0$, and $\langle\rho_{2Q_y}\rangle = 0$ which spontaneously breaks translational and C_4 lattice rotational symmetry. (3) A unidirectional PDW phase with no uniform SC component, i.e., with $\langle\Delta_0\rangle = 0$, $|\langle\Delta_{Q_x}\rangle| = |\langle\Delta_{-Q_x}\rangle| \neq 0$ (from which it follows that $\langle\rho_{2Q_x}\rangle \neq 0$), and $|\langle\Delta_{\pm Q_y}\rangle| = 0$. Both the CDW + SC and PDW states are found to break spin-rotational symmetry, as well, through formation of unidirectional SDW order with a modulated amplitude such that the superconducting component of the order has its maximum amplitude at nodes of the SDW order and is minimal (vanishes in the case of the PDW) where the SDW amplitude is maximal. (To describe the magnetic components of these orders, we would need to include additional fields, which we have neglected for simplicity elsewhere in this paper.)

Naturally, it is not true that these three distinct phases are exactly degenerate; the CDW + SC phase achieves the lowest variational energy. However, the ground-state energy per site of the CDW + SC is lower than the PDW only by roughly $\Delta E = 0.001tx$, and lower than the uniform SC by roughly $\Delta E = 0.01tx$. These differences are so small that it is not clear that they are significant (within the accuracy of the variational ansatz), and in any case one would expect that small changes to the model could easily tip the balance one way or the other. At the rough intuitive level, this near degeneracy reflects the fact that locally all three phases look pretty similar in that they all look like a uniform *d*-wave superconductor, with or without coexisting antiferromagnetism depending on the local doped hole concentration.

A few other aspects of the results are significant as well: (1) The periodicity of the CDW order for either of the striped phases is not determined by Fermi-surface nesting features; rather the preferred density of holes per unit length of stripe n_s is a function of the value of J/t (i.e., is determined by the strength of the interactions), ranging from about $n_s = 0.35$ for $J/t = 0.2$ to $n_s = 1$ (corresponding to insulating stripes) for $J/t = 0.8$. (2) The SDW component of the order suffers a π phase shift across the row of sites at which the CDW order is maximal; thus, for even period CDW order, the SDW period is twice that of the CDW (which is the same as the period of the SC order in the case of the PDW), while for odd period CDW order, the SDW period is equal to that of the CDW. (3) In the context of the cuprates, there has been considerable discussion of whether striped states or checkerboard states (bidirectional CDW states that preserve the C_4 rotational symmetry of the underlying lattice) are preferred; an earlier version of this calculation (Corboz *et al.*, 2011) indicated that the checkerboard phase is never preferred. An insulating diagonal striped phase (with $n_s = 1$) was also found to have relatively low

[5]Jaefari, Lal, and Fradkin (2010) also found a topological term that could lead to a deconfined quantum critical point, but which in this case is inaccessible due to the presence of spatially anisotropic perturbations (Senthil and Fisher, 2006).

variational energy, but never competitive with the vertical stripe phases.

D. Pair density waves in model systems

There is no material system in which we know for certain that PDW order occurs. Thus, it is worth addressing as a point of principle whether this phase occurs in any theoretically tractable microscopic model.

1. PDW phases in Hubbard-like ladders

We already noted in Sec. IV.B that DMRG studies of Hubbard and t-J two-leg ladders revealed that these systems have a spin gap over a significant hole doping range, and that they exhibit strong superconducting correlations. What is less widely recognized is that, in many cases, what is formed is a 1D version of a PDW, in which uniform SC order parameter correlations fall exponentially with distance, while there exist charge $2e$ finite momentum ($2k_F$) superconducting correlations (Δ_{2k_F}) and uniform charge $4e$ (Δ_{4e}) correlations which exhibit power-law falloff of spatial correlations and a divergent $T=0$ susceptibility.

To begin with, we consider a highly asymmetric ladder—the Kondo-Heisenberg (KH) chain. Its 3D cousin is often used as a model of heavy-fermion systems. The 1D version consists of an interacting electron gas [a Luttinger liquid (LL) gapless in both its spin and charge sectors] and a spin chain (with exchange coupling J_H), coupled to each other by the Kondo exchange interaction J_K. The KH chain has been studied by many using diverse methods (Zachar, Kivelson, and Emery, 1996; Sikkema, Affleck, and White, 1997; Coleman *et al.*, 1999; Zachar, 2001; Zachar and Tsvelik, 2001; Berg, Fradkin, and Kivelson, 2010). From DMRG studies (Sikkema, Affleck, and White, 1997) it is known that there is a broad range of parameters J_H/J_K and electron densities in which the KH chain has a spin gap, corresponding to the "Kondo-singlet" regime of the heavy-fermion literature.

However, both DMRG stimulations (Berg, Fradkin, and Kivelson, 2010) and bosonized effective theories (Zachar, 2001; Zachar and Tsvelik, 2001; Berg, Fradkin, and Kivelson, 2010) show that in the spin-gap regime the correlators of all fermion bilinears are short ranged, including the Néel order parameter of the spin chain, the SDW order parameter of the LL, all the fermionic pair fields that describe possible uniform SC order parameters of the LL (both singlet and triplet), as well as the particle-hole CDW order parameters of the LL. Specifically, Berg, Fradkin, and Kivelson (2010) showed (using both DMRG simulations and bosonization) that the most prominent long-range SC correlations involve Δ_{4e} and a composite PDW order parameter of the form $\Delta_{2k_F} = \mathbf{\Delta} \cdot \mathbf{N}$, where $\mathbf{\Delta}$ is the uniform *triplet* SC order parameter of the LL and $\mathbf{N}$ is the Néel order parameter of the spin chain. The PDW order parameter inherits the ordering wave vector $Q = \pi/a$ (where a is the lattice spacing of the spin chain) of the short-range Néel correlations of the spin chain. Similarly, in this phase there are four-fermion CDW order parameters with power-law correlators. Thus, the Kondo-singlet regime of the KH chain is a PDW that cannot be described by a conventional condensation of Cooper pairs with finite momentum.

Turning to the more usual (symmetric) two-leg ladder, Jaefari and Fradkin (2012), extending the results and methods of Wu, Liu, and Fradkin (2003), found that, in addition to d-wave superconductivity coexisting with stripe charge order, these ladders also have commensurate PDW phases. In the weak-coupling regime, in which bosonization is most accurate, the PDW phases arise as follows: The electronic structure of a noninteracting two-leg ladder has a bonding and an antibonding band. For certain values of the electron density (and for strong enough repulsive interactions) the bonding band is at a commensurate filling and umklapp processes open a charge gap on that band. Except for the special case of a half-filled bonding band, for general commensurate filling this Mott insulator is a commensurate charge-density wave (with ordering wave vector $Q/2$) that coexists with a gapless spin sector whose low-energy behavior is that of a spin-$1/2$ antiferromagnetic chain with Néel quasi-long-range order characterized by wave vector Q. In this regime, the spin sectors of the bonding and antibonding bands are coupled by a Kondo-type exchange coupling, which is a marginally relevant perturbation that drives the ladder into a state with a full spin gap.

Thus, in this regime the low-energy degrees of freedom of the two-leg ladder are quite similar to those found in the KH chain. Indeed, Jaefari and Fradkin (2012) found two distinct SC phases: (a) a uniform d-wave SC which coexists with a CDW wave and (b) a commensurate PDW phase whose order parameters are composite operators of the uniform triplet superconductor of the antibonding band and the SDW (Néel) order parameter for the bonding band. Here, too, both bilinear order parameters are short ranged but their product has power-law correlations.

2. Occurrence of the PDW in mean-field theory

Inhomogeneous superconducting states closely related to the PDW have been found in several mean-field theories of superconducting states in strongly correlated systems (Himeda, Kato, and Ogata, 2002; Raczkowski *et al.*, 2007; Yang *et al.*, 2009; Loder, Kampf, and Kopp, 2010; Zelli, Kallin, and Berlinsky, 2011; Lee, 2014; Soto-Garrido and Fradkin, 2014). Loder, Kampf, and Kopp (2010) and Loder *et al.* (2011) recently showed that it is possible to obtain a PDW SC state (with and without an SDW component) using a BCS mean-field theory. Their work uses a $t-t'$ tight binding model with a realistic Fermi surface appropriate for the cuprate superconductors and an *attractive* nearest neighbor interaction V. Naturally, since no generic band structure has a divergent PDW susceptibility, Loder and co-workers found that this state occurs only for fairly large attractive interactions. On the other hand, for such large values of the attractive interaction, the applicability of the BCS mean-field theory is questionable. This problem is well known in the conventional Larkin-Ovchinnikov state (Larkin and Ovchinnikov, 1964) and the related Fulde-Ferrell time-reversal-breaking SC state (Fulde and Ferrell, 1964). [Similar issues arise in spin-imbalanced Fermi gases (Radzihovsky and Vishwanath, 2009; Radzihovsky, 2011).]

Putting these caveats aside, the results of Loder, Kampf, and Kopp (2010) and Loder *et al.* (2011) are quite suggestive.

As expected, for small V they found uniform d-wave superconductivity. However, in the vicinity of $x = 1/8$ hole doping, for a range of interactions $1 \lesssim V \lesssim 3$ they found a PDW phase which coexists with charge and spin-stripe order, while for $V \gtrsim 3$, an intertwined charge and spin-stripe state survives (without superconductivity). Similar results were found earlier in variational Monte Carlo simulations of the t-J model by Himeda, Kato, and Ogata (2002) and by Raczkowski *et al.* (2007), and in a mean-field theory of the t-J model by Yang *et al.* (2009).

A potentially significant feature of these mean-field results is the nature of the quasiparticle spectrum in the PDW state. In a PDW state without spin-stripe order, there is a "pseudo-Fermi surface" of Bogoliubov quasiparticles with multiple pockets of increasing complexity as the ordering period changes (Baruch and Orgad, 2008; Berg, Fradkin, and Kivelson, 2009b; Berg *et al.*, 2009; Loder *et al.*, 2011; Zelli, Kallin, and Berlinsky, 2011; Lee, 2014). However, the presence of even a small amount of uniform d-wave SC component gaps out all these pockets, leaving only the usual nodal states of a d-wave superconductor. Coexisting spin-stripe order apparently gaps the spectrum of Bogoliubov quasiparticles. Also interesting is the finding by Zelli, Kallin, and Berlinsky (2011, 2012) that the pockets of Bogoliubov quasiparticles of the PDW state can give rise to quantum oscillations of the magnetization in the mixed state of a PDW superconductor.

3. Amperian pairing and the PDW state

A recent mean-field theory by Lee (2014), based on the concept of Amperian pairing (Lee, Lee, and Senthil, 2007), found a PDW state with pockets of Bogoliubov quasiparticles. In this RVB approach (Baskaran, Zou, and Anderson, 1987; Lee, Nagaosa, and Wen, 2006), the electron is expressed as a composite of a spinless charged boson (the holon) and a charge neutral spin-1/2 fermion (the spinon). A necessary accompaniment to this decomposition is that the holons and spinons are coupled through a strongly fluctuating gauge field, typically with a U(1) gauge group. In the simplest version, known as the Baskaran-Zou-Anderson (BZA) mean-field state (Baskaran, Zou, and Anderson, 1987), this theory assumes that the spinons form a Fermi surface while the holons Bose condense. In this picture, the superconducting state arises once the spinons form pairs with d-wave symmetry and themselves condense (Kotliar, 1988). Amperian pairing refers to a pairing of spinons triggered by singular forward-scattering interactions mediated by the gauge bosons of this theory (Lee, Lee, and Senthil, 2007). The variational wave functions used by Himeda, Kato, and Ogata (2002) and by Raczkowski *et al.* (2007) in the context of the t-J model are in fact Gutzwiller-projected BZA mean-field states generalized to allow for periodically modulated pairing as in the PDW state. Lee (2014) argued that the RVB mean-field theory also allows for a spinon-pair condensate with finite momentum tied to the nesting wave vector of the mean-field theory spinon Fermi surface. From the perspective of broken symmetries, this state is identical to the PDW state. In agreement with what we advocate here, and in our earlier papers (Berg, Fradkin, and Kivelson, 2009a; Berg *et al.*, 2009), in Lee's theory the uniform SC and the PDW are taken to be the dominant orders, while other orders, e.g., charge-stripe order, are subdominant (see Sec. III). In the resulting PDW state, the original BZA Fermi surface is gapped out by the spinon condensate leaving behind pockets of charge neutral spinons. The emerging picture has qualitatively the same features as found in the Bogoliubov–de Gennes mean-field theory (Baruch and Orgad, 2008; Berg *et al.*, 2009). Specifically, Lee (2014) argued that this explains puzzling features of the observed angle-resolved photoemission spectroscopy (ARPES) spectra in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (He *et al.*, 2011).

4. Thermal stabilization of the PDW phase

The variational treatment of the t-J model discussed in Sec. IV.C shows that the ground-state energy of the PDW is very close to the true ground-state energy; no convincing evidence has been adduced that it is the true ground state. However, the PDW has larger low temperature entropy than any of the competing phases (Lee, 2014). Specifically, while any of the phases with a uniform d-wave component of the order parameter has, at most, nodal points in k space at which there are gapless quasiparticle excitations, the PDW supports Fermi pockets with a nonzero density of states. This gives rise to a contribution to the low temperature entropy of the PDW of the form $S_{\rm PDW} = \alpha_S x k_B T/\sqrt{\Delta_0 t} + \mathcal{O}(T^2)$, where α_S is a number of the order of 1 and Δ_0 is the scale of the antinodal gap; all other states have entropies that vanish at least in proportion to T^2. Even if the PDW phase has slightly higher energy density than the true ground-state phase by an amount $\Delta E \approx \alpha_E x t$, then for small enough α_E, the PDW will nonetheless be the equilibrium phase for a range of T above $T_- \approx \Delta_0\sqrt{\alpha_E/\alpha_S}[t/\Delta_0]^{3/4} \sim 0.03[t/\Delta_0]^{3/4}$. It is easy to envisage circumstances (especially where Δ_0/t is not too small), for which this transition occurs at temperatures above but comparable to the uniform $T_{\rm SC}$.

E. Superconductivity and nematic order

Charge nematic order has been seen in transport experiments in very underdoped samples of $YBa_2Cu_3O_{6+y}$ (YBCO) (Ando *et al.*, 2002), in inelastic neutron scattering on underdoped YBCO in the regime where there is no spin gap (Hinkov *et al.*, 2008), and above $T_{\rm SC}$ for a broad range of hole doping in Nernst effect measurements (Daou *et al.*, 2010). Scanning tunneling microscopy (STM) experiments documented nematic order on long length scales in underdoped $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Lawler *et al.*, 2010) and provided evidence for a nematic quantum critical point near optimal doping in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Fujita *et al.*, 2014), although this interpretation has been challenged (da Silva Neto *et al.*, 2013). Charge nematic order was conjectured by two of us in 1998 (Kivelson, Fradkin, and Emery, 1998), including the possible occurrence of such a nematic quantum critical point (QCP) inside the high T_c superconducting phase. For a recent review of electronic nematic order, see Fradkin *et al.* (2010).

However, the relation between charge nematic order and high temperature superconductivity has remained unclear. Part of the problem is that most theories of nematicity have been formulated either in terms of a Pomeranchuk instability in a

Fermi liquid (Oganesyan, Kivelson, and Fradkin, 2001) or in terms of the coupling of a Fermi liquid to a somehow preexisting nematic-order-parameter field theory (Metlitski and Sachdev, 2010), whereas the nematic state observed in experiments on high temperature superconductors is best described as a system of "fluctuating stripes" (Kivelson *et al.*, 2003), i.e., as a state in which stripe charge order has been melted (thermally or quantum mechanically) leaving a uniform nematic phase (Kivelson, Fradkin, and Emery, 1998). Nevertheless, several theories have been proposed which suggest that, inside the nematic phase, quantum nematic fluctuations may give rise to a SC state (Kee and Kim, 2004), and that superconductivity which is primarily the result of other pairing interactions (e.g., spin fluctuations) can be enhanced in the neighborhood of a nematic QCP (Lederer *et al.*, 2015; Metlitski *et al.*, 2015). The relation if any between nematicity and PDW order is even less clear. However, it has been shown recently that near a QCP of an (as-yet undetected) spin-triplet nematic state (Wu *et al.*, 2007) a host of SC states arise, including PDW phases (Soto-Garrido and Fradkin, 2014).

V. INTERTWINED ORDER IN THE CUPRATES

Here we review experimental evidence for many of the types of order that have already been discussed. In particular, we emphasize the close associations between various distinct orders.

A. SDW, SC, and PDW orders

Perhaps the most striking evidence of an intimate connection between SDW and SC orders is observed in the "214" cuprates derived from La_2CuO_4. For example, in $La_2CuO_{4+\delta}$ with stage-4 interstitial order (Lee *et al.*, 2004) SDW and SC orders both onset at 40 K in zero magnetic field. While the application of a c-axis magnetic field causes a reduction in T_{SC}, it changes T_{SDW} relatively little, although it enhances the SDW order. The story is similar in $La_{2-x}Sr_xCuO_4$ with $x = 0.10$, where the application of a c-axis magnetic field induces SDW order that onsets at the zero-field T_{SC} (Lake *et al.*, 2002).

The most compelling evidence of PDW order is found in $La_{2-x}Ba_xCuO_4$. Unlike the typical cuprate, the highest $T_c \approx 32$ K occurs at $x \approx 0.09$, and there is a deep minimum in the T_c vs x curve at $x = 1/8$ [the "1/8 anomaly"; see, e.g., Hücker *et al.* (2011)]. However, even at $x = 1/8$, quasi-2D SC correlations onset above 40 K (Li *et al.*, 2007; Tranquada *et al.*, 2008); the existence of such a 2D regime can be most readily understood as being a consequence of the frustration of interlayer Josephson coupling (Himeda, Kato, and Ogata, 2002; Berg *et al.*, 2007, 2009) associated with PDW order. Indeed, at $x = 1/8$ $La_{2-x}Ba_xCuO_4$ (LBCO) exhibits a cascade of phase transitions and crossovers, as shown in Fig. 2. In order of descending temperatures, this system has a charge order transition at $T_{CDW} = 54$ K where charge stripe (CDW) order onsets (roughly coinciding with a structural transition). At $T_{SDW} = 42$ K long-range SDW order onsets. As a function of decreasing T, ρ_{ab} drops by an order of magnitude just below T_{SDW}, then levels off before again dropping precipitously to immeasurably small values as T approaches $T_{KT} = 16$ K. Meanwhile, ρ_c begins to increase significantly below T_{CDW}, reaching a maximum value at around 30 K, and finally dropping precipitously toward zero as T approaches $T_{3D} = 10$ K. For $T_{KT} > T > T_{3D}$, LBCO behaves as if the layers were decoupled, leaving a 2D superconductor without detectable c-axis Josephson coupling. Moreover, the full (3D) Meissner state is observed only below $T_c = 4$ K.

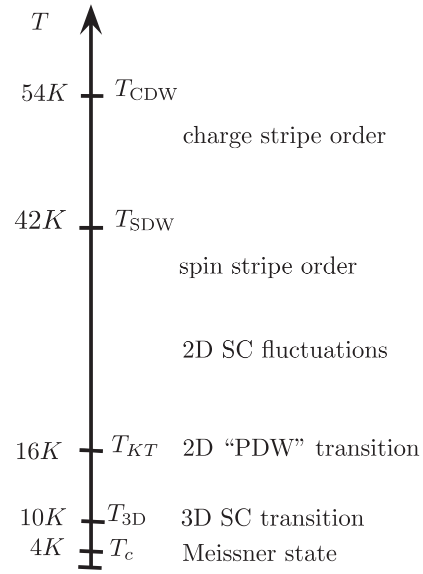

FIG. 2. Scales and phases of $La_{2-x}Ba_xCuO_4$ near $x = 1/8$: PDW fluctuations begin below the onset of charge order T_{CDW} and become pronounced below T_{SDW}. The presumably PDW 2D SC phase occurs below T_{KT}. The regime between T_{3D} and T_c is expected to be an XY glass, and the 3D d-wave SC lies below T_c. See text for details.

The first evidence for layer decoupling actually came from a study of c-axis optical conductivity in $La_{1.85-y}Nd_ySr_{0.15}CuO_4$ (Tajima *et al.*, 2001), where it was observed that the Josephson plasma resonance (JPR) essentially disappeared when the Nd concentration was tuned into the stripe-ordered regime. It was later demonstrated that the application of a c-axis magnetic field to superconducting $La_{1.9}Sr_{0.1}CuO_4$ causes a rapid reduction in the JPR frequency (Schafgans, LaForge *et al.*, 2010), while resulting in relatively little change in the in-plane superfluid density (Schafgans, Homes *et al.*, 2010).

Further evidence for a close association between CDW, SDW, and SC order comes from pump-probe studies. $La_{1.8-x}Eu_{0.2}Sr_xCuO_4$ has the same low temperature structure as $La_{2-x}Ba_xCuO_4$, exhibits CDW and SDW order, but has more strongly suppressed SC order (Fink *et al.*, 2011). Nevertheless, it has been demonstrated that pumping a crystal of $x = 1/8$ doping with a very short burst of 80-meV photons can induce the appearance of SC order, as indicated by probing the c-axis infrared reflectivity, in a time as short as 1 to 2 ps after the pump (Fausti *et al.*, 2011). In a related experiment (Först *et al.*, 2014) on $La_{2-x}Ba_xCuO_4$ with $x = 1/8$, it has now been shown that the pump causes the CDW order to melt within ~0.4 ps; the crystal symmetry shows a slight response, but only on a delayed time scale of

15 ps. The fast turn on of the bulk superconductivity in a sample with a stripe-ordered ground state suggests that strong superconducting correlations are also present in that ground state. The fact that the same pump melts the charge stripes is compatible with the idea that PDW order in the ground state frustrates the interlayer superconducting phase coherence. Disrupting the static CDW order also affects the PDW, removing the frustration and resulting in bulk superconductivity.

For the pump to be effective at inducing interlayer coherence in the $x = 0.125$ samples, the polarization must be parallel to the CuO_2 planes, within which it couples to Cu-O bond-stretching phonons. Intriguingly, a new experiment on $La_{2-x}Ba_xCuO_4$ with $x = 0.115$ demonstrates that a near-infrared pump pulse with polarization perpendicular to the planes can induce a c-axis JPR at a temperature as high as 45 K, well above the bulk T_{SC} of 13 K, but below T_{CDW} (Nicoletti *et al.*, 2014).

Continuing with $La_{2-x}Ba_xCuO_4$ but moving to $x = 0.095$, bulk SC onsets together with weak SDW order at 32 K (Wen *et al.*, 2012a). In this case, weak CDW order appears at the same temperature, constrained by the structural transition (Wen *et al.*, 2012b). Enhancement of SDW and CDW order by an applied magnetic field occurs at the expense of the SC order (Wen *et al.*, 2012a; Hücker *et al.*, 2013); nevertheless, the coincident onset temperatures indicate a close connection between these orders. In $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, there is evidence for a similar onset of quasi-2D superconductivity at a temperature well above the bulk T_{SC} (Ding *et al.*, 2008); however, the onset of SDW order occurs at a temperature ~20 K higher (Ichikawa *et al.*, 2000).

An interesting situation occurs in electrochemically oxygen-doped $La_{2-x}Sr_xCuO_{4+\delta}$ (Mohottala *et al.*, 2006). Here, for $x \lesssim 1/8$, the excess oxygen content appears to tune itself so that the net hole concentration is approximately $1/8$. Measurements by muon spin rotation (μSR) spectroscopy indicate that the volume is phase separated into superconducting and magnetically ordered regions (Mohottala *et al.*, 2006). Nevertheless, neutron diffraction measurements on crystals with $x = 0.04$, 0.065, and 0.09 found that SDW peaks appear simultaneously with the SC order at ~39 K for all samples (Udby *et al.*, 2013). Regardless of whether these orders are entirely segregated, their energy scales are remarkably similar.

Evidence of anomalous 2D superconducting fluctuations in $YBa_2Cu_3O_{6+y}$ exist, although their association with PDW order is far less clear than in $La_{2-x}Ba_xCuO_4$. Optical evidence of a JPR within a bilayer persisting to temperatures well above T_{SC} has been presented by Dubroka *et al.* (2011). Interestingly, there seems to be some correlation between the onset temperature of the JPR and the onset of CDW and/or nematic order. Recent magnetization studies in high fields by Yu *et al.* (2014) have likewise been interpreted as evidence of significant PDW correlations. Finally, pump-probe studies similar to those reported in $La_{2-x}Ba_xCuO_4$ have been carried out in $YBa_2Cu_3O_{6+y}$, with results that, while less clear-cut in terms of magnitude and persistence, are still reminiscent of the former (Hu *et al.*, 2014; Kaiser *et al.*, 2014). It is a key issue to determine if PDW-type correlations exist in YBCO and more generally in hole-doped cuprates. Interestingly, the latest experiment (Först *et al.*, 2014) on underdoped $YBa_2Cu_3O_{6.6}$ indicates that the pump conditions that enhance the coherent interlayer transport at $T > T_{SC}$ also depress the CDW order of the type that will be discussed in Sec. V.E.

B. CDW and SDW orders

For a number of 214 compounds, CDW order develops at a temperature that is generally higher than the SDW transition. For $La_{2-x}Ba_xCuO_4$ and $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, the CDW order is limited by a structural transition that breaks the effective fourfold symmetry of the Cu-O bonds (Axe and Crawford, 1994; Ichikawa *et al.*, 2000; Hücker *et al.*, 2011); however, in $La_{1.8-x}Eu_{0.2}Sr_xCuO_4$, where the structural transition takes place at $T > 120$ K, the maximum T_{CDW} is a modest 80 K (Fink *et al.*, 2011). The fact that $T_{CDW} > T_{SDW}$ indicates that the CDW order is not secondary to the SDW (Zachar, Kivelson, and Emery, 1998), in contrast to the situation in chromium (Pynn *et al.*, 1976). This does not mean that the SDW and CDW are not strongly correlated with one another—neutron-scattering experiments show that the SDW fluctuations become virtually gapless as soon as CDW order is established (Fujita *et al.*, 2004; Tranquada *et al.*, 2008).

In $La_{2-x}Sr_xCuO_4$, where the average crystal structure makes all Cu-O bonds equivalent, relatively strong SDW order is observed only for $x \approx 0.12$ (Yamamoto *et al.*, 1998; Kimura *et al.*, 2000). Nevertheless, a nuclear quadrupole resonance (NQR) study detected a pattern of intensity loss in the normal state of underdoped $La_{2-x}Sr_xCuO_4$ that matches the behavior observed in CDW-ordered 214 cuprates (Hunt *et al.*, 1999). While the direct cause of intensity loss is likely from SDW correlations (Julien *et al.*, 2001), CDW order has now been detected by x-ray diffraction for x near 0.12 and $T < 85$ K (Wu *et al.*, 2012; Christensen *et al.*, 2014; Croft *et al.*, 2014; Thampy *et al.*, 2014). Interestingly, both the SDW and CDW wave vectors are rotated ~3° from the Cu-O bond directions (Kimura *et al.*, 2000; Thampy *et al.*, 2014). From a symmetry perspective (Robertson *et al.*, 2006), this rotation is a necessary consequence of the incommensurate character of the SDW and the orthorhombicity of $La_{2-x}Sr_xCuO_4$. However, the magnitude of the effect surely reflects the fact that diagonal SDW order dominates for $x < 0.055$, where $La_{2-x}Sr_xCuO_4$ is insulating (Birgeneau *et al.*, 2006).[6] For $T < T_{SC}$, the application of a c-axis magnetic field enhances both the SDW and CDW order, in much the same way as in LBCO for dopings sufficiently far from $x = 1/8$ that stripe order is weak in zero field (Wen *et al.*, 2012a; Hücker *et al.*, 2013).

A significant feature of the CDW order in 214 cuprates is that the wave vector is locked to that of the SDW, with $\boldsymbol{q}_{CDW} = 2\boldsymbol{q}_{SDW}$. To distinguish this feature, we use the label CDW1 to denote it in comparisons below.

[6] Recall that in variational studies of the t-J model, insulating diagonal stripes were found to have energy only slightly larger than that of vertical stripes and so could be easily stabilized in an orthorhombic environment (Sec. IV.C).

C. CDW and SDW quantum critical points

Several studies have documented possible signatures of quantum critical behavior near the doping at which CDW order disappears in 214 systems (Doiron-Leyraud and Taillefer, 2012). In $La_{1.8-x}Eu_{0.2}Sr_xCuO_4$, μSR studies detected SDW order over a broad range of x, ending near $x_c \gtrsim 0.2$ (Klauss *et al.*, 2000). Oxygen-isotope effect (Suryadijaya, Sasagawa, and Takagi, 2005), Hall effect (Takeshita *et al.*, 2004), and resonant soft-x-ray diffraction (Fink *et al.*, 2011) studies suggest that CDW order disappears at a similar x_c. Superconductivity is observed for $0.14 < x < 0.27$, with T_{SC} forming a dome centered on $x \sim 0.21$ (Suryadijaya, Sasagawa, and Takagi, 2005). This looks similar to cases of SC order appearing at a quantum critical point, with the major difference that the magnitude of T_{SC} (20 K) is comparable to the maxima of $T_{SDW} \approx 27$ K and $T_{CDW} \approx 80$ K.

In $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, neutron and x-ray diffraction measurements suggest that there should be a similar x_c where stripe order disappears (Niemöller *et al.*, 1999; Ichikawa *et al.*, 2000), and the maximum of T_{SC} occurs at a similar position (Daou *et al.*, 2009b). Measurements of the in-plane resistivity in a magnetic field sufficient to suppress the superconductivity indicate linear-T behavior for $x = 0.24$ down to low temperature, but an upturn below 40 K for $x = 0.20$, with a related difference in the Hall effect (Daou *et al.*, 2009b). These behaviors, together with results on the thermopower, are consistent with the presence of a quantum critical point at $x = x_c$ (Daou *et al.*, 2009a).

D. CDW and nematic orders

Spectroscopic imaging scanning tunneling microscopy (SI-STM) has provided evidence for short-range CDW correlations in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Hoffman *et al.*, 2002; Howald *et al.*, 2003; Kohsaka *et al.*, 2007; Parker *et al.*, 2010; Fujita *et al.*, 2014; da Silva Neto *et al.*, 2014), $Bi_{2-y}Pb_ySr_{2-z}La_zCuO_{6+x}$ (Wise *et al.*, 2008), and $Ca_{2-x}Na_xCuO_2Cl_2$ (Kohsaka *et al.*, 2007). Besides having a short correlation length, another difference from the 214 cuprates is that the CDW wave vector decreases with doping (Wise *et al.*, 2008; da Silva Neto *et al.*, 2014), scaling roughly like the antinodal $2k_F$ measured by angle-resolved photoemission spectroscopy but larger in magnitude (Meng *et al.*, 2011; Comin *et al.*, 2014). Because of this difference, we denote the order as CDW2.

While the translational-symmetry breaking of the charge is local, a long-range rotational symmetry breaking associated with an electronic nematic state (Kivelson, Fradkin, and Emery, 1998) has been identified at low temperature in underdoped $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Lawler *et al.*, 2010; Mesaros *et al.*, 2011). [Note that this identification is not without controversy (da Silva Neto *et al.*, 2013).] With doping, the nematic order and CDW2 correlations both disappear at $x_c \approx 0.19$, the point at which the low temperature antinodal pseudogap appears to close (Tallon and Loram, 2001; Gor'kov and Teitel'baum, 2006; Vishik *et al.*, 2012; Fujita *et al.*, 2014). Studies of quantum oscillations in $YBa_2Cu_3O_{6+y}$ found a mass divergence very nearby, at $x_c \approx 0.18$, suggesting a QCP (Ramshaw *et al.*, 2015). Notably, T_{SC} and H_{c2} are maximized here.

While static AF order is not relevant to this behavior, there is an interesting connection with the energy range in which dynamic AF correlations remain strong. In the parent AF insulator phase, the spin waves are well defined (Headings *et al.*, 2010) because they exist at energies ($\lesssim$0.3 eV) far below the gap for charge excitations [$\sim$1.5 eV (Basov and Timusk, 2005)]. This situation changes with hole doping. Experimentally it has been observed that the momentum-integrated magnetic spectral weight remains comparable to that of the parent insulator for energies below the antinodal pseudogap energy, becoming much weaker above that scale (Stock *et al.*, 2010; M. Fujita *et al.*, 2012). With doping, the magnetic spectral weight close to the AF wave vector becomes quite weak for $x \gtrsim 0.2$ (M. Fujita *et al.*, 2012). Thus, there is at least a strong association between nematic order and AF spectral weight, consistent with the idea that related electronic textures are necessary to sustain even short-range AF correlations for substantial x.

To get a measure of the temperature scale associated with nematic order, we must turn to $YBa_2Cu_3O_{6+y}$, where a study of in-plane anisotropy in the Nernst effect has suggested that nematic order develops at a temperature comparable with T^* determined from in-plane resistivity (Daou *et al.*, 2010). Of course, in orthorhombic $YBa_2Cu_3O_{6+y}$, the fourfold symmetry of the planes is already broken by the presence of the Cu-O chains; nevertheless, the temperature dependence of the anisotropy in the Nernst effect is quite distinct from that of the orthorhombic strain. The relatively sharp onset (Xia *et al.*, 2008; He *et al.*, 2011; Karapetyan *et al.*, 2012) of a Kerr signal in multiple families of cuprates at similar temperatures to those at which charge order begins to be detectable is probably associated (Hosur *et al.*, 2013; Varma, 2014; Hosur *et al.*, 2015) with some pattern of point-group symmetry breaking, as well.

At the onset of the anisotropic behavior, the Nernst coefficient is found to be negative (Daou *et al.*, 2010). As the temperature drops and approaches T_{SC}, a positive contribution to the Nernst coefficient develops that is associated with superconducting fluctuations (Wang, Li, and Ong, 2006). In 214 cuprates, a positive contribution to the Nernst coefficient is also detected in association with CDW order (Cyr-Choiniere *et al.*, 2009; Hess *et al.*, 2010; Li *et al.*, 2011), although the magnitude of this contribution is depressed for $x \approx 1/8$. The trend is somewhat different in $YBa_2Cu_3O_{6+y}$ with $x = 0.12$ (Chang *et al.*, 2011), where suppression of the superconductivity with a strong magnetic field leaves the Nernst coefficient strongly negative. This occurs in the regime where various measures of CDW order have been reported (Wu *et al.*, 2011; Chang *et al.*, 2012; Ghiringhelli *et al.*, 2012), as we discuss next.

E. CDW and SC orders

In $YBa_2Cu_3O_{6+y}$, at least two CDW phase boundaries have been detected. Starting with the case of zero magnetic field, short-range CDW order onsets gradually below a transition temperature that has a maximum of $\sim$150 K for $x \sim 0.12$ (Achkar *et al.*, 2012; Chang *et al.*, 2012; Ghiringhelli *et al.*, 2012). The onset temperature and strength of the order both decrease as doping approaches the regime of quasistatic magnetic order at $x \lesssim 0.08$ and the regime of optimum doping at $x \gtrsim 0.14$ (Ghiringhelli *et al.*, 2012; Blackburn *et al.*, 2013b; Blanco-Canosa *et al.*, 2014; Hücker *et al.*, 2014). The intensity grows on approaching T_{SC} from above, and then decreases somewhat on cooling below T_{SC}. Application of a c-axis magnetic field enhances the CDW intensity for $T < T_{SC}$, but has no impact for $T > T_{SC}$ (Chang *et al.*, 2012; Blackburn *et al.*, 2013b). Initially, there were some questions as to whether the response detected by x-ray scattering might be dynamic, with integration over the fluctuations by coarse energy resolution; however, recent characterizations of the inelastic spectrum (Blackburn *et al.*, 2013a; Le Tacon *et al.*, 2014) and the comparative strength of the scattering (Thampy *et al.*, 2013) indicate that the CDW correlations are static. The fact that a broadening of the Cu and O NMR lines is seen (Wu *et al.*, 2015) with similar T dependence as the x-ray signal further corroborates the static character of the CDW correlations.

The general doping dependence of the CDW order, with a maximum at $x \sim 0.12$, and the enhancement by a magnetic field that also suppresses SC order, are similar to features observed in 214 cuprates. In contrast, there is a significant difference in the CDW wave vectors. First, there are distinct modulations along the principal axes, $\mathbf{q}_1 = (\delta_1, 0, 0.5)$ and $\mathbf{q}_2 = (0, \delta_2, 0.5)$, with δ_2 slightly greater than δ_1 and both having a magnitude close to 0.3 (Chang *et al.*, 2012; Ghiringhelli *et al.*, 2012; Blackburn *et al.*, 2013b). While the intensities of the CDW peaks at $\mathbf{q}_1$ and $\mathbf{q}_2$ can be comparable for a range of x, they can also be quite different, as in the phase with Cu-O chain order characterized as ortho-II (Blackburn *et al.*, 2013b; Blanco-Canosa *et al.*, 2013).

The second difference from 214 cuprates has to do with the doping dependence of the CDW wave vectors, which decrease with x. This behavior is similar to that seen by STM in, for example, $Bi_2Sr_2CaCu_2O_{8+\delta}$, and, in fact, x-ray scattering experiments have demonstrated CDW peaks in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Hashimoto *et al.*, 2014; da Silva Neto *et al.*, 2014) and $Bi_2Sr_{2-x}La_xCuO_{6+\delta}$ (Comin *et al.*, 2014) at wave vectors identical to those inferred from STM measurements on the same samples. As a result, this order corresponds to CDW2. Similar CDW2 order has also been detected in underdoped $HgBa_2CuO_{6+\delta}$ by x-ray scattering (Tabis *et al.*, 2014).

Incommensurate AF correlations have a finite excitation gap across most of the doping range where CDW order is observed (Dai *et al.*, 2001; Hinkov *et al.*, 2010). In fact, the spin gap appears to open on cooling at essentially the same temperature as T_{CDW}, based on nuclear magnetic resonance (NMR) measurements of the spin-lattice relaxation rate $1/T_1$ (Baek *et al.*, 2012; Hücker *et al.*, 2014). The wave vector of the lowest-energy AF correlations grows with x in a manner qualitatively similar to that seen in both $La_{2-x}Sr_xCuO_4$ and $Bi_{2-x}Sr_{2+x}CuO_{6+\delta}$ (Enoki *et al.*, 2013). Substitution of Zn into planar Cu sites in $YBa_2Cu_3O_{6+y}$ induces local, static, short-range SDW correlations while suppressing the CDW2 intensity (Blanco-Canosa *et al.*, 2013). Also, for $YBa_2Cu_3O_{6+y}$ with $x \lesssim 0.08$, where CDW2 order fades away, the spin gap collapses, and the low-energy spin correlations have uniaxially modulated short-ranged SDW character (Haug *et al.*, 2010).

Considering the similar doping ranges for CDW1 and CDW2, together with their opposite relationships with SDW order, as well as the universal presence of dynamic AF correlations (of correlated-insulator character) across the underdoped regime, it appears as if CDW1 and CDW2 are dual characters of the same underlying electronic texture, possibly nematic. However, only one of these aspects is realized at a time.

The current interest in CDW order was originally stimulated by observations of quantum oscillations in various transport properties measured in $YBa_2Cu_3O_{6+y}$ as a function of the magnetic field (Doiron-Leyraud *et al.*, 2007; LeBoeuf *et al.*, 2007; Sebastian *et al.*, 2008). The low observed oscillation frequency implies small pockets, presumably due to Fermi-surface reconstruction (Taillefer, 2009; Sebastian, Harrison, and Lonzarich, 2012), stimulating analyses in terms of stripe (Millis and Norman, 2007) and CDW order (Yao, Lee, and Kivelson, 2011); such a connection continues to influence the interpretation of quantum oscillation studies (Sebastian *et al.*, 2014). The temperature dependence of transport properties measured in high magnetic fields strongly resembles the low-field behavior of stripe-ordered systems (Laliberté *et al.*, 2011). Direct evidence for CDW order (CDW3, distinct from CDW2) at high magnetic fields and low temperature has been obtained through NMR measurements (Wu *et al.*, 2011, 2013b). In the CDW2 phase seen by x-ray scattering, NMR measurements detect broadening of NMR lines from planar O and Cu sites (Wu *et al.*, 2015), whereas, in the CDW3 phase, a line splitting develops above an onset field of at least 10 T. The onset temperature for CDW3 is comparable to the zero-field T_{SC}. Measurements of sound velocity (LeBoeuf *et al.*, 2013) suggest that there is a thermodynamic phase transition associated with CDW3; however, the transition fields determined by the sound velocity and NMR studies differ by a significant amount (17 vs 10 T, respectively). The relationship of CDW3 to CDW1 and CDW2 remains to be determined; however, the spin correlations remain gapped in CDW3, except near $x = 0.08$ (Wu *et al.*, 2011, 2013a).

VI. SUPERCONDUCTING VERSUS PSEUDOGAP

No issue in cuprate physics, it seems, has been more intensely debated than the nature and origin of the pseudogap Δ_{pg}. Many of the differing perspectives have been reviewed by others (Timusk and Statt, 1999; Tallon and Loram, 2001; Norman, Pines, and Kallin, 2005; Hüfner *et al.*, 2008). Here we summarize some of the empirical observations regarding superconducting and pseudogap behavior in the cuprates and then discuss how the concept of intertwined orders might provide a consistent interpretation.

A. Spectroscopic characterizations

Empirically, two characteristic energy gaps have been identified in the cuprates (Deutscher, 1999; Hüfner *et al.*, 2008). One of these, Δ_c, is a measure of the coherent

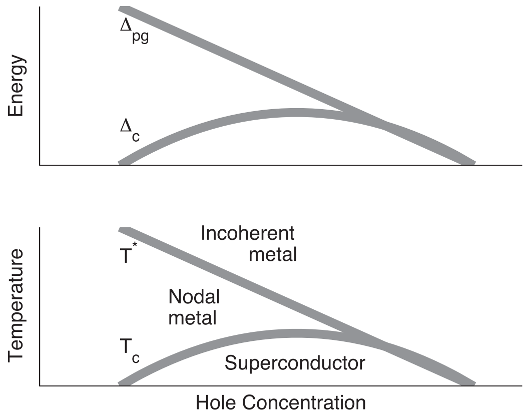

FIG. 3. Schematic diagram summarizing doping dependence of gaps and characteristic temperatures. Above optimal doping, the experimental identification of the pseudogap can differ depending on whether one is looking at measurements in the normal or superconducting state; the behavior shown is consistent temperature-dependent tunneling measurements. From Deutscher, 1999.

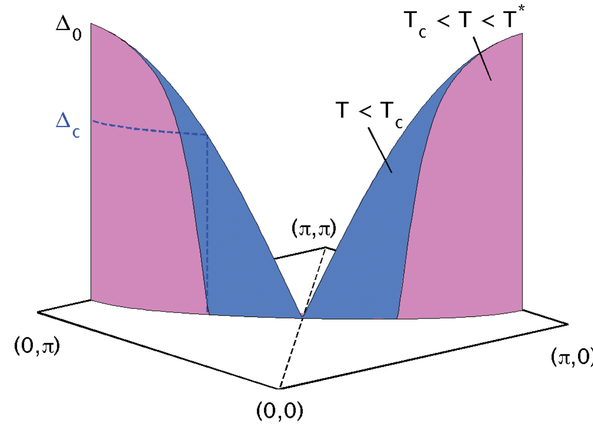

FIG. 4 (color online). Schematic diagram summarizing ARPES measurements of gaps around the Fermi surface in underdoped cuprates. The combined gray regions indicate the d-wave gap, with magnitude Δ_0, observed at $T < T_{SC}$. At T_{SC}, the dark gray part of the gap closes, leaving a gapless arc of states in the near-nodal region and a pseudogap (light gray) in the antinodal region. The pseudogap loses definition at $T \sim T^*$. The energy scale for coherent pairing Δ_c indicated by tunneling empirically corresponds to the magnitude of the d-wave gap at the wave vector corresponding to the end of the nodal arc.

superconducting gap, as obtained, for example, from Andreev reflection in point-contact spectroscopy with current flowing along an in-plane Cu-O bond direction. As a function of doping, Δ_c forms a dome, with $2\Delta_c/k_B T_{SC} \sim 4$–6 (Deutscher, 1999), as indicated schematically in Fig. 3 (top). The other gap, the pseudogap Δ_{pg}, can be measured by tunneling along the c axis. In the overdoped regime, $\Delta_{pg} \sim \Delta_c$, but as x decreases, Δ_{pg} grows monotonically (Deutscher, 1999; Hüfner *et al.*, 2008). Δ_c is detected only for $T < T_{SC}$, whereas Δ_{pg} can be observed up to $T \sim T^*$ (Timusk and Statt, 1999; Hüfner *et al.*, 2008), as indicated in Fig. 3 (bottom).

Further information on the gaps and their relationship is provided by ARPES (Damascelli, Shen, and Hussain, 2003; Hashimoto, Vishik *et al.*, 2014), much of which has been done on $Bi_2Sr_2CaCu_2O_{8+\delta}$, because of its excellent cleavability. Spectroscopic imaging with STM provides further information (Fischer *et al.*, 2007; K. Fujita *et al.*, 2012). Where ARPES averages over a substantial surface area, STM is able to map the microscopic variation of states.

With the possible exception of samples with $x \lesssim 0.08$, ARPES experiments on $Bi_2Sr_2CaCu_2O_{8+\delta}$ deep in the superconducting state ($T \ll T_{SC}$) exhibit a Fermi-surface gap with d-wave symmetry. For nearly optimal doping, the gap has the simple angular dependence $\Delta(\mathbf{k}) \approx \Delta_0[\cos(k_x) - \cos(k_y)]$ as indicated in Fig. 4. While the near-nodal gap appears to be approximately x independent for a broad range of doping below optimal (Vishik *et al.*, 2012; da Silva Neto *et al.*, 2013), the antinodal gap increases with decreasing x. Sharp "quasiparticle" peaks with energy $\Delta(\mathbf{k})$ and width small compared to the antinodal gap energy exist along the entire Fermi surface, again with the possible exception of highly underdoped samples (Vishik *et al.*, 2012; Zhao *et al.*, 2013). Likewise, in single-layer $(Bi,Pb)_2(Sr,La)_2CuO_{6+\delta}$, quasiparticle peaks are harder to identify, but by looking at the difference between the spectra above and below T_c, coherent quasi-particle-like features have been identified around the entire Fermi surface, but only for doping above optimal (Kondo *et al.*, 2009). However, for $x \lesssim 0.18$ a rather different picture emerges from an analysis of the quasiparticle interference (QPI) signal measured in STM. From this it is inferred that there are no coherent quasiparticles in the antinodal regime (beyond the AF Brillouin zone boundary) and, moreover, that there is effectively a nodal arc—despite the fact that one is deep in the superconducting state (Fujita *et al.*, 2014; He *et al.*, 2014).

For underdoped and even slightly overdoped samples, although the gap in the antinodal portion of the Fermi surface unambiguously persists (Ding *et al.*, 1996; Loeser *et al.*, 1996; Kanigel *et al.*, 2006) for a range of temperatures above T_{SC} and below T^*, simple measures indicate that the gap closes at T_{SC} along a finite "Fermi arc" (Norman *et al.*, 1998) in the near-nodal region, as indicated in Fig. 4. The antinodal Δ_0 thus corresponds to Δ_{pg}, while the coherent gap Δ_c seemingly corresponds to $\Delta(k)$ at the wave vectors corresponding to the ends of the normal-state Fermi arc (Pushp *et al.*, 2009; Rameau *et al.*, 2011; Reber *et al.*, 2013). However, the arc ends are probably not very well defined; indeed, it is unclear whether the arcs are produced by the vanishing of the nodal gap (Lee *et al.*, 2007), or simply indicate the portion of the Fermi surface in which the scattering rate is greater than the gap (Kondo *et al.*, 2013; Reber *et al.*, 2013). The near-nodal scattering rate appears to be strongly T dependent near T_c in this analysis. Indeed, while the energy of the antinodal quasiparticles does not change to any detectable extent upon approach to T_c, the coherent spectral weight [i.e., the peak in the spectral function at $\Delta(\mathbf{k})$] vanishes at T_c or slightly above (Fedorov *et al.*, 1999).

As mentioned previously, there are significant differences in aspects of the quasiparticle spectrum measured in ARPES and those inferred from the QPI analysis of the STM spectrum. Reconciling these, which we will not attempt here, is a major open issue. Still, in the superconducting state, STM shows that, in real space, the gap behavior is spatially uniform for $E < \Delta_c$, but inhomogeneous at higher energies, especially near Δ_{pg} (Howald, Fournier, and Kapitulnik, 2001; Pan *et al.*, 2001; Pushp *et al.*, 2009).

B. Temperature dependence and broader context

The temperature dependence of the pseudogap can be somewhat ambiguous, so it is instructive to consider results from a variety of experimental probes. For example, c-axis optical conductivity provides a useful probe of Δ_{pg}, as the contribution from quasiparticles in the nodal region is suppressed. The temperature dependence of $\sigma_c(\omega)$ in underdoped $YBa_2Cu_3O_{6+y}$ suggests that Δ_{pg} does not change but rather the states gradually fill in as the temperature approaches T^* (Homes *et al.*, 1993). A similar impression is given by an analysis of the temperature dependence of the Hall constant in $La_{2-x}Sr_xCuO_4$ (Gor'kov and Teitel'baum, 2006). Fitting with a temperature-independent term (from near-nodal carriers) plus a thermally activated component yields a temperature-independent gap that is quantitatively consistent with Δ_{pg} determined by ARPES. STM measurements indicate that the pseudogap is detectable (with a spatially varying magnitude) in the entire field of view for a significant range of $T > T_{SC}$, but becomes confined to increasingly rare regions as the temperature approaches T^* (Gomes *et al.*, 2007). In ARPES, the pseudogap onsetting at T^* has long been associated with the antinodal region. [For recent references, see He *et al.* (2011) and Kondo *et al.* (2013).]

Most of the spectroscopic measures of the pseudogap contain no direct information on the origin of the gap. There are a few measurements that provide evidence for pairing correlations. One of the key features of superconducting pairs is that they involve a mixing of particle and hole states. This results in particle-hole symmetry near the Fermi energy. ARPES measurements of Yang *et al.* (2008, 2011) on underdoped $Bi_2Sr_2CaCu_2O_{8+\delta}$ found evidence of such particle-hole symmetry for antinodal states far above T_{SC}. The mixing of particle and hole states also impacts the dispersion of the particle states, and this characteristic dispersion has been seen for antinodal states above T_{SC} in the same system (Kanigel *et al.*, 2008). On the other hand, ARPES studies of single-layer $Bi_{1.5}Pb_{0.55}Sr_{1.6}La_{0.4}CuO_{6+\delta}$ (Hashimoto *et al.*, 2010) found significant particle-hole asymmetry in a similar analysis. In underdoped $YBa_2Cu_3O_{6+y}$, a different signature of superconducting correlations in the normal state has been reported. Measurements of optical conductivity for light polarized along the c axis found indications of a transverse Josephson plasma resonance at temperatures as high as 180 K (Dubroka *et al.*, 2011), and enhanced coherent transport at $T \gg T_{SC}$ has been achieved by optically pumping apical oxygen vibrations (Hu *et al.*, 2014).

It is also relevant to note that there is another significant gap in the problem. The undoped cuprate parent compounds are charge-transfer insulators, with an optical gap ($\sim$1.5 eV) limited by the energy difference between Cu $3d_{x^2-y^2}$ and O $2p_\sigma$ states rather than the larger on-site Coulomb repulsion U that is responsible for magnetic moments on the Cu sites (Zaanen, Sawatzky, and Allen, 1985; Emery, 1987). On doping holes into the CuO_2 planes, the optical gap does not collapse; rather, infrared reflectivity studies demonstrate the coexistence of the charge-transfer gap with finite optical conductivity that is transferred into the gap. Integrating the conductivity within the gap, the effective carrier density grows in proportion to x (Cooper *et al.*, 1990; Uchida *et al.*, 1991), rather than $1 - x$ as predicted by conventional band theory. At high temperatures, the conduction electrons are completely incoherent, as indicated by the absence of a Drude peak in the optical conductivity (Takeya *et al.*, 2002; Lee *et al.*, 2005). A Drude peak develops on cooling, with a weight that is proportional to, but smaller than, x within the underdoped regime (Uchida *et al.*, 1991; Padilla *et al.*, 2005). The Drude peak is likely associated with near-nodal states and develops as the pseudogap becomes apparent in the antinodal region; hence, the region $T_{SC} < T < T^*$ is labeled "nodal metal" in Fig. 3 (bottom). Of course, these features are associated with energy scales much smaller than the charge-transfer gap and, especially, U.

C. Interpreting the pseudogap

The extent to which this gap, especially at temperatures above T_{SC}, reflects "d-wave superconducting pairing without phase coherence" versus a distinct "second gap" associated with some other ordering phenomenon (or "Mottness") has been endlessly debated. This is commonly referred to in the literature as the one gap versus two gaps dichotomy. We add to this debate in Sec. VIII. Here we note that a key issue is whether there exists a crossover temperature below which the amplitude of the order parameter (or parameters) is well defined. More specifically, can one phenomenologically associate with an experimentally determined pseudogap temperature T^* in the phase diagram of the cuprates a crossover scale below which many of the spectroscopic characteristics of an ordered phase begin to be apparent, but without any associated long-distance correlations, much less a broken symmetry? In theory, it is sometimes possible to identify a crossover temperature $T_{mf} > T_{SC}$ at which a local amplitude of the order parameter develops; often T_{mf} is identified with a mean-field transition temperature calculated in one way or another. This is possible, for instance, in an array of weakly Josephson coupled superconducting grains. However, it is important to realize that in generic problems neither the notion of a locally defined amplitude of an order parameter nor T_{mf} are well defined even in theory.

From the studies of underdoped samples, there are certainly features that suggest two distinct gaps. The near-nodal gap seen in ARPES is undoubtedly a superconducting gap, given its singular angle dependence and the fact that it closes, in some sense, more or less at T_{SC}. Conversely, therefore, it is natural to posit that the antinodal gap, which hardly varies as T crosses T_{SC}, is not a superconducting gap. Moreover, the x dependence of Δ_c contrasts markedly with that of Δ_{pg}. Finally, from STM measurements, it appears that Δ_{pg} tends to be largest in local regions in which evidence of CDW order is strongest, and weakest where CDW correlations are weak (Kohsaka *et al.*, 2008), inviting an association between the pseudogap and CDW order.

In contrast, the argument for a gap with a single origin begins at optimal doping. The simple d-wave form of the gap at $T \ll T_{SC}$, and the fact that even the energy widths of the quasiparticle peaks do not vary substantially as a function of position along the Fermi surface, make a very compelling case that it is a uniform d-wave superconducting gap. However, in $Bi_2Sr_2CaCu_2O_{8+\delta}$, even for $x = x_{opt}$, the antinodal gap survives to $T > T_{SC}$ without any significant change in magnitude, i.e., it becomes the pseudogap. If it is a superconducting gap below T_{SC}, and its magnitude remains roughly the same above T_{SC}, it is difficult to imagine it is unrelated to superconducting pairing.

Returning to low T, where the antinodal gap increases with decreasing x, one might be tempted to associate the "extra" gap size with the growth of a second order parameter. Assuming the two gaps add in quadrature, this would mean that $\Delta_{pg} = \sqrt{|\Delta_0|^2 + |\Delta_{other}|^2}$, which still implies that the largest contribution to the gap comes from superconductivity as long as $\Delta_{pg} < \sqrt{2}\Delta_0$, as is true for all but the smallest values of x. From a theoretical perspective, in the weak-coupling limit the superconducting instability would be strongly suppressed and even the dominant pairing symmetry would be changed were Δ_{pg} entirely associated with a partial gapping of the Fermi surface produced by a nonsuperconducting order (Cho *et al.*, 2013; Mishra *et al.*, 2014).

In a loose sense, an interplay between uniform d-wave superconductivity and a PDW is precisely what is needed to account for the one-gap, two-gap dichotomy: On the one hand, both orders imply local d-wave-like pairing. On the other hand, globally they produce distinct patterns of broken symmetry, and, in particular, the PDW has an associated CDW component. Indeed, as has been noted (Baruch and Orgad, 2008; Berg, Chen, and Kivelson, 2008), the PDW state results in a single-particle spectrum that resembles that seen in the pseudogap phase immediately above T_{SC}, with a relatively large gap in the antinodal regions (see Fig. 4), and a Fermi pocket in the nodal region whose back side [for reasons that were already clear in early d-density-wave (DDW) calculations (Chakravarty and Kee, 2008) of the same quantities] has relatively little spectral weight, giving it the appearance of a Fermi arc.

As discussed previously, the strongest evidence of PDW order comes from transport measurements in roughly $1/8$ doped $La_{2-x}Ba_xCuO_4$, for which fortunately ARPES data are available (Valla *et al.*, 2006; He *et al.*, 2009). The ARPES measurements show a clear antinodal gap that changes little on warming into the disordered state; however, below 40 K, there is also a d-wave-like gap along the nodal arc (although no coherent quasiparticle peaks are seen). In a weak-coupling analysis, the SDW order that is present would not cause a gap along the near-nodal arc (Baruch and Orgad, 2008), so that one would be led to conclude that the near-nodal gap is due to the existence of a uniform d-wave component of the order. This, then, creates a problem for the explanation of the frustrated interlayer Josephson coupling. However, the SDW order involves substantial Cu moments (Luke *et al.*, 1991; Hücker, Gu, and Tranquada, 2008) and so cannot be properly described in a weak-coupling picture. Turning to another experimental example, a d-wave-like gap (plus a uniform energy offset) has been reported in an ARPES study of $Bi_2Sr_{2-y}La_yCuO_{6+\delta}$ for hole concentrations $x < 0.10$ (Peng *et al.*, 2013). This regime is insulating (Ono and Ando, 2003) and a neutron-scattering study found evidence for diagonal spin-stripe correlations for similar dopings in the closely related $Bi_{2+y}Sr_{2-y}CuO_{6+\delta}$ (Enoki *et al.*, 2013). Hence, there is circumstantial evidence that SDW order may cause an (incoherent) d-wave-like gap along the near-nodal arc. However, further work is needed to resolve these issues.

VII. DETECTABLE SIGNATURES OF PDW ORDER

To date, the evidence of the existence of a PDW state, even in $La_{2-x}Ba_xCuO_4$, is indirect. Indeed, even evidence of the existence of conceptually similar Fulde-Ferrell-Larkin-Ovchinnikov states in partially magnetized superconductors has been challenging to obtain (Mayaffre *et al.*, 2014). There are, however, a number of clear experimental signatures which, if observed, would constitute unambiguous evidence of the existence of a PDW. These signatures have been discussed previously by us in some detail in Berg *et al.* (2009); here, for completeness, we briefly enumerate some of the most promising ideas.

(1) Existence of a uniform charge $4e$ condensate: Since in most geometries the coupling between the PDW order itself and any external superconductor vanishes when spatially averaged, the leading harmonic in the Josephson effects will also vanish. However, higher order (four-electron) tunneling processes will always produce phase locking to the accompanying composite charge $4e$ components $\Delta_{4e,a}$ of the PDW. There are several ways this can be detected, for instance, in any sort of experiment involving Josephson oscillations in which the relevant Josephson relation is $\hbar\omega = 4eV$, or if a sliver of $La_{2-x}Ba_xCuO_4$ is used as the weak link in a superconducting quantum interference device (SQUID) ring of a conventional superconductor, then the relevant flux quantum should be $hc/4e = \phi_0/2$.

(2) Tests of interlayer frustration: The frustration of the interlayer Josephson coupling apparent in transport and optics in $La_{2-x}Ba_xCuO_4$ and other candidate cuprates is the strongest existing evidence of a PDW state. This interpretation can be tested by purposeful perturbations which change the symmetry of the PDW state in such a way as to reduce the interlayer frustration. As already mentioned, the photoinduced onset of interlayer coherence observed in association with the transient melting of the CDW order may already be an example of such a test. Other proposed tests involve relieving the interlayer frustration through the application of a suitably oriented in-plane magnetic field (Yang, 2013), or possibly by reorienting the stripes by the application of uniaxial strain.

(3) Half-quantum vortices: One of the defining features of the PDW is the intertwining of the CDW and SC components of the order parameter. Since the pairing field must be a single-valued function of position, the phase of the order parameter must be single-valued modulo 2π. As a consequence, it is easy to show that wherever there is a dislocation in the CDW order, there must be an accompanying half-quantum vortex—a

vortex that can be viewed as the fundamental vortex of the uniform charge $4e$ superconducting order. This is one of the possible topological defects in a PDW, the one with characteristics that are a unique reflection of the broken symmetry.

(4) CDW $1Q$ order: As stressed, a composite CDW order with wave vector $\mathbf{K}_a = 2\mathbf{Q}_a$ should be detectable by STM and/or x-ray (or indirectly by neutron) diffraction whenever a PDW with wave vector $\mathbf{Q}_a$ is present.[7] However, if below a critical temperature T_c a uniform SC component develops as well (and if the PDW is not entirely quenched at the same point, which would be possible only if the transition were first order) then below T_c a *subharmonic* component of the CDW order parameter with ordering vector $\mathbf{K}'_a = \mathbf{Q}_a$ should develop with an amplitude proportional to the uniform SC order parameter.

VIII. DISCUSSION

Generically, a critical point is reached by varying a single parameter T for instance in the theories we have been discussing. A multicritical point can arise when a second parameter is varied in such a way as to cause two lines of second order transitions to intersect.

Competing orders generically refers to the physics close to a multicritical point. In terms of a Landau-Ginzburg-Wilson effective field theory, the multicritical point occurs where the mean-field ordering temperatures T_a^0 of two or more orders (labeled by an index a) are equal. That they "compete" implies that the appropriate biquadratic terms [e.g., in Eq. (3.1), γ_j with $j = 1$–3] are positive. Under still more highly fine-tuned circumstances, a multicritical point can exhibit a higher symmetry. For instance, the tetracritical point at which $T_{\rm CDW} = T_{\rm SC}$ in Fig. 1 occurs at a unique temperature and doping concentration T_P and x_P; this tetracritical point could have a larger $O(4)$ symmetry unifying the CDW and SC orders, but this requires fine-tuning at least one additional parameter.

Intertwined orders refers to the case in which $T_{\rm SC}^0 \sim T_{\rm CDW}^0$ over a range of situations, i.e., in the case of the cuprates, over a range of doping concentrations and material families. Where this occurs, it must have its origin in a feature of the microscopic physics as it has no natural explanation simply in terms of robust and generic features of coupled order parameters. Indeed, any such observation carries with it the suggestion that the same features of the microscopic physics that are responsible for one order also give rise to the other. In other words, there likely exists a high-energy scale at which an amplitude of the order develops which cannot really be associated with one or the other order, as it is somehow a precursor to all of them. Then, at lower scales, small energy effects favor one or the other pattern of ordering.

In particular, we have in mind the notion that below T^*, to an extent that varies smoothly as a function of x, local pairing correlations as well as local CDW and antiferromagnetic spin correlations begin to grow, in much the way CDW and SC correlations develop below $T^* \sim \Delta_s$ in the quasi-1D model analyzed in Sec. VIII.B. However, as we now discuss, there are good reasons to question the validity of this perspective.

A. Identification of T^* with local pairing critiqued

That there is a degree of local CDW order over a considerable portion of the pseudogap regime of the phase diagram of the cuprates is now an established fact. Moreover, as discussed extensively in Secs. III.B and VI, the idea that there are local superconducting pairing correlations which are in one way or another responsible for the d-wave character of the pseudogap and various subtle precursor indications of superconductivity without long-range phase coherence is one that has been advocated in various versions. Thus, the idea of identifying the pseudogap temperature T^* with the onset of a generalized order parameter amplitude which has both CDW and SC (and possibly AF) character is surely appealing. In one way or another, it underlies most attempts to come to grips with the phenomenology of intertwined orders.

However, there have been at least as many papers which have critiqued this idea (or supposedly "proven" it false). For the most part, these papers studied a property which might be expected to exhibit a signature of local SC (or other) order; then correcting for other contributions (which can, itself, require a rather complicated analysis) they infer from the lack of a clear fluctuational contribution that the conjectured local order does not exist. For example, constraints on the existence of pairing without phase coherence at temperatures well above $T_{\rm SC}$ have been adduced from measurements of terahertz conductivity (Bilbro *et al.*, 2011), Nernst effect (Wang, Li, and Ong, 2006), and magnetoresistance (Rullier-Albenque, Alloul, and Rikken, 2011). These constraints are serious; on the other hand, as far as we know, there exists no reliable microscopic theory capable of making even semiquantitative predictions of the consequences of local pairing (or even of sharply defining what this means). Thus, in critiquing this idea, we instead focus on rather broad phenomenological features of the phase diagram.

For the most part, the pseudogap temperature T^* is defined in one of several somewhat arbitrary ways to extract an explicit number to characterize the scale of T at which the behavior of a particular measured quantity changes from one sort to another. There thus may be cause to question particular values or even trends in the canonical T^* curves. For the purpose of discussion, we set aside these ambiguities and accept the canonical curves, originally defined such that $T^*(x)$ is the local maximum of the magnetic susceptibility χ, as being representative of the pseudogap crossover scale.

With this identification, a glaring issue is apparent. At $x = 0$, T^* clearly reflects the local growth of antiferromagnetic correlations; in the 2D AF Heisenberg model, χ has a maximum at $T^* \sim J/2$, where the AF correlation length is around two lattice constants (Chakravarty, Halperin, and Nelson, 1988). Given that T^* appears to be a continuous function of x, for small x this identification must remain valid, implying that T^* has nothing to do with the onset of either SC or CDW correlations. Conversely, in many of the hole-doped cuprates for $x \gtrsim 0.1$, while there is still evidence of AF

[7]Modulations in the SC gap can be indirectly observed in STM (in the absence of Josephson tunneling). Evidence of such modulations was given by Fang *et al.* (Fang *et al.*, 2004; Baruch and Orgad, 2008).

tendencies (local moments) at short distances, there is an associated spin gap which grows in magnitude as T^* decreases, making any direct association between T^* and AF order appear unnatural; on the other hand, it is in the range of doping near $x = 1/8$ that the best evidence exists for both short-range CDW order and local SC correlations.

Much of the trouble may come from a naive expectation that a state with short-range order behaves something like a corresponding ordered phase if probed at short distances and times. Conceptually, near a critical point, the correlations at distances small compared to the correlation length but large compared to the lattice constant look critical, i.e., the properties are neither those of the ordered nor of the disordered phase (Kivelson *et al.*, 2003). More specifically, as summarized by example below, the issue of what is meant by a local magnitude of an order parameter can be much more subtle than any naive intuition would suggest. However, excuses aside, whether or not local pairing (or singlet formation) onsets in any well-defined sense at temperatures of order T^* is clearly one of the central unresolved issues in the field.

B. Phase diagrams of intertwined orders

The physics of strong correlations is largely solved in 1D, and much can be said about the problem in quasi-1D. It is thus worthwhile summarizing features of this limit. In Fig. 5, we show the generic phase diagram for an array of weakly coupled LE liquids—a more generic version of the phase diagram already discussed in Fig. 1. Here K_c is the charge Luttinger exponent within a single LE chain, which (along with the spin gap Δ_s) is an appropriate long-distance measure of the nature of the intrachain interactions. There is much here that is reminiscent of the phase diagram of the cuprates. At high T there is a non-Fermi-liquid normal state—in this case it is a set of nearly decoupled LLs. There is a crossover scale $T^* \sim \Delta_s$ below which a pseudogap opens in the spin fluctuation spectrum. Finally, at low T, there is a complex phase diagram in which two ordered phases compete and possibly coexist. In this problem, the ordering temperatures are well separated from T^*, as they are proportional to a positive power of the interchain couplings, and so are (by construction) parametrically small.

One aspect of this problem that is worth emphasizing is that both the SC and CDW susceptibilities are typically small for $T > T^*$ and then begin to grow rapidly for $T < T^*$

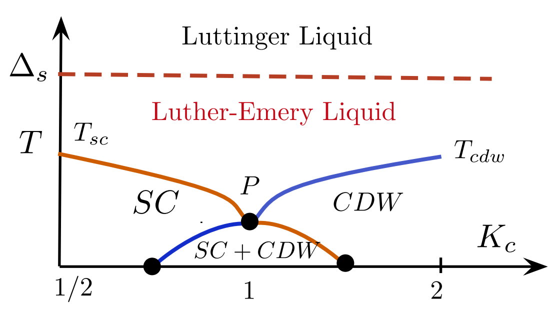

FIG. 5 (color online). Sketch of the phase diagram with SC and CDW orders in quasi-1D strongly correlated systems with a spin gap Δ_s. P is a multicritical point.

(as discussed in Sec. IV.B). Moreover, in this range of T, both susceptibilities are proportional to Δ_s, so there is no obvious way to more correctly identify Δ_s as a precursor pairing or CDW gap. Rather, the LE liquid is a precursor state in which the opening of the spin gap is conducive to both CDW and SC order.

Turning from the microscopic to more phenomenological macroscopic considerations, there is a well-defined sense in which it is more natural to have multiple orders coexisting in 2D (or quasi-2D) than in higher dimension. For instance, the natural minimal quasi-2D theory of a superconducting and an unidirectional incommensurate CDW is obtained by the usual assumption of the Kosterlitz-Thouless (KT) theory (Chaikin and Lubensky, 1995): At low temperatures the amplitudes of the order parameters vary smoothly as a function of parameters, while the phase fields θ of the SC order parameter, and ϕ of the sliding incommensurate CDW order parameter, have strong thermal fluctuations. In this limit the free energy density with $U(1) \times U(1)$ symmetry is a quadratic functional of the gradients of θ and ϕ. Instead of a fixed point, this system has a fixed "plane" parametrized by T/ρ_s and κ/ρ_s. Since gauge invariance forbids any coupling between the two phase fields, vortices and dislocations proliferate independently from each other and their respective Kosterlitz-Thouless critical temperatures are $T_{SC} \simeq (2/\pi)\rho_s$ and $T_{CDW} \simeq (2/\pi)\kappa$.

An analysis of this type was done for the thermal melting of the PDW state by Berg, Fradkin, and Kivelson (2009a). The general setting was discussed in Sec. III.B. To simplify matters, here we restrict ourselves to the case of unidirectional PDW order along the x axis (i.e., in a nematic phase). The thermal fluctuations are now controlled by the effective NLSM in Eq. (3.6) with the terms corresponding to the y component of the PDW (θ_y and ϕ_y) set equal to zero. There are thus three phase fields: θ_0 of the uniform d-wave SC, and θ_x and ϕ_x for the SC and CDW components of the x directed PDW. The corresponding stiffnesses (ignoring anisotropies) are $\rho_s \equiv K_0$, $\rho_{PDW} = K_1 = K_2$, and $\kappa = K_3 = K_4$. Note that V, when relevant, locks the phase θ_0 to θ_x (mod π).

There are now four types of topological excitations: (a) the vortex of the d-wave SC phase θ_0, (b) the vortex of the PDW phase θ_x, (c) the half-vortex of θ_x bound to a single dislocation of ϕ_x, and (d) the double dislocation of the CDW phase field ϕ_x. Various sequences of vortex unbinding can lead to a variety of complex phase diagrams; an example is sketched in Fig. 6 for the case where ρ_s is assumed to be small compared to ρ_{PDW} and κ. The considerations [which are described in more detail in Berg, Fradkin, and Kivelson (2009a)] which lead to this phase diagram are as follows: (a) At low temperature, there is a fully ordered "striped SC," in which θ_0 and θ_x are locked to each other, and there is coexisting CDW order. (b) By assumption, the uniform SC order melts above a low temperature T_d, where the θ_0 vortices unbind through a KT transition. Above T_d, the large fluctuations of θ_0 render V irrelevant as well. The resulting phase is a pure PDW. (c) If κ/ρ_{PDW} is large, the next transition involves the proliferation of θ_x vortices, leading to a KT transition to a pure CDW phase. Then, at still higher temperatures, the proliferation of ϕ_x dislocations leads to a uniform nonsuperconducting nematic phase. (d) If κ/ρ_{PDW} is small, the first

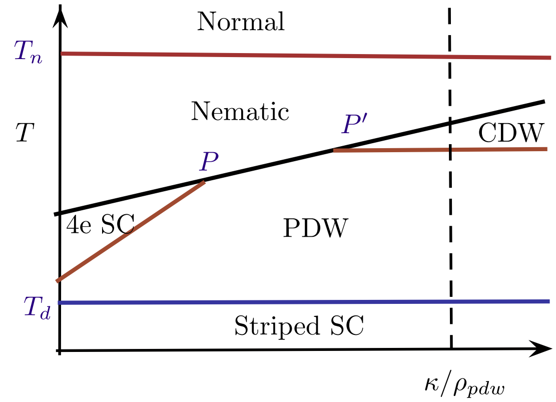

FIG. 6 (color online). Qualitative phase diagram for the melting of a unidirectional PDW state coexisting with d-wave SC order. See text for details. $La_{2-x}Ba_xCuO_4$ is presumably along the dark broken vertical line.

transition from the PDW phase involves the proliferation of ϕ_x double dislocations, resulting in a KT transition to a uniform superconducting phase—one, however, with the charge $4e$ condensate from Eq. (3.2). Then, at higher temperatures, the proliferation of θ_x half-vortices (reflecting the charge of the condensate) leads again to the uniform nonsuperconducting nematic phase. (e) At intermediate values of $\kappa/\rho_{\mathrm{PDW}}$, the lowest-energy topological excitations are the bound state of the half-vortex and a single dislocation, the proliferation of which results in a direct KT transition to the pure nematic phase. In all cases, at still higher temperatures, the proliferation of disclinations (not included in the NLSM we have discussed) will lead to an Ising transition from the nematic phase to a uniform normal phase, as indicated by T_n in Fig. 6.

Some details of this pattern may be altered significantly by changing parameters [i.e., the stiffnesses but also the strength of the coupling to the lattice (Barci and Fradkin, 2011)]. The phase diagram of Fig. 6 has several multicritical points (denoted by P and P') which have a larger "emergent" [SU(3)] symmetry (Berg, Fradkin, and Kivelson, 2009a). It is important to note that in this problem the natural U(1) symmetries of the order parameters alone produce a phase diagram with a complex set of phases and critical temperatures that are comparable to each other without invoking any fine-tuned multicritical point with a large emergent symmetry.

Finally, it is important to mention that evidence, both theoretical and experimental, has been adduced concerning the existence of other new phases of matter in the cuprates including DDW order (Chakravarty *et al.*, 2001), intraunit cell orbital current order (OAF) (Varma, 2006), and various forms of topologically ordered phases. It goes without saying that establishing which of these phases actually exists in this family of materials is of central importance. From the present perspective, the existence of any of these phases would primarily serve to further emphasize the intrinsic complexity of the phase diagram. While some of these orders, for instance, some forms of OAF order, can be constructed as composite orders in terms of the fields already present in our analysis, others (such as DDW) would involve the introduction of yet further fields.

C. Critique of theories with emergent symmetries

Theories invoking large emergent symmetry groups have been proposed in the context of high T_{SC} superconductors, starting with S. C. Zhang's proposal to unify d-wave superconductivity (which has a complex order parameter field) with Néel antiferromagnetic order (which has a three-component real order parameter field) in a larger $SO(5)$ symmetry (Zhang, 1997; Demler, Hanke, and Zhang, 2004). Generalizations of this concept have since been pursued, including a model relating nematic order to d-density wave and d-wave superconductivity [where the larger symmetry is SO(6)] by Kee, Doh, and Grzesiak (2008) and Kee (2010), and, most germane to the present discussion, theories of Efetov, Meier, and Pepin (2013), Sachdev and La Placa (2013), and Hayward *et al.* (2014) which envisage an SO(6) or SU(2) symmetry relating charge-density-wave order and superconductivity.

What is very attractive about these approaches is that they intertwine the various orders to such an extent that they become indistinguishable at short distances. However, typically, systems are less symmetric at low energies and long wavelengths than at the microscopic level—this observation underlies, for example, the standard model of particle physics that unifies the electromagnetic, weak, and strong interactions at high energies. Nevertheless, emergent symmetries are not unheard of. Both the Kondo impurity and the two-channel Kondo impurity problems exhibit emergent SU(2) spin-rotational symmetry at low energies, even if this symmetry is strongly broken in the microscopic model. The two-leg Hubbard ladder exhibits a fully gapped phase which, for small U (where all the gaps are exponentially small), has an emergent SO(8) symmetry (Lin, Balents, and Fisher, 1998).[8] Several (Fernandes and Schmalian, 2010; Davis and Lee, 2013; Efetov, Meier, and Pepin, 2013; Sachdev and La Placa, 2013) suggested that an emergent symmetry unifying CDW and SC order can arise from singular induced interactions at "hot spots" on the Fermi surface, such as occur in close proximity to an antiferromagnetic QCP of a Fermi liquid.

The behavior of such multicritical points with enhanced symmetries is well understood and has been studied since the early 1970s (Aharony and Bruce, 1974; Nelson and Fisher, 1975; Kosterlitz, Nelson, and Fisher, 1976; Aharony, 2003). The main results of these renormalization group studies is that, except for a special case which does not concern us here, the enhanced symmetry is fragile, since at the fixed point of such multicritical points various symmetry-breaking operators are relevant. In particular, at an SO(4) invariant multicritical point, terms that break the symmetry to $\mathrm{O}(2)\times\mathrm{O}(2)$ are strongly relevant.

Calabrese and co-workers further investigated the stability of the multicritical points associated with breaking of a larger symmetry group to smaller subgroups, $\mathrm{O}(n_1+n_2)\mapsto \mathrm{O}(n_1)\times\mathrm{O}(n_2)$, using a five-loop ϵ expansion (Calabrese *et al.*, 2003). They found that, for $N=n_1+n_2\geq 3$, the

[8]Intertwined orders in $d=2$ (with possibly enhanced symmetries) have been found in weakly correlated systems with band structures with quadratic crossings (Sun *et al.*, 2009; Murray and Vafek, 2014; Vafek, Murray, and Cvetkovic, 2014).

symmetry-breaking perturbations render the O(N) symmetric fixed point unstable and that, for $N \geq 5$, RG flows drive the system to a critical decoupled fixed point at which the O(n_1) and O(n_2) symmetries are decoupled. For $N = 3$ (and possibly also for $N = 4$) the actual critical behavior is controlled by the so-called biconical fixed point. Furthermore, the crossover away from the unstable O(N) fixed point is quite rapid reflecting the fact that the symmetry-breaking perturbations are strongly relevant. In particular (and this is what matters to our analysis), near the multicritical point the actual resulting T_c's rapidly diverge from each other as a power law of the form $|T_{\rm SC} - T_{\rm CDW}| \propto |r_{\rm SC} - r_{\rm CDW}|^{1/\phi}$. The crossover exponent ϕ is determined by the scaling law $\phi = \nu(d - \Delta_2)$, where ν is the correlation length exponent at the O(N)-symmetric fixed point, d is the dimension of space, and Δ_2 is the scaling dimension of the quadratic symmetry-breaking operator $|\Delta(\boldsymbol{x})|^2 - |\rho_{\boldsymbol{Q}}(\boldsymbol{x})|^2$. Current best estimates (Calabrese *et al.*, 2003) for the crossover exponent in $d = 3$ dimensions yield $\phi \simeq 1.35$ for $N = 4$ (and somewhat larger values for $N \geq 5$) which implies that the splitting of the critical temperatures is larger than a linear function of the quadratic symmetry-breaking field. In addition, a further complication is that, over a significant range of parameters, these phase transitions have a strong tendency to become fluctuation-induced first order transitions.

Many of these issues were discussed extensively in the context of the SO(5) theory [see, e.g., Aharony (2003)]. The upshot of this analysis is that, even if the transition is actually continuous, the fixed point with high symmetry is generally unstable and that for the critical temperatures of the competing orders to be similar in magnitude requires extremely delicate fine-tuning of the microscopically determined control parameters. This is the key message from this analysis.

The high temperature superconductors are quasi-two-dimensional systems. However, the fine-tuning problem, if anything, is worse in 2D (and quasi-2D) systems with a large continuous global symmetry, such as O(N) with $N > 2$. In strictly 2D, such systems cannot have a phase transition at any finite temperature. Close to $d = 2$ they are well described by a nonlinear σ model, which describes the long-distance fluctuations of a classical N-component Heisenberg model, whose order parameter is an N-component unit vector $\boldsymbol{n}(\boldsymbol{x})$ (such that $||\boldsymbol{n}||^2 = 1$). Renormalization group analysis of the two-dimensional nonlinear σ model (Polyakov, 1975; Brézin and Zinn-Justin, 1976) shows that thermal fluctuations are marginally relevant ("asymptotically free") perturbations at the $T = 0$ fixed point, and consequently that this system is in its disordered (high temperature) phase for all values of the temperature T. As a result, at low temperatures the correlation length scales as $\xi(T) \sim a \exp[2\pi K/(N-2)T]$, where K is the helicity modulus and a is the lattice spacing.

This gives a broad fluctuational regime which is exponentially sensitive to the magnitude of coupling parameters of the symmetry-breaking operators and/or to three-dimensional couplings. An important example is the case in which there is a small symmetry-breaking term of (dimensionless) magnitude h which explicitly breaks an O(N) symmetry (with $N > 2$) down to O(2) $\cong$ U(1). This problem was extensively discussed by Affleck (1986) (in the context of easy-plane symmetry breaking in quantum antiferromagnetic spin chains) and, more recently, by Fellows *et al.* (2012) in the context of competing orders. A key consequence of the marginal relevance of temperature in the O(N) symmetric theory is that the (Kosterlitz-Thouless) critical temperature of the O(2)-invariant system has a logarithmic dependence on the symmetry-breaking field h (Affleck, 1986; Fellows *et al.*, 2012)

$$T_{\rm KT} \sim \frac{K(N-2)}{4\pi} \frac{1}{\ln(1/|h|)}. \tag{8.1}$$

Since the critical temperature of the O(N)-symmetric model is zero, the finite value of $T_{\rm KT}$ reflects the significance of even very small symmetry-breaking terms.

Finally, we discuss how these considerations apply to the multiple orders considered by Hayward *et al.* (2014) in the context of the cuprates, as representative of the various proposals for emergent higher symmetries in the cuprates introduced at the beginning of this section. The underlying problem has a $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \times Z_2$ symmetry, where the first U(1) is associated with the superconducting order, and the final two U(1)'s and the Z_2 are associated with the CDW order, and correspond to translational symmetry in the x and y directions and rotation by $\pi/2$ about the z axis, respectively. Hayward *et al.* assumed that there is an approximate much larger SO(6), although they do explicitly take into account terms that differentiate the superconducting and CDW components of the order parameter, which thus break the symmetry down to $\mathrm{U}(1) \times \mathrm{SO}(4)$. However, the assumed SO(4) symmetry is still nongeneric, and this, along with the assumption that interlayer coupling can be neglected, allowing the system to be treated as 2D, is what is responsible for the central feature of the scenario of Hayward *et al.* (2014), resulting in a CDW correlation length that never diverges at any nonzero temperature. Again, terms that violate either assumption lead to a CDW ordering temperature that is small only in proportion to the inverse of the logarithm of the term's magnitude.

As a possible rejoinder to this critique, several studies (Efetov, Meier, and Pepin, 2013; Sachdev and La Placa, 2013; Meier *et al.*, 2014) have proposed that a near-perfect symmetry between CDW and SC orders can result from a higher level organization associated with close proximity to a metallic quantum critical point associated with large Q antiferromagnetic order. These theories consider a metallic Fermi liquid which is not too strongly coupled to the quantum critical fluctuations, so that they have a strong effect only on the electronic quasiparticles residing very close to a set of hot spots on the Fermi surface (Abanov and Chubukov, 2000; Vekhter and Chubukov, 2004; Chubukov, Galitski, and Yakovenko, 2005; Tsvelik and Chubukov, 2014) [for a review, see Wang and Chubukov (2014)]. It is unclear whether the weak-coupling focus on hot spots is reasonable in realistic, strongly coupled systems. Indeed, quantum Monte Carlo studies of Berg, Metlitski, and Sachdev (2012) of a metallic antiferromagnetic quantum critical point found clear evidence of induced superconductivity, but no reported evidence of the growth of CDW correlations of comparable strength.

Nevertheless, even if induced CDW order can also arise as a consequence of scattering by quantum critical AF fluctuations, it is far from clear why the proposed emergent symmetry should, at finite temperatures especially, be immune to the already discussed divergent flows of the symmetry-breaking terms associated with classical critical phenomena. Moreover, for this scenario to apply, the system must at least be fine-tuned to the near proximity of an antiferromagnetic quantum critical point. In many of the hole-doped cuprates in much of the range of dopings where SC and CDW orders appear to be intertwined, the antiferromagnetic correlation length measured in neutron scattering is no more than one to two lattice constants,[9] i.e., nowhere near being quantum critical.

D. Ineluctable complexity

The intrinsic complexity of the phase diagrams of correlated materials, the cuprates in particular, is by now self-evident. At the microscopic level, this is clearly a result of the quantum frustration of such systems: the kinetic energy favors highly delocalized uniform density fluid states, while the interaction energy favors localized, spatially inhomogeneous crystalline states. Understanding each phase that occurs with great precision is certainly a worthwhile undertaking. What is presented here is a step toward understanding the origin of the complexity itself. The intertwining of CDW, SDW, and SC order in the cuprates is particularly striking and well documented and is probably (Emery and Kivelson, 1993; Dagotto, 2005; Fradkin and Kivelson, 2012) associated with a rather general local tendency to phase separation.

We presented highly suggestive, but by no means conclusive, evidence for the existence of a PDW phase in the cuprates. If confirmed, this represents the discovery of a new phase of matter, which would be significant independent of any other implications. More broadly, as a state that tangibly intertwines CDW and SC (and, in some cases, SDW order as well) in an explicit, testable fashion (i.e., associated with a pattern of broken symmetry), it has the potential for providing a unifying starting point for studying the broader issues of intertwined orders in the cuprates.

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[9]The equal-time spin correlation length ξ is obtained by integrating the magnetic $S(\mathbf{Q},\omega)$ over ω, determining the half width at half maximum of the resulting $S(\mathbf{Q})$, and taking the inverse of that width. For $La_{1.86}Sr_{0.14}CuO_4$, such an analysis gave $\xi \approx a$ (Hayden *et al.*, 1996). For optimally doped $YBa_2Cu_3O_{6+y}$ and $Bi_2Sr_2CaCu_2O_{8+\delta}$, one can estimate an upper limit from the half-width of the resonance peak in the superconducting state, yielding $\xi \lesssim 1.3a$ (Dai *et al.*, 2001) and $\xi \lesssim a$ (Fong *et al.*, 1999), respectively.

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