Baryon resonances in large N_c QCD

N. Matagne[*]

Service de Physique Nucléaire et Subnucléaire, University of Mons, Place du Parc, B-7000 Mons, Belgium

Fl. Stancu[†]

University of Liège, Institute of Physics B5, Sart Tilman, B-4000 Liège 1, Belgium

(published 10 March 2015)

The current status and open challenges of large N_c QCD baryon spectroscopy are reviewed. After introducing the $1/N_c$ expansion method, the latest achievements for the ground state properties are revisited. Next the applicability of this method to excited states is presented using two different approaches with their advantages and disadvantages. Selected results for the spectrum and strong and electromagnetic decays are described. Also further developments for the applicability of the method to excited states are presented, based on the qualitative compatibility between the quark excitation picture and the meson-nucleon scattering picture. A quantitative comparison between results obtained from the mass formula of the $1/N_c$ expansion method and quark models brings convincing support to quark models and the implications of different large N_c limits are discussed. The SU(6) spin-flavor structure of the large N_c baryon allows a convenient classification of highly excited resonances into SU(3) multiplets and predicts mass ranges for the missing partners.

DOI: 10.1103/RevModPhys.87.211 PACS numbers: 11.15.Pg, 11.30.Ly, 14.20.−c, 02.20.Qs

CONTENTS

I. Introduction 211
II. Large N_c Baryons 214
III. Spin-flavor Symmetry 214
IV. Baryon Operator Expansion 216
V. Ground State Observables 216
  A. Masses 217
  B. Axial vector couplings 217
  C. Magnetic moments 217
  D. Charge radii and quadrupole moments 218
VI. Excited Symmetric Spin-flavor States 218
VII. Excited Mixed Symmetric Spin-flavor States 220
  A. The symmetric core + excited quark procedure 220
  B. The totally symmetric orbital-spin-flavor wave function procedure 221
VIII. Spectrum Calculations for Mixed Symmetric States 222
  A. The lowest negative parity $[\mathbf{70}, 1^-]$ multiplet 222
  B. Highly excited negative parity states 225
  C. Positive parity mixed symmetric states 225
IX. Heavy Baryon Masses in the Combined $1/N_c$ and $1/m_Q$ Expansion 227
X. Mass Formula in the $1/N_c$ Expansion Versus the Quark Model 227
XI. The Quark Excitation Versus the Meson-nucleon Resonance Picture 228
XII. Baryon Masses in the Combined $1/N_c$ and Chiral Expansions 230
XIII. Strong Decays 230
  A. Radially excited states 231
  B. The 56-plet 231
  C. The $[\mathbf{70}, 1^-]$-plet 232
XIV. Photoproduction Amplitudes in the $1/N_c$ Expansion 232
XV. Different Large N_c Limits 233
XVI. Exotics 234
XVII. Conclusions 235
Acknowledgments 235
Appendix A: The Generalized Wigner-Eckart Theorem and Isoscalar Factors of SU(6) 236
Appendix B: Symmetry Properties of Isoscalar Factors 239
Appendix C: Matrix Elements of SU(4) Generators 240
Appendix D: Isoscalar Factors of the Permutation Group 241
References 243

I. INTRODUCTION

Understanding the baryon structure directly from quantum chromodynamics (QCD), the theory of strong interactions, is a basic problem of hadronic physics. In 1974 two new papers heralded a new era in low-energy QCD. One was the paper by 't Hooft (1974) who proposed a perturbative expansion in QCD, in powers of $1/N_c$, where N_c is the number of colors. The other was Wilson's paper (Wilson, 1974) who discretized the continuum Euclidean space on a grid, laying the foundation of lattice calculations.

Tremendous progress has been achieved since 1974 in lattice QCD which has reproduced the ground state baryon masses at a few percent level and lattice results of several groups are in agreement. However, the extraction of resonant states remains a very difficult problem. There are large statistical and systematic errors. Traditionally all these states are treated as stable states but exploratory steps have been made in the direction of resonant states.

[*]Nicolas.Matagne@umons.ac.be
[†]fstancu@ulg.ac.be

For a recent review see, for example, Mohler (2012) and Lang and Verduci (2013). Also most studies are restricted to the first positive and negative parity resonances of total angular momentum $J = 1/2$, namely, the Roper $N(1440)1/2^+$ and the $N(1535)1/2^-$ resonance, respectively (Alexandrou *et al.*, 2014). However, it was at least possible to show that the number of each spin and flavor states in the lowest energy bands is in agreement with the expectations based on a weakly broken $\mathrm{SU}(6) \times \mathrm{O}(3)$ symmetry (Edwards *et al.*, 2013), used in quark models and in the treatment of excited states in large N_c QCD, as presented in this paper.

On the other hand, the $1/N_c$ expansion of QCD, proposed by 't Hooft, which has been extended by Witten (1979a) and applied to baryons (Witten, 1979b), has a clear phenomenological success. It offers the possibility of studying various baryon properties in a more direct way. Presently it is considered to be a model-independent, powerful, and systematic tool for baryon spectroscopy. This method is based on the discovery that, for N_f flavors, the ground state baryons display an exact contracted $\mathrm{SU}(2N_f)$ spin-flavor (SF) symmetry in the large N_c limit of QCD (Gervais and Sakita, 1984a, 1984b; Dashen and Manohar, 1993a, 1993b). Such a symmetry follows from consistency conditions on a meson-baryon scattering amplitude which must be satisfied for the theory to be unitary. As a consequence, at $N_c \to \infty$ the ground state baryons are degenerate. At large, but finite N_c, the spin-flavor symmetry is broken and the mass splitting starts at order $1/N_c$. As shown by Dashen, Jenkins, and Manohar (Dashen and Manohar, 1993a, 1993b; Jenkins, 1993a, 1993b, 1993c; Dashen, Jenkins, and Manohar, 1994, 1995), the consistency conditions restrict the form of subleading $1/N_c$ corrections, so that definite predictions can be made. An operator reduction rule simplifies the $1/N_c$ expansion.

The $1/N_c$ expansion method is closer to QCD than the quark models so that it provides a deeper understanding of the success of various quark models. This means that many results obtained in the nonrelativistic quark model, the bag model, or the Skyrme model can be proven in large N_c QCD to order $1/N_c$ or $1/N_c^2$, as we shall discuss. Being based on group theory it allows one to classify baryonic states, make predictions for the not yet discovered members of SU(6) multiplets, and study their properties.

The lattice QCD and the $1/N_c$ expansion can be combined together. Lattice simulations with a varying number of colors are extremely useful for confirming the validity of the $1/N_c$ expansion. So far, one was able to demonstrate that the results of the real world where $N_c = 3$ are already "close" to $N_c = \infty$ (Teper, 1998). A summary of such recent lattice studies and the extrapolation to the 't Hooft limit can be found in a recent comprehensive review paper (Lucini and Panero, 2013).

In addition, the existing lattice simulations at $N_c = 3$ for ground state baryons were able to test important features of the $1/N_c$ expansion results, in particular, the baryon mass relations. Lattice data display both the $1/N_c$ expansion and SU(3) flavor-symmetry breaking hierarchies (Jenkins *et al.*, 2010).

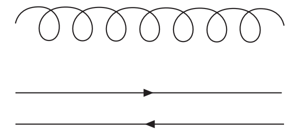


FIG. 1. A gluon in the traditional and double line notations.

Since 1974 large N_c QCD played an important role in phenomenology as well as in a number of theoretical developments in gauge theories as, for example, the fundamental problems of confinement and spontaneous symmetry breaking (SSB). The status of large N_c QCD 30 years later after its introduction by 't Hooft can be found in Goity *et al.* (2005). For pure fundamental aspects, as, for example, anti–de Sitter/conformal field theory (AdS/CFT) duality, gravity, and string theory approaches to flavor physics, phenomenology of quark-gluon plasma, etc., one can consult the proceedings of a workshop held in 2011 (Veneziano, 2011).

Presently there are several excellent reviews on large N where one can see that the $\mathrm{SU}(N)$ field theories simplify when N becomes large and the solutions to these theories possess an expansion in $1/N$. We refer the interested reader, for example, to Manohar's lectures (Manohar, 1998), partly based on the treatment proposed by Coleman (1985) with examples of theories with fields which transform according either to the vector representation (one-index representation) or to the adjoint representation (a two-index representation), which can be used in the case of QCD. Manohar's lectures also rely on Witten's papers, directly related to QCD (Witten, 1979a, 1979b). Also, several properties of large N_c QCD were described by Bhaduri (1988) from general arguments, where contact was made with the Skyrme model of the baryon, mentioned again in Sec. XI, as equivalent to the nonrelativistic quark model in the large N_c limit.

The adjoint representation carries two indices—the upper one labels the basis vectors of the fundamental representation such as for quarks, and the lower index corresponds to its complex conjugate such as for antiquarks, as described in Sec. XV. In this representation the gluon field therefore has two indices. This inspired 't Hooft to introduce the double line notation for gluons (Fig. 1) which provides a simple way to keep track of the color index contraction and find the combinatoric factors in a Feynman diagram and the N_c counting rules.

For example, consider the one-loop gluon vacuum polarization diagram (Fig. 2). From the right part of Fig. 2 it is easy to determine its combinatoric factor depending on N_c. Indeed, the color quantum numbers of the initial and final states are specified but not the inner index k which leads to a combinatoric factor equal to N_c for this Feynman diagram. At $N_c \to \infty$ the contribution of this diagram would be infinite.

To obtain a finite limit for this process, one can renormalize the theory by introducing a new coupling constant $g/\sqrt{N_c}$ instead of g. Then

$$\frac{g}{\sqrt{N_c}} \to 0 \quad \text{when } N_c \to \infty, \tag{1}$$

where g is fixed when N_c becomes large. In the one-loop gluon vacuum polarization we have two vertices and one

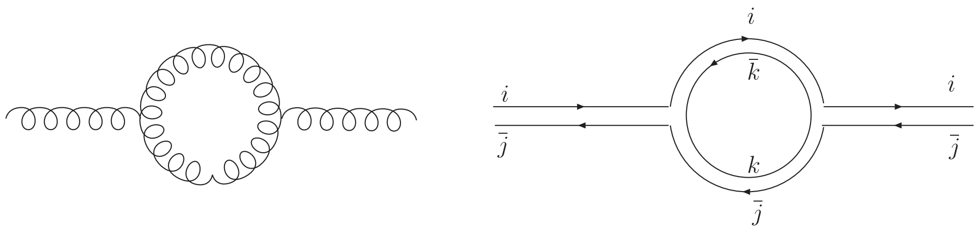


FIG. 2. The gluon vacuum polarization diagram in the standard (left) and double line notations of 't Hooft (right).

combinatoric factor N_c. With this renormalization, the order of the Feynman diagram in Fig. 2 becomes

$$\left(\frac{g}{\sqrt{N_c}}\right)^2 N_c = g^2 \tag{2}$$

independent of N_c as expected, and thus finite. For the combinatoric factors of more complex Feynman diagrams the interested reader is referred to Witten's paper (Witten, 1979b). The main conclusion is that the leading Feynman diagrams are planar and contain a minimum number of quark loops.

As mentioned, the application of the $1/N_c$ expansion method to baryon spectroscopy combines Witten's developments (Witten, 1979b) and the discovery that for N_f flavors the ground state baryons display an exact contracted $SU(2N_f)$ spin-flavor symmetry in the large N_c limit of QCD (Gervais and Sakita, 1984a, 1984b; Dashen and Manohar, 1993a, 1993b). Therefore the large N_c analysis for baryons is quite subtle, more subtle than that for mesons. The counting rules were studied by Witten (1979b). They were first used to study the ground state baryon masses, described by the symmetric representation **56** of SU(6) and other properties as, for example, axial vector couplings, magnetic moments, heavy-quark baryons, etc. (Dashen and Manohar, 1993a, 1993b; Jenkins, 1993a, 1993b, 1993c; Carone, Georgi, and Osofsky, 1994; Dashen, Jenkins, and Manohar, 1994, 1995; Luty and March-Russell, 1994; Jenkins and Lebed, 1995; Dai *et al.*, 1996).

The success of the $1/N_c$ expansion method in describing ground state baryons raised the question of whether excited baryons can be described as well by the same method. It is not obvious that the consistency condition used for the ground state is applicable to excited baryons because excited baryons are not generically stable asymptotic states even at large N_c. Witten has shown that the characteristic width of an excited baryon is N_c^0 (Witten, 1979b). Based on the argument that for some class of states the width goes like N_c^{-1}, Pirjol and Yan (1998a, 1998b) showed that a contracted SU(4) symmetry also exists for those excited states. From requiring the pion-excited baryon scattering amplitude to satisfy Witten's large N_c counting rules, they derived consistency conditions analogous to those obtained by Dashen, Jenkins, and Manohar (Dashen and Manohar, 1993a, 1993b; Jenkins, 1993a, 1993b, 1993c; Dashen, Jenkins, and Manohar, 1994, 1995) for s-wave baryons. Pirjol and Yan also showed that the solutions to large N_c consistency conditions coincide with the predictions of the nonrelativistic quark model for excited states.

The legitimacy of the procedure used by Pirjol and Yan was later questioned by Cohen and Lebed (2003a, 2003b), inasmuch as the characteristic width of an excited baryon is N_c^0 according to Witten (1979b). Cohen and Lebed tried to support the applicability of the $1/N_c$ expansion method by studying the compatibility between the scattering picture and the quark model picture, using quark operators as defined by Dashen and Manohar (1993a, 1993b), Jenkins (1993a, 1993b, 1993c), and Dashen, Jenkins, and Manohar (1994, 1995). They showed that the two pictures share the same pattern of degeneracy, giving rise to degenerate sets of resonances, identical in their quantum numbers J and I in both pictures at fixed grand spin K, as discussed in Sec. XI. From there they concluded that the two pictures are generically compatible.

In practice the extension of the $1/N_c$ expansion method to excited states is also based on the observation that these states can be approximately classified as $SU(2N_f)$ multiplets, and that the resonances can be grouped into excitation bands, $N = 1, 2, \ldots$, as in quark models, each band containing a number of $SU(6) \times O(3)$ multiplets.

The symmetric multiplets of these bands are similar to the ground state from group theory point of view. Therefore they were analyzed by analogy to the ground state. In this case the mass splitting starts at order $1/N_c$ as well.

That is why this review is largely devoted to mixed symmetric states for which, by contrast to symmetric states, the splitting starts at order N_c^0. The problem is in fact that these states are technically more difficult to study and two distinct procedures have been proposed so far. The first procedure was based on the Hartree approximation, in the spirit of Witten's arguments (Goity, 1997). In this procedure the system of N_c quarks is split into a ground state core which creates a mean field and an excited quark moving in this field. Then the Pauli principle is fulfilled by the core wave function only, but not by the total wave function (Carlson *et al.*, 1998, 1999). As we shall see, the numerous applications of this procedure were mostly restricted to the $N = 1$ band, but concerned the study of both the baryon spectra and their electromagnetic and strong decays. We understand that the technical advantage of this method was that the matrix elements of the $SU(2N_f)$ generators, needed in the calculations of spectra and decays, were known at that time for symmetric states only, but not for mixed symmetric spin-flavor states. A disadvantage is that the number of terms included in an operator describing an observable becomes generally large and it is difficult to select the dominant ones.

Later an alternative procedure, based on the identity of all quarks in the system, was proposed by Matagne and Stancu (2008a). There is no physical reason to separate the excited quark from the rest of the system. The total wave function is completely antisymmetric, the orbital-spin-flavor part is symmetric, being combined with an antisymmetric color part. The orbital and spin-flavor parts have the same mixed symmetry. The method can straightforwardly be applied to all bands, including more than one excited quark. The analytic form of the matrix elements of the $\mathrm{SU}(2N_f)$ generators for the necessary mixed symmetric spin-flavor states have been obtained as described by Matagne and Stancu (2009). A quantitative analysis was performed for a number of $\mathrm{SU}(2N_f) \times \mathrm{O}(3)$ multiplets for which data exist. They correspond to the excitation bands with $N = 1$, 2, or 3. In the nonstrange sector it covers a resonance mass region up to about 2.5 GeV. So far only the spectrum has been analyzed. There is an expectation that the decays will also be considered within this approach.

One important goal of the $1/N_c$ expansion method was to understand whether the success of the nonrelativistic quark model has a natural explanation in large N_c QCD. Various studies presented here prove the compatibility between quark models and the $1/N_c$ mass formula. An interesting outcome is the similarity between the Regge trajectories resulting from both the $1/N_c$ expansion method and the quark models.

This paper is organized as follows. In Sec. II we introduce the definition of large N_c baryons according to 't Hooft and Witten. In Sec. III we sketch the derivation of the contracted $\mathrm{SU}(2N_f)$ spin-flavor symmetry and recall the resulting $\mathrm{su}(N_f)_c$ algebra. The baryon operator expansion method is described in Sec. IV. The latest results on the ground state baryons, as, for example, the magnetic moments, are the subject of Sec. V. After introducing the formalism of the $1/N_c$ expansion, Secs. VI–VIII are devoted to the study of excited states, with a special emphasis on the two distinct approaches to treating mixed symmetric spin-flavor states. The extension to heavy baryon masses is considered in Sec. IX. Section X contains considerations about the compatibility between the $1/N_c$ expansion and a quark model mass formula. Important qualitative support to the $1/N_c$ expansion method applied to excited states is brought in Sec. XI by a comparison between the quark excitation picture to order N_c^0 and the meson-nucleon scattering picture. The combined $1/N_c$ and chiral expansions are updated in Sec. XII. The present status of the strong and electromagnetic decays is summarized in Secs. XIII and XIV. A short discussion of various large N_c limits, including that of 't Hooft, is given in Sec. XV. Some of the appendixes are devoted to the extended Wigner-Eckart theorem and the derivation of isoscalar factors of SU(6) generators needed in this work. General analytic expressions are reproduced. They could perhaps be applied to other fields, in particular, to systems where the hypercharge is a good quantum number.

II. LARGE N_c BARYONS

According to Witten, large N_c baryons are colorless bound states composed of N_c valence quarks described by a completely antisymmetric color wave function of the form

$$C^A = \varepsilon_{i_1 i_2 i_3 \cdots i_{N_c}} q^{i_1} q^{i_2} q^{i_3} \cdots q^{i_{N_c}}. \tag{3}$$

Then the total wave function of such a system can be obtained by combining C^A with the orbital part $\psi_{\ell m}$, the spin part χ, and the flavor part ϕ by using Clebsch-Gordan (CG) coefficients of the permutation group S_{N_c} (Stancu, 1996) to obtain a totally antisymmetric wave function written symbolically as

$$\Psi = \psi_{\ell m} \chi \phi C^A. \tag{4}$$

Because C^A is antisymmetric the product $\psi_{\ell m}\chi\phi$ must be symmetric. For the ground state $\psi_{\ell m}$ is symmetric, inasmuch as all identical quarks are in an s state. Therefore the product $\chi\phi$ must be symmetric which makes the study of the ground state rather easy. For excited states described by mixed symmetric orbital states the product $\chi\phi$ must have the same mixed symmetry, as discussed in Sec. VII.B.

The number of quarks inside a large N_c baryon grows as N_c. Witten proposed to describe such a system by a Hartree approximation where each quark experiences the same average potential. In this approximation one has

$$M_{\text{baryon}} \sim \mathcal{O}(N_c). \tag{5}$$

On the other hand, the size of the baryon is governed by the confinement scale $\Lambda_{\text{QCD}}^{-1} \simeq 1$ fm which is fixed. Thus the quark density must increase with N_c. Corrections to the Hartree approximation follow from the spin-flavor structure of baryons discussed below.

III. SPIN-FLAVOR SYMMETRY

Gervais and Sakita (1984a, 1984b) and Dashen and Manohar (1993a, 1993b) derived a set of consistency conditions for the pion-nucleon coupling constants in the large N_c limit of QCD. The arguments were based on the large N_c counting rules for meson-baryon scattering, analyzed by Witten (1979b), who showed that the baryon mass, Eq. (5), and the axial vector coupling constant g_A are $\mathcal{O}(N_c)$ and that the pion decay constant f_π is $\mathcal{O}(N_c{}^{1/2})$. Then the pion-nucleon vertex $g_A \vec{q}/f_\pi$, where $\vec{q}$ is the pion momentum, grows as $N_c{}^{1/2}$ at fixed pion energy.

In the large N_c limit, as the baryon is infinitely heavy compared with the pion, the time component of the baryon-pion coupling vanishes. Then the space components of the axial vector current matrix element can be written as

$$\langle B|\bar{q}\gamma^i\gamma_5 T^a q|B\rangle = g N_c \langle B|X^{ia}|B\rangle, \tag{6}$$

with the coupling constant g factored out so that g and $\langle B|X^{ia}|B\rangle$ are of order $\mathcal{O}(N_c^0)$, which means that in this definition X^{ia} is an operator defined on nucleon states which has a finite large N_c limit.

Considering the direct + crossed diagrams, Fig. 3, the pion-baryon scattering amplitude becomes

$$\mathcal{A}(\pi\mathcal{B} \to \pi\mathcal{B}) \propto -i \frac{N_c^2 g^2}{f_\pi^2} \frac{q^i q'^j}{q^0} [X^{ia}, X^{jb}], \tag{7}$$

where the initial and final baryons are on shell and X^{ia} is the baryon-meson vertex operator acting on the spin-flavor

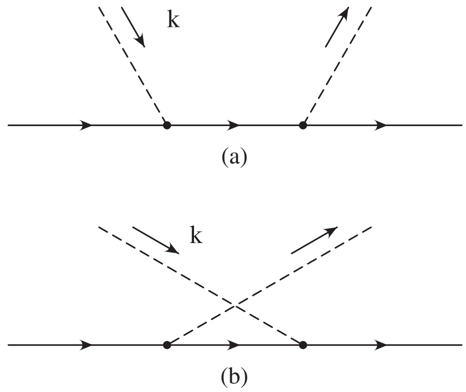


FIG. 3. Leading-order diagrams for the scattering $B + \pi \to B + \pi$.

nucleon states $\mathcal{B}$. When $N_f = 2$, for example, in the product of X's the sum runs over all the possible values of the spin and isospin intermediate baryon states. According to Gervais and Sakita (1984a, 1984b) and Dashen and Manohar (1993a, 1993b) there must be other states that cancel the order N_c of the amplitude above so that the total amplitude is of the order of 1 and consistent with unitarity. These states form an infinite tower of degenerate baryon states which are the solutions of the following consistency condition:

$$N_c[X^{ia}, X^{jb}] \leq \mathcal{O}(1). \tag{8}$$

If one makes the expansion

$$X^{ia} = X_0^{ia} + \frac{1}{N_c} X_1^{ia} + \frac{1}{N_c^2} X_2^{ia} + \cdots, \tag{9}$$

the constraint (8) requires

$$[X_0^{ia}, X_0^{jb}] = 0, \tag{10}$$

when $N_c \to \infty$. In this limit, the spin operators S^i $(i = 1, 2, 3)$, the flavor operators T^a $(a = 1, 2, \ldots, N_f)$, together with the spin-flavor operators X^{ia} can be identified with the generators of a contracted spin-flavor group $\mathrm{SU}(2N_f)_c$, where N_f is the number of flavors. Its algebra is

$$\begin{aligned} &[S^i, T^a] = 0, \qquad [S^i, S^j] = i\varepsilon^{ijk} S^k, \\ &[T^a, T^b] = i f^{abc} T^c, \qquad [S^i, X_0^{ja}] = i\varepsilon^{ijk} X_0^{ka}, \\ &[T^a, X_0^{ib}] = i f^{abc} X_0^{ic}, \qquad [X_0^{ia}, X_0^{jb}] = 0. \end{aligned} \tag{11}$$

On the other hand, the $\mathrm{su}(2N_f)$ algebra reads

$$\begin{aligned} &[S^i, S^j] = i\varepsilon^{ijk} S^k, \quad [T^a, T^b] = i f^{abc} T^c, \quad [S^i, T^a] = 0, \\ &[S^i, G^{ja}] = i\varepsilon^{ijk} G^{ka}, \quad [T^a, G^{jb}] = i f^{abc} G^{jc}, \\ &[G^{ia}, G^{jb}] = \frac{i}{4}\delta^{ij} f^{abc} T^c + \frac{i}{2}\varepsilon^{ijk}\left(\frac{1}{N_f}\delta^{ab} S^k + d^{abc} G^{kc}\right). \end{aligned} \tag{12}$$

Thus the contracted algebra $\mathrm{su}(2N_f)_c$ is obtained from the commutation relations (12) by taking the limit

$$X_0^{ia} = \lim_{N_c \to \infty} \frac{G^{ia}}{N_c}. \tag{13}$$

The first formal notion of the operation of group or algebra contraction was given by Segal (1951) who considered a sequence of Lie groups. His approach was more general than that of Inönü and Wigner (1953) who introduced the definition from the physicist's point of view in 1953. In their paper they investigate, in some generality, in which sense groups can be limiting cases of other groups. The observation was that the classical mechanics is a limiting case of relativistic mechanics when the velocity of light becomes infinite. Inönü and Wigner called *contraction* the operation of obtaining a new group by a new singular transformation of the infinitesimal elements of the old group. The contracted infinitesimal elements form an Abelian invariant subgroup of the contracted group. In this way the Lorentz group contracts to the Galilei group. In field theory the limit can be related to coupling constants (Hermann, 1966) as in Eq. (1). In the present case the parameter of the singular transformation should be $1/\sqrt{N_c}$. In the limit $N_c \to \infty$ one obtains the operators (13) as elements of an algebra nonisomorphic to the original one.

Dashen and Manohar (1993a, 1993b) solved Eq. (10) for $N_f = 2$ finding in this way the simplest irreducible representations of $\mathrm{SU}(4)_c$. Dashen, Jenkins, and Manohar (1994, 1995) classified all possible representations of the contracted spin-flavor algebra using the theory of induced representations (Mackey, 1968). This theory gives a complete classification of all irreducible representations of a semidirect product $\mathcal{G} \wedge \mathcal{A}$ of a compact Lie group $\mathcal{G}$ and an Abelian invariant subgroup $\mathcal{A}$. In large N_c QCD $\mathcal{G}$ is the direct product $\mathrm{SU}(2) \otimes \mathrm{SU}(N_f)$ and the group $\mathcal{A}$ generates an Abelian invariant subalgebra, the elements of which are X_0^{ia} defined in Eq. (10), as first pointed out by Gervais and Sakita (1984a, 1984b).

The basis vectors of the induced representations form infinite towers of degenerate (S, I) baryon states, each tower corresponding to a value of the grand spin $\vec{K} = \vec{I} + \vec{S}$, like in the Skyrme model or the nonrelativistic quark model in the large N_c limit. The identification with physical states can be made by assuming that $K = n_s/2$, where n_s is the number of strange quarks in a baryon. The simplest irreducible representation for two flavors is a tower of states with $K = 0$ and $S = I = 1/2, 3/2, 5/2, \ldots$.

Dashen and Manohar (1993a, 1993b) and Jenkins (1993a, 1993b, 1993c) found that subleading $1/N_c$ corrections to the Hartree approximation, Eq. (5), are constrained by additional large N_c conditions and they proposed a systematic expansion in powers of $1/N_c$ for ground state baryons.

A diagrammatic and group-theoretical analysis of baryons in the $1/N_c$ expansion had also been used by Carone, Georgi, and Osofsky (1994) and independently by Luty and March-Russell (1994) shortly before similar results were obtained by Dashen, Jenkins, and Manohar (1995). Carone, Georgi, and Osofsky (1994) and Luty and March-Russell (1994) have used the quark representation approach which is closely tied to the intuitive picture of baryons as quark bound states.

The connection to the nonrelativistic quark model is hinted at from the work of Gervais and Sakita (1984a, 1984b) in the following way. To obtain the quark representations of the group $\mathcal{G}$ defined above one has to consider the symmetric state of N_c quarks, reduce it to the representation of the direct product $SU(2) \otimes SU(N_f)$, and take the limit $N_c \to \infty$.

IV. BARYON OPERATOR EXPANSION

According to Sec. III there are two kinds of representations of the spin-flavor algebra of baryons in large N_c. One is the Skyrme model representation as obtained by Dashen, Jenkins, and Manohar (1994, 1995) using the theory of induced representations. At $N_c \to \infty$ the spectrum consists of infinite towers of states of given spin S and isospin I combined to a fixed value of the grand spin K. The other is the quark representation proposed by Carone, Georgi, and Osofsky (1994) and Luty and March-Russell (1994) which is closely related to the nonrelativistic quark model and is convenient to study decays as well.

The Skyrme and the nonrelativistic quark model representations of the large N_c spin-flavor algebra for baryons are identical in the $N_c \to \infty$ limit. At finite N_c they differ in their organization of $1/N_c$ corrections but give equivalent results at a given order in $1/N_c$. The Skyrme representations of the contracted algebra are infinite dimensional. The quark representations use the algebra (12) so that the representations are finite. For the ground state it consists of a tower of states that terminates at spin $S = N_c/2$. The connection between the Skyrme and quark representations is discussed by Dashen, Jenkins, and Manohar (1995).

Next we summarize the analysis of Dashen, Jenkins, and Manohar (1995) of the baryon operator expansion realized in the quark representation. Quark operators can be classified as n-body quark operators $\mathcal{O}^{(n)}$. By definition each $\mathcal{O}^{(n)}$ acts on n quarks, where $0 \leq n \leq N_c$. The $SU(2N_f)$ generators are one-body operators. One can construct $\mathcal{O}^{(n)}$ for any n starting from the generators of $SU(2N_f)$. For doing this, it is convenient to express them in terms of the representations of the direct product $SU(2) \times SU(N_f)$. One has

$$\begin{aligned} S^i &= \sum_{j=1}^{N_c} q_j^\dagger (S^i \times \mathbb{1}) q_j \quad (3,1), \\ T^a &= \sum_{j=1}^{N_c} q_j^\dagger (\mathbb{1} \times T^a) q_j \quad (1, N_f^2 - 1), \\ G^{ia} &= \sum_{j=1}^{N_c} q_j^\dagger (S^i \times T^a) q_j \quad (3, N_f^2 - 1), \end{aligned} \tag{14}$$

where $q_j^\dagger$ and q_j are creation and annihilation quark operators with j the quark line number. The operators $q_j^\dagger$, q_j obey the Bose statistics because the ground state baryons of N_c quarks are in a completely symmetric spin-flavor state, as mentioned in Sec. II. The brackets on the right-hand side of the definitions (14) denote the $(SU(2), SU(N_f))$ dimensional notation of the spin S^i, flavor T^a, and spin-flavor generators G^{ia} of $SU(2N_f)$ in the decomposition $SU(2) \times SU(N_f)$. Thus one can see that the SU(2) baryon spin generator S^i is equal to the sum of spin generators of the N_c quarks forming the baryon. A similar remark holds for the isospin and G^{ia}. Note that, for a finite N_c, in the quark representation G^{ia} recovers the form $S^i T^a$ operator of the axial current.

The latter remark requires some useful comments. We closely follow Jenkins (1998). From Eqs. (9) and (13) one can see that the operators X_0^{ia} and G^{ia}/N_c differ at the subleading order $1/N_c$. The ambiguity in the choice of the spin-flavor generator arises from the fact that the contracted spin-flavor algebra for baryons is exact only in the large N_c limit, which means that matrix elements of the spin-flavor generators G^{ia} are known to leading order in $1/N_c$ up to a normalization factor. When X_0^{ia} is used, the operator basis of the $1/N_c$ expansion is the same as the operator basis of the large N_c Skyrme model, whereas the operator basis constructed from the algebra (12) is the same as the operator basis of the large N_c nonrelativistic quark model. Both operator bases parametrize the same large N_c physics encoded in coefficients entering the expansion formula as introduced next.

Here we deal with the quark model representation. In such a case any QCD operator which transforms as an irreducible representation of $SU(2) \times SU(N_f)$ can be written as an expansion in n-body quark operators $\mathcal{O}^{(n)}$, which transform under the same irreducible representation. For the ground state baryons one has (Jenkins, 1993a, 1993b, 1993c; Carone, Georgi, and Osofsky, 1994; Dashen, Jenkins, and Manohar, 1994, 1995; Luty and March-Russell, 1994)

$$\mathcal{O}_{\rm QCD} = \sum_n c^{(n)} \frac{1}{N_c^{n-1}} \mathcal{O}^{(n)}, \tag{15}$$

where $c^{(n)}$ are unknown dynamical coefficients. Each $\mathcal{O}^{(n)}$ operator is accompanied by a factor of $1/N_c^{n-1}$, which comes from the fact that one needs at least $n-1$ gluon exchanges at the quark level to generate n-body effective operators in the $1/N_c$ expansion out of one-body QCD operators. A generic n-body operator can be written as a homogeneous nth degree polynomial in the generators S^i, T^a, and G^{ia} (up to an occasionally zero-degree rescaling term). As we shall see, the matrix elements of the baryon observables can be calculated in terms of the matrix elements of the baryon spin-flavor generators S^i, T^a, and G^{ia}. Quark operator identities can be used to construct a linearly independent complete operator basis of n-body operators. Operator reduction rules to simplify the $1/N_c$ expansion have also been derived (Dashen, Jenkins, and Manohar, 1995).

The extension to strange baryons, where SU(3) is broken, will be presented in the following, as well as the extension to excited states.

The method has been applied to baryon masses, axial currents, magnetic moments, etc. Comprehensive reviews can be found, for example, in Jenkins (1998, 2001). Therefore we shall not give details of these achievements; we shall recall only a few basic results and mention the latest developments.

V. GROUND STATE OBSERVABLES

It was later noticed that SU(3) flavor breaking cannot be neglected relative to $1/N_c$ corrections. The $1/N_c$ operator expansion can be generalized to include SU(3) breaking as shown next.

A. Masses

As far as the masses are concerned, the original $1/N_c$ expansion (15) has been combined with a perturbative flavor breaking by Jenkins and Lebed (1995). The SU(3) symmetry breaking was implemented to $\mathcal{O}(\epsilon)$, where the parameter ϵ represents the quark mass difference divided by the chiral symmetry breaking scale, which is of the order of 1 GeV. This leads to a generalized form of the mass operator used currently in calculating the spectra of both nonstrange and strange baryons. The generalized mass operator is

$$M = \sum_i c_i O_i + \sum_i d_i B_i. \tag{16}$$

The first sum contains the operators O_i which are SU(N_f) invariants, and the operators B_i in the second sum break SU(3) explicitly and have zero expectation values for nonstrange baryons.

In Eq. (16) O_1 is the leading SF singlet operator proportional to N_c and O_i with $i > 1$ brings $1/N_c$ corrections which estimate the amount of SF symmetry breaking.

For further purposes, the operators O_i of Eq. (15) with $i > 1$ are here defined such as to be applied to orbitally excited baryons as well, besides the ground state baryons. They are SU(2) scalar products

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{\mathrm{SF}}^{(k)}, \tag{17}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{\mathrm{SF}}^{(k)}$ is a k-rank tensor in SU(2) spin, but invariant in SU(N_f). For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms. The rank $k = 1$ tensor has as components the generators L^i of SO(3). The components of the $k = 2$ tensor operator of SO(3) are

$$L^{(2)ij} = \tfrac{1}{2}\{L^i, L^j\} - \tfrac{1}{3}\delta_{i,-j}\vec{L}\cdot\vec{L}, \tag{18}$$

which, like L^i, act on the orbital wave function $|\ell m\rangle$ of the whole system of N_c quarks. Examples will be given throughout the paper.

Presently we illustrate Eq. (15) with the simple case of ground state nonstrange baryons. We use the operator identities (Dashen, Jenkins, and Manohar, 1995)

$$\{S^i, S^i\} + \{T^a, T^a\} + \{G^{ia}, G^{ia}\} = \tfrac{3}{2}N_c(N_c + 4), \tag{19}$$

$$\{T^a, T^a\} = \{S^i, S^i\}, \tag{20}$$

where the right-hand side in the first identity is the eigenvalue of the SU(4) Casimir operator for a spin-flavor symmetric state described by the partition $[N_c]$. Then one can reduce the mass operator to the simple form

$$M = c_1 N_c + c_3 \frac{1}{N_c} S^2 + \mathcal{O}\left(\frac{1}{N_c^3}\right), \tag{21}$$

where we use a notation for c_i to be consistent with the rest of the paper. Thus the mass splitting starts at order $1/N_c$. Usually higher orders $\mathcal{O}(1/N_c^3)$ are neglected. Using the nucleon mass $m_N \simeq 940$ MeV and the $\Delta(1232)$ resonance mass one can obtain

$$c_1 \simeq 289 \text{ MeV}, \qquad c_3 \simeq 292 \text{ MeV}. \tag{22}$$

The coefficient c_1 is close to the constituent mass of the quarks u or d and c_3 reproduces the hyperfine $\Delta - N$ splitting.

As already mentioned, the corrections due to the flavor-symmetry breaking are proportional to the parameter $\epsilon \sim 0.25$ (Jenkins and Lebed, 1995). The $1/N_c$ expansion, including flavor-symmetry breaking, predicted a hierarchy of spin and flavor-symmetry relations for QCD baryons that is observed in nature. It also provided a quantitative understanding of why some SU(3) flavor-symmetry relations in the baryon sector, as, for example, the Gell-Mann-Okubo mass formula or the Coleman-Glashow mass relation (Coleman and Glashow, 1961), are satisfied to a greater precision than expected from flavor-symmetry breaking suppression factors alone (Jenkins and Lebed, 2000). A detailed updated summary, including isospin breaking, can be found in Jenkins (2001).

B. Axial vector couplings

In the exact SU(3) symmetry limit the baryon axial vector current operator A^{ia} is a rank one tensor operator in SU(2) spin and in SU(3) it transforms as a flavor adjoint. Its group structure is of the form of G^{ia} of Eq. (14). An extended analysis in the $1/N_c$ expansion has included up to three-body operators in Dashen, Jenkins, and Manohar (1995). The expansion at linear order in SU(3)-flavor breaking involves additional SU(2)-spin rank one operators in different flavor representations (Dashen, Jenkins, and Manohar, 1995). A comparison with the experimental data was performed by Dai *et al.* (1996). The present status is summarized in Jenkins (2001).

C. Magnetic moments

In the exact SU(3) flavor-symmetry limit the baryon magnetic moment operator is a rank one irreducible tensor operator in SU(2) spin and an octet in SU(3) flavor.

The $1/N_c$ expansion has been analyzed by several authors (Dashen, Jenkins, and Manohar, 1994, 1995; Jenkins and Manohar, 1994; Luty, March-Russell, and White, 1995; Dai *et al.*, 1996; Lebed and Martin, 2004a; Flores-Mendieta, 2009; Jenkins, 2012; Ahuatzin *et al.*, 2014). Using the method of Lebed (1995) to classify static observables the complete set of 27 linearly independent operators of the octet and decuplet ground state baryons organized in powers of $1/N_c$ in their matrix elements was given in Table I of Lebed and Martin (2004a) for the component $i = 3$. Operator demotions have been taken into account in the $1/N_c$ power dependence.

The "operator demotion" was first defined by Carlson *et al.* (1999). In a demotion one identifies a linear combination of operators whose matrix elements are a higher order in powers of $1/N_c$ than those of the component operators, so that only one of the components represents an independent operator at the starting order. The result depends on the particular states used for evaluating the matrix elements. The example given in

Carlson *et al.* (1999) clearly clarifies the procedure, which is possible when it happens that the matrix elements of the SU(6) generators contain both leading and subleading orders in N_c. A look at the tables given in Appendix B supports this statement.

The most updated analysis can be found in Jenkins (2012) where it was shown that the combined expansion in $1/N_c$ and SU(3) flavor breaking is needed to understand the hierarchy of baryon magnetic moments found in nature. The 27 linearly independent operators were written in the basis (U_3, Q) defined by the chain $SU(3) \supset SU_U(2)$, in terms of the U spin, instead of the usual (T^3, T^8) basis related to the chain $SU(3) \supset SU_I(2)$, where I is the isospin. The reason is that the magnetic moments M^{iQ} of light baryons are proportional to the quark charge matrix $T^Q = \mathrm{diag}(2/3, -1/3, -1/3)$, where the charge operator reads

$$T^Q = T^3 + \frac{1}{\sqrt{3}} T^8. \tag{23}$$

This commutes with the third component of U defined as

$$U^3 = -\frac{1}{2} T^3 + \frac{\sqrt{3}}{2} T^8, \tag{24}$$

both T^Q and U^3 being operators which are linear combinations of SU(3) generators corresponding to vanishing roots in Cartan's classification; see Eqs. (8.105d) of Stancu (1996). Instead of using the standard (I_3, Y) coordinates one can draw a weight diagram (U^3, T^Q), where each U multiplet contains baryons of identical charge and the T^Q operator changes the charge by one unit in passing from one U multiplet to another. By analogy to T^Q one can introduce the operator

$$G^{iQ} = G^{i3} + \frac{1}{\sqrt{3}} G^{i8} \tag{25}$$

and define the magnetic moments M^{iQ} by the linear combination (Jenkins, 2012)

$$M^{iQ} = a G^{iQ} + b \frac{1}{N_c} S^i T^Q, \tag{26}$$

up to order $1/N_c$, with coefficients a and b to fit experimental data.

There are 27 magnetic moments of ground state baryons, nine for the octet including the $\Sigma^0 \to \Lambda$ transition magnetic moment and ten magnetic moments for the decuplet plus eight decuplet-octet transition magnetic moments. The two new experimental results on decuplet-octet transition magnetic moments $\Lambda\Sigma^{*0}$ and $\Sigma\Sigma^{*+}$ (Keller *et al.*, 2011, 2012) were added to the analysis made in Jenkins (2012). The conclusion was that further progress in understanding the hierarchy of baryon magnetic moments requires additional experimental measurements. A significant SU(3) breaking was found and this breaking is expected to be enhanced in the magnetic moments relative to that of other observables, which makes studying magnetic moments particularly useful.

D. Charge radii and quadrupole moments

Studies of baryon charge radii and quadrupole moments in the $1/N_c$ expansion have been performed by Buchmann and Lebed (2000) and Buchmann, Hester, and Lebed (2002) for two flavors and extended in Buchmann and Lebed (2003) to three flavors. The charge radius is the first moment of a Coulomb monopole transition amplitude. The calculations have been made in the simple single-photon exchange ansatz which requires only two operators to describe both the charge radii and quadrupole moment observables. In 2003 only the charge radii of p, n, and Σ^- were known experimentally. For the other baryons predictions were made. On the other hand, only the $\Delta \to N$ quadrupole transition matrix element has been measured and calculated in the above mentioned $1/N_c$ expansion. An important feature is that the leading order of the diagonal quadrupole moment is $\mathcal{O}(N_c^0)$.

The quadrupole moments and charge radii are related. In the one-gluon exchange picture the following relation between the quadrupole transition moment $Q_{\Delta^+ p}$ and the neutron charge radius r_n^2 has been obtained (Buchmann, Hester, and Lebed, 2002):

$$Q_{\Delta^+ p} = \frac{1}{\sqrt{2}} r_n^2 \frac{N_c}{N_c + 3} \sqrt{\frac{N_c + 5}{N_c - 1}}, \tag{27}$$

which simplifies at $N_c = 3$ and can be obtained within other frameworks.

VI. EXCITED SYMMETRIC SPIN-FLAVOR STATES

In the $1/N_c$ expansion method the analysis of the masses of resonances which can be assigned to the SU(6) symmetric irreducible representation, denoted in the following by the partition $[N_c]$, can be easily performed by analogy to the ground state. The most convenient framework is to use Eq. (16) and $SU(2N_f)$ algebra with operators that act on symmetric spin-flavor states obtained as inner products of SU(2) and $SU(N_f)$ basis vectors. Applications to the $[\mathbf{56}, 2^+]$ and $[\mathbf{56}, 4^+]$ have been considered by Goity, Schat, and Scoccola (2003) and Matagne and Stancu (2005a), respectively. A wave function of an orbitally symmetric state $|\ell m\rangle$, spin S, S_3, and total angular momentum J, J_3 obtained by using CG coefficients, takes the general form

$$|[N_c]\ell S; JJ_3; (\lambda\mu)YII_3\rangle = \sum_{m,S_3} \left(\begin{array}{cc|c} \ell & S & J \\ m & S_3 & J_3 \end{array} \right) |\ell m\rangle |[N_c]SS_3; (\lambda\mu), YII_3\rangle, \tag{28}$$

where $|[N_c]SS_3; (\lambda\mu)YII_3\rangle$ is a symmetric spin-flavor state under N_c permutations, $(\lambda\mu)$ labels an SU(3) irrep and the quantum numbers Y, I, I_3 stand for the hypercharge, isospin, and its projection, labeling the basis vectors of a given $(\lambda\mu)$ irrep. For example, the states (28) of the $[\mathbf{56}, 2^+]$ multiplet are two SU(3) octets $^2 8_{3/2}$ and $^2 8_{5/2}$ and four decuplets $^4 10_{1/2}$, $^4 10_{3/2}$, $^4 10_{5/2}$, and $^4 10_{7/2}$, and for the $[\mathbf{56}, 4^+]$ multiplet , they are two SU(3) octets $^2 8_{7/2}$ and $^2 8_{9/2}$ and four decuplets $^4 10_{5/2}$, $^4 10_{7/2}$, $^4 10_{9/2}$, and $^4 10_{11/2}$.

TABLE I. Operators of Eq. (16) and coefficients c_i and d_i in MeV resulting from numerical fits to data obtained for the symmetric multiplets $[\mathbf{56}, 2^+]$ (Goity, Schat, and Scoccola, 2003) and $[\mathbf{56}, 4^+]$ (Matagne and Stancu, 2005a).

Operator	$[\mathbf{56}, 2^+]$	$[\mathbf{56}, 4^+]$
$O_1 = N_c \mathbb{1}$	541 ± 4	736 ± 30
$O_2 = \frac{1}{N_c} L^i S^i$	18 ± 16	4 ± 40
$O_3 = \frac{1}{N_c} S^i S^i$	241 ± 14	135 ± 90
$B_1 = -\mathcal{S}$	206 ± 18	110 ± 67
$B_2 = \frac{1}{N_c} L^i G^{i8} - \frac{1}{2\sqrt{3}} O_2$	104 ± 64	
$B_3 = \frac{1}{N_c} S^i G^{i8} - \frac{1}{2\sqrt{3}} O_3$	223 ± 68	
$\chi^2_{\rm dof}$	$\simeq 0.7$	$\simeq 0.26$

The operators O_i can be obtained in a straightforward manner by using their definition (17). The operators B_i must have zero expectation values for nonstrange baryons; see, for example, Table I and Eq. (30).

The first corrections to the leading term O_1 of order N_c start at order $1/N_c$ as for the ground state. The most dominant contributions to the mass formula given by Eq. (16) are expected from the operators shown in Table I. Their matrix elements are easy to calculate [for details see, for example, Matagne and Stancu (2005a)]. The angular momentum components L^i act on the whole system so that the eigenvalue of the spin-orbit term O_2 becomes

$$\langle O_2 \rangle = \frac{1}{2N_c}[J(J+1) - \ell(\ell+1) - S(S+1)], \qquad (29)$$

in agreement with the results of Goity, Schat, and Scoccola (2003).

The SU(3) breaking operator B_2 can be rewritten as

$$B_2 = -\frac{\sqrt{3}}{2N_c} \vec{L} \cdot \vec{S}_s, \qquad (30)$$

where $\vec{S}_s$ is the spin operator acting on the strange quarks. Its matrix elements can be calculated as indicated in Matagne and Stancu (2005a).

The analytic form of the first term $S_i G_{i8}$ of B_3 was derived from the matrix elements of the SU(6) generators for totally symmetric spin-flavor states (Matagne and Stancu, 2006a). This is

$$\langle S^i G^{i8} \rangle = \frac{1}{4\sqrt{3}}\left[3I(I+1) - S(S+1) + \frac{3}{4}\mathcal{S}(2-\mathcal{S})\right], \qquad (31)$$

where $\mathcal{S}$ is the strangeness. It can be rewritten in terms of the number of strange quarks $N_s = -\mathcal{S}$ in order to recover the expression introduced in Jenkins and Lebed (1995).

It was also found (Matagne and Stancu, 2006a) that the expectation values of O_2, O_3, B_2, and B_3 satisfy

$$\frac{\langle B_2 \rangle}{\langle B_3 \rangle} = \frac{\langle O_2 \rangle}{\langle O_3 \rangle}, \qquad (32)$$

for every J, in both the octet and the decuplet. This can be used as a check of the analytic expressions of these operators in terms of N_c given in Goity, Schat, and Scoccola (2003) and Matagne and Stancu (2005a).

In the numerical fit for resonances belonging to the $[\mathbf{56}, 2^+]$ multiplet (Goity, Schat, and Scoccola, 2003), ten experimentally known resonances with a status of three or four stars were used and predictions were made for another 14 resonances. At higher energies, namely, the multiplet $[\mathbf{56}, 4^+]$, the experimental situation is poorer so that in Matagne and Stancu (2005a) only five resonances were used in the fit (with a status of one, two, three, or four stars) and 19 masses were predicted. From Table I one can see that the coefficient c_1 of the leading operator O_1 has by far the largest value in both cases. It is interesting to see that this coefficient is larger for $[\mathbf{56}, 4^+]$ than for $[\mathbf{56}, 2^+]$. It hints at a dependence of c_1 with energy or equivalently with the band number N. One then expects a Regge trajectory-type behavior in terms of N (Matagne and Stancu, 2013); for an illustration see Sec. X.

The coefficient c_2 of the spin-orbit operator O_2 has a small value, which decreases with the excitation energy. The smallness of the spin-orbit contribution supports the quark model calculations, where the spin-orbit term is usually neglected. The decrease in energy is in agreement with the intuitive picture of Glozman (2002) according to which, at high energies, the spin dependent interactions are expected to vanish as a consequence of the chiral symmetry restoration.

The breaking of the spin-flavor symmetry is essentially given by the operator O_3 which represents the hyperfine interaction and turns out to be the most important after O_1. The coefficient c_3 is a measure of the splitting between octets and decuplets, as for the ground state described by the coefficients given in Eq. (22). Although within numerical errors the values of c_3 for $[\mathbf{56}, 2^+]$ and $[\mathbf{56}, 4^+]$ are compatible with each other, the central values show a decrease with the band number N, or else with the excitation energy, as mentioned above.

In general, the SU(3) flavor breaking is dominated by $B_1 = -\mathcal{S}$. It gives a mass shift of about 200 MeV per unit of strangeness in $[\mathbf{56}, 2^+]$. The operators B_2 and B_3 can provide the Λ-Σ splitting in octets and were included in the numerical fit of Goity, Schat, and Scoccola (2003). Matagne and Stancu (2005a) ignored them in the fit because of the lack of data. Then, including only B_1, a mass shift of about 110 MeV per unit strangeness, with rather large error bars, has been obtained. From Goity, Schat, and Scoccola (2003) there is an indication that the contributions of B_2 and B_3 to the mass sometimes roughly cancel mutually and sometimes they add to an unexpected large number, so that the higher J states are lighter, which is unexpected. In conclusion, more data are desired for strange excited resonances, for both $N = 2$ and $N = 4$ bands.

To illustrate the discussion, in Table II we reproduce the results of Matagne and Stancu (2005a) for the partial contribution and the total mass predicted by the $1/N_c$ expansion, Eq. (16), for the $[\mathbf{56}, 4^+]$ multiplet. As a matter of fact, the resonance $\Sigma(2455)^{**}$ marked as "bumps" in the 2013 Review of Particle Physics (Beringer *et al.*, 2012) could possibly be assigned to $\Sigma_{5/2}(2478)$ of Table II.

Finally note that the operator B_2, through its off-diagonal matrix elements, induces a mixing between octet and decuplet

TABLE II. The partial contribution and the total mass (MeV) predicted by Eq. (16) as compared with the empirically known masses for resonances assigned to the $[\mathbf{56}, 4^+]$ multiplet. From Matagne and Stancu, 2005a.

	$1/N_c$ expansion results Partial contribution (MeV)						
	c_1O_1	c_2O_2	c_3O_3	$b_1\bar{B}_1$	Total (MeV)	Empirical (MeV)	Name, status
$N_{7/2}$	2209	−3	34	0	2240 ± 97		
$\Lambda_{7/2}$				110	2350 ± 118		
$\Sigma_{7/2}$				110	2350 ± 118		
$\Xi_{7/2}$				220	2460 ± 166		
$N_{9/2}$	2209	2	34	0	2245 ± 95	2245 ± 65	$N(2220)^{****}$
$\Lambda_{9/2}$				110	2355 ± 116	2355 ± 15	$\Lambda(2350)^{***}$
$\Sigma_{9/2}$				110	2355 ± 116		
$\Xi_{9/2}$				220	2465 ± 164		
$\Delta_{5/2}$	2209	−9	168	0	2368 ± 175		
$\Sigma_{5/2}$				110	2478 ± 187		
$\Xi_{5/2}$				220	2588 ± 220		
$\Omega_{5/2}$				330	2698 ± 266		
$\Delta_{7/2}$	2209	−5	168	0	2372 ± 153	2387 ± 88	$\Delta(2390)^{*}$
$\Sigma'_{7/2}$				110	2482 ± 167		
$\Xi'_{7/2}$				220	2592 ± 203		
$\Omega_{7/2}$				330	2702 ± 252		
$\Delta_{9/2}$	2209	1	168	0	2378 ± 144	2318 ± 132	$\Delta(2300)^{**}$
$\Sigma'_{9/2}$				110	2488 ± 159		
$\Xi'_{9/2}$				220	2598 ± 197		
$\Omega_{9/2}$				330	2708 ± 247		
$\Delta_{11/2}$	2209	7	168	0	2385 ± 164	2400 ± 100	$\Delta(2420)^{****}$
$\Sigma_{11/2}$				110	2495 ± 177		
$\Xi_{11/2}$				220	2605 ± 212		
$\Omega_{11/2}$				330	2715 ± 260		

states at fixed J. Accordingly, in Table II the states Σ_J and Σ'_J are defined as

$$|\Sigma_J\rangle = |\Sigma_J^{(8)}\rangle \cos\theta_J^\Sigma + |\Sigma_J^{(10)}\rangle \sin\theta_J^\Sigma, \tag{33}$$

$$|\Sigma'_J\rangle = -|\Sigma_J^{(8)}\rangle \sin\theta_J^\Sigma + |\Sigma_J^{(10)}\rangle \cos\theta_J^\Sigma. \tag{34}$$

The masses of the physical states become

$$M(\Sigma_J) = M(\Sigma_J^{(8)}) + d_2\langle\Sigma_J^{(8)}|B_2|\Sigma_J^{(10)}\rangle \tan\theta_J^\Sigma, \tag{35}$$

$$M(\Sigma'_J) = M(\Sigma_J^{(10)}) - d_2\langle\Sigma_J^{(8)}|B_2|\Sigma_J^{(10)}\rangle \tan\theta_J^\Sigma, \tag{36}$$

where $M(\Sigma_J^{(8)})$ and $M(\Sigma_J^{(10)})$ are the diagonal matrix of the mass operator (16). The expression of the mixing angle can be found in Matagne and Stancu (2005a) together with a discussion about the fitting procedure. Similar relations hold for Ξ.

VII. EXCITED MIXED SYMMETRIC SPIN-FLAVOR STATES

In fact, the first application of the large N_c method was a phenomenological analysis of strong decays of $\ell = 1$ orbitally excited baryons (Carone *et al.*, 1994). An important purpose was to show that the success of the nonrelativistic quark model has a natural explanation in large N_c QCD. It was based on the Hartree approximation suggested by Witten (1979b).

Presently there are two procedures of applying the $1/N_c$ expansion to the study of the mixed symmetric states. We describe them shortly and consider applications in the following sections.

A. The symmetric core + excited quark procedure

The first, called in the following the symmetric core + excited quark procedure, is also inspired by the Hartree picture and is based on the separation of the N_c-quark system into a ground state symmetric core of $N_c - 1$ quarks and an excited quark. Within this procedure, the study of the matrix elements of the mass operators relevant at the lowest nontrivial order (Goity, 1997) was followed by the first phenomenological analysis of electromagnetic transitions (Carlson and Carone, 1998a, 1998b), and by an analysis of the nonstrange $\ell = 1$ baryon masses of the $N = 1$ band (Carlson *et al.*, 1998, 1999), later extended to strange baryons (Goity, Schat, and Scoccola, 2002; Schat, Goity, and Scoccola, 2002).

In the symmetric core + excited quark procedure each $SU(2N_f)$ generator is split into two parts,

$$S^i = S_c^i + s^i, \qquad T^a = T_c^a + t^a, \qquad G^{ia} = G_c^{ia} + g^{ia}, \tag{37}$$

TABLE III. The 12 linearly independent spin-singlet flavor-singlet operators for SU(4), in powers of $1/N_c$ in their matrix elements. For $F > 2$, and ignoring possible coherence in matrix elements of T^a_c, one must include $(1/N_c^2)tS_cG_c$ and $(1/N_c^2)\ell^i g^{ia} S_c^j G_c^{ja}$ in row N_c^{-1}. From Carlson *et al.*, 1998, 1999.

Order	Operator
N_c^1	N_c
N_c^0	ℓs, $\frac{1}{N_c}\ell t G_c$, $\frac{1}{N_c}\ell^{(2)} g G_c$
N_c^{-1}	$\frac{1}{N_c}tT_c$, $\frac{1}{N_c}\ell S_c$, $\frac{1}{N_c}\ell g T_c$, $\frac{1}{N_c}S_c^2$, $\frac{1}{N_c}sS_c$, $\frac{1}{N_c}\ell^{(2)}sS_c$, $\frac{1}{N_c^2}\ell^{(2)}t\{S_c, G_c\}$, $\frac{1}{N_c^2}\ell^i g^{ja}\{S_c^j, G_c^{ia}\}$

where the operators carrying a lower index c act on a symmetric ground state core and s^i, t^a, and g^{ia} act on the excited quark.

The procedure has the algebraic advantage that it reduces the problem of the knowledge of the matrix elements of the $SU(2N_f)$ generators S^i, T^a, and G^{ia}, acting on the whole system, to the knowledge of the matrix elements of S_c^i, T_c^a, and G_c^{ia}, acting on symmetric states of partition $[N_c - 1]$, which are simpler to find than the matrix elements of the $[N_c - 1, 1]$ mixed symmetric states. In fact, they were already derived for SU(4) (Pirjol and Yan, 1998a, 1998b) at the time the procedure was proposed.

Then the operator reduction rules for the ground state (Dashen, Jenkins, and Manohar, 1995) may be used for the core operators. However, the number of terms to be included in operators describing observables remains usually very large as compared to the experimental data. The list of 12 linearly independent spin-singlet flavor-singlet operators for SU(4), in powers of $1/N_c$ in their matrix elements, shown in Table III, was constructed in Carlson *et al.* (1998, 1999).

Later on the method was formally supported by Pirjol and Schat (2008) in a large N_c quark model described in a permutation group context and an application to $\ell = 1$ mixed symmetric states was considered. Starting from an exact wave function for the whole system of N_c quarks, Pirjol and Schat (2008) performed a matching calculation of a general two-body quark-quark interaction onto operators of the $1/N_c$ expansion. The separation of the system into a core + excited quark was made on purpose by introducing Eqs. (37). The main result is a mass formula where the coefficients are defined by linear combinations of radial overlap integrals containing the form factors of the quark-quark interaction. These definitions imply constraints on the dynamical coefficients because they are expressed in terms of common integrals. The Pauli principle is fulfilled provided these constraints are satisfied. But in practice the coefficients are varied independently so that the Pauli principle is fulfilled only within the symmetric core and one recovers the Hartree approximation.

Moreover, one should note that the symmetric core + excited quark procedure is simple for mixed symmetric states with one excited quark, i.e., those belonging to the $N = 1$ band. For $N > 1$ bands, where more than one quark is excited, the technique becomes more complicated as shown for mixed symmetric multiplets of the $N = 2$ band (Matagne and Stancu, 2005b).

A simpler approach is desired. This is described in Sec VII.B. In this approach the Pauli principle is fulfilled for the entire system of N_c quarks, so that the orbital-spin-flavor wave function is totally symmetric. This method requires and provides the matrix elements of $SU(2N_f)$ generators between states of mixed symmetry of partition $[N_c - 1, 1]$. The procedure is valid for any number of excited quarks which do not need to be separated from the whole system, and it can conveniently be applied to any excitation band having $N \geq 1$.

B. The totally symmetric orbital-spin-flavor wave function procedure

We remind the reader that we deal with a system of N_c quarks having ℓ units of orbital excitation. Therefore the orbital (O) wave function must have a mixed symmetry $[N_c - 1, 1]$, which describes the lowest excitations in a baryon.

If the color wave function is antisymmetric, the orbital-spin-flavor wave part must be symmetric. Then the spin-flavor (FS) part must have the same symmetry as the orbital part in order to obtain a totally symmetric state in the orbital-spin-flavor space.

Matagne and Stancu (2008a), as an alternative, proposed an approach where the separation of the system into a symmetric core of $N_c - 1$ quarks and an excited quark is neither necessary nor desired. In that case one deals with $SU(2N_f)$ generators acting on the whole system of N_c quarks and the number of independent operators needed in the mass formula is generally smaller than the number of the experimental data. The resulting mass formula is therefore more physically transparent and its simple form allows applications to multiplets belonging to any band with $N \geq 1$, even in cases where the data are more scarce. Examples will be given later on. First we discuss the difference between the two procedures.

In the exact orbital-spin-flavor wave function both the orbital and the spin-flavor parts of the total wave function are described by the partition $[f] = [N_c - 1, 1]$. By inner product rules of the permutation group one can form a totally symmetric orbital-spin-flavor wave function described by the partition $[N_c]$ as

$$|[N_c]\rangle = \frac{1}{\sqrt{d_{[N_c-1,1]}}} \sum_Y |[N_c - 1, 1]Y\rangle_O |[N_c - 1, 1]Y\rangle_{FS}, \tag{38}$$

where $d_{[N_c-1,1]} = N_c - 1$ is the dimension of the representation $[N_c - 1, 1]$ of the permutation group S_{N_c} and Y labels a Young tableau (or a Yamanouchi symbol). The sum is performed over all possible standard Young tableaux. In each term the first basis vector represents the orbital space (O) and the second the spin-flavor space (FS). In this sum there is only one Y (the normal Young tableau) where the N_cth particle is in the second row and $N_c - 2$ terms where the N_cth particle is in the first row. In the symmetric core + excited quark procedure the latter terms are ignored. An example is given in Appendix D.

In our approach, the system of N_c quarks is described by the wave function (38). We therefore treat the quarks as identical, whether they are excited or not. Assuming that the whole system has an orbital angular momentum ℓ we identify the orbital part in Eq. (38) with a spherical harmonic $|\ell m_\ell\rangle$ and the spin-flavor part in SU(6) with a basis vector $|[f](\lambda\mu)YII_3;SS_3\rangle$ of SU(6) defined as adequate inner products of spin $|SS_3\rangle$ and SU(3)-flavor states $|(\lambda\mu)YII_3\rangle$ which span the invariant subspace of an SU(3) irrep $(\lambda\mu)$.

Following Matagne and Stancu (2011a) the most general form of such a symmetric orbital-spin-flavor wave function in $\mathrm{SU}(6)\times\mathrm{O}(3)$, having a total angular momentum J and projection J_3, is given by

$$|\ell S;JJ_3;(\lambda\mu)YII_3\rangle = \sum_{m_\ell,S_3}\left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & S_3 & J_3 \end{array}\right)|\ell m_\ell\rangle|[f]SS_3;(\lambda\mu)YII_3\rangle, \tag{39}$$

where the first factor is the usual Clebsch-Gordan coefficient of SU(2). In the present case we have $[f]=[N_c-1,1]$ which does not need to be specified for $|\ell m_\ell\rangle$. This form is similar to that of symmetric states given in Eq. (28). For spectrum calculations or other observables one needs to know the matrix elements of the $\mathrm{SU}(2N_f)$ generators, S^i, T^a, and G^{ia}, between the states (39). They are explicitly given in Appendix A, for both $[f]=[N_c]$ and $[f]=[N_c-1,1]$, together with some of their properties. The results for SU(4) were derived by Hecht and Pang (1969) in the context of nuclear physics and for SU(6) the isoscalar factors were mostly obtained by Matagne and Stancu (2009) and completed by Matagne and Stancu (2011a).

To summarize, the relation between the two approaches is remotely similar to that between Hartree and Hartree-Fock approaches. The symmetric core + excited quark approach is simpler, limiting the application of the Pauli principle to a symmetric core of N_c-1 quarks. The procedure of this section is more complicated from a group theory point of view, but rigorously takes into account the Pauli principle for the entire system of N_c quarks.

The symmetric core + excited quark approach has the merit of being the first proposed but the separation of the system into a symmetric core and an excited quark leads to an excessively large number of independent operators, making difficult the choice of dominant operators and the understanding of their physical meaning. Even when the fit looks acceptable some of the contributions to the mass cancel mutually, which may suggest that the decomposition of the system was not necessary. As we see in Sec. VIII.A there appear peculiar situations, as, for example, the case of $\Lambda(1405)$, where the entire spin-spin interaction is removed by construction, because of the approximation made in the wave function of the symmetric core + excited quark approach. Furthermore, except for one case, the approach has not been applied to higher mass resonances ($N>1$ band) most probably because of being cumbersome (the core is no more in the ground state), as explained in Sec. VIII.C.

VIII. SPECTRUM CALCULATIONS FOR MIXED SYMMETRIC STATES

Next we present a summary of results for resonances described as mixed symmetric states of either negative or positive parity.

A. The lowest negative parity $[\mathbf{70},1^-]$ multiplet

In the baryon spectrum, the $[\mathbf{70},1^-]$ multiplet has been most extensively studied, being the best experimentally known negative parity mixed symmetric multiplet. For $N_f=2$ there are numerous studies as, for example, Carone *et al.* (1994), Goity (1997), Carlson *et al.* (1998, 1999), Carlson and Carone (1998a, 1998b), and Pirjol and Yan (1998a, 1998b).

The above studies were in the spirit of the Hartree approximation where the system of N_c quarks was split into a ground state core of N_c-1 quarks and an excited quark, as described in Sec. VII.A. This means that each generator of $\mathrm{SU}(2N_f)$ was written as a sum of two terms, one acting on the excited quark and the other on the core, as in Eq. (37). Then, as mentioned, the number of the coefficients c_i in the mass formula is too large compared to the available data on resonance masses and cannot be uniquely determined in a numerical fit, as has been done for the lowest negative parity nonstrange baryons (Carlson *et al.*, 1998, 1999). Accordingly, the choice of the most dominant operators in the mass formula (16) became out of control which implies that important physical effects can be missed, as discussed next.

Several fits were performed by Carlson *et al.* (1998, 1999) for nonstrange baryons. From our point of view the most interesting one is the result given in Table VII of that paper, which is consistent with the mechanism of the Goldstone-boson exchange (GBE) model (Glozman and Riska, 1996; Glozman *et al.*, 1998). In this fit the operator $(1/N_c)\ell^{(2)}gG_c$ plays a crucial role and is related to a pion exchange between the excited quark and a core quark.

To our knowledge the $N_f=3$ case was considered only by Goity, Schat, and Scoccola (2002) and Schat, Goity, and Scoccola (2002) in the symmetric core + excited quark procedure, where first order corrections in SU(3) symmetry breaking were also included. For both $N_f=2$ and $N_f=3$ cases, the conclusion was that the splitting starts at order N_c^0. The list of dominant operators and the best fit coefficients in the mass formula (16) is shown in Table IV. The fit was made to 19 empirical quantities (17 masses and 2 mixing angles) associated with resonances with three or more stars status and it gives $\chi^2_{\mathrm{dof}}=1.29$.

There are 11 operators of type O_i and four of type B_i included in the mass formula. One can see that the coefficients c_3 and c_4 are large, consistent with the SU(4) case where it was found that the operator $\propto(1/N_c)\ell^{(2)}gG_c$ plays a crucial role, as mentioned. The contribution of the spin operators O_6 and O_7 is large, as expected, but there is some mutual cancellation. Some operators of a more complex nature such as O_9, O_{10}, and O_{11} contribute also substantially, but the total contribution somewhat cancels out. One should notice the absence of the flavor term $t\cdot T_c$, never included in the

TABLE IV. The dominant operators and the best fit coefficients for the masses of nonstrange and strange baryons belonging to the $[\mathbf{70}, 1^-]$ multiplet with $\chi^2_{\rm dof} = 1.29$. From Schat, Goity, and Scoccola, 2002.

Operator	Fitted coefficient (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 = 449 \pm 2$
$O_2 = l_h s_h$	$c_2 = 52 \pm 15$
$O_3 = \frac{3}{N_c} l^{(2)}_{hk} g_{ha} G^c_{ka}$	$c_3 = 116 \pm 44$
$O_4 = \frac{4}{N_c+1} l_h t_a G^c_{ha}$	$c_4 = 110 \pm 16$
$O_5 = \frac{1}{N_c} l_h S^c_h$	$c_5 = 74 \pm 30$
$O_6 = \frac{1}{N_c} S^c_h S^c_h$	$c_6 = 480 \pm 15$
$O_7 = \frac{1}{N_c} s_h S^c_h$	$c_7 = -159 \pm 50$
$O_8 = \frac{1}{N_c} l^{(2)}_{hk} s_h S^c_k$	$c_8 = 6 \pm 110$
$O_9 = \frac{1}{N_c^2} l_h g_{ka} \{S^c_k, G^c_{ha}\}$	$c_9 = 213 \pm 153$
$O_{10} = \frac{1}{N_c^2} t_a \{S^c_h, G^c_{ha}\}$	$c_{10} = -168 \pm 56$
$O_{11} = \frac{1}{N_c^2} l_h g_{ha} \{S^c_k, G^c_{ka}\}$	$c_{11} = -133 \pm 130$
$B_1 = t_8 - \frac{1}{2\sqrt{3}N_c} O_1$	$d_1 = -81 \pm 36$
$B_2 = T^c_8 - \frac{N_c-1}{2\sqrt{3}N_c} O_1$	$d_2 = -194 \pm 17$
$B_3 = \frac{1}{N_c} d_{8ab} g_{ha} G^c_{hb} + \frac{N_c^2-9}{16\sqrt{3}N_c^2(N_c-1)} O_1 + \frac{1}{4\sqrt{3}(N_c-1)} O_6 + \frac{1}{12\sqrt{3}} O_7$	$d_3 = -150 \pm 301$
$B_4 = l_h g_{h8} - \frac{1}{2\sqrt{3}} O_2$	$d_4 = -82 \pm 57$

analyses based on the symmetric core + excited quark approach in SU(6).

We remind the reader that in the symmetric core + excited quark approach the total flavor operator was written as the sum of three terms $T \cdot T = T_c \cdot T_c + 2t \cdot T_c + 3/4$, each thought to be linearly independent. The first term acts on the core and its matrix elements are identical to those of $S_c \cdot S_c$ when the SU(4) spin-flavor state is symmetric. Then its contribution cannot be distinguished from that of $S_c \cdot S_c$. The last term, the constant $3/4$, can be absorbed in the leading-order term but to our understanding $t \cdot T_c$ cannot be ignored.

The calculated masses were compared with the quark model results of Isgur and Karl (1978) based on an oscillator confinement, where the oscillator parameter was fitted to the $N = 1$ band. In the work of Isgur and Karl the hyperfine interaction is represented by the spin-spin and tensor parts of the Fermi-Breit Hamiltonian derived from the one gluon exchange (OGE) (De Rujula, Georgi, and Glashow, 1975). The spin-orbit part is neglected. The resonance $\Lambda(1405)$ appeared by about 100 MeV too high, as in more recent studies, based on the more realistic linear confinement and the flavor dependent Goldstone-boson exchange interaction (Glozman *et al.*, 1998), which reproduces the correct level ordering of the Roper and the first negative parity nonstrange baryons, impossible to obtain in models based on the one gluon-exchange interaction.

Goity, Schat, and Scoccola (2002) and Schat, Goity, and Scoccola (2002) explained the lightness of $\Lambda(1405)$ and $\Lambda(1520)$, seen as spin-orbit partners, by the fact that the spin-spin terms $(1/N_c) S_c \cdot S_c$ and $(1/N_c) s \cdot S_c$ do not contribute to their masses because the core has $S_c = 0$. This is the effect of the simplicity of their wave function where the part corresponding to the spin $S_c = 1$ is missing, as inferred by the arguments of Sec. VII.B. However, the spin-spin interaction cannot be neglected, even though it is of the order of $1/N_c$, because it is the leading term that splits N and Δ. In octets and decuplets the spin-spin interaction survives, despite the approximate wave function. It causes their masses to rise because there the core has a nonzero spin component.

The spin-orbit splitting is explained as the combined effect of the operators O_4, O_5, O_9, and O_{11}. The large error bars of the coefficients of the operators O_8 and B_3 make these operators irrelevant in the mass formula.

The $[\mathbf{70}, 1^-]$ lowest multiplet was also analyzed within the framework described in Sec. VII.B, based on the totally symmetric orbital-spin-flavor wave function, first in SU(4) (Matagne and Stancu, 2008a) and next in SU(6) (Matagne and Stancu, 2011a). The list of dominant operators and the numerical results for c_i and d_i obtained by Matagne and Stancu (2011a) are presented in Table V. One can see that the number of operators used in the fit is considerably smaller than that of Table IV. The one-body spin-orbit operator O_2 is the same as in Table IV. The spin operator O_3 and the flavor

TABLE V. Operators and their coefficients in the mass formula (16), obtained from three distinct numerical fits. The values of c_i and d_i are indicated under the heading Fit n $(n = 1, 2, 3)$, in each case (Matagne and Stancu, 2011a).

Operator	Fit 1 (MeV)	Fit 2 (MeV)	Fit 3 (MeV)
$O_1 = N_c \mathbb{1}$	489 ± 4	492 ± 4	492 ± 4
$O_2 = \ell^i s^i$	24 ± 6	6 ± 6	6 ± 5
$O_3 = \frac{1}{N_c} S^i S^i$	129 ± 10	123 ± 10	123 ± 10
$O_4 = \frac{1}{N_c} [T^a T^a - \frac{1}{12} N_c (N_c + 6)]$	145 ± 16	134 ± 16	135 ± 16
$O_5 = \frac{3}{N_c} L^i T^a G^{ia}$	-19 ± 7	3 ± 7	4 ± 3
$O_6 = \frac{15}{N_c} L^{(2)ij} G^{ia} G^{ja}$	9 ± 1	9 ± 1	9 ± 1
$O_7 = \frac{1}{N_c^2} L^i G^{ja} \{S^j, G^{ia}\}$	129 ± 33	6 ± 33	
$B_1 = -\mathcal{S}$	138 ± 8	138 ± 8	137 ± 8
$B_2 = \frac{1}{N_c} \sum_{i=1}^{3} T^i T^i - O_4$	-59 ± 18	-40 ± 18	-40 ± 18
$\chi^2_{\rm dof}$	1.7	0.9	0.84

operator O_4 are two body. The operator O_4 in SU(3) was defined in Matagne and Stancu (2011a) such as to be applicable to flavor singlets as well. For octets and decuplets it gives the same matrix elements as the isospin operator $(1/N_c)T^aT^a$ in SU(4), of the order of $1/N_c$. For flavor singlets the order of the matrix elements of O_4 is N_c^0. The operators O_5 and O_6 are two body, but G^{ia} sums coherently in both and introduces a factor N_c except for the ^{28} multiplets. The operator O_7 is three body and has a more complex form, but it contains the generator G^{ia} 2 times so that the order of its matrix elements becomes N_c^0. However, looking at Table V and comparing Fit 2 and Fit 3, where O_7 has been removed in the latter from the mass formula, one can see that its role is negligible. The SU(3) breaking operator B_1 represents the total strangeness and B_2 was defined to account for the Λ-Σ splitting. The diagonal and off-diagonal matrix elements of O_i as a function of N_c can be found in Matagne and Stancu (2011a). The isoscalar factors of Tables XV–XVII, and XVIII of Appendix A were used to obtain their analytic expressions.

The numerical fit included the 17 resonances from the Particle Data Group 2010 (Nakamura *et al.*, 2010), with a status of three and four stars and two mixing angles. Fit 1 is based on the experimental value $M(\Lambda(1405)) = 1407$ MeV which gives $\chi^2_{\rm dof} = 1.7$. To improve the fit we took the value 1500 MeV for the mass of $\Lambda(1405)$, inspired by quark model studies where usually $M(\Lambda(1405))$ appears too high, as mentioned. This is the result of Fit 2 where $\chi^2_{\rm dof}$ lowers to 0.9.

The $\Lambda(1405)$ resonance is a long-standing problem. Deeper dynamical arguments are necessary to understand its exceptionally low mass [for a review see, for example, Hyodo and Jido (2012)].

Table VI reproduces the partial contribution and the total mass obtained by using the coefficients of Fit 1. One can see that in flavor singlets the contribution of the spin operator O_3 is not particularly large but the flavor operator O_4 brings an

TABLE VI. The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion obtained from Fit 1. The last two columns give the empirically known masses (Nakamura *et al.*, 2010) and the resonance name and status. From Matagne and Stancu, 2011a.

	Partial contributions (MeV)											
	c_1O_1	c_2O_2	c_3O_3	c_4O_4	c_5O_5	c_6O_6	c_7O_7	d_1B_1	d_2B_2	Total (MeV)	Experiment (MeV)	Name, status
$N_{1/2}$	1467	−8	32	36	19	0	−31	0	0	1499 ± 10	1538 ± 18	$S_{11}(1535)^{****}$
$\Lambda_{1/2}$								138	15	1668 ± 9	1670 ± 10	$S_{01}(1670)^{****}$
$\Sigma_{1/2}$								138	−25	1628 ± 10		
$\Xi_{1/2}$								276	0	1791 ± 13		
$N_{3/2}$	1467	4	32	36	−10	0	16	0	0	1542 ± 10	1523 ± 8	$D_{13}(1520)^{****}$
$\Lambda_{3/2}$								138	15	1698 ± 8	1690 ± 5	$D_{03}(1690)^{****}$
$\Sigma_{3/2}$								138	−25	1658 ± 9	1675 ± 10	$D_{13}(1670)^{****}$
$\Xi_{3/2}$								276	0	1821 ± 11	1823 ± 5	$D_{13}(1820)^{***}$
$N'_{1/2}$	1467	−20	162	36	48	−18	42	0	0	1648 ± 11	1660 ± 20	$S_{11}(1650)^{****}$
$\Lambda'_{1/2}$								138	15	1784 ± 16	1785 ± 65	$S_{01}(1800)^{***}$
$\Sigma'_{1/2}$								138	−25	1745 ± 17	1765 ± 35	$S_{11}(1750)^{***}$
$\Xi'_{1/2}$								276	0	1907 ± 20		
$N'_{3/2}$	1467	−8	162	36	19	15	−17	0	0	1675 ± 10	1700 ± 50	$D_{13}(1700)^{***}$
$\Lambda'_{3/2}$								138	15	1826 ± 12		
$\Sigma'_{3/2}$								138	−25	1787 ± 13		
$\Xi'_{3/2}$								276	0	1949 ± 16		
$N_{5/2}$	1467	12	162	36	−29	−4	25	0	0	1669 ± 10	1678 ± 8	$D_{15}(1675)^{****}$
$\Lambda_{5/2}$								138	15	1822 ± 10	1820 ± 10	$D_{05}(1830)^{****}$
$\Sigma_{5/2}$								138	−25	1782 ± 11	1775 ± 5	$D_{15}(1775)^{****}$
$\Xi_{5/2}$								276	0	1945 ± 14		
$\Delta_{1/2}$	1467	8	32	181	38	0	−24	0	0	1702 ± 18	1645 ± 30	$S_{31}(1620)^{****}$
$\Sigma''_{1/2}$								138	34	1875 ± 16		
$\Xi''_{1/2}$								276	59	2037 ± 22		
$\Omega_{1/2}$								413	74	2190 ± 29		
$\Delta_{3/2}$	1467	−4	32	181	−19	0	12	0	0	1668 ± 20	1720 ± 50	$D_{33}(1700)^{****}$
$\Sigma''_{3/2}$								138	34	1841 ± 16		
$\Xi''_{3/2}$								276	59	2003 ± 21		
$\Omega_{3/2}$								413	74	2156 ± 27		
$\Lambda''_{1/2}$	1467	−24	32	−108	0	0	−38	138	−44	1421 ± 14	1407 ± 4	$S_{01}(1405)^{****}$
$\Lambda''_{3/2}$	1467	12	32	−108	0	0	19	138	−44	1515 ± 14	1520 ± 1	$D_{03}(1520)^{****}$
$N_{1/2} - N'_{1/2}$	0	−8	0	0	−10	−55	18	0	0	−55		
$N_{3/2} - N'_{3/2}$	0	−12	0	0	−15	17	28	0	0	18		

TABLE VII. Operators and their coefficients in the mass formula obtained from four numerical fits of highly excited negative parity resonances of the $N = 3$ band (Matagne and Stancu, 2012a). The values of c_i and d_i are indicated under the heading Fit n $(n = 1, 2, 3, 4)$.

Operator	Fit 1 (MeV)	Fit 2 (MeV)	Fit 3 (MeV)	Fit 4 (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 = 672 \pm 8$	$c_1 = 673 \pm 7$	$c_1 = 672 \pm 8$	$c_1 = 673 \pm 7$
$O_2 = \ell^i s^i$	$c_2 = 18 \pm 19$	$c_2 = 17 \pm 18$	$c_2 = 19 \pm 9$	$c_2 = 20 \pm 9$
$O_3 = \frac{1}{N_c} S^i S^i$	$c_3 = 121 \pm 59$	$c_3 = 115 \pm 46$	$c_3 = 120 \pm 58$	$c_3 = 112 \pm 42$
$O_4 = \frac{1}{N_c}[T^a T^a - \frac{1}{12} N_c (N_c + 6)]$	$c_4 = 202 \pm 41$	$c_4 = 200 \pm 40$	$c_4 = 205 \pm 27$	$c_4 = 205 \pm 27$
$O_5 = \frac{3}{N_c} L^i T^a G^{ia}$	$c_5 = 1 \pm 13$	$c_5 = 2 \pm 12$		
$O_6 = \frac{15}{N_c} L^{(2)ij} G^{ia} G^{ja}$	$c_6 = 1 \pm 6$		$c_6 = 1 \pm 5$	
$B_1 = -\mathcal{S}$	$d_1 = 108 \pm 93$	$d_1 = 108 \pm 92$	$d_1 = 109 \pm 93$	$d_1 = 108 \pm 92$
$\chi^2_{\rm dof}$	1.23	0.93	0.93	0.75

essential contribution in lowering the masses of $\Lambda(1405)$ and $\Lambda(1520)$. The spin-orbit partners $N_J - N'_J$ $(J = 1/2, 3/2)$ receive contributions from the operators O_2, O_5, O_6, and O_7 via their off-diagonal matrix elements.

The global conclusion is that both the spin O_3 and the flavor operator O_4 contribute dominantly to the spin-flavor breaking. In particular, the flavor operator contributes to the masses of decuplets and flavor singlets with a coefficient of the same order as that of the spin operator in octets. Thus, in the symmetric core + excited quark approach, even if the contribution of $T_c \cdot T_c$ is identified to that of $S_c \cdot S_c$ there is no reason to ignore the isospin term $t \cdot T_c$, as a part of $T \cdot T$, as explained earlier.

B. Highly excited negative parity states

In the approach proposed by Matagne and Stancu (2011a), based on the exact wave function, as described in Sec. VII.B, the number of linearly independent operators in the mass formula is considerably reduced as compared to the ground state core + excited quark procedure. Thus it was possible to analyze highly excited states belonging to the $N = 3$ band (Matagne and Stancu, 2012a) where the experimental data are still scarce.

The $N = 3$ band contains eight $SU(6) \times O(3)$ multiplets. According to the notation of Stancu and Stassart (1991) these are $[\mathbf{56}, 1^-]$, $[\mathbf{56}, 3^-]$, $[\mathbf{70}', 1^-]$, $[\mathbf{70}'', 1^-]$, $[\mathbf{70}, 2^-]$, $[\mathbf{70}, 3^-]$, $[\mathbf{20}, 1^-]$, and $[\mathbf{20}, 3^-]$, where $[\mathbf{70}', 1^-]$ and $[\mathbf{70}'', 1^-]$ correspond to radial excitations. This classification provides 45 nonstrange states (one state $N_{9/2^-}$, one state $\Delta_{9/2^-}$, five states $N_{7/2^-}$, two states $\Delta_{7/2^-}$, eight states $N_{5/2^-}$, four states $\Delta_{5/2^-}$, nine states $N_{3/2^-}$, five states $\Delta_{3/2^-}$, seven states $N_{1/2^-}$, and three states $\Delta_{1/2^-}$). The analysis of Matagne and Stancu (2012a) included all mixed symmetric multiplets $[\mathbf{70}, \ell^-]$ ($\ell = 1, 2$, and 3) of the band.

Experimentally in the 1900–2400 MeV mass region the 2010 Particle Data Group (Nakamura *et al.*, 2010) provided the following resonances: $N_{19}(2250)^{****}$, $N_{17}(2190)^{****}$, $\Lambda_{07}(2100)^{****}$, $N_{15}(2220)^{**}$, $N_{13}(2080)^{*}$, $N_{11}(2090)^{*}$, $\Delta_{37}(2220)^{*}$, and $\Delta_{35}(2350)^{*}$, which may be interpreted as belonging to mixed symmetric multiples $[\mathbf{70}, \ell^-]$ ($\ell = 1, 2$, and 3), in agreement with Stancu and Stassart (1991). To them two new ones were added, namely, $N_{15}(2060)$ and $N_{13}(2120)$, proposed by Anisovich *et al.* (2011, 2012), which presently acquired a two-star status (Beringer *et al.*, 2012). Note that to the latter a mass of 2150 MeV was associated as initially reported by Anisovich *et al.* (2012).

Four distinct numerical fits were performed by including the operators $O_1, \ldots, O_6$ and B_1 of Table V, by analogy with the $N = 1$ band, from which the operator O_7 has been neglected. The results of the numerical fits are exhibited in Table VII. One can see that the contributions of O_5 and O_6, depending on the angular momentum, are negligible, but the coefficient of the spin-orbit operator, although small, remains important to the fit. The spin operator O_3 brings a dominant contribution to ^4N resonances and the isospin operator O_4 brings an even larger contribution to the masses of Δ and Λ resonances. As Table VIII shows, in the latter case its sign is negative and improves the agreement to the experiment.

Therefore, like in the $N = 1$ band, one can see that the isospin operator neglected in the symmetric core + excited quark approach is important and definitely crucial in fitting the mass of the $\Lambda_{07}(2100)^{****}$ resonance.

C. Positive parity mixed symmetric states

Here we present results for the masses of nonstrange and strange baryon resonances thought to belong to the $[\mathbf{70}, 0^+]$ and $[\mathbf{70}, 2^+]$ multiplets of the $N = 2$ band.

Although tedious in extending the symmetric core + excited quark approach to more than one excited quark an effort has been made to apply it to the $N = 2$ band (Matagne and Stancu, 2005b, 2006b), where the orbital wave function contains a term where two quarks are excited to the p shell. For example, using the quark model notation ρ and λ for mixed symmetric three-quark states with the pair 1,2 in an antisymmetric and a symmetric state, respectively, one can write the orbital wave function for $\ell = 2$ as

$$|[N_c - 1, 1]2^+\rangle_{\rho,\lambda} = \sqrt{\frac{1}{3}}|[N_c - 1, 1]_{\rho,\lambda}(0s)^{N_c - 1}(0d)\rangle + \sqrt{\frac{2}{3}}|[N_c - 1, 1]_{\rho,\lambda}(0s)^{N_c - 2}(0p)^2\rangle, \quad (40)$$

where the two quarks in the p shell are coupled to $\ell = 2$. In the first term a quark is excited in the d shell so it can be treated as in the $[\mathbf{70}, 1^-]$ multiplet. The second term can be treated as an excited quark coupled to an excited core and one can use the fractional parentage technique developed by

TABLE VIII. Partial contributions and the total mass (MeV) predicted by the $1/N_c$ expansion, obtained from Fit 4 of Table VII. The last two columns indicate the empirically known masses and the resonance name and status (whenever known).

	Partial contributions (MeV)							
	c_1O_1	c_2O_2	c_3O_3	c_4O_4	d_1B_1	Total (MeV)	Experiment (MeV)	Name, status
$^4N[\mathbf{70},3^-]_{9/2}$	2018	29	140	51	0	2238 ± 46	2275 ± 75	$G_{19}(2250)^{****}$
$^2N[\mathbf{70},3^-]_{7/2}$	2018	10	28	51	0	2107 ± 17	2150 ± 50	$G_{17}(2190)^{****}$
$^4N[\mathbf{70},3^-]_{5/2}$	2018	−23	140	51	0	2186 ± 41	2180 ± 80	$D_{15}(2200)^{**}$
$^2N[\mathbf{70},3^-]_{5/2}$	2018	−39	28	51	0	2058 ± 14	2060 ± 15	$D_{15}(2060)$
$^4N[\mathbf{70},3^-]_{3/2}$	2018	−39	140	51	0	2170 ± 42	2150 ± 60	$D_{13}(2150)$
$^2N[\mathbf{70}',1^-]_{3/2}$	2018	3	28	51	0	2101 ± 14	2081 ± 20	$D_{13}(2080)^{*}$
$^2N[\mathbf{70}',1^-]_{1/2}$	2018	−7	28	51	0	2091 ± 12	2100 ± 20	$S_{11}(2090)^{*}$
$^2\Delta[\mathbf{70},3^-]_{7/2}$	2018	−10	28	256	0	2292 ± 25	2200 ± 80	$G_{37}(2220)^{*}$
$^2\Delta[\mathbf{70},2^-]_{5/2}$	2018	−7	28	256	0	2295 ± 25	2305 ± 26	$D_{35}(2350)^{*}$
$^2\Lambda[\mathbf{70},3^-]_{7/2}$	2018	29	28	−153	108	2030 ± 82	2030 ± 82	$G_{07}(2100)^{****}$

Matagne and Stancu (2005b). In this case the construction of the orbital part of the wave function becomes rather complicated which is the case for all bands with $N \geq 2$. The first application (Matagne and Stancu, 2005b) was made to nonstrange baryons using the SU(4) algebra to construct the operators O_i in the mass formula (16). The method was extended by Matagne and Stancu (2006b) to include strange baryons as well. In this case the contribution of operators of type B_i was added according to Eq. (16). There are many linearly independent operators which can be constructed from the excited quark and the excited core operators. To make the method applicable the list has been restricted to those thought to be the most dominant. This is shown in Table IX, where ℓ_q is the angular momentum of the excited quark. The flavor operator O_6 was included. Its contribution is important, especially for flavor singlets. Although listed and discussed in this paper, the operator O_4 was ignored in the fit because of scarcity of data.

There are two operators B_i with one acting on the excited quark, and the other on the core. Their contribution mutually cancels to a large extent. The $\Lambda\Sigma$ splitting obtained in the sector ^{48} is disturbingly large such that it provides for Λ and Ξ baryons nearly equal masses.

The $[\mathbf{70},\ell^+]$ baryons have been revisited (Matagne and Stancu, 2013) by using the procedure described in Sec. VII.B, where the operators act on the entire system. In that analysis the wave function (39) has been used. The list of dominant operators is given in Table X. Note that O_2 is a single particle operator, having the same matrix elements as Matagne and Stancu (2006b). The matrix elements of O_3 and O_4 are easy to calculate. The matrix elements of O_5 and O_6 were obtained from Eq. (B2) and (B4) of Matagne and Stancu (2011a) and the corresponding isoscalar factors of Tables XV–XVII, and XVIII of Appendix A.

The closed analytic form of the matrix elements of O_i, as a function of N_c, are not presented here except for flavor singlets, Table XI, needed for the discussion. Those for octets and decuplets can be found in Tables II and III of Matagne and Stancu (2013), respectively.

There is a single operator that generates the flavor breaking $B_1 = -\mathcal{S}$, the same for all sectors, where $\mathcal{S}$ is the strangeness. In such a case there is no $\Lambda\Sigma$ splitting but the mass sequence with increasing number of strange quarks looks more natural in octets and decuplets compared to the results of Matagne and Stancu (2006b) based on the symmetric core + excited quark approach.

This analysis was also motivated by the fact that the recent multichannel partial wave analysis of Anisovich *et al.* (2012) has revealed the existence of new positive parity resonances presently reported by the Particle Data Group (Beringer *et al.*, 2012).

TABLE IX. List of operators and the coefficients resulting from the fit with $\chi^2_{\rm dof} \simeq 1.0$, for nonstrange and strange baryons belonging to the $[\mathbf{70},\ell^+]$ multiplets ($\ell = 0$ and 2). From Matagne and Stancu, 2006b.

Operator	Fitted coefficient (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 = 556 \pm 11$
$O_2 = \ell^i_q s^i$	$c_2 = -43 \pm 47$
$O_3 = \frac{3}{N_c}\ell_q^{(2)ij} g^{ia} G_c^{ja}$	$c_3 = -85 \pm 72$
$O_4 = \frac{4}{N_c+1}\ell^i_q t^a G_c^{ia}$	
$O_5 = \frac{1}{N_c}(S^i_c S^i_c + s^i S^i_c)$	$c_5 = 253 \pm 57$
$O_6 = \frac{1}{N_c} t^a T^a_c$	$c_6 = -25 \pm 86$
$B_1 = t^8 - \frac{1}{2\sqrt{3}N_c} O_1$	$d_1 = 365 \pm 169$
$B_2 = T^8_c - \frac{N_c-1}{2\sqrt{3}N_c} O_1$	$d_2 = -293 \pm 54$

TABLE X. List of dominant operators and their coefficients in the mass formula (16) obtained in three distinct numerical fits. From Matagne and Stancu, 2013.

Operator	Fit 1	Fit 2	Fit 3
$O_1 = N_c \mathbb{1}$	616 ± 11	616 ± 11	616 ± 11
$O_2 = \ell^i s^i$	150 ± 239	52 ± 44	243 ± 237
$O_3 = \frac{1}{N_c} S^i S^i$	149 ± 30	152 ± 29	136 ± 29
$O_4 = \frac{1}{N_c}[T^a T^a - \frac{1}{12}N_c(N_c+6)]$	66 ± 55	57 ± 51	86 ± 55
$O_5 = \frac{3}{N_c} L^i T^a G^i$	-22 ± 5		-25 ± 52
$O_6 = \frac{15}{N_c} L^{(2)ij} G^{ia} G^{ja}$	14 ± 5	14 ± 5	
$B_1 = -\mathcal{S}$	23 ± 38	24 ± 38	-22 ± 35
$\chi^2_{\rm dof}$	0.61	0.52	2.27

TABLE XI. Matrix elements of O_i for flavor-singlet resonances included in the analysis of Matagne and Stancu (2013).

	O_1	O_2	O_3	O_4	O_5	O_6
$^21[\mathbf{70},2^+]\frac{5}{2}^+$	N_c	$\frac{2}{3}$	$\frac{3}{4N_c}$	$-\frac{2N_c+3}{4N_c}$	$-\frac{N_c-3}{2N_c}$	0
$^21[\mathbf{70},2^+]\frac{3}{2}^+$	N_c	-1	$\frac{3}{4N_c}$	$-\frac{2N_c+3}{4N_c}$	$\frac{3(N_c-3)}{4N_c}$	0
$^21[\mathbf{70},0^+]\frac{1}{2}^+$	N_c	0	$\frac{3}{4N_c}$	$-\frac{2N_c+3}{4N_c}$	0	0

Like for the $N = 1$ and 3 bands, one can see that both the spin and flavor operators O_3 and O_4, respectively, acting on the entire system, bring similar contributions to the mass, although c_4 is smaller than c_3, but what matters is $c_i\langle O_i\rangle$. The operator O_4, having negative matrix elements (see Table XI), helps in providing a good agreement of the mass of $\Lambda(1810)1/2^{+***}$ interpreted as the flavor singlet $^2\Lambda'[\mathbf{70},0^+]1/2$. Table X shows that the operator O_5 is not important for a good fit but O_6 is crucial in obtaining a good $\chi^2_{\rm dof}$.

IX. HEAVY BARYON MASSES IN THE COMBINED $1/N_c$ AND $1/m_Q$ EXPANSION

The heavy-quark limit was first discussed by Witten (1979b). Later on the masses of ground state baryons containing a single heavy quark $Q = c, b$ were studied in a combined $1/m_Q$ and $1/N_c$ expansion and SU(3) flavor-symmetry breaking by Jenkins (1996b, 1997). The combined limit $m_c \to \infty$, $m_b \to \infty$, $N_c \to \infty$ for fixed m_c/m_b and $N_c\Lambda_{\rm QCD}/m_b$ has led to a light quark ℓ and a heavy quark h spin-flavor symmetry $\mathrm{SU}(6)_\ell \times \mathrm{SU}(4)_h$. For finite m_Q and N_c this symmetry is violated by effects of the order of $1/N_c$ and $1/N_c m_Q$. A hierarchy of mass splittings was predicted together with the masses of all of the unknown charmed baryons, as, for example, Σ_c^*, Ξ_c', and Ω_c^* and of all unknown bottom baryons. The masses of the bottom baryons Σ_b, Σ_b^*, and Ξ_b observed ten years later were in good agreement with the theoretical predictions (Jenkins, 2008).

Model-independent predictions for excitation energies and other observables of isoscalar heavy baryons were discussed in a combined heavy quark and large N_c expansions (Aziza Baccouche *et al.*, 2001).

The mass spectrum of the $\ell = 1$ charmed baryons was also studied in the $1/N_c$ method and the heavy-quark effective theory and certain mass relations were derived (Lee, Liu, and Song, 2000). The simplicity of the approach stems from the fact that the light quark system is in the ground state and the heavy quark is orbitally excited. This is an improvement over previous studies made by Lee, Liu, and Song (1998).

X. MASS FORMULA IN THE $1/N_c$ EXPANSION VERSUS THE QUARK MODEL

It is important to see whether or not there is a compatibility between the model-independent $1/N_c$ expansion and the quark models, which successfully describe baryon spectroscopy. As mentioned in Sec. VII, in the first application of the large N_c method to a phenomenological analysis of strong decays of $\ell = 1$ orbitally excited baryons (Carone *et al.*, 1994), the basic purpose was to understand whether the success of the nonrelativistic quark model has a natural explanation in large N_c QCD.

The above application was based on the Hartree approximation suggested by Witten (1979b) which inspired the symmetric core + excited quark procedure of Sec. VII.A. Subsequently the validity of this procedure was formally supported by Pirjol and Schat (2008) in a permutation group context by trying to match a large N_c quark model Hamiltonian with the baryon mass formula (16) of the $1/N_c$ expansion, including orbitally excited baryons, where some operators O_i contain angular momentum components. Only light baryons were considered, i.e., the su(4) algebra. The derivation confirmed the consistency between the order $\mathcal{O}(1/N_c)$ of the corresponding operators O_i shown in Table III and those resulting from large N_c quark models. Moreover, an explicit comparison of the Hamiltonian eigenvalues was made for both the OGE (De Rujula, Georgi, and Glashow, 1975) and the GBE (Glozman and Riska, 1996) models.

Later on Pirjol and Schat (2010) tried to give more insight into the spin-flavor structure of the hyperfine interaction used in quark models. They found that both OGE and GBE quark models are compatible with the $\ell = 1$ nonstrange baryon data.

Independently, a connection between a semirelativistic quark model and the mass formula of the $1/N_c$ expansion was established for light nonstrange baryons (Semay *et al.*, 2007) and for light nonstrange + strange baryons (Semay, Buisseret, and Stancu, 2007), extended afterward to heavy baryons (Semay, Buisseret, and Stancu, 2008); for a detailed review see, for example, Buisseret *et al.* (2008). A clear correspondence was found between various terms of the quark model eigenvalues and those of the $1/N_c$ expansion mass formula.

The spin-independent Hamiltonian used in Semay *et al.* (2007), Semay, Buisseret, and Stancu (2007, 2008), and Buisseret *et al.* (2008) had a relativistic kinetic term and a Y-junction confinement interaction potential to which a Coulomb interaction term and a quark self-energy were added. Using the auxiliary field technique (Silvestre-Brac, Semay, and Buisseret, 2012) one can obtain an analytic expression for the mass of light qqq or heavy qqQ baryons including SU(3) breaking. A key quantity is the band number N in a harmonic oscillator picture, shown to be a good quantum number within the approximations considered in treating the quark model Hamiltonian. This allowed one to compare the dependence of various c_i coefficients as a function of N resulting from the quark model and the $1/N_c$ expansion results described above. For example, for light baryons the quark model mass formula can be shortly written as

$$M_{qqq} = M_0 + n_s \Delta M_s, \tag{41}$$

where the first term holds for exact SU(3) flavor and the second term represents the breaking contribution. One can make the following identification with the mass formula (16):

$$c_1^2 = M_0^2/9, \qquad n_s\Delta M_s = \sum_i d_i B_i, \tag{42}$$

where the number of strange quarks is $n_s = 0, 1, 2$, or 3. The quantities M_0 and ΔM_s are functions of the band number N. Using the analytic form of M_0 in terms of quark model parameters σ, α_0, and f, defined, for example, in Semay *et al.* (2007), one can write

$$c_1^2 = \frac{2\pi}{9}\sigma(N+3) - \frac{4\pi}{9\sqrt{3}}\sigma\alpha_0 - \frac{f\sigma}{3}, \tag{43}$$

with the following choice of parameters (Buisseret *et al.*, 2008):

$$\sigma = 0.163\ \mathrm{GeV}^2, \qquad \alpha_0 = 0.4, \qquad f = 3.6. \tag{44}$$

This expression is plotted as a function of the band number N in Fig. 4 where it is compared with large N_c results. One can see that there is a rather good quantitative agreement between the large N_c and the quark model results for c_1^2. In the quark model c_1^2 contains the effect of the kinetic and of the confinement parts of the spin-independent Hamiltonian and the nonperturbative QCD at large distances becomes dominated by confinement. The present agreement between the quark model results and large N_c QCD brings further support to quark models.

Actually the quantity c_1^2 is linear in the band number N, suggesting a Regge-type behavior obtained from the analytic form of M_0, which is

$$M_0^2 \propto 2\pi\sigma(N+3) \tag{45}$$

containing the quark model parameter σ responsible for the slope.

On the other hand, a Regge-type behavior of the leading spin-flavor singlet term of the large N_c mass formula was

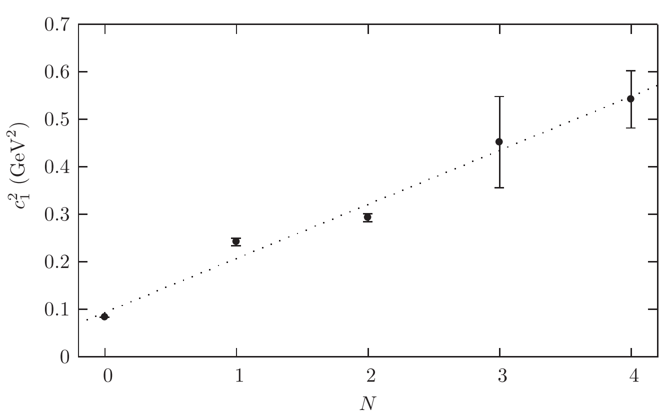


FIG. 4. Comparison between the quark model and large N_c results for c_1^2 (GeV^2) as a function of the band number N. The dotted line represents the quark model mass equation (43) with the parameters (44) from Buisseret *et al.* (2008) and the points with error bars indicate large N_c results: at $N = 0$ the value of c_1 was from Eq. (22), at $N = 1$ from Matagne and Stancu (2011a), at $N = 2$ from Goity, Schat, and Scoccola (2003) describing the multiplet $[\mathbf{56}, 2^+]$ (see Table I), at $N = 3$ from Matagne and Stancu (2012a), fit 3 corresponding to the multiplets $[\mathbf{70}, \ell^-]$ ($\ell = 1, 2, 3$), and at $N = 4$ from Matagne and Stancu (2005a) describing the multiplet $[\mathbf{56}, 4^+]$ (see Table I).

discussed by Goity and Matagne (2007) and Matagne and Stancu (2013), where two distinct nearly parallel Regge trajectories have been found, the lower one for the symmetric **56**-plets and the upper one for the mixed symmetric **70**-plets. It would be interesting to understand such an effect. In addition, one could try to see if baryons and mesons lead to degenerate Regge slopes in agreement with the predictions of Armoni and Patella (2009) where massive mesons and baryons become supersymmetric partners in the large N_c limit.

XI. THE QUARK EXCITATION VERSUS THE MESON-NUCLEON RESONANCE PICTURE

The contracted $\mathrm{SU}(2N_f)_c$ spin-flavor symmetry is a consequence of large N_c consistency conditions imposed on the meson-baryon scattering amplitudes (Gervais and Sakita, 1984a, 1984b; Dashen and Manohar, 1993a, 1993b). Therefore it seems natural to inquire about the compatibility between the quark excitation picture presented in Secs. VII and VIII and the meson-nucleon resonance picture.

According to Gervais and Sakita (1984a, 1984b) and Dashen and Manohar (1993a, 1993b) in large N_c QCD the pion-baryon couplings must satisfy a set of consistency conditions which require the existence of an infinite tower of degenerate baryon states with $I = J$ and also determine the ratios of the pion-baryon coupling constants, which turn out to be identical to those given by the Skyrme model. This implies that the large N_c QCD displays a contracted spin-flavor symmetry $\mathrm{SU}(2N_f)_c$. The symmetry is a property of ground state baryons. As a matter of fact, the meson sector does not display such a symmetry.

There is no *a priori* justification of extending this symmetry to excited states, which do not become stable at large N_c and where, in addition, an extension to $\mathrm{SU}(2N_f) \times \mathrm{O}(3)$ symmetry is necessary for introducing angular momentum components.

In fact, Witten (1979b) showed that the characteristic width of an excited baryon is of the order of N_c^0, while stable states are assumed in Gervais and Sakita (1984a, 1984b) and Dashen and Manohar (1993a, 1993b). Pirjol and Yan (1998a, 1998b) were the first to analyze the consistency condition for excited baryons. Their procedure is similar to that of Dashen and Manohar (1993a, 1993b). The pions are scattered off excited baryons and one must assume that the target is stable. The target was described by a mixed symmetric representation of $\mathrm{SU}(2N_f)$, where they claimed that the pion-nucleon coupling goes as $N_c^{-1/2}$ to produce narrow resonances. Functional forms of relations satisfying the consistency conditions were motivated from a simple nonrelativistic quark model.

The legitimacy of this procedure was questioned by Cohen *et al.* (2004a). They found that the existence of states with a width which goes as N_c^{-1} is an artifact of the simple quark model used in Pirjol and Yan (1998a, 1998b).

To evade the difficulties of an extension of the techniques from the ground state to excited states (Pirjol and Yan, 1998a, 1998b), Cohen and Lebed (2003a, 2003b, 2003c, 2005) proposed to study the scattering process in large N_c and compare the findings with the quark excitation picture, named the *quark-shell picture*, and based, as mentioned, on the extended symmetry $SU(2N_f) \times O(3)$. For simplicity we concentrate on the SU(4) case.

The starting point was the linear relations of the S matrices $S^{\pi}_{LL'RR'IJ}$ and S^{η}_{LRJ} of π and η scattering off a ground state baryon. They are given by the following equations:

$$S^{\pi}_{LL'RR'IJ} = \sum_K (-1)^{R'-R}\sqrt{(2R+1)(2R'+1)}(2K+1) \times \begin{Bmatrix} K & I & J \\ R' & L' & 1 \end{Bmatrix} \begin{Bmatrix} K & I & J \\ R & L & 1 \end{Bmatrix} s^{\pi}_{KLL'} \tag{46}$$

and

$$S^{\eta}_{LRJ} = \sum_K \delta_{KL}\delta(LRJ)s^{\eta}_K \tag{47}$$

in terms of the reduced amplitudes $s^{\pi}_{KL'L}$ and s^{η}_K, respectively. These equations were first derived in the context of the chiral soliton model (Hayashi *et al.*, 1984; Mattis and Peskin, 1985; Mattis, 19861989a, 1989b; Mattis and Mukerjee, 1988), where the mean field breaks the rotational and isospin symmetries, so that J and I are not conserved but the grand spin K is conserved and excitations can be labeled by K. These relations are exact in large N_c QCD and are independent of any model assumption. The notation is as follows. For π scattering R and R' are the spin of the incoming and outgoing baryons, respectively ($R = 1/2$ for N and $R = 3/2$ for Δ), L and L' are the partial wave angular momentum of the incident and final π, respectively (the orbital angular momentum L of η remains unchanged), I and J represent the total isospin and total angular momentum associated with a given resonance, and K is the magnitude of the grand spin $\vec{K} = \vec{I} + \vec{J}$. The $6j$ coefficients imply four triangle rules $\delta(LRJ)$, $\delta(R1I)$, $\delta(L1K)$, and $\delta(IJK)$.

Equations (46) and (47) help to relate scattering amplitudes in various channels with K amplitudes and look for common poles, i.e., resonances. These poles should correspond to degenerate towers of states. The quantum numbers of the channels are the quantum numbers of an N_c quark system given by a large N_c quark model. Thus the quantum numbers of an N_c quark system are the important degrees of freedom of the quark-shell picture.

According to Cohen and Lebed, if the pattern of degeneracy resulting from Eqs. (46) and (47) is the same as that of the quark-shell picture it means that the two pictures are compatible and the extension of the $1/N_c$ expansion method to excited states is justified. The compatibility is illustrated next for $N_f = 2$.

The quark-shell picture requires the introduction of a Hamiltonian model with an $SU(2N_f) \times O(3)$ symmetry containing operators up to order N_c^0.

(1) In the symmetric core + excited quark procedure (Sec. VII.A), there are three operators up to the order of N_c^0, namely,

$$O_1 = N_c \mathbb{1}, \quad O_2 = \ell \cdot s, \quad O_3 = \frac{15}{N_c}\ell^{(2)} \cdot g \cdot G_c, \tag{48}$$

which generate the Hamiltonian

$$H = c_1 N_c \mathbb{1} + c_2 \ell \cdot s + c_3 \frac{15}{N_c}\ell^{(2)} \cdot g \cdot G_c. \tag{49}$$

The only three distinct eigenvalues of this Hamiltonian can be obtained analytically. For $\ell = 1$ they were given in Cohen and Lebed (2003a, 2003b) and Pirjol and Schat (2003). Note that the normalization of O_3 is different in Cohen and Lebed (2003a, 2003b) and Pirjol and Schat (2003) which is reflected in the corresponding analytic expressions of the eigenvalues. A similar analysis based on the Hamiltonian (49) has been extended to $\ell = 3$ in Matagne and Stancu (2011b).

(2) In the exact basis (Sec. VII.B), there are also three operators with matrix elements up to the order of $\mathcal{O}(N_c^0)$. Using the notation of Matagne and Stancu (2012b)) they are

$$O_1 = N_c \mathbb{1}, \quad O_2 = \ell \cdot s, \quad O_6 = \frac{15}{N_c} L^{(2)} \cdot G \cdot G, \tag{50}$$

which generates the Hamiltonian

$$H = c_1 N_c \mathbb{1} + c_2 \ell \cdot s + c_6 \frac{15}{N_c} L^{(2)} \cdot G \cdot G. \tag{51}$$

The first two terms are the same as in Eq. (49) but in O_6 the SO(3) tensor $L^{(2)}$ and the SU(4) operator G act on the whole system. Interestingly, the corresponding Hamiltonian has analytical solutions too. These are

$$m'_0 = c_1 N_c - c_2 - \tfrac{25}{4} c_6, \tag{52}$$

$$m'_1 = c_1 N_c - \tfrac{1}{2} c_2 + \tfrac{25}{8} c_6, \tag{53}$$

$$m'_2 = c_1 N_c + \tfrac{1}{2} c_2 - \tfrac{5}{8} c_6. \tag{54}$$

Then for $\ell = 1$ the following degenerate sets of resonances were found:

$$N_{1/2}, \quad \Delta_{3/2}, \quad (s^{\eta}_0), \quad (m'_0), \tag{55}$$

$$N_{1/2}, \ \Delta_{1/2}, \ N_{3/2}, \ \Delta_{3/2}, \ \Delta_{5/2}, \ (s^{\pi}_{100}, s^{\pi}_{122}), \ (m'_1), \tag{56}$$

$$\Delta_{1/2}, \ N_{3/2}, \ \Delta_{3/2}, \ N_{5/2}, \ \Delta_{5/2}, \ \Delta_{7/2}, \ (s^{\pi}_{222}, s^{\eta}_2), \ (m'_2), \tag{57}$$

where on the right side we indicate the associated amplitudes $s^{\pi}_{KL'L}$ or s^{η}_K of Eq. (46) or (47) followed by the mass of each degenerate set. Thus the degenerate sets are identical to those obtained from the meson-baryon picture. In addition, the degenerate sets in the exact basis are identical to those found in Cohen and Lebed (2003a, 2003b) and Pirjol and Schat (2003), which means that the same quantum numbers are involved. The masses m'_i of Eqs. (52)–(54) shown here are naturally different from m_i of the above references because the Hamiltonian is different in structure and it contains different dynamical coefficients. However, it has similar large N_c properties.

The conclusion is that any resonance that exists must fall into multiplets that become degenerate in both mass and width (or equivalently coupling constant) at large N_c. The pattern of degeneracy is fully fixed by the contracted $SU(2N_f)$ symmetry. For $N_f = 2$ each set of degenerate states is defined by a single quantum number $K = 0$, 1, and 2 for Eqs. (55), (56), and (57), respectively.

For $\ell = 3$, described within the symmetric core plus excited quark procedure, we refer to our analysis (Matagne and Stancu, 2011b) which confirms the compatibility between the two pictures once more. In addition we supported the triangular rule $\delta(K\ell 1)$ proposed in Cohen and Lebed (2003c) according to which one can associate a common $K = 2$ to both $\ell = 1$ and $\ell = 3$. In some sense the quark-shell picture, where ℓ is conserved, brings an alternative information to the resonance picture, which may be more relevant for experimentalists, because it implies an energy dependence via the ℓ dependence which measures the orbital excitation.

The inclusion of strange quarks complicates the analysis. One must consider only those states within a multiplet with the same values of isospin and strangeness as for $N_c = 3$. The problem was discussed qualitatively by Cohen and Lebed (2005). Analyzing the $SU(3) \times SU(2)$ content of the **70** irrep of SU(6), 20 multiplets were found, with five distinct masses, corresponding to $K = 0, 1/2, 1, 3/2$, and 2. This is a model-independent result. A simple Hamiltonian expressed in terms of the symmetric core plus excited quark procedure containing the operators

$$O_1 = N_c \mathbb{1}, \quad O_2 = \ell \cdot s, \quad O_3 = \frac{3}{N_c} \ell^{(2)} \cdot g \cdot G_c,$$
$$O_4 = \ell \cdot s + \frac{4}{N_c + 1} \ell \cdot t \cdot G_c, \quad O_5 = \frac{1}{N_c}\left(t \cdot T - \frac{\mathbb{1}}{12}\right) \tag{58}$$

also gives five distinct masses, which suggests that the compatibility between the quark-shell picture and the meson-nucleon scattering picture can be achieved. Note that the additional operator O_5 acting only on flavor, usually omitted in the symmetric core + excited quark procedure (Carlson *et al.*, 1998, 1999), is crucial in the compatibility issue. The compatibility has not yet been studied explicitly in the exact basis procedure for $N_f = 3$.

Finally, we mention that, in the nonstrange sector, the compatibility between the two pictures was claimed on a general group-theoretical argument by Cohen and Lebed (2003c) for completely symmetric, mixed symmetric, and completely antisymmetric flavor-spin states of N_c quarks having angular momentum up to $\ell = 3$.

A bridge between the quark models in large N_c and the solitonic approach of the Skyrme model was established by Diakonov, Petrov, and Vladimirov (2013) within a relativistically invariant formalism to take into account $q\bar{q}$ pairs. In this work it was pointed out that the advantage of the large N_c limit is that the baryon physics simplifies considerably which allows one to take into full account important relativistic and field-theory effects which are often ignored.

XII. BARYON MASSES IN THE COMBINED $1/N_c$ AND CHIRAL EXPANSIONS

Based on the idea that the combined $1/N_c$ expansion and chiral perturbation theory (ChPT) can constrain the low-energy interactions of baryons with mesons, a $1/N_c$ expansion of the chiral Lagrangian was formulated by Jenkins (1996a) quite early, for the lowest-lying baryons. The expansion parameters are $1/N_c$ and $m_q/\Lambda_{\rm QCD}$ with the double limit $1/N_c \to 0$ and $m_q/\Lambda_{\rm QCD} \to 0$ and the ratio $(1/N_c)/(m_q/\Lambda_{\rm QCD})$ held fixed. The two limits cannot be taken independently from each other. The chiral Lagrangian correctly implements the pseudoscalar meson nonet symmetry and the contracted spin-flavor symmetry introduced in Sec. III. It describes the interaction of the spin-1/2 baryon octet and the spin-3/2 baryon decuplet with the pseudoscalar nonet. Strong CP violation was included.

Within the same framework the combined $1/N_c$ and chiral expansions was recently considered in Cordon and Goity (2013) based on the important conjecture that the two expansions do not commute (Adkins and Nappi, 1984). The dynamics underlying the noncommutativity is due to the behavior of the Δ resonance (Cohen and Broniowski, 1992; Dashen, Jenkins, and Manohar, 1994; Cohen, 1996).

We recall that ChPT is an effective field theory that makes use of an expansion in powers of momenta p (Leutwyler, 1995). The baryon mass splitting is taken to be $\mathcal{O}(p)$ in this expansion, named ξ expansion in Cordon and Goity (2013). Results for baryon masses and axial couplings were obtained in an expansion where $1/N_c = \mathcal{O}(\xi) = \mathcal{O}(p)$, thought to be the most realistic for studying baryons at $N_c = 3$. Applications to lattice QCD were presented. It would be interesting to extend the work from $N_f = 2$ to $N_f = 3$. Results for the axial currents with three flavors, in a similar framework, were presented in Flores-Mendieta, Hernandez-Ruiz, and Hofmann (2012).

XIII. STRONG DECAYS

Besides the spectrum, the strong decays of baryons represent an important field of application of the large N_c method. As for the spectrum, one can perform an operator analysis. So far only a few papers were devoted to the study of strong decays within this framework. As mentioned at the beginning of Sec. VII the first application of the large N_c method was a phenomenological analysis of strong decays of $\ell = 1$ orbitally excited baryons (Carone *et al.*, 1994). This work was intended to show that the success of nonrelativistic quark models has an explanation in large N_c QCD. For this purpose it was enough to consider a restricted basis of operators at subleading order in $1/N_c$. This study was followed by the analysis of strong decays of the Roper resonance (Carlson and Carone, 2000).

A complete analysis to $\mathcal{O}(1/N_c)$ of strong decays of nonstrange baryons belonging to the 20-plet of SU(4) was given in Goity, Schat, and Scoccola (2005) followed by the analysis of positive parity nonstrange resonances (Goity and Scoccola, 2005) and nonstrange + strange resonances of the **56**-plet of SU(6) (Goity, Jayalath, and Scoccola, 2009).

Finally the study of negative parity baryon decays was extended to SU(6) in Jayalath *et al.* (2011).

Note that all the above cited studies of strong decays of negative parity mixed symmetric states in the framework of the operator analysis rely on the Hartree approximation (Carone *et al.*, 1994), or its implementation according to Sec. VII.A.

Another framework to study strong decays is based on the scattering amplitudes. This is the approach introduced in Sec. XI and used in Cohen and Lebed (2003a, 2003b, 2003c, 2005) and Cohen *et al.* (2004a). The analytic structure of these amplitudes was used to prove the compatibility between the operator analysis and the meson-nucleon scattering picture. In the operator analysis only terms of the order of N_c^0 have been used and the name was the quark-shell picture. The comparison was therefore only qualitative. Higher order terms were introduced by Pirjol and Schat (2003).

Below we describe a few results obtained in the operator analysis approach for the strong decay widths.

A. Radially excited states

For historical and pedagogical purposes we first shortly present the study of strong decay widths of the lowest-lying radially excited baryons of Carlson and Carone (2000), with special attention to the Roper resonance. The large N_c mass formula was written under the form of a Gürsey-Radicati type. The analysis was free of any assumption regarding the interaction potential and the quark wave functions. The decay was assumed to take place via a single quark interaction vertex so that the transition operator can be expressed in terms of SU(6) generators as

$$\mathcal{H}_{\rm eff} \propto G^{ia} k^i \pi^a, \tag{59}$$

where G^{ia} is the SU(6) generator defined by Eq. (14), k^i is the meson momentum component i, and π^a is the meson field operator. The SU(6) operator acts on the excited quark. The matrix elements of the operator (59) between the excited baryon B_i and the final baryon B_f + meson give the transition amplitude

$$\langle \Psi(B_f, \pi^a)|\mathcal{H}_{\rm eff}|\Psi(B_i)\rangle = f(k) k^j \langle B_f|G^{ja}|B_i\rangle, \tag{60}$$

where $f(k)$ is a function that parametrizes the momentum dependence of the amplitude which encodes the baryon structure and therefore the binding potential. For a harmonic-type confinement this function has a simple analytic form (Koniuk and Isgur, 1980) and the above factorization takes place in general if recoil effects of the emitting quark are ignored (Sartor and Stancu, 1986b). Carlson and Carone (2000) found a simple functional form $f(k) = (2.8 \pm 0.2)/k$ to fit the data best while the harmonic type confinement led to an exponential decrease with k (Koniuk and Isgur, 1980). The above analysis has been stimulated by the successful large N_c study of strong decays of the **70**-plet (Carone *et al.*, 1994) which preceded the more involved study of strong decays of Goity, Schat, and Scoccola (2005) and Goity and Scoccola (2005). A good choice of the profile function $f(k)$ as above can help in including the largest part of the momentum dependence in the leading terms of the large N_c expansion of the transition operator.

Next we present the more elaborate, rather recent studies.

B. The 56-plet

Goity and Scoccola (2005) analyzed the multiplets $[\mathbf{56}', 0^+]$ and $[\mathbf{56}, 2^+]$ in SU(4). One interprets the Roper resonance as belonging to $[\mathbf{56}', 0^+]$.

The transition operators, classified in multiplets of the $O(3) \times SU(2N_f)$ group (Goity and Scoccola, 2005), are reproduced in Table XII. The list contains one-body and two two-body operators, with the order specified in the last column. The name $O^{[\ell_P,1]}$ contains the partial wave ℓ_P of the decay channel and 1 is the isospin of the emitted pion.

The operators ξ^ℓ_m are components of a tensor of rank ℓ in SO(3), responsible for the transition between an excited state with orbital angular momentum ℓ and the ground state. They were normalized to have matrix elements of the form

$$\langle 0|\xi^\ell_{m'}|\ell m\rangle = (-)^{\ell - m}\delta_{m,-m'}, \tag{61}$$

consistent with the Wigner-Eckart theorem provided the reduced matrix element on the right-hand side is $\langle 0||\xi^\ell||\ell\rangle = \sqrt{2\ell + 1}$. The operators S and G are SU(4) generators. From these generators one constructs spin-flavor tensors $(\mathcal{G}^{[S_P,I_P]}_{[S_{3P},I_{3P}]})_q$, where S_P, S_{3P} are the spin and its projection and I_P, I_{3P} are the isospin and its projection, the same as the isospin and its projection for the emitted meson. The quantity q numbers the operators generally considered in an operator expansion study $q = 1, 2, \ldots$. The adequate coupling for the partial wave ℓ_P of the meson emission defines the operators

$$\left(B^{[\ell_P,I_P]}_{[m_P,I_{3P}]}\right)_q = \sum_m \langle \ell, m; S_P S_{3P}|\ell_P, m_P\rangle \xi^\ell_m \left(\mathcal{G}^{[S_P,I_P]}_{[S_{3P},I_{3P}]}\right)_q, \tag{62}$$

which were used to construct a transition operator in the $1/N_c$ expansion as

$$B^{[\ell_P,I_P]}_{[m_P,I_{3P}]} = \left(\frac{k_P}{\Lambda}\right)^{\ell_P} \sum_q C_q^{[\ell_P,I_P]}(k_P) (B^{[\ell_P,I_P]}_{[m_P,I_{3P}]})_q, \tag{63}$$

containing a desired number of terms each having a coefficient $C_q^{[\ell_P,I_P]}(k_P)$ to be fit from data. The quantity $(k_P/\Lambda)^{\ell_P}$ was designed to capture the main momentum dependence of the dynamical coefficients $C_q^{[\ell_P,I_P]}(k_P)$ which were taken in practice as momentum independent. The constant Λ was chosen to be equal to 200 MeV. Another alternative would

TABLE XII. Basis operators for pion decay of $[\mathbf{56}', 0^+]$ and $[\mathbf{56}, 2^+]$ resonances in SU(4). From Goity and Scoccola, 2005.

	Name	Operator	Order
One-body	$O_1^{[\ell_P,1]}$	$\frac{1}{N_c}(\xi^\ell G)^{[\ell_P,1]}$	$\mathcal{O}(N_c^0)$
Two-body	$O_2^{[\ell_P,1]}$	$\frac{1}{N_c^2}(\xi^\ell([S,G])^{[1,1]})^{[\ell_P,1]}$	$\mathcal{O}(1/N_c)$
	$O_3^{[\ell_P,1]}$	$\frac{1}{N_c^2}(\xi^\ell(\{S,G\})^{[2,1]})^{[\ell_P,1]}$	$\mathcal{O}(1/N_c)$

be to introduce a momentum dependence through a profile function as in Carlson and Carone (2000). The two ways are equally valid, as long as no explicit dynamics is involved.

In this notation the transition operator (59) of Carlson and Carone (2000) corresponds to the $O_1^{[\ell_P,1]}$ operator listed in Table XII.

Then the strong decay width in a nonrelativistic kinematics is defined as

$$\Gamma^{[\ell_P,I_P]} = \frac{k_P}{8\pi^2}\frac{M_B}{M_B^*}\frac{|B(\ell_P,I_P,S,I,J^*,I^*,S^*)|^2}{(2J^*+1)(2I^*+1)}, \tag{64}$$

where $B(\ell_P,I_P,S,I,J^*,I^*,S^*)$ are the reduced matrix elements of the strong decay operator $B^{[\ell_P,I_P]}_{[m_P,I_{3P}]}$ defined above, with J^*,I^*,S^* the quantum numbers of the decaying resonance of mass M_{B^*} and S,I the spin and isospin of the baryon ground state of mass M_B.

The reduced matrix elements $B(\ell_P,I_P,S,I,J^*,I^*,S^*)$ are defined by the generalized Wigner-Eckart theorem introduced in Appendix C with notations adapted to the present case.

Goity and Scoccola (2005) calculated the decay widths in the p, f, and h partial waves for a number of resonances with increasing masses starting from the Roper $N(1440)1/2^+$ to $\Delta(2420)9/2^+$. The decay channels were πN and $\pi\Delta$. The results of Goity and Scoccola (2005) indicate that the pion decays are qualitatively well described at leading order described by the operator $O_1^{[\ell_P,1]}$, which explains why the simple picture of the quark model works qualitatively well and also justifies the choice of Carlson and Carone (2000). However, the Roper resonance requires important next-to-the-leading corrections as considered in Carlson and Carone (2000), a result also consistent with the quark model studies, as, e.g., Sartor and Stancu (1986b), which give too small a width. Also the predicted suppression of the η decay is consistent with the experimental results obtained so far.

In the extension to SU(6), Goity, Jayalath, and Scoccola (2009) followed a similar procedure to SU(4) to construct the spin-flavor transition operators. The SU(3) conserving operators correspond to those of Table XII but written in SU(6) notations. The SU(6) basis contains an additional SU(3) symmetry breaking (SB) operator

$$\mathcal{G}_{\rm SB} \equiv \frac{1}{N_c}(d_{8ab} - \delta_{ab}/\sqrt{3})G_{ib} \tag{65}$$

of order $\mathcal{O}((m_s - m_{u,d})/\sqrt{N_c})$ which is necessary to carefully distinguish between the emission of pions and K mesons. The conclusions are similar to those obtained in the SU(4) case.

C. The $[\mathbf{70},1^-]$-plet

As mentioned, the first analysis of the strong decays of the lowest negative parity baryons was made in Goity, Schat, and Scoccola (2005), where the multiplet belongs to the irreducible representation 20 of SU(4). Jayalath *et al.* (2011) extended the study of negative parity baryon decays to SU(6); see these papers for details. The construction of transition operators is similar to that of the mass operator, using operators acting on the excited quark or on the core. This implies a larger number of transition operators than for the 56-plet. Both studies indicate that the one-body operators are dominant in the S- and D-partial wave decay widths, which again support the quark model picture based on the spectator model, where the pseudoscalar meson is emitted from the excited quark. The two-body operators are crucial for an overall good description. They are thought to encode the longer range dynamics of the decay. However the calculated widths of the $N(1535) \to \eta N$ and $N(1650) \to \eta N$ are too small at leading order. The SU(3) breaking effects turn out to be unnaturally large as the next-to-leading order analysis has shown.

An exhaustive combined analysis of the masses, strong decay widths, and photocouplings (see below) was recently performed by de Urreta, Goity, and Scoccola (2014) for the lowest nonstrange negative parity resonances belonging to the $[70,1^-]$ multiplet of $\mathrm{SU}(4)\times\mathrm{O}(3)$ including an updated input for the $N_{1/2}$ baryons. The conclusion was that the composition of the spin-$1/2$ and spin-$3/2$ states, which involve two mixing angles, is in agreement with the nonrelativistic quark model of Isgur and Karl obtained from the analysis of strong decays alone (Isgur and Karl, 1978).

XIV. PHOTOPRODUCTION AMPLITUDES IN THE $1/N_c$ EXPANSION

The first analysis of the helicity amplitudes in the $1/N_c$ expansion was devoted to negative parity baryons by Carlson and Carone 1998a, 1998b). Regarding positive parity baryons, a particular case, the decay $\Delta^+ \to p\gamma$, was studied a few years later by Jenkins, Ji, and Manohar (2002), where the ratio of the helicity amplitudes $A_{3/2}/A_{1/2}$ was found to be $\sqrt{3} + \mathcal{O}(1/N_c^2)$, compatible with experiment and with quark models where the ratio is $\sqrt{3}$; see, for example, Eqs. (C49) and (C50) of Sartor and Stancu (1986a). Therefore this study showed that the ratio between the electric quadrupole $E2$ and the magnetic moment $M1$ is of the order of $1/N_c^2$. In this work the isovector electromagnetic current operator was expanded in powers of the photon momentum k^j:

$$J^{ia}_{\rm EM} \propto \mu^{ia} + Q^{(ij)a}k^j + \cdots, \tag{66}$$

where $i,j = 1,2,3$ are SU(2) spin and $a = 1,2,3$ isospin indices. The $M1$ and $E2$ transition amplitudes are

$$M1 = e\sqrt{k^3}\langle N|\mu^{33}|\Delta\rangle, \qquad E2 = \frac{e}{12}(k^3)^{3/2}\langle N|Q^{(20)}|\Delta\rangle, \tag{67}$$

where, in the second equation, the quadrupole moment operator $Q^{(ij)a}$ has been expressed in terms of the spherical component $\ell = 2$, $m_\ell = 0$ for $a = 3$. The $1/N_c$ expansion was applied to μ^{ia} and $Q^{(ij)a}$ operators.

Shortly after, this study was extended to hyperon radiative decays by Lebed and Martin (2004b) who calculated the radiative widths.

A few years later the photoproduction amplitudes of positive parity baryons were thoroughly studied by Goity and Scoccola (2007). This type of analysis was extended to negative parity photoproduction amplitudes by Scoccola, Goity, and Matagne (2008) by systematically building a complete basis of current operators to subleading order in $1/N_c$. The conclusion was that the one-body operators are dominant and the subleading corrections in $1/N_c$ are important and suggest evidence for the need of two-body operators.

An alternative, model-independent approach to study nonstrange resonances was based on the large N_c consistency conditions (Gervais and Sakita, 1984a, 1984b; Dashen and Manohar, 1993a, 1993b) to derive linear relations among partial wave amplitudes for the elastic $\pi N \to \pi N$ and the inelastic $\pi N \to \pi\Delta$ processes (Cohen *et al.*, 2004b). The leading-order relations were derived in the context of chiral soliton models in Hayashi *et al.* (1984), Mattis and Peskin (1985), Mattis (1986, 1989a, 1989b) and Mattis and Mukerjee (1988) as mentioned in Sec. XI. Their rederivation based on group structure was obtained by Cohen and Lebed (2003a, 2003b, 2003c). Cohen *et al.* (2004b) introduced next-to-leading order and the predictions made were confirmed by experiment.

The method was extended to pion photoproduction in Cohen *et al.* (2005). The corrections to order $1/N_c$ and $1/N_c^2$ give a remarkable agreement with the experiment.

The approach of Cohen *et al.* (2005) was later modified to provide a model-independent expansion for the electromagnetic multipole amplitudes of the pion electroproduction process $e^-N \to e^-\pi N$ (Lebed and Yu, 2009). The results seem to be more ambiguous.

XV. DIFFERENT LARGE N_c LIMITS

Soon after 't Hooft's generalization of QCD from $N_c = 3$ to arbitrarily large N_c ('t Hooft, 1974), it was pointed out by Corrigan and Ramond (1979) that there is an ambiguity in the generalization of the quark content of $SU(N_c)$ to $N_c > 3$. The argument was that the quarks can appear in other representations than the fundamental representation of $SU(N_c)$. Therefore one can construct distinct theories that agree at $N_c = 3$ but differ at $N_c \to \infty$. Then each distinct extrapolation leads to a distinct $1/N_c$ expansion for the observables under study.

So far several inequivalent large N_c generalizations have been proposed. Some of them have been discussed by Bolognesi (2007). If the quarks are in the fundamental representation one can construct a totally antisymmetric color state as defined by Eq. (3), which must be combined with a symmetric orbital-spin-flavor part, as already mentioned.

As an alternative, Corrigan and Ramond (1979) proposed a description of baryons as formed from quarks transforming under the fundamental representation and "larks" [antiquarks in SU(3)] transforming under the antisymmetric $N(N-1)/2$ representation; see Table XIII.

Actually there are three possible two-index representations for $SU(N)$. They are called tensors of rank $N^{(n,m)}$ (Stancu, 1996), where $n + m = 2$ in this case. They are exhibited in Table XIII. The superscripts refer to symmetry and subscripts to antisymmetry. The number of linearly independent components of each tensor gives the dimension of the corresponding irrep, denoted by n_q. Each tensor has a number of constraints given by its properties. This number has to be subtracted from N^2. Then the number of independent components is generally smaller than N^{n+m}. For example, the symmetric two-index irrep has the property

$$T^{ij} = T^{ji}, \tag{68}$$

which gives $C_N^2 = N(N-1)/2$ constraints. Then the dimension of the symmetric two-index irrep is $N^2 - N(N-1)/2 = N(N+1)/2$. The dimension of the antisymmetric irrep is naturally $N(N-1)/2$. The sum of the two must be equal to N^2 to be consistent with the direct product of two fundamental representations of $SU(N)$ which can be decomposed as

$$\square \times \square = \square\square + \begin{array}{c}\square\\[-4pt]\square\end{array}. \tag{69}$$

The two-index irreducible tensor T^i_j must satisfy the trace condition (Stancu, 1996)

$$T^1_1 + T^2_2 + \cdots + T^N_N = 0. \tag{70}$$

From here it follows that the dimension of the representation T^i_j is $N^2 - 1$. Thus T^i_j corresponds to the adjoint representation. This two-index tensor can be constructed from two one-index tensors T^i and T_j as

$$T^i_j = T^i T_j. \tag{71}$$

We recall that the contravariant tensor T^i can represent quarks and the covariant tensor T_j can describe antiquarks so that one can make the identification

$$T^i = q^i, \qquad T_j = \bar{q}_j. \tag{72}$$

The idea of Corrigan and Ramond was extended by Bolognesi (2007) to the two-index symmetric and antisymmetric representations in an effective Lagrangian approach.

Independently Armoni, Shifman, and Veneziano (2003a, 2003b) used the two-index antisymmetric representation to define a new $1/N_c$ expansion at a fixed number of N_f flavors. For $N_f = 1$, in the large N_c limit, their approach is equivalent to the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. In particular, Armoni, Shifman, and Veneziano (2003a) predicted exactly degenerate parity doublets. It would be interesting to find out if there are useful implications for baryons within this context. In addition, by using the equivalence to $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, Armoni and Patella (2009) showed that mesons and baryons become

TABLE XIII. Two-index irreducible representations of $SU(N)$ defined in terms of irreducible tensors, where n_q (last column) is the number of linearly independent components of each tensor.

Tensor	Rank	n_q
T^{ij}	(2,0)	$\frac{N(N+1)}{2}$
T_{ij}	(0,2)	$\frac{N(N-1)}{2}$
T^i_j	(1,1)	$N^2 - 1$

asymptotically superpartners, which may explain the coincidence of their Regge slopes.

By using either a mean-field approach (Witten, 1979b; Bolognesi, 2007), a diagrammatic method (Cherman and Cohen, 2006; Cherman, Cohen, and Lebed, 2009; Cohen, Shafer, and Lebed, 2010), or within the framework of a semirelativistic constituent quark model (Buisseret and Semay, 2010) it was proven that in the antisymmetric case the mass of such baryons scales as N_c^2. In the symmetric case the mass scales as N_c^2 also (Buisseret, Matagne, and Semay, 2012). Then it follows that both the two-index symmetric and antisymmetric representations lead to the same limit for the baryon masses at $N_c \to \infty$ and that the fundamental representation used by 't Hooft (1974) and the two-index antisymmetric representation of quarks lead to the same results at $N_c = 3$.

The results of Buisseret and Semay (2010) and Buisseret, Matagne, and Semay (2012) imply that in the Corrigan-Ramond limit, where $n_q = 3$, the baryon mass is of the order of $\mathcal{O}(1)$, like for mesons, in agreement with Cherman, Cohen, and Lebed (2009). We should stress that in the 't Hooft limit a large number of QCD properties have a simple understanding. However, there are cases where the 't Hooft limit is not sufficient (Harada, Sannino, and Schechter, 2004). In exchange, the Corrigan-Ramond limit has a richer structure and is convenient to study QCD at higher matter density (Frandsen, Kouvaris, and Sannino, 2006).

One should note that in Buisseret and Semay (2010) only the spin-independent part of the Hamiltonian was considered. The spin contribution was analyzed in a later work (Buisseret, Matagne, and Semay, 2012) for the ground state of light baryons in three inequivalent large N_c limits and it was proven that it scales as $S(S+1)/n_q$ in all cases. Then Table XIII implies that the spin contribution to the perturbative expansion resulting from one gluon exchange in the two-index representation is of order $\mathcal{O}(1/N_c^2)$, while in 't Hooft's limit, the subleading order is $\mathcal{O}(1/N_c)$ in agreement with results based on the spin-flavor symmetry (Dashen, Jenkins, and Manohar, 1994, 1995). Contrary, in the Goldstone-boson exchange model (Glozman and Riska, 1996; Glozman *et al.*, 1998) the contribution of the spin-flavor hyperfine splitting requires more attention in a perturbative expansion, when the coupling constants of the exchanged bosons are also considered in a large N_c limit.

The large N_c antisymmetric limit also implies the emergence of an $SU(2N_f)$ spin-flavor symmetry and predicts equally successful baryon mass relations as those derived in the standard $1/N_c$ expansion (Dashen, Jenkins, and Manohar, 1994, 1995), but with different $1/N_c$ suppression factors (Cherman, Cohen, and Lebed, 2012). Accordingly, Cherman, Cohen, and Lebed (2012) concluded that the large N_c baryons in the fundamental and antisymmetric two-index representations are about equally close to the $N_c = 3$ world, at least for the ground state baryon masses.

Studies based on the flavor adjoint representation were performed by Bolognesi (2007), Bolognesi and Shifman (2007), and Auzzi, Bolognesi, and Shifman (2008).

XVI. EXOTICS

As is well known, the existing quark models, inspired by QCD, can describe the properties of baryons as three quark systems qqq and of mesons as quark-antiquark $q\bar{q}$ pairs. These models predict the existence of new resonances, called exotics, which are formed from more than three quarks or antiquarks ($q^m\bar{q}^n, m+n > 3$). For a review see, for example, Stancu (2002), and references therein.

It was natural to inquire about the existence of exotics in large N_c. To our knowledge the problem was first raised by Cohen and Lebed (2004) in conjunction with the presently controversial pentaquark Θ^+, a $q^4\bar{s}$ system ($q = u, d$), of total angular momentum $J = 1/2$, isospin $I = 0$, strangeness $\mathcal{S} = +1$, and a mass of about 1.5 GeV, having a narrow width less than 15 MeV (Diakonov, Petrov, and Polyakov, 1997), the observation of which was announced by the LEPS Collaboration in 2003 (Nakano *et al.*, 2003). This was followed by a number of observations with either positive or negative results, reviewed, for example, by Stancu (2005). Cohen and Lebed argued that a large N_c analysis by itself cannot predict the mass of Θ^+, but it can predict the existence of degenerate partners with $\mathcal{S} = +1$, the quantum numbers of which can be related to poles in the KN scattering amplitude. They used the SU(3) extension (Mattis and Mukerjee, 1988) of the formalism presented in Sec. XI, where the baryons are described as resonances in the meson-nucleon scattering. If, for example, one imposes the theoretical assumption that Θ^+ is a state with $J = 1/2$ the degenerate partners should have $I = 1$, $J = 1/2, 3/2$ and $I = 2$, $J = 3/2, 5/2$. One does not expect the widths of these partners to be similar to that of Θ^+. However, Cohen and Lebed noted that large N_c neither implies nor precludes the existence of exotics.

Shortly afterward Jenkins and Manohar (2004a) also stated that large N_c spin-flavor symmetry does not predict that exotic baryons exist. They introduced the notion of exoticness E (Jenkins and Manohar, 2004b), as the minimal value for which the flavor baryon representation can be constructed from $qqq(q\bar{q})^E$ in SU(6). They derived the quantum numbers of exotics in both the quark and Skyrme models, the results being identical to $N_c \to \infty$, as for ordinary baryons. They proposed an $1/N_c$ mass expansion for exotic baryons and transition operators between baryons with different values of E.

In the quark representation described in Sec. IV the degenerate partners predicted by Cohen and Lebed belong to the SU(3) multiplets **27** for $I = 1$ and to **35** for $I = 2$, respectively, while Θ^+ was considered as a member of an antidecuplet $\overline{\mathbf{10}}$ with spin 1/2 (Diakonov, Petrov, and Polyakov, 1997). These three SU(3) representations have $E = 1$.

The work of Jenkins and Manohar was extended to one more important irreducible representation of the contracted spin-flavor symmetry by Pirjol and Schat (2007), who constructed a complete set of positive parity pentaquarks with one unit of orbital angular momentum, which in the large N_c limit falls into two towers with $K = 1/2$ and $K = 3/2$ of the contracted SU(4) symmetry.

After much interest for the pentaquark θ^+ and its charmed partner θ_c^0 [a $uudd\bar{c}$ system belonging to an SU(3) antisextet, a submultiplet of the $\bar{\mathbf{60}}$ irreducible representation of SU(4) (Wu and Ma, 2004; Stancu, 2005)] during the period of 2003 to 2005, there followed overwhelming evidence that they do not exist [see C. G. Wohl in the Particle Data Group (Beringer *et al.*, 2012)]. However, the common feature of most of the experimental results was that they were nondedicated experiments until 2004. Later on, dedicated high statistics experiments were performed [for a review, see, e.g., Liu, Mao, and Ma (2014)]. Recently a narrow peak structure at about 1.54 GeV in the missing mass of K_S in the reaction $\gamma + p \to pK_S K_L$ was observed (Amaryan *et al.*, 2012). In this experiment one tries to exploit the quantum mechanical interference between the channels $\gamma p \to \theta^+ \bar{K}^0 \to pK_S K_L$ and $\gamma p \to p\phi \to pK_S K_L$, where the latter can enhance the small amplitude of the θ^+ channel.

More recently the pentaquark θ^+ was reconsidered in a new theory of collective excitation as due to a Gamow-Teller transition (as in nuclear physics) but as a transition of the s quark from the highest filled level to excited u, d quark levels in a mean field (Diakonov, Petrov, and Vladimirov, 2013). In this way θ^+ was recovered with the same mass as was first predicted (Diakonov, Petrov, and Polyakov, 1997).

As a consequence of Coleman's conclusion in his Erice lectures (Coleman, 1985) that "in the large N limit, quadrilinears make meson pairs and nothing else" recently Weinberg (2013) argued that exotic mesons consisting of two quarks and two antiquarks are not ruled out in large N_c QCD. He suggested that the real question is the decay rate of a tetraquark. Weinberg's suggestion was subsequently supported and analyzed by Knecht and Peris (2013), Lebed (2013), and Cohen and Lebed (2014a, 2014b).

XVII. CONCLUSIONS

The $1/N_c$ expansion of QCD can provide a qualitative and, to a large extent, a quantitative understanding of a large number of hadronic phenomena. It has been proven to be an appropriate tool for studying hadron spectroscopy in a model-independent way. A great advantage is that it helps to organize and relate the observables at each order in $1/N_c$.

Previous reviews have shown that the ground state baryons satisfy the hierarchy predicted by this expansion, whenever necessary combined with a perturbative treatment of SU(3)-flavor breaking. It had also successfully predicted the masses of heavy-quark baryons and can help in the discovery of the remaining bottom baryons. The description of axial vector couplings, magnetic moments, charge radii, and quadrupole moments was also successful.

Here we have mostly been concerned with the baryon excited states. The physics of excited states gets sorted out hierarchically in powers of $1/N_c$ as well (de Urreta, Goity, and Scoccola, 2014). The presently known approaches are based on an extension of the spin-flavor symmetry to $\mathrm{SU}(2N_f) \times \mathrm{O}(3)$ symmetry. They seem to successfully explain most of the measured baryon masses. The quantitative calculations allow one to group resonances in octets, decuplets, and singlets formed of excited states. Many of these are predictions, which may be used in the experimental discovery of unknown baryons, in particular, of excited hyperons.

The $1/N_c$ mass operator has been compared to quark models and the comparison gives strong support to quark models and a better understanding of the coefficients of the mass formula which encode the quark dynamics. The leading-order term, proportional to N_c, can be understood as representing the contribution of the kinetic and of the confinement parts of a quark model Hamiltonian. This term then naturally increases with the excitation energy, or else, with the band number. The deviation from spin-flavor symmetry is given by corrections in powers of $1/N_c$ and a dominant part is the spin term $S \cdot S$, containing a spin-spin interaction, compatible with the one used in one-gluon exchange models. Especially for $N_f = 3$ flavors, a novelty is that the contribution of the pure flavor term $T \cdot T$ is as important in decuplets and flavor singlets, as it is the spin term in octets. The rewriting of $T \cdot T$ in terms of the spin and spin-flavor terms by using the Casimir operator of $\mathrm{SU}(2N_f)$ may bring more support of models containing a Goldstone-boson exchange interaction. A quantitative analysis is highly desirable.

The present studies of strong decay widths and photoproduction amplitudes, made in the symmetric core + excited quark approach, require subleading order corrections in order to fit experimental data. In particular, the transition operators include terms corresponding to the pseudoscalar meson emission, customarily used in the quark model description of decays, but also other higher order terms, unknown in quark model studies, the meaning of which could perhaps give a better insight into the transitions amplitude described by quark models. Similar studies for mixed symmetric spin-flavor states based on the totally antisymmetric wave function approach described in Sec. VIII are desirable. They could help to extend the analysis of decays to highly excited resonances belonging to bands with $N > 1$.

So far all studies have been devoted to a fixed $\mathrm{SU}(6) \times \mathrm{O}(3)$ multiplet of a given band N. On the other hand, it has been shown that the excitation band number N could be used to obtain Regge-type trajectories for the spin-independent part of the mass formula in both large N_c and quark model calculations. A global fit of resonances belonging to several multiplets of the same band N would be interesting to perform. It could settle the issue whether symmetric and mixed symmetric multiplets can lead to the same trajectory or should lead to distinct trajectories as shown in Sec. X.

The $1/N_c$ expansion method is receiving support from lattice QCD calculations showing that $N_c = 3$ is not too far from a larger N_c. A recent study concentrates on subleading corrections of hyperfine type (Cordon, DeGrand, and Goity, 2014). More accurate results are desired. We hope that the interplay between large N_c QCD and lattice calculations will further enlighten the understanding of excited baryons.

ACKNOWLEDGMENTS

We are indebted to Christian Lang for useful information on lattice QCD results. One of us (F. S.) is grateful to Ileana Guiasu for her constant interest in the subject which led to many stimulating discussions. This research was supported by the Fond de la Recherche Scientifique—FNRS under Grant No. 4.4501.05.

APPENDIX A: THE GENERALIZED WIGNER-ECKART THEOREM AND ISOSCALAR FACTORS OF SU(6)

Here we follow the derivation of the isoscalar factors of SU(6) as given by Matagne and Stancu (2009) and completed by Matagne and Stancu (2011a). They provide the diagonal and off-diagonal matrix elements of the SU(6) generators needed in calculating the spectra and transition amplitudes of strong and electromagnetic decays.

The SU(6) generators are components of an irreducible SU(6) tensor operator which span the invariant subspace of the adjoint representation denoted here by the partition $[21^4]$, or otherwise by its dimensional notation **35**. As for any other irreducible representation its matrix elements can be expressed in terms of a generalized Wigner-Eckart theorem, which factorizes each matrix element into products of Clebsch-Gordan coefficients and a reduced matrix element. To write the Wigner-Eckart theorem in its general form we redefine the generators forming the algebra (12) as

$$E^i = \frac{S^i}{\sqrt{N_f}}, \qquad E^a = \frac{T^a}{\sqrt{2}}, \qquad E^{ia} = \sqrt{2}G^{ia}, \tag{A1}$$

where $N_f = 3$ for SU(6) and $N_f = 2$ for SU(4) (Appendix C). Note that the generic name for every generator will remain E^{ia} (Hecht and Pang, 1969).

First we discuss the SU(6) case. An irrep of SU(6) is denoted by the partition $[f]$ and the SU(3) irreps are labeled by $(\lambda\mu)$ following Elliott (1958a, 1958b), equivalent to (p,q) in particle physics (Lichtenberg, 1970). Then one can write the matrix element of every SU(6) generator E^{ia} as

$$\langle [f](\lambda'\mu')Y'I'I'_3S'S'_3|E^{ia}|[f](\lambda\mu)YII_3SS_3\rangle = \sqrt{C^{[f]}(\mathrm{SU}(6))}\left(\begin{array}{cc|c} S & S^i & S' \\ S_3 & S^i_3 & S'_3 \end{array}\right)\left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right) \times \sum_{\rho=1,2}\left(\begin{array}{cc||c} (\lambda\mu) & (\lambda^a\mu^a) & (\lambda'\mu') \\ YI & Y^aI^a & Y'I' \end{array}\right)_\rho \left(\begin{array}{cc||c} [f] & [21^4] & [f] \\ (\lambda\mu)S & (\lambda^a\mu^a)S^i & (\lambda'\mu')S' \end{array}\right)_\rho, \tag{A2}$$

where $C^{[f]}(\mathrm{SU}(6))$ is the SU(6) Casimir operator eigenvalue associated with the irreducible representation $[f]$, followed by the familiar Clebsch-Gordan coefficients of SU(2) spin and SU(2) isospin. The sum over ρ contains products of isoscalar factors of SU(3) and SU(6), respectively. The label ρ is necessary whenever one has to distinguish between irreps $[f'] = [f]$ with multiplicities $m_{[f]}$ larger than 1 in the Clebsch-Gordan series (Matagne and Stancu, 2011a)

$$[f] \times [21^4] = \sum_{[f']} m_{[f']}[f']. \tag{A3}$$

The two values for ρ in both SU(6) and SU(3) reflect the multiplicity problem already appearing in the direct product of SU(3) irreducible representations

$$\begin{aligned}(\lambda\mu)\times(11) = {}& (\lambda+1,\mu+1) + (\lambda+2,\mu-1) \\ & + (\lambda\mu)_1 + (\lambda\mu)_2 + (\lambda-1,\mu+2) \\ & + (\lambda-2,\mu+1) + (\lambda+1,\mu-2) \\ & + (\lambda-1,\mu-1),\end{aligned} \tag{A4}$$

where (11) labels the SU(3) adjoint representation. One can see the representation $(\lambda\mu)$, which is one of the factors on the left-hand side, appears twice on the right-hand side. To distinguish between the two $(\lambda\mu)$'s one introduces the index ρ, which then takes two values, for both the SU(3) and SU(6) isoscalar factors. More details can be found in Matagne and Stancu (2011a).

In Eq. (A2) the Casimir operator eigenvalue for the symmetric representation with $[f] = [N_c]$ is

$$C^{[N_c]}(\mathrm{SU}(6)) = \frac{5N_c(N_c+6)}{12}, \tag{A5}$$

and for the mixed symmetric representation with $[f] = [N_c-1,1]$ is

$$C^{[N_c-1,1]}(\mathrm{SU}(6)) = \frac{N_c(5N_c+18)}{12}. \tag{A6}$$

The general analytic expressions of isoscalar factors of SU(3) needed for this analysis can be taken from the nuclear physics studies of Hecht (1965) where one has to replace λ and μ by their definition in terms of N_c. Their properties are summarized in Appendix C.

For convenience we reproduce our results for the isoscalar factors of SU(6) entering the generalized Wigner-Eckart theorem, Eq. (A2). The tables shown give the analytic expressions of the isoscalar factors in terms of N_c and spin. They can be used either in the calculation of masses of baryons and decay observables from the real world ($N_c = 3$), of electromagnetic moment relations (Lebed, 1995), or in the analysis of the compatibility between the $1/N_c$ expansion and the pion-nucleon scattering results, where one has to include states with $N_c \geq 3$; see Sec. XI.

We separately exhibit our results for the symmetric $[N_c]$ representation in Table XIV and for the mixed symmetric

TABLE XIV. Isoscalar factors of SU(6) generators defined by Eq. (A2), related to the product $[N_c] \times [21^4] \to [N_c]$.

$(\lambda_1\mu_1)S_1$	$(\lambda_2\mu_2)S_2$	ρ	$\left(\begin{array}{cc\|c} [N_c] & [21^4] & [N_c] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 & (\lambda\mu)S \end{array}\right)_\rho$
$(\lambda+2,\mu-1)S+1$	(11)1	/	$-\sqrt{\frac{3}{2}}\sqrt{\frac{2S+3}{2S+1}}\sqrt{\frac{(N_c-2S)(N_c+2S+6)}{5N_c(N_c+6)}}$
$(\lambda\mu)S$	(11)1	1	$4(N_c+3)\sqrt{\frac{2S(S+1)}{5N_c(N_c+6)[N_c(N_c+6)+12S(S+1)]}}$
$(\lambda\mu)S$	(11)1	2	$-\sqrt{\frac{3}{2}}\sqrt{\frac{(N_c-2S)(N_c+4-2S)(N_c+2+2S)(N_c+6+2S)}{5N_c(N_c+6)[N_c(N_c+6)+12S(S+1)]}}$
$(\lambda-2,\mu+1)S-1$	(11)1	/	$-\sqrt{\frac{3}{2}}\sqrt{\frac{2S-1}{2S+1}}\sqrt{\frac{(N_c+4-2S)(N_c+2+2S)}{5N_c(N_c+6)}}$
$(\lambda\mu)S$	(00)1	/	$\sqrt{\frac{4S(S+1)}{5N_c(N_c+6)}}$
$(\lambda\mu)S$	(11)0	1	$\sqrt{\frac{N_c(N_c+6)+12S(S+1)}{10N_c(N_c+6)}}$
$(\lambda\mu)S$	(11)0	2	0

representation $[N_c-1,1]$ in Tables XV–XVII, and XVIII. In each case one can check that the isoscalar factors satisfy the following orthogonality relation:

$$\sum_{\rho,(\lambda\mu)S,(\lambda^a\mu^a)S^i} \left(\begin{array}{cc\|c} [f] & [21^4] & [f_1] \\ (\lambda\mu)S & (\lambda^a\mu^a)S^i & (\lambda_1\mu_1)S_1 \end{array}\right)_\rho \left(\begin{array}{cc\|c} [f] & [21^4] & [f_2] \\ (\lambda\mu)S & (\lambda^a\mu^a)S^i & (\lambda_2\mu_2)S_2 \end{array}\right)_\rho = \delta_{f_1f_2}\delta_{\lambda_1\lambda_2}\delta_{\mu_1\mu_2}\delta_{S_1S_2}. \tag{A7}$$

TABLE XV. Isoscalar factors of the SU(6) generators [Eqs. (A1) and (A2)], corresponding to the $^2 8$ multiplet of $N_c = 3$.

$(\lambda_1\mu_1)S_1$	$(\lambda_2\mu_2)S_2$	ρ	$\left(\begin{array}{cc\|c} [N_c-1,1] & [21^4] & [N_c-1,1] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 & (\lambda\mu)S \end{array}\right)_\rho$
$(\lambda\mu)S+1$	(11)1	1	$-\frac{3\sqrt{2S(2S+3)(N_c+2S+2)}}{\sqrt{(S+1)(2S+1)[N_c(N_c+6)+12S(S+1)](5N_c+18)}}$
$(\lambda\mu)S+1$	(11)1	2	$\frac{N_c}{S+1}\sqrt{\frac{3(2S+3)(N_c-2S+4)(N_c+2S+6)}{2(2S+1)(N_c-2S)[N_c(N_c+6)+12S(S+1)](5N_c+18)}}$
$(\lambda\mu)S$	(11)1	1	$\{12S(S+1)+N_c[4S(S+1)-3]\}\sqrt{\frac{2}{S(S+1)[N_c(N_c+6)+12S(S+1)]N_c(5N_c+18)}}$
$(\lambda\mu)S$	(11)1	2	$\frac{4S^2(S+1)^2-2N_cS(S+1)-(S^2+S-1)N_c^2}{2S(S+1)}\sqrt{\frac{6(N_c-2S+4)(N_c+2S+6)}{(N_c-2S)(N_c+2S+2)[N_c(N_c+6)+12S(S+1)]N_c(5N_c+18)}}$
$(\lambda\mu)S-1$	(11)1	1	$-3\sqrt{\frac{2(S+1)(2S-1)(N_c-2S)}{S(2S+1)(N_c(N_c+6)+12S(S+1))(5N_c+18)}}$
$(\lambda\mu)S-1$	(11)1	2	$\frac{N_c}{S}\sqrt{\frac{3(2S-1)(N_c-2S+4)(N_c+2S+6)}{2(2S+1)(N_c+2S+2)(N_c(N_c+6)+12S(S+1))(5N_c+18)}}$
$(\lambda+2,\mu-1)S+1$	(11)1	/	$-\frac{1}{S+1}\sqrt{\frac{3S(S+2)(2S+3)(N_c-2S-2)(N_c+2S+2)(N_c+2S+6)}{2(2S+1)(N_c+2S+4)N_c(5N_c+18)}}$
$(\lambda+2,\mu-1)S$	(11)1	/	$\frac{1}{S+1}\sqrt{\frac{3(2S+3)(N_c+2S+2)(N_c+2S+6)}{2(2S+1)(N_c+2S+4)(5N_c+18)}}$
$(\lambda+1,\mu-2)S+1$	(11)1	/	$-2\sqrt{\frac{3S(2S+3)(N_c-2S-2)}{(S+1)(2S+1)(N_c-2S)(N_c+2S+4)(5N_c+18)}}$
$(\lambda+1,\mu-2)S$	(11)1	/	$-2\sqrt{\frac{3(N_c-2S-2)}{(S+1)(2S+1)(N_c-2S)(N_c+2S+4)(5N_c+18)}}$
$(\lambda-1,\mu-1)S$	(11)1	/	$\sqrt{\frac{12(N_c+2S)}{S(2S+1)(N_c-2S+2)(N_c+2S+2)(5N_c+18)}}$
$(\lambda-1,\mu-1)S-1$	(11)1	/	$-2\sqrt{\frac{3(S+1)(N_c+2S)(2S-1)}{S(2S+1)(N_c-2S+2)(N_c+2S+2)(5N_c+18)}}$
$(\lambda-2,\mu+1)S$	(11)1	/	$\frac{1}{S}\sqrt{\frac{3(2S-1)(N_c-2S)(N_c-2S+4)}{2(2S+1)(N_c-2S+2)(5N_c+18)}}$
$(\lambda-2,\mu+1)S-1$	(11)1	/	$-\frac{1}{S}\sqrt{\frac{3(S-1)(S+1)(2S-1)(N_c-2S)(N_c+2S)(N_c-2S+4)}{2(2S+1)(N_c-2S+2)N_c(5N_c+18)}}$
$(\lambda\mu)S$	(11)0	1	$\sqrt{\frac{N_c(N_c+6)+12S(S+1)}{2N_c(5N_c+18)}}$
$(\lambda\mu)S$	(11)0	2	0
$(\lambda\mu)S$	(00)1	/	$\sqrt{\frac{4S(S+1)}{N_c(5N_c+18)}}$

TABLE XVI. Isoscalar factors of the SU(6) generators, corresponding to the ^{48} multiplet of $N_c = 3$.

$(\lambda_1\mu_1)S_1$	$(\lambda_2\mu_2)S_2$	ρ	$\left(\begin{array}{cc} [N_c-1,1] & [21^4] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 \end{array}\middle\Vert\begin{array}{c} [N_c-1,1] \\ (\lambda-2,\mu+1)S \end{array}\right)_\rho$
$(\lambda-2,\mu+1)S$	(11)1	1	$[N_c(4S-3)+6S]\sqrt{\frac{2(S+1)}{S[N_c(N_c+6)+12(S-1)S]N_c(5N_c+18)}}$
$(\lambda-2,\mu+1)S$	(11)1	2	$-\frac{N_c-2S}{S}\sqrt{\frac{3(S-1)(S+1)(N_c-2S+6)(N_c+2S)(N_c+2S+4)}{2(N_c-2S+2)[N_c(N_c+6)+12(S-1)S]N_c(5N_c+18)}}$
$(\lambda\mu)S+1$	(11)1	/	$-\sqrt{\frac{3}{2}}\sqrt{\frac{2S+3}{2S+1}}\sqrt{\frac{(N_c-2S)(N_c+2S+4)}{N_c(5N_c+18)}}$
$(\lambda\mu)S$	(11)1	/	$-\frac{1}{S}\sqrt{\frac{3}{2}}\sqrt{\frac{(N_c-2S)(N_c+2S+4)}{(N_c+2S+2)(5N_c+18)}}$
$(\lambda\mu)S-1$	(11)1	/	$\frac{N_c+4S^2}{S}\sqrt{\frac{3(N_c+2S+4)}{2(2S-1)(2S+1)(N_c+2S+2)N_c(5N_c+18)}}$
$(\lambda-2,\mu+1)S-1$	(11)1	1	$\frac{3\sqrt{2(S-1)(N_c+2S)}}{\sqrt{S[N_c(N_c+6)+12(S-1)S](5N_c+18)}}$
$(\lambda-2,\mu+1)S-1$	(11)1	2	$-\frac{N_c}{S}\sqrt{\frac{3(N_c-2S+6)(N_c+2S+4)}{2(N_c-2S+2)[N_c(N_c+6)+12(S-1)S](5N_c+18)}}$
$(\lambda-1,\mu-1)S$	(11)1	/	$-2\sqrt{\frac{3(S+1)(N_c-2S)(N_c+2S)}{S(N_c-2S+2)(N_c+2S+2)N_c(5N_c+18)}}$
$(\lambda-1,\mu-1)S-1$	(11)1	/	$2(S-1)\sqrt{\frac{3(N_c-2S)(N_c+2S)}{S(2S-1)(N_c-2S+2)(N_c+2S+2)N_c(5N_c+18)}}$
$(\lambda-3,\mu)S-1$	(11)1	/	$-2\sqrt{\frac{3(S-1)(N_c+2S-2)}{(2S-1)(N_c-2S+4)N_c(5N_c+18)}}$
$(\lambda-4,\mu+2)S-1$	(11)1	/	$-\sqrt{\frac{3}{2}}\sqrt{\frac{2S-3}{2S-1}}\sqrt{\frac{(N_c+2S)(N_c-2S+2)(N_c-2S+6)}{(N_c-2S+4)N_c(5N_c+18)}}$
$(\lambda-2,\mu+1)S$	(11)0	1	$\sqrt{\frac{N_c(N_c+6)+12(S-1)S}{2N_c(5N_c+18)}}$
$(\lambda-2,\mu+1)S$	(11)0	2	0
$(\lambda-2,\mu+1)S$	(00)1	/	$\sqrt{\frac{4S(S+1)}{N_c(5N_c+18)}}$

TABLE XVII. Isoscalar factors of the SU(6) generators, corresponding to the ^{210} multiplet of $N_c = 3$.

$(\lambda_1\mu_1)S_1$	$(\lambda_2\mu_2)S_2$	ρ	$\left(\begin{array}{cc} [N_c-1,1] & [21^4] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 \end{array}\middle\Vert\begin{array}{c} [N_c-1,1] \\ (\lambda+2,\mu-1)S \end{array}\right)_\rho$
$(\lambda+4,\mu-2)S+1$	(11)1	/	$-\sqrt{\frac{3(2S+5)(N_c+2S+4)(N_c+2S+8)(N_c-2S-2)}{2(2S+3)(N_c+2S+6)N_c(5N_c+18)}}$
$(\lambda+3,\mu-3)S+1$	(11)1	/	$2\sqrt{\frac{3(S+2)(N_c-2S-4)}{(2S+3)(N_c+2S+6)N_c(5N_c+18)}}$
$(\lambda+1,\mu-2)S+1$	(11)1	/	$-2(S+2)\sqrt{\frac{3(N_c+2S+2)(N_c-2S-2)}{(S+1)(2S+3)(N_c-2S)(N_c+2S+4)N_c(5N_c+18)}}$
$(\lambda+2,\mu-1)S+1$	(11)1	1	$3\sqrt{\frac{2(S+2)(N_c-2S-2)}{(S+1)(N_c(N_c+6)+12(S+1)(S+2))(5N_c+18)}}$
$(\lambda+2,\mu-1)S+1$	(11)1	2	$-\frac{N_c}{S+1}\sqrt{\frac{3(N_c-2S+2)(N_c+2S+8)}{2(N_c+2S+4)(N_c(N_c+6)+12(S+1)(S+2))(5N_c+18)}}$
$(\lambda+2,\mu-1)S$	(11)1	1	$[N_c(4S+7)+6(S+1)]\sqrt{\frac{2S}{(S+1)[N_c(N_c+6)+12(S+1)(S+2)]N_c(5N_c+18)}}$
$(\lambda+2,\mu-1)S$	(11)1	2	$-\frac{N_c+2S+2}{S+1}\sqrt{\frac{3S(S+2)(N_c-2S-2)(N_c-2S+2)(N_c+2S+8)}{2(N_c+2S+4)[N_c(N_c+6)+12(S+1)(S+2)]N_c(5N_c+18)}}$
$(\lambda+1,\mu-2)S$	(11)1	/	$2\sqrt{\frac{3S(N_c+2S+2)(N_c-2S-2)}{(S+1)(N_c-2S)(N_c+2S+4)N_c(5N_c+18)}}$
$(\lambda\mu)S+1$	(11)1	/	$\frac{N_c+4(S+1)^2}{S+1}\sqrt{\frac{3(N_c-2S+2)}{2(2S+1)(2S+3)(N_c-2S)N_c(5N_c+18)}}$
$(\lambda\mu)S$	(11)1	/	$-\frac{1}{S+1}\sqrt{\frac{3(N_c+2S+2)(N_c-2S+2)}{2(N_c-2S)(5N_c+18)}}$
$(\lambda\mu)S-1$	(11)1	/	$-\sqrt{\frac{3(2S-1)(N_c+2S+2)(N_c-2S+2)}{2(2S+1)N_c(5N_c+18)}}$
$(\lambda+2,\mu-1)S$	(11)0	1	$\sqrt{\frac{N_c(N_c+6)+12(S+1)(S+2)}{2N_c(5N_c+18)}}$
$(\lambda+2,\mu-1)S$	(11)0	2	0
$(\lambda+2,\mu-1)S$	(00)1	/	$\sqrt{\frac{4S(S+1)}{N_c(5N_c+18)}}$

TABLE XVIII. Isoscalar factors of the SU(6) generators, corresponding to the ^{21} multiplet of $N_c = 3$.

$(\lambda_1\mu_1)S_1$	$(\lambda_2\mu_2)S_2$	ρ	$\left(\begin{array}{cc} [N_c-1,1] & [21^4] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 \end{array}\middle\Vert\begin{array}{c} [N_c-1,1] \\ (\lambda-1,\mu-1)S \end{array}\right)_\rho$
$(\lambda+1,\mu-2)S+1$	(11)1	/	$-\sqrt{\frac{3(2S+3)(N_c-2S-2)(N_c-2S+2)(N_c+2S+4)}{2(N_c-2S)(2S+1)N_c(5N_c+18)}}$
$(\lambda+1,\mu-2)S$	(11)1	/	$-\sqrt{\frac{3(N_c-2S-2)(N_c-2S+2)(N_c+2S+4)}{2S(2S+1)(N_c-2S)N_c(5N_c+18)}}$
$(\lambda-1,\mu-1)S$	(11)1	1	$[N_c(4S-3)+6S]\sqrt{\frac{2(S+1)}{S[N_c^2+12(S^2-1)]N_c(5N_c+18)}}$
$(\lambda-1,\mu-1)S$	(11)1	2	$-\{N_c(N_c+6)-4[S(S-1)-3]\}\sqrt{\frac{3(2S-1)(S+1)(N_c-2S-2)(N_c+2S-2)}{2S(2S+1)(N_c-2S+2)(N_c+2S+2)[N_c^2+12(S^2-1)]N_c(5N_c+18)}}$
$(\lambda-1,\mu-1)S-1$	(11)1	1	$3\sqrt{\frac{2N_c(2S-1)}{S[N_c^2+12(S^2-1)](5N_c+18)}}$
$(\lambda-1,\mu-1)S-1$	(11)1	2	0 if $S=1/2$
$(\lambda-1,\mu-1)S-1$	(11)1	2	$-[N_c(N_c+6)-12(S^2-1)]\sqrt{\frac{3(N_c-2S-2)(N_c+2S-2)}{2S(2S+1)(N_c-2S+2)(N_c+2S+2)[N_c^2+12(S^2-1)]N_c(5N_c+18)}}$ if $S \geq 1$
$(\lambda\mu)S+1$	(11)1	/	$\sqrt{\frac{6(2S+3)(N_c+2S+4)}{(2S+1)(N_c-2S)N_c(5N_c+18)}}$
$(\lambda\mu)S$	(11)1	/	$\frac{1}{S}\sqrt{\frac{6(N_c+2S+4)}{(N_c-2S)(N_c+2S+2)(5N_c+18)}}$
$(\lambda\mu)S-1$	(11)1	/	$\frac{S+1}{S}\sqrt{\frac{6(2S-1)(N_c+2S+4)}{(2S+1)(N_c+2S+2)N_c(5N_c+18)}}$
$(\lambda-2,\mu+1)S$	(11)1	/	$\frac{1}{S}\sqrt{\frac{6(S+1)(N_c-2S+4)(2S-1)}{(N_c-2S+2)N_c(5N_c+18)}}$
$(\lambda-2,\mu+1)S-1$	(11)1	/	$\frac{1}{S}\sqrt{\frac{6(N_c-2S+4)(S-1)(2S-1)}{(N_c-2S+2)(N_c+2S)(5N_c+18)}}$
$(\lambda-3,\mu)S-1$	(11)1	/	0 if $S=1/2$
$(\lambda-3,\mu)S-1$	(11)1	/	$-\sqrt{\frac{3(N_c+2S-2)(N_c+2S+2)(N_c-2S+4)(S-1)}{2S(N_c+2S)N_c(5N_c+18)}}$ if $S \geq 1$
$(\lambda-1,\mu-1)S$	(11)0	1	$\sqrt{\frac{N_c^2+12(S^2-1)}{2N_c(5N_c+18)}}$
$(\lambda-1,\mu-1)S$	(11)0	2	0
$(\lambda-1,\mu-1)S$	(00)1	/	$\sqrt{\frac{4S(S+1)}{N_c(5N_c+18)}}$

We note that the analytic expressions obtained for the isoscalar factors of the symmetric representation $[N_c]$ were obtained by Matagne and Stancu (2006a).

APPENDIX B: SYMMETRY PROPERTIES OF ISOSCALAR FACTORS

In Table XIX we indicate the values of λ and μ for various physical sectors as a function of N_c. The flavor singlet case ^21_J is discussed in more detail in Matagne and Stancu (2009).

We recall that the isoscalar factors of SU(3) obey the following orthogonality relation:

$$\sum_{Y''I''Y^aI^a}\left(\begin{array}{cc} (\lambda''\mu'') & (11) \\ Y''I'' & Y^aI^a \end{array}\middle\Vert\begin{array}{c} (\lambda'\mu') \\ YI \end{array}\right)_\rho\left(\begin{array}{cc} (\lambda''\mu'') & (11) \\ Y''I'' & Y^aI^a \end{array}\middle\Vert\begin{array}{c} (\lambda\mu) \\ Y'I' \end{array}\right)_\rho = \delta_{\lambda'\lambda}\delta_{\mu'\mu}\delta_{Y'Y}\delta_{I'I}, \tag{B1}$$

which can be easily checked. For completeness also note that the isoscalar factors obey the following symmetry property:

$$\left(\begin{array}{cc} (\lambda\mu) & (11) \\ YI & -Y^aI^a \end{array}\middle\Vert\begin{array}{c} (\lambda'\mu') \\ Y'I' \end{array}\right) = (-)^{1/3[\mu'-\mu-\lambda'+\lambda+(3/2)Y^a]+I'-I}\sqrt{\frac{\dim(\lambda'\mu')(2I+1)}{\dim(\lambda\mu)(2I'+1)}}\left(\begin{array}{cc} (\lambda'\mu') & (11) \\ Y'I' & Y^aI^a \end{array}\middle\Vert\begin{array}{c} (\lambda\mu) \\ YI \end{array}\right), \tag{B2}$$

where $\dim(\lambda\mu) = (1/2)(\lambda+1)(\mu+1)(\lambda+\mu+2)$ is the dimension of the irrep $(\lambda\mu)$ of SU(3).

TABLE XIX. Values of λ and μ as a function of N_c for all sectors of physical interest.

	λ	μ
^28_J	1	$\frac{N_c-1}{2}$
^48_J	1	$\frac{N_c-1}{2}$
^210_J	3	$\frac{N_c-3}{2}$
^21_J	0	$\frac{N_c-3}{2}$

The SU(6) isoscalar factors satisfy to the following symmetry property:

$$\begin{pmatrix} [f] & [21^4] & \Vert & [f] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 & \Vert & (\lambda\mu)S \end{pmatrix} = (-1)^{1/3(\mu_1-\mu-\lambda_1+\lambda)}(-1)^{S_1-S}\sqrt{\frac{\dim(\lambda_1\mu_1)(2S_1+1)}{\dim(\lambda\mu)(2S+1)}} \times \begin{pmatrix} [f] & [21^4] & \Vert & [f] \\ (\lambda\mu)S & (\lambda_2\mu_2)S_2 & \Vert & (\lambda_1\mu_1)S_1 \end{pmatrix}. \tag{B3}$$

APPENDIX C: MATRIX ELEMENTS OF SU(4) GENERATORS

Here we reproduce the matrix elements of the SU(4) generators for the symmetric irrep $[N_c]$. In Matagne and Stancu (2006a) they were written as a particular case of Eq. (A2). The form presented below is entirely compatible with that given in Hecht and Pang (1969). Thus in the case of $SU(4) \supset SU(2) \times SU(2)$ the analog of Eq. (A2) becomes

$$\langle [N_c]I'I'_3S'S'_3|E^{ia}|[N_c]II_3SS_3\rangle = \sqrt{C^{[N_c]}(SU(4))}\begin{pmatrix} [N_c] & [21^2] & \Vert & [N_c] \\ IS & I^aS^i & \Vert & I'S' \end{pmatrix} \times \begin{pmatrix} S & S^i & | & S' \\ S_3 & S^i_3 & | & S'_3 \end{pmatrix}\begin{pmatrix} I & I^a & | & I' \\ I_3 & I^a_3 & | & I'_3 \end{pmatrix}, \tag{C1}$$

where

$$C^{[N_c]}(SU(4)) = [3N_c(N_c+4)]/8 \tag{C2}$$

is the SU(4) Casimir operator eigenvalue for the symmetric irrep $[N_c]$. Also note that a symmetric state of SU(4) has $I = S$. We recall that the su(4) algebra is a particular case of the algebra (12), where $N_f = 2$

$$\begin{aligned} &[S^i, S^j] = i\varepsilon^{ijk}S^k, \qquad [T^a, T^b] = i\varepsilon^{abc}T^c, \\ &[S^i, T^a] = 0, \qquad [S^i, G^{ia}] = i\varepsilon^{ijk}G^{ka}, \\ &[T^a, G^{ib}] = i\varepsilon^{abc}G^{ic}, \\ &[G^{ia}, G^{jb}] = \frac{i}{4}\delta^{ij}\varepsilon^{abc}T^c + \frac{i}{2}\delta_{ab}\varepsilon_{ijk}S_k. \end{aligned} \tag{C3}$$

The tensor operators E^{ia} are related to S^i, T^a, and G^{ia} ($i = 1, 2, 3$; $a = 1, 2, 3$) by

$$E^i = \frac{S^i}{\sqrt{2}}, \qquad E^a = \frac{T^a}{\sqrt{2}}, \qquad E^{ia} = \sqrt{2}G^{ia}. \tag{C4}$$

This is a particular case of Eqs. (A1), where we now take $N_f = 2$. In Eq. (C1) they are identified by $I^aS^i = 01$, 10, and 11, respectively. Now we want to obtain the SU(4) isoscalar factors as particular cases of the SU(6) results with $Y^a = 0$. In SU(4) the hypercharge of a system of N_c quarks takes the value $Y = N_c/3$. By comparing Eqs. (A2) and (C1) we obtained the relation

$$\begin{pmatrix} [N_c] & [21^2] & \Vert & [N_c] \\ IS & I^aS^i & \Vert & I'S' \end{pmatrix} = r^{I^aS^i}\sqrt{\frac{C^{[N_c]}(SU(6))}{C^{[N_c]}(SU(4))}}\sum_{\rho=1,2}\begin{pmatrix} (\lambda\mu) & (\lambda^a\mu^a) & \Vert & (\lambda'\mu') \\ \frac{N_c}{3}I & 0I^a & \Vert & \frac{N_c}{3}I' \end{pmatrix}_\rho \times \begin{pmatrix} [N_c] & [21^4] & \Vert & [N_c] \\ (\lambda\mu)S & (\lambda^a\mu^a)S^i & \Vert & (\lambda'\mu')S' \end{pmatrix}_\rho, \tag{C5}$$

where

$$r^{I^aS^i} = \begin{cases} \sqrt{\frac{3}{2}} & \text{for } I^aS^i = 01, \\ 1 & \text{for } I^aS^i = 10, \\ 1 & \text{for } I^aS^i = 11, \end{cases} \tag{C6}$$

due to Eqs. (C1) and (C4) and taking into account that in SU(6) one has $E^i = S^i/\sqrt{3}$ while in SU(4) one has $E^i = S^i/\sqrt{2}$. In Eq. (C5) we have made the replacement

$$\lambda = 2I, \qquad \mu = \frac{N_c}{2} - I; \qquad \lambda' = 2I', \qquad \mu' = \frac{N_c}{2} - I', \tag{C7}$$

and took

$$(\lambda^a\mu^a) = \begin{cases} (00) & \text{for } I^a = 0, \\ (11) & \text{for } I^a = 1. \end{cases} \tag{C8}$$

In this way we have recovered the SU(4) isoscalar factors presented in Table A4.2 of Hecht and Pang (1969) up to a phase factor. In doing these analytic calculations we have made use of the isoscalar factors of SU(3) obtained by Hecht (1965), Table 4. These coefficients were derived in a nuclear physics context but they can be easily rewritten in terms of N_c due to Eqs. (C7).

By introducing these isoscalar factors into the matrix elements (C1) and the corresponding values of I^a and S^i according to the definition (C6), we have recovered the expressions given in Eqs. (A1)–(A3) of Carlson *et al.* (1998, 1999),

$$\langle [N_c]S' = I'; m'_1, \alpha'_1|G^{ia}|S = I; m_1, \alpha_1[N_c]\rangle = \frac{1}{4}\sqrt{\frac{2I+1}{2I'+1}}\sqrt{(N_c+2)^2 - (I'-I)^2(I'+I+1)^2} \times \begin{pmatrix} S & 1 & | & S' \\ m_1 & i & | & m'_1 \end{pmatrix}\begin{pmatrix} I & 1 & | & I' \\ \alpha_1 & a & | & \alpha'_1 \end{pmatrix}, \tag{C9}$$

$$\langle [N_c]S' = I'; m'_1, \alpha'_1|T^a|S = I; m_1, \alpha_1[N_c]\rangle = \sqrt{I(I+1)}\begin{pmatrix} I & 1 & | & I \\ \alpha_1 & a & | & \alpha'_1 \end{pmatrix}\delta_{I'I}\delta_{S'S}\delta_{m'_1m_1}, \tag{C10}$$

$$\langle [N_c]S' = I'; m_1', \alpha_1' | S^i | S = I; m_1, \alpha_1 [N_c] \rangle$$
$$= \sqrt{I(I+1)} \begin{pmatrix} S & 1 & \Big| & S \\ m_1 & i & \Big| & m_1' \end{pmatrix} \delta_{I'I} \delta_{S'S} \delta_{\alpha_1' \alpha_1}. \quad \text{(C11)}$$

We recall that $S = I$ for a symmetric representation. Note that the matrix elements of G^{ia} in Carlson *et al.* (1998 1999) refer to the symmetric representation $[N_c - 1]$ describing a core of $N_c - 1$ quarks, while here we consider a system of N_c quarks; hence we have the term $N_c + 2$ instead of $N_c + 1$ under the square root in Eq. (C9). As an example, putting $N_c = 3$ in Eq. (C9) one can recover the first four rows of Table 2 of Carlson and Carone (2000) by taking into account the relation between Clebsch-Gordan and $3j$ coefficients.

APPENDIX D: ISOSCALAR FACTORS OF THE PERMUTATION GROUP

Here we recall the definition of isoscalar factors of the permutation group S_n. We denote a basis vector in the invariant subspace of the irrep $[f]$ of S_n by $|[f]Y\rangle$, where Y is the corresponding Young tableau or Yamanouchi symbol. A basis vector obtained from the inner product of two irreps $[f']$ and $[f'']$ is defined by the sum over products of basis vectors of $|[f']Y'\rangle$ and $|[f'']Y''\rangle$ at fixed $[f']$ and $[f'']$:

$$|[f]Y\rangle = \sum_{Y'Y''} S([f']Y'[f'']Y''|[f]Y)|[f']Y'\rangle|[f'']Y''\rangle, \quad \text{(D1)}$$

where $S([f']Y'[f'']Y''|[f]Y)$ are CG coefficients of S_n. Any CG coefficient can be factorized into an isoscalar factor, here called a K matrix (Stancu, 1996), and a CG coefficient of S_{n-1}. To apply the factorization property it is necessary to specify the row p of the nth particle and the row q of the $(n-1)$th particle. The remaining particles are distributed in a Young tableau denoted by y. Then the isoscalar factor K associated with a given CG of S_n is defined as

$$S([f']p'q'y'[f'']p''q''y''|[f]pqy)$$
$$= K([f']p'[f'']p''|[f]p) S([f'_{p'}]q'y'[f''_{p''}]q''y''|[f_p]qy), \quad \text{(D2)}$$

where the right-hand side contains a CG coefficient of S_{n-1} containing $[f_p]$, $[f'_{p'}]$, and $[f''_{p''}]$ which are the partitions obtained from $[f]$ after the removal of the nth particle. The K matrix obeys the following orthogonality relations:

$$\sum_{p'p''} K([f']p'[f'']p''|[f]p) K([f']p'[f'']p''|[f_1]p_1) = \delta_{ff_1}\delta_{pp_1}, \quad \text{(D3)}$$

$$\sum_{fp} K([f']p'[f'']p''|[f]p) K([f']p_1'[f'']p_1''|[f]p) = \delta_{p'p_1'}\delta_{p''p_1''}. \quad \text{(D4)}$$

We consider a system of N_c quarks having a total spin S. The group SU(2) allows only partitions with a maximum of two rows, in this case with $N_c/2 + S$ boxes in the first row and $N_c/2 - S$ in the second row. So, one has

$$[f'] = \left[\frac{N_c}{2} + S, \frac{N_c}{2} - S\right]. \quad \text{(D5)}$$

Then one can write a symmetric state of N_c particles with spin S as the linear combination

$$|[N_c]1\rangle = c_{11}^{[N_c]}(S)|[f']1\rangle|[f']1\rangle + c_{22}^{[N_c]}(S)|[f']2\rangle|[f']2\rangle. \quad \text{(D6)}$$

The isoscalar factors used to construct the spin-flavor symmetric state (D6) are

$$c_{11}^{[N_c]} = K([f']1[f']1|[N_c]1),$$
$$c_{22}^{[N_c]} = K([f']2[f']2|[N_c]1). \quad \text{(D7)}$$

The isoscalar factors needed to construct the state of mixed symmetry $[N_c - 1, 1]$ from the same inner product are

$$c_{11}^{[N_c-1,1]} = K([f']1[f']1|[N_c - 1, 1]2),$$
$$c_{22}^{[N_c-1,1]} = K([f']2[f']2|[N_c - 1, 1]2). \quad \text{(D8)}$$

The above coefficients and the orthogonality relation (D4) give

$$c_{11}^{[N_c-1,1]} = -c_{22}^{[N_c]}, \qquad c_{22}^{[N_c-1,1]} = c_{11}^{[N_c]}. \quad \text{(D9)}$$

When the N_cth particle is located in different rows in the flavor and spin parts the needed coefficients are

$$c_{12}^{[N_c-1,1]} = K([f']1[f']2|[N_c - 1, 1]2) = 1,$$
$$c_{21}^{[N_c-1,1]} = K([f']2[f']1|[N_c - 1, 1]2) = 1, \quad \text{(D10)}$$

which are identical because of the symmetry properties of K. The identification of the so-called "elements of orthogonal basis rotation" of Carlson *et al.* (1998, 1999) with the above isoscalar factors is the following. For the symmetric states one has

$$c_{11}^{[N_c]} = c_{0-}^{\text{SYM}}, \qquad c_{22}^{[N_c]} = c_{0+}^{\text{SYM}}, \quad \text{(D11)}$$

and for the mixed symmetric states there is

$$c_{11}^{[N_c-1,1]} = c_{0-}^{\text{MS}}, \qquad c_{22}^{[N_c-1,1]} = c_{0+}^{\text{MS}}, \quad \text{(D12)}$$

$$c_{12}^{[N_c-1,1]} = c_{++}^{\text{MS}}, \qquad c_{21}^{[N_c-1,1]} = c_{--}^{\text{MS}}. \quad \text{(D13)}$$

The coefficients $(c_{pp}^{[N_c]})^2$ $(p = 1,\ 2)$ can be defined in the context of $SU(6) \supset SU(2) \times SU(3)$ as squares of isoscalar factors of S_n. We write the matrix elements of the generators S_i in two different ways. One is to use the Wigner-Eckart theorem for SU(2):

$$\langle [N_c](\lambda'\mu')Y'I'I_3'; S'S_3'|S^i|[N_c](\lambda\mu)YII_3; SS_3\rangle$$
$$= \delta_{SS'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{YY'}\delta_{II'}\delta_{I_3I_3'}\sqrt{C(\text{SU}(2))} \begin{pmatrix} S & 1 & \Big| & S' \\ S_3 & i & \Big| & S_3' \end{pmatrix}. \quad \text{(D14)}$$

The other is to calculate the matrix elements of S^i by using the fact that this is a one-body operator

$$S^i = \sum_{k=0}^{N_c} s^i(k), \tag{D15}$$

where $s_i(k)$ is a single particle operator acting on the particle k. Then for a symmetric state one can write

$$\langle S^i \rangle = N_c \langle s^i(N_c) \rangle. \tag{D16}$$

We define the spin state

$$|S_1, 1/2; SS_3; p\rangle = \sum_{m_1,m_2} \begin{pmatrix} S_1 & 1/2 & \bigg| & S \\ m_1 & m_2 & \bigg| & S_3 \end{pmatrix} |S_1, m_1\rangle |1/2, m_2\rangle \tag{D17}$$

in terms of an SU(2)-spin CG coefficient with $S_1 = S - 1/2$ for $p = 1$ and $S_1 = S + 1/2$ for $p = 2$.

For a symmetric state similar to (D6) one obtains

$$\begin{aligned} &\langle S_1 1/2; SS'_3; p | s_i(N_c) | S_1 1/2; SS_3; p \rangle \\ &= \sqrt{\frac{3}{4}} \sum_{m_1 m_2 m'_2} \begin{pmatrix} S_1 & 1/2 & \bigg| & S \\ m_1 & m_2 & \bigg| & S_3 \end{pmatrix} \begin{pmatrix} S_1 & 1/2 & \bigg| & S \\ m_1 & m'_2 & \bigg| & S'_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1 & \bigg| & 1/2 \\ m_2 & i & \bigg| & m'_2 \end{pmatrix} \\ &= (-)^{2S} \sqrt{\frac{3}{2}(2S+1)} \begin{pmatrix} S & 1 & \bigg| & S \\ S_3 & i & \bigg| & S'_3 \end{pmatrix} \begin{Bmatrix} 1 & S & S \\ S_1 & 1/2 & 1/2 \end{Bmatrix}. \end{aligned} \tag{D18}$$

Using all this algebra we obtain the equality

$$\sqrt{S(S+1)} = (-)^{2S} N_c \sqrt{\frac{3}{2}} \sqrt{2S+1} \left[\left(c_{22}^{[N_c]} \right)^2 \begin{Bmatrix} 1 & S & S \\ S + 1/2 & 1/2 & 1/2 \end{Bmatrix} - \left(c_{11}^{[N_c]} \right)^2 \begin{Bmatrix} 1 & S & S \\ S - 1/2 & 1/2 & 1/2 \end{Bmatrix} \right], \tag{D19}$$

which is an equation for the unknown quantities. The other equation is the normalization relation (D3)

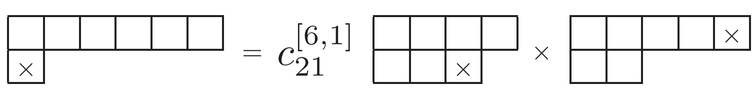
$$(c_{11}^{[N_c]})^2 + (c_{22}^{[N_c]})^2 = 1. \tag{D20}$$

We found

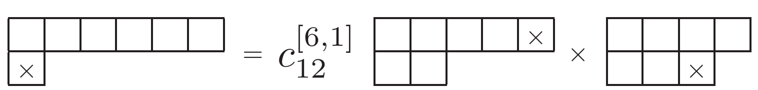
$$\begin{aligned} c_{11}^{[N_c]}(S) &= \sqrt{\frac{S[N_c + 2(S+1)]}{N_c(2S+1)}}, \\ c_{22}^{[N_c]}(S) &= \sqrt{\frac{(S+1)(N_c - 2S)}{N_c(2S+1)}}, \end{aligned} \tag{D21}$$

as in Carlson *et al.* (1998, 1999).

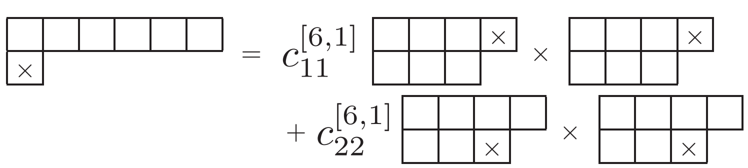
In Eqs. (D22)–(D25), we illustrate the application of isoscalar factors for mixed symmetric states of a system with $N_c = 7$ (Matagne, 2006). In each inner product the first Young diagram corresponds to spin and the second to flavor. Accordingly, one can see that Eq. (D22) stands for $^2 10$, Eq. (D23) for $^4 8$, Eq. (D24) for $^2 8$, and Eq. (D25) for $^2 1$, in the sense of Table XIX. Each inner product contains the corresponding isoscalar factors and the position of the N_cth particle is marked with a cross. In the right-hand side, from the location of the cross one can read off the values of p and of p'. The equations are



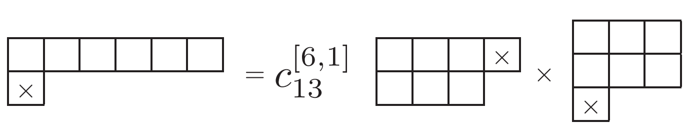
(D22)



(D23)



(D24)



(D25)

The above example is a particular case of the approximate spin-flavor wave function used in the approach of the symmetric core + excited quark of Carlson *et al.* (1998, 1999). One can see that the N_cth particle is always in the second row ($p = 2$) of the spin-flavor wave function and all the terms with the N_cth particle in the first row are missing from the exact wave function (38). Using group theoretical arguments, the relation between the exact wave function and the approximate one as used in Carlson *et al.* (1998, 1999) was thoroughly discussed in Matagne and Stancu (2008b, 2010).

REFERENCES

Adkins, G. S., and C. R. Nappi, 1984, Nucl. Phys. **B233**, 109.

Ahuatzin, G., R. Flores-Mendieta, M. A. Hernandez-Ruiz, and C. P. Hofmann, 2014, Phys. Rev. D **89**, 034012.

Alexandrou, C., T. Korzec, G. Koutsou, and T. Leontiou, 2014, Phys. Rev. D **89**, 034502.

Amaryan, M. J., *et al.*, 2012, Phys. Rev. C **85**, 035209.

Anisovich, A. V., R. Beck, E. Klempt, V. A. Nikonov, A. V. Sarantsev, and U. Thoma, 2012, Eur. Phys. J. A **48**, 15.

Anisovich, A. V., E. Klempt, V. A. Nikonov, A. V. Sarantsev, and U. Thoma, 2011, Eur. Phys. J. A **47**, 153.

Armoni, A., and A. Patella, **2009**, J. High Energy Phys. 07, 073.

Armoni, A., M. Shifman, and G. Veneziano, 2003a, Nucl. Phys. **B667**, 170.

Armoni, A., M. Shifman, and G. Veneziano, 2003b, Phys. Rev. Lett. **91**, 191601.

Auzzi, R., S. Bolognesi, and M. Shifman, 2008, Phys. Rev. D **77**, 125029.

Aziza Baccouche, Z., C.-K. Chow, T. D. Cohen, and B. A. Gelman, 2001, Phys. Lett. B **514**, 346.

Beringer, J., *et al.* (Particle Data Group), 2012, Phys. Rev. D **86**, 010001.

Bhaduri, R. K., 1988, *Models of the nucleon: from quarks to solitons*, Lecture notes and supplements in physics (Addison-Wesley, Reading, MA), p. 203.

Bolognesi, S., 2007, Phys. Rev. D **75**, 065030.

Bolognesi, S., and M. Shifman, 2007, Phys. Rev. D **75**, 065020.

Buchmann, A. J., J. A. Hester, and R. F. Lebed, 2002, Phys. Rev. D **66**, 056002.

Buchmann, A. J., and R. F. Lebed, 2000, Phys. Rev. D **62**, 096005.

Buchmann, A. J., and R. F. Lebed, 2003, Phys. Rev. D **67**, 016002.

Buisseret, F., N. Matagne, and C. Semay, 2012, Phys. Rev. D **85**, 036010.

Buisseret, F., and C. Semay, 2010, Phys. Rev. D **82**, 056008.

Buisseret, F., C. Semay, Fl. Stancu, and N. Matagne, 2008, *Bled Workshops in Physics, DMFA, Zalonistvo, Ljubljana*, Vol. 9, No. 1, p. 9.

Carlson, C. E., and C. D. Carone, 1998a, Phys. Lett. B **441**, 363.

Carlson, C. E., and C. D. Carone, 1998b, Phys. Rev. D **58**, 053005.

Carlson, C. E., and C. D. Carone, 2000, Phys. Lett. B **484**, 260.

Carlson, C. E., C. D. Carone, J. L. Goity, and R. F. Lebed, 1998, Phys. Lett. B **438**, 327.

Carlson, C. E., C. D. Carone, J. L. Goity, and R. F. Lebed, 1999, Phys. Rev. D **59**, 114008.

Carone, C. D., H. Georgi, L. Kaplan, and D. Morin, 1994, Phys. Rev. D **50**, 5793.

Carone, C. D., H. Georgi, and S. Osofsky, 1994, Phys. Lett. B **322**, 227.

Cherman, A., and T. D. Cohen, **2006**, J. High Energy Phys. 12, 035.

Cherman, A., T. D. Cohen, and R. F. Lebed, 2009, Phys. Rev. D **80**, 036002.

Cherman, A., T. D. Cohen, and R. F. Lebed, 2012, Phys. Rev. D **86**, 016002.

Cohen, T. D., 1996, Rev. Mod. Phys. **68**, 599.

Cohen, T. D., and W. Broniowski, 1992, Phys. Lett. B **292**, 5.

Cohen, T. D., D. C. Dakin, R. F. Lebed, and D. R. Martin, 2005, Phys. Rev. D **71**, 076010.

Cohen, T. D., D. C. Dakin, A. Nellore, and R. F. Lebed, 2004a, Phys. Rev. D **69**, 056001.

Cohen, T. D., D. C. Dakin, A. Nellore, and R. F. Lebed, 2004b, Phys. Rev. D **70**, 056004.

Cohen, T. D., and R. F. Lebed, 2003a, Phys. Rev. Lett. **91**, 012001.

Cohen, T. D., and R. F. Lebed, 2003b, Phys. Rev. D **67**, 096008.

Cohen, T. D., and R. F. Lebed, 2003c, Phys. Rev. D **68**, 056003.

Cohen, T. D., and R. F. Lebed, 2004, Phys. Lett. B **578**, 150.

Cohen, T. D., and R. F. Lebed, 2005, Phys. Rev. D **72**, 056001.

Cohen, T. D., and R. F. Lebed, 2014a, Phys. Rev. D **89**, 054018.

Cohen, T. D., and R. F. Lebed, 2014b, Phys. Rev. D **90**, 016001.

Cohen, T. D., D. L. Shafer, and R. F. Lebed, 2010, Phys. Rev. D **81**, 036006.

Coleman, S., 1985, *Aspects of Symmetry* (Cambridge University Press, Cambridge), Chap. 8.

Coleman, S., and S. L. Glashow, 1961, Phys. Rev. Lett. **6**, 423.

Cordon, A. C., T. DeGrand, and J. L. Goity, 2014, Phys. Rev. D **90**, 014505.

Cordon, A. C., and J. L. Goity, 2013, Phys. Rev. D **87**, 016019.

Corrigan, E., and P. Ramond, 1979, Phys. Lett. **87B**, 73.

Dai, J., R. Dashen, E. Jenkins, and A. V. Manohar, 1996, Phys. Rev. D **53**, 273.

Dashen, R. F., E. E. Jenkins, and A. V. Manohar, 1994, Phys. Rev. D **49**, 4713; **51**, 2489(E) (1995).

Dashen, R. F., E. E. Jenkins, and A. V. Manohar, 1995, Phys. Rev. D **51**, 3697.

Dashen, R. F., and A. V. Manohar, 1993a, Phys. Lett. B **315**, 425.

Dashen, R. F., and A. V. Manohar, 1993b, Phys. Lett. B**315**, 438.

De Rujula, A., H. Georgi, and S. L. Glashow, 1975, Phys. Rev. D **12**, 147.

de Urreta, E. G., J. L. Goity, and N. N. Scoccola, 2014, Phys. Rev. D **89**, 034024.

Diakonov, D., V. Petrov, and M. V. Polyakov, 1997, Z. Phys. A **359**, 305.

Diakonov, D., V. Petrov, and A. A. Vladimirov, 2013, Phys. Rev. D **88**, 074030.

Edwards, R. G., N. Mathur, D. G. Richards, and S. J. Wallace, 2013, Phys. Rev. D **87**, 054506.

Elliott, J. P., 1958a, Proc. R. Soc. A **245**, 128.

Elliott, J. P., 1958b, Proc. R. Soc. A **245**, 562.

Flores-Mendieta, R., 2009, Phys. Rev. D **80**, 094014.

Flores-Mendieta, R., M. A. Hernandez-Ruiz, and C. P. Hofmann, 2012, Phys. Rev. D **86**, 094041.

Frandsen, M. T., C. Kouvaris, and F. Sannino, 2006, Phys. Rev. D **74**, 117503.

Gervais, J. L., and B. Sakita, 1984a, Phys. Rev. Lett. **52**, 87.

Gervais, J. L., and B. Sakita, 1984b, Phys. Rev. D **30**, 1795.

Glozman, L. Y., 2002, Phys. Lett. B **541**, 115.

Glozman, L. Y., W. Plessas, K. Varga, and R. F. Wagenbrunn, 1998, Phys. Rev. D **58**, 094030.

Glozman, L. Y., and D. O. Riska, 1996, Phys. Rep. **268**, 263.

Goity, J. L., 1997, Phys. Lett. B **414**, 140.

Goity, J. L., C. Jayalath, and N. N. Scoccola, 2009, Phys. Rev. D **80**, 074027.

Goity, J. L., R. L. Lebed, A. Pich, C. L. Schat, and N. N. Scoccola, 2005, Eds., *Proceedings of the Workshop on Large N_c QCD 2004, Trento, Italy* (World Scientific, Singapore).

Goity, J. L., and N. Matagne, 2007, Phys. Lett. B **655**, 223.
Goity, J. L., C. Schat, and N. Scoccola, 2005, Phys. Rev. D **71**, 034016.
Goity, J. L., C. L. Schat, and N. N. Scoccola, 2002, Phys. Rev. D **66**, 114014.
Goity, J. L., C. L. Schat, and N. N. Scoccola, 2003, Phys. Lett. B **564**, 83.
Goity, J. L., and N. N. Scoccola, 2005, Phys. Rev. D **72**, 034024.
Goity, J. L., and N. N. Scoccola, 2007, Phys. Rev. Lett. **99**, 062002.
Harada, M., F. Sannino, and J. Schechter, 2004, Phys. Rev. D **69**, 034005.
Hayashi, A., G. Eckart, G. Holzwart, and H. Walliser, 1984, Phys. Lett. **147B**, 5.
Hecht, K. T., 1965, Nucl. Phys. **62**, 1.
Hecht, K. T., and S. C. Pang, 1969, J. Math. Phys. (N.Y.) **10**, 1571.
Hermann, R., 1966, *Lie groups for physicists*, The Mathematical Physics Monograph Series, Princeton University (Benjamin, New York), Chap. 11.
Hyodo, T., and D. Jido, 2012, Prog. Part. Nucl. Phys. **67**, 55.
Inönü, E., and E. P. Wigner, 1953, Proc. Natl. Acad. Sci. U.S.A. **39**, 510.
Isgur, N., and G. Karl, 1978, Phys. Rev. D **18**, 4187.
Jayalath, C., J. L. Goity, E. Gonzalez de Urreta, and N. N. Scoccola, 2011, Phys. Rev. D **84**, 074012.
Jenkins, E. E., 1993a, Phys. Lett. B **315**, 431.
Jenkins, E. E., 1993b, Phys. Lett. B **315**, 441.
Jenkins, E. E., 1993c, Phys. Lett. B **315**, 447.
Jenkins, E. E., 1996a, Phys. Rev. D **53**, 2625.
Jenkins, E. E., 1996b, Phys. Rev. D **54**, 4515.
Jenkins, E. E., 1997, Phys. Rev. D **55**, R10.
Jenkins, E. E., 1998, Annu. Rev. Nucl. Part. Sci. **48**, 81.
Jenkins, E. E., 2001, arXiv:hep-ph/0111338.
Jenkins, E. E., 2008, Phys. Rev. D **77**, 034012.
Jenkins, E. E., 2012, Phys. Rev. D **85**, 065007.
Jenkins, E. E., X. D. Ji, and A. V. Manohar, 2002, Phys. Rev. Lett. **89**, 242001.
Jenkins, E. E., and R. F. Lebed, 1995, Phys. Rev. D **52**, 282.
Jenkins, E. E., and R. F. Lebed, 2000, Phys. Rev. D **62**, 077901.
Jenkins, E. E., and A. V. Manohar, 1994, Phys. Lett. B **335**, 452.
Jenkins, E. E., and A. V. Manohar, **2004**a, J. High Energy Phys. 06, 039.
Jenkins, E. E., and A. V. Manohar, 2004b, Phys. Rev. Lett. **93**, 022001.
Jenkins, E. E., A. V. Manohar, J. W. Negele, and A. Walker-Loud, 2010, Phys. Rev. D **81**, 014502.
Keller, D., *et al.* (CLAS Collaboration), 2011, Phys. Rev. D **83**, 072004.
Keller, D., *et al.* (CLAS Collaboration), 2012, Phys. Rev. D **85**, 052004.
Knecht, M., and S. Peris, 2013, Phys. Rev. D **88**, 036016.
Koniuk, R., and N. Isgur, 1980, Phys. Rev. D **21**, 1868; **23**, 818(E) (1981).
Lang, C. B., and V. Verduci, 2013, Acta Phys. Pol. B Proc. Suppl. **6**, 967.
Lebed, R. F., 1995, Phys. Rev. D **51**, 5039.
Lebed, R. F., 2013, Phys. Rev. D **88**, 057901.
Lebed, R. F., and D. R. Martin, 2004a, Phys. Rev. D **70**, 016008.
Lebed, R. F., and D. R. Martin, 2004b, Phys. Rev. D **70**, 057901.
Lebed, R. F., and L. Yu, 2009, Phys. Rev. D **80**, 076006.
Lee, J.-P., C. Liu, and H. S. Song, 1998, Phys. Rev. D **59**, 034002.
Lee, J.-P., C. Liu, and H. S. Song, 2000, Phys. Rev. D **62**, 096001.
Leutwyler, H., 1995, arXiv:hep-ph/9406283.
Lichtenberg, D. B., 1970, *Unitary symmetry and elementary particles* (Academic Press, New York).
Liu, T., Y. Mao, and B.-Q. Ma, 2014, Int. J. Mod. Phys. A **29**, 1430020.
Lucini, B., and M. Panero, 2013, Phys. Rep. **526**, 93.
Luty, M. A., and J. March-Russell, 1994, Nucl. Phys. **B426**, 71.
Luty, M. A., J. March-Russell, and M. White, 1995, Phys. Rev. D **51**, 2332.
Mackey, G. W., 1968, *Induced representations of groups and quantum mechanics* (Benjamin, New York).
Manohar, A. V., 1998, "Large N QCD," Les Houches Lectures Session LXVIII: Probing the Standard Model of Particle Interactions, edited by F. David and R. Gupta, arXiv:hep-ph/9802419.
Matagne, N., 2006, "Baryon resonances in large N_c QCD," Ph.D. thesis, University of Liège, arXiv:hep-ph/0701061.
Matagne, N., and Fl. Stancu, 2005a, Phys. Rev. D **71**, 014010.
Matagne, N., and Fl. Stancu, 2005b, Phys. Lett. B **631**, 7.
Matagne, N., and Fl. Stancu, 2006a, Phys. Rev. D **73**, 114025.
Matagne, N., and Fl. Stancu, 2006b, Phys. Rev. D **74**, 034014.
Matagne, N., and Fl. Stancu, 2008a, Nucl. Phys. **A811**, 291.
Matagne, N., and Fl. Stancu, 2008b, Phys. Rev. D **77**, 054026.
Matagne, N., and Fl. Stancu, 2009, Nucl. Phys. **A826**, 161.
Matagne, N., and Fl. Stancu, 2010, Phys. Rev. D **81**, 116002.
Matagne, N., and Fl. Stancu, 2011a, Phys. Rev. D **83**, 056007.
Matagne, N., and Fl. Stancu, 2011b, Phys. Rev. D **84**, 056013.
Matagne, N., and Fl. Stancu, 2012a, Phys. Rev. D **85**, 116003.
Matagne, N., and Fl. Stancu, 2012b, Phys. Rev. D **86**, 076007.
Matagne, N., and Fl. Stancu, 2013, Phys. Rev. D **87**, 076012.
Mattis, M. P., 1986, Phys. Rev. Lett. **56**, 1103.
Mattis, M. P., 1989a, Phys. Rev. D **39**, 994.
Mattis, M. P., 1989b, Phys. Rev. Lett. **63**, 1455.
Mattis, M. P., and M. Mukerjee, 1988, Phys. Rev. Lett. **61**, 1344.
Mattis, M. P., and M. E. Peskin, 1985, Phys. Rev. D **32**, 58.
Mohler, D., 2012, *Proc. Sci.*, LATTICE2012, 003, arXiv:1211.6163.
Nakamura, K., *et al.* (Particle Data Group), 2010, J. Phys. G **37**, 075021.
Nakano, T., *et al.* (LEPS Collaboration), 2003, Phys. Rev. Lett. **91**, 012002.
Pirjol, D., and C. Schat, 2003, Phys. Rev. D **67**, 096009.
Pirjol, D., and C. Schat, 2007, Phys. Rev. D **75**, 076004.
Pirjol, D., and C. Schat, 2008, Phys. Rev. D **78**, 034026.
Pirjol, D., and C. Schat, 2010, Phys. Rev. D **82**, 114005.
Pirjol, D., and T. M. Yan, 1998a, Phys. Rev. D **57**, 1449.
Pirjol, D., and T. M. Yan, 1998b, Phys. Rev. D **57**, 5434.
Sartor, R., and Fl. Stancu, 1986a, Phys. Rev. D **33**, 727.
Sartor, R., and Fl. Stancu, 1986b, Phys. Rev. D **34**, 3405.
Schat, C. L., J. L. Goity, and N. N. Scoccola, 2002, Phys. Rev. Lett. **88**, 102002.
Scoccola, N. N., J. L. Goity, and N. Matagne, 2008, Phys. Lett. B **663**, 222.
Segal, I. E., 1951, Duke Math. J. **18**, 221.
Semay, C., F. Buisseret, N. Matagne, and Fl. Stancu, 2007, Phys. Rev. D **75**, 096001.
Semay, C., F. Buisseret, and Fl. Stancu, 2007, Phys. Rev. D **76**, 116005.
Semay, C., F. Buisseret, and Fl. Stancu, 2008, Phys. Rev. D **78**, 076003.
Silvestre-Brac, B., C. Semay, and F. Buisseret, 2012, J. Phys. Math. **4**, P120601.
Stancu, Fl., 1996, *Group theory in subnuclear physics*, Oxford Studies in Nuclear Physics Vol. 19 (Oxford University Press, Oxford), p. 1.
Stancu, Fl., 2002, Few-Body Syst. Suppl. **13**, 225.
Stancu, Fl., 2005, Int. J. Mod. Phys. A **20**, 209.

Stancu, Fl., and P. Stassart, 1991, Phys. Lett. B **269**, 243.
Teper, M. J., 1998, Phys. Rev. D **59**, 014512.
't Hooft, G., 1974, Nucl. Phys. B **72**, 461.
Veneziano, G., 2011, *Proceedings of the workshop on Large-N gauge theories, Florence, Italy*, http://ggi-www.fi.infn.it.
Weinberg, S., 2013, Phys. Rev. Lett. **110**, 261601.
Wilson, K. G., 1974, Phys. Rev. D **10**, 2445.
Witten, E., 1979a, Nucl. Phys. **B149**, 285.
Witten, E., 1979b, Nucl. Phys. **B160**, 57.
Wu, B., and B. Q. Ma, 2004, Phys. Rev. D **70**, 034025.