Radiative corrections in precision electroweak physics: A historical perspective

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(published 19 February 2013)

The aim of this article is to review the important role played by radiative corrections in precision electroweak physics, in the framework of both the Fermi theory of weak interactions and the standard theory of particle physics. Important theoretical developments, closely connected with the study and applications of the radiative corrections, are also reviewed. The role of radiative corrections in the analysis of some important signals of new physics is also discussed.

DOI: 10.1103/RevModPhys.85.263 PACS numbers: 12.15.Lk

CONTENTS

I. INTRODUCTION

The aim of this article is to review, from a historical perspective, the important role played by radiative corrections (RC) in precision electroweak physics, in the framework of both the original Fermi theory of weak interactions and the renormalizable standard theory of particle physics, usually referred to as the standard model. Those two areas are discussed in Secs. II and III, respectively.

Studies of such corrections are closely connected with important developments in theoretical particle physics, which are also reviewed. The role of radiative corrections in the analysis of some important signals of new physics is also discussed.

As shown in the Table of Contents, six subsections are based on the Fermi theory of weak interactions and 21 subsections are based on the standard theory of particle physics. They review important and interesting subjects in electroweak physics. On the other hand, in view of the magnitude of the area, encompassing more than 50 years of physics, it was not possible to cover every conceivable subject. Taking this into account, we apologize beforehand for the omission of important and interesting developments that lie beyond the scope of this article.

There are a number of excellent reviews of gauge theories in general and the standard theory of particle physics, in particular. Among them are the following: Abers and Lee (1973), Bég and Sirlin (1974), Weinberg (1974), Taylor (1976), Faddeev and Slavnov (1980), Aoki *et al.* (1982), Bég and Sirlin (1982), Quigg (1983), Cheng and Li (1984),

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Ellis and Peccei (1986), Pokorski (1987), Alexander *et al.* (1988), Altarelli, Kleiss, and Verzegnassi (1989), Einhorn (1991), Jegerlehner (1991), Donoghue, Golowich, and Holstein (1992), Bailin and Love (1993), Hollik (1993), Sirlin (1994a), Bardin, Hollik, and Passarino (1995), Langacker (1995), Merritt *et al.* (1995), Bardin and Passarino (1999), Gunion *et al.* (2000), Sirlin (2000), Böhm, Denner, and Joos (2001), Aitchison and Hey (2003), Sirlin, Marciano, and Chatterjee (2003), Paschos (2007), Jegerlehner (2008), and Langacker (2010).

II. RADIATIVE CORRECTIONS IN THE FERMI THEORY OF WEAK INTERACTIONS

The powerful and highly successful relativistic methods developed by Feynman, Schwinger, Tomonaga, Dyson, and others to evaluate the radiative corrections in quantum electrodynamics[1] were first applied to the weak interactions in the mid-1950s. In particular, Behrends, Finkelstein, and Sirlin (1956) studied the $\mathcal{O}(\alpha)$ electromagnetic corrections to muon decay in the framework of the four-fermion Fermi theory of weak interactions.

We recall that in this theory the interaction Lagrangian density for muon decay is given by

$$\mathcal{L} = -g_i[\bar{\psi}_e \Gamma^i \psi_\mu][\bar{\psi}_{\nu_\mu} \Gamma_i \psi_{\nu_e}] + \text{H.c.}, \tag{1}$$

where i runs over the scalar (S), vector (V), tensor (T), axial vector (A), and pseudoscalar (P) interactions. Explicitly, we have[2]

$$\begin{aligned} \Gamma^S &= 1, \qquad (\Gamma^V)^\mu = \gamma^\mu, \\ (\Gamma^T)^{\mu\nu} &= \frac{\sigma^{\mu\nu}}{\sqrt{2}} = \frac{i}{2\sqrt{2}}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \\ (\Gamma^A)^\mu &= i\gamma^\mu\gamma_5, \qquad \Gamma^P = i\gamma_5. \end{aligned} \tag{2}$$

Equation (1) is the interaction Lagrangian density in the charge-retention order in which leptons of equal charge are placed in the same covariant. $\mathcal{L}$ can be written also in the charge-exchange order

$$\mathcal{L} = -\tilde{g}_i[\bar{\psi}_{\nu_\mu} \Gamma^i \psi_\mu][\bar{\psi}_e \Gamma_i \psi_{\nu_e}] + \text{H.c.}, \tag{3}$$

where $\tilde{g}_i$ are related to g_i by Fierz (1937) transformations.

While Eq. (1) is convenient for actual calculations in the Fermi theory, Eq. (3) conforms more closely with current formulations in which μ decay arises from charged current interactions.

The $\mathcal{O}(\alpha)$ radiative corrections to muon decay in the Fermi theory arise from the interchange of a virtual photon between the μ and the e, the electromagnetic field renormalizations of these particles, and the inner bremsstrahlung contributions.

An important result is that in the charge-retention order of Eq. (1), the $\mathcal{O}(\alpha)$ corrections to muon decay are ultraviolet (UV) convergent only for the vector and axial vector interactions (Behrends, Finkelstein, and Sirlin, 1956). This can be readily understood by analogy with quantum electrodynamics (QED). It is well known that in the scattering of an electron by an external potential, the UV divergence of the vertex part cancels against those in the wave-function renormalizations of the external legs (by the Ward identity). For the vector coupling in muon decay in the charge-retention order, we have an analogous situation, except for the fact that the muon and electron have different masses. However, as the coefficients of these divergences are independent of the fermion masses, they also cancel in muon decay. The corrections involving the axial vector coupling in the charge-retention order can be obtained from those in the vector case by means of the formal transformation $\psi_e \to \psi'_e = \gamma_5\psi_e$, $m_e \to -m_e$ in the Lagrangian density. Thus, they differ only from the vector case by the change $m_e \to -m_e$ and, consequently, the UV divergences cancel also for the axial vector coupling. In contrast, for the S, T, and P interactions of the charge-retention order, the analogy with QED is no longer valid and the $\mathcal{O}(\alpha)$ corrections are logarithmically ultraviolet divergent.

A. Nonconservation of parity. The two-component theory of the neutrino

Lee and Yang (1956) proposed the revolutionary idea that parity is not conserved in the weak interactions and this was soon verified by elegant experiments. In order to accommodate parity nonconservation, Eq. (1) was generalized to

$$\mathcal{L} = -[\bar{\psi}_e \Gamma^i \psi_\mu][\bar{\psi}_{\nu_\mu} \Gamma_i (g_i + g'_i\gamma_5) \psi_{\nu_e}] + \text{H.c.}, \tag{4}$$

with an analogous modification of Eq. (3).

To lowest order, Eq. (4) leads to the following expression for the energy-angle distribution of e^- (e^+) from the decay of a polarized μ^- (μ^+) at rest:

$$\begin{aligned} dN(x,\theta) = \frac{d^3p}{(2\pi)^4}\frac{m_\mu E_0 A}{6}\Big\{ & 3(1-x) \\ & + 2\rho\left[\frac{4}{3}x - 1 - \frac{1}{3}\frac{m_e^2}{E_0^2 x}\right] + 3\eta\frac{m_e}{E_0}\frac{(1-x)}{x} \\ & \mp P\beta\xi\cos\theta\left[1 - x + 2\delta\left(\frac{4x}{3} - 1 - \frac{1}{3}\frac{m_e^2}{m_\mu E_0}\right)\right]\Big\}, \end{aligned} \tag{5}$$

where the upper and lower signs refer to μ^- and μ^+, respectively, θ is the angle between the $e^\mp$ momentum and the spin direction of the $\mu^\mp$; $x = E/E_0$, where E is the $e^\mp$ energy and $E_0 = (m_\mu^2 + m_e^2)/2m_\mu$ its maximum value; p is the $e^\mp$ momentum, $\beta = p/E$, and P is the degree of polarization of $\mu^\mp$. The parameter ρ that describes the energy distribution of $e^\mp$ from unpolarized muons was introduced long ago by Michel (1950) and is generally referred to as the Michel parameter. The parameters ξ and δ, which are currently employed to describe the effects of parity nonconservation, were introduced by Kinoshita and Sirlin (1957a, 1957b). Alternative expressions to Eq. (5), using different

[1]See, for example, Schwinger (1958), Feynman (1962), Kinoshita (1990), and Schweber (1994).

[2]In this paper we used the notational conventions and γ matrices of Bjorken and Drell (1965). We also used "natural units" $\hbar = c = 1$. In Eq. (1) it is understood that the contravariant and covariant indices are contracted and summed from 0 to 3 as in $[\gamma^\mu][\gamma_\mu]$, $[\sigma^{\mu\nu}][\sigma_{\mu\nu}]$, etc.

parametrizations, were obtained by Bouchiat and Michel (1957) and Larsen, Lubkin, and Tausner (1957). Since $E_0 \gg m_e$, the terms proportional to m_e^2 in the cofactors of ρ and δ are very small. For the same reason, the term proportional to η is potentially significant only in the very low-energy part of the spectrum. For a more detailed discussion of theoretical and experimental aspects of muon decay, and the relations between the parameters A, ρ, η, ξ, δ and the couplings g_i and g_i' see, for example,[3] Kinoshita and Sirlin (1957a, 1957b), Berman and Sirlin (1962), Sachs and Sirlin (1975), Sirlin (1980b), and Fetscher and Gerber (2010).

In 1957, Landau (1957), Lee and Yang (1957), and Salam (1957) reintroduced the two-component theory of the neutrino, an elegant formulation that had been long abandoned because it leads to parity nonconservation. This theory can be regarded as a special case of the four-component theory of a massless neutrino, subject to the subsidiary condition

$$a_- \psi_\nu = \psi_\nu \tag{6}$$

or

$$a_+ \psi_\nu = \psi_\nu, \tag{7}$$

where

$$a_\mp = \frac{1 \mp \gamma_5}{2} \tag{8}$$

are the left and right chiral projectors.

If Eq. (6) is satisfied, the massless neutrino has helicity $h = -1$ and the corresponding antineutrino has $h = 1$. If Eq. (7) is satisfied, the signs are reversed. From measurements of the polarization and angular distribution of high-energy positrons in μ^+ decays, it was concluded that $\bar{\nu}_e$ and ν_μ have opposite helicities. Moreover, the helicity of $\bar{\nu}_e$ in β decay was found to be positive. These observations led to the conclusion that both $\bar{\nu}_e$ and $\bar{\nu}_\mu$ have $h = +1$, correspondingly ν_e and ν_μ have $h = -1$, and therefore Eq. (6) holds.

Comparing Eq. (6) with the Lagrangian density in Eq. (4) one readily finds

$$g_S = g_S' = g_T = g_T' = g_P = g_P' = 0, \tag{9}$$

$$g_i' \equiv -g_i \qquad (i = V, A). \tag{10}$$

Namely, in the two-component neutrino theory only the vector and axial vector couplings of the charge-retention order survive, precisely the interactions for which the $\mathcal{O}(\alpha)$ radiative corrections had been previously found to be convergent (Behrends, Finkelstein, and Sirlin, 1956).

Comparison of Eqs. (9) and (10) with the general expressions relating ρ, δ, η, and ξ to the coupling constants further leads to the important conclusions:

$$\rho = \delta = \tfrac{3}{4}, \tag{11}$$

$$\xi = -\frac{g_V g_A^* + g_A g_V^*}{|g_V|^2 + |g_A|^2}, \tag{12}$$

$$\eta = \frac{1}{2}\left[\frac{|g_A|^2 - |g_V|^2}{|g_A|^2 + |g_V|^2}\right]. \tag{13}$$

Thus, in the two-component theory of the neutrino the parameters ρ and δ are sharply predicted, while ξ and η depend only on g_V and g_A.

B. Radiative corrections to muon decay in the two-component theory of the neutrino: Cancellation of mass singularities in integrated observables

In comparing theory with experiment in muon decay, it is important to evaluate the $\mathcal{O}(\alpha)$ corrections since they play a significant role. Including those corrections in the framework of the two-component theory of the neutrino, one obtains the following expression for the energy-angle distributions of $e^-(e^+)$ in the decay of a polarized $\mu^-(\mu^+)$ at rest (Kinoshita and Sirlin, 1959a):

$$\begin{aligned} dN(x,\theta) = {} & \frac{d^3p}{(2\pi)^4}\frac{m_\mu E_0}{3}2(|g_V|^2 + |g_A|^2)\Big\{3 - 2x \\ & - \frac{m_e^2}{E_0^2 x} + \frac{6\eta m_e}{E_0}\frac{(1-x)}{x} + \frac{\alpha}{2\pi}f(x) \\ & \pm P\beta\xi\cos\theta\left[1 - 2x + \frac{m_e^2}{m_\mu E_0} + \frac{\alpha}{2\pi}g(x)\right]\Big\}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} f(x) = {} & (6-4x)R(x) + 6(1-x)\ln x + \frac{(1-x)}{3x^2} \\ & \times[(5 + 17x - 34x^2)(\omega + \ln x) - 22x + 34x^2], \end{aligned} \tag{15}$$

$$\begin{aligned} g(x) = {} & (2-4x)R(x) + (2-6x)\ln x \\ & - \frac{1-x}{3x^2}\Big[(1 + x + 34x^2)(\omega + \ln x) + 3 - 7x \\ & - 32x^2 + 4(1-x)^2\frac{\ln(1-x)}{x}\Big], \end{aligned} \tag{16}$$

$$\begin{aligned} R(x) = {} & 2\mathrm{Li}_2(x) - \frac{\pi^2}{3} - 2 + \omega\left[\frac{3}{2} + 2\ln\left(\frac{1-x}{x}\right)\right] \\ & - (2\ln x - 1)\ln x + \left(3\ln x - 1 - \frac{1}{x}\right)\ln(1-x), \end{aligned} \tag{17}$$

$$\omega = \ln\left(\frac{m_\mu}{m_e}\right), \tag{18}$$

and

[3] In several early papers, including Kinoshita and Sirlin (1957a, 1957b), and Sachs and Sirlin (1975), γ_5 was defined with a sign opposite to the one employed in the present article.

$$\mathrm{Li}_2(x) = -\int_0^x dt \frac{\ln(1-t)}{t} \tag{19}$$

is the dilogarithm function.[4] In Eqs. (15) and (16), we neglected terms of $\mathcal{O}(\alpha m_e/E)$, although all the contributions of $\mathcal{O}(\alpha)$ not proportional to $\cos\theta$ have been exactly evaluated (Behrends, Finkelstein, and Sirlin, 1956; Grotch, 1968; Nir, 1989). The terms $m_e^2/E_0^2 x$ and $m_e^2/m_\mu E_0$ in Eq. (14) are very small and frequently omitted in the literature.

Integrating Eq. (14) over all values of p and θ, one obtains the expression for the muon lifetime τ_μ, including $\mathcal{O}(\alpha)$ corrections

$$\frac{1}{\tau_\mu} = \frac{(|g_V|^2 + |g_A|^2) m_\mu^5}{192\pi^3}\left[1 - \frac{8m_e^2}{m_\mu^2} + 4\eta\frac{m_e}{m_\mu}\right] \times\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right], \tag{20}$$

where we neglected terms of order $(m_e/m_\mu)^4$, $\eta(m_e/m_\mu)^3$, and $\alpha m_e/m_\mu$.

The $\mathcal{O}(\alpha)$ radiative corrections have a large effect on the $e^\mp$ spectrum in μ decay. In fact, they decrease the decay probability for large x and increase it for small x. In order to estimate the magnitude of this effect, it was been pointed out that if the $e^\mp$ spectrum in Eq. (14) is fitted with an effective uncorrected formula of the Michel type [cf. Eq. (5)] over the range $0.3 \lesssim x \lesssim 0.95$, the parameter ρ_{eff} obtained in this manner is $\rho_{\mathrm{eff}} \approx 0.71$ rather than the value 3/4 of the two-component theory (Kinoshita and Sirlin, 1959a). Similar observations hold for the parameter δ that governs the x dependence of the $\cos\theta$ part of the decay probability. Since current determinations of ρ and δ agree with the predictions of Eq. (11) at the 0.035% and 0.046% levels, respectively (Bayes *et al.*, 2011), it is clear that the radiative corrections play a crucial role in verifying the validity of the two-component theory of the neutrino.

On the other hand, the $\mathcal{O}(\alpha)$ corrections to the muon lifetime given in Eq. (20) amount to only -4.2×10^{-3}. The reason why the corrections to the electron spectrum are quite large while the corrections to τ_μ are rather small has been traced to the cancellation of "mass singularities" in integrated observables, discovered by Kinoshita and Sirlin (1959a). In the case of muon decay, it implies that the corrections to the lifetime and the integrated asymmetry are finite in the mathematical limit $m_e \to 0$. The properties discussed above can be nicely illustrated by considering the terms proportional to the large parameter $\omega = \ln(m_\mu/m_e) \approx 5.332$ in the corrections to the spectrum [cf. Eqs. (14), (15), and (17)]. They are proportional to

$$\frac{\alpha}{2\pi}\omega dx\left\{(6-4x)x^2\left[\frac{3}{2} + 2\ln\left(\frac{1-x}{x}\right)\right] + \frac{(1-x)}{3}[5 + 17x - 34x^2]\right\}, \tag{21}$$

and contain the electron mass singularity since ω diverges in the $m_e \to 0$ limit. When integrated over the full spectrum, i.e., in the range $1 \geq x \geq 0$, Eq. (21) vanishes, leading to the cancellation mentioned above. Furthermore, the expression between curly brackets is negative in the upper part of the spectrum ($x \gtrsim 0.68$) and positive for $x \lesssim 0.68$. Using Eq. (16), one readily verifies that the terms proportional to $(\alpha/2\pi)\omega$ in the $\cos\theta$ term of Eq. (14) also cancel when integrated over the full range $1 \geq x \geq 0$. The cancellation of mass singularities has also been verified in the $\mathcal{O}(\alpha)$ contributions to $1/\tau_\mu$ proportional to g_S^2, g_T^2, and g_P^2 in the general Fermi theory, as well as in the corrections to the β-decay lifetime in the framework of the V-A theory (see Sec. II.D). Furthermore, it has provided one of the main motivations for the Kinoshita-Lee-Nauenberg (KLN) theorem (Kinoshita, 1962; Lee and Nauenberg, 1964).

An observable for which the $\mathcal{O}(\alpha)$ corrections become extremely large is the asymmetry of low-energy $e^\mp$ (Kinoshita and Sirlin, 1957c). Their effect on the asymmetry parameter ξ is also discussed by Kinoshita and Sirlin (1959a). Another important result of the two-component neutrino theory was the prediction of the photon spectrum and rate in radiative muon decay $\mu \to e + \nu + \bar{\nu} + \gamma$ before its detection (Kinoshita and Sirlin, 1959b). As an example, for photons of energy $\geq 20m_e$, the branching ratio was predicted to be 1.2%.

As emphasized, the two-component theory of the neutrino leads to the definite predictions $\rho = \delta = 3/4$ [cf. Eq. (11)]. In order to measure with high precision these basic parameters (as well as ξ, η, and A) in the four-component neutrino framework of the general Fermi theory [cf. Eq. (5)], one approach has been to employ the fractional radiative corrections of the two-component neutrino theory which, as discussed, are finite and well defined. Specifically (Sherwood, 1967), the expression between curly brackets not involving $\cos\theta$ in Eq. (5) is multiplied by

$$1 + [(\alpha/2\pi)f(x)]/[3 - 2x - m_e^2/E_0^2 x + 6\eta(m_e/E_0)(1-x)/x],$$

while the expression proportional to $\cos\theta$ is multiplied by

$$1 + [(\alpha/2\pi)g(x)]/[1 - 2x + m_e^2/m_\mu E_0].$$

Comparison with Eq. (14) shows that these factors are indeed the corresponding fractional corrections in the two-component neutrino theory. The justification for this procedure is that, to a high degree of precision, the current experimental information is consistent with pure V, A, V', and A' interactions. Possible deviations which in the four-component neutrino framework involve quadratic expressions in g_i, $g_i'(i = S, T, P)$ are expected to be very small and can therefore be treated at the tree level. The products of these small deviations with $(\alpha/2\pi)f(x)$ and $(\alpha/2\pi)g(x)$ are of second order in the small quantities and, therefore, are not considered significant.

At present, very precise measurements of ρ, δ, ξ, and η are carried out in the TWIST (TRIUMF weak interaction symmetry test) experiment at TRIUMF (Canada's national laboratory for particle and nuclear physics) (Bayes *et al.*, 2011), and an accurate determination of τ_μ was made by the Mulan Collaboration at PSI (Webber *et al.*, 2011a, 2011b).

[4]See, for example, Lewin (1958).

C. The V-A theory

The discovery of parity nonconservation led to another important development: by greatly increasing the number of observables available for experimental and theoretical study, it opened the way for the determination of the basic phenomenological interaction. This led Sudarshan and Marshak (1957, 1958) and Feynman and Gell-Mann (1958) to propose a universal V-A Fermi interaction for charged current processes, such as muon decay, β decay, and the semileptonic decays of hyperons.

In the case of muon decay, this theory implies the validity of Eqs. (9) and (10) and furthermore states that

$$g_A = -g_V. \tag{22}$$

Using the Fierz transformations (Fierz, 1937), Eqs. (9), (10), and (22) lead to the following coupling constants $\tilde{g}_i$, $\tilde{g}'_i$ in the charge-exchange order:

$$\tilde{g}_S = \tilde{g}'_S = \tilde{g}_T = \tilde{g}'_T = \tilde{g}_P = \tilde{g}'_P = 0, \tag{23}$$

$$\tilde{g}_V = -\tilde{g}_A = g_V = -\tilde{g}'_V = \tilde{g}'_A. \tag{24}$$

Defining $G_\mu \equiv \sqrt{2} g_V$, Eqs. (9), (10), and (22)–(24) lead to

$$\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu] \times [\bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_{\nu_e}] + \text{H.c.} \tag{25}$$

$$= -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_\mu] \times [\bar{\psi}_{\nu_\mu}\gamma_\mu(1-\gamma_5)\psi_{\nu_e}] + \text{H.c.} \tag{26}$$

Thus, the interaction Lagrangian for muon decay in the V-A theory has a simple and elegant form that involves a single coupling constant and is preserved in passing from the charge-retention to the charge-exchange order. Equations (9), (10), and (22) lead also to the sharp predictions

$$\rho = \delta = 3/4, \tag{27}$$

$$\eta = 0, \tag{28}$$

$$\xi = 1, \tag{29}$$

as can be readily verified by inserting Eq. (22) into Eqs. (12) and (13).

With the neglect of strong interaction (SI) effects, in the original version of the V-A theory other four-fermion interaction processes were described by Lagrangian densities of the same form as Eq. (25). For example, for $n \to p + e^- + \bar{\nu}_e$, the basic process for β decay, the Lagrangian density was postulated to be of the form,

$$\mathcal{L}_{\beta\,\text{decay}} = -\frac{G_V}{\sqrt{2}}[\bar{\psi}_p\gamma^\mu(1-\gamma_5)\psi_n][\bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_{\nu_e}] + \text{H.c.}, \tag{30}$$

where G_V is the vector coupling constant in β decay.

D. Radiative corrections to muon decay in the V-A theory and the Fermi constant

Taking into account Eqs. (22) and (27)–(29), we see that in the V-A theory the energy-angle distributions of e^- (e^+) in muon decay are simply obtained by setting $|g_A| = g_V = G_\mu/\sqrt{2}$, $\eta = 0$, and $\xi = 1$ in the two-component theory expression [Eq. (14)]. In particular, the $\mathcal{O}(\alpha)$ corrections are still governed by the functions $f(x)$ and $g(x)$. Furthermore, using the transformation $\psi_e \to \psi'_e = \gamma_5\psi_e$, $m_e \to -m_e$ discussed in Sec. II.A, it can be shown that in the V-A theory there are no contributions to the differential decay rate [Eq. (14)] that involve odd powers of m_e (Roos and Sirlin, 1971). This implies that corrections of $\mathcal{O}((\alpha/\pi)m_e/m_\mu)$ are absent and that the leading mass-dependent corrections to the differential decay rate are of $\mathcal{O}((\alpha/\pi)m_e^2/m_\mu^2\ln(m_\mu^2/m_e^2))$. On the other hand, in the calculation of integrated observables such as the total decay rate, the integration over the electron or positron momentum gives rise to corrections of $\mathcal{O}(\alpha)$ proportional to $(m_e/m_\mu)^3$, as well as even powers of m_e/m_μ (van Ritbergen and Stuart, 1999).

Radiative corrections of $\mathcal{O}(\alpha^2)$ to the electron spectrum were evaluated by Arbuzov, Czarnecki, and Gaponenko (2002), Arbuzov and Melnikov (2002), Arbuzov (2003), and Anastasiou, Melnikov, and Petriello (2007).

Recently, the TWIST Collaboration (Bayes *et al.*, 2011) reported very accurate measurements of the parameters ρ, δ, and $\mathcal{P}_\mu^\pi\xi$ in the four-component neutrino framework of the general Fermi theory ($\mathcal{P}_\mu^\pi$ is the initial degree of polarization of the muon from π decay):

$$\rho = 0.749\,77 \pm 0.000\,12(\text{stat}) \pm 0.000\,23(\text{syst}), \tag{31}$$

$$\delta = 0.750\,49 \pm 0.000\,21(\text{stat}) \pm 0.000\,27(\text{syst}), \tag{32}$$

$$\mathcal{P}_\mu^\pi\xi = 1.000\,84 \pm 0.000\,29(\text{stat})^{+0.001\,65}_{-0.000\,63}(\text{syst}). \tag{33}$$

These results are in good agreement with the predictions of the V-A theory, Eqs. (27) and (29) and $\mathcal{P}_\mu^\pi = 1$, at a high level of precision. As mentioned, the RC play a crucial role in the analysis. They also use these results to derive interesting bounds for the combinations $|(g_R/g_L)\zeta|$ and $(g_L/g_R)m_2$ in the generalized left-right symmetry model (g_L and g_R are the gauge couplings of W_L and W_R, ζ is the mixing angle when W_L and W_R are expressed in terms of the mass eigenstates W_1 and W_2, and m_2 is the mass of W_2).

The radiative corrections to the muon lifetime τ_μ have been the subject of great interest and detailed studies. In fact, the argument given at the end of Sec. II.A can be generalized: it has been shown that to leading order in G_μ, but all orders in α, the radiative corrections to muon decay in the V-A theory are finite after mass and charge renormalization (Berman and Sirlin, 1962). The detailed calculations now reach the two-loop level and lead to

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}F(x)[1+\delta_\mu], \tag{34}$$

where $x = m_e^2/m_\mu^2$, $F(x) = 1 - 8x - 12x^2\ln x + 8x^3 - x^4$ is a tree-level phase-space factor, and δ_μ is the radiative correction.

Neglecting very small terms proportional to powers of m_e/m_μ, we have

$$\delta_\mu = \frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)\left[1+\frac{2\alpha}{3\pi}\ln\left(\frac{m_\mu}{m_e}\right)\right]+6.700\left(\frac{\alpha}{\pi}\right)^2+\cdots. \tag{35}$$

The $\mathcal{O}(\alpha)$ term has been known since the end of the 1950s (Berman, 1958; Kinoshita and Sirlin, 1959a), the logarithmic term of $\mathcal{O}(\alpha^2)$ was derived in 1971 (Roos and Sirlin, 1971), and the last term in 1999 (Steinhauser and Seidensticker, 1999; van Ritbergen and Stuart, 1999, 2000), about 40 years after the one-loop correction. The two terms of $\mathcal{O}(\alpha^2)$ nearly cancel each other. Including very small one- and two-loop contributions proportional to powers of m_e/m_μ (van Ritbergen and Stuart, 1999; Pak and Czarnecki, 2008), we have

$$\delta_\mu = -4.199\,48\times10^{-3} + 1.06\times10^{-6}, \tag{36}$$

where the first and second terms stand for the one- and two-loop contributions, respectively. This reveals that when the corrections are expressed in terms of α, as in Eq. (35), the $\mathcal{O}(\alpha^2)$ effects are very small, and the original $\mathcal{O}(\alpha)$ calculation turns out to be accurate. Alternatively, δ_μ is frequently written in the form (Steinhauser and Seidensticker, 1999; van Ritbergen and Stuart, 1999, 2000)

$$\delta_\mu = \frac{\alpha(m_\mu)}{2\pi}\left(\frac{25}{4}-\pi^2\right)+6.700\left(\frac{\alpha(m_\mu)}{\pi}\right)^2+C(x)+\cdots, \tag{37}$$

where $\alpha(m_\mu)=1/135.902\,628\,3\ldots$ is the running $\alpha(\mu)$ parameter at the m_μ scale. In this second form the logarithmic term of $\mathcal{O}(\alpha^2)$ has been absorbed in the $\mathcal{O}(\alpha(m_\mu))$ contribution, and the $\mathcal{O}(\alpha^2(m_\mu))$ effects are $\approx 3.6\times10^{-5}$, considerably larger than in Eq. (36). The correction δ_μ has also been studied using optimization methods that select the optimal scale in $\alpha(\mu)$, permit one to analyze the scheme dependence of the calculations, and estimate the unknown terms of $\mathcal{O}(\alpha^3(m_\mu))$ (Ferroglia, Ossola, and Sirlin, 1999). This analysis leads to an estimated error of $\approx 2.6\times10^{-7}$ in δ_μ due to the truncation of the perturbative series.

$C(x)$ in Eq. (37) denotes very small RC proportional to powers of x. Specifically,

$$\begin{aligned}C(x) =& \frac{\alpha(m_\mu)}{\pi}[x(-12\ln x-9-4\pi^2+16\pi^2x^{1/2})+\mathcal{O}(x^2)]\\ &-\left(\frac{\alpha(m_\mu)}{\pi}\right)^2 0.0784+\cdots.\end{aligned} \tag{38}$$

The terms of $\mathcal{O}(\alpha(m_\mu)x^l/\pi)$ $(l=1,\ 3/2)$ were derived by van Ritbergen and Stuart (1999). Their expression differs from that in Eq. (38) because of the factorization of $F(x)$ in our Eq. (34), which was not employed by van Ritbergen and Stuart. For clarity, we point out that to the stated level of accuracy our result for $1/\tau_\mu$ based on Eqs. (34), (37), and (38) through the terms of $\mathcal{O}(\alpha(m_\mu)x^l/\pi)$ is equivalent to that obtained in their 1999 paper. The contribution of $\mathcal{O}((\alpha(m_\mu)/\pi)^2)$ was derived years later (Pak and Czarnecki, 2008) and amounts to -4.3×10^{-7}. An interesting feature is that its leading contribution is linear in m_e/m_μ: $-[\alpha(m_\mu)/\pi]^2(5/4)\pi^2x^{1/2}=-3.27\times10^{-7}$.

Because of the high precision of the τ_μ measurement (Webber *et al.*, 2011a, 2011b) and the theoretical clarity of Eqs. (34), (35), (37), and (38), G_F, the universal Fermi constant of the weak interactions, is identified with G_μ. Inserting the experimental value $\tau_\mu = 2\,196\,980.3(2.2)$ ps, Eqs. (34), (37), and (38) lead to $\delta_\mu = -4.198\,18\times10^{-3}$ and

$$G_F = G_\mu = 1.166\,378\,8(7)\times10^{-5}\ \mathrm{GeV}^{-2}, \tag{39}$$

an important 0.6 ppm determination (Webber *et al.*, 2011a, 2011b).

We note that the evaluation of δ_μ in the α and $\alpha(m_\mu)$ schemes, namely, $\delta_\mu=-4.19842\times10^{-3}$ [see Eq. (36)] and $\delta_\mu=-4.19818\times10^{-3}$, respectively, differ by -2.4×10^{-7}. This difference is consistent with the estimate of the third order coefficient in the $\alpha(m_\mu)$ expansion on the basis of the optimization methods, namely, $(c_3)_{\mathrm{est}}\approx-20$ (Ferroglia, Ossola, and Sirlin, 1999). The effect of this difference on the determination of G_F [see Eq. (39)] is also small in comparison with the current experimental error.

We also note that, in some theoretical discussions of $1/\tau_\mu$, a factor $(1+3m_\mu^2/M_W^2)$ that represents the tree-level correction from the W-boson propagator is applied to the right-hand side (rhs) of Eq. (34). Since this factor does not arise in the Fermi theory framework, it is not included in our Eq. (34). It was pointed out by van Ritbergen and Stuart (1999) that, in the standard theory calculations, it can be more naturally included in the electroweak correction Δr [cf. Eq. (54)]. More generally, it can be included in the expressions of the form $G_F(1-\mathrm{EWC})$, where EWC denotes a generic electroweak correction such as $\Delta\hat{r}$, $\Delta\hat{r}_W$, and Δr_{eff} [cf. Eqs. (57), (58), and (66)]. On the other hand, it is useful to observe that this factor would amount to an addition of only $\approx 5\times10^{-7}$ to such electroweak correction, which is negligible at the current level of accuracy.

E. The universality of the weak interactions and the conserved vector current hypothesis

The principle of universality of the weak interactions is a concept of enduring significance. In fact, it has motivated, at least in part, several important developments in particle physics.

The origin of the idea can be traced to 1947–1949, when several authors (Pontecorvo, 1947; Klein, 1948; Puppi, 1948, 1949; Lee, Rosenbluth, and Yang, 1949; Tiomno and Wheeler, 1949) noted that the basic processes $\mu^-\to e^-+\nu_\mu+\bar{\nu}_e$, $n\to p+e^-+\bar{\nu}_e$, and $\mu^-+p\to n+\nu_\mu$ are characterized approximately by the same coupling constant, of magnitude $\approx 10^{-5}$ GeV^{-2}. On this basis they proposed a universal weak interaction among the doublets (ν_e, e), (ν_μ, μ), and (p, n). In 1951, Enrico Fermi stated that this similarity is probably not accidental and has a deep meaning not understood at the time (Fermi, 1951). He also suggested a possible analogy with the universality of electric charge.

In their paper, Feynman and Gell-Mann (1958) compared G_μ with G_V, the vector coupling in β decay extracted from ^{14}O decay, a superallowed $(0^+\to0^+)$ Fermi transition in which only the vector current contributes to zeroth order in α. They found $G_V = G_\mu$ within roughly 1%. The result was surprising, since even if one assumed $G_V = G_\mu$ at the Lagrangian level as a manifestation of universality, a close equality was not expected because nucleons in β decay are affected by strong interactions, while this is not the case for the leptons in muon decay. This prompted Feynman and Gell-Mann (1958) to invoke the conserved vector current (CVC) hypothesis, previously discussed by Gershtein and Zeldovich (1955). Specifically, the hadronic vector current in β decay is assumed to be conserved in the presence of the strong interactions. Since conservation laws are generally associated with symmetries of the theory, they further identified it with the $\Delta I_3 = 1$ isospin current. The near equality $G_V \approx G_\mu$ could then be understood on the basis of two concepts: the principle of universality that states $G_V = G_\mu$ at the Lagrangian level, and CVC that implies that the strong interactions do not renormalize G_V at $q^2 = 0$ in the limit of isospin invariance.

CVC, in turn, had another important consequence. If the strangeness conserving ($\Delta S = 0$) vector current is conserved, it would be natural to assume that the strangeness nonconserving ($\Delta S = 1$) vector current in semileptonic decays is also conserved in some suitable limit. This was one of the main motivations for the search for higher partial symmetries of the strong interactions. A number of possibilities were considered (Behrends *et al.*, 1962), culminating with the phenomenologically successful $\mathrm{SU}(3)_{\mathrm{flavor}}$ symmetry (Gell-Mann, 1962; Gell-Mann and Ne'eman, 1964). Gell-Mann also noted that a normalization of the hadronic currents is necessary in order to precisely define the concept of universality. This was an important motivation for current algebra (Gell-Mann, 1964a). In fact, the nonlinearity of the basic current algebra relation

$$[J_0^a(x), J_0^b(y)]_{x_0=y_0} = i f^{abc} J_0^c(x)\delta^3(\vec{x}-\vec{y}), \tag{40}$$

where $f^{abc}(a,b,c = 1,\ldots,8)$ are the SU(3) structure constants, determines the normalization of the hadronic currents. $\mathrm{SU}(3)_{\mathrm{flavor}}$ also led to the fundamental concept of quarks (Gell-Mann, 1964b; Zweig, 1964) and the quark model of hadrons.

F. Radiative corrections to β decay in the V-A theory

When the CVC hypothesis was formulated, it was natural to suspect that the $\approx 1\%$ difference between G_V and G_μ was due to electromagnetic corrections. Here we have in mind electromagnetic corrections not contained in Fermi's Coulomb function which is automatically included in the theory of β decay. However, when the $\mathcal{O}(\alpha)$ corrections to the decay probability of neutron β decay were calculated by Berman (1958) and Kinoshita and Sirlin (1959a) in the V-A theory [cf. Eq. (30)], a striking result was found: contrary to the case of muon decay, the $\mathcal{O}(\alpha)$ corrections to β decay were logarithmically divergent. In particular, the detailed expression found by Kinoshita and Sirlin (1959a) for the $\mathcal{O}(\alpha)$ corrections to the electron or positron spectrum is given by

$$\Delta P d^3p = \frac{\alpha}{2\pi} P^0 d^3p \left\{ 6\ln\left(\frac{\Lambda}{m_p}\right) + g(E, E_m) + \frac{9}{4} \right\}, \tag{41}$$

$$\begin{aligned} g(E,E_m) = {} & 3\ln\left(\frac{m_p}{m_e}\right) - \frac{3}{4} - \frac{4}{\beta}\mathrm{Li}_2\left(\frac{2\beta}{1+\beta}\right) \\ & + 4\left[\frac{\tanh^{-1}\beta}{\beta} - 1\right]\left[\frac{(E_m - E)}{3E} - \frac{3}{2} \right. \\ & \left. + \ln\left\{\frac{2(E_m-E)}{m_e}\right\}\right] + \frac{\tanh^{-1}\beta}{\beta}\left[2(1+\beta^2) \right. \\ & \left. + \frac{(E_m-E)^2}{6E^2} - 4\tanh^{-1}\beta\right], \end{aligned} \tag{42}$$

where p and E are the momentum and energy of the electron or positron, E_m is the end-point energy, $\beta = p/E$, m_p is the proton mass, Λ is the ultraviolet cutoff, and

$$P^0 d^3p = \frac{8G_V^2}{(2\pi)^4}(E_m - E)^2 d^3p \tag{43}$$

is the uncorrected spectrum. In deriving Eq. (41), strong interactions have been neglected, so these results represent the corrections to the β decay of "bare nucleons" devoid of hadronic structure. Very small contributions of $\mathcal{O}(E/m_p)$ have also been neglected.

The reason why the corrections to β decay are divergent in the V-A theory while those for muon decay are finite, can be understood in two ways:

(i) In contrast to the muon decay case, starting with the interaction Lagrangian of Eq. (30) appropriate to β decay, it is not possible to bring the two charged particles into the same covariant while retaining only V and A interactions. Thus, the analogy with QED discussed in Sec. II.A is lost in the case of β decay and the corrections are divergent.

(ii) Using a current algebra formulation, it can be shown that in the V-A theory the divergent part of the corrections to Fermi transitions is of the form

$$\frac{\alpha}{2\pi} P^0 d^3p\, 3[1 + 2\bar{Q}]\ln(\Lambda/M), \tag{44}$$

where $\bar{Q}$ is the average charge of the underlying hadronic fields in the process and M is a relevant mass. In the case of Eq. (30), the underlying fields are the neutron and proton so that $\bar{Q} = 1/2$ and the divergent part is $(\alpha/2\pi)P^0 d^3p\, 6\ln(\Lambda/M)$, in agreement with Eq. (41). In the case of muon decay, the roles of p and n are played by ν_μ and μ^-, so that $\bar{Q} = -1/2$ and Eq. (44) vanishes, consistent with the fact that the corrections to muon decay are finite in the V-A theory. It is interesting to note that in the corrections proportional to $|M_F|^2$, where M_F is the Fermi matrix element, the terms $3\ln(\Lambda/M)$ and $6\bar{Q}\ln(\Lambda/M)$ in Eq. (44) arise from the vector and axial vector currents, respectively. Similarly, in Eq. (41) $3\ln(\Lambda/m_p) + g(E, E_m)$ is the contribution from the vector current while the remaining $3\ln(\Lambda/m_p) + 9/4$ emerges from the axial vector current. Thus, although the axial vector current does not contribute to the Fermi matrix element at the tree level, it plays an important role in $\mathcal{O}(\alpha)$.

The finding that the radiative corrections to β decay in the V-A theory are divergent, while those to muon decay are convergent, created a serious theoretical problem since both processes are fundamental observables. Originally, Feynman, Berman, Kinoshita, and Sirlin thought that this conundrum was due to the fact that strong interactions had been ignored in the calculations of the β-decay corrections. In fact, it was easy to imagine that strong interactions could give rise to form factors that would cut off the high-energy contributions of the virtual photons. If so, Λ in Eqs. (41) and (44) was expected to be of the order of magnitude of the nucleon mass $M_N \approx 1$ GeV. The same point of view was strongly advocated by Källén (1967). A further complication at the time was that for $\Lambda \gtrsim 1$ GeV the radiative corrections increased the difference between G_V and G_μ. The situation, as it existed in 1960, was summarized by Feynman (1960).

The statement of universality was significantly changed when Cabibbo (1963) proposed his theory of semileptonic decays, constructed on the basis of $SU(3)_{\text{flavor}}$ currents. Rather than stating $G_V(\Delta S = 0) = G_\mu$, the principle of universality was expressed as

$$G_V(\Delta S = 0) = G_\mu \cos\theta_c,\\ G_V(\Delta S = 1) = G_\mu \sin\theta_c, \tag{45}$$

where θ_c is the Cabibbo angle. Thus, in the new framework we had

$$G_\mu^2 = G_V^2(\Delta S = 0) + G_V^2(\Delta S = 1). \tag{46}$$

Equation (45) had two important consequences: by adjusting appropriately $\sin\theta_c$, it successfully described the fact that $\Delta S = 1$ semileptonic decays are significantly suppressed relative to $\Delta S = 0$ processes and, furthermore, the radiative corrections with $\Lambda \gtrsim 1$ GeV had an effect that was at least in the right direction to comply with Eqs. (45) and (46).

In the 1960s there were other developments that also contributed significantly to the analysis of universality. Behrends and Sirlin (1960) showed that if the conservation of SU(2) vector currents (such as the isospin currents) is broken by mass splittings, their matrix elements at zero momentum transfer are not renormalized to first order in the symmetry-breaking parameters. They also conjectured the generalization of this theorem to higher symmetries. The results were confirmed by Terent'ev (1963) on the basis of a different argument. Ademollo and Gatto (1964) independently derived the analogous theorem for SU(3) vector currents. This nonrenormalization theorem plays an important role in the analysis of universality: in the SU(2) case it applies to β decay, while in the SU(3) context it is relevant for the $\Delta S = 1$ semileptonic decays.

In 1966, there was another important and surprising development. Bjorken (1966), using current algebra methods, reached the conclusion that the strong interactions do not tame the logarithmic divergence of the radiative corrections to the Fermi transitions in β decay. Thus, according to this approach, the cutoff Λ did not arise from the strong interactions. The analysis was extended by Abers, Norton, and Dicus (1967), who studied the divergent part of the corrections to the Fermi amplitude arising from the axial vector current. In their work, they applied the Bjorken-Johnson-Low limit (Bjorken, 1966; Johnson and Low, 1966) with a simplified, canonical evaluation of the relevant commutators. Sirlin (1967a), using a different approach, showed that the function $g(E, E_m)$, which describes the corrections to the electron or positron spectrum in β decay [cf. Eq. (42)], is valid in the presence of the strong interactions, provided one neglects small contributions of $\mathcal{O}(\alpha E/M)$, where M is a relevant hadronic mass. The approach employed by Sirlin (1967a), the so-called $1/k$ method, consists of separating out, in a gauge-invariant manner, the contributions that behave as $1/k$ as $k \to 0$ in the hadronic parts of the Feynman integrals, where k is the virtual photon four-momentum. Such contributions are not affected by the strong interactions and lead to the function $g(E, E_m)$. The remaining contributions are shown to fall into two classes: constant amplitudes, independent of E and E_m, which are affected by the strong interactions, but can be absorbed by suitable redefinitions of the vector and axial vector coupling constants g_V and g_A, and very small terms of $\mathcal{O}(\alpha E/M)$ which are neglected. This method was extended to treat other observables such as the longitudinal polarization of electrons or positrons (Sirlin, 1967a) and the asymmetry from polarized nuclei in β decay (Shann, 1971; Yokoo, Suzuki, and Morita, 1973; Garcia and Maya, 1978; Gluck and Toth, 1992). The current algebra formulation and the $1/k$ method finally overlapped when, in a subsequent paper, Abers *et al.* (1968) were able to obtain not only the divergent parts, but also the corrections to the energy spectrum described by the function $g(E, E_m)$. In fact, the current algebra formulation led to the important conclusion that, neglecting very small contributions of $\mathcal{O}(\alpha E/M)$, the $\mathcal{O}(\alpha)$ corrections to the Fermi amplitude arising from the vector current are not affected by the strong interactions, and it appeared that the divergent contributions involving the axial vector current were also known. Although other methods to evaluate the radiative corrections to β decay were pursued, most notably by Källén (1967), the current algebra formulation became the prevalent approach.

Thus, in 1967 the situation regarding the radiative corrections to β decay was both interesting and perplexing. On the one hand, the current algebra approach had been the basis of great technical progress. On the other hand, there was the great difficulty that in the V-A local Fermi theory the corrections are divergent. At the time, two different solutions to this serious problem were suggested: (i) Cabibbo, Maiani, and Preparata (1967a, 1967b) and Johnson, Low, and Suura (1967) proposed to modify the space-space commutators of the current algebra of hadronic currents in such a way that the radiative corrections to β decay become convergent. (ii) Sirlin (1967b) proposed that the solution to the dilemma lies instead in an extension of the Fermi theory involving charged intermediate bosons $W^\pm$. The argument was that in this framework the leading divergent contributions to muon and β decay are the same, so that they can be absorbed in a universal renormalization of G_μ and G_V, as discussed by Sirlin (1967b) and Abers *et al.* (1968). This approach, however, was not complete since the intermediate boson theory employed was not renormalizable and, as a consequence, logarithmic divergences with very small coefficients were not canceled. An additional limitation was that in this theory the effective cutoff was $\Lambda \approx m_W$, and its magnitude was unknown at the time.

Analogous results were previously obtained by Lee (1962), Shaffer (1962, 1963), Bailin (1964, 1965), and Dorman (1964), who studied the radiative corrections in the intermediate boson framework in the case of bare nucleons, devoid of strong interactions. The situation was summarized by Sirlin (1968).

As explained in Sec. III, the solution of the serious problem affecting the radiative corrections to β decay had to wait until the emergence of the standard theory, a renormalizable theory of electroweak interactions.

Recently, a close, analytic correction for the $\mathcal{O}(\alpha)$ radiative correction to the $\bar{\nu}_e$ (ν_e) spectrum in allowed β decay was derived (Sirlin, 2011). The motivation of this calculation is that knowledge of the $\bar{\nu}_e$ (ν_e) spectrum is currently important for reactor studies of neutrino oscillations. One finds

$$dP_\nu = dP_\nu^0\left[1 + \left(\frac{\alpha}{2\pi}\right)h(\hat{E}, E_m)\right], \tag{47}$$

where

$$dP_\nu^0 = A\hat{p}\,\hat{E}\,F(Z, \hat{E})K^2 dK \tag{48}$$

is the zeroth order spectrum,

$$\begin{aligned} h(\hat{E}, E_m) = {}& 3\ln\left(\frac{m_p}{m_e}\right) + \frac{23}{4} - \frac{8}{\hat{\beta}}\,\mathrm{Li}_2\left(\frac{2\hat{\beta}}{1+\hat{\beta}}\right) \\ & + 8\left(\frac{\tanh^{-1}\hat{\beta}}{\hat{\beta}} - 1\right)\ln\left(\frac{2\hat{E}\,\hat{\beta}}{m_e}\right) \\ & + 4\frac{\tanh^{-1}\hat{\beta}}{\hat{\beta}}\left[\frac{7 + 3\hat{\beta}^2}{8} - 2\tanh^{-1}\hat{\beta}\right], \end{aligned} \tag{49}$$

where m_p is the proton mass, K is the $\bar{\nu}_e$ energy, $\hat{E} = E_m - K$, E_m the end-point energy of the electron in the β decay, $\hat{p} = \sqrt{\hat{E}^2 - m_e^2}$, $\hat{\beta} = \hat{p}/\hat{E}$, $F(Z, E)$ is the Fermi Coulomb function, A is a constant independent of K, and $\mathrm{Li}_2(z)$ is the dilogarithm function defined in Eq. (19). As in the case of the $\mathcal{O}(\alpha)$ correction to the e^- spectrum [cf. Eq. (42)], the function $h(\hat{E}, E_m)$ is valid in the presence of the strong interactions, provided small contributions of $\mathcal{O}(\alpha\hat{E}/M)$ are neglected.

Including the $\mathcal{O}(\alpha)$ radiative corrections, the theoretical expressions for the e^- and $\bar{\nu}_e$ spectra in allowed β decay can be written in the form

$$\frac{dP_e}{dE} = f_e(E, E_m), \qquad \frac{dP_\nu}{dK} = f_\nu(K, E_m), \tag{50}$$

where

$$f_e(E, E_m) = ApE(E_m - E)^2 F(Z, E)\left[1 + \frac{\alpha}{2\pi}g(E, E_m)\right], \tag{51}$$

$$f_\nu(K, E_m) = A\hat{p}\,\hat{E}\,K^2 F(Z, \hat{E})\left[1 + \frac{\alpha}{2\pi}h(\hat{E}, E_m)\right], \tag{52}$$

$h(\hat{E}, E_m)$ is defined in Eq. (49) and $g(E, E_m)$, the function that describes the $\mathcal{O}(\alpha)$ radiative correction to the e^- spectrum, is shown in Eq. (42).

Comparing Eqs. (51) and (52), neglecting contributions of $\mathcal{O}(\alpha^2)$, and recalling $\hat{E} = E_m - K$, one finds (Sirlin, 2011)

$$f_\nu(K, E_m) = f_e(\hat{E}, E_m)\left[1 + \frac{\alpha}{2\pi}[h(\hat{E}, E_m) - g(\hat{E}, E_m)]\right]. \tag{53}$$

Equation (53) describes the conversion from the e^- spectrum in a specific decay into the corresponding $\bar{\nu}_e$ spectrum when the $\mathcal{O}(\alpha)$ radiative corrections are included. This conversion procedure is the method currently employed to determine the $\bar{\nu}_e$ spectrum from the measured electron spectrum. In turn, as mentioned, knowledge of the $\bar{\nu}_e$ spectrum is important for reactor studies of neutrino oscillations.

An interesting theoretical property of $h(\hat{E}, E_m)$ is that its $m_e \to 0$ limit converges and leads to a simple expression (m_e is the electron mass). This is in sharp contrast with the behavior of $g(E, E_m)$ that diverges as $m_e \to 0$. This important difference can be explained in the following way (Sirlin, 2011). For given K, as $m_e \to 0$ all collinear e-γ configurations become energy degenerate and generally give rise to mass singularities. An elementary but powerful theorem in quantum mechanics on degenerate systems and mass singularities, due to Lee and Nauenberg (1964), leads to the conclusion that these singularities are canceled in the power series expansions of transition probabilities if the latter are summed over an appropriate ensemble of such degenerate states. In the derivation of the radiative corrections to the $\bar{\nu}_e$ (ν_e) spectrum, one performs the d^3p and d^3k integrations, where p and k are the electron and photon momenta, so indeed one sums over the set of collinear e-γ configurations that become energy degenerate in the $m_e \to 0$ limit. Therefore, according to this theorem, $h(\hat{E}, E_0)$ should be free of $\ln m_e$ singularities, as found in the explicit calculation. In contrast, this is not the case in the derivation of the radiative corrections to the e^- spectrum, since the d^3p integration is not carried out. As a consequence, the Lee-Nauenberg theorem is not applicable to $g(E, E_m)$ and, as is well known, this function diverges in the $m_e \to 0$ limit. Analogous examples of mass singularities in the $\mathcal{O}(\alpha)$ radiative corrections to the differential spectra, and their cancellation in the lifetimes, integrated asymmetries and some partial decay rates in muon and β decays were extensively discussed by Kinoshita and Sirlin (1959a).

Pion β decay $\pi^+ \to \pi^0 + e^+ + \nu_e$ and its charge conjugate $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$ are processes of special interest, since their interpretation is devoid of the complications of nuclear structures that affect nuclear β decays. In this sense, they may be regarded as the simplest examples of superallowed $0 \to 0$ Fermi transitions. On the other hand, their branching ratio, $(1.036 \pm 0.006) \times 10^{-8}$ (Počanić *et al.*, 2004; Nakamura *et al.*, 2010), is very small and, consequently, the measurement of their decay rate is much less precise than in the nuclear transitions.

Recently, Passera, Philippides, and Sirlin (2011) compared the radiative corrections involving the weak hadronic vector current in pion β decay, as evaluated in the V-A theory in two different frameworks: (i) the current algebra formulation, in which quarks are the fundamental underlying fields, and (ii) the elementary approach in which pions are regarded as the fundamental fields. The comparison of the two calculations revealed a small difference that was shown to arise from a specific short-distance contribution that depends on the algebra satisfied by the weak and electromagnetic currents.[5]

[5]The fact that this particular contribution is model dependent was already pointed out by Abers *et al.* (1968).

In fact, the space-space components of the algebra are different in (i) and (ii) and this was shown to explain the discrepancy discussed previously. The results were also compared with a recent calculation based on chiral perturbation theory (χPT) (Cirigliano *et al.*, 2003). Taking into account its theoretical error, the χPT calculation was found to be consistent with those based on either (i) or (ii). Passera, Philippides, and Sirlin (2011) also discussed the important differences between the radiative corrections to pion β decay as evaluated in the V-A and standard theories.

III. RADIATIVE CORRECTIONS IN THE STANDARD THEORY OF PARTICLE PHYSICS

The standard theory (ST) originally proposed by Glashow (1961), Weinberg (1967), and Salam (1968) emerged, with important contributions from other physicists, in the period 1967–1974. At present, it is a gauge theory of the electromagnetic, weak, and strong interactions based on the $SU(2)_L \times U(1) \times SU(3)_C$ symmetry group. Here $SU(2)_L \times U(1)$ is the symmetry group of the EW sector and $SU(3)_C$ is that of quantum chromodynamics (QCD), the current theory of the strong interactions.

As shown by 't Hooft (1971), 't Hooft and Veltman (1972a, 1972b), Lee (1972), Lee and Zinn-Justin (1972, 1973), Becchi, Rouet, and Stora (1974, 1976), Zinn-Justin *et al.* (1975), and others, it is a renormalizable theory. This implies that the EWC in this theory can be evaluated by perturbative field theoretic methods, since the ultraviolet divergences found in the calculations can be absorbed as unobservable contributions to the masses and couplings of the theory. In the domain in which the strong interaction running coupling $\alpha_s(\mu)$ is small, the same is true of the QCD corrections.

In 1972, dimensional regularization, an ingenious method to regularize ultraviolet divergences, was proposed by 't Hooft and Veltman (1972a), Ashmore (1972), and Bollini and Giambiagi (1972). It is particularly useful in the context of gauge theories such as the ST. Dimensional regularization of infrared divergences was proposed by Gastmans and Meuldermans (1973) and Marciano and Sirlin (1975a), and that of mass singularities by Marciano (1975). Dimensional regularization of infrared and mass singularities is widely used at present, particularly in QCD calculations.

Once the renormalizability of the ST was recognized, it was natural to study the EW and QCD corrections of the theory. The aims of these studies are as follows:

(i) To verify the ST at the level of its quantum corrections.
(ii) To search for discrepancies that may signal the presence of new physics beyond the ST.

In the EW sector, these are essentially the objectives of what is now called precision electroweak physics.

A. Early developments

Already in the 1970s there were a number of important developments:

(i) The evaluation of one-loop EWC to $g_\mu - 2$ (Bars and Yoshimura, 1972; Fujikawa, Lee, and Sanda, 1972; Jackiw and Weinberg, 1972).
(ii) Weinberg (1973) showed that there are no violations of $\mathcal{O}(\alpha)$ to parity and strangeness conservation in strong interaction amplitudes.
(iii) Gaillard and Lee (1974) studied processes which are forbidden at the tree level, but occur via loop effects, and showed that the GIM mechanism (Glashow, Iliopoulos, and Maiani, 1970) generally suppresses neutral current amplitudes of $\mathcal{O}(G_F\alpha)$.
(iv) Bollini, Giambiagi, and Sirlin (1973) studied the cancellation of ultraviolet divergences in fundamental natural relations of the ST.
(v) Using a simplified version of the ST involving integer-charged quarks, and neglecting the effect of the strong interactions, Sirlin (1974) showed explicitly that the one-loop EWC to β decay are indeed finite in the ST and that the "cutoff" is given by M_Z. This leads to large EWC of $\mathcal{O}(4\%)$, a result that has important consequences in the test of the universality of the weak interactions. Indeed, this result was one of the early "smoking guns" of the EW sector of the ST at the level of its quantum corrections. On the other hand, as discussed in Sec. III.J, an evaluation of the EWC in the "real" ST, based on fractionally charged quarks, and taking into account the effect of the strong interactions, had to wait until the development of the current algebra formulation of radiative corrections in gauge theories (Sirlin, 1978).
(vi) Veltman (1977) and Chanowitz, Furman, and Hinchliffe (1978) discovered that heavy particles do not generally decouple in the EWC of the ST, and that a heavy top quark gives contributions of $\mathcal{O}(G_F M_t^2)$ to the ρ parameter, defined as the ratio of the neutral and charged current coupling constants at zero momentum transfer.

B. Input parameters

Three precisely measured constants play a particularly important role as input parameters in electroweak physics:

(i) The fine structure constant $\alpha = 1/137.035\,999\,679(94)$ (Nakamura *et al.*, 2010), with a relative error $\pm 6.9 \times 10^{-4}$ parts per million (ppm), obtained most precisely from $g_{(e)} - 2$.
(ii) The Fermi constant $G_F = G_\mu = 1.166\,378\,8(7) \times 10^{-5}$ GeV^{-2}, with a relative error of 0.6 ppm (see Sec. II.D).
(iii) $M_Z = 91.1876 \pm 0.0021$ GeV (Nakamura *et al.*, 2010), with a relative error of 23 ppm.

This precise determination of M_Z required sophisticated experimental techniques and an accurate study of the Z line shape, in which QED and EW corrections play an important role [see, for example, Berends *et al.* (1989)]

C. The on-shell scheme of renormalization

Toward the end of the 1970s it seemed likely that experimental physicists would search for the W and Z intermediate vector bosons of the ST and hopefully measure their masses. This motivated the idea of studying at the loop level the relation between M_W, M_Z, G_F, α, and the EW mixing parameter $\sin^2\theta_W$, as well as other fundamental parameters of the theory, such as the quark masses and the Higgs boson mass M_H. The hope was that this analysis would lead to more accurate predictions for M_W and M_Z. At the time, G_F and α were accurately known, and $\sin^2\theta_W$ was determined with less precision from ν-N deep inelastic scattering via the neutral and charged currents. Thus, it became clear that it was necessary to evaluate the EWC to the last two processes to extract $\sin^2\theta_W$, and to muon decay to obtain the relation with G_F and α.

Since this required the analysis of a number of processes involving neutral and charged currents, in order to facilitate the evaluation of the corresponding EWC, Sirlin (1980a) proposed a simple, physically motivated framework to renormalize the EW sector of the ST. This approach, with important contributions from other physicists,[6] is currently known as the on-shell (OS) scheme. In the same 1980 paper, the OS scheme was applied to evaluate the one-loop EWC to muon decay in the ST. The analysis leads to the basic OS relations (Sirlin, 1980a, 1984)

$$s^2c^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2(1-\Delta r)}, \tag{54}$$

$$s^2 = \sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}, \tag{55}$$

where $G_F = G_\mu$ is the Fermi constant discussed in Secs. II.D and III.B, and Δr is the EWC to muon decay. From Eq. (55) we see that in the OS scheme the EW mixing parameter $\sin^2\theta_W$ is simply defined in terms of the physical masses M_W and M_Z to all orders in perturbation theory. In two subsequent papers, the OS scheme was applied to the study of the EWC to ν-N deep inelastic scattering via the neutral current (Marciano and Sirlin, 1980) and via the charged current (Sirlin and Marciano, 1981). This trilogy of papers achieved the aim of establishing contact, at the level of the EWC, between the theory and the expected measurements of M_W and M_Z. In fact, using Eqs. (54) and (55) and the information from ν-N scattering, they led to more accurate predictions of M_W and M_Z before the actual measurements. As the experiments on ν-N scattering improved, the role of the EWC became more important. A detailed analysis (Amaldi *et al.*, 1987) led to the estimates $M_W = 80.2 \pm 1.1$ GeV, $M_Z = 91.6 \pm 0.9$ GeV, with central values that differ from the current ones by about 0.2 and 0.4 GeV, respectively. We point out that this closeness is rather accidental (for example, the top-quark mass used in calculations at the time was much smaller than its present value). Nonetheless, the early predictions were useful because they provided what turned out to be realistic mass ranges for the experimental searches of the W and Z bosons. Furthermore, as shown by Amaldi *et al.* (1987), they also turned out to be in good agreement with the early measurements of the W and Z masses.

[6]See, for example, Aoki *et al.* (1982), Böhm, Spiesberger, and Hollik (1986), Consoli, Hollik, and Jegerlehner (1989), and Hollik (1990).

The EWC Δr in Eq. (54) depends on various physical parameters of the ST such as $\alpha, M_W, M_Z, M_H, M_f, \alpha_s(M_Z), \ldots$, where M_H is the Higgs boson mass, M_f is a generic fermion mass, and $\alpha_s(M_Z)$ is the QCD running coupling evaluated at the scale $\mu = M_Z$. It follows from Eqs. (54) and (55) that Δr is a physical observable. Equations (54) and (55) can be viewed as the relation between the physical parameters of the Fermi theory (low-energy effective theory), namely, G_F and α, and those of the ST (underlying theory), namely, $\alpha, M_W, M_Z, M_H, M_f, \ldots$, at the level of the quantum corrections. These relations are currently used to calculate $M_W = M_W(M_H)$, leading to very sharp constraints on M_H.

The on-shell scheme is also used in the ZFITTER program (Bardin *et al.*, 2001; Arbuzov *et al.*, 2006) and the GFITTER project (Flacher *et al.*, 2009), extensively employed in the analysis of the electroweak precision observables.

D. The modified minimal subtraction scheme of renormalization

Another important and useful approach is the modified minimal subtraction ($\overline{\text{MS}}$) renormalization framework, in which the electroweak mixing parameter is identified with the running coupling $\sin^2\theta_W(\mu) = e^2(\mu)/g^2(\mu)$ evaluated at the $\mu = M_Z$ scale. [Here g is the $SU(2)_L$ gauge coupling.]

In this scheme, the renormalization of $\sin^2\theta_W(\mu)$ and the various couplings is implemented by the $\overline{\text{MS}}$ prescription (Bardeen *et al.*, 1978; Buras, 1980). At the one-loop level, this involves subtracting

$$\delta = \frac{1}{n-4} + \frac{1}{2}[\gamma_E - \ln(4\pi)] \tag{56}$$

from the EWC, where the first term is the characteristic pole in dimensional regularization and $\gamma_E = 0.5772\ldots$ is Euler's constant. Since at the one-loop level δ always appears in combination with $\ln(1/\mu)$ where μ is the 't Hooft mass scale, an equivalent procedure is to rescale μ according to $\mu = \mu' e^{\gamma/2}/(4\pi)^{1/2}$, subtract only the $(n-4)^{-1}$ pole term, and then set μ', rather μ, at the relevant mass scale. This second formulation can be conveniently generalized to higher-order EWC and one can define the $\overline{\text{MS}}$ renormalization procedure as the subtraction of the pole terms $(n-4)^{-m}$ ($m \geq 1$), and the identification of the rescaled parameter μ' with the relevant mass scale.

Although masses can also be defined as running parameters, a hybrid scheme in which couplings and $\sin^2\theta_W(\mu)$ are renormalized by $\overline{\text{MS}}$ subtractions, but masses are still the physical ones, has proved to be useful and is frequently employed.

An early application of the $\overline{\text{MS}}$ scheme (Marciano and Sirlin, 1981) was the derivation of precise SU(5) predictions for the neutral current amplitude $\sin^2\theta_W^{\text{exp}}(q^2)$ [defined in Eq. (60)], and M_W and M_Z.

It was also employed in the early papers by Llewellyn Smith and Wheater (1981) and Wheater and Llewellyn Smith (1982) on the EWC to deep inelastic neutrino and electron scattering.

Two important relations in this scheme were derived by Sirlin (1989) and Fanchiotti and Sirlin (1990):

$$\hat{s}^2\hat{c}^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2(1-\Delta\hat{r})}, \tag{57}$$

$$\hat{s}^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_W^2(1-\Delta\hat{r}_W)}, \tag{58}$$

where $\hat{s}^2 \equiv \sin^2\hat{\theta}_W(M_Z)$ is the $\overline{\mathrm{MS}}$ electroweak mixing parameter evaluated at the scale $\mu = M_Z$, $\hat{c}^2 = 1-\hat{s}^2$, and $\Delta\hat{r}$ and $\Delta\hat{r}_W$ are the corresponding EWC. In 1989, Eq. (57) and the early M_Z measurements at the CERN Large Electron-Positron (LEP) collider were applied to significantly improve the determination of $\sin^2\hat{\theta}_W(M_Z)$ (Sirlin, 1989). In fact, $\hat{\alpha}(M_Z)$, $\sin^2\hat{\theta}_W(M_Z)$, and $\alpha_s(M_Z)$ provide the initial values for the renormalization group equations (RGEs) satisfied by the running $\mathrm{SU}(2)_L \times \mathrm{U}(1) \times \mathrm{SU}_C(3)$ gauge couplings, which play a crucial role in the study of grand unified theories (GUTs) and in the discovery of supersymmetric grand unification [see, for example, Langacker and Polonsky (1993, 1995)]. In particular, the 1989 analysis (Sirlin, 1989) found that the improved value of $\sin^2\hat{\theta}_W(M_Z)$ was indeed consistent with supersymmetric grand unification.

A modification of the renormalization prescription for $\sin^2\hat{\theta}_W(M_Z)$ was proposed by Marciano and Rosner (1990) and Marciano (1991, 1993, 1995). According to this prescription, aside from the $1/(n-4)$ pole terms, the contributions from particles of mass $m > M_Z$ that do not decouple in the $m \to \infty$ limit are also subtracted from the amplitude multiplying $\sin^2\hat{\theta}_W(M_Z)$ and are therefore absorbed in this renormalized parameter. The aim of this prescription is to obtain values of $\sin^2\hat{\theta}_W(M_Z)$ from the on-resonance observables which are insensitive to heavy particles of mass $m > M_Z$, a property that facilitates the analysis of the evolution of $\sin^2\hat{\theta}_W(M_Z)$ to the GUT scale.

The neutral current vertex of the Z boson into a fermion-antifermion pair $(f\bar{f})$ has the form

$$\langle f\bar{f}|J_Z^\mu|0\rangle = V_f(q^2)\bar{u}_f\gamma_\mu\left[\frac{I_{3f}(1-\gamma_5)}{2} - \hat{k}_f(q^2)\hat{s}^2 Q_f\right]v_f, \tag{59}$$

where $V_f(q^2)$, $\hat{k}_f(q^2)$, and its OS counterpart $k_f(q^2)$ are electroweak form factors. I_{3f} and Q_f denote the third component of the weak isospin and the charge of fermion f, respectively.

In terms of the $\hat{k}_f$ and k_f form factors, the neutral current amplitude $\sin^2\theta_W^{\mathrm{exp}}(q^2)$ discussed by Marciano and Sirlin (1981) is

$$\sin^2\theta_W^{\mathrm{exp}}(q^2) \equiv \hat{k}_f(q^2)\hat{s}^2 = k_f(q^2)s^2. \tag{60}$$

The $\overline{\mathrm{MS}}$ and OS definitions of the electroweak mixing angle are related by (Degrassi, Fanchiotti, and Sirlin, 1991)

$$\hat{s}^2 = s^2\left(1 + \frac{c^2}{s^2}\Delta\hat{\rho}\right), \tag{61}$$

$$\Delta\hat{\rho} = \mathrm{Re}\left[\frac{A_{WW}(M_Z^2)}{M_W^2} - \frac{A_{ZZ}(M_Z^2)}{M_Z^2\hat{\rho}}\right]_{\overline{\mathrm{MS}}}, \tag{62}$$

where $A_{WW}(q^2)$ and $A_{ZZ}(q^2)$ are the W-W and Z-Z transverse self-energies, $\hat{\rho} = (1-\Delta\hat{\rho})^{-1}$, and $\overline{\mathrm{MS}}$ denotes the modified minimal subtraction renormalization and the choice $\mu = M_Z$.

The $\overline{\mathrm{MS}}$ scheme is also used in the radiative correction program GAPP (Erler, 1999a), extensively employed by Erler and Langacker in their biannual reviews of the electroweak model and constraints on new physics [see, for example, Erler and Langacker (2010)].

Early studies of the QCD contributions to EWC include Djouadi and Verzegnassi (1987), Djouadi (1988), Kniehl (1990), and Halzen and Kniehl (1991). The incorporation of QCD effects in the basic EWC $\Delta\hat{r}$, $\Delta\hat{r}_W$, and Δr was implemented by Fanchiotti, Kniehl, and Sirlin (1993) (see also references therein). They include perturbative $\mathcal{O}(\alpha\alpha_s)$ contributions and $t\bar{t}$ threshold effects. Here α_s is evaluated at a relevant mass scale such as M_Z or M_t.

E. The effective electroweak mixing parameter

Another useful version of the electroweak mixing parameter is $s_{\mathrm{eff}}^2 \equiv \sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$, extensively employed by the electroweak working group (EWWG) to analyze the data at the Z resonance. Here eff and lept are abbreviations for effective and leptonic, respectively. It is defined by (Rolandi, 1992)

$$1 - 4\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}} = \frac{g_V^l}{g_A^l}, \tag{63}$$

where g_V^l and g_A^l are the effective vector and axial vector couplings of the $Z \to l\bar{l}$ amplitude at resonance ($q^2 = M_Z^2$) and l stands for a charged lepton.

The relations between s_{eff}^2 and $\hat{s}^2$, s^2 were obtained by Gambino and Sirlin (1994a)

$$s_{\mathrm{eff}}^2 = \mathrm{Re}\,\hat{k}_l(M_Z^2)\hat{s}^2 = \mathrm{Re}\,k_l(M_Z^2)s^2, \tag{64}$$

where $\hat{k}_l(q^2)$ and $k_l(q^2)$ are the electroweak form factors introduced in Eq. (59) and the following lines. Because of a fortuitous cancellation of EWC, $\mathrm{Re}\,\hat{k}_l(M_Z^2)$ is very close to 1. Applying the Marciano-Rosner renormalization prescription (cf. Sec. III.D), Gambino and Sirlin (1994a) found

$$\Delta = s_{\mathrm{eff}}^2 - \hat{s}^2 \approx 3\times 10^{-4}. \tag{65}$$

They also pointed out that, if this prescription is not applied, so that the complete top-quark contribution is included in the calculation of $\hat{k}_l(M_Z^2)$, the difference becomes even smaller, namely, $\Delta \approx 1\times 10^{-4}$ for $M_t = 173.2$ GeV.

Combining Eqs. (57) and (64), and writing $\mathrm{Re}\,\hat{k}_l(M_Z^2) = 1 + (\hat{e}^2/\hat{s}^2)\Delta\hat{k}(M_Z^2)$, one finds (Ferroglia, Ossola, and Sirlin, 2001)

$$s_{\mathrm{eff}}^2 c_{\mathrm{eff}}^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2(1-\Delta r_{\mathrm{eff}})}, \tag{66}$$

$$\Delta r_{\mathrm{eff}} = \Delta\hat{r} + \frac{e^2}{s_{\mathrm{eff}}^2}\Delta\hat{k}\left(1 - \frac{s_{\mathrm{eff}}^2}{c_{\mathrm{eff}}^2}\right)(1+x_t) + \cdots, \tag{67}$$

where $x_t = 3G_F M_t^2/\sqrt{2}\,8\pi^2$ is the leading contribution to $\Delta\hat{\rho}$. Equation (67) includes the complete one-loop EWC, as well as the two-loop contributions enhanced by factors $(M_t^2/M_Z^2)^n$ $(n = 1, 2)$. We note that Eq. (66) has a form analogous to Eqs. (54) and (57). The one-loop approximation to Eq. (67) had been previously applied to discuss the mass scale of new physics in the Higgs-less scenario (Kniehl and Sirlin, 2000).

The asymptotic behaviors for large M_t, M_H, of the basic corrections Δr, $\Delta\hat{r}$, and $\Delta r_{\rm eff}$ are instructive. At the one-loop level, we have

$$\Delta r \sim -\frac{3\alpha}{16\pi s^4}\frac{M_t^2}{M_Z^2} + \frac{11\alpha}{24\pi s^2}\ln\left(\frac{M_H}{M_Z}\right) + \cdots, \qquad (68)$$

$$\Delta r_{\rm eff} \approx \Delta\hat{r} \sim -\frac{3\alpha}{16\pi\hat{s}^2\hat{c}^2}\frac{M_t^2}{M_Z^2} + \frac{\alpha}{2\pi\hat{s}^2\hat{c}^2}\left(\frac{5}{6}-\frac{3}{4}\hat{c}^2\right)\ln\left(\frac{M_H}{M_Z}\right) + \cdots. \qquad (69)$$

Equations (68) and (69) reveal a quadratic dependence on M_t, and a logarithmic dependence on M_H. The asymptotic behaviors in M_t and M_H have opposite signs, a fact that helps to explain a well-known $M_t - M_H$ correlation, namely, increasing (decreasing) values of M_t favor increasing (decreasing) values of M_H. The cofactor of M_t^2/M_Z^2 in Δr is approximately larger by $c^2/s^2 \approx 3.5$ than in $\Delta\hat{r}$, $\Delta r_{\rm eff}$. This implies that Δr is significantly more sensitive to M_t than $\Delta\hat{r}$ and $\Delta r_{\rm eff}$.

The asymptotic behavior for large M_t of the neutral current amplitude is

$$\text{N.C.ampl.} \sim \frac{G_F}{1-x_t}, \qquad (70)$$

where x_t is defined after Eq. (67).

Additional contributions to Δr and $\Delta r_{\rm eff}$ lead to shifts $\delta M_W/M_W \approx -0.205\delta(\Delta r)$, $\delta s^2_{\rm eff}/s^2_{\rm eff} \approx 1.52\delta(\Delta r_{\rm eff})$.

Current values for the three versions of the electroweak mixing parameter discussed above are

$$\sin^2\theta_W = 0.222\,90(29), \qquad \sin^2\theta_{\rm eff}^{\rm lept} = 0.231\,53(16),$$
$$\sin^2\hat{\theta}_W^2(M_Z) = 0.231\,23(16). \qquad (71)$$

The value of $\sin^2\theta_W$ was obtained using Eq. (55) and the experimental values $M_Z = 91.1876(21)$ GeV (see Sec. III.B) and $M_W = 80.385(15)$ GeV (the average of the Fermilab Tevatron and CERN LEP2 measurements). The $\sin^2\theta_{\rm eff}^{\rm lept}$ value is the average of the values obtained from all the asymmetries measured at LEP and at the Stanford Linear Collider (SLC), and dates back to 2005. The value of $\sin^2\hat{\theta}_W(M_Z)$ has been derived from that of $\sin^2\theta_{\rm eff}^{\rm lept}$ taking into account Eq. (65).

The values of the QCD coupling and the $\overline{\rm MS}$ fine structure constant at the scale $\mu = M_Z$ are given by (Dissertori *et al.*, 2010; Erler and Langacker, 2010)

$$\alpha_s(M_Z) = 0.1184(7), \quad \hat{\alpha}(M_Z) = [127.916(15)]^{-1}. \qquad (72)$$

F. Renormalization schemes: General observations

As discussed in Secs. III.C–III.E, the EWC have been evaluated in specific renormalization schemes. An interesting feature is that each scheme is associated with a specific definition of the renormalized electroweak mixing parameter.

Two of the most frequently employed schemes are as follows:

(i) The OS scheme, discussed in Sec. III.C. It is "physical," since it identifies renormalized couplings and masses with physical, scale-independent observables, such as G_F, α, M_W, M_Z, M_H, M_f, ... It has also provided the framework for very accurate calculations such as the complete two-loop evaluation of Δr and $\sin^2\theta_{\rm eff}^{\rm lept}$ (cf. Sec. III.N). As mentioned in Sec. III.C, it is used in the ZFITTER and GFITTER programs, extensively employed by the LEP EW and GFITTER groups in the analysis of the precision electroweak observables.

(ii) The $\overline{\rm MS}$ scheme, discussed in Sec. III.D. It has good convergence properties. In fact, in this scheme one essentially subtracts the pole terms, and therefore the calculations follow closely the structure of the unrenormalized theory. In this way it avoids large finite corrections that are often induced by renormalization. It employs scale-dependent couplings such as $\alpha(\mu)$, $\hat{s}^2(\mu)$, which play a crucial role in the study of grand unification. On the other hand, the use of such couplings generally leads to a residual scale dependence in the evaluation of observables, due to the truncation of the perturbative series. As explained in Sec. III.D, it is used in the GAPP program, extensively employed by Erler and Langacker in their biannual contributions to the Review of Particle Physics.

More recently, a novel approach, called the effective scheme, was proposed by Ferroglia, Ossola, and Sirlin (2001). It employs scale-independent parameters such as $s^2_{\rm eff}$, G_F, M_W, M_Z, ... Consequently, the calculation of observables is strictly scale independent in finite orders of perturbation theory. Furthermore, it shares the good convergence properties of the $\overline{\rm MS}$ scheme, a fact that is related to the numerical closeness of $s^2_{\rm eff}$ and $\hat{s}^2(M_Z)$ [cf. Eq. (65)].

The comparative evaluation of the EWC using different renormalization schemes is often very useful, because it provides an estimate of the theoretical error due to the truncation of the perturbative series.

G. The running of $\alpha(\mu)$ and $\sin^2\theta_W(\mu)$

An important contribution to the EWC is associated with the running of α to the M_Z scale via vacuum polarization contributions, an effect usually parametrized as

$$\frac{\alpha(M_Z)}{\alpha} = \frac{1}{1-\Delta\alpha}. \qquad (73)$$

The leptonic contribution is

$$\Delta\alpha_l = 314.976\,86 \times 10^{-4} \simeq 0.031\,50. \qquad (74)$$

This result includes three-loop contributions evaluated by Steinhauser (1998). The contribution of the five lightest quarks $(u-b)$ is evaluated using dispersion relations involving the experimental cross section for $e^+e^- \to$ hadrons at low $\sqrt{s}$ and perturbative QCD (PQCD) at large $\sqrt{s}$. Important studies of these effects were carried out by

Eidelman and Jegerlehner (1995) and Jegerlehner (2003) (and references therein). Recent, accurate values include $\Delta\alpha_h^{(5)} = 0.027\,50(33)$ (Burkhardt and Pietrzyk, 2011), $\Delta\alpha_h^{(5)} = 0.027\,626(138)$ (Hagiwara *et al.*, 2011), and $\Delta\alpha_h^{(5)} = 0.027\,57(10)$ (Davier *et al.*, 2011). The smaller error in the last reference is partly due to the use of PQCD in the $\sqrt{s}$ range between 1.8 and 3.7 GeV. Combining the result obtained in that reference with $\Delta\alpha_l$ [cf. Eq. (74)], one finds the accurate value

$$\Delta\alpha = \Delta\alpha_l + \Delta\alpha_h^{(5)} = 0.059\,07(10). \tag{75}$$

Equation (75) does not include the top-quark contribution, which is evaluated perturbatively, amounts to

$$\Delta\alpha_{\rm top} = -0.72 \times 10^{-4}, \tag{76}$$

and is usually taken into account together with other M_t-dependent EWC.

Running versions of the electroweak mixing parameter were proposed by Czarnecki and Marciano (2000) and by Ferroglia, Ossola, and Sirlin (2004). For $q^2 < 0$, Czarnecki and Marciano define

$$\sin^2\theta_W(Q^2) = \kappa(Q^2)\sin^2\hat{\theta}_W(M_Z), \tag{77}$$

where $Q^2 = -q^2$ and $\kappa(Q^2)$ is identified with the $\overline{\rm MS}$ form factor $\hat{k}_e(-Q^2)$ [cf. Eq. (59) in the case $f = e$]. Czarnecki and Marciano (2000) found that the EWC lead to $\kappa(0) = 1.0301 \pm 0.0025$, and pointed out that this $+3\%$ increase in the value of the electroweak mixing parameter, appropriate for low Q^2, gives rise to a 38% reduction in the left-right polarization asymmetry $A_{\rm LR}$ in Møller scattering. The reason is that $A_{\rm LR}$ is proportional to $1 - 4\sin^2\theta_W(Q^2)$, a factor close to zero, and a small shift in the value of the electroweak mixing parameter has a pronounced effect. In the same work, $\sin^2\theta_W(Q^2)$ was evaluated and displayed over a large range $0 \le Q \le 1$ TeV, where $Q \equiv \sqrt{Q^2}$.

Ferroglia, Ossola, and Sirlin (2004) proposed an alternative "running" version of the electroweak mixing parameter. Specifically, they define

$$\sin^2\hat{\theta}_W(q^2) = \left(1 - \frac{\hat{c}}{\hat{s}}\frac{a_{\gamma Z}(q^2, M_Z)}{q^2}\right)\hat{s}^2, \tag{78}$$

where $\hat{s}^2 = \sin^2\hat{\theta}_W(M_Z)$ (cf. Sec. III.D) and $a_{\gamma Z}(q^2, M_Z)$ is the "pinch technique" (PT) γZ self-energy evaluated at $\mu = M_Z$.

We recall that the pinch technique (Cornwall, 1981, 1982; Cornwall and Papavassiliou, 1989; Papavassiliou, 1990) is a prescription that combines the conventional self-energies with "pinch parts" from vertex and box diagrams in such a manner that the modified self-energies are gauge independent and are endowed with desirable theoretical properties. The PT self-energies in the electroweak sector of the ST were derived by Degrassi and Sirlin (1992). In the same paper it was shown that the pinch parts can be identified with amplitudes involving appropriate equal-time commutators of currents, which explains the fact that they are process independent and are not affected by the strong interactions. Ferroglia, Ossola, and Sirlin (2004) evaluated and displayed $\sin^2\hat{\theta}_W(q^2)$ in both the spacelike ($q^2 < 0$) and timelike ($q^2 > 0$) domains, appropriate to e^--e^- and e^+-e^- colliders, respectively. In the second case $a_{\gamma Z}(q^2, M_Z)$ is generally complex and $\sin^2\hat{\theta}_W(q^2)$ is defined by the real part of the rhs of Eq. (78).

Interestingly, they obtained

$$1 - 4\sin^2\hat{\theta}_W(0) = 0.0452 \pm 0.023, \tag{79}$$

which is close to $0.0450 \pm 0.023 \pm 0.0010$, the result previously found by Czarnecki and Marciano (1996) for the complete one-loop EWC to $A_{\rm LR}$ at $Q^2 = 0.025$ GeV^2 and $y \equiv Q^2/s = 1/2$, appropriate to the SLC experiment E158 (Kumar *et al.*, 1995). Setting $q^2 = M_Z^2$ in Eq. (78), one finds $\sin^2\hat{\theta}_W(M_Z^2) = 0.230\,48$, which is lower than $\sin^2\theta_{\rm eff}^{\rm lept} = 0.231\,53$ by 0.45%; although not in precise agreement, the two parameters are rather close.

It is also interesting to note that the running both of α and of the weak mixing angle has been derived directly in the $\overline{\rm MS}$ scheme (Erler, 1999b; Erler and Ramsey-Musoff, 2005, respectively).

H. The M_t prediction

An important example of the successful interplay between theory and experiment was the prediction of the top-quark mass M_t and its subsequent measurement.

Before 1995, the top quark could not be produced directly, but it was possible to estimate its mass because of its virtual contributions to the EWC. In 1994, a global analysis by the EWWG led to the indirect determination (Pietrzyk, 1994)

$$M_t = 177 \pm 11^{+18}_{-19}\ {\rm GeV}, \tag{80}$$

where the central value corresponds to $M_H = 300$ GeV, the first error is experimental, and the second represents the shift in the central value assuming $M_H = 65$ GeV (− 19 GeV), or $M_H = 1$ TeV (+ 18 GeV). This can be compared with the current experimental value $M_t = 173.2 \pm 0.9$ GeV (Tevatron Electroweak Working Group, 2011).

This successful prediction was possible because of the sensitive M_t dependence of the basic EWC [cf. Eqs. (68) and (69)].

I. Evidence for electroweak corrections

(a) Evidence for EWC beyond the running of α (Sirlin, 1994b). Using the experimental values of α, G_F, M_Z, $M_W = 80.385 \pm 0.015$ GeV (LEP Electroweak Working Group, 2012; Tevatron Electroweak Working Group, 2012), and Eqs. (54) and (55), one finds

$$(\Delta r)_{\rm exp} = 0.035\,06 \pm 0.000\,90. \tag{81}$$

The contribution to Δr from the running of α is $\Delta\alpha = 0.059\,07 \pm 0.000\,10$ [cf. Eq. (75)]. Thus, the contribution to Δr beyond the running of α is

$$(\Delta r)_{\rm exp} - \Delta\alpha = -0.024\,01 \pm 0.000\,91, \tag{82}$$

which differs from 0 by 26σ.

An alternative argument is to compare the values of $\sin^2\theta_{\rm eff}^{\rm lept} = 0.231\,53 \pm 0.000\,16$ and $\sin^2\theta_W = 0.222\,90 \pm 0.000\,29$ [cf. Eq. (71) and following discussion]. The difference is $0.008\,63 \pm 0.000\,33$, also 26σ, and it arises from EWC not including $\Delta\alpha$. Indeed, the difference is mainly due to the EWC $c^2\Delta\hat{\rho}$ in Eq. (61).

(b) Evidence for electroweak bosonic corrections (EWBC) (Gambino and Sirlin, 1994b). They include loops involving the bosonic sectors W, Z, H, and unphysical scalars. They are subleading numerically relative to the fermionic contributions, but important conceptually. Strong evidence for the EWBC can be found by measuring $\Delta r_{\rm eff}$. Using the experimental values of α, G_F, M_Z, $\sin^2\theta_{\rm eff}^{\rm lept}$, and Eq. (66) one finds

$$(\Delta r_{\rm eff})_{\rm exp} = 0.060\,59 \pm 0.000\,45. \tag{83}$$

Subtracting the contribution of the EWBC from the theoretical expression for $\Delta r_{\rm eff}$ given in Eq. (67), but retaining the fermionic EWC, the theoretical value is $(\Delta r_{\rm eff})^{(f)}_{\rm theor} = 0.050\,45 \pm 0.000\,56$. The difference $(\Delta r_{\rm eff})_{\rm exp} - (\Delta r_{\rm eff})^{(f)}_{\rm theor} = 0.01014 \pm 0.00072$ provides an estimate of the EWBC to $\Delta r_{\rm eff}$. Thus, they differ from 0 by 14σ.

J. Precise test of Cabibbo-Kobayashi-Maskawa unitarity

Since the Cabibbo-Kobayashi-Maskawa (CKM) matrix V_{ij} is unitary, a fundamental prediction is that

$$\sum_j |V_{ij}|^2 = 1, \qquad \sum_i |V_{ij}|^2 = 1. \tag{84}$$

In particular, in the three-generation case, the elements of the first row must satisfy the equality

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1. \tag{85}$$

$|V_{ud}|^2$, the dominant term in Eq. (85), is obtained most precisely from the $0^+ \to 0^+$ superallowed Fermi transitions in β decay. Using the current algebra formulation to evaluate the $\mathcal{O}(\alpha)$ EWC in the standard theory, one finds the following expression for the probability of these important transitions (Sirlin, 1978):

$$Pd^3p = P^0 d^3 p\left\{1 + \frac{\alpha}{2\pi}\left[3\ln\left(\frac{M_Z}{m_p}\right) + g(E, E_m) + 6\bar{Q}\ln\left(\frac{M_Z}{M}\right) + 2C + \mathcal{A}_{\bar{g}}\right]\right\}, \tag{86}$$

$$P^0 d^3 p = \frac{G_F^2 (V_{ud})^2}{8\pi^4} |M_F|^2 F(Z, E)(E_m - E)^2 d^3 p, \tag{87}$$

where p, E, and E_m are the momentum, energy, and end-point energy of the electron or positron in the decay, $F(Z, E)$ is the Fermi Coulomb function, $g(E, E_m)$ is defined in Eq. (42) in Sec. II.F, and M_F is the matrix element of the weak hadronic vector current between the initial and final nuclei. For isotriplet transitions $|M_F|^2 = 2$.

The terms between square brackets in Eq. (86) represent the $\mathcal{O}(\alpha)$ corrections not contained in $F(Z, E)$ in the approximation of neglecting contributions of $\mathcal{O}((\alpha/\pi)E/m_p)$. The first two terms in that expression arise from the weak hadronic vector current and are not affected by the SI. In particular, the proton mass m_p cancels in the sum. We recall that the function $g(E, E_m)$ describes the $\mathcal{O}(\alpha)$ radiative corrections to the electron or positron spectrum in β decay in the presence of the SI (cf. Sec. II.F). The third term is a short-distance contribution to the Fermi amplitude arising from the weak hadronic axial vector current and $\bar{Q}$ is the average charge of the fundamental doublet involved in the transition. In the ST this is the u-d doublet and $\bar{Q} = (2/3 - 1/3)/2 = 1/6$. M is a hadronic mass of $\mathcal{O}(1\ \text{GeV})$. The $2C$ term is the corresponding nonasymptotic part and $\mathcal{A}_{\bar{g}} \approx -0.34$ is a very small asymptotic QCD contribution proportional to $\alpha_s(M_Z)$. Although the axial vector current does not contribute to the Fermi amplitude at the tree level, we see that it gives rise to an important EWC in Eq. (86).

The EWCs to β decay are dominated by a large logarithmic term, $(3\alpha/2\pi)\ln(M_Z/2E_m)$. For example, in the superallowed ^{14}O decay, $E_m = 2.3$ MeV, and this contribution amounts to 3.4%. As we will see, such large correction is phenomenologically crucial to verify Eq. (85). As mentioned in Sec. III.A, this result was one of the early smoking guns of the EW sector of the ST at the level of its quantum corrections.

Contributions of $\mathcal{O}(Z\alpha^2)$ and $\mathcal{O}(Z^2\alpha^3)$ are denoted by δ_2 and δ_3. In particular, in the mid-1980s a reevaluation of δ_2 played an important role in the test of the CVC hypothesis. In fact, at the time the analysis of eight accurately measured superallowed Fermi transactions showed a significant departure from CVC expectations. Simple theoretical arguments strongly suggested that the problem arose in the evaluation of the two-loop δ_2 that had been done numerically long before. The correction was then evaluated analytically by Sirlin and Zucchini (1986) and Sirlin (1987) and, when applied to the eight transitions, led to good agreement with CVC, a result confirmed by a new numerical evaluation (Jaus and Rasche, 1987). One finds that δ_2 varies from 0.22% for ^{14}O decay to 0.50% for the ^{54}Co transition, while δ_3 is much smaller (Jaus and Rasche, 1987).

There is also a correction δ_c that describes the lack of perfect overlap between the wave functions of the parent and daughter nuclei due to Coulomb forces and configuration mixing effects in the shell-model wave functions, as well as a nuclear-structure-dependent correction $\delta_{\rm NS}$. They have been extensively discussed in the literature [see Towner and Hardy (2008) and references therein].

Over the years, a number of refinements have been incorporated in the evaluation of the EWC. For example, leading logarithmic contributions $\mathcal{O}(\alpha^n \ln^n(M_Z/m_p))$ and $\mathcal{O}(\alpha^n \ln^n(m_p/2E_m))$ have been summed via a renormalization group analysis by Marciano and Sirlin (1986) and Czarnecki, Marciano, and Sirlin (2004). They lead to the replacements

$$1 + \left(\frac{2\alpha}{\pi}\right)\ln\left(\frac{M_Z}{m_p}\right) \to S(m_p, M_Z) = 1.022\,48, \tag{88}$$

$$1 + \left(\frac{3\alpha}{2\pi}\right)\ln\left(\frac{m_p}{2E_m}\right) \to L(2E_m, m_p), \tag{89}$$

where $(3\alpha/2\pi)\ln(m_p/2E_m)$ is a leading contribution to $g(E, E_m)$. In the case of neutron β decay, for example, $L(2E_m, m_p) = 1.020\,94$. Sirlin (1982) showed that all semileptonic processes mediated by the W boson are enhanced by a short-distance EWC analogous to Eq. (88), namely, of the form $1 + (2\alpha/\pi)\ln(M_Z/M) \rightarrow S(M, M_Z)$, where M is a relevant hadronic mass. Interesting examples include the hadronic decays of the τ (Marciano and Sirlin, 1988), π_{l2} decays (Marciano and Sirlin, 1993), and muon capture (Czarnecki, Marciano, and Sirlin, 2007), where short-distance effects of this type play an important role in the EWC. More recently, Marciano and Sirlin (2006) developed a new method to compute hadronic effects on EWC to low-energy weak interaction semileptonic processes. It employs high-order perturbative QCD results originally derived for the Bjorken sum rule for polarized electroproduction, as well as a large N QCD-motivated interpolating function that matches long- and short-distance EWC. When applied to the superallowed Fermi transitions, it improves the evaluation of the axial vector current contribution in Eq. (86) and reduces by a factor of 2 the theoretical loop uncertainty in the extraction of V_{ud}.

A critical survey (Hardy and Towner, 2009) examined 20 superallowed $0^+ \rightarrow 0^+$ β decays. The analysis leads to the evaluation of the $\mathcal{F}t$ values for the 20 transitions, where $\mathcal{F}$ is a phase-space factor that includes the Fermi Coulomb function, the electroweak corrections, and the nuclear corrections δ_c and $\delta_{\rm NS}$, and t is the partial half-life.

The CVC hypothesis predicts that the $\mathcal{F}t$ values should be the same for all these transitions, a demanding test that is well satisfied by the results. For the weighted average of the 13 most accurate $\mathcal{F}t$ values (those with errors less than $\pm 0.4\%$) they obtained $\overline{\mathcal{F}t} = 3071.87 \pm 0.83$ s, a result that leads to the important determination

$$|V_{ud}| = 0.974\,25(22). \tag{90}$$

The value of $|V_{us}|$ can be determined from K_{l3} decays and that of $|V_{us}|/|V_{ud}|$ from the ratio of $K^+ \rightarrow \mu^+\nu$ and $\pi^+ \rightarrow \mu^+\nu$ decay rates. Combining the two inputs they found

$$|V_{us}| = 0.225\,34(93). \tag{91}$$

Inserting Eqs. (90) and (91) and $|V_{ub}| = (3.93 \pm 0.35) \times 10^{-3}$ (Amsler, 2008), Hardy and Towner (2009) obtained

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.999\,95(61), \tag{92}$$

an impressive 0.06% test of the three-generation ST at the level of its quantum corrections. It is interesting to note that the overall EWCs to Eq. (92) are of $\mathcal{O}(4\%)$, i.e., 66 times larger than the 0.061% error.

EWCs of $\mathcal{O}(\alpha)$ to neutron β decay were included in the classic work of Wilkinson (1982). A number of refinements were introduced by Czarnecki, Marciano, and Sirlin (2004). Since the axial vector current is not conserved, in the case of the Gamow-Teller amplitude the current algebra analysis of the EWC does not lead to a simple expression, independent of the SI, in contrast with the corrections involving the vector current [cf. Eq. (86) and the discussion following that equation]. The strategy followed was to define $g_A = G_A/G_V$ ($G_V \equiv G_F|V_{ud}|$) in terms of the neutron lifetime τ_n by means of

$$\frac{1}{\tau_n} = \frac{G_F^2|V_{ud}|^2}{2\pi^3} m_e^5(1 + 3g_A^2)f(1 + \mathrm{RC}), \tag{93}$$

where $f = 1.6887$ is a phase-space factor that includes the Coulomb Fermi function contribution, as well as smaller corrections, and $1 + \mathrm{RC}$ is identified with the well-known EWC involving the vector current. This implies that some EWC are absorbed in this definition of g_A and, therefore, G_A. An interesting point is that the correction $1 + (\alpha/2\pi)g(E, E_m)$ (Sirlin, 1967a) and the short-distance contribution $1 + 2(\alpha/\pi)\ln(M_Z/m_p)$ (Sirlin, 1982) affect both the Fermi and Gamow-Teller transitions, so they are well described by the factorization of the EWC in Eq. (93). It follows that the same is true of the large logarithmic term $(3\alpha/2\pi)\ln(m_p/2E_m)$ contained in $(\alpha/2\pi)g(E, E_m)$.

They proceeded then to evaluate a number of higher-order EWC to Eq. (93): the sum of the corrections of $\mathcal{O}(\alpha^n \ln^n(M_Z/m_p))$ and $\mathcal{O}(\alpha^n \ln^n(m_p/2E_m))$ according to Eqs. (88) and (89), the contribution of δ_2, and next-to leading log corrections of $\mathcal{O}(\alpha^2\ln(M_Z/m_p))$ and $\mathcal{O}(\alpha^2\ln(m_p/m_f))$ arising from fermion vacuum polarization insertions in loops with photon propagators. The analysis led to

$$|V_{ud}|^2(1 + 3g_A^2)\tau_n = 4908 \pm 4 \text{ s}. \tag{94}$$

Using the experimental averages $\tau_n = 885.7(7)$ s and $g_A = 1.2720(18)$ (from the polarized neutron decay asymmetry), Eq. (94) leads to

$$|V_{ud}| = 0.9729(12) \qquad \text{(neutron decay)}, \tag{95}$$

which is consistent with Eq. (90), but much less precise. Equation (94) can also be applied to calculate g_A using the accurate $|V_{ud}|$ value from the superallowed Fermi transitions and the experimental value of τ_n. In this way they obtained the precise prediction

$$g_A = 1.2703(8), \tag{96}$$

which was compared with the experimental values derived from the asymmetry.

Over the years, the test of unitarity of the CKM matrix shown in Eq. (85) has been used to set bounds on certain forms of new physics. The strategy is to attribute to the new physics the deviation from unity of the experimental value of $\sum_{i=1}^3 |V_{ui}|^2$, so that exact CKM unitarity is satisfied; see, for example, Sirlin (1995).

(i) Fourth generation. For a long time, the determination of $\sum_{i=1}^3 |V_{ui}|^2$ led to values smaller than unity by about 2σ. This suggested the possibility of a fourth generation (Marciano and Sirlin, 1986) and the derivation of an upper bound for $V_{ub'}$, where b' denotes the additional down quark. Since the current result [Eq. (92)] is in excellent agreement with three-generation unitarity, at present this test does not provide a signal for a fourth generation. Nonetheless, if a fourth generation exists, from Eq. (92) one finds $|V_{ub'}| \leq 0.03$ (90% C.L.), which is not very restrictive since $|V_{ub}| \simeq 4 \times 10^{-3}$.

(ii) Z' bosons. In some models with additional U(1) factors, the new Z' bosons have different couplings to quarks and leptons and, consequently, give rise to EWC involving box diagrams that distinguish μ and semileptonic decays (Marciano and Sirlin, 1987). As a consequence, the experimental value of $\sum_{i=1}^3 |V_{ui}|^2$ is modified by a contribution that depends on the ratio $M_{Z'}/M_W$, where $M_{Z'}$ is the Z' mass. The analysis leads to lower bounds for $M_{Z'}$. Typically, they are of the order of a few hundred GeV and are not competitive with the bounds from direct searches, precision electroweak data, or atomic parity violation (Erler *et al.*, 2009; del Aguila, de Blas, and Perez-Victoria, 2010; Erler and Langacker, 2010), which are of $\mathcal{O}(1\ \text{TeV})$.

(iii) Compositness. It is frequently discussed in terms of a residual four-fermion interaction with a coupling $1/\Lambda^2$, where Λ represents the composite mass scale. If we assume that the new interaction involves only particles of the same generation, it would affect β transitions but not muon decay. If we further assume that it is of the form of Eq. (30) with $G_V/\sqrt{2}$ replaced by $1/\Lambda^2$, $G_V^2/G_\mu^2 = V_{ud}^2$ is modified to $V_{ud}^2(1 + 2\sqrt{2}/G_V\Lambda^2)$. Using Eq. (92) one then obtains the bound $2\sqrt{2}V_{ud}/G_\mu\Lambda^2 < 9.7\times 10^{-4}$ or $\Lambda > 16$ TeV (90% C.L.).

(iv) Left-right symmetry. In the "manifest" left-right symmetry models (Bég *et al.*, 1977), there are two small parameters: the mixing angle ζ that relates the W_1 and W_2 mass eigenstates to the left- and right-handed fields W_L and W_R, and $\delta = (m_1/m_2)^2$, where m_i ($i = 1, 2$) are the corresponding masses. Corrections linear in the small parameter δ contribute to G_V and G_μ, but cancel in their ratio. This can be shown using the results of Bég *et al.* (1977). In particular, if terms of second and higher order in the small parameters ζ and δ are neglected, one finds $G_V/G_\mu = (1-\zeta)V_{ud}$, with analogous shifts for the other semileptonic decays. [For other predictions in the manifest left-right symmetric model, see also Holstein and Treiman (1977).] As a consequence, Eq. (92) becomes

$$\sum_{j=1}^{3}|V_{uj}|^2 = 0.999\,95 \pm 0.000\,61 + 2\zeta(V_{ud})^2. \tag{97}$$

Thus, CKM unitarity [Eq. (85)] leads to

$$\zeta = (0.3 \pm 3.2)\times 10^{-4}. \tag{98}$$

K. Electroweak corrections to muon capture

The study of muon capture by nuclei, $\mu^- N \to \nu_\mu N'$, has played an important role in the development of weak interaction physics; see, for example, Primakoff (1959), Mukhopadhyay (1977), and Gorringe and Fearing (2003).

In 2007, the MuCap Collaboration (Andreev *et al.*, 2007) reported a precise measurement of the $1S$ singlet capture rate in hydrogen:

$$\Gamma(\mu^- p \to \nu_\mu n)_{1S}^{\text{singlet}} = 725.0 \pm 13.7 \pm 10.7\ \text{s}^{-1}. \tag{99}$$

A major aim of the experiment is an accurate determination of the induced pseudoscalar coupling $g_P(q^2)$ in the matrix element of the axial vector current between nucleon states:

$$\langle n|A_\alpha|p\rangle = \bar{u}_n(p_2)\left[g_A(q^2)\gamma_\alpha\gamma_5 + g_P(q^2)\frac{q_\alpha}{m_\mu}\gamma_5\right]u_p(p_1), \tag{100}$$

where $q = p_2 - p_1$. On the theoretical side, the partially conserved axial current and chiral perturbation theory predict (Kaiser, 2003 and references therein)

$$g_P(q_0^2) = 8.2 \pm 0.2, \tag{101}$$

where $q_0^2 = -0.88 m_\mu^2$, as appropriate for μ^- capture in H. Comparing Eq. (99) with the theoretical expression used at the time for the capture rate (which did not take into account the EWC), it was found that $g_P^{\text{exp}}(q_0^2) = 6.0 \pm 1.2$, which is about 2σ below the prediction in Eq. (101).

In order to advance the theory of muon capture to a higher level of precision, Czarnecki, Marciano, and Sirlin (2007) incorporated the EWC in the theoretical expression for the capture rates. They found that they enhance the capture rates for H and ^{3}He by 2.8% and 3.0%, respectively. It turns out that the g_P values extracted by comparing the theoretical and experimental results are very sensitive to the effect of the EWC. In fact, in the case of H, when the EWC are included, they found

$$g_P^{\text{exp}}(q_0^2) = 7.3 \pm 1.2 \quad (\text{H}), \tag{102}$$

an increase of $g_P^{\text{exp}}(q_0^2)$ by about +22%. Furthermore, Eq. (102) agrees, within the error, with the theoretical prediction of Eq. (101). The implications of the EWC in the case of ^{3}He capture, $\mu^{-\,3}\text{He} \to \nu^3\text{H}$, were also analyzed.

L. Electroweak corrections to neutrino-lepton scattering

Before the advent of the ST, the QED corrections to the process $\nu_e + e \to \nu_e + e$ were studied by Lee and Sirlin (1964). After the emergence of the ST, neutrino-lepton scattering became a subject of special interest. Aside from the fact that they are fundamental processes, they provide instructive and interesting examples of scattering reactions in the weak interactions. In particular, their theory is relatively simple: at the tree level, they are not affected by the strong interactions and, at the one-loop EW level, they are less sensitive to strong interactions than νN and eN scattering, and $e^+ + e^- \to f + \bar{f}$ annihilation.

Including the EW and QED corrections of $\mathcal{O}(\alpha)$, and using the $\overline{\text{MS}}$ scheme of renormalization, the differential cross section for $\nu_\mu + e \to \nu_\mu + e$ is given by (Sarantakos, Sirlin, and Marciano, 1983)

$$\begin{aligned}\frac{d\sigma}{dz} = \frac{2G_F^2(\rho_{\text{N.C.}}^{(\nu;l)})^2(p_1\cdot p_2)}{\pi[1-(q^2/M_Z^2)]^2}\Big\{&\varepsilon_-^2(q^2)\left[1+\frac{\alpha}{\pi}f_-(z)\right]\\ &+\varepsilon_+^2(q^2)(1-z)^2\left[1+\frac{\alpha}{\pi}f_+(z)\right]\\ &-\varepsilon_+(q^2)\varepsilon_-(q^2)\frac{m_e^2}{(p_1\cdot p_2)}z\left[1+\frac{\alpha}{\pi}f_{+-}(z)\right]\Big\},\end{aligned} \tag{103}$$

where p_1 is the four-momentum of the incident neutrino, p_2 and p_2' are the four-momenta of the initial and final electrons, $q^2 = (p_2 - p_2')^2$,

$$z = -\frac{q^2}{2(p_1 \cdot p_2)} = \frac{E_e' - m_e}{E_\nu}, \tag{104}$$

$$\rho_{\rm N.C.}^{(\nu;l)} = 1 + \frac{\hat{\alpha}}{4\pi\hat{s}^2}\left\{\frac{3}{4\hat{s}^2}\ln c^2 - \frac{7}{4} + \frac{2\hat{c}_Z}{\hat{c}^2} + \frac{3}{4}\xi\left[\frac{\ln(c^2/\xi)}{c^2-\xi} + \frac{1}{c^2}\frac{\ln\xi}{1-\xi}\right] + \frac{3}{4}\frac{M_t^2}{M_W^2}\right\}, \tag{105}$$

$$\varepsilon_-(q^2) = \tfrac{1}{2}[1 - 2\hat{\kappa}^{(\nu_\mu;l)}(q^2)\hat{s}^2], \tag{106}$$

$$\varepsilon_+(q^2) = -\hat{\kappa}^{(\nu_\mu;l)}(q^2)\hat{s}^2, \tag{107}$$

$$\hat{\kappa}^{(\nu_\mu;l)}(q^2) = 1 - \frac{\alpha}{2\pi\hat{s}^2}\left[\sum_i (C_{3i}Q_i - 4\hat{s}^2Q_i^2)J_i(q^2) - 2J_\mu(q^2) + \ln c\left(\frac{1}{2} - 7\hat{c}^2\right) + \frac{\hat{c}^2}{3} + \frac{1}{2} + \frac{\hat{c}_\gamma}{\hat{c}^2}\right], \tag{108}$$

$$J_i(q^2) = \int_0^1 dx x(1-x)\ln\left(\frac{m_i^2 - q^2x(1-x)}{M_Z^2}\right), \tag{109}$$

$$\hat{c}_Z = \frac{19}{8} - \frac{7}{2}\hat{s}^2 + 3\hat{s}^4, \tag{110}$$

$$\hat{c}_\gamma = \frac{19}{8} - \frac{17}{4}\hat{s}^2 + 3\hat{s}^4. \tag{111}$$

In these expressions terms of $\mathcal{O}(\alpha q^2/M_Z^2)$ have been neglected. In this approximation, we see from Eq. (105) that $\rho_{\rm N.C.}^{(\nu;l)}$ is independent of q^2. It is also independent of the ν and charged lepton flavors. In contrast, $\hat{\kappa}^{(\nu_\mu;l)}(q^2)$ depends on q^2. It also depends on the incident neutrino flavor via the term $-2J_\mu(q^2)$ in Eq. (108) (which arises from the "ν_μ charge radius" diagrams). As in previous sections, $s^2 = 1 - c^2 = \sin^2\theta_W$ [cf. Eq. (55)] and $\hat{s}^2 = 1 - \hat{c}^2 = \sin^2\hat{\theta}_W(M_Z)$ (cf. Sec. III.D). In Eq. (105), $\hat{\alpha} = \hat{\alpha}(M_Z) \simeq 1/127.9$ is the $\overline{\rm MS}$ QED coupling at scale $\mu = M_Z$ and $\xi = M_H^2/M_Z^2$. In Eq. (108), the sum is over the charged leptons and quarks (in the quark sector $\sum_i = 3\sum_f$ where f denotes the flavors and the factor 3 represents the color degrees of freedom), and m_i, Q_i, and C_{3i} are the mass, charge (in units of the proton charge e_p), and twice the third component of the weak isospin of the ith fermion, respectively. In Eq. (104), E_e' and E_ν are the energies of the outgoing electron and the incident neutrino in the rest frame of the incoming electron. Thus, in that frame, $z = T/E_\nu$, where T is the kinetic energy of the scattered electron.

The expressions for $\hat{\kappa}^{(\nu_\mu;l)}(q^2)$ in Eq. (108) and $J_i(q^2)$ in Eq. (109) have been updated from those presented by Sarantakos, Sirlin, and Marciano (1983) to take into account the use of $\sin^2\hat{\theta}_W(M_Z)$ in Eqs. (106) and (107), while the early work employed $\sin^2\hat{\theta}_W(M_W)$.

The functions $f_-(z)$, $f_+(z)$, and $f_{+-}(z)$ in Eq. (103) describe QED corrections. The first two functions have been evaluated analytically in the relativistic approximation (Sarantakos, Sirlin, and Marciano, 1983), assuming m_e/E_e, m_e/E_ν, and $m_e/(E_{\rm max} - E_e) \ll 1$. Exact expressions for $f_-(z)$ and $f_+(z)$ can be obtained from Ram (1967); $f_{+-}(z)$ was evaluated exactly by Passera (2001). However, these expressions are long and complicated and are best treated using numerical tabulations.

The differential cross sections for $\bar{\nu}_\mu + e \to \bar{\nu}_\mu + e$, $\nu_e + e \to \nu_e + e$, and $\bar{\nu}_e + e \to \bar{\nu}_e + e$ are obtained from the $\nu_\mu + e \to \nu_\mu + e$ case by making simple changes explained by Sarantakos, Sirlin, and Marciano (1983). In particular, in $\nu_e + e \to \nu_e + e$ there are two distinct classes of contributions, one involving the neutral currents as in $\nu_\mu + e \to \nu_\mu + e$, and the other mediated by the W boson.

If the tree-level propagator factors $(1 - q^2/M_W^2)^{-2}$ and $(1 - q^2/M_Z^2)^{-2}$ are ignored (i.e., if $q^2/M_W^2 \ll 1$), in passing from $\nu_\mu + e \to \nu_\mu + e$ to $\nu_e + e \to \nu_e + e$ the only changes are as follows:

(i) $\varepsilon_-(q^2)$ in Eq. (106) is changed to

$$\varepsilon_-(q^2) = \frac{1}{2}[1 - \hat{\kappa}^{(\nu_e;l)}(q^2)\hat{s}^2] - \frac{1}{\rho_{\rm N.C.}^{(\nu;l)}}, \tag{112}$$

where $\rho_{\rm N.C.}^{(\nu;l)}$ is defined in Eq. (105),

(ii) $\varepsilon_+(q^2)$ in Eq. (107) is changed to

$$\varepsilon_+(q^2) = -\hat{\kappa}^{(\nu_e;l)}(q^2)\hat{s}^2. \tag{113}$$

(iii) $\hat{\kappa}^{(\nu_e;l)}(q^2)$ is obtained from $\hat{\kappa}^{(\nu_\mu;l)}(q^2)$ by replacing $-2J_\mu(q^2) \to -2J_e(q^2)$ in Eq. (108).

We note that the additional $(-\rho_{\rm N.C.}^{(\nu;l)})^{-1}$ term in Eq. (112) reflects the tree-level contribution of the W mediated amplitude, and the change in (iii) arises from the charge radius diagrams that depend on the neutrino flavor.

The results discussed in this section have been applied to the study of the electron recoil-energy spectra and the total cross sections for neutrino-electron scattering by solar neutrinos (Bahcall, Kamiokowski, and Sirlin, 1995). This paper also presents simple modifications of the relativistic expressions for the QED functions $f_-(z)$ and $(1-z)^2 f_+(z)$ so that they can be applied approximately in the nonrelativistic domain. An approximate expression for $f_{+-}(z)$ (a function that had not been calculated previously) is also included.

As mentioned, the expressions in Eqs. (103)–(113) have been derived in the $\overline{\rm MS}$ scheme of renormalization. If the analysis is carried out, instead, in the OS scheme, the expression for $\rho_{\rm N.C.}^{(\nu;l)}$ is essentially the same as Eq. (105), except that $\hat{s}^2$, $\hat{c}^2$, and $\hat{\alpha}$ are changed to s^2, c^2, and α. On the other hand, the OS form factor $\kappa^{(\nu_\mu;l)}(q^2)$ (Marciano and Sirlin, 1980; Sarantakos, Sirlin, and Marciano, 1983) that multiplies $\sin^2\theta_W$ in the EWC has a considerably more complex structure than the $\overline{\rm MS}$ form factor $\hat{\kappa}^{(\nu_\mu;l)}(q^2)$ given in Eq. (108). In particular, in $\mathcal{O}(\alpha)$, $\kappa^{(\nu_\mu;l)}(q^2)$ depends on M_H, while $\hat{\kappa}^{(\nu_\mu;l)}(q^2)$ does not. This more complex structure can be traced to the contributions of the counterterm $(c^2/s^2){\rm Re}[A_{ZZ}(M_Z^2)/M_Z^2 - A_{WW}(M_W^2)/M_W^2]$ present in $\kappa^{(\nu_\mu;l)}(q^2)$ [we recall that $A_{ZZ}(q^2)$ and $A_{WW}(q^2)$ are the Z and W transverse self-energies].

M. Electron-positron annihilation

Since LEP was an e^--e^+ collider, the study of the annihilation process into fermion-antifermion pairs, $e^- + e^+ \to f + \bar{f}$, became a subject of great interest.

Passarino and Veltman (1979) examined the EW and QED corrections to $e^- + e^+ \to \mu^- + \mu^+$. They also introduced a method to reduce one-loop tensor integrals to scalar ones, which has been frequently employed in the calculation of the EWC to several important processes. Since that time, detailed studies of EW, QED, and QCD corrections to $e^- + e^+ \to f + \bar{f}$ were carried out by several; see, for example, Ellis and Peccei (1986) and references therein; Alexander *et al.* (1988) and references therein; and Altarelli, Kleiss, and Verzegnassi (1989), Kühn (1989), and Bardin, Hollik, and Passarino (1995) and references therein. Degrassi and Sirlin (1991) analyzed the EWC to cross sections, asymmetries, and Z partial widths using both the on-shell and the $\overline{\rm MS}$ renormalization frameworks. The results of the partial widths and asymmetries for some final-state modes were then compared numerically with those obtained in the formulation of Consoli, Hollik, and Jegerlehner (1989) and Hollik (1990). The corrections to the $Zb\bar{b}$ vertex involved a significant M_t^2 dependence and played an important role in the indirect determination of the top-quark mass before the discovery of this fundamental particle (Akhundov, Bardin, and Riemann, 1986; Beenakker and Hollik, 1988; Bernabeu, Pich, and Santamaria, 1991).

The asymmetries measured at LEP and SLC are of special interest because they provide the most precise determination of $\sin^2\theta_{\rm eff}^{\rm lept}$ (cf. Sec. III.E). They include (a) the measurement at LEP of the forward-backward asymmetries $A_{\rm FB}^{0,f}$ (for $f = e, \mu, \tau, s$, and b), the τ^- polarization asymmetry P_τ in $e^- + e^+ \to \tau^- + \tau^+$, and the forward-backward asymmetry $Q_{\rm FB}^{\rm had}$ between positive and negative charge in hadronic Z events; (b) the measurements at SLC of the left-right e^--polarization asymmetry $A_{\rm LR}^0$ and the combined forward-backward e^--polarization asymmetries $A_{\rm LR}^{\rm 0FB}$, separately analyzed for hadronic and leptonic final states. For a recent discussion, see Erler and Langacker, 2010, particularly Sec. 10.4.

For a long time, there has been an intriguing difference at the 3σ level between the values of $\sin^2\theta_{\rm eff}^{\rm lept}$ derived from the leptonic and hadronic asymmetries. In fact, one finds $(s_{\rm eff}^2)_l = 0.231\,13(21)$ from the leptonic asymmetries $(A_{\rm FB}^{0,l}, A^0(P_\tau), A_{\rm LR}^0, A_{\rm LR}^{0,{\rm FB}})$ ($l = e$, μ, and τ) and $(s_{\rm eff}^2)_h = 0.232\,22(27)$ from the hadronic asymmetries $(A_{\rm FB}^{0,q}, Q_{\rm FB}^{\rm had})$ ($q = s$, c, and b). Furthermore, the results within each group are in good agreement with each other. The intriguing question remains of whether the difference between $(s_{\rm eff}^2)_l$ and $(s_{\rm eff}^2)_h$ is due to a statistical fluctuation or arises from new physics involving perhaps the third generation of quarks. The second scenario is difficult to implement because of the constraints imposed by the $Z \to b\bar{b}$ branching ratio. In the first case, a possible approach to take into account the difference is to enlarge the error, as discussed by Gurtu (1996), Degrassi *et al.* (1998), and Ferroglia *et al.* (2002). For example, if the $\sin^2\theta_{\rm eff}^{\rm lept}$ error is increased by a factor $[\chi^2/{\rm DOF}]^{1/2}$ following the Particle Data Group prescription (Barnett *et al.*, 1996), one obtains the value $\tilde{s}_{\rm eff}^2 = 0.231\,53(25)$. The discrepancy discussed above is of particular significance for the indirect estimate of M_H, which is sensitive to the precise value of $\sin^2\theta_{\rm eff}^{\rm lept}$. Since this issue has not been resolved, the usual procedure is to employ the average value obtained from all the asymmetries.

N. Estimates of the Higgs boson mass

The Higgs boson is the fundamental missing piece of the ST. Thus, an important question is to what extent can M_H be estimated using the precision electroweak data and the theoretical expressions for the relevant observables, which depend on M_H via EWC. In fact, such estimates may provide useful information for explorations at the CERN Large Hadron Collider (LHC), since one of its main objectives is the search for this fundamental particle.

At the one-loop level, for large M_H, the dependence of the EWC on M_H is proportional to $\ln(M_H/M_Z)$ [cf. Eqs. (68) and (69)], a slowly varying function.[7] Thus, precise calculations are needed. Theorists distinguish two classes of errors: (a) parametric, such as $\delta M_W, \delta s_{\rm eff}^2, \delta M_t, \delta\Delta\alpha_h^{(5)}, \ldots$; (b) uncertainties due to the truncation of the perturbative series (i.e., uncalculated higher-order effects). As mentioned at the end of Sec. III.F, estimates of the second class of errors are often obtained by comparing the evaluation of the EWC using different renormalization schemes. In the case when the expansion parameters are scale dependent, as in the $\overline{\rm MS}$ scheme of renormalization, errors of the second class are frequently estimated by examining the scale dependence of the calculations.

The comparison of the accurate experimental values of M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$ with their theoretical calculations have been subjects of particular interest, since they provide important information about M_H.

Over the years, a number of higher-order EWCs were incorporated in the theoretical calculations. Contributions of $\mathcal{O}(\alpha)$, $\mathcal{O}(\alpha^n \ln^n M_Z/M_W)$, and $\mathcal{O}(\alpha^2 \ln M_Z/M_f)$ (where f is a generic quark or lepton) were analyzed in the period 1979–1984. Those of $\mathcal{O}(\alpha^2 (M_t/M_W)^4)$, $\mathcal{O}(\alpha\alpha_s)$, and $\mathcal{O}(\alpha\alpha_s^2 (M_t/M_W)^2)$ were studied from the late 1980s to the middle 1990s. EWCs of $\mathcal{O}[\alpha^2 (M_t/M_W)^2]$ were evaluated by Degrassi, Gambino, and Vicini (1996), Degrassi, Gambino, and Sirlin (1997), Degrassi *et al.* (1998), and Degrassi and Gambino (2000); (see also references in those papers).

Simple analytic formulas for the theoretical calculation of $\sin^2\theta_{\rm eff}^{\rm lept}$, M_W, and the leptonic partial widths Γ_l of the Z boson were presented by Ferroglia *et al.* (2002). They accurately reproduced the results of the detailed calculations in the on-shell, $\overline{\rm MS}$, and effective schemes (cf. Sec. III.F) as functions of M_H, M_t, $\Delta\alpha_h^{(5)}$, and $\alpha_s(M_Z)$, over the range $20 \le M_H \le 300$ GeV. In particular, they incorporated the complete one-loop EWC, as well as the two-loop contributions enhanced by factors $(M_t^2/M_Z^2)^n$ ($n = 1, 2$) that had been studied previously. These simple formulas were applied to

[7] It is interesting to note that the evaluation of higher-order corrections to the ρ parameter has a long history, starting with van der Bij and Veltman (1984), where the contributions of $\mathcal{O}(\alpha^2 M_H^2)$ were obtained. The important two-loop QCD and EW contributions to the ρ parameter were evaluated by Djouadi and Verzegnassi (1987), Fleischer *et al.* (1994), and Chetyrkin, Kühn, and Steinhauser (1995). Later developments include calculations, at the three- and four-loop levels, of pure EW and mixed EW and QCD corrections in the large M_H or M_t limits (van der Bij *et al.*, 2001; Faisst *et al.*, 2003; Boughezal, Tausk, and van der Bij, 2005; Boughezal and Czakon, 2006; Chetyrkin *et al.*, 2006).

estimate M_H and its 95% C.L. upper bound M_H^{95} using either $(s^2_{\rm eff})_{\rm exp}$, $(M_W)_{\rm exp}$, or, simultaneously, $(s^2_{\rm eff})_{\rm exp}$, $(M_W)_{\rm exp}$, and $(\Gamma_l)_{\rm exp}$ as input parameters.

An important advance has been the calculation of the complete two-loop contribution to Δr in the OS scheme of renormalization. It includes the fermionic contribution, which involves diagrams with one or two closed fermion loops (Freitas *et al.*, 2000, 2002) and the purely bosonic two-loop contribution[8] (Awramik and Czakon, 2002; Awramik *et al.*, 2003; Onishchenko and Veretin, 2003). Since Δr is the quantum correction in the relation of M_W with α, G_F, and M_Z, this result directly provides the two-loop EWC in the theoretical calculation of M_W.

Another important achievement has been the calculation, also in the OS scheme, of the complete two-loop EWC in the theoretical evaluation of $\sin^2\theta_{\rm eff}^{\rm lept}$ (Awramik *et al.*, 2004; Hollik, Meier, and Uccirati, 2005, 2006, 2007; Awramik, Czakon, and Freitas, 2006a, 2006b).

Simple analytic formulas that accurately incorporate the contribution of the one- and two-loop EWCs in the theoretical calculations of M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$, as functions of M_H, M_t, $\Delta\alpha$, $\alpha_s(M_Z)$, and M_Z, were given, respectively, by Awramik *et al.* (2004) and Awramik, Czakon, and Freitas (2006b).

Next we illustrate the application of these accurate formulas to the estimate of the Higgs boson mass M_H and its 95% C.L. upper bound M_H^{95}. We use as inputs $M_W = 80.385(15)$ GeV (LEP Electroweak Working Group, 2012; Tevatron Electroweak Working Group, 2012), $M_Z = 91.1876(21)$ GeV (see Sec. III.B), $M_t = 173.2(0.9)$ GeV (Tevatron Electroweak Working Group, 2011), $\sin^2\theta_{\rm eff}^{\rm lept} = 0.231\,53(16)$ [cf. Eq. (71)], $\Delta\alpha = 0.059\,07(10)$ [cf. Eq. (75)], and $\alpha_s(M_Z) = 0.1184(7)$ (Dissertori *et al.*, 2010). On this basis, we obtain the following estimates:

$$M_H = 98^{+25}_{-21}\ \text{GeV}, \quad M_H^{95} = 142\ \text{GeV} \quad (M_W + s^2_{\rm eff}), \tag{114}$$

$$M_H = 81^{+28}_{-24}\ \text{GeV}, \quad M_H^{95} = 131\ \text{GeV} \quad (M_W), \tag{115}$$

$$M_H = 129^{+53}_{-38}\ \text{GeV}, \quad M_H^{95} = 226\ \text{GeV} \quad (s^2_{\rm eff}). \tag{116}$$

Equation (114) was obtained by means of a χ^2 analysis based on theoretical expressions for both M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$, while Eqs. (115) and (116) were derived from the separate application of the M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$ formulas, respectively.

As a comparison, a recent standard global fit to the EW data (Baak *et al.*, 2012) employs the inputs $M_W = 80.399(23)$ GeV, $M_Z = 91.1875(21)$ GeV, $M_t = 173.3(1.1)$ GeV, $\sin^2\theta_{\rm eff}^{\rm lept} = 0.23153(16)$, $\Delta\alpha = 0.05899(10)$, and $\alpha_s(M_Z) = 0.1193(28)$, and derives the estimate $M_H = 96^{+31}_{-24}$ GeV, $M_H^{95} = 169$ GeV (this last value includes the effect of the estimated theoretical error).

Since the inputs in our calculations are somewhat different (particularly in the case of M_W for which we use a more recent and precise value), for comparison purposes we repeat our calculation of Eq. (114) employing the same inputs as in the global fit. This leads to $M_H = 103^{+32}_{-26}$ GeV, $M_H^{95} = 160$ GeV, which can be compared with the values $M_H = 96^{+31}_{-24}$ GeV, $M_H^{95} = 169$ GeV obtained in the global fit with the same input parameters. Thus, we see that the estimates obtained by combining the theoretical expressions for M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$ are rather close to those obtained in the global fit, an observation that illustrates the importance and sensitivity of these two observables in the prediction of M_H and M_H^{95}.

We note that the central values of M_H in both Eq. (114) and the global fit are well below the 95% C. L. lower bound

$$(M_H)_{\rm L.B.} = 114.4\ \text{GeV}, \tag{117}$$

inferred from the direct experimental searches of the Higgs boson at LEP and the Tevatron. On the other hand, the two M_H estimates are compatible with Eq. (117) when their errors are taken into account.

In Sec. III.M, we pointed out that, for a long time, there has been an intriguing difference, at the 3σ level, between the values of $\sin^2\theta_{\rm eff}^{\rm lept}$ derived from the leptonic and hadronic asymmetries, namely, $(s^2_{\rm eff})_l = 0.231\,13(21)$ and $(s^2_{\rm eff})_h = 0.23222(27)$. In order to illustrate the potential effect of this dichotomy, we give the M_H and M_H^{95} estimates obtained by separately using these values, as well as their combinations with the theoretical expression for M_W:

$$M_H = 54^{+33}_{-21}\ \text{GeV}, \qquad M_H^{95} = 117\ \text{GeV} \qquad [(s^2_{\rm eff})_l], \tag{118}$$

$$M_H = 71^{+23}_{-18}\ \text{GeV}, \qquad M_H^{95} = 111\ \text{GeV} \qquad [M_W, (s^2_{\rm eff})_l], \tag{119}$$

$$M_H = 513^{+387}_{-212}\ \text{GeV} \qquad [(s^2_{\rm eff})_h], \tag{120}$$

$$M_H = 117^{+32}_{-27}\ \text{GeV}, \qquad M_H^{95} = 173\ \text{GeV} \qquad [M_W, (s^2_{\rm eff})_h]. \tag{121}$$

We see that the estimates based on $(s^2_{\rm eff})_l$, either by itself [see Eq. (118)] or in combination with M_W [see Eq. (119)], are very low. In fact, at the 1σ level, they disagree with $(M_H)_{\rm L.B.}$ [see Eq. (117)]. We also note that M_H^{95} in Eq. (118) is barely compatible with $(M_H)_{\rm L.B.}$, while its value in Eq. (119) is lower. Thus, in an hypothetical scenario in which the $(s^2_{\rm eff})_l$-$(s^2_{\rm eff})_h$ discrepancy were to

[8] For clarity, we point out that in the recent higher-order calculations Δr is introduced by $s^2c^2 = (\pi\alpha/\sqrt{2}G_F M_Z^2)(1+\Delta r)$, with Δr in the numerator, which coincides with the expression originally derived by Sirlin (1980a). Of course, at the one-loop level, this expression and Eq. (54) are equivalent.

settle on the leptonic side, for example, by bringing the value of $(s^2_{\rm eff})_h$ close to the present determination of $(s^2_{\rm eff})_l$, a serious discrepancy would arise between the M_H, M_H^{95} estimates and $(M_H)_{\rm L.B.}$.

In contrast, $(s^2_{\rm eff})_h$ leads to considerably larger estimates of M_H and M_H^{95} [cf. Eqs. (120) and (121)]. We note that in Eq. (120) we have not included the value of M_H^{95}. The reason is that the range of validity of the simple analytic formulas is 10 GeV $\leq M_H \leq$ 1 TeV, while their application to Eq. (120) leads to a value of M_H^{95} considerably larger than 1 TeV.

Finally we consider the estimates of M_H, M_H^{95} based on $\tilde{s}^2_{\rm eff} = 0.231\,53(25)$, the value obtained from the weighted average of $(s^2_{\rm eff})_l$ and $(s^2_{\rm eff})_h$ by enlarging the error according to the Particle Data Group prescription (cf. the discussion toward the end of Sec. III.M):

$$M_H = 129^{+89}_{-54}\ \text{GeV}, \qquad M_H^{95} = 302\ \text{GeV} \qquad (\tilde{s}^2_{\rm eff}), \tag{122}$$

$$M_H = 90^{+27}_{-22}\ \text{GeV}, \qquad M_H^{95} = 137\ \text{GeV} \qquad (M_W, \tilde{s}^2_{\rm eff}). \tag{123}$$

As expected, the central value in Eq. (122) is the same as in Eq. (116), but the errors and M_H^{95} are larger. On the other hand, the central value and M_H^{95} in Eq. (123) are smaller than in Eq. (114). The reason is that the increased error in $\tilde{s}^2_{\rm eff}$ gives greater weight to the M_W contribution, which favors smaller values of M_H and M_H^{95}.

In February 2012, the ATLAS Collaboration at LHC (ATLAS Collaboration, 2012a) reported that their combined search for the ST Higgs boson excludes the M_H ranges 112.9–115.5, 131–238, and 251–466 GeV at 95% C.L. Thus, subject to that exclusion, the still-allowed domains are 115.5–131, 238–251, and ≥ 466 GeV. On the same day, the CMS Collaboration at LHC (CMS Collaboration, 2012a) reported that their combined search excludes the M_H range 127–600 GeV at 95% C.L. and 129–525 GeV at 99% C.L. Thus, subject to the 95% C.L. exclusion, the still-allowed regions are 114.4–127 and ≥ 600 GeV. At the same time, the ATLAS Collaboration reported an excess of events above the expected ST background around $M_H \sim 126$ GeV with a local significance of 3.5σ, while the CMS Collaboration found an excess at $M_H = 124$ GeV with a local significance of 3.1σ. Both collaborations expect to collect a considerable amount of additional data in 2012 in order to ascertain whether the observed excesses represent real signals of the Higgs boson or they simply reflect statistical fluctuations of the ST background. For the moment, we observe that, when the 1σ errors are taken into account, the estimates in both Eq. (114) and the global fit are compatible with a Higgs boson in the neighborhood of $M_H = 125$ GeV.

There are also interesting theoretical upper and lower bounds for M_H, $M_{\rm max}(\Lambda)$, and $M_{\rm min}(\Lambda)$, where Λ is the scale up to which the ST is assumed to be valid. $M_{\rm max}(\Lambda)$ is obtained from the requirement that the Higgs self-coupling does not exhibit a Landau pole below Λ. $M_{\rm min}(\Lambda)$ is obtained from considerations of vacuum stability. If $\Lambda = M_P$, $M_{\rm max}(M_P) \approx 175$ GeV [cf. Bezrukov *et al.* (2012) and references therein]. Since the recent Higgs boson searches at LHC exclude the range 129–525 GeV at 99% C.L., this result indicates that, in the absence of new physics, the ST is a weakly coupled theory up to M_P. Recent analyses of $M_{\rm min}(M_P)$ include Bezrukov *et al.* (2012) and Elias-Miró *et al.* (2012). Bezrukov *et al.* found $M_{\rm min} = 129 \pm 6$ GeV, which overlaps with the allowed region in the recent searches. Elias-Miró *et al.* derive both stability and metastability bounds. For their central values, they found $M_{\rm min}(M_P) = 130 \pm 3$ GeV and $M_{\rm min}^{\rm metas}(M_P) = 111 \pm 3$ GeV. The metastability bound is derived by requiring that the lifetime of the electroweak vacuum is larger than the age of the Universe. Combining the results explained above, and assuming that the Higgs boson is discovered in the range $115.5 \leq M_H \leq 127$ GeV currently allowed by the direct searches at the LHC, Bezrukov *et al.* (2012) concluded the following:

(a) a new energy scale between the Fermi and Planck scales is not necessarily required,
(b) in the absence of such scale, the EW theory remains weakly coupled up to M_P,
(c) and the EW vacuum has a lifetime larger than the age of the Universe.

On 4 July 2012, the ATLAS (ATLAS Collaboration, 2012b) and CMS (CMS Collaboration, 2012b) Collaborations at the LHC announced the discovery at the 5σ level of a boson in the mass interval 124–126 GeV. There is a widespread belief in the physics community that this is the long-sought Higgs boson. To ascertain whether this is the case, further analyses are in progress to determine whether the spin of the newly discovered particle is indeed 0 as befits the Higgs boson, and whether its production and decay rates conform with the ST expectations.

O. The muon $g_\mu - 2$

The anomalous magnetic moment of the muon, $a_\mu = (g_\mu - 2)/2$, is one of the most interesting and precisely measured observables in particle physics. In fact, since each sector of the ST contributes in a significant way to its theoretical prediction, the a_μ measurement by the E821 experiment at the Brookhaven National Laboratory (Bennett *et al.*, 2002, 2004, 2006; Roberts, 2010), with a remarkable precision of 0.5 ppm, permits one to test the entire ST and examine possible new physics effects (Czarnecki and Marciano, 2001; Stöckinger, 2007). It is important to note that even more precise measurements are planned at the Fermilab experiment P989 and J-PARC with anticipated errors that are smaller than the current one by factors of 4 and 5.4, respectively.

The ST prediction of a_μ includes QED, EW, and hadronic (leading- and higher-order) contributions $a_\mu^{\rm ST} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm HLO} + a_\mu^{\rm HHO}$. The QED contribution, computed to four loops and estimated to five,[9] currently stands at $a_\mu^{\rm QED} = 116\,584\,718.08(15)\times 10^{-11}$ (Laporta and Remiddi, 1993, 1996; Kinoshita and Nio, 2004, 2006a, 2006b; Kataev, 2006; Aoyama *et al.*, 2007, 2008, 2008a, 2008b, 2010, 2011a, 2011b, 2011c, 2012a, 2012b; Passera, 2007; Aoyama, Hayakawa, Kinoshita, and Nio, 2010), while the EW effects, suppressed by a factor $(m_\mu/M_W)^2$, amount to $a_\mu^{\rm EW} = 154(2)\times 10^{-11}$ (Czarnecki, Krause, and Marciano, 1995, 1996; Degrassi and Giudice, 1998; Czarnecki *et al.*, 2003).

Recent calculations of the hadronic leading-order contribution, based on the hadronic e^+e^- annihilation data, include $a_\mu^{\rm HLO} = 6949.1(42.7)\times 10^{-11}$ (Hagiwara *et al.*, 2011), $a_\mu^{\rm HLO} = 6903(53)\times 10^{-11}$ (Jegerlehner and Nyffeler, 2009), and $a_\mu^{\rm HLO} = 6923(42)\times 10^{-11}$ (Davier *et al.*, 2011). The three results agree within errors. A recent analysis by Jegerlehner and Szafron (2011) found good agreement between the calculations based on the e^+e^- annihilation and τ decays data leading to $a_\mu^{\rm HLO} = 6909.6(46.5)\times 10^{-11}$.

The higher-order hadronic contribution is divided into two parts: $a_\mu^{\rm HHO} = a_\mu^{\rm HHO}({\rm vp}) + a_\mu^{\rm HHO}({\rm lbl})$. The first one $a_\mu^{\rm HHO}({\rm vp}) = -98(1)\times 10^{-11}$ (Hagiwara *et al.*, 2007) is the $\mathcal{O}(\alpha^3)$ contribution of diagrams containing hadronic vacuum polarization insertions. The second one, also of $\mathcal{O}(\alpha^3)$, is the hadronic light-by-light contribution; since it cannot be derived from data, its evaluation is based on specific models. Two of the most recent determinations, $116(39)\times 10^{-11}$ (Jegerlehner and Nyffeler, 2009; Nyffeler, 2009) and $105(26)\times 10^{-11}$ (Prades, de Rafael, and Vainshtein, 2009), are in good agreement. If one adds the latter to $a_\mu^{\rm HLO} = 6949.1(42.7)\times 10^{-11}$ and the rest of the ST contributions, one obtains $a_\mu^{\rm ST} = 116\,591\,828(50)\times 10^{-11}$. The difference with the experimental value $a_\mu^{\rm exp} = 116\,592\,089(63)\times 10^{-11}$ (Roberts, 2010) is $\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm ST} = 261(80)\times 10^{-11}$, i.e., $+3.3\sigma$ (all errors have been added in quadrature). A somewhat larger discrepancy 3.6σ is obtained if one employs $a_\mu^{\rm HLO} = 6923(42)\times 10^{-11}$.

It has been pointed out that supersymmetry (SUSY) may provide a natural explanation for the $(3\text{–}4)\sigma$ discrepancy between $a_\mu^{\rm exp}$ and $a_\mu^{\rm ST}$ [for a review, see Stöckinger (2007)]. Assuming, for simplicity, a single mass $M_{\rm SUSY}$ for the supersymmetric particles that contribute to $a_\mu^{\rm SUSY}$, one finds (Kosower, Krauss, and Sakai, 1983; Yuan *et al.*, 1984; Moroi and Moroi, 1996; Ibrahim and Nath, 2000; Heinemeyer, Stöckinger, and Weiglein, 2004a, 2004b)

$$a_\mu^{\rm SUSY} \simeq {\rm sgn}(\mu)\times 130\times 10^{-11}\left(\frac{100\,{\rm GeV}}{M_{\rm SUSY}}\right)^2 \tan\beta, \qquad (124)$$

where $\tan\beta > 3\text{–}4$ is the ratio of the two scalar vacuum expectation values and ${\rm sgn}(\mu)$ is the sign of the μ term in SUSY models. Assuming that $a_\mu^{\rm SUSY}$ cancels the discrepancy, so that $a_\mu^{\rm SUSY} = \Delta a_\mu$, and using, for example, the value $\Delta a_\mu = 261(80)\times 10^{-11}$, one finds ${\rm sgn}(\mu) = +$ and

$$M_{\rm SUSY} \simeq 71^{+14}_{-9}\sqrt{\tan\beta}\ {\rm GeV}. \qquad (125)$$

For $\tan\beta \sim 4\text{–}50$, Eq. (125) leads to the rough estimate $124 \le M_{\rm SUSY} \le 601$ GeV. On the other hand, signals of supersymmetric particles have not been uncovered so far. Other new physics explanations of the a_μ discrepancy have also been discussed (Czarnecki and Marciano, 2001).

In an alternative approach, not involving new physics, Passera, Marciano, and Sirlin (2008, 2009, 2010) considered whether an increase in the hadroproduction cross section $\sigma(s)$ in low-energy e^+e^- collisions, due to hypothetical experimental errors, could bridge the a_μ discrepancy. They found that this is unlikely in view of the current experimental error estimates. If, nonetheless, this turns out to be the explanation of the discrepancy, it has an interesting consequence: the increase in $\sigma(s)$ also increases $\Delta\alpha^{(5)}_{\rm had}(M_Z)$ which, in turn, affects the estimate of M_H. They found that, in this hypothetical scenario, the 95% C.L. upper bound on the Higgs boson mass is reduced to about 135 GeV which, in conjunction with $(M_H)_{\rm LB} = 114.4$ GeV, leaves a narrow window for the mass of this fundamental particle. This window is slightly larger than the range allowed by the very recent LHC direct searches (cf. the previous to last pararaph in Sec. III.N).

P. Atomic parity violation

The interference of the electromagnetic and weak neutral current amplitudes leads to parity violating effects in atomic transitions that have been the subject of ingenious experiments and detailed theoretical studies.

The pseudoscalar component of the electron-quark interaction, arising from the Z boson exchange at $q^2 = 0$, is usually expressed in the form

$$\mathcal{H}_{\rm PV} = \frac{G_\mu}{\sqrt{2}}\{[C_{1u}\bar{u}\gamma^\mu u + C_{1d}\bar{d}\gamma^\mu d][\bar{e}\gamma_\mu\gamma_5 e] + [C_{2u}\bar{u}\gamma^\mu\gamma_5 u + C_{2d}\bar{d}\gamma^\mu\gamma_5 d][\bar{e}\gamma_\mu e] + \cdots\}, \qquad (126)$$

where the ellipsis represents heavy-quark contributions ($q = s$, c, b, and t).

The C_{2i} ($i = u$, d) are suppressed by a factor $1 - 4\sin^2\hat{\theta}_W(M_Z) \simeq 0.075$ that arises from the electron's vector coupling to the Z boson. Also, the C_{1i} ($i = u$, d) terms are of primary importance for heavy atoms because they add up coherently over all quarks in the nucleus. As a consequence, parity violating effects are dominated by contributions proportional to the weak charge

$$Q_W(Z, A) \equiv 2[(A + Z)C_{1u} + (2A - Z)C_{1d}], \qquad (127)$$

where Z and A are the atomic and mass numbers of the atom.

The dominance of the C_{1i} ($i = u$, d) terms is also theoretically fortunate, because the corresponding hadronic currents are conserved and therefore are not affected by the strong interactions at $q = 0$.

As pointed out by Bouchiat and Bouchiat (1974), parity violating effects in heavy atoms scale roughly as Z^3 (one Z factor reflects the coherence effect in Q_W, while the others arise from the electron wave function and momentum near the nucleus).

[9] After this paper was submitted for publication, the calculations of the five loop contributions to a_e and a_μ were completed (Aoyama *et al.*, 2012c, 2012d), leading to $a_\mu^{\rm QED} = 116\,584\,718.845(37)\times 10^{-11}$ and $\Delta a_\mu = 260(80)\times 10^{-11}$.

Electroweak corrections of $\mathcal{O}(\hat{\alpha})$ to the C_{1i} and C_{2i} ($i = u, d$) coefficients were evaluated in the $\overline{\text{MS}}$ scheme by Lynn (1982), and Marciano and Sirlin (1983, 1984).

For the dominant coefficients C_{1i} ($i = u, d$), Marciano and Sirlin expressed their results in the form

$$C_{1u} = \frac{\rho'_{\text{PV}}}{2}\left[1 - \frac{8}{3}\kappa'_{\text{PV}}(0)\sin^2\hat{\theta}(M_W)\right], \tag{128}$$

$$C_{1d} = -\frac{\rho'_{\text{PV}}}{2}\left[1 - \frac{4}{3}\kappa'_{\text{PV}}(0)\sin^2\hat{\theta}(M_W)\right]. \tag{129}$$

The constants ρ'_{PV} and $\kappa'_{\text{PV}}(0)$ contain the $\mathcal{O}(\hat{\alpha})$ EWC, which depend on M_t, M_H, M_W, and M_Z, and are normalized so that $\rho'_{\text{PV}} = \kappa'_{\text{PV}}(0) = 1$ at the tree level. The detailed expressions for ρ'_{PV} and $\kappa'_{\text{PV}}(0)$ are given by Marciano and Sirlin (1983, 1984). A more recent version of these results, that employs $\sin^2\hat{\theta}_W(M_Z)$ instead of $\sin^2\hat{\theta}_W(M_W)$, was presented by Marciano (1995).

Measurements of atomic parity violation have been made in bismuth, lead, thallium, and cesium [for reviews, see Masterson and Wieman (1995), Bouchiat and Bouchiat (1997), and Ginges and Flambaum (2004)]. The most precise so far have been measurements of Q_W in cesium, at the 0.4% level. The analysis of the data requires detailed atomic physics calculations (Blundell, Johnson, and Sapirstein, 1995; Porsev, Beloy, and Derevianko, 2009) and QED corrections (Ginges and Flambaum, 2004). A recent result (Porsev, Beloy, and Derevianko, 2009) is $Q_W(\text{Cs}) = -73.16(29)_{\text{exp}} \times (20)_{\text{th}}$, in impressive agreement with the ST expectation $Q_W(\text{Cs})_{\text{ST}} = -73.15(2)$ (Erler and Langacker, 2010).

Q_W is insensitive to the T parameter and thus provides a direct probe of the S parameter, as emphasized by Marciano and Rosner (1990) and Marciano (1991, 1995) (cf. Sec. III.T).

Recently, sharp lower bounds on the mass of Z' bosons associated with interesting models beyond the ST have been derived from atomic parity violation measurements (Diener, Godfrey, and Turan, 2012). They also set constraints on the Z' couplings.

Q. Radiative corrections in flavor physics: The $b \to s\gamma$ case

Over the years, flavor physics played a crucial role in shaping our understanding of the interactions of elementary particles. The study of weak decays, including flavor and CP violating meson decays, led physicists to discover the GIM mechanism (Glashow, Iliopoulos, and Maiani, 1970) and the CKM matrix (Cabibbo, 1963; Kobayashi and Maskawa, 1973), both of which are essential elements in establishing the particle content of the ST.

In recent years, flavor physics observables were measured with great accuracy at several experimental facilities. Currently, one of the experiments at LHC, named LHCb, is primarily devoted to the measurement of the properties of hadrons containing a bottom quark. A second forthcoming experiment at CERN, called NA62, will measure very rare decays of charged kaons. Two new super-B factories will be built in Frascati (Italy) and at KEK (Japan). While experiments at high-energy colliders allow physicists to search for new physics beyond the ST by attempting to produce new particles, precise flavor physics experiments exploit the high luminosity of flavor factories in order to search for the effects of new physics in rare events. In this sense, the direct searches at high-energy colliders are complementary to the indirect searches at flavor factories, which are sensitive to energy scales as high as $\sim 10^4$–10^5 TeV.

An extended description of all of the observables in weak decays goes beyond the scope of the present review; the interested reader can find a comprehensive introduction to this topic in the classic Les Houches lectures by Buras (1998). Here we focus on a single representative example: the inclusive radiative decay of the B meson mediated by the partonic decay process $b \to s\gamma$. There are three reasons for this choice:

(i) As all flavor-changing neutral current (FCNC) processes, the $b \to s\gamma$ decay is a loop-induced process in the ST. As such, it is sensitive to new physics contributions, which can be of the same order in the coupling constants as the leading-order contribution in the ST.

(ii) As will be shown, inclusive decays are theoretically clean processes since they are not sensitive to nonperturbative effects and can be calculated with great accuracy within perturbation theory.

(iii) The measurements of this process, which was carried out at CLEO (Cornell), BELLE (KEK Tsukuba), and *BABAR* (Stanford), are precise; in order to match the current experimental accuracy it was necessary to consider, in calculating the branching ratio, the effects of next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) QCD corrections, as well as the effect of NLO electroweak corrections.

At the hadron level, the processes of interest are the inclusive radiative decays of B mesons into a photon and an arbitrary hadronic state of total strangeness -1, $\bar{B} \to X_s\gamma$, where $\bar{B}$ denotes a $\bar{B}^0$ or B^- meson, while X_s indicates an inclusive hadronic state not containing charmed particles. At the parton level, these processes are induced by a FCNC decay of the b quark contained in the $\bar{B}$ meson. The b quark decays into a photon and a strange quark plus other partons, collectively indicated by the symbol X_s^{parton}. In the ST, such a decay first takes place at one-loop order through diagrams involving heavy particles; for example, through a triangle loop with two virtual top quarks and a virtual W boson. Such diagrams are now commonly referred to as "penguin" diagrams.

In contrast with the exclusive decay modes, inclusive decays of B mesons are theoretically clean observables; in fact, it is possible to prove that the decay width $\Gamma(\bar{B} \to X_s\gamma)$ is well approximated by the partonic decay rate $\Gamma(b \to X_s^{\text{parton}}\gamma)$:

$$\Gamma(\bar{B} \to X_s\gamma) = \Gamma(b \to X_s^{\text{parton}}\gamma) + \Delta^{\text{nonpert}}. \tag{130}$$

The second term on the rhs of Eq. (130) represents nonperturbative corrections. The latter are small, since they are suppressed at least by a factor $(\Lambda_{\text{QCD}}/m_b)^2$, where m_b is the b-quark mass and $\Lambda_{\text{QCD}} \sim 200$ MeV. Equation (130) is known as the *heavy-quark expansion* [reviews of this topic and on heavy-quark effective theory can be found in Neubert (1994) and Manohar and Wise (2000)].

The partonic process can be studied within the context of perturbative QCD. However, the first-order QCD corrections to the partonic process are very large. The large corrections originate from hard gluon exchanges between quark lines of the one-loop electroweak graphs. In general, Feynman diagrams involving different mass scales depend on logarithms of the ratios of these scales. If there is a strong hierarchy among the mass scales, then the logarithms are numerically large. In the case of QCD corrections to the partonic process $b \to X_s^{\rm parton}\gamma$, the mass scales involved are M_W, M_t, and m_b. M_W and M_t are of the same order of magnitude $\mu_W \sim 100$ GeV, while the b-quark mass is considerably smaller: $m_b \sim 5$ GeV. Consequently, one finds that $\alpha_s(m_b) \times \ln(\mu_W^2/m_b^2) \sim 1$; therefore, the perturbative expansion is spoiled, and the large logarithmic corrections must be resummed to all orders.

The easiest way to implement the resummation of the logarithms is to work within the context of a renormalization-group-improved effective theory with five active quarks. In such a theory, the heavy degrees of freedom involved in the decay under study are integrated out. By means of an operator product expansion, it is possible to factorize the contribution of the short-distance and long-distance dynamics in the decay of the B meson. In the ST, the short-distance dynamics is characterized by mass scales of the order of the top-quark or W-boson mass, while the long-distance dynamics is characterized by the b-quark mass. The boundary between short distance and long distance is chosen at a low-energy scale μ_b, such that $m_b \sim \mu_b \ll M_W$. The scale μ_b is unphysical, and therefore physical quantities cannot depend on it: This fact is employed in order to obtain RGEs satisfied by the various factors in the calculation. The large logarithmic corrections are resummed by solving these RGEs.

The Lagrangian employed in calculating the $b \to X_s^{\rm parton}\gamma$ decay rate can be written as

$$\mathcal{L} = \mathcal{L}_{\rm QED\otimes QCD}(u, d, c, s, b) + \sum_{i=1}^{8} \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} C_i(\mu, \mu_W) Q_i(\mu) + \mathcal{O}\left(\frac{m_b}{M_W}\right). \quad (131)$$

In Eq. (131) $\mathcal{L}_{\rm QED\otimes QCD}$ represents the usual QED and QCD Lagrangians with five active quark flavors, while Q_i are eight effective operators of dimensions five and six. Operators with dimensions larger than 6 are suppressed by inverse powers of the W-boson mass and are ignored. The short-distance dynamics is encoded in the "coupling constants" that multiply the effective operators, which are called *Wilson coefficients* and are indicated by C_i in Eq. (131). The Wilson coefficients are the only elements of the Lagrangian which depend on the heavy particle masses M_W and m_t. The eight effective operators Q_i appearing in the Lagrangian in Eq. (131) are listed, for example, in Misiak and Steinhauser (2007).

Any perturbative calculation of the $b \to X_s^{\rm parton}\gamma$ decay rate within the context of the renormalization-group-improved perturbation theory applied to the Lagrangian in Eq. (131) involves three different steps:

(1) The first step, called *matching*, consists of fixing the value of the Wilson coefficients at the high-energy scale $\mu_W \sim M_W$, m_t. This is achieved by requiring that Green's functions calculated in the full ST and in the effective theory provide the same result up to terms suppressed by the ratio between the external momenta and μ_W. At the scale μ_W, QCD corrections are free of large logarithmic corrections and can therefore be evaluated in finite-order perturbation theory.

(2) Second, once the value of the Wilson coefficient at the electroweak scale has been obtained from the matching step, it is then necessary to obtain the value of the Wilson coefficients at the low-energy scale $\mu_b \sim m_b$. This can be achieved by solving the system of RGE satisfied by the Wilson coefficient. The RGE system has the following form:

$$\mu \frac{d}{d\mu} C_i(\mu) = \gamma_{ji}(\mu) C_j(\mu), \quad (132)$$

where the summation over j is implied. The matrix γ in Eq. (132) is the anomalous dimension matrix of the effective operators. The elements of the matrix have perturbative expansions in powers of α_s. Since the various operators mix under renormalization, this step of the calculation is called *mixing*. By solving the RGE, it is possible to resum the large logarithms of the ratio μ_W/μ_b to all orders in α_s in the Wilson coefficients.

(3) Finally, it is necessary to calculate on-shell matrix elements of the partonic process in the effective theory. QCD radiative corrections to the matrix elements do not include large logarithms, since the dependence on the heavy degrees of freedom is completely encoded within the Wilson coefficients.

Radiative decays of the B meson were first experimentally observed in the exclusive $B \to K^*\gamma$ decay mode by the CLEO Collaboration at Cornell in 1993. Nowadays, the branching ratio of the inclusive decay $\bar{B} \to X_s\gamma$ has been measured by several collaborations. The current world average obtained by averaging the CLEO, BELLE, and *BABAR* measurements (Asner *et al.*, 2010) is

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm exp}_{E_\gamma > E_0} = (3.55 \pm 0.24 \pm 0.09) \times 10^{-4}. \quad (133)$$

In Eq. (133), the first error is due to statistical and systematic uncertainty, while the second is due to the theoretical input on the b-quark Fermi motion. In order to eliminate irreducible backgrounds, experimental collaborations impose a lower cut on the photon energy. The value in Eq. (133) refers to a lower cut $E_0 = 1.6$ GeV.

The measurement in Eq. (133) has an experimental error of 7% and must be compared with an equally accurate theoretical prediction within the ST. In renormalization-group-improved perturbation theory, N^{m}LO QCD calculations of this process involve the resummation of $\alpha_s^n \ln^{n-m}(m_b^2/\mu_W^2)$ logarithms, as well as the evaluation of $\mathcal{O}(\alpha_s^m)$ corrections to the Wilson coefficients at the scale μ_W and to the matrix elements. In order to obtain theoretically reliable predictions and to match the current experimental accuracy, it was necessary to evaluate both the NLO (i.e., $m = 1$) and NNLO (i.e., $m = 2$) QCD corrections, as well as the NLO electroweak corrections [i.e., $\mathcal{O}(\alpha\alpha_s^n \ln^n(m_b^2/\mu_W^2))$].

The fascinating history of the calculation of the radiative corrections to the $\bar{B} \to X_s\gamma$ process was recently reviewed by

Buras (2011). The calculation of the LO QCD (i.e., $m = 0$) corrections in renormalization-group-improved perturbation theory was carried out in the period 1988–1993. An interesting technical feature of this calculation is that, in order to obtain the anomalous dimensions, one needs to evaluate two-loop Feynman diagrams already at LO QCD. Once these LO QCD corrections became available (Ciuchini *et al.*, 1993, 1994; Cella *et al.*, 1994a, 1994b), it was pointed out that their renormalization scale dependence is very large (Ali, Greub, and Mannel, 1993): by varying μ_b in the range $m_b/2 < \mu_b < 2m_b$, the predicted branching ratio changed by $\sim 60\%$. Consequently, the evaluation of the NLO QCD corrections was necessary (Buras *et al.*, 1994).

The evaluation of the NLO QCD corrections was a challenging task involving several groups in the calculation of the matching, mixing, and matrix elements; it was completed at the beginning of the last decade. Comprehensive reviews of this effort, along with complete lists of references to the contribution of the various groups, were written by Buras and Misiak (2002) and Hurth (2003). It is worth emphasizing that NLO determinations of the branching ratio include electroweak effects of $\mathcal{O}(\alpha)$ (Czarnecki and Marciano, 1998; Baranowski and Misiak, 2000; Gambino and Haisch, 2000).

While the calculation of the NLO QCD and electroweak corrections considerably reduces the scale dependence of the $\bar{B} \to X_s \gamma$ branching ratio in the ST, Gambino and Misiak (2001) pointed out that this calculation is affected by a $\sim 10\%$ theoretical uncertainty related to the choice of the charm quark mass in the two-loop matrix elements of the four-quark operators. Consequently, in order to reduce this uncertainty, an evaluation of the NNLO QCD corrections became necessary. A first estimate of the NNLO branching ratios, including all the numerically dominant effects, was completed by Misiak *et al.* (2007) and Misiak and Steinhauser (2007). Reviews of the NNLO calculation, including references to the contributions of various groups, can be found, for example, in Ferroglia (2008), Haisch (2008), and Misiak (2011).

One finds that the NLO QCD, NNLO QCD, and NLO electroweak contributions amount to approximately 30%, 10%, and 4% of the LO QCD result, respectively. The predicted value in the ST was found to be

$$\mathcal{B}(\bar{B} \to X_s \gamma)^{\mathrm{ST}}_{E_\gamma > 1.6\ \mathrm{GeV}} = (3.15 \pm 0.23) \times 10^{-4}, \tag{134}$$

which agrees with the world average of the experimental measurements within 1.2σ. The error on the theoretical estimate is about 7% and was obtained by combining four different uncertainties in quadrature: parametric uncertainty (3%), uncertainty due to missing higher-order corrections (3%), uncertainty due to nonperturbative corrections (5%), and uncertainty due to the m_c-interpolation ambiguity in the calculation of Misiak and Steinhauser (2007, 2010) (3%). The result in Eq. (134) is affected by a theoretical uncertainty which is approximately of the same magnitude as the experimental one. Additional perturbative NNLO corrections to the branching ratio were recently evaluated by Ewerth (2008), Asatrian *et al.* (2010), Ferroglia and Haisch (2010), and Misiak and Poradzinski (2011); although these corrections are not included in the calculation leading to Eq. (134), their numerical impact is expected to be marginal. Additional work within perturbation theory is still required to eliminate the m_c-interpolation ambiguity (Boughezal, Czakon, and Schutzmeier, 2007).

The current theoretical error is dominated by the uncertainty associated with nonperturbative effects, estimated to be about 5% (Misiak *et al.*, 2007). The nonperturbative uncertainty primarily arises from corrections of $\mathcal{O}(\alpha_s \Lambda_{\mathrm{QCD}}/m_b)$, which are difficult to evaluate; they were analyzed by Lee, Paz, and Neubert (2007) and Benzke *et al.* (2010).

New physics contributions to the partonic process can modify the matching conditions for the Wilson coefficients of the operators in the low-energy effective theory and can also induce new operators besides those already present in the ST. Therefore, the good agreement between the ST prediction and the measured value of the $\bar{B} \to X_s \gamma$ branching ratio sets strong constraints on the parameters of some new physics models. For example, an analysis of the decay within the type II two-Higgs-doublet model leads one to set a lower bound on the mass of the charged Higgs boson: $M_{H^\pm} > 295$ GeV at 95% confidence level (Misiak *et al.*, 2007).

R. Unstable particles

In the early 1990s, Sirlin (1991a) found that the conventional definitions of the mass and width of the Z^0 vector boson, namely,

$$M^2 = M_0^2 + \mathrm{Re}\, A(M^2), \tag{135}$$

$$M\Gamma = -\frac{\mathrm{Im}\, A(M^2)}{1 - \mathrm{Re}\, A'(M^2)}, \tag{136}$$

where M_0 is the bare mass, M is the on-shell mass, and $A(s)$ is the transverse self-energy, are gauge dependent in NNLO, i.e., at the two- and three-loop levels, respectively. By extension, analogous conclusions hold true for other unstable particles. This led to a serious theoretical problem because, in the context of gauge theories, a fundamental requirement is that physical observables should be gauge independent.

The original argument was based on the observation that the complex-valued position $\bar{s}$ of the propagator's pole must be gauge independent, since it is a singularity of the analytically extended S matrix. In the case of bosons, the inverse propagator is proportional to

$$\Pi(s) = s - M_0^2 - A(s), \tag{137}$$

where $s = q^2$ is the square of the four-momentum transfer. Thus, the pole position is

$$\bar{s} = M_0^2 + A(\bar{s}). \tag{138}$$

Writing $\bar{s} = m_2^2 - i m_2 \Gamma_2$, where m_2 and Γ_2 are real, gauge-independent parameters, one has

$$m_2^2 = M_0^2 + \mathrm{Re}\, A(\bar{s}), \tag{139}$$

$$m_2 \Gamma_2 = -\mathrm{Im}\, A(\bar{s}). \tag{140}$$

If one expands $A(\bar{s})$ about m_2^2 and retains only leading terms in Γ_2, Eqs. (139) and (140) lead back to Eqs. (135) and (136). On the other hand, if terms of higher order in Γ_2 are retained, the comparison of Eqs. (139) and (140) with

Eqs. (135) and (136) show that indeed M^2 and Γ are gauge dependent in higher orders. At the two-loop level, the analysis shows that the gauge dependence of M^2 occurs only in a restricted range of the gauge parameter ξ and, as a consequence, it is bounded. In fact, later Passera and Sirlin (1996) showed that the maximum shift in M due to the gauge dependence at the two-loop level is about 2 MeV. Although a small effect, it is of the same magnitude as the 2.1 MeV experimental error. However, at the three-loop level and higher, the gauge dependence is unbounded, so that M and Γ [cf. Eqs. (135) and (136)] are not only inconsistent with basic principles, but their numerical values depend in an arbitrary manner on the choice of ξ.

In fact, the comparison of the pole definitions of the mass and width (m_2, Γ_2) with the conventional ones (M, Γ) leads to the conclusion that the gauge dependences of the latter are numerically very large, particularly in the case of a heavy Higgs boson (Kniehl and Sirlin, 1998a, 1998b).

At this stage, it is instructive to point out the conceptual difference between the gauge-independent parameter m_2^2 and the gauge-dependent M^2. While m_2^2 is the real part of the zero of the inverse propagator, M^2 is the zero of the real part, an important difference.

In a second 1991 contribution, Sirlin (1991b) analyzed specific physical amplitudes and derived an independent proof of the need for additional higher-order gauge-dependent counterterms in Eq. (135), a result that gives additional support to the arguments and conclusions of the first paper.

It has also been emphasized that Eq. (136) leads to serious unphysical singularities if $A(s)$ is not analytic in the neighborhood of M^2. This occurs when M^2 is very close to a physical threshold, as discussed by Fleischer and Jegerlehner (1981), Bardin *et al.* (1991), Kniehl (1991, 1992a, 1992b, 1994), Bhattacharya and Willenbrock (1993), and Kniehl, Palisoc, and Sirlin (2000, 2002), or, in the resonance region, when the unstable particle is coupled to massless quanta, as in the cases of the W vector boson and unstable quarks. In particular, it was pointed out that the on-shell mass of an unstable quark has an unbounded gauge dependence of $\mathcal{O}(\alpha_s(\xi_g - 3)\Gamma)$, where ξ_g is the gluon gauge parameter and Γ is the width (Passera and Sirlin, 1998a, 1998b; Sirlin, 1999).

In order to solve the serious problems raised by the gauge dependence of M and Γ [cf. Eqs. (135) and (136)], Sirlin (1991a) proposed to define the mass and width of the Z^0 vector boson by means of the gauge-independent parameters

$$m_1 = \sqrt{m_2^2 + \Gamma_2^2}, \qquad \Gamma_1 = \frac{m_1 \Gamma_2}{m_2}. \tag{141}$$

As emphasized in the same work, the advantage of the (m_1, Γ_1) definitions relative to the (m_2, Γ_2) is that m_1 and Γ_1 can be identified with the Z^0 mass and width measured at LEP.

Formal proofs of the gauge independence of $\bar{s}$ and the gauge dependence of M and Γ, based on the Nielsen identities that describe the gauge dependence of Green's functions (Nielsen, 1975), have been presented by Gambino and Grassi (2000) and Grassi, Kniehl, and Sirlin (2001, 2002).

Applying the Nielsen identities to $\Pi(s, \xi_k)$ [cf. Eq. (137)], one finds

$$\frac{\partial}{\partial \xi_l}\Pi(s, \xi_k) = 2\Lambda_l(s, \xi_k)\Pi(s, \xi_k), \tag{142}$$

where we indicated explicitly the dependence on the gauge parameters ξ_k and $\Lambda_l(s, \xi_k)$ is a complex, amputated, one particle irreducible, two point Green's function of $\mathcal{O}(g^2)$ involving the gauge field, its Becchi-Rouet-Stora-Tyutin variation, and the gauge fermion.

As $\bar{s}$ is the zero of $\Pi(s, \xi_k)$, it follows that

$$\Pi(\bar{s}, \xi_k) = 0. \tag{143}$$

Differentiating Eq. (143) with respect to ξ_l:

$$\frac{\partial \bar{s}}{\partial \xi_l}\frac{\partial}{\partial \bar{s}}\Pi(\bar{s}, \xi_k) + \frac{\partial}{\partial \xi_l}\Pi(\bar{s}, \xi_k) = 0. \tag{144}$$

Equations (142) and (143) imply that the second term on the left-hand side (lhs) of Eq. (144) vanishes. As $(\partial/\partial\bar{s})\Pi(\bar{s}, \xi_k) = 1 + \mathcal{O}(g^2)$, Eq. (144) leads to

$$\frac{\partial \bar{s}}{\partial \xi_l} = 0, \tag{145}$$

which expresses the gauge independence of $\bar{s}$. It is important to note that this conclusion is valid to all orders in perturbation theory.

Instead, taking the real part of Eq. (142):

$$\frac{\partial}{\partial \xi_l}\mathrm{Re}\Pi(s, \xi_k) = 2[\mathrm{Re}\,\Lambda_l(s, \xi_k)\mathrm{Re}\,\Pi(s, \xi_k) - \mathrm{Im}\,\Lambda_l(s, \xi_k)\mathrm{Im}\,\Pi(s, \xi_k)]. \tag{146}$$

Recalling that the on-shell M^2 is the zero of the $\mathrm{Re}\,\Pi(s, \xi_k)$, it follows that

$$\mathrm{Re}\,\Pi(M^2, \xi_k) = 0. \tag{147}$$

Differentiating Eq. (147) with respect to ξ_l and using Eqs. (146) and (147), one obtains

$$\frac{\partial M^2}{\partial \xi_l}\mathrm{Re}\Pi'(M^2, \xi_k) - 2\mathrm{Im}\,\Lambda_l(M^2, \xi_k)\mathrm{Im}\,\Pi(M^2, \xi_k) = 0, \tag{148}$$

where the prime stands for a derivative with respect to M^2. Noting that $\mathrm{Re}\,\Pi'(M^2, \xi_k) = \mathcal{O}(1)$ and that both $\mathrm{Im}\,\Lambda_l(M^2, \xi_k)$ and $\mathrm{Im}(M^2, \xi_k)$ are $\mathcal{O}(g^2)$, Eq. (148) implies that $\partial M^2/\partial \xi_l = \mathcal{O}(g^4)$. Thus, M^2 is gauge dependent at the two-loop level, i.e., in NNLO, the same conclusion reached by Sirlin (1991a, 1991b).

Similarly, for the conventional expression of the width [cf. Eq. (136)], a somewhat lengthier derivation leads in leading order to

$$\frac{d}{d\xi_l}\frac{\mathrm{Im}\,\Pi(M^2, \xi_k)}{\mathrm{Re}\,\Pi'(M^2, \xi_k)} = 2\{\mathrm{Im}\,\Lambda_l(M^2, \xi_k)[\mathrm{Im}\,\Pi(M^2, \xi_k)]^2\}' + \mathcal{O}(g^8), \tag{149}$$

where $d/d\xi_l$ stands for the total derivative with respect to ξ_l. Since $\mathrm{Im}\,\Lambda_l$ and $\mathrm{Im}\,\Pi$ are both of $\mathcal{O}(g^2)$, Eq. (149) implies that Eq. (136) is gauge dependent in $\mathcal{O}(g^6)$, i.e., in NNLO, in agreement with the earlier conclusions (Sirlin, 1991a, 1991b).

S. Renormalization of the Cabibbo-Kobayashi-Maskawa matrix

An important problem associated with the CKM matrix (Cabibbo, 1963; Kobayashi and Maskawa, 1973) is its renormalization. An early analysis (Marciano and Sirlin, 1975b) focused on the renormalization of UV divergences in the two-generation case. Since the CKM matrix is one of the fundamental cornerstones of the weak interactions and, by extension, of the ST, it is important to develop renormalization schemes that treat both the finite and divergent contributions with well-defined renormalization conditions. Over the last two decades several papers addressed this basic problem at various levels of generality and complexity (Denner and Sack, 1990; Kniehl and Pilaftsis, 1996; Gambino, Grassi, and Madricardo, 1999; Barroso, Brucher, and Santos, 2000; Kniehl, Madricardo, and Steinhauser, 2000; Diener and Kniehl, 2001; Yamada, 2001; Espriu, Manzano, and Talavera, 2002; Pilaftsis, 2002; Zhou, 2003, 2004; Denner, Kraus, and Roth, 2004; Liao, 2004; Kniehl and Sirlin, 2006a, 2006b, 2009; Almasy, Kniehl, and Sirlin, 2009).

The main difficulties in the CKM renormalization arise from external-leg mixing self-energy corrections. For instance, for an outgoing quark, these EWC are of the form

$$\Delta\mathcal{M}_{ii'}^{\mathrm{leg}} = \bar{u}_i(p)\Sigma_{ii'}(\not{p})\frac{1}{\not{p}-m_{i'}}, \tag{150}$$

where i denotes the outgoing quark of momentum p and mass m_i, i' the virtual quark of mass $m_{i'}$, and $\Sigma_{ii'}(\not{p})$ the self-energy.

In the following, we outline the strategies followed in two of the most recently proposed on-shell schemes to renormalize the CKM matrix at the one-loop level.

(A) Using a simple procedure based on Dirac algebra, Kniehl and Sirlin (2006a, 2006b) separated the contributions to $\Sigma_{ii'}(\not{p})/(\not{p}-m_{i'})$ into two classes: (1) gauge-independent self-mass (sm) contributions proportional to $(\not{p}-m_{i'})^{-1}$ with a cofactor that involves the chiral projectors $a_\pm = (1\pm\gamma_5)/2$, but not $\not{p}$; (2) gauge-dependent wave-function renormalization (wfr) contributions in which the virtual quark propagator $(\not{p}-m_{i'})^{-1}$ has been canceled. Furthermore, using the unitarity relation $V_{lm}V_{mn}^\dagger = \delta_{ln}$ satisfied by the CKM matrix elements V_{lm}, one finds that the wfr have an important property: all the gauge-dependent and all the UV-divergent wfr contributions to the physical amplitude $W\to q_i\bar{q}_j$ depend only on an overall factor V_{ij} and the external quark masses m_i and m_j, a property shared by the one-loop proper vertex contributions. This leads to the cancellation of the gauge dependence and UV divergence of the wfr contributions to $W\to q_i+\bar{q}_j$ with those arising from the one-loop vertex corrections, exactly as in the unmixed, single generation case.

The renormalization of the sm contributions is implemented using the mass counterterms

$$\bar{\psi}^Q(\delta m^{Q(+)}a_+ + \delta m^{Q(-)}a_-)\psi^Q \quad (Q=U,D), \tag{151}$$

where U (D) stands for the up (down) quarks, and $\delta m^{Q(\pm)}$ are nondiagonal matrices subject to the Hermiticity condition $\delta m^{Q(+)} = \delta m^{Q(-)\dagger}$.

The UV-divergent sm contributions obey the Hermiticity condition, so they can be canceled by the $\delta m^{Q(\pm)}$ in all ii' channels. However, this is not the case for some of the finite parts. For this reason, they used a specific renormalization prescription: the $\delta m^{Q(\pm)}$ were adjusted to cancel the full sm contributions in all diagonal ($i=i'$) channels, as well as the uc, ut, and ct channels for the U quarks and the sd, bd, and bs channels for the D quarks. This implies that there are residual sm contributions in the reverse cu, tu, tc, ds, db, and sb channels, but they are finite, gauge independent, and very small. In fact, since these residual sm contributions converge in the limit $m_{i'}\to m_i$, they may be regarded as additional finite and gauge-independent contributions to wfr that happen to be small. An attractive feature of this renormalization prescription is that the external-leg sm contributions are fully canceled when the external particle is a u, d, $\bar{u}$, or $\bar{d}$ quark, a useful property since V_{ud} is by far the most precisely determined CKM matrix element [cf. Eq. (90)].

The renormalization procedure outlined presents interesting similarities with the approach followed by Feynman (1949, 1962) in QED. Thus, it may be regarded as a generalization of Feynman's approach to the case in which the self-energy $\Sigma_{ii'}(\not{p})$ contains nondiagonal as well as diagonal components.

In the same work, Kniehl and Sirlin (2006a, 2006b) showed that an equivalent and interesting formulation of the same renormalization scheme is obtained by diagonalizing the complete mass matrix $m - \delta m^{Q(+)}a_+ - \delta m^{Q(-)}a_-$ (m is the diagonal, renormalized mass matrix) by biunitarity transformations acting on the up and down quark spaces. This procedure generates an explicit CKM counterterm matrix δV, which automatically satisfies the following important properties: it is gauge independent, preserves unitarity in the sense that both the renormalized and bare CKM matrices V and $V_0 = V - \delta V$ are unitary at the one-loop level, and leads to renormalized amplitudes that are nonsingular in the limit in which any two quarks become mass degenerate. In this alternative formulation, the off-diagonal UV-divergent sm contributions are canceled by δV while, as usual, the diagonal sm contributions are canceled by the mass counterterms that are also diagonal.

The renormalization scheme outlined has been generalized to the case of an extended lepton sector that includes Dirac and Majorana neutrinos in the framework of the seesaw mechanism (Almasy, Kniehl, and Sirlin, 2009).

(B) A second on-shell renormalization scheme (Kniehl and Sirlin, 2009) is based on explicit mass counterterm matrices

$$\delta m_{ii'}^Q = \delta m_{ii'}^{Q(+)}a_+ + \delta m_{ii'}^{Q(-)}a_- \quad (Q=U,D), \tag{152}$$

where $\delta m_{ii'}^{Q(+)}$ and $m_{ii'}^{Q(-)}$ are defined in terms of the Lorentz-invariant self-energy functions and obey two important properties: (i) they are gauge independent, and (ii) they automatically satisfy the Hermiticity condition $\delta m_{ii'}^{Q(+)} = m_{ii'}^{Q(-)\dagger}$ of the mass matrix. The second property implies that they can be applied directly to all diagonal and off-diagonal amplitudes and, in this sense, they are "flavor democratic" since they do not single out particular flavor channels. As in the case of scheme (A), diagonalization of the complete mass matrices leads to a gauge-independent CKM counterterm matrix δV that preserves unitarity and now satisfies another highly desirable theoretical property, namely, "flavor democracy."

T. *S*, *T*, and *U* parameters

"New physics," i.e., physics beyond the ST, may contribute to EWC. If the new physics is associated with a high-mass scale and contributes mainly to the self-energies, the idea has been proposed to parametrize its contributions in terms of three amplitudes S, T, and U introduced by Peskin and Takeuchi (1990); see also Lynn, Peskin, and Stuart (1986), Holdom and Terning (1990), Kennedy and Langacker (1990, 1991), Altarelli and Barbieri (1991), Golden and Randall (1991), and Peskin and Takeuchi (1992).

In the $\overline{\rm MS}$ scheme we have (Marciano and Rosner, 1990; Marciano, 1991, 1995; Sirlin, 1993, 1994a)

$$\Delta\hat{r} = (\Delta\hat{r})_{\rm ST} + \frac{\hat{\alpha}}{4\hat{s}^2\hat{c}^2} S_Z - \hat{\alpha}T, \tag{153}$$

$$\Delta\hat{r}_W = (\Delta\hat{r}_W)_{\rm ST} + \frac{\hat{\alpha}}{4\hat{s}^2} S_W, \tag{154}$$

$$\frac{\hat{\alpha}}{4\hat{s}^2\hat{c}^2} S_Z = \left[\frac{A_{ZZ}(M_Z^2) - A_{ZZ}(0)}{M_Z^2}\right]_{\overline{\rm MS}}^{\rm new}, \tag{155}$$

$$\frac{\hat{\alpha}}{4\hat{s}^2} S_W = \left[\frac{A_{WW}(M_W^2) - A_{WW}(0)}{M_W^2}\right]_{\overline{\rm MS}}^{\rm new}, \tag{156}$$

$$\hat{\alpha}T = \left[\frac{A_{WW}(0)}{M_W^2} - \frac{A_{ZZ}(0)}{M_Z^2}\right]_{\overline{\rm MS}}^{\rm new}. \tag{157}$$

In Eqs. (155)–(157), the A functions are the unrenormalized self-energies defined according to the conventions of Marciano and Sirlin (1980), $\overline{\rm MS}$ means that the $\overline{\rm MS}$ renormalization has been implemented and $\mu = M_Z$ chosen, and "new" denotes new physics contributions. In Eqs. (155) and (156), we applied the $\overline{\rm MS}$ renormalization prescription for $\hat{\alpha}(M_Z)$ and $\sin^2\hat{\theta}_W(M_Z)$ proposed by Marciano and Rosner (1990), which excludes new-heavy-physics contributions in $A_{\gamma\gamma}^{\rm new}(q^2)$ and $A_{\gamma Z}^{\rm new}(q^2)$. Consequently, these two self-energies are not included in the definitions of S_Z and S_W. We recall that $\hat{s}^2 = 1 - \hat{c}^2 = \sin^2\hat{\theta}_W(M_Z)$ is the $\overline{\rm MS}$ electroweak mixing parameter evaluated at the scale $\mu = M_Z$, $\Delta\hat{r}$ and $\Delta\hat{r}_W$ are the EWC in Eqs. (57) and (58), respectively (cf. Sec. III.D), while $(\Delta\hat{r})_{\rm ST}$ and $(\Delta\hat{r}_W)_{\rm ST}$ are their values in the ST. $\hat{\alpha}$ is the $\overline{\rm MS}$ fine structure constant at $\mu = M_Z$ (cf. Sec. III.E).

Alternatively, one defines $S \equiv S_Z$, $U \equiv S_W - S_Z$. T and U are primarily sensitive to isodoublet mass splittings (generally, $U \ll T$), while S probes contributions from mass-degenerate fermion doublets.

In conjunction with Eqs. (57) and (58), the modifications of $\Delta\hat{r}$ and $\Delta\hat{r}_W$ displayed in Eqs. (153) and (154), induce the linear shifts

$$\hat{s}^2 = (\hat{s}^2)_{\rm ST} + \frac{\hat{\alpha}}{4(\hat{c}^2 - \hat{s}^2)}[S - 4\hat{s}^2\hat{c}^2 T], \tag{158}$$

$$M_W = (M_W)_{\rm ST}\left[1 + \frac{\hat{\alpha}\hat{c}^2}{2(\hat{c}^2 - \hat{s}^2)}T + \frac{\hat{\alpha}}{8\hat{s}^2}U - \frac{\hat{\alpha}}{4(\hat{c}^2 - \hat{s}^2)}S\right], \tag{159}$$

where EWC of $\mathcal{O}(\alpha^2)$ have been neglected.

Combining Eqs. (158) and (159), one can solve for S_W:

$$\frac{\hat{\alpha}}{4\hat{s}^2} S_W = 2B + C, \tag{160}$$

where

$$B = \frac{M_W}{(M_W)_{\rm ST}} - 1, \qquad C = \frac{\hat{s}^2}{(\hat{s}^2)_{\rm ST}} - 1. \tag{161}$$

In the case $U = 0$, i.e., $S_W = S_Z \equiv S$, we also have

$$\hat{\alpha}\hat{c}^2 T = 2[B + (\hat{s}^2)_{\rm ST} C], \tag{162}$$

where we neglected a second-order term $C^2[(\hat{s}^2)_{\rm ST}/\hat{s}^2]$ on the rhs of the equation. In Eqs. (158)–(162), $(M_W)_{\rm ST}$ and $(\hat{s}^2)_{\rm ST}$ are calculated using the EWC of the ST (cf. Sec. III.N) and a chosen reference value for M_H, while M_W is identified with the measured mass of the W boson. In turn, $\hat{s}^2$ is evaluated using the experimental value of $\sin^2\theta_{\rm eff}^{\rm lept}$, obtained from the Z-pole asymmetries and applying Eq. (65).

In order to obtain the dependence of the neutral current observables on S and T, one expresses the corresponding amplitudes in terms of G_F and the ST EWC evaluated at the chosen reference value for M_H, multiplies them by $\rho(0)^{\rm new} = 1 + \hat{\alpha}T$, and inserts the expression for $\hat{s}^2$ given in Eq. (158). In particular, the weak charge $Q_W({\rm Cs})$, measured in atomic parity violation, is insensitive to T and thus provides a direct probe of S (Marciano and Rosner, 1990; Marciano, 1991, 1995).

A recent global analysis (Baak *et al.*, 2012) employs the reference values $M_{H,\rm ref} = 120$ GeV and $M_{t,\rm ref} = 173$ GeV and obtains

$$S = 0.04 \pm 0.10, \qquad T = 0.05 \pm 0.11, \qquad U = 0.08 \pm 0.11, \tag{163}$$

while, assuming $U = 0$, the results are

$$S|_{U=0} = 0.07 \pm 0.09, \quad T|_{U=0} = 0.10 \pm 0.08. \tag{164}$$

We see that the results in Eq. (163) are in good agreement with the ST predictions $S = T = U = 0$. By comparison, a fourth generation of mass-degenerate fermions leads to $S = 4/6\pi \approx 0.21$ [cf. Bertolini and Sirlin (1984)], while technicolor models roughly contribute $S \approx (0.05\text{–}0.10)N_T N_D + 0.12$, where N_T and N_D are the number of technicolors and isodoublets, respectively (Marciano, 1995). Therefore, for one generation with $N_T = N_D = 4$ one expects $S \approx 0.9\text{–}1.7$, values significantly larger than the S result shown in Eq. (163). Bertolini and Sirlin (1991) showed that Majorana neutrinos can give large negative contributions to the T parameter, i.e., of opposite sign to those from the top quark and Dirac neutrinos.

An alternative formulation of the S, T, U analysis is based on the ϵ_i ($i = 1, 2, 3, b$) parameters, defined in terms of the physical quantities M_W, Γ_l, $A_{\rm FB}^{(l)}$, and $\Gamma_{b\bar{b}}$ (Altarelli and Barbieri, 1991; Altarelli *et al.*, 1992; Altarelli, Barbieri, and Caravaglios, 1993a, 1993b; Altarelli, 1994).

The applications of the S, T, and U formalism focused mainly on new physics fermionic contributions to the self-energies. On the other hand, new physics bosonic contributions are also of general interest. However, this poses a theoretical problem: as pointed out by Degrassi, Kniehl, and Sirlin (1993), in contrast with the fermionic case, the bosonic contributions to the S, T, and U parameters, defined in terms of the conventional self-energies [cf. Eqs. (155)–(157)], are gauge dependent in the ST; furthermore, T and U are divergent unless a constraint is imposed among the gauge parameters. It is natural to expect that the same theoretical problems arise in the bosonic new physics contributions. In order to circumvent this problem, Degrassi, Kniehl, and Sirlin (1993) proposed to replace the conventional self-energies in Eqs. (155)–(157) by their pinch-technique counterparts (Cornwall, 1981, 1982; Cornwall and Papavassiliou, 1989; Papavassiliou, 1990; Degrassi and Sirlin, 1992), which are gauge independent. Thus, this modification leads to a gauge-independent formulation of S, T, and U in the bosonic sector.

U. Supersymmetry

In Sec. III.T we pointed out that a recent global fit leads to values of the S, T, and U parameters that are in good agreement with the ST expectations $S = T = U = 0$. Thus, at present, the analysis of the precision electroweak data does not lead to clear signals of new physics beyond the ST.

However, there are powerful theoretical arguments that strongly suggest the presence of new physics. The most obvious one is that the ST does not incorporate gravity, one of the fundamental forces of nature. In fact, the unification of gravity with the ST, in particular, and quantum mechanics in general, is one of the most important unsolved problems in theoretical particle physics. At present, there is a widespread belief among theorists that string theory provides the most hopeful framework to achieve this major goal. On the other hand, string theory leads to a landscape with an enormous number of possible vacua (Bousso and Polchinski, 2000; Susskind, 2003), without clear selection criteria, except perhaps for anthropic arguments.

Another powerful argument, involving RC, is the Higgs boson mass hierarchy problem. This involves the important fact that the RC to M_H^2 are quadratically divergent. Thus, the relation of the physical, renormalized mass M_H, and the bare mass M_H^0, is of the form

$$M_H^2 = (M_H^0)^2 + O(\lambda, g^2, h_F^2)\Lambda^2 + \cdots, \tag{165}$$

where g is the $\mathrm{SU}(2)_L$ gauge couplings, λ is the quartic Higgs self-coupling, $h_F = m_f/v$, m_f is the mass of fermion f, $v = (1/\sqrt{2}G_F)^{1/2} = 246$ GeV is the vacuum expectation value of the Higgs field, and Λ is the cutoff introduced to regularize the UV divergence. The second term in Eq. (165) is the quadratically divergent part of the self-mass RC and the ellipsis represents $\ln\Lambda$ contributions as well as finite terms.[10]

Within the ST, the presence of the $\mathcal{O}(\Lambda^2, \ln\Lambda) + \cdots$ terms on the rhs of Eq. (165) does not cause difficulties: as in all renormalizable theories, they are canceled by the divergent part of the mass counterterm $-\delta M^2 = (M_H^0)^2 - M_H^2$. In such an approach, $(M_H^0)^2$ and the RC are regarded as unobservable quantities and only M_H^2 has a physical meaning. However, if we assume that the ST is embedded in a larger theory that cuts off the momentum integral in the RC at its own finite scale, Λ acquires a physical meaning. Specifically, Λ in Eq. (165) is then identified with the scale of the new physics. For example, if the new physics beyond the ST is gravity, Λ is identified with the Planck mass $\Lambda = M_P = G_N^{-1/2} = 1.2221 \times 10^{19}$ GeV, where G_N is Newton's gravitational constant.

To illustrate the effect of these considerations on Eq. (165), we consider a leading quadratically divergent contribution to the RC arising from the diagram $H \to \text{top loop} \to H$. Employing Eq. (8.6) in Langacker (2010) with $m_t = 173.2$ GeV and $\nu = v = 246$ GeV, we find that this diagram contributes $\approx -3.8 \times 10^{-2}\Lambda^2$. Using the gravity scale, $\Lambda = 1.221 \times 10^{19}$ GeV, one obtains a RC $\approx -5.6 \times 10^{36}$ GeV^2. Since, in absolute value, this is enormously larger than the expected value of M_H^2, there must be an extraordinarily fine-tuned cancellation between $(M_H^0)^2$ and the RC. As an illustration, if we assume $M_H = 125$ GeV, the level of the required fine-tuning is

$$\frac{(M_H^0)^2 - 5.6 \times 10^{36}\ \mathrm{GeV}^2}{5.6 \times 10^{36}\ \mathrm{GeV}^2} = \frac{(125)^2}{5.6 \times 10^{36}} = 2.8 \times 10^{-33},$$

namely 3 parts in 10^{33}. Such fine-tuning is generally regarded as unnatural. On the other hand, if we demand a relatively small level of fine-tuning, the same RC employed before leads to a value of M_H rather close to Λ. For example, if we assume that the level of fine-tuning is 10%, we have $M_H = 0.75 \times 10^{18}$ GeV $\approx M_P/16$. This is usually referred to as the hierarchy problem. Namely, assuming a relatively small level of fine-tuning, the quadratically divergent RC push the value of M_H from the electroweak scale to a value within an order of magnitude of the gravitational scale.

The same problems occur when one considers the RC to the vacuum expectation value of the Higgs field $\langle 0|H|0\rangle$:

$$v^2 = v_0^2 + O(\lambda, g^2, h_F^2)\Lambda^2 + \cdots, \tag{166}$$

where $v = 246$ GeV defines the electroweak scale. Again, if $\Lambda = M_P$, very large RC emerge, so that an unnaturally fine-tuned cancellation between v_0 and the RC must take place. On the other hand, if one demands a relatively small level of fine-tuning, the value of the weak scale v moves close to M_P.

[10]Rules of correspondence between the poles' positions in dimensional regularization and UV cutoffs in four-dimensional calculations with L loops were stated by Veltman (1981) for quadratic divergences, and derived by Ossola and Sirlin (2003), on the basis of a heuristic argument, for quadratic and higher divergences.

One frequently invoked solution of the Higgs boson mass hierarchy problem is TeV scale supersymmetry. As is well known, this theory, based on elegant symmetry principles, postulates that every fermion (boson) particle has boson (fermion) supersymmetric partners with the same quantum numbers and masses. Since mass-degenerate partners of known particles have not been found, it is clear that in nature supersymmetry is broken.

An important property of supersymmetry is that the quadratically divergent RC to M_H^2 arising from the fermion and boson loops cancel each other, leaving only much smaller supersymmetry-breaking contributions. Thus, if TeV scale supersymmetry is an approximate symmetry of nature, such cancellation would provide an elegant solution to the Higgs boson mass hierarchy problem, based on fundamental symmetry principles.

In fact, over the last several years, supersymmetric scenarios such as the minimal-supersymmetric standard model (MSSM) have emerged as leading candidates for theoretical frameworks beyond the ST. It involves five Higgs bosons: two neutral CP-even scalars h and H ($M_h < M_H$), one neutral CP-odd pseudoscalar A, and one charged pair $H^\pm$.

An interesting property is that, at the tree level, $M_h \leq M_Z$, which is ruled out by direct searches at 95% C.L. This is also in contrast with the ST, where there is no tree-level upper limit on M_H, except for perturbativity and unitarity bounds. On the other hand, for large stop masses, there are sizable RC, dominated by top and stop loops, that significantly increase the upper bound for M_h. At present, the analysis yields $M_h \lesssim$ 135 GeV (Haber, 2010). Thus, we see that RC indeed play a crucial role in ensuring the phenomenological consistency of the MSSM.

In the MSSM, supersymmetric contributions decouple if the superpartners' masses are much larger than M_Z. In that regime, the fits are of the same general quality as in the case of the ST. If some of them are of $\mathcal{O}(M_Z)$, the fits are worse, leading to constraints in the MSSM parameter space.

Another important result of supersymmetry is that the unification of gauge couplings is much more successful when they are extrapolated using the MSSM β functions, with the couplings intersecting at $M_{\rm GUT} \sim 3 \times 10^{16}$ GeV, than when employing the ST β functions. On the other hand, at present the agreement is not perfect: using $\hat{\alpha}(M_Z)$ and $\sin^2\hat{\theta}_W(M_Z)$ as inputs, one finds the prediction $\alpha_s(M_Z) \approx 0.13$, which is slightly larger than the observed value ≈ 0.12.

As discussed in Sec. III.O, the possible contribution of supersymmetric partners of low mass may provide a natural explanation for the $\sim 3.5\sigma$ discrepancy between the experimental and ST values of $a_\mu = (g_\mu - 2)/2$.

Notwithstanding the impressive successes of supersymmetry, it is important to remember that the existence of supersymmetric partners, its most direct and compelling prediction, has not been established so far.

It is also important to note that a much more egregious hierarchy problem emerges in the analysis of the cosmological constant $\Lambda_c = 8\pi G_N \rho$, where ρ is the vacuum energy density of the Universe. Assuming that the observed acceleration of the Universe is due to Λ_c, the observed vacuum energy density is $\rho = \mathcal{O}(10^{-47}\ \text{GeV}^4)$, while estimates of the contribution to ρ of elementary particles range roughly from $\mathcal{O}(\text{TeV}^4)$ in TeV supersymmetry to $\mathcal{O}((10^{19}\ \text{GeV})^4) = \mathcal{O}(10^{76}\ \text{GeV}^4)$ if the UV cutoff in the quartically divergent integrals is identified with M_P. Thus, there is mismatch of roughly 59 to 123 orders of magnitude between the estimates of Λ_c from particle physics and the observed value. This implies that a cancellation between the bare cosmological constant Λ_c^0 and the contributions from elementary particles would require an extremely large and unnatural level of fine-tuning. At the moment, it seems that there are no compelling explanations for the observed value of Λ_c, based on fundamental principles. In their absence, anthropic arguments are sometimes invoked: namely, the value of Λ_c should be in the relatively small range that allows the formation of galaxies, a crucial requirement for the existence of life itself (Weinberg, 1989). Such anthropic arguments may serve, for example, as a selection criterion to choose among the multitude of vacua in the string landscape. Nonetheless, if a more fundamental explanation of the observed value of Λ_c is not found, it seems clear that the requirement of natural fine-tuning faces a great challenge in the Λ_c hierarchy problem.

ACKNOWLEDGMENTS

The authors are indebted to W. J. Marciano and M. Passera for interesting observations and valuable information. They also thank Fern Simes for her help in the preparation of the manuscript. The work of A. S. was supported in part by the National Science Foundation Grant No. PHY-0758032. The work of A. F. was supported in part by the PSC-CUNY Award No. 64133-00 42 and by the National Science Foundation Grant No. PHY-1068317.

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