Dual pairing of symmetry and dynamical groups in physics

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This article reviews many manifestations and applications of dual representations of pairs of groups primarily in atomic and nuclear physics. Examples are given to show how such paired representations are powerful aids in understanding the dynamics associated with shell-model coupling schemes and in identifying the physical situations for which a given scheme is most appropriate. In particular, they suggest model Hamiltonians that are diagonal in the various coupling schemes. The dual pairing of group representations has been applied profitably in mathematics to the study of invariant theory. Parallel applications to the theory of symmetry and dynamical groups in physics are shown to be equally valuable. In particular, the pairing of the representations of a discrete group with those of a continuous Lie group or those of a compact Lie group with those of a noncompact Lie group makes it possible to infer many properties of difficult groups from those of simpler groups. This review starts with the representations of the symmetric and unitary groups, which are used extensively in the many-particle quantum mechanics of bosonic and fermionic systems. It gives a summary of the many solutions and computational techniques for solving problems that arise in applications of symmetry methods in physics and which result from the famous Schur-Weyl duality theorem for the pairing of these representations. It continues to examine many chains of symmetry groups and dual chains of dynamical groups associated with several coupling schemes in atomic and nuclear shell models and the valuable insights and applications that result.

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CONTENTS

I. INTRODUCTION

The value of exploiting the symmetries and algebraic structures of a physical system in attempting to understand its properties is nowadays much more widely appreciated than when distinguished physicists were making disparaging remarks about *das Gruppenpest*. In fact, it has become evident that group theory and the related theory of Lie algebras underlie quantum mechanics and provide the essential language for the interpretation of physical phenomena in quantum mechanical terms.

A common strategy in seeking to understand a physical system is to start by accumulating a large body of data that relates to the phenomena of interest and examining it from many perspectives until it falls into recognizable patterns. The second step is to interpret the patterns in terms of a phenomenological model; this provides some predictive capability and facilitates the search for new data that can be used to refine and develop the model. The challenge is then to understand the model, and hence the data it describes, in terms of a fundamental theory of the system.

The problems faced in each of these steps are tailor-made for applications of group theory, the study of symmetries and algebraic structures, which provides a natural language for describing the properties of physical systems and the relationships between physical and mathematical models of such systems. It is nowadays well recognized that solvable models invariably have simple algebraic structures; it is why they are solvable. Moreover, the challenge of mapping a phenomenological algebraic model into a much grander algebraic scheme, such as that of many-particle quantum mechanics, is appropriately viewed as a problem in group representation theory.

A central problem considered in this review is to take a successful phenomenological model of some subdynamics of a many-particle system and give it a microscopic interpretation by identifying it with a submodel of many-particle quantum mechanics. Another is to derive simple phenomenological models that exhibit the dynamics associated with particular coupling schemes for the microscopic theory. It turns out that the algebraic methods of group theory and, in particular, the complementary concepts of symmetry groups and dynamical groups provide the basic tools needed for these objectives.

A. Algebraic models in quantum mechanics

In quantum mechanics, the observables of a model are represented by Hermitian linear operators on a Hilbert space. In this review, we focus on algebraic models for which there exist a basic set of observables that span a finite-dimensional Lie algebra that we denote by $\mathfrak{g}$. Thus, if $\hat{X}$ and $\hat{Y}$ are operators in $\mathfrak{g}$ representing model observables and $\hat{Z}$ is defined by the commutation relation

$$[\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X} = \mathrm{i}\hat{Z}, \tag{1}$$

then $\hat{Z}$ is also an element of $\mathfrak{g}$. (Note that the factor $\mathrm{i} = \sqrt{-1}$ is needed in quantum mechanics because the commutator $[\hat{X}, \hat{Y}]$ is skew Hermitian when $\hat{X}$ and $\hat{Y}$ are Hermitian.) Other observables for the model are now given by Hermitian polynomials in the elements of $\mathfrak{g}$ with commutation relations inferred from the identity

$$[A, BC] = [A, B]C + B[A, C]. \tag{2}$$

We then say that the full algebra of observables for such a model is finitely generated, i.e., it is generated by the finite-dimensional subalgebra $\mathfrak{g}$.

Now observe that many-particle quantum mechanics is an algebraic model with a finitely generated Lie algebra of observables. For example, the position and momentum observables $\{x_{ni}, p_{ni}; n = 1, \ldots, N, i = 1, 2, 3\}$ of an N-particle system of spinless particles in three-space $\mathbb{R}^3$ are represented in quantum mechanics as operators $\{\hat{x}_{ni}, \hat{p}_{ni}; n = 1, \ldots, N, i = 1, 2, 3\}$ on a Hilbert space $\mathbb{H}$ of normalizable wave functions according to the basic equations of quantum mechanics

$$\hat{x}_{nj}\Psi(x) = x_{nj}\Psi(x), \qquad \hat{p}_{nj}\Psi(x) = -\mathrm{i}\hbar\frac{\partial}{\partial x_{nj}}\Psi(x). \tag{3}$$

These basic observables obey commutation relations

$$\begin{aligned}&[\hat{x}_{nj}, \hat{x}_{mk}] = [\hat{p}_{nj}, \hat{p}_{mk}] = 0,\\ &[\hat{x}_{nj}, \hat{p}_{mk}] = \mathrm{i}\hbar\delta_{n,m}\delta_{j,k},\end{aligned} \tag{4}$$

which close on a Heisenberg-Weyl Lie algebra. The Hilbert space of the N-particle system then carries a unitary irreducible representation (irrep) of this algebra, i.e., an irrep in which $\hat{x}_{ni}$ and $\hat{p}_{ni}$ are Hermitian operators. For a system of many particles with intrinsic spin, it is necessary to augment this algebra by the addition of suitable spin operators.

B. Symmetry groups and dynamical groups

A *symmetry group of a system* is, by definition, a group of transformations of the system that leave its Hamiltonian invariant. For example, a symmetry group for a system with a Hamiltonian that is rotationally invariant is the rotation group SO(3) [or SU(2) if particles with intrinsic spin are involved]. A given Hamiltonian may have more than one symmetry group and a given system may have many possible

choices of Hamiltonian. Thus, a system may have several possible symmetry groups. Note that by a "system" we mean any self-contained model or theory of a physical system; thus different models of a physical system are regarded as distinct systems.

Variations on the definition of a dynamical group for a quantum mechanical system have been given, for example, by Iachello and Arima (1987), Bohm, Ne'eman, and Barut (1988), Isacker (1999), and Wulfman (2011). We adopt the definition that *a dynamical group for a system* is a Lie group of transformations of the Hilbert space $\mathbb{H}$ of the system such that $\mathbb{H}$ carries an irreducible unitary representation of the dynamical group. Essentially this ensures that the dynamical group is rich enough to relate all parts of the system. This is appropriate because if a system has parts that are not related in some way, then the parts are more usefully regarded as belonging to distinct systems.

It is also useful to define *a dynamical group for a Hamiltonian* $\hat{H}$ of a system to be a Lie group of transformations of the Hilbert space $\mathbb{H}$ of the system such that $\mathbb{H}$ is a sum of irreducible unitary representations of this group with each irreducible subspace being spanned by eigenstates of $\hat{H}$. The irreps of the dynamical group for $\hat{H}$ then describe the states of the system but fail to describe the relationships between states belonging to different irreps; for this one needs the full dynamical group of the system.

Consider, for example, a two-level BCS model of superconductivity (Bardeen, Cooper, and Schrieffer, 1957) with a Hamiltonian

$$\hat{H} = \sum_{k=1}^{2} \varepsilon_k \hat{S}_0^k - \sum_{i,k=1}^{2} g_{ik} \hat{S}_+^i \hat{S}_-^k, \tag{5}$$

where for each $k=1, 2$, $\hat{S}_\pm^k = \hat{S}_x^k \pm \mathrm{i}\hat{S}_y^k$, $\hat{S}_0^k = \hat{S}_z^k$, and $\{\hat{S}_x^k, \hat{S}_y^k, \hat{S}_z^k\}$ are basis elements of an $\mathrm{su}(2)^k$ (so-called quasispin) Lie algebra. When the model Hilbert space is the space of a single irreducible $\mathrm{SU}(2)^1 \times \mathrm{SU}(2)^2$ representation, then the group $G = \mathrm{SU}(2)^1 \times \mathrm{SU}(2)^2$ is a dynamical group for both the model and the Hamiltonian. However, when the Hilbert space for the model carries a sum of two or more inequivalent irreps of G, then G is a dynamical group for the Hamiltonian $\hat{H}$ but not for the model with this Hamiltonian. It is also seen that the subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)^1 \times \mathrm{SU}(2)^2$ with a single infinitesimal generator $\hat{S}_0 = \hat{S}_0^1 + \hat{S}_0^2$ is a symmetry group for this Hamiltonian.

Note that a system can have many dynamical groups. A particularly useful choice is one for which the important observables of the system have simple expressions in terms of its Lie algebra $\mathfrak{g}$, e.g., as linear or quadratic polynomials of the algebra's operators. Different choices may suit different situations, as illustrated in the following.

C. Coupling schemes

Dynamical groups are used in the construction of basis states for the Hilbert space of a system. A good basis is one for which the representation theory of a dynamical group, and especially that of its Lie algebra, facilitates the calculation of matrix elements of physically relevant operators. For large (especially infinite-dimensional) Hilbert spaces, it is also useful if truncations of the Hilbert space to subspaces spanned by suitable subsets of basis states give accurate approximations for states of interest.

Desirable basis states are given by a *coupling scheme* defined by a subgroup chain

$$G \supset K \supset G_{\mathrm{sym}}, \tag{6}$$

where G is a dynamical group for the system, K is a dynamical group for some class of Hamiltonians for this system, and G_{sym} is a symmetry group of transformations that leave these Hamiltonians invariant. Basis states for the system in a given irrep of G are then given by state vectors $\{|\alpha\lambda\beta\kappa\nu\rangle\}$ for a unitary irrep of G, where λ labels an irrep of K that occurs with multiplicity indexed by α, κ labels an irrep of G_{sym} with multiplicity index β, and ν indexes a basis for the irrep κ. From the definitions of K and G_{sym}, it then follows that α and λ are good quantum numbers for the eigenstates of any Hamiltonian $\hat{H}$ for which K is a dynamical group. Also κ and ν are good quantum numbers when G_{sym} is a symmetry group for $\hat{H}$. To determine the eigenstates of $\hat{H}$, it then remains to diagonalize the α- and ν-independent matrices

$$M^{\lambda\kappa}_{\beta\beta'} = \langle\alpha\lambda\beta\kappa\nu|\hat{H}|\alpha\lambda\beta'\kappa\nu\rangle. \tag{7}$$

Special cases arise when the spectrum of a Hamiltonian does not depend on the multiplicity indices α and β. This can happen because the irreps of G_{sym} in $\mathbb{H}$ are uniquely defined by the quantum numbers λ and κ so that multiplicity indices are not needed, or because one is interested in a class of Hamiltonians for which the basis states $\{|\alpha\lambda\beta\kappa\nu\rangle\}$ are eigenstates for any choices of α and β. In either case,

$$\hat{H}|\alpha\lambda\beta\kappa\nu\rangle = E_{\lambda\kappa}|\alpha\lambda\beta\kappa\nu\rangle \tag{8}$$

for a subset of Hamiltonians, and the coupling scheme is said to diagonalize Hamiltonians in this subset. A subgroup chain is often said to define a so-called *dynamical symmetry* for the class of Hamiltonians that it diagonalizes (Iachello and Arima, 1987). Thus coupling schemes defined by subgroup chains make it possible to determine the spectra of corresponding classes of Hamiltonians by purely algebraic methods. Conversely, the interpretation of a coupling scheme defined by a subgroup chain in terms of the class of Hamiltonians that it diagonalizes provides important insights into its physical significance.

In general, there will be more than one coupling scheme for a given system with a given symmetry group G_{sym}, inasmuch as there may be more than one intermediate group K between G and G_{sym} in the subgroup chain (6). However, in general, because of multiplicities, there may not exist a coupling scheme, defined by a subgroup chain $G \supset K \supset G_{\mathrm{sym}}$, for which an arbitrary Hamiltonian for a system with symmetry group G_{sym} will automatically be diagonal. The challenge is then to diagonalize the Hamiltonian in a basis defined by some convenient coupling scheme. The careful choice of coupling scheme is often important, particularly in infinite-dimensional spaces such as those of the atomic and nuclear shell models, for which approximations are inevitable.

A coupling scheme defined by a subgroup chain is best understood by an example. Consider the BCS model of

superconductivity with Hamiltonians of the form (5) for which a dynamical group is given by $G = \mathrm{SU}(2)^1 \times \mathrm{SU}(2)^2$ and a symmetry group by $G_{\mathrm{sym}} = \mathrm{U}(1)$, where U(1) is the group with infinitesimal generator $\hat{S}_0 = \hat{S}_0^1 + \hat{S}_0^2$. This model has two coupling schemes defined by the subgroup chains

$$\mathrm{SU}(2)^1 \times \mathrm{SU}(2)^2 \supset \mathrm{U}(1)^1 \times \mathrm{U}(1)^2 \supset \mathrm{U}(1), \tag{9}$$

$$\mathrm{SU}(2)^1 \times \mathrm{SU}(2)^2 \supset \mathrm{SU}(2) \supset \mathrm{U}(1), \tag{10}$$

where SU(2) is the group with Lie algebra spanned by $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$ with $\hat{S}_a = \hat{S}_a^1 + \hat{S}_a^2$ for $a = x$, y, and z. The first of these coupling schemes diagonalizes the subset of Hamiltonians of the form

$$\hat{H}_1 = \sum_{k=1}^{2} [\varepsilon_k \hat{S}_0^k - g_k \hat{S}_+^k \hat{S}_-^k], \tag{11}$$

and the second diagonalizes Hamiltonians of the form

$$\hat{H}_2 = \varepsilon \sum_{k=1}^{2} \hat{S}_0^k - g \sum_{i,k=1}^{2} \hat{S}_+^i \hat{S}_-^k. \tag{12}$$

In the two-level BCS model, the first coupling scheme describes the weakly coupled limit of two paired fermion states and models the normal phase of a superconductor, whereas the second coupling scheme describes the strong-coupling limit and models the superconducting phase.

The study of coupling schemes and the Hamiltonians they diagonalize is a profitable way to expose the dynamical content of a system. The interacting boson model (Iachello and Arima, 1987), which has a U(6) dynamical group, has been well studied in this way. It is a model with three coupling schemes that diagonalize subsets of rotationally invariant Hamiltonians. A remarkable result of these studies is the observation that, for arbitrary rotationally invariant Hamiltonians and relatively large boson numbers, the low-energy eigenstates of this model exist predominantly in one of the three possible phases characteristic of its dynamical symmetries. Similar results have also been observed in other systems (Rowe, 2004b). This is remarkable because it happens even when the Hamiltonians contain significant interaction terms that mix different dynamical symmetries. Reviews of such studies have been given in Sec. 7 of Rosensteel and Rowe (2005) and by Cejnar, Jolie, and Casten (2010) in which they are interpreted in terms of *quasidynamical symmetries* (see Sec. IX.C).

D. Complementary symmetry groups and dynamical groups

It is useful if a symmetry group and a dynamical group for a Hamiltonian $\hat{H}$ are subgroups of a dynamical group for the whole system. However, although a dynamical group for $\hat{H}$ usually contains a symmetry group of the Hamiltonian as a subgroup, as in Eq. (6), it will not generally contain a maximal symmetry group. As a consequence, a given choice of a dynamical group may not make it possible to take full advantage of the symmetries of the Hamiltonian in seeking its eigenstates. An optimal symmetry group is as close to a maximal symmetry group as possible, whereas an optimal dynamical group for a Hamiltonian is the simplest that enables the spectrum of the Hamiltonian to be computed easily. Augmenting the symmetry group while decreasing the dynamical group becomes possible if one gives up the constraint that it must be a subgroup of the dynamical group. However, if this is to be useful, the symmetry group and the dynamical group must remain compatible in such a way that the two groups can effectively complement each other. Such a compatibility is achieved if the symmetries of the Hamiltonian are a combination of subgroups of a dynamical group and groups that commute with the dynamical group.

The rationale for seeking commuting dynamical and symmetry groups is as follows. Suppose that κ labels a unitary irrep of a symmetry group G_{Sym} of a Hamiltonian $\hat{H}$ and ν labels a basis for this irrep. Let $\mathbb{H}^{(\kappa\nu)}$ denote the subspace of all states in $\mathbb{H}$ with the quantum numbers κ and ν. Then if it should happen that each subspace $\mathbb{H}^{(\kappa\nu)}$ carries a unitary representation of a group G_{Dyn}, this group will be of major assistance in determining the spectra and other properties of any Hamiltonian for which the commuting group G_{Sym} is a symmetry group. Moreover, the identification of a group G_{Dyn} whose action on a Hilbert space of interest commutes with that of a desired symmetry group provides a potentially useful way of constructing simply solvable models.

E. Dual pairs of group representations

Suppose a given Hilbert space $\mathbb{H}$ carries commuting representations of G_{Sym} and G_{Dyn}. A particularly valuable situation arises when the unitary representation of G_{Dyn} carried by every subspace $\mathbb{H}^{(\kappa\nu)}$, as defined above, is irreducible and defined uniquely by κ. In this case, a basis for the whole Hilbert space is given by a set $\{\psi_{\mu\kappa\nu}\}$, where ν indexes a basis for an irrep κ of G_{Sym} and μ indexes a basis for the irrep of G_{Dyn}, which can now also be labeled by κ. The pair of groups G_{Sym} and G_{Dyn} are then said to have *dual representations* on the Hilbert space $\mathbb{H}$.

Situations of this kind might appear to be rare. In fact, as this review shows, they are common and occur for all the standard coupling schemes of the atomic and nuclear shell models. They prove to be of profound importance. Moreover, it is possible to benefit from the widespead study of the dual pairing of group representations in the mathematical field known as *invariant theory* (see Sec. II). From such studies, several of which were initiated in nuclear physics, some remarkable relationships have been discovered between the properties of very different groups that happen to have dual representations on some Hilbert spaces of relevance to physics. Some of these properties are well known. For example, it is common practice to speak of a unitary representation of a many-particle system as having a given symmetry, where the symmetry referred to is a representation of the symmetric group of permutations of the particles, i.e., a group with representations that are dual to those of a unitary group. This duality, known as Schur-Weyl duality, leads to the Young diagram and many other powerful techniques that give simple solutions to problems that arise in physics.

Dual representations of a pair of Lie groups are defined precisely as follows.

Two groups G_1 and G_2 are said to have dual representations on a space $\mathbb{H}$ if the following conditions are met: (i) $\mathbb{H}$ is the carrier space for fully reducible representations of both G_1 and G_2; (ii) the actions of G_1 and G_2 on $\mathbb{H}$ commute; (iii) the representation κ of the direct product group $G_1 \times G_2$ on $\mathbb{H}$, defined by the actions of G_1 and G_2 on $\mathbb{H}$, is multiplicity free; and (iv) each irrep of G_1 that occurs in the decomposition of $\mathbb{H}$ is paired with a single irrep of G_2, and vice versa.

Condition (i) restricts consideration to representations of G_1 and G_2 that are expressible as direct sums of irreps. Thus, for present purposes, we exclude representations of some noncompact groups that are direct integrals of irreps. For the purposes of this review, we also restrict consideration to representations that are unitary.

Condition (iii) states that, in the decomposition of the representation κ of $G_1 \times G_2$ on $\mathbb{H}$ to a direct sum of irreps, no irrep appears more than once. Condition (iv) guarantees that, in such a decomposition, it is possible to identify, with a common label, paired irreps of the two groups [e.g., by the angular-momentum label l in the SU(1, 1) × SO(3) example given below].

Simple proofs of the duality theorems on which this review is based are given elsewhere (Rowe, Repka, and Carvalho, 2011).

F. A simple example

Suppose we want to determine the spectrum of a central-force Hamiltonian for a particle moving in ordinary three-space or for the relative motion of a diatomic molecule moving about its center of mass. A standard practice is to seek eigenfunctions of the Hamiltonian in a basis of spherical harmonic-oscillator wave functions. Basis wave functions for the Hilbert space $\mathbb{H}$ of a spherical harmonic oscillator are given by products

$$\psi_{nlm}(r, \theta, \varphi) = R_{nl}(r) Y_{lm}(\theta, \varphi), \tag{13}$$

where $\{R_{nl}\}$ are radial wave functions and $\{Y_{lm}\}$ are spherical harmonics. The subset of wave functions $\{\psi_{nlm}\}$ with n fixed span a Hilbert space $\mathbb{H}^{(n)} \subset \mathbb{H}$ that carries an irrep of the U(3) symmetry group of a harmonic-oscillator Hamiltonian. The subset $\{\psi_{nlm}\}$ with fixed values of n and l span a Hilbert space $\mathbb{H}^{(nl)} \subset \mathbb{H}^{(n)} \subset \mathbb{H}$ that carries an irrep of the rotation group SO(3). Thus, the basis wave functions $\{\psi_{nlm}\}$ are those of the coupling scheme defined by the irreps of the symmetry groups of a harmonic oscillator in the subgroup chain

$$\begin{array}{ccccc} \mathrm{U}(3) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2). \\ n & & l & & m \end{array} \tag{14}$$

As well as being a symmetry group for the spherical harmonic oscillator, the group U(3) is also a dynamical group inasmuch as its irreps are spanned by eigenstates of this Hamiltonian. We now enquire as to whether or not there is a group that commutes with the SO(3) symmetry group that could serve as a dynamical group for a general central-force Hamiltonian. We find that SU(1,1) is such a group and that it has a unitary irrep on each of the Hilbert spaces $\mathbb{H}^{(lm)}$ spanned by the harmonic-oscillator wave functions $\{\psi_{nlm}\}$ with fixed values of l and m. Thus, the representation theory of the group SO(3) and its SO(2) subgroup determines the spherical harmonics with the quantum numbers l and m. Moreover, the representation theory of the dynamical group SU(1,1) and its Lie algebra can be used to determine the radial wave functions for a general central-force Hamiltonian.

The group SU(1,1) is defined as follows. From the vector operators $\hat{\mathbf{r}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ for a particle in three-space, we can form the SO(3)-invariant (i.e., scalar) operators $\hat{r}^2 = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}$ and $\hat{p}^2 = \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}$. A linear combination of these scalars is a harmonic-oscillator Hamiltonian

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m\omega^2 \hat{r}^2, \tag{15}$$

and the commutator

$$[\hat{r}^2, \hat{p}^2] = 2i\hbar(\hat{\mathbf{r}} \cdot \hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \tag{16}$$

is another SO(3) scalar. Further commutators produce no new operators, which means that the operators $\{\hat{r}^2, \hat{p}^2, \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot \hat{\mathbf{r}}\}$ are a basis for a Lie algebra. This is the Lie algebra su(1,1) of the group SU(1,1) which is now observed to be a dynamical group for any central-force Hamiltonian

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \hat{V}(r), \tag{17}$$

where V is a rotationally invariant potential energy.

The remarkable property of this SU(1,1) group is that, although its elements commute with those of SO(3), its irreps on the Hilbert space of a particle in three-space are uniqely defined by the SO(3) angular-momentum quantum number l. Moreover, any wave function in $\mathbb{H}$ that has angular momentum l must belong to irreps of both SO(3) and SU(1,1) labeled by l. This is a deep result which has its origins in the centrifugal coupling of the radial and rotational motions of the particle.

G. Outline of the review

Section II gives an historical review of the major contributions to the development and applications of dual group representations of which we are aware.

Section III presents the duality relationship, known as Schur-Weyl duality, between the unitary and symmetric groups on the Hilbert spaces of many particles. This duality gives rise to the Young diagram methods and relationships between characters which are of enormous practical importance in the use of these groups in physics.

Section IV shows the power of Schur-Weyl duality in deriving the branching rules and tensor products that are needed in atomic and nuclear physics.

Section V presents duality relationships between pairs of unitary groups. These duality relationships are applied to the construction of fully antisymmetric combinations of space and spin wave functions and of space, spin, and isospin wave functions. Such methods are needed in atomic, nuclear, and elementary particle physics.

Section VI introduces methods of second quantization which provide simple ways of ensuring that wave functions of multiple identical bosons or fermions are automatically symmetric or antisymmetric, respectively.

Section VII applies duality techniques to many-boson systems. It shows the underlying algebraic structures of central-force problems and the main models of nuclear collective motion, e.g., the Bohr model, interacting boson model, and the microscopic symplectic model.

Section VIII applies duality techniques to many-fermion systems. It shows the relationship between pairing models and corresponding single-j- and multi-j-shell coupling schemes. Similar relationships are shown for LST coupling models.

Section IX gives a brief review of other related developments in group theoretical methods.

Section X gives a summary and some possibilities for further development and/or pursuits of methods described in the review.

II. HISTORICAL PERSPECTIVE

Dual pairs of group representations were first used in the context of invariant theory by Schur (1901, 1927). In constructing the finite-dimensional irreps of general linear groups, Schur discovered what is now called Schur-Weyl duality, which he reported in his 1901 doctoral dissertation (Schur, 1901). Weyl developed the theory and applied it to diverse physical problems (Weyl, 1946). In his book, originally published in 1928, Weyl (1950) gave wide publicity to Young's work [cf. Robinson (1977)] on the symmetric groups (including coining the term Young tableau), extended the work of Schur (e.g., to include Weyl's character formula), and applied Schur's discovery in quantum mechanical contexts.

Subsequent to these seminal works, Schur-Weyl duality has been used to develop branching rules and tensor-product decompositions for a wide range of groups of interest to the physics community. In a series of publications, Littlewood (1950) used Schur-Weyl duality and particularly Schur function techniques to advance the theory of group characters. Work done on branching rules up to the mid-1960s was summarized by Whippman (1965).

In more recent times, King (1975), Wybourne (1993), Macdonald (1995, 1998), and colleagues further developed the character theory of Lie groups and, in the process, exhibited the power of Schur function techniques. The publications from King (1970, 1971, 1975), Black, King, and Wybourne (1983), and Black and Wybourne (1983) provide entry points into their work on the character theory of compact Lie groups. Schur function techniques have also been used to extend the application of Schur-Weyl duality to the character theory of noncompact Lie groups (King and Wybourne, 1985, 1998, 2000a, 2000b; Rowe, Wybourne, and Butler, 1985; Grudzinski and Wybourne, 1996; Thibon, Toumazet, and Wybourne, 1997), Hecke algebras (Wybourne, 1991; King, 1993), and supersymmetry (Borodin and Rozhkovskaya, 1995; Cheng and Wang, 2000, 2001; Cheng and Zhang, 2004).

Haase and Butler (1984a, 1984b) used Schur-Weyl duality to derive relationships among coupling coefficients of symmetric and unitary groups. Also noteworthy is the work of D'Hoker (1984) and Koike and Terada (1987), in which Young diagram techniques for unitary groups are systematically extended to various classical Lie groups, and the work of Brauer (1937) which extended the theory from the symmetric group to the commutants of orthogonal and symplectic groups on tensor-product spaces with applications to quantum groups, knots, and links (Benkart, 1996).

In addition to the many examples already listed, a recent application of duality to error correction in quantum computing was given by Junge, Kim, and Kribs (2005). This underscores the fact that Schur-Weyl duality is not only useful in the construction of wave functions with specified permutation symmetries, it is also useful when one wants to classify the symmetry properties of more general composite systems.

The second duality relationship to be discovered was the duality between the representations of pairs of unitary groups, the so-called unitary-unitary duality. The application of this duality relationship, which follows directly from Schur-Weyl duality, has been widely used in the classification of nuclear shell-model states following the introduction of Wigner's U(4) supermultiplet group (Wigner, 1937). In this classification, representations of the U(4) supermultiplet group, which contains a U(2) isospin subgroup and a U(2) intrinsic-spin subgroup for spin-1/2 nucleons, are combined with contragredient irreps of a U(n) group of transformations of spatial wave functions to form the totally antisymmetric states required for a many-nucleon system.

To our knowledge, the next duality relationship to be discovered was the compact symplectic-symplectic duality. The utility of compact symplectic groups in the atomic shell model was brought to the attention of the physics community by Racah (1943). It was later applied to the classification of nuclear shell-model states in jj coupling by Flowers (1952a, 1952b, 1952c), who introduced extra group theoretical structures to account for the nucleon's isospin degrees of freedom. In the process, Flowers recognized and exploited many duality relationships. The symplectic-symplectic duality theorem underlying these relationships was formulated and proved, using character theory, by Helmers (1961), who described what he had discovered as *group complementarity*. Independently, Kerman (1961) introduced the related and much used concept of a quasispin group which proved to be a special case of Helmers' theorem. The concept of USp-USp duality was reviewed in works directed to physicists by Parikh (1978) and Lipkin (2002). It has been widely applied to fermion-pair coupling phenomena, in both nuclear systems and superconductivity. Recent applications are described in Lorazo (1970), Engel *et al.* (1997), Van Isacker (1999), Palchikov, Dobeš, and Jolos (2001), Rowe and Rosensteel (2001), Dean and Hjorth-Jensen (2003), Sviratcheva, Georgieva, and Draayer (2003), and Rosensteel and Rowe (2003).

A duality relationship between the representations of pairs of orthogonal groups was similarly identified in the classification of nuclear shell-model states in LS coupling. In this case, the discovery emerged from two quite distinct methods for the construction of shell-model basis states that diagonalize a simple LS-coupling pairing Hamiltonian. Bayman

(1960) gave the basis states in terms of the irreps of a symmetry group for the Hamiltonian and, independently, Flowers and Szpikowski (1964) gave the same basis states in terms of the irreps of a distinct dynamical group. Although the duality of these complementary approaches has long been understood, the orthogonal-orthogonal duality theorem underlying them has only recently been formulated and proved in physics (Kota and Alcarás, 2006; Rowe and Carvalho, 2007; Rowe, Repka, and Carvalho, 2011).

Independent of the discoveries in physics, the theory of dual group representations has been developed in mathematics and given a rigorous basis within the framework of invariant theory. The oscillator *Weil representation* of the noncompact symplectic groups was particularly influential in this development. Following the construction of the oscillator representations of the noncompact symplectic groups by Segal (1959), Shale (1962), and Weil (1964), the well-known $SU(1,1) \times O(3)$ duality relationship (discussed in the Introduction) was recognized as a special case of a more general symplectic-orthogonal duality relationship. This duality relationship was introduced into physics following its discovery by Moshinsky and Quesne (1969, 1970), based on results in Chacón's thesis (Chacón, 1969), and derived, together with other such relationships, by Kashiwara and Vergne (1978). In fact, apart from Schur-Weyl duality, the duality relations described in this review are all special cases of the so-called *dual reductive pairs* identified in a paper by Howe (1989) that was written in 1976 and widely circulated but not published at that time. Examples of dual reductive pairs were given by Gelbart (1973) and Howe (1985) and many authors have contributed to the subject. In addition to the duality relationships considered in this review, more are known in mathematics. Reviews of invariant theory and dual group representations in mathematics are given by Howe (1995), Li (2000), and Goodman (2004). For example, there are duality relationships between $U(p,q)$ and $U(n)$ (Kashiwara and Vergne, 1978), between the compact symplectic and noncompact $SO^*(2n)$ group (Leung and Ton-That, 1995), between pairs of noncompact groups $[O(p,q), Sp(m,\mathbb{R})]$ (Adams, 1983), and also between $O(p,q)$ and an orthosymplectic group. The latter dual pair is shown by Lu and Howe (2010) to be relevant to Maxwell's equations. Howe duality has been extended to the realm of exceptional Lie groups (Dvorsky and Sahi, 1998) and quantum groups (Green, 1999), and has also been applied to gauge theories (Schmidt, 1999) and the quantization of constrained systems (Landsman, 1999). Its use in deriving branching rules for the harmonic series of $Sp(n,\mathbb{R})$ was initiated by Rowe, Wybourne, and Butler (1985) and followed by a similar derivation of branching rules for $U(p,q)$ and $SO^*(2n)$ by King and Wybourne (1985). Branching rules for many classical pairs of Lie groups have been derived by Howe, Tan, and Willenbring (2004).

III. THE SYMMETRIC AND UNITARY GROUPS

The symmetric group S_N is the group of permutations of the indices that label the particles of an N-particle system, and the unitary groups are transformations that preserve the orthogonality relationships of quantum mechanical states. These groups are indispensable in quantum mechanics. The symmetric group is a fundamental symmetry of the quantum mechanics of identical particles. Moreover, the Schur-Weyl duality theorem shows that the subgroup of all unitary transformations of a many-particle system that commute with the symmetric group is the so-called group of one-body unitary transformations. It also shows that this subgroup and the symmetric group have dual representations on a many-particle Hilbert space. Thus, it identifies the group of one-body unitary transformations as the fundamental dynamical group of many-particle quantum mechanics.

Elementary particles are considered to be either bosonlike or fermionlike, which means that their many-particle wave functions are totally symmetric or totally antisymmetric, respectively, under permutation. However, wave functions often have several components. For example, fermion wave functions may be combinations of spatial, spin, and isospin wave functions which need not be separately antisymmetric. Thus, in calculations, it is necessary to keep track of the way the separate components transform under permutation of the particle indices so that they can be put together in antisymmetric combinations. This awe-inspiring task is much simplified by use of the Schur-Weyl theorem.

Suppose a space of single-particle wave functions is n-dimensional and that $U(n)$ is the group of unitary transformations of this space. The Schur-Weyl theorem shows that there is a duality relationship between the irreps of the unitary group $U(n)$ and those of the symmetric group S_N for an N-particle system. Thus, not only do the fully antisymmetric states of a many-fermion system carry a corresponding irrep of $U(n)$ but the states of any specified permutation symmetry carry a $U(n)$ irrep. This is a remarkable result that provides basic tools for the application of group theory to many-particle systems and, in particular, to the development and interpretation of phenomena in terms of the various shell-model coupling schemes. In fact, as this section shows, Schur-Weyl duality implies that many of the well-known properties of the discrete S_N group can be used to infer corresponding properties for the continuous $U(n)$ Lie group and vice versa. Thus, numerous parallel techniques have been developed for the simultaneous study of the symmetric and unitary groups, e.g., within the framework of Young diagram methods and character theory.

This section makes substantial use of the review of Young's diagram and related Schur function techniques given by Wybourne (1970). Sections III.A–III.C closely follow those of Rowe and Wood (2010).

A. The Schur-Weyl duality theorem

Let $\mathbb{H}_n$ denote an n-dimensional Hilbert space and let

$$\mathbb{H}_n^N = \mathbb{H}_n^{\otimes N} = \underbrace{\mathbb{H}_n \otimes \cdots \otimes \mathbb{H}_n}_{N \text{ copies}} \tag{18}$$

denote the tensor product of N copies of $\mathbb{H}_n$. The following Schur-Weyl theorem is naturally understood if $\mathbb{H}_n$ and $\mathbb{H}_n^N$ are interpreted as spaces of single-particle and N-particle wave functions, respectively.

Theorem 1 (Schur-Weyl duality): The groups S_N and $U(n)$ have dual representations on $\mathbb{H}_n^N$.[1]

Let $\{\psi_1, \psi_2, \ldots, \psi_n\}$ denote an orthonormal basis of single-particle wave functions for $\mathbb{H}_n$. The group $U(n)$ then has a defining n-dimensional irrep $\hat{U}^{\{1\}}$ on $\mathbb{H}_n$ given by transformations of its single-particle wave functions

$$\hat{U}^{(1)}(g)\psi_\nu = \sum_\mu \psi_\mu g_{\mu\nu}, \qquad g \in U(n). \tag{19}$$

The corresponding Hilbert space $\mathbb{H}_n^N$ has an orthonormal basis given by the N-particle wave functions

$$\Psi_{\nu_1\cdots\nu_N} = \psi_{\nu_1} \otimes \psi_{\nu_2} \otimes \cdots \otimes \psi_{\nu_N}, \tag{20}$$

and carries a reducible $U(n)$ tensor-product representation $\hat{U}^N$ given for $g \in U(n)$ by

$$\hat{U}^{(N)}(g)\Psi_{\nu_1\cdots\nu_N} = \sum_{\mu_1\cdots\mu_N} \Psi_{\mu_1\cdots\mu_N} g_{\mu_1\nu_1} g_{\mu_2\nu_2} \cdots g_{\mu_N\nu_N}. \tag{21}$$

For any positive integer N, this $U(n)$ representation and its irreducible subrepresentations are referred to as *tensor representations* of degree N.

The Hilbert space $\mathbb{H}_n^N$ also carries a reducible representation $\hat{P}$ of the symmetric group S_N, defined by the permutations of the N indices of the $\{\Psi_{\nu_1\nu_2\cdots\nu_N}\}$ basis; e.g., if $\pi_{12} \in S_N$ is the permutation that exchanges particles 1 and 2, then

$$\hat{P}(\pi_{12})\Psi_{\nu_1\nu_2\cdots\nu_N} = \Psi_{\nu_2\nu_1\cdots\nu_N}. \tag{22}$$

A proof of the Schur-Weyl duality theorem can be found, for example, in Chapter V of Weyl and Robertson (1950), Chapter 5 of Sternberg (1994), and Chapter 6 of Fulton and Harris (1991). Here we highlight and explain the main points of the theorem, some of which are immediately evident. For example, it is readily ascertained that the actions of S_N and $U(n)$ commute:

$$\hat{P}(\pi)\hat{U}^{\{N\}}(g)\Psi_{\nu_1\cdots\nu_N} = \hat{U}^{\{N\}}(g)\hat{P}(\pi)\Psi_{\nu_1\cdots\nu_N}, \tag{23}$$

for all $g \in U(n)$ and all $\pi \in S_N$. Thus, the direct product group $S_N \times U(n)$ has a reducible representation $\hat{T}$ on $\mathbb{H}_n^N$ for $N > 1$ and $n > 1$, defined by

$$\hat{T}(\pi, g)\Psi_{\nu_1\cdots\nu_N} = \hat{P}(\pi)\hat{U}^{\{N\}}(g)\Psi_{\nu_1\cdots\nu_N}, \tag{24}$$

for $\pi \in S_N$ and $g \in U(n)$. Every irrep of $S_N \times U(n)$ is then expressed as an "outer product" $\hat{T} = \hat{\mathcal{P}} \times \hat{\mathcal{U}}$, where $\hat{\mathcal{P}}$ and $\hat{\mathcal{U}}$ are, respectively, irreps of S_N and $U(n)$, and

$$\hat{T}(\pi, g) = \hat{\mathcal{P}}(\pi) \times \hat{\mathcal{U}}(g), \quad \pi \in S_N, \quad g \in U(n). \tag{25}$$

(In the mathematics literature, the irrep $\hat{\mathcal{P}} \times \hat{\mathcal{U}}$ is denoted $\hat{\mathcal{P}} \otimes \hat{\mathcal{U}}$. However, we avoid this notation because of the potential confusion with the use of $\otimes$ for the standard tensor product for irreps of a single group.)

[1]The theorem is somewhat more general than stated here. This is because $GL(n, \mathbb{C})$ is the complex extension of $U(n)$ and, as a consequence, the representations of $U(n)$ extend to (nonunitary) representations of $GL(n, \mathbb{C})$. However, for present purposes, we restrict consideration to the $U(n) \subset GL(n, \mathbb{C})$ subgroup.

By condition (iv) of the definition of duality, the Schur-Weyl theorem affirms that all irreps occurring in the decomposition of $\hat{T}$ are of the form

$$\hat{T}^\lambda = \hat{P}^{(\lambda)} \times \hat{U}^{\{\lambda\}}, \tag{26}$$

where $\hat{P}^{(\lambda)}$ and $\hat{U}^{\{\lambda\}}$ are, respectively, S_N and $U(n)$ irreps that are uniquely defined by a common label λ. Moreover, every irrep of S_N appears in the above decomposition, provided that $N \leq n$. Some S_N irreps do not occur if $N > n$; for example, it is not possible to form a totally antisymmetric N-particle wave function with fewer than N linearly independent single-particle wave functions. Thus, we learn from the Schur-Weyl theorem that every S_N irrep $\hat{P}^{(\lambda)}$, with $N \leq n$, is uniquely associated with a corresponding tensor irrep $\hat{U}^{\{\lambda\}}$ of $U(n)$. Conversely, every tensor irrep $\hat{U}^{\{\lambda\}}$ of degree N of $U(n)$ is uniquely associated with an S_N irrep $\hat{P}^{(\lambda)}$. Although well known and often taken for granted, we emphasize again that these are remarkable results because they mean that much of the representation theory of a family of continuous Lie groups, namely, the unitary groups and their subgroups, can be inferred from the representation theory of the finite symmetric groups.

B. Characterization of $U(n) \times S_N$ representations

A $U(n)$ tensor irrep is characterized in two standard ways: by its highest weight relative to a Cartan subalgebra and by its S_N symmetry. The relationship between these alternative characterizations exposes the duality relationship between S_N and $U(n)$ representations.

The group $U(n)$ of $n \times n$ unitary matrices has the property that a matrix $g \in U(n)$ can be expressed as an exponential $g = e^{iX}$, where X is an $n \times n$ Hermitian matrix. Moreover, every physical observable is represented in quantum mechanics by a Hermitian operator. Thus, it is customary in physics to define the Lie algebra $u(n)$ of the group $U(n)$ as the set of Hermitian $n \times n$ matrices.

Let $C_{\mu\nu}$ denote a matrix that has the entry 1 at the intersection of row μ with column ν and 0 everywhere else, i.e.,

$$(C_{\mu\nu})_{ij} = \delta_{\mu,i}\delta_{j,\nu}. \tag{27}$$

These matrices have commutation relations

$$[C_{\mu\nu}, C_{\kappa\lambda}] = \delta_{\nu,\kappa}C_{\mu\lambda} - \delta_{\mu,\lambda}C_{\kappa\nu} \tag{28}$$

and span the *complex extension* of the $u(n)$ Lie algebra. A basis for $u(n)$ is then given, in terms of them, by the Hermitian linear combinations

$$C_{\mu\nu} + C_{\nu\mu}, \quad i(C_{\mu\nu} - C_{\nu\mu}), \quad 1 \leq \mu, \nu \leq n. \tag{29}$$

They have N-particle tensor representations given by

$$C_{\mu\nu} \to \hat{C}_{\mu\nu} = \sum_{i=1}^N \psi_\mu(i)\frac{\partial}{\partial\psi_\nu(i)}, \tag{30}$$

where $\partial/\partial\psi_\nu(i)$ is a functional derivative with respect to a single-particle wave function $\psi_\nu(i)$ in the ith factor in the tensor-product space $\mathbb{H}_n^N$ of Eq. (18).

Useful basis states for an irrep of U(n) and its Lie algebra u(n) are given by the simultaneous eigenstates of the subset of commuting operators that represent the diagonal matrices of a Cartan subalgebra of u(n). We call these operators *Cartan operators*. A basis of Cartan operators for the tensor representations and corresponding raising and lowering operators are given by

$$\{\hat{C}_{\nu\nu}, \nu = 1, \ldots, n\} \qquad \text{(Cartan operators)}, \tag{31}$$

$$\{\hat{C}_{\mu\nu}, 1 \le \mu < \nu \le n\} \qquad \text{(raising operators)}, \tag{32}$$

$$\{\hat{C}_{\mu\nu}, 1 \le \nu < \mu \le n\} \qquad \text{(lowering operators)}. \tag{33}$$

If a state $|\Psi\rangle$ of a U(n) representation is an eigenstate of the Cartan operators, i.e.,

$$\hat{C}_{\nu\nu}|\Psi\rangle = \lambda_\nu|\Psi\rangle, \qquad \nu = 1, \ldots, n, \tag{34}$$

then the set of eigenvalues $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is said to be the *weight* of the state $|\Psi\rangle$.

A weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is said to be higher than a weight $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ if $\lambda_1 > \lambda'_1$, or if $\lambda_1 = \lambda'_1$ and $\lambda_2 > \lambda'_2$, etc. For convenience, commas and trailing zeros are usually omitted in specifying a weight when it would be unambiguous to do so. Thus, a highest weight (2,1,0,0) is written as (21), and a highest weight (1,0,0,0,0) is written as (1). However, a weight (21,0,0) is written explicitly.

Because U(n) is compact, all its irreps are unitary and finite dimensional. Thus, a U(n) irrep has a highest weight. An important property of Lie algebra structure theory is that a finite-dimensional irrep of any semisimple or reductive Lie algebra has a state with a uniquely defined highest weight; in the case of the unitary algebra u(n), the components λ_i of the highest weight λ satisfy the inequality

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n, \tag{35}$$

where, for a tensor irrep, $\lambda_n \ge 0$. It follows that a U(n) irrep is completely defined by its highest weight.

Consider, for example, the highest-weight state for the $N = 3$ irrep of U(3) with highest weight $(1^3) \equiv (111)$. It must be a linear combination of wave functions of the form $\{\psi_i(1)\psi_j(2)\psi_k(3)\}$, where i, j, and k index three distinct single-particle states. The particular linear combination that is annihilated by the raising operators is given (to within a normalization factor) by the fully antisymmetric wave function

$$\Psi^{(1^3)}_{\text{h.wt.}}(1, 2, 3) = \mathfrak{D}^{123}_{123}, \tag{36}$$

where $\mathfrak{D}^{123}_{123}$ is the so-called *Slater determinant* of a 3×3 matrix

$$\mathfrak{D}^{123}_{123} = \begin{vmatrix} \psi_1(1) & \psi_1(2) & \psi_1(3) \\ \psi_2(1) & \psi_2(2) & \psi_2(3) \\ \psi_3(1) & \psi_3(2) & \psi_3(3) \end{vmatrix}. \tag{37}$$

On the other hand, highest-weight states for the $N = 3$ irreps {3} and {21} have wave functions given by

$$\Psi^{(3)}_{\text{h.wt.}}(1, 2, 3) = \mathfrak{D}^1_1\mathfrak{D}^1_2\mathfrak{D}^1_3, \tag{38}$$

$$\Psi^{(21)}_{\text{h.wt.}}(1, 2, 3) = \mathfrak{D}^{12}_{12}\mathfrak{D}^1_3, \tag{39}$$

where $\mathfrak{D}^1_i = \psi_1(i)$ and

$$\mathfrak{D}^{12}_{ij} = \begin{vmatrix} \psi_1(i) & \psi_1(j) \\ \psi_2(i) & \psi_2(j) \end{vmatrix}. \tag{40}$$

A highest-weight state for an arbitrary U(n) irrep is similarly expressed in terms of Slater determinants

$$\mathfrak{D}^{12\cdots k}_{i_1 i_2 \cdots i_k} = \begin{vmatrix} \psi_1(i_1) & \psi_1(i_2) & \cdots & \psi_1(i_k) \\ \psi_2(i_1) & \psi_2(i_2) & \cdots & \psi_2(i_k) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_k(i_1) & \psi_k(i_2) & \cdots & \psi_k(i_k) \end{vmatrix}. \tag{41}$$

For example, a highest-weight state for the $N = 7$ irrep of U(3) with highest weight (421) is given by

$$\Psi^{(421)}_{\text{h.wt.}}(1, 2, 3, 4, 5, 6, 7) = \mathfrak{D}^{123}_{123}\mathfrak{D}^{12}_{45}\mathfrak{D}^1_6\mathfrak{D}^1_7. \tag{42}$$

It can be seen that the wave function $\Psi^{(421)}_{\text{h.wt.}}$ is a sum of tensor products of seven single-particle wave functions corresponding to four particles with wave function ψ_1, two particles with wave function ψ_2, and one particle with wave function ψ_3. Thus, it is of weight (421). Moreover, from its structure as a product of highest-weight Slater determinants, it is of highest weight.

The above construction gives a highest-weight state for any U(n) tensor irrep. However, there are many such highest-weight states in the N-particle Hilbert space $\mathbb{H}^N_n$ for a given highest weight, λ. These many highest-weight states correspond to the many possible permutations of the particle indices. For example, the wave function

$$\Psi^{(421)}_{\text{h.wt.}}(1, 2, 7, 6, 5, 4, 3) = \mathfrak{D}^{123}_{127}\mathfrak{D}^{12}_{65}\mathfrak{D}^1_4\mathfrak{D}^1_3, \tag{43}$$

obtained by permuting the particle indices in Eq. (42) is of U(n) highest weight (421) but is distinct from that given by Eq. (42). Clearly the set of all such wave functions of U(n) highest weight (421), obtained by permutations of the particle indices, spans a representation of the symmetric group S_7. More significantly, the Schur-Weyl theorem shows that this S_7 representation is irreducible and dual to the U(n) irrep with highest weight (421). In general, the Schur-Weyl theorem implies that the highest weight λ for any tensor irrep of U(n) on $\mathbb{H}^N_n$ defines a dual irrep of S_N with $N = |\lambda|$, where $|\lambda| = \sum_{i=1}^n \lambda_i$.

The above example shows how an integer highest weight λ, i.e., one with non-negative integer components that satisfy the inequality (35), determines both a U(n) tensor irrep and an S_N irrep, where N is the sum $N = |\lambda| = \sum_{i=1}^n \lambda_i$ of the components of λ. Such a highest weight is described as an *ordered partition* of N and we write $\lambda \vdash N$. If such a partition has p nonzero parts, with $p \le n$, it is said to have length $l(\lambda) = p$.

C. Classification of S_N and U(n) irreps by Young diagrams

The duality relationship between S_N irreps and U(n) tensor irreps is clarified by Young diagram techniques. As the above $N = 7$, $n = 3$ example illustrates, both a tensor irrep of U(n) and the corresponding S_N irrep are characterized by an

ordered partition $\lambda \vdash N$. It is conventional to label the S_N and $U(n)$ irreps corresponding to an ordered partition λ by (λ) and $\{\lambda\}$, respectively. The Schur-Weyl theorem can then be expressed as the statement that the representation $\hat{T}^N_n$ of the direct product group $S_N \times U(n)$ carried by the tensor-product space $\mathbb{H}^N_n$ is a direct sum of irreps given by

$$\hat{T}^N_n = \bigoplus_{\lambda \vdash N}^{l(\lambda)\le n} (\lambda) \times \{\lambda\}. \tag{44}$$

The irreps of S_N and $U(n)$ corresponding to an ordered partition $\lambda \vdash N$ are equivalently characterized by a so-called Young diagram $Y^{(\lambda)}$, which is an array of left adjusted boxes with λ_1 boxes in the first row, λ_2 in the second row, ..., λ_p in the pth row. For example, the partition $\lambda = (421)$ is identified with the Young diagram

$$Y^{(421)} := \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \end{array}\,. \tag{45}$$

The boxes of a Young diagram can be regarded as containers for indices that label particles for S_N and single-particle wave functions for $U(n)$. Such numbered diagrams are called *Young tableaux*. For example, for the wave function $\Psi^{(421)}_{\text{h.wt.}}(1, 2, 3, 4, 5, 6, 7)$ of the highest-weight state given by Eq. (42), we can put the particle-number indices $1, \ldots, 7$ into an S_7 diagram and the single-particle state indices into a U(3) diagram as follows:

$$\begin{array}{ccc} S_7 & & U(3) \\ \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} & \times & \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \end{array}, \tag{46}$$

where the columns of the tableaux correspond to the determinantal factors in the wave function of Eq. (42):

$$\mathfrak{D}^{123}_{123} \sim \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \times \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \qquad \mathfrak{D}^{12}_{45} \sim \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \end{array} \times \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \tag{47}$$

$$\mathfrak{D}^{1}_{6} \sim \boxed{6} \times \boxed{1}, \qquad \mathfrak{D}^{1}_{7} \sim \boxed{7} \times \boxed{1}. \tag{48}$$

The tableaux in Eq. (46) are special because the first represents a so-called *leading state* of the S_7 irrep (421) and the second represents a state of highest weight for the U(3) irrep $\{421\}$, as constructed above. The leading state is defined, somewhat arbitrarily, to be one for which the particle indices $1, 2, \ldots, 7$ are entered sequentially down columns, starting with the first column. The Young tableau for a $U(n)$ highest-weight state is one for which the integer i fills all boxes of row i. Other basis states are obtained by putting the numbers into the boxes in different ways, subject to the condition that all the particle indices must be distinct. However, simple rules must be followed to avoid getting an overcomplete set. For example, inspection of the wave function $\Psi^{(421)}_{\text{h.wt.}}$ shows that the numbers in corresponding columns of the S_7 and U(3) tableaux, respectively, give the particle and state indices of the single-particle wave functions of a Slater determinant. Interchanging their order, in either tableau, can at most change the sign of the wave function. To obtain a linearly independent set of states, it is therefore appropriate to impose the rule that the numbers in any column must always increase strongly from top to bottom (increasing strongly simply means that no number is repeated, whereas increasing weakly means not decreasing). A second, less obvious rule, is that, to obtain linearly independent states, the numbers in any row of boxes should also increase [weakly in the case of a $U(n)$ tableau and strongly in the case of S_N] from left to right. Note also that, while state indices may be repeated, the particle indices must all be distinct.

It is easy to check that the above rules work out in given situations. For example, the four-dimensional space $\mathbb{H}^{N=2}_{n=2}$ is spanned by three states of the $\lambda = (2)$ irrep of $S_2 \times U(2)$ and one state of the $\lambda = (11)$ irrep:

$$\begin{array}{ccccc} S_2 & \times & U(2) & & \\ \boxed{1}\boxed{2} & \times & \boxed{1}\boxed{1} & \sim & \psi_1(1)\psi_1(2), \\ \boxed{1}\boxed{2} & \times & \boxed{1}\boxed{2} & \sim & \psi_1(1)\psi_2(2) + \psi_2(1)\psi_1(2), \\ \boxed{1}\boxed{2} & \times & \boxed{2}\boxed{2} & \sim & \psi_2(1)\psi_2(2), \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} & \times & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} & \sim & \begin{vmatrix} \psi_1(1) & \psi_1(2) \\ \psi_2(1) & \psi_2(2) \end{vmatrix}. \end{array} \tag{49}$$

Note that to determine the state with the Young tableau $\boxed{1}\boxed{2}$ for U(2), one applies the lowering operator $\hat{C}_{21}$, defined by Eq, (30), to the state with tableau $\boxed{1}\boxed{1}$, i.e.,

$$\hat{C}_{21}\psi_1(1)\psi_1(2) = \psi_1(1)\psi_2(2) + \psi_2(1)\psi_1(2). \tag{50}$$

Similarly, the eight-dimensional space $\mathbb{H}^{N=3}_{n=2}$ is spanned by four states of the $\lambda = (3)$ irrep of $S_3 \times U(2)$ and four states of the $\lambda = (21)$ irrep:

$$\begin{array}{cccc} S_3 & \times & U(2) & \qquad S_3 \times U(2) \\ \boxed{1}\boxed{2}\boxed{3} & \times & \boxed{1}\boxed{1}\boxed{1}, & \boxed{1}\boxed{2}\boxed{3} \times \boxed{1}\boxed{1}\boxed{2}, \\ \boxed{1}\boxed{2}\boxed{3} & \times & \boxed{1}\boxed{2}\boxed{2}, & \boxed{1}\boxed{2}\boxed{3} \times \boxed{2}\boxed{2}\boxed{2}, \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \hline \end{array} & \times & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \hline \end{array}, \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \hline \end{array} & \times & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \hline \end{array}. \end{array} \tag{51}$$

D. The relationship between S_N and $U(n)$ characters

A first application of Schur-Weyl duality is to obtain an algorithm for deriving characters of the unitary groups from those of the symmetric groups. Such characters are frequently needed because $U(n)$ plays a central role in the decomposition of tensor products of various group representations and in the reduction of an irrep of a group on restriction to a subgroup. These are frequently occurring problems in the application of symmetry to many-particle physics.

The character χ of a representation $\hat{T}$ of a group G is a complex-valued function $\chi: G \to \mathbb{C}$ whose value for a group element $g \in G$ is the trace

$$\chi(g) = \mathrm{Tr}[\hat{T}(g)]. \tag{52}$$

Characters take the same values for all elements of a group that belong to a common conjugacy class, because, if two elements g_1 and g_2 of a group G are conjugate, they are related by $g_1 = g g_2 g^{-1}$ for some $g \in G$. The identity $\mathrm{Tr}[\hat{T}(g_1)] = \mathrm{Tr}[\hat{T}(g g_2 g^{-1})] = \mathrm{Tr}[\hat{T}(g)\hat{T}(g_2)\hat{T}(g)^{-1})] = \mathrm{Tr}[\hat{T}(g_2)]$ then implies that

$$\chi(g_1) = \chi(g_2). \tag{53}$$

For the symmetric group S_N, all elements that belong to the same conjugacy class have the same *cycle structure*. This is readily seen by expressing a general permutation as a product of cyclic permutations. An arbitrary element $g \in S_N$ can be expressed as

$$g = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ g_1 & g_2 & g_3 & \cdots \end{pmatrix},$$

which denotes the permutation of a set of N objects in which the object in slot i is moved to slot g_i. Consider, for example, the particular element

$$g' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

of S_5. It corresponds to a sequence of two cyclic permutations $g' = (1 \to 3 \to 5 \to 1)(2 \to 4 \to 2)$ that can be expressed efficiently with the notation

$$g' \equiv (1, 3, 5)(2, 4). \tag{54}$$

Such a product of a three-cycle and a two-cycle is said to have a (3,2) cycle structure. It is then readily seen that the conjugate of g' by g (for any $g \in S_5$) is the element

$$g g' g^{-1} = (g_1, g_3, g_5)(g_2, g_4) \tag{55}$$

that has the same cycle structure as g'. The converse is also true: two elements of S_N with the same cycle structure are conjugate. Thus, we identify a class ρ of S_N with a so-called cycle structure $(\rho_1, \rho_2, \ldots)$, where ρ_i is the length of a cycle. For example, the class containing all permutations $\{g(1, 2, 3)(4, 5)g^{-1}; g \in S_5\}$ has a cycle structure (3,2). It may also be noted that if the lengths of the cycles defining a class are ordered such that

$$\rho_1 \geq \rho_2 \geq \rho_3 \geq \cdots, \tag{56}$$

then the class label $\rho \equiv (\rho_1, \rho_2, \ldots)$ is an ordered partition of N. It follows from this result that the number of classes of the group S_N is equal to the number of ordered partitions of N which, in turn, is equal to the number of inequivalent irreps of S_N. However, a partition defines a class whether or not it is ordered. Thus, it is often useful to denote a class by the set $\rho \equiv \{i^{r_i}\}$, where r_i denotes the number of cycles of length i.

The values of an S_N character $\{\chi_\rho\}$ for the classes of S_N are conventionally displayed in tables in which each row corresponds to an irrep of S_N. Thus, for example, because S_3 has three irreps, labeled by $\lambda = (3)$, (21), and (1^3), and three classes, $\rho = (1^3)$, (2,1), and (3), its character table is a 3×3 array $\{\chi^\lambda_\rho\}$ as shown in Table I. The irreducible S_N characters satisfy the orthogonality relations

TABLE I. The character table for S_3 (n_ρ is the number of group elements in the class ρ).

Class	(1^3)	(2,1)	(3)
n_ρ	1	3	2
$\chi^{(3)}$	1	1	1
$\chi^{(21)}$	2	0	−1
$\chi^{(1^3)}$	1	−1	1

$$\sum_{\rho \vdash N} n_\rho \chi^\lambda_\rho \chi^{\lambda'}_\rho = N! \delta_{\lambda,\lambda'}, \tag{57}$$

where n_ρ is the number of elements of S_N in the class ρ.

The number n_ρ is determined as follows. The number of elements of the group S_N is equal to $N!$, which is the number of ways of ordering the integers $1, 2, \ldots, N$. Thus, the number of elements of S_N in a class is the total number of inequivalent ways of distributing the N particle indices over the cycle structure of the class. There are a total of $N!$ distributions. However, different distributions that correspond simply to a permutation of cycles of the same length correspond to the same element of S_N. For example, the distributions (5,6,7)(1,2)(3,4) and (5,6,7)(3,4)(1,2) over the cycle structure (3,2,2) denote identical permutations. Thus, it is necessary to divide $N!$ by the number of orderings of the cycles of the same length. If r_i denotes the number of cycles of length i in the cycle structure ρ, this number is $r_1! r_2! \cdots r_N!$. Also, all cyclic permutations of the numbers within a cycle correspond to the same permutation, e.g., (1,3,5), (3,5,1), and (5,1,3) are identical permutations as can be seen from the definition of Eq. (54). The number of identical permutations obtained in this way is $1^{r_1} 2^{r_2} \cdots N^{r_N}$. Thus, the number of permutations n_ρ in the class $\rho \equiv \{i^{r_i}\}$ of S_N is given by

$$n_\rho = \frac{N!}{r_1! r_2! \cdots r_N! 1^{r_1} 2^{r_2} \cdots N^{r_N}}. \tag{58}$$

We now consider the characters of the group $U(n)$. First recall that every $U(n)$ matrix can be brought to diagonal form by a unitary transformation. Such a unitary transformation does not change the trace of a matrix. Consequently, every $U(n)$ matrix is conjugate to a diagonal matrix of the form given by

$$z = \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{pmatrix}, \qquad |z_i| = 1. \tag{59}$$

And each $U(n)$ class contains a representative diagonal matrix with diagonal entries that we denote by a set of complex numbers $z = \{z_i\}$. For such a $U(n)$ matrix, it is seen from Eq. (19) that

$$\hat{U}^{(1)}(z)\psi_\nu = z_\nu \psi_\nu, \tag{60}$$

and from Eq. (20) for an N-particle tensor-product wave function that

$$\hat{U}^{(N)}(z)\Psi_{\nu_1 \nu_2 \cdots \nu_N} = \left(\prod_{j=1}^{N} z_{\nu_j}\right)\Psi_{\nu_1 \nu_2 \cdots \nu_N}. \tag{61}$$

Thus, for an N-particle state $\Psi^{(\lambda)}$ of weight λ, of a U(n) tensor representation, it is determined that

$$\hat{U}^{(N)}(z)\Psi^{(\lambda)} = (z_1)^{\lambda_1}(z_2)^{\lambda_2}\cdots(z_n)^{\lambda_n}\Psi^{(\lambda)}. \tag{62}$$

The character of a U(n) representation is therefore a function of the n complex variables $\{z_i\}$. For example, the fundamental U(n) irrep $\{1\}$ has character

$$s_1 = \sum_{i=1}^{n} z_i. \tag{63}$$

The character s_λ of a general tensor irrep of U(n), with highest weight regarded as a partition $\lambda \vdash N$, is now obtained from the character χ^λ of the dual S_N irrep as follows. First determine the character of the reducible $S_N \times U(n)$ representation $\hat{T}$ on $\mathbb{H}_n^N$. This is easy in the basis defined by Eq. (20) because the action of a group element $(\pi, z) \in S_N \times U(n)$, with z diagonal, is simply to permute the basis functions and multiply them by z-dependent factors. For example, the identity

$$\hat{T}(\pi_{12}, z)\Psi_{\nu_1\nu_2\nu_3} = z_{\nu_1}z_{\nu_2}z_{\nu_3}\Psi_{\nu_2\nu_1\nu_3} \tag{64}$$

implies that (after normalizing, as necessary, so that $\{\Psi_{\nu_1\nu_2\nu_3}\}$ is an orthonormal basis)

$$\langle\Psi_{\nu_1\nu_2\nu_3}, \hat{T}(\pi_{12}, z)\Psi_{\nu_1\nu_2\nu_3}\rangle = \delta_{\nu_1,\nu_2}z_{\nu_1}^2 z_{\nu_3}, \tag{65}$$

and hence

$$\sum_{\nu_1,\nu_2,\nu_3}\langle\Psi_{\nu_1\nu_2\nu_3}, \hat{T}(\pi_{12}, z)\Psi_{\nu_1\nu_2\nu_3}\rangle = p_2(z)p_1(z), \tag{66}$$

where $p_k(z)$, for any positive integer k, is the so-called *power sum*

$$p_k(z) = \sum_{i=1}^{n} z_i^k. \tag{67}$$

It will be noted that the permutation $\pi_{12} \in S_3$ belongs to the class with cycle structure (2,1). In general, one finds that the character Ξ^N of the reducible representation $\hat{T}$ of $S_N \times U(n)$ on $\mathbb{H}^N$ has values

$$\Xi^N(\rho, z) = p_\rho(z), \tag{68}$$

where, for $\rho = \{i^{r_i}\}$,

$$p_\rho(z) = [p_1(z)]^{r_1}[p_2(z)]^{r_2}[p_3(z)]^{r_3}\cdots. \tag{69}$$

Now, if s_λ denotes the character of the U(n) irrep of highest weight $\lambda \vdash N$, it follows from Eq. (44) of the Schur-Weyl theorem that

$$\Xi^N(\rho, z) = \sum_{\lambda\vdash N}\chi_\rho^\lambda s_\lambda(z). \tag{70}$$

Thus, we obtain the simple identity

$$\sum_{\lambda\vdash N}\chi_\rho^\lambda s_\lambda(z) = p_\rho(z). \tag{71}$$

Use of the orthogonality relation (57) together with Eq. (71) leads to the well-known expression for U(n) characters

$$s_\lambda(z) = \frac{1}{N!}\sum_{\rho\vdash N} n_\rho \chi_\rho^\lambda p_\rho(z), \tag{72}$$

first derived by Schur (1901).

Characters of the U(n) polynomial irreps, denoted by s_λ, are known as *Schur functions* or *S functions*. They were studied as sets of orthogonal symmetric polynomials long before they were determined by Schur to be characters of the unitary groups (Macdonald, 1995). Some of the above relationships also appear to have been known long ago in different contexts. For example, as pointed out by Ledermann (1987) and Fulton and Harris (1991), Eq. (71) can also be derived from a formula given by Frobenius (1900) for S_N characters. What is remarkable is how simply and easily Schur's formula, Eq. (72), follows from a consideration of the dual actions of the groups S_N and U(n) on the space $\mathbb{H}_n^N$.

It is worth noting that Eq. (72) has the remarkable and valuable property of not only relating characters of S_N and U(n) irreps for all positive integer values of N and n, but of doing so in an N- and n-independent way. Thus, many results arising from character theory and Schur-Weyl duality are N and n independent.

The particular significance of Schur-Weyl duality for the nuclear and atomic shell model is that it enables the Pauli constraints on systems of identical particles to be taken into account by simply restricting to appropriate combinations of unitary group representations, as the following section will show. Further information about the many varied uses of Schur functions in physics are given in the works of Wybourne (1970) and King (1975).

IV. APPLICATIONS OF SCHUR-WEYL DUALITY

In addition to the character formula (72), the one-to-one correspondence between irreps of the unitary and symmetric groups also implies a linkage between other important operations on irreps of these groups, e.g., branching rules and tensor products, which we now consider. Such linkages, which are thematic of duality, enable results derived for the irreps of one group to be applied to the irreps of the other. Later sections will include examples from other dual-pair situations.

Recall that a representation $\hat{T}$ of a group G automatically defines a representation of any subgroup $H \subseteq G$ known as the restriction of the representation $\hat{T}$ to the subgroup. However, even if a representation $\hat{T}$ is irreducible as a representation of G, its restriction to a subgroup $H \subset G$ is generally reducible. Branching rules give expansions of the restriction of irreps of a group as direct sums of subgroup irreps. For example, on restriction to an SO(2) subgroup, an irrep $[L]$ of the rotation group SO(3), labeled by a positive integer-valued angular-momentum quantum number L, branches to a direct sum of $(2L + 1)$ SO(2) irreps labeled by integer-valued components of the angular momentum $M = -L, -L + 1, \ldots, +L$ about the axis of SO(2) rotations. This well-known result is expressed formally by the branching rule

$$\mathrm{SO}(3)\downarrow\mathrm{SO}(2) : [L]\downarrow\bigoplus_{M=-L}^{L}[M]. \tag{73}$$

A. Branching rules for symmetric and unitary groups

This section shows that several branching rules for symmetric and unitary groups are expressed directly in terms of the coefficients appearing in decompositions of S_N and $U(n)$ tensor products.

Let (κ), (τ), and (λ) denote irreps of S_N, each defined by an ordered partition of N. The reduction of the tensor products of pairs of S_N irreps then defines a set of $\{\gamma^{\lambda}_{\kappa\tau}\}$ coefficients, with integer values, by the expansion

$$(\kappa)\otimes(\tau)=\bigoplus_{\lambda}\gamma^{\lambda}_{\kappa\tau}(\lambda). \tag{74}$$

Similarly, if $\{\kappa\}$, $\{\tau\}$, and $\{\lambda\}$ denote $U(n)$ tensor irreps, defined by ordered partitions of integers, the tensor products of such $U(n)$ irreps determine integer-valued $\{\Gamma^{\lambda}_{\kappa\tau}\}$ coefficients (known as Littlewood-Richardson coefficients) by the expansion

$$\{\kappa\}\otimes\{\tau\}=\bigoplus_{\lambda}\Gamma^{\lambda}_{\kappa\tau}\{\lambda\}. \tag{75}$$

In the latter case, if $\kappa\vdash N_1$, $\tau\vdash N_2$, and $\lambda\vdash N_3$, then $\Gamma^{\lambda}_{\kappa\tau}$ is zero unless $N_3=N_1+N_2$, i.e., unless $|\lambda|=|\kappa|+|\tau|$. The above equations can be expressed equivalently in terms of characters, i.e.,

$$\chi^{\kappa}_{\rho}\chi^{\tau}_{\rho}=\sum_{\lambda}\gamma^{\lambda}_{\kappa\tau}\chi^{\lambda}_{\rho}, \tag{76}$$

$$s_{\kappa}(z)s_{\tau}(z)=\sum_{\lambda}\Gamma^{\lambda}_{\kappa\tau}s_{\lambda}(z). \tag{77}$$

Because the characters of S_N and $U(n)$ are known, the coefficients $\gamma^{\lambda}_{\kappa\tau}$ and $\Gamma^{\lambda}_{\kappa\tau}$ can be computed and various methods exist for their computation. For example, the $\gamma^{\lambda}_{\kappa\tau}$ coefficients are readily computed by use of the orthogonality property of S_N characters and the $\Gamma^{\lambda}_{\kappa\tau}$ coefficients can be determined by manipulation of Young diagrams using the so-called Littlewood-Richardson rule (Hamermesh, 1962).

We now consider branching rules.

Theorem 2: The coefficients appearing in the expansions of the branching rules

$$S_{N_1+N_2}\downarrow S_{N_1}\times S_{N_2}:\quad (\lambda)\downarrow\bigoplus_{\kappa\vdash N_1,\tau\vdash N_2}\Gamma^{\lambda}_{\kappa\tau}(\kappa)\times(\tau), \tag{78}$$

$$U(mn)\downarrow U(m)\times U(n):\quad \{\lambda\}\downarrow\bigoplus_{\kappa\vdash|\lambda|,\tau\vdash|\lambda|}\gamma^{\lambda}_{\kappa\tau}\{\kappa\}\times\{\tau\}, \tag{79}$$

$$U(m+n)\downarrow U(m)\times U(n):\quad \{\lambda\}\downarrow\bigoplus_{N_1+N_2=|\lambda|}\ \bigoplus_{\mu\vdash N_1,\nu\vdash N_2}\Gamma^{\lambda}_{\mu\nu}\{\mu\}\times\{\nu\}, \tag{80}$$

for the symmetric and unitary groups are identical to the coefficients in the S_N and $U(n)$ tensor-product reductions of Eqs. (74) and (75).

Proof: First we prove the branching rule (78) starting with the observation that if $N=N_1+N_2$ then

$$\mathbb{H}^{N}_{n}=\mathbb{H}^{N_1}_{n}\otimes\mathbb{H}^{N_2}_{n}. \tag{81}$$

The space $\mathbb{H}^{N_1}_{n}$ carries a representation

$$\hat{T}^{N_1}_{n}=\bigoplus_{\kappa\vdash N_1}(\kappa)\times\{\kappa\} \tag{82}$$

of the group $S_{N_1}\times U(n)$ and the space $\mathbb{H}^{N_2}_{n}$ carries a representation

$$\hat{T}^{N_2}_{n}=\bigoplus_{\tau\vdash N_2}(\tau)\times\{\tau\} \tag{83}$$

of the group $S_{N_2}\times U(n)$. Thus, $\mathbb{H}^{N}_{n}$ carries a representation

$$\hat{T}^{N_1,N_2}_{n}=\bigoplus_{\kappa\tau}(\kappa)\times(\tau)\times(\{\kappa\}\otimes\{\tau\})=\bigoplus_{\kappa\tau\lambda}\Gamma^{\lambda}_{\kappa\tau}(\kappa)\times(\tau)\times\{\lambda\}, \tag{84}$$

with $\kappa\vdash N_1$ and $\tau\vdash N_2$, of the product group $S_{N_1}\times S_{N_2}\times U(n)$. We also know that the Hilbert space $\mathbb{H}^{N}_{n}$ carries a representation

$$\hat{T}^{N}_{n}=\bigoplus_{\lambda}(\lambda)\times\{\lambda\} \tag{85}$$

of the group $S_N\times U(n)$. Combining Eqs. (84) and (85), we obtain the branching rule

$$S_N\times U(n)\downarrow S_{N_1}\times S_{N_2}\times U(n):\ \hat{T}^{N}_{n}\downarrow\hat{T}^{N_1,N_2}_{n}, \tag{86}$$

and, hence, the branching rule for $S_N\downarrow S_{N_1}\times S_{N_2}$ given by Eq. (78). This branching rule is equivalently expressed in terms of characters by the identity

$$\chi^{\lambda}_{\rho_1\rho_2}=\sum_{\kappa,\tau}\Gamma^{\lambda}_{\kappa\tau}\chi^{\kappa}_{\rho_1}\chi^{\tau}_{\rho_2}, \tag{87}$$

where ρ_1 and ρ_2 are, respectively, classes of S_{N_1} and S_{N_2}, and $\rho_1\rho_2$ is regarded as a class of $S_N=S_{N_1+N_2}$.

To prove the branching rule of Eq. (79), consider the product Hilbert space

$$\mathbb{H}^{N}_{mn}=(\mathbb{H}_m\otimes\mathbb{H}_n)^{N}=\mathbb{H}^{N}_{m}\otimes\mathbb{H}^{N}_{n}, \tag{88}$$

where $\mathbb{H}_m$ is of dimension m and $\mathbb{H}_n$ is of dimension n.

According to the Schur-Weyl theorem, the Hilbert spaces $\mathbb{H}^{N}_{m}$ and $\mathbb{H}^{N}_{n}$ carry representations of $S_N\times U(m)$ and $S_N\times U(n)$ given, respectively, by

$$\hat{T}^{N}_{m}=\bigoplus_{\kappa\vdash N}^{l(\kappa)\le m}(\kappa)\times\{\kappa\},\qquad \hat{T}^{N}_{n}=\bigoplus_{\tau\vdash N}^{l(\tau)\le n}(\tau)\times\{\tau\}. \tag{89}$$

It follows that $\mathbb{H}^{N}_{mn}$ carries a representation

$$\hat{T}^{N}_{m,n}=\bigoplus_{\kappa\vdash N}^{l(\kappa)\le m}\ \bigoplus_{\tau\vdash N}^{l(\tau)\le n}[(\kappa)\otimes(\tau)]\times\{\kappa\}\times\{\tau\}=\bigoplus_{\lambda}\ \bigoplus_{\kappa\vdash N}^{l(\kappa)\le m}\ \bigoplus_{\tau\vdash N}^{l(\tau)\le n}\gamma^{\lambda}_{\kappa\tau}(\lambda)\times\{\kappa\}\times\{\tau\} \tag{90}$$

of the direct product group $S_N\times U(m)\times U(n)$, where $\gamma^{\lambda}_{\kappa\tau}$ is the coefficient of λ in the reduction of the tensor product $(\kappa)\otimes(\tau)$ given by Eq. (74). We also know from the Schur-Weyl theorem that the Hilbert space $\mathbb{H}^{N}_{mn}$ carries a representation of $S_N\times U(nm)$ given by

$$\hat{T}^N_{mn} = \bigoplus_{\lambda\vdash N}^{l(\lambda)\le mn} (\lambda)\times\{\lambda\}. \tag{91}$$

From these two expressions we obtain the branching rule

$$\mathrm{S}_N\times\mathrm{U}(mn)\downarrow \mathrm{S}_N\times\mathrm{U}(m)\times\mathrm{U}(n) : \hat{T}^N_{mn}\downarrow\hat{T}^N_{m,n}, \tag{92}$$

and, hence, the branching rule for $\mathrm{U}(mn)\downarrow\mathrm{U}(m)\times\mathrm{U}(n)$ given by Eq. (79).

Finally, to prove the third branching rule, Eq. (80), first observe that when restricted to the subgroup $\mathrm{U}(m)\times\mathrm{U}(n)$, the $\mathrm{U}(m+n)$ character for the irrep $\{\lambda\}$ is given by $s_\lambda(x,y)$, where (x,y) denotes the elements $(x_1,\ldots,x_m,y_1,\ldots,y_n)$ of a diagonal $\mathrm{U}(m+n)$ matrix. From Eq. (72) we have the identity

$$s_\lambda(x,y) = \frac{1}{N!}\sum_\rho n_\rho \chi^\lambda_\rho p_\rho(x,y). \tag{93}$$

Now by making use of Eq. (58) for n_ρ and Eq. (69) for p_ρ, we have

$$s_\lambda(z) = \frac{1}{N!}\sum_\rho \frac{N!}{r_1!r_2!r_3!\cdots 1^{r_1}2^{r_2}3^{r_3}\cdots} \times\chi^\lambda_\rho[p_1(z)]^{r_1}[p_2(z)]^{r_2}[p_3(z)]^{r_3}\cdots, \tag{94}$$

where r_i denotes the number of cycles of length i in the cycle structure $(1^{r_1},2^{r_2},3^{r_3},\ldots)$ of the class ρ. Thus, with $z=(x,y)$ and the definition $p_k(z)=\sum_{i=1}^n z_i^k$ [cf. Eq. (67)], it follows that

$$p_k(z) = p_k(x,y) = p_k(x)+p_k(y), \tag{95}$$

and that

$$[p_k(z)]^{r_k} = \sum_{s_k+t_k=r_k}\frac{r_k!}{s_k!t_k!}[p_k(x)]^{s_k}[p_k(y)]^{t_k}. \tag{96}$$

Hence,

$$p_\rho(x,y) = \sum_{\sigma,\tau}\frac{r_1!r_2!r_3!\cdots}{s_1!s_2!s_3!\cdots t_1!t_2!t_3!\cdots}p_\sigma(x)p_\tau(y), \tag{97}$$

where the sum is over all classes σ with cycle structure $(1^{s_1},2^{s_2},3^{s_3},\ldots)$ of the symmetric group S_{N_1} and all classes τ with cycle structure $(1^{t_1},2^{t_2},3^{t_3},\ldots)$ of the symmetric group S_{N_2} for any N_1, N_2 satisfying $N_1+N_2=N$ and $s_k+t_k=r_k$ for all k. Inserting Eq. (97) into Eq. (93), with the identity

$$n_\sigma = \frac{N_1!}{s_1!s_2!\cdots s_N!1^{s_1}2^{s_2}\cdots N^{s_N}}, \tag{98}$$

from Eq. (58), yields

$$s_\lambda(x,y) = \sum_{\sigma,\tau}\frac{n_\sigma n_\tau}{N_1!N_2!}\chi^\lambda_{\sigma\tau}p_\sigma(x)p_\tau(y), \tag{99}$$

and, with the expansion (87), we obtain

$$s_\lambda(x,y) = \sum_{N_1+N_2=N}\sum_{\substack{\sigma,\tau\\ \mu\vdash N_1,\nu\vdash N_2}}\frac{n_\sigma n_\tau}{N_1!N_2!}\Gamma^\lambda_{\mu\nu}\chi^\mu_\sigma\chi^\nu_\tau p_\sigma(x)p_\tau(y) = \sum_{N_1+N_2=N}\sum_{\mu\vdash N_1,\nu\vdash N_2}\Gamma^\lambda_{\mu\nu}s_\mu(x)s_\nu(y), \tag{100}$$

where the second equality follows from Eq. (72). This is the expression of the branching rule (80) in terms of characters and completes the proof of Theorem 2. ■

The branching rules given by Theorem 2 are remarkable because they show that coefficients defined for the tensor products of one group determine the branching rules of a different group. Even Eq. (79), which at a superficial glance might appear to be simply the inverse of Eq. (75), is seen to involve outer products of distinct unitary groups in contrast to Eq. (75) which is concerned with tensor products of a single unitary group.

It is useful to note that the above relationships between tensor products and branching rules for the symmetric and unitary groups hold for all values of N and n. However, one must be mindful of the fact that results of calculations involving symmetric group characters may lead to labels for $\mathrm{U}(n)$ irreps that do not exist for particular n values. Consider, for example, the following application of the Littlewood-Richardson rule: $\{1^2\}\otimes\{1\}=\{21\}\oplus\{1^3\}$. This is correct for all $\mathrm{U}(n)$ for which $n\ge 3$. But $\{1^3\}$ does not exist as an irrep of $\mathrm{U}(2)$. The irrep label $\{1^3\}$ is therefore discarded and the correct relation for $\mathrm{U}(2)$ is $\{1^2\}\otimes\{1\}=\{21\}$.

B. Symmetrized tensor products: Plethysms

The concept of a plethysm as a symmetrized tensor power was conceived by Littlewood (1936) in a natural generalization of the Schur-Weyl theorem. Littlewood denoted the plethysm operation symbolically by $\{\kappa\}\otimes\{\lambda\}$. However, in the following, we use the symbol $\textcircled{P}$ instead of $\otimes$, which we reserve for a tensor product. Plethysm was introduced into the mainstream of physics by Smith and Wybourne (1967, 1968) and Wybourne (1970) and subsequently employed by many for branching rule calculations and other applications, notably in atomic and nuclear spectroscopy.

Suppose, for example, that $\mathbb{H}$ is a space of single-particle wave functions that carries an irrep of a group such as $\mathrm{U}(3)$ and one wants to know what representation of $\mathrm{S}_N\times\mathrm{U}(3)$ is carried by the corresponding tensor-product space $\mathbb{H}^N=\mathbb{H}^{\otimes N}$ of N-particle wave functions. When $\mathbb{H}$ is three dimensional and the irrep of $\mathrm{U}(3)$ carried by $\mathbb{H}$ is the defining representation, the answer is already given by the Schur-Weyl theorem. In general it is given by an expansion of irreps determined by the plethysm operation. However, as in the decomposition of a tensor product of SU(2) representations as sums of irreps obtained by the use of Clebsch-Gordan coefficients, it is not generally important to know how to derive the coefficients in the expansion of a plethysm because computer programs are available for that purpose. It is more important to understand what the coefficients are and how to use them. Thus, the value of plethysms lies in the availability of computer programs to evaluate them; cf. Carvalho and D'Agostino (2001a, 2001b), and references therein. Thus, as soon as the answer to a problem is expressed as a plethysm, it is effectively solved.

If $\mathbb{H}_m$ is an m-dimensional Hilbert space, it carries the defining irrep $\{1\}$ of the group $\mathrm{U}(m)$ and the tensor product $\mathbb{H}^N_m$ of N copies of $\mathbb{H}_m$ carries a reducible representation of $\mathrm{S}_N\times\mathrm{U}(m)$, denoted here by $\hat{T}^N_1$. According to the

Schur-Weyl theorem, this representation is a direct sum of irreps given by

$$\hat{T}_1^N = \bigoplus_{\lambda \vdash N} (\lambda) \times \{\lambda\}, \tag{101}$$

where the sum over partitions of N is restricted to λ with no more than m parts. It is then meaningful to regard the U(m) irrep $\{\lambda\}$ as the cofactor of the S_N irrep (λ) in the expansion (101). Thus, we say that $\{\lambda\}$ is a symmetrized tensor power of the irrep $\{1\}$, and denote it by the so-called plethysm

$$\{1\} \circledcirc \{\lambda\} = \mathbf{P}^{(\lambda)} \hat{T}_1^N = \{\lambda\}, \tag{102}$$

where $\mathbf{P}^{(\lambda)}$ is a projection operator that picks out the cofactor of the S_N irrep (λ) in a representation of $S_N \times U(m)$. This equation is equivalent to Eq. (101). However, its value is that it leads to a powerful generalization of the Schur-Weyl theorem. For, if $\mathbb{H}_m$ is also the carrier space for an m-dimensional irrep $\{\kappa\}$ of a group U(n) for some $n < m$ and $\hat{T}_\kappa^N$ is the corresponding reducible representation of $S_N \times U(n)$ carried by $\mathbb{H}_m^N$ then, by definition, the cofactor of the S_N irrep (λ) in the expansion of this irrep defines a general plethysm for any $\lambda \vdash N$ by

$$\{\kappa\} \circledcirc \{\lambda\} = \mathbf{P}^{(\lambda)} \hat{T}_\kappa^N. \tag{103}$$

A plethysm defined in this way can be evaluated using S_N characters. Let Ξ_κ^N denote the character of the $S_N \times U(n)$ representation $\hat{T}_\kappa^N$ carried by $\mathbb{H}_m^N$. Using the orthogonality relationship (57) for S_N characters,

$$\chi^\lambda \cdot \chi^\kappa = \frac{1}{N!} \sum_\rho n_\rho \chi_\rho^\lambda \chi_\rho^\kappa = \delta_{\lambda,\kappa}, \tag{104}$$

Equation (103) is expressed in terms of characters by

$$s_\kappa \circledcirc s_\lambda = \chi^\lambda \cdot \Xi_\kappa^N. \tag{105}$$

For example, with the character Ξ_1^N given by Eq. (68), one regains the previously derived identity

$$s_\lambda = s_1 \circledcirc s_\lambda = \chi^\lambda \cdot \Xi_1^N = \frac{1}{N!} \sum_\rho n_\rho \chi_\rho^\lambda p_\rho. \tag{106}$$

We also obtain a useful and insightful expression for the plethysm $s_\kappa \circledcirc s_\lambda$ from the observation that the unitary irrep $\{\kappa\}$ of U(n) on $\mathbb{H}_m$ can be regarded as a map from U(n) to U(m), i.e., to the fundamental m-dimensional irrep $\{1\}$ of U(m). Recall that the character of a U(m) matrix is given by its trace. Thus, the character s_1 of the fundamental U(m) irrep $\{1\}$ is a function

$$s_1(z) = \sum_{i=1}^m z_i \tag{107}$$

of a set of variables $\{z_1, \ldots, z_m\}$ corresponding to the diagonal entries of U(m) matrices. Because each class of a unitary group contains a diagonal matrix, we can restrict consideration to subsets of diagonal matrices. Thus, we consider a diagonal U(n) matrix

$$M^{(n)}(x) = \mathrm{diag}[x_1, x_2, \ldots, x_n]. \tag{108}$$

In a suitable basis for an m-dimensional U(n) irrep $\{\kappa\}$, this matrix maps to the $m \times m$ matrix

$$M^{(m)}(z^{(\kappa)}(x)) = \mathrm{diag}[z_1^{(\kappa)}(x), z_2^{(\kappa)}(x), \ldots, z_m^{(\kappa)}(x)]. \tag{109}$$

It follows that the character of the irrep $\{\kappa\}$ is given by

$$s_\kappa(x) = \sum_{i=1}^m z_i^{(\kappa)}(x) = s_1(z^{(\kappa)}(x)). \tag{110}$$

It also follows that $\Xi_\kappa^N(\rho, x) = \Xi_1^N(\rho, z^{(\kappa)}(x))$ and hence, because $\chi^\lambda \cdot \Xi_1^N = s_\lambda$, that

$$[s_\kappa \circledcirc s_\lambda](x) = s_\lambda(z^{(\kappa)}(x)). \tag{111}$$

As an example, let $n = 3$ and $\{\kappa\} = \{2\}$. The U(3) irrep $\{2\}$ has dimension $m = 6$ and character given by

$$s_2(x) = \sum_{i \le j}^3 x_i x_j = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_1 x_3. \tag{112}$$

It is equal to the U(6) character $s_1(z) = \sum_i^6 z_i^{(2)}(x)$ with

$$\begin{aligned} &z_1^{(2)}(x) = x_1^2, \quad z_2^{(2)}(x) = x_2^2, \quad z_3^{(2)}(x) = x_3^2, \\ &z_4^{(2)}(x) = x_1 x_2, \quad z_5^{(2)}(x) = x_2 x_3, \quad z_6^{(2)}(x) = x_1 x_3. \end{aligned} \tag{113}$$

Thus, the plethysm $s_2 \circledcirc s_2$ for U(3) is given by

$$\begin{aligned} [s_2 \circledcirc s_2](x) &= s_2(z^{(2)}(x)) = \sum_{i \le j}^6 z_i^{(2)}(x) z_j^{(2)}(x) \\ &= x_1^4 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^3 x_2 + x_1^2 x_2 x_3 \\ &\quad + x_1^3 x_3 + \cdots. \end{aligned} \tag{114}$$

Then from a knowledge of the S functions and their orthogonality properties (Macdonald, 1995), one obtains

$$s_2 \circledcirc s_2 = s_4 + s_{2^2}. \tag{115}$$

Plethysms provide powerful tools for numerous operations arising in the applications of symmetry to quantum mechanical systems. An early use of plethysms in the nuclear shell model by Elliott (1958a, 1958b) made use of laborious hand calculations by Ibrahim (1951, 1952) to determine which SU(3) irreps occur in the nuclear $(2s1d)$ shell and with what S_N symmetries. Such calculations can now be carried out quickly and easily by use of the available computer codes. As discussed in Sec. V, a knowledge of the S_N symmetry of an SU(3) irrep is required in order that the SU(3) wave functions, in nuclear physics, can be combined with spin and isospin wave functions of complementary S_N symmetry to form totally antisymmetric states. Because a U(3) irrep remains irreducible on restriction to its SU(3) subgroup, the first problem was to classify the U(3) irreps that occur in the $(2s1d)$ shell by their $S_N \times U(3)$ symmetries. A single nucleon in the $(2s1d)$ shell belongs to the six-dimensional U(3) irrep $\{2\}$ and so the N-nucleon states with S_N symmetry (κ) with $\kappa \vdash N$ span a U(3) representation given by $\{2\} \circledcirc \{\kappa\}$. For example, using the code of Carvalho and D'Agostino (2001b), we obtain the U(3) plethysm

$$\begin{aligned} \{2\} \circledcirc \{32\} = &\{541\} \oplus \{442\} \oplus \{532\} \oplus \{64\} \oplus 2\{622\} \\ &\oplus \{73\} \oplus \{721\} \oplus \{82\} \oplus \{631\}. \end{aligned} \tag{116}$$

Thus, the space of $N = 5$ nucleons in the $(2s1d)$ shell with S_5 symmetry (32) is a sum of U(3) irreps $\{\lambda_1\lambda_2\lambda_3\}$ and, hence, of SU(3) irreps $(\lambda\mu) = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$ given by the SU(3) plethysm

$$(2) \circledcirc \{32\} = (13) \oplus (02) \oplus (21) \oplus (24) \oplus 2(40) \oplus (43) \oplus (51) \oplus (62) \oplus (32), \tag{117}$$

in agreement with a result given by Elliott (1958a, 1958b). [Note that the round brackets in this equation denote SU(3) irreps rather than S_N ireps as used elsewhere.]

An important application of plethysms is to the evaluation of branching rules. The above definition of a plethysm shows that, whereas a Hilbert space $\mathbb{H}_m$ that carries an m-dimensional irrep $\{\kappa\}$ of a group U(n) is also the carrier space for an irrep $\{1\}$ of the group U(m), a subspace $\mathbb{H}_m^{\{\lambda\}} \subset \mathbb{H}_m^N$ that carries a U(m) irrep $\{\lambda\}$, with $\lambda \vdash N$ is the carrier space for a (generally reducible) representation $\{\kappa\} \circledcirc \{\lambda\}$ of U(n). Thus, if U(n) is a subgroup of U(m) with an m-dimensional irrep κ, the restriction of the U(m) irrep $\{\lambda\}$ to U(n) is given by the plethysm $\{\kappa\} \circledcirc \{\lambda\}$. In other words, the branching rule

$$\mathrm{U}(m) \downarrow \mathrm{U}(n) : \{1\} \downarrow \{\kappa\} \tag{118}$$

implies the general rule

$$\mathrm{U}(m) \downarrow \mathrm{U}(n) : \{\lambda\} \downarrow \{\kappa\} \circledcirc \{\lambda\}. \tag{119}$$

In fact, the above example of computing the symmetrized tensor products of SU(3) irreps was viewed by Elliott (1958a, 1958b) as a calculation of the U(6) $\downarrow$ U(3) branching rules required for the classification of $(2s1d)$-shell states that reduce the subgroup chain

$$\mathrm{U}(6) \supset \mathrm{U}(3) \supset \mathrm{SO}(3) \supset \mathrm{SO}(2). \tag{120}$$

Thus, because U(3) is a subgroup of the U(6) group whose defining six-dimensional irrep $\{1\}$ satisfies the branching rule U(6) $\downarrow$ U(3) : $\{1\} \downarrow \{2\}$ it follows, for example, that the U(6) irrep $\{32\}$ restricts to the U(3) representation $\{2\} \circledcirc \{32\}$.

Plethysms are also used for calculating the properties of other groups that are subgroups or contain subgroups of general linear or unitary groups. For example, if a group G contains some U(n) group as a subgroup one can restrict the characters of G to U(n) and thereby express them as sums of S functions. In this way, calculations involving characters of irreps of a given group can be evaluated by means of operations on S functions and the results reexpressed in terms of characters of the group under study. Conversely, if the group is a subgroup of a unitary group, the characters of its irreps can be regarded as linear combinations of S functions with their arguments restricted to the subgroup. For example, on restriction of U(3) to its SO(3) subgroup, we have the branching rules

$$\mathrm{U}(3) \downarrow \mathrm{SO}(3) : \{2\} \downarrow [L = 2] \oplus [L = 0], \tag{121}$$

$$: \{0\} \downarrow [L = 0]. \tag{122}$$

Thus, if $\overline{\{2\}}$ is now used to denote the restriction of the U(3) irrep $\{2\}$ to SO(3), we can make the identification

$$[L = 2] \equiv \overline{\{2\}} - \overline{\{0\}}. \tag{123}$$

This device was introduced by Littlewood (1944a). It makes the most sense when the formulas for plethysms are written in terms of S functions because, while it is not clear what the negative of a group representation means, the negative of a Schur function is well defined.

In many applications there is a need to apply plethysms sequentially and to representations which may be expressed as tensor products of other representations or linear combinations of irreps. Moreover, as Eq. (123) illustrates, one encounters applications in which combinations of irreps occur with negative coefficients. Rules for expressing plethysms of algebraic combinations of Schur functions in terms of sums and products of simple plethysms were determined by Littlewood (1944b) and extended by Smith and Wybourne (1967, 1968) and others. The following rules, expressed in terms of functions A, B, and C, which may be any combinations (sums and sums of products) of S functions, are taken from the book of Wybourne (1970). The first rule

$$(A \circledcirc B) \circledcirc C = A \circledcirc (B \circledcirc C) \tag{124}$$

shows that while plethysms are not commutative, they are associative. It is next observed that a plethysm is distributive on the right with respect to addition, subtraction, and multiplication, i.e., it satisfies the identities

$$A \circledcirc (BC) = (A \circledcirc B)(A \circledcirc C), \tag{125}$$

$$A \circledcirc (B \pm C) = A \circledcirc B \pm A \circledcirc C. \tag{126}$$

Plethysms are not distributive on the left but obey the combination rules

$$(A + B) \circledcirc \{\lambda\} = \sum_{\mu\nu} \Gamma^{\lambda}_{\mu\nu} (A \circledcirc \{\mu\})(B \circledcirc \{\nu\}), \tag{127}$$

$$(A - B) \circledcirc \{\lambda\} = \sum_{\mu\nu} (-1)^{N_\nu} \Gamma^{\lambda}_{\mu\nu} (A \circledcirc \{\mu\})(B \circledcirc \{\tilde{\nu}\}), \tag{128}$$

$$(AB) \circledcirc \{\lambda\} = \sum_{\mu\nu} \gamma^{\lambda}_{\mu\nu} (A \circledcirc \{\mu\})(B \circledcirc \{\nu\}), \tag{129}$$

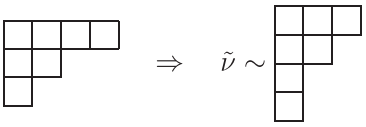
where $\gamma^{\lambda}_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ are, respectively, the coefficients appearing in the S_N and U(n) branching rules (74) and (75), $N_\nu = \sum_i \nu_i$, and $\{\tilde{\nu}\}$ denotes the irrep defined by the partition of N_ν conjugate to ν as defined by Eq. (132). The above rules are consistent with the relationships, which follow from Eq. (105),

$$s_\kappa \circledcirc (s_\mu s_\nu) = \sum_\lambda \Gamma^{\lambda}_{\mu\nu} s_\kappa \circledcirc s_\lambda, \tag{130}$$

$$(s_\mu s_\nu) \circledcirc s_\lambda = \left(\sum_\kappa \Gamma^{\kappa}_{\mu\nu} s_\kappa \right) \circledcirc s_\lambda. \tag{131}$$

The partition $\tilde{\nu} \vdash N_\nu$ appearing in Eqs. (128) and (129) is conjugate to the partition $\nu \vdash N_\nu$, defined such that the Young diagram for $\tilde{\nu}$ is obtained from the diagram for ν by interchanging rows with columns; e.g.,

$$\nu \sim \quad \Rightarrow \quad \tilde{\nu} \sim \ . \tag{132}$$



The characters of the S_{N_ν} irreps (ν) and $(\tilde{\nu})$ are related by the equation

$$\chi^{\tilde{\nu}}_{\rho} = \chi^{1^N}_{\rho}\chi^{\nu}_{\rho} = (-1)^{\rho}\chi^{\nu}_{\rho}, \tag{133}$$

with $(-1)^{\rho} = \pm 1$, according as ρ is a class of even or odd permutations.

V. UNITARY-UNITARY DUALITY

An important use of Schur-Weyl duality is to derive rules for constructing basis wave functions for systems of indistinguishable particles when the particle-wave functions belong to tensor-product spaces. For example, each particle might have space and spin wave functions; its wave functions would then belong to a tensor product of spatial and spin Hilbert spaces. If the particles are bosonlike, with integer intrinsic spins, then according to the spin-statistics theorem, their total wave functions should be symmetric under permutations. On the other hand, if they are fermionlike, and so have half-odd integer spins, they should be antisymmetric under odd permutations. The spin wave functions and the spatial wave functions separately can have other symmetries so long as their combinations are symmetric (for bosons) or antisymmetric (for fermions). The use of Schur-Weyl duality to construct symmetric and antisymmetric many-particle wave functions in tensor-product spaces gives rise to $U(n) \times U(m)$ duality relationships which play a central role in the construction of coupling schemes for many-particle calculations in atomic and subatomic physics.

A. The unitary-unitary duality theorem

Let

$$\mathbb{H}_{mn} = \mathbb{H}_m \otimes \mathbb{H}_n \tag{134}$$

denote a tensor product of Hilbert spaces, of (say) spatial and spin wave functions for a single particle, of dimension m and n, respectively. Then $\mathbb{H}_m$ carries the m-dimensional irrep $\{1\}$ of $U(m)$, $\mathbb{H}_n$ carries the n-dimensional irrep $\{1\}$ of $U(n)$, and $\mathbb{H}_{mn}$ carries the mn-dimensional irrep $\{1\}$ of $U(mn)$. We distinguish these irreps by $\hat{U}^{\{1\}}_m$, $\hat{U}^{\{1\}}_n$, and $\hat{U}^{\{1\}}_{mn}$, respectively.

The question now arises: How does one build up symmetric and antisymmetric many-particle basis wave functions for irreps of $U(m) \times U(n)$ from single-particle wave functions in $\mathbb{H}_{mn}$? The answer is given by the unitary-unitary duality theorem. Let $\mathbb{H}^N_{mn}$ denote the tensor product of N copies of $\mathbb{H}_{mn}$

$$\mathbb{H}^N_{mn} = \underbrace{\mathbb{H}_{mn} \otimes \cdots \otimes \mathbb{H}_{mn}}_{N \text{ copies}} = \mathbb{H}^N_m \otimes \mathbb{H}^N_n. \tag{135}$$

According to the Schur-Weyl theorem, the subspace $\mathbb{H}^{\{N\}}_{mn}$ of fully symmetric N-particle wave functions in $\mathbb{H}^N_{mn}$ carries an irrep $(N) \times \{N\}$ of the direct product group $S_N \times U(mn)$. However, the fully symmetric irrep (N) of S_N is one-dimensional. It follows that the subspace of fully symmetric N-particle wave functions $\mathbb{H}^{\{N\}}_{mn} \subset \mathbb{H}^N_{mn}$ is the carrier space for the $U(mn)$ irrep $\{N\}$. Similarly, the subspace $\mathbb{H}^{\{1^N\}}_{mn}$ of fully antisymmetric N-particle wave functions in $\mathbb{H}^N_{mn}$ is the carrier space for the $U(mn)$ irrep $\{1^N\}$. The converse of these observations is given by the following theorem.

Theorem 3 (unitary-unitary duality): The groups $U(m)$ and $U(n)$ have dual representations on the fully symmetric and fully antisymmetric subspaces of $\mathbb{H}^N_{mn}$ in accordance with the branching rules:

$$U(mn) \downarrow U(m) \times U(n) : \{N\} \downarrow \bigoplus_{\kappa \vdash N} \{\kappa\} \times \{\kappa\}, \tag{136}$$

$$: \{1^N\} \downarrow \bigoplus_{\kappa \vdash N} \{\kappa\} \times \{\tilde{\kappa}\}, \tag{137}$$

where the sum in Eq. (136) extends over all partitions $\kappa \vdash N$ of length $l(\kappa) \leq \min(m, n)$ and the sum in Eq. (137) extends over all partitions $\kappa \vdash N$ for which $l(\kappa) \leq m$ and $l(\tilde{\kappa}) \leq n$, where $\tilde{\kappa}$ is defined by Eq. (132).

A direct proof of Theorem 3 is given by Rowe, Repka, and Carvalho (2011). The following proof shows that it is implied by the Schur-Weyl theorem.

Proof: According to the Schur-Weyl theorem, the Hilbert space $\mathbb{H}^N_{mn}$ carries the $S_N \times U(mn)$ representation $\hat{T}^N_{mn} = \bigoplus_{\tau \vdash N}^{l(\tau) \leq mn} (\tau) \times \{\tau\}$. Thus, according to the branching rule of Eq. (79), this representation restricts to the representation

$$\hat{T}^N_{m,n} = \bigoplus_{\tau} \bigoplus_{\kappa \vdash N}^{l(\kappa) \leq m} \bigoplus_{\lambda \vdash N}^{l(\lambda) \leq n} \gamma^{\tau}_{\kappa\lambda}(\tau) \times \{\kappa\} \times \{\lambda\} \tag{138}$$

of $S_N \times U(m) \times U(n)$. The component of this representation with S_N symmetry (τ) is then

$$\hat{T}^{(\tau)}_{m,n} = \bigoplus_{\kappa \vdash N}^{l(\kappa) \leq m} \bigoplus_{\lambda \vdash N}^{l(\lambda) \leq n} \gamma^{\tau}_{\kappa\lambda}(\tau) \times \{\kappa\} \times \{\lambda\}. \tag{139}$$

The tensor product of two S_N irreps $(\kappa) \otimes (\lambda)$ contains a copy of the identity irrep (N) if and only if $\lambda = \kappa$ (Hamermesh, 1962). Similarly, it contains a copy of the antisymmetric irrep (1^N) if and only if $\lambda = \tilde{\kappa}$ (the conjugate of κ). Moreover, no tensor product of two S_N irreps contains more than one copy of either (N) or (1^N). Thus,

$$\gamma^N_{\kappa\lambda} = \delta_{\kappa,\lambda}, \qquad \gamma^{1^N}_{\kappa\lambda} = \delta_{\kappa,\tilde{\lambda}}, \tag{140}$$

and we obtain

$$\hat{T}^{(N)}_{m,n} = \bigoplus_{\kappa \vdash N}^{l(\kappa) \leq m, l(\lambda) \leq n} (N) \times \{\kappa\} \times \{\kappa\}, \tag{141}$$

$$\hat{T}^{(1^N)}_{m,n} = \bigoplus_{\kappa \vdash N}^{l(\kappa) \leq m, \kappa_1 \leq n} (1^N) \times \{\kappa\} \times \{\tilde{\kappa}\}. \tag{142}$$

Comparing these results with those of the Schur-Weyl expression for the representation of $S_N \times U(mn)$ carried by the symmetric and antisymmetric components of $\mathbb{H}^N_{mn}$, for which

$$\hat{T}^{(N)}_{mn} = (N) \times \{N\}, \qquad \hat{T}^{(1^N)}_{mn} = (1^N) \times \{1^N\}, \tag{143}$$

leads to the results of the theorem. ■

Corollary 3: Let $\mathbb{H}^{\mathrm{S}}_{mn} = \bigoplus_{N=0}^{\infty} \mathbb{H}^{\{N\}}_{mn}$ denote the sum of the Hilbert spaces that carry the fully symmetric irreps $\{N\}$ of $\mathrm{U}(mn)$ and let $\mathbb{H}^{\mathrm{AS}}_{mn} = \bigoplus_{N=0}^{\infty} \mathbb{H}^{\{1^N\}}_{mn}$ denote the sum of the Hilbert spaces that carry the fully antisymmetric irreps $\{1^N\}$ of $\mathrm{U}(mn)$. The groups $\mathrm{U}(m)$ and $\mathrm{U}(n)$ then have dual representations on $\mathbb{H}^{\mathrm{S}}_{mn}$ given by $\bigoplus_{\kappa}\{\kappa\} \times \{\kappa\}$ and on $\mathbb{H}^{\mathrm{AS}}_{mn}$ given by $\bigoplus_{\kappa}\{\kappa\} \times \{\tilde{\kappa}\}$.

Proof: The corollary follows from the observation that each $\{\kappa\}$ is a partition of a non-negative integer N, i.e., $\kappa \vdash N$, and occurs once and only once in the sum. ■

As the following examples illustrate, unitary-unitary duality can be employed directly at an operational level to construct appropriately symmetrized wave functions for an N-particle system.

B. Symmetric and antisymmetric space-spin wave functions

The above duality relationships show that if $\mathbb{H}_m$ and $\mathbb{H}_n$ are, respectively, Hilbert spaces of spatial and spin wave functions for a particle, then the combinations of these wave functions, appropriate for a system of N bosons (with integer spins), belong to the fully symmetric subspace of the tensor-product space $\mathbb{H}^N_{mn} = \mathbb{H}^N_m \otimes \mathbb{H}^N_n$. This subspace contains only wave functions that are invariant under any permutation $P \in \mathrm{S}_N$.

We first consider a simple case in which $\{\psi^{\kappa}_i\}$ and $\{\varphi^{\lambda}_j\}$ denote orthonormal bases for S_N irreps (κ) and (λ), respectively. There is known to be precisely one S_N-invariant bilinear combination of these basis functions for each $\lambda = \kappa$. If the bases correspond, it is given by $\Phi^{\kappa} = \sum_i \psi^{\kappa}_i \otimes \varphi^{\kappa *}_i$, where the $*$ denotes complex conjugation.[2] This follows from the observation that, under a permutation $P \in \mathrm{S}_N$,

$$\Phi^{\kappa} \to \hat{P}\Phi^{\kappa} = \sum_{ijk} \psi^{\kappa}_j \otimes \varphi^{\kappa *}_k M^{(\kappa)}_{ji}(P) M^{(\kappa)*}_{ki}(P), \tag{144}$$

where $M^{(\kappa)}(P)$ is the matrix representing the permutation $P \in \mathrm{S}_N$, relative to the basis $\{\psi^{\kappa}_i\}$ or $\{\varphi^{\kappa}_i\}$. Thus, because the irrep (κ) is unitary, we determine that $\hat{P}\Phi^{\kappa} = \Phi^{\kappa}$.

It is important to note that it is always possible, and indeed natural, to choose bases for S_N irreps such that the matrices $M^{(\kappa)}(P)$ are real. The S_N invariants are then given by $\Phi^{\kappa} = \sum_i \psi^{\kappa}_i \otimes \varphi^{\kappa}_i$.

Now, for arbitrary values of m and n, let $\{\psi^{\kappa}_{i\alpha}\}$ denote an orthonormal basis for $\mathbb{H}^N_m$, where $\kappa \vdash N$ labels an $\mathrm{S}_N \times \mathrm{U}(m)$ irrep, i indexes a basis for the S_N irrep (κ), and α indexes a basis for the $\mathrm{U}(m)$ irrep $\{\kappa\}$. Let $\{\varphi^{\lambda}_{j\beta}\}$ denote a similarly defined basis for $\mathbb{H}^N_n$ which reduces the group $\mathrm{S}_N \times \mathrm{U}(n)$. The product functions $\{\psi^{\kappa}_{i\alpha} \otimes \varphi^{\lambda}_{j\beta}\}$ are then a basis for the tensor-product space $\mathbb{H}^N_m \otimes \mathbb{H}^N_n$. Thus, from the above results, an orthonormal basis for the totally symmetric subspace of $\mathbb{H}^N_m \otimes \mathbb{H}^N_n$, appropriate for a system of bosons, is given by the linear combinations

$$\Psi^{\kappa}_{\alpha\beta} = \frac{1}{\sqrt{d^{\kappa}}} \sum_i \psi^{\kappa}_{i\alpha} \otimes \varphi^{\kappa}_{i\beta}, \tag{145}$$

with κ running over the ordered partitions of N for which $l(\kappa) \le m$ and $l(\kappa) \le n$, and d^{κ} is the dimension of the S_N irrep (κ). These $\Psi^{\kappa}_{\alpha\beta}$ wave functions have the useful property that they reduce the subgroup chain

$$\begin{array}{ccccc} \mathrm{U}(mn) & \supset & \mathrm{U}(m) & \times & \mathrm{U}(n) \\ N & & \kappa & & \kappa \end{array}. \tag{146}$$

The bilinear combinations of N-particle spatial wave functions in $\mathbb{H}^N_m$ with N-particle spin wave functions in $\mathbb{H}^N_n$, appropriate for fermions, span the subspace of the tensor-product space $\mathbb{H}^N_m \otimes \mathbb{H}^N_n$ that is antisymmetric under the action of the symmetric group S_N.

As noted following Theorem 3, for every S_N irrep corresponding to a partition $\kappa \vdash N$ there is a so-called conjugate irrep, corresponding to the partition $\tilde{\kappa} \vdash N$ defined by Eq. (132). Bases for such conjugate irreps $\{\psi^{\kappa}_i\}$ and $\{\varphi^{\tilde{\kappa}}_i\}$ are naturally put into one-to-one correspondence, such that the matrices for these irreps are related by the identity

$$M^{(\tilde{\kappa})}_{ki}(P) = (-1)^P M^{(\kappa)}_{ki}(P), \tag{147}$$

with $(-1)^P = \pm 1$ according to whether P is an even or odd permutation. It is also observed that, if $\{\psi^{\kappa}_i\}$ and $\{\varphi^{\lambda}_j\}$ are orthonormal bases for S_N irreps, it is only possible to form antisymmetric bilinear combinations of these basis functions $\{\psi^{\kappa}_{i\alpha} \otimes \varphi^{\lambda}_{j\beta}\}$ if $\lambda = \tilde{\kappa}$. The transformation of the combination $\Phi^{\kappa} = \sum_i \psi^{\kappa}_i \otimes \varphi^{\tilde{\kappa}}_i$ under a permutation $P \in \mathrm{S}_N$ is then given by

$$\Phi^{\kappa} \to \hat{P}\Phi^{\kappa} = \sum_{ijk} \psi^{\kappa}_j \otimes \varphi^{\tilde{\kappa}}_k M^{(\kappa)}_{ji}(P) M^{(\tilde{\kappa})}_{ki}(P). \tag{148}$$

Thus, by choosing phases such that the matrices $M^{(\kappa)}(P)$ are both real and unitary, it follows that $P\Phi^{\kappa} = (-1)^P \Phi^{\kappa}$. Similarly, for arbitrary values of m and n, an orthonormal basis for the antisymmetric subspace of $\mathbb{H}^N_m \otimes \mathbb{H}^N_n$ is given by the linear combinations

$$\Psi^{\kappa}_{\alpha\beta} = \frac{1}{\sqrt{d^{\kappa}}} \sum_i \psi^{\kappa}_{i\alpha} \otimes \varphi^{\tilde{\kappa}}_{i\beta} \tag{149}$$

that reduce the subgroup chain

$$\begin{array}{ccccc} \mathrm{U}(mn) & \supset & \mathrm{U}(m) & \times & \mathrm{U}(n), \\ 1^N & & \kappa & & \tilde{\kappa} \end{array} \tag{150}$$

with κ running over the ordered partitions of N for which $l(\kappa) \le m$ and $l(\tilde{\kappa}) \le n$.

C. Antisymmetric space-spin-isospin wave functions

The $\mathrm{U}(m) \times \mathrm{U}(n)$ duality of representations can be applied to situations where products of more than two wave functions occur. Such situations arise, for example, in the nuclear shell model when nucleon wave functions are products of three components, spatial, spin, and isospin, or in elementary particle physics when many-quark systems, for example, have flavor, spin, and color degrees of freedom.

To illustrate the role of duality in such cases, consider a single-nucleon Hilbert space,

$$\mathbb{H} = \mathbb{H}_L \otimes \mathbb{H}_S \otimes \mathbb{H}_T, \tag{151}$$

[2]Note that S_N irreps are self-contragredient and, as a consequence, their Hilbert spaces are invariant under complex conjugation.

that is a tensor product of spaces $\mathbb{H}_L$ (for spatial wave functions), $\mathbb{H}_S$ (for spin wave functions), and $\mathbb{H}_T$ (for isospin wave functions). Because a nucleon has spin $1/2$ and isospin $1/2$, the spaces $\mathbb{H}_S$ and $\mathbb{H}_T$ have dimension two, and each carries a defining irrep of U(2). Denoting the dimension of $\mathbb{H}_L$ by n, it follows that $\mathbb{H}$ is of dimension $4n$, that it carries the defining irrep $\{1\}$ of U($4n$), and that it remains irreducible on restriction to the U(n) × U(4) subgroup.

A typical shell-model problem is to define a basis for the nuclear subspace $\mathbb{H}^{\{1^N\}}$ of fully antisymmetric wave functions in $\mathbb{H}^N$ and to classify such a basis by the quantum numbers associated with irreps of U(n), U$(2)_S$, and U$(2)_T$ and other useful groups. There are many possible coupling schemes.

The so-called *Wigner supermultiplet scheme* (Wigner, 1937; Hecht and Pang, 1969) of *LST* coupling starts with the four-dimensional spin-isospin space

$$\mathbb{H}_{ST} = \mathbb{H}_S \otimes \mathbb{H}_T, \tag{152}$$

which carries the standard irrep of U(4). The construction of a basis for the totally antisymmetric subspace of

$$\mathbb{H}^N = \mathbb{H}_L^N \otimes \mathbb{H}_{ST}^N \tag{153}$$

is thereby reduced to the standard problem, discussed in the first application, for which U(n) × U(4) duality applies. Thus, the branching rule of the duality theorem, Eq. (137),

$$\mathrm{U}(4n) \downarrow \mathrm{U}(n) \times \mathrm{U}(4) : \{1^N\} \downarrow \bigoplus_{\lambda \vdash N} \{\lambda\} \times \{\tilde{\lambda}\}, \tag{154}$$

implies that the fully antisymmetric irreps of U(n) × U(4) are given by the tensor product $\{\lambda\} \times \{\tilde{\lambda}\}$ irreps carried by subspaces of the fully antisymmetric subspace $\mathbb{H}^{\{1^N\}} \subset \mathbb{H}^N$.

A natural basis for U(4) is one that reduces the subgroup chain

$$\begin{array}{ccccccccc} U(4) & \supset & \mathrm{SU}(2)_S & \times & \mathrm{SU}(2)_T & \supset & \mathrm{U}(1)_S & \times & \mathrm{U}(1)_T. \\ \tilde{\lambda} & & S & & T & & M_S & & M_T \end{array} \tag{155}$$

A desirable choice for U(n) is one that reduces the subgroup chain

$$\begin{array}{ccccc} \mathrm{U}(n) & \supset & \mathrm{SO}(3)_L & \supset & \mathrm{SO}(2)_L, \\ \{\lambda\} & & L & & M_L \end{array} \tag{156}$$

where SO$(3)_L$ is the standard rotation group. These choices then give basis states for $\mathbb{H}^{\{1^N\}}$ that reduce the chain

$$\begin{array}{ccccccccc} \mathrm{U}(4n) & \supset & \mathrm{U}(n) \times \mathrm{U}(4) & \supset & \mathrm{SO}(3)_L \times \mathrm{SU}(2)_S \times \mathrm{SU}(2)_T. \\ 1^N & & \lambda \quad \tilde{\lambda} & \beta & L, M_L & & S, M_S & & T, M_T \end{array} \tag{157}$$

Note that the label β is included to denote the additional labels needed to provide a complete classification of basis states. Additional labels are provided, for example, by including an intermediate subgroup between U(n) and SO$(3)_L$. One possibility is to include the group O(n) in the chain

$$\mathrm{U}(n) \supset \mathrm{O}(n) \supset \mathrm{SO}(3)_L \supset \mathrm{SO}(2)_L. \tag{158}$$

An alternative, for a suitable choice of $\mathbb{H}_L$, is to include in the chain the group U(3), as in Elliott's shell model of nuclear rotational states (Elliott, 1958a, 1958b),

$$\mathrm{U}(n) \supset \mathrm{U}(3) \supset \mathrm{SO}(3)_L \supset \mathrm{SO}(2)_L, \tag{159}$$

where U(3) is the symmetry group of the spherical harmonic oscillator.

D. Unitary-unitary duality in boson systems

The above examples of the use of unitary-unitary duality for fermions have parallels in bosonic systems with multiple degrees of freedom. For example, in the interacting boson model with two kinds of boson, corresponding to neutron pairs and proton pairs of which each carries an irrep $\{1\}$ of U(6), the states of N such bosons carry an irrep $\{N\}$ of U(12). The N boson states can then be classified by the irrep labels in the subgroup chain U(12) ⊃ U(6) × U$_F$(2) which, in accordance with Theorem 3, are given by the branching rule of Eq. (136) for which $\kappa = \{N/2 + F, N/2 - F\}$, where F is known as F spin [see, for example, Isacker *et al.* (1986) for more details].

VI. METHODS OF SECOND QUANTIZATION

The techniques of second quantization were invented for the quantization of fields. However, they prove to be equally powerful and insightful in the many-body quantum mechanics of indistinguishable particles and in the theory of Lie algebras. At the time the terminology was introduced, it was common to regard standard quantum mechanics, in which the dynamics of particles was replaced by wave mechanics, as first quantization. On the other hand, field theory, which considers particles as the quanta of fields, was regarded as second quantization.

Quantization of the electromagnetic field was achieved by Born, Heisenberg, and Jordan (1926). Their theory can be understood, at an elementary level, as extending the Hamiltonian for a system of harmonic oscillators

$$\hat{H} = \sum_\nu \hbar\omega_\nu \left(c_\nu^\dagger c^\nu + \frac{1}{2} \right), \tag{160}$$

with raising and lowering operators that satisfy commutation relations

$$[c_\mu^\dagger, c_\nu^\dagger] = [c^\mu, c^\nu] = 0, \qquad [c^\nu, c_\mu^\dagger] = \delta_\mu^\nu, \tag{161}$$

to an infinite number of oscillators characterized by continuously variable frequencies. The raising operators for states of the electromagnetic field are then interpreted as creation operators for photons. This provides the fundamental link between the wave and particle theories of light. Note that we use lower indices for creation operators $\{c_\nu^\dagger\}$ and upper indices for annihilation operators $\{c^\nu\}$. This is to emphasize the fact that, whereas the creation operators transform as a basis for the standard irrep of a unitary group U(n), where n is the number of indices, the annihilation operators transform as a basis for the contravariant irrep; i.e., if a creation operator $c_\nu^\dagger$ transforms under an element $g \in \mathrm{U}(n)$ according to the equation $c_\nu^\dagger \to \sum_\mu c_\mu^\dagger g_{\mu\nu}$, then the corresponding annihilation operator transforms $c^\nu \to \sum_\mu c^\mu g_{\mu\nu}^*$.

At first sight, it would appear that the quantum mechanical interpretation of the particle-wave duality for fermions

(e.g., electrons) of nonzero rest mass is different. Unlike quantum electrodynamics, nonrelativistic quantum mechanics begins with particles and by means of the Schrödinger equation assigns wave functions to them to describe their states. Thus, it was not obvious what could be achieved by a second quantization of the fields (i.e., wave functions) to regain the particles. In retrospect, it is now realized that a field theory of electrons is provided by the Dirac equation (Dirac, 1928a, 1928b). Moreover, the derivation of the Dirac equation by factorization of the relativistic Hamiltonian for an electron has similarities with the factorization of the harmonic-oscillator Hamiltonian which gives the quantization of the electromagnetic field. However, because of the somewhat misleading particle-hole interpretation of the Dirac equation, the parallel was not recognized at the time. The vital step toward a fermionic field theory was taken a few years later by Jordan and Wigner (1928), who introduced operators $\{a_\nu^\dagger\}$ and $\{a^\nu\}$ that, respectively, create and annihilate fermions. Their success was based on a recognition that the Pauli principle, which states that two identical fermions cannot occupy the same state, is automatically accommodated if the fermion operators are required to satisfy anticommutation relations

$$\{a_\mu^\dagger, a_\nu^\dagger\} = \{a^\mu, a^\nu\} = 0, \qquad \{a_\mu^\dagger, a^\nu\} = \delta_\mu^\nu, \tag{162}$$

rather than commutation relations. This follows simply from the observation that

$$\{a_\nu^\dagger, a_\nu^\dagger\} = 2a_\nu^\dagger a_\nu^\dagger = 0. \tag{163}$$

One of the many advantages of using the methods of second quantization in nonrelativistic quantum mechanics is that it ensures the exchange symmetries of identical particles are respected without the need for labeling indistinguishable particles and symmetrizing (or antisymmetrizing) their wave functions. Thus, when expressed in terms of creation operators that commute with one another, the many-boson wave functions for identical bosons are automatically symmetric under exchange. Similarly, when expressed in terms of creation operators that anticommute with one another, the many-fermion wave functions for identical fermions are automatically antisymmetric under exchange. For example, the sets of three-boson and three-fermion states

$$|\mu\nu\kappa\rangle_B = c_\mu^\dagger c_\nu^\dagger c_\kappa^\dagger |0\rangle, \tag{164}$$

$$|\mu\nu\kappa\rangle_F = a_\mu^\dagger a_\nu^\dagger a_\kappa^\dagger |0\rangle, \tag{165}$$

where $|0\rangle$ is the zero-particle vacuum state are automatically symmetric and antisymmetric, respectively; e.g., $|\mu\nu\kappa\rangle_B = |\nu\mu\kappa\rangle_B$ and $|\mu\nu\kappa\rangle_F = -|\nu\mu\kappa\rangle_F$.

Another huge advantage of the second-quantization formalism is that it provides a powerful framework for the manipulation of Lie algebras and their representations from which the duality relationships discussed in this review emerge naturally. Consider a basis $\{X_{\mu,\nu}\}$ for the complex extension of the Lie algebra u(n) of the group U(n) with commutation relations

$$[X_{\mu,\nu}, X_{\mu',\nu'}] = \delta_{\nu,\mu'} X_{\mu,\nu'} - \delta_{\nu',\mu} X_{\mu',\nu}. \tag{166}$$

It follows from the boson commutation relations Eq. (161) that this Lie algebra has a *boson realization*

$$X_{\mu,\nu} \to \hat{X}_{\mu,\nu} = c_\mu^\dagger c^\nu. \tag{167}$$

This realization automatically extends the defining irrep $\{1\}$ of U(n) on the Hilbert space $\mathbb{H}$ to the symmetric irreps $\{N\}$ on symmetric subspaces $\mathbb{H}^{\{N\}} \subset \mathbb{H}^N$ for positive integer values of N. Thus, whereas the one-boson states $\{c_\nu^\dagger|0\rangle, \nu = 1, \ldots, n\}$ are a basis for the U(n) irrep $\{1\}$, the set of N-boson states $\{c_{\nu_1}^\dagger c_{\nu_2}^\dagger, \ldots, c_{\nu_N}^\dagger|0\rangle, \nu_1, \nu_2, \ldots, \nu_N = 1, \ldots, n\}$ are a basis for the U(n) irrep $\{N\}$.

It follows from these results that the unitary-unitary duality theorem has a natural expression in the language of second quantization. This is seen for the boson operators by regarding $\{c_{i\nu}^\dagger\}$ and $\{c^{i\nu}\}$, with commutation relations

$$[c^{i\mu}, c_{j\nu}^\dagger] = \delta_j^i \delta_\nu^\mu, \quad [c_{i\mu}^\dagger, c_{j\nu}^\dagger] = [c^{i\mu}, c^{j\nu}] = 0, \tag{168}$$

as creation and annihilation operators for the single-particle states of the tensor-product space $\mathbb{H} = \mathbb{H}_m \otimes \mathbb{H}_n$, with $i = 1, \ldots, m$ indexing a basis for $\mathbb{H}_m$ and $\nu = 1, \ldots, n$ indexing a basis for $\mathbb{H}_n$. The Lie algebras u(mn), u(m), and u(n) are then realized by the operators

$$\hat{X}_{i\mu,j\nu}^{(mn)} = c_{i\mu}^\dagger c^{j\nu}, \qquad \hat{X}_{i,j}^{(m)} = \sum_\nu c_{i\nu}^\dagger c^{j\nu},$$
$$\hat{X}_{\mu,\nu}^{(n)} = \sum_i c_{i\mu}^\dagger c^{i\nu}. \tag{169}$$

Highest-weight states for the N-boson U(m) × U(n) irreps given by the unitary-unitary branching rule

$$\mathrm{U}(mn) \downarrow \mathrm{U}(m) \times \mathrm{U}(n) : \{N\} \downarrow \bigoplus_{\kappa \vdash N} \{\kappa\} \times \{\kappa\} \tag{170}$$

are constructed as follows. Let $i = 1, 2, \ldots, m$ and $\nu = 1, 2, \ldots, n$, respectively, index basis states for the $\{1\}$ irreps of U(m) and U(n) in order of decreasing weight. And let $\mathfrak{B}^K$ denote the determinant of boson operators

$$\mathfrak{B}^K = \begin{vmatrix} c_{11}^\dagger & c_{12}^\dagger & \cdots & c_{1K}^\dagger \\ c_{21}^\dagger & c_{22}^\dagger & \cdots & c_{2K}^\dagger \\ \vdots & \vdots & \cdots & \vdots \\ c_{K1}^\dagger & c_{K2}^\dagger & \cdots & c_{KK}^\dagger \end{vmatrix} \tag{171}$$

defined in parallel with the Slater determinants of Eq. (37). If $|0\rangle$ is the boson vacuum state and $\tilde{\kappa}$ is the partition conjugate to κ, then the state

$$|\{\kappa\}; \text{h.wt.}\rangle = \mathfrak{B}^{\tilde{\kappa}_1} \mathfrak{B}^{\tilde{\kappa}_2} \mathfrak{B}^{\tilde{\kappa}_3} \cdots |0\rangle \tag{172}$$

is observed to be of highest weight κ relative to both U(m) and U(n).

Similar results for the antisymmetric representations of U(mn) are obtained for the fermion realizations starting from the observation that the U(n) commutation relations (166) are also satisfied by the fermion realization

$$X_{\mu,\nu} \to \hat{X}_{\mu,\nu} = a_\mu^\dagger a^\nu. \tag{173}$$

However, while the one-fermion states $\{a_\nu^\dagger|0\rangle, \nu = 1, \ldots, n\}$ are a basis for the U(n) irrep $\{1\}$, the N-fermion states $\{a_{\nu_1}^\dagger a_{\nu_2}^\dagger, \ldots, a_{\nu_N}^\dagger|0\rangle, \nu_1, \nu_2, \ldots, \nu_N = 1, \ldots, n\}$ now span the U(n) irrep $\{1^N\}$.

In parallel with Eq. (169), the Lie algebras u(mn), u(m), and u(n) also have fermion realizations

$$\hat{X}^{(mn)}_{i\mu,j\nu} = a^\dagger_{i\mu}a^{j\nu}, \quad \hat{X}^{(m)}_{i,j} = \sum_\nu a^\dagger_{i\nu}a^{j\nu}, \quad \hat{X}^{(n)}_{\mu,\nu} = \sum_i a^\dagger_{i\mu}a^{i\nu}. \tag{174}$$

However, the analogous extension of the irrep $\{1\}$ of U(mn) with fermions gives an antisymmetric irrep $\{1^N\}$ of U(mn) on $\mathbb{H}^{\{1^N\}}$ for $N > 1$, which according to the Schur-Weyl theorem satisfies the branching rule

$$\mathrm{U}(mn) \downarrow \mathrm{U}(m) \times \mathrm{U}(n) : \{1^N\} \downarrow \bigoplus_{\kappa \vdash N} \{\kappa\} \times \{\tilde{\kappa}\}, \tag{175}$$

where $\tilde{\kappa}$ is the partition conjugate to κ.

Highest-weight states for these U(m) × U(n) irreps are constructed in terms of fermion operators as follows. As for the boson basis, let $i = 1, 2, \ldots, m$ and $\nu = 1, 2, \ldots, n$ index basis states for the U(m) and U(n) $\{1\}$ irreps, respectively, in order of decreasing weight. Then, let $\mathfrak{F}^K$ denote the product of fermion operators

$$\mathfrak{F}^K_i = a^\dagger_{i1}a^\dagger_{i2}\cdots a^\dagger_{iK}. \tag{176}$$

Unlike its boson counterpart, this simple product already satisfies the antisymmetry requirement. Thus, the state

$$|\{\kappa\}; \text{h.wt.}\rangle = \mathfrak{F}^{\tilde{\kappa}_1}_1 \mathfrak{F}^{\tilde{\kappa}_2}_2 \mathfrak{F}^{\tilde{\kappa}_3}_3 \cdots |0\rangle \tag{177}$$

is observed to be of highest weight relative to both U(m) and U(n).

In addition to the unitary-unitary duality (stated in Theorem 3, see Sec. V.A), we will see in the following sections how the powerful formalism of second quantization plays a relevant role in identifying other dualities of importance in physical applications.

VII. DUAL REPRESENTATIONS ON HARMONIC-OSCILLATOR BOSON SPACES

In many-body theory, one is primarily interested in many-fermion systems. However, the position and momentum coordinates of fermions obey the boson commutation relations of a Heisenberg-Weyl Lie algebra. For example, nucleons in a nucleus are described in a zero-order approximation as independent particles in a harmonic-oscillator potential. Thus, in nonrelativistic quantum mechanics, in which nucleons are neither created nor destroyed in their interactions with one another, the excitations of a system of nucleons can be described in terms of harmonic-oscillator quanta. Similarly, the vibrational excitations of a condensed matter system are often appropriately described in terms of phonons which, similar to harmonic-oscillator quanta, are bosonic. Moreover, composite systems of tightly bound fermions, such as alpha particles, behave as bosons at low densities (see Sec. IX.A). The essential quality of a boson is that its creation and annihilation operators obey the same boson commutation relations as those of harmonic-oscillator quanta.

Several pairs of groups can be found with dual representations on a given multidimensional harmonic-oscillator space. Paramount to these dualities is the following theorem.

Theorem 4 (symplectic-orthogonal duality): The groups O(N) and Sp(m, $\mathbb{R}$) have dual representations on the Hilbert space of the Nm-dimensional oscillator.

Note that the group denoted here by Sp(m, $\mathbb{R}$) is the real noncompact symplectic group of rank m. Many authors denote this same group by Sp(2m, $\mathbb{R}$). Note also that when N is odd the Sp(m, $\mathbb{R}$) representation is a *projective representation*, i.e., a double-valued (spinor) representation. It is a genuine representation of the twofold cover of Sp(m, $\mathbb{R}$) known as a metaplectic group. Nonetheless, here both the genuine and projective oscillator representations of Sp(m, R) will be referred to without qualification as ''representations.''

Proofs of this and other duality theorems were given by Kashiwara and Vergne (1978) and Howe (1989). Less mathematically sophisticated proofs of this and the other duality theorems featured in this review were also given recently (Rowe, Repka, and Carvalho, 2011) in which complete sets of extremal (highest or lowest weight) states were identified for dual pairs of representations. Special cases are proven below to illustrate the significance of this theorem.

From Theorem 4 (together with the unitary-unitary duality theorem), it follows that the Hilbert space of an Nm-dimensional harmonic oscillator carries dual representations of the pairs of groups shown as direct products in the following chains:

$$\begin{array}{ccc} \mathrm{Sp}(Nm,\mathbb{R}) & \times & \mathrm{O}(1) \\ \cup & & \cap \\ \mathrm{Sp}(N,\mathbb{R}) & \times & \mathrm{O}(m) \\ \cup & & \cap \\ \mathrm{U}(N) & \times & \mathrm{U}(m) \\ \cup & & \cap \\ \mathrm{O}(N) & \times & \mathrm{Sp}(m,\mathbb{R}) \\ \cup & & \cap \\ \mathrm{O}(1) & \times & \mathrm{Sp}(Nm,\mathbb{R}). \end{array} \tag{178}$$

All the above direct product groups have realizations on the space of an Nm-dimensional harmonic oscillator. This space can be viewed as that of N particles in an m-dimensional harmonic-oscillator space or as that of m particles in an N-dimensional space. Sometimes it is useful to think of the space of Nm simple harmonic oscillators or of a single Nm-dimensional harmonic oscillator. The many possibilities for the interpretation of N and m mean that the above chains of duality relationships have many applications, examples of which are explored in this section.

The subspace of one-quantum states of the Nm-dimensional harmonic oscillator is spanned by the states $\{c^\dagger_{\alpha k}|0\rangle; \alpha = 1, \ldots, N, k = 1, \ldots, m\}$. This space carries an irrep $\{1\}$ of the group U(Nm) as studied in Sec. V.A. Because the boson operators $\{c^\dagger_{\alpha k}\}$ are symmetric under exchange, the space of W harmonic-oscillator quanta carries an irrep $\{W\}$ of U(Nm). Thus, by the Corollary to Theorem 3, the Hilbert space of the Nm-dimensional harmonic oscillator carries a dual representation of the direct product U(N) × U(m) given by $\bigoplus_\kappa \{\kappa\} \times \{\kappa\}$.

A remarkable characteristic of the duality relationships between the pairs of groups listed in Eq. (178) is the seesaw relationship between the paired subgroup chains. For example, whereas the representation of U(N) is dual to that of U(m) on the Nm-dimensional harmonic-oscillator space,

the representation of the orthogonal subgroup O(N) of U(N) is dual to that of an Sp(m, $\mathbb{R}$) group of which U(m) is a subgroup. We find that such chains of dual pairs of group representations provide powerful tools for a useful classification of states of a Hilbert space.

A. O(1) × Sp(m, $\mathbb{R}$) duality

Starting with the lowest pair in the chain and setting $N = 1$, we have a dual representation of the groups O(1) and Sp(m, $\mathbb{R}$) on the Hilbert space of the m-dimensional harmonic oscillator. The group O(1) is a discrete group with only two elements: the identity element and an element represented as a parity inversion operator. Thus, O(1) has two distinct one-dimensional irreps: one spanned by a state of positive parity and the other by a state of negative parity. However, each of these O(1) irreps occurs in the m-dimensional harmonic-oscillator space an infinite number of times. Thus, according to Theorem 4, all the positive parity states of the m-dimensional harmonic oscillator carry an irrep of the noncompact group Sp(m, $\mathbb{R}$) and all the negative parity states carry another irrep. These Sp(m, $\mathbb{R}$) irreps are understood as follows.

Let $\{c_i^\dagger, c^i; i = 1, \ldots, m\}$ denote the creation and annihilation operators of quanta for the m-dimensional harmonic oscillator. The Lie algebra of the group Sp(m, $\mathbb{R}$) then has a complex extension spanned by subsets of raising and lowering operators

$$\hat{\mathcal{A}}_{ij} = c_i^\dagger c_j^\dagger, \qquad \hat{\mathcal{B}}_{ij} = c^i c^j, \tag{179}$$

together with the commutators of these operators

$$[\hat{\mathcal{B}}_{ij}, \hat{\mathcal{A}}_{kl}] = \delta_{j,k} c_i^\dagger c^l + \delta_{j,l} c_i^\dagger c^k + \delta_{i,k} c^l c_j^\dagger + \delta_{i,l} c^k c_j^\dagger. \tag{180}$$

Thus, it has a u(m) subalgebra spanned by Hermitian linear combinations of the operators

$$\hat{\mathcal{C}}_{ij} = \tfrac{1}{2}(c_i^\dagger c^j + c^j c_i^\dagger) = c_i^\dagger c^j + \tfrac{1}{2}\delta_{i,j}. \tag{181}$$

A state $|\phi\rangle$ for an irrep of this Lie algebra that satisfies the following equations:

$$\hat{\mathcal{B}}_{ij}|\phi\rangle = 0, \qquad 1 \le i, j \le m, \tag{182}$$

$$\hat{\mathcal{C}}_{ij}|\phi\rangle = 0, \qquad 1 \le i < j \le m, \tag{183}$$

$$\hat{\mathcal{C}}_{ii}|\phi\rangle = \lambda_i|\phi\rangle, \qquad 1 \le i \le m, \tag{184}$$

is then a lowest-weight state for an Sp(m, $\mathbb{R}$) irrep with lowest weight $\lambda = \{\lambda_1, \ldots, \lambda_m\}$. Such an Sp($m$, $\mathbb{R}$) irrep is denoted by the symbol $\langle\lambda\rangle$.

It is now seen that the two Sp(m, $\mathbb{R}$) irreps on the Hilbert space of the m-dimensional harmonic oscillator have lowest-weight states given by the harmonic-oscillator ground state $|0\rangle$ and the one-quantum state $c_1^\dagger|0\rangle$. The corresponding irreps are then denoted by $\langle(\frac{1}{2})^m\rangle$ and $\langle\frac{3}{2}, (\frac{1}{2})^{m-1}\rangle$. To simplify the notation, Sp(m, $\mathbb{R}$) irreps are sometimes denoted more simply by the U(1) and SU(m) quantum numbers $\langle\lambda_m(\lambda_1 - \lambda_m, \lambda_2 - \lambda_m, \ldots)\rangle$. Then, the above two irreps are denoted by

$$\langle\tfrac{1}{2}(0)\rangle \equiv \langle(\tfrac{1}{2})^m\rangle, \qquad \langle\tfrac{1}{2}(1)\rangle \equiv \langle\tfrac{3}{2}, (\tfrac{1}{2})^{m-1}\rangle. \tag{185}$$

B. O(N) × Sp(1, $\mathbb{R}$) duality

In this section we consider the U(N) × U(m) and O(N) × Sp(m, $\mathbb{R}$) dualities with $m = 1$. These direct product groups then simplify to U(N) × U(1) and O(N) × SU(1, 1) [with Sp(1, $\mathbb{R}$) isomorphic to SU(1,1)]. Thus, we consider the paired subgroups of the chains

$$\begin{array}{ccc} \mathrm{U}(N) & \times & \mathrm{U}(1) \\ \cup & & \cap \\ \mathrm{O}(N) & \times & \mathrm{SU}(1,1) \end{array} \tag{186}$$

and show that they relate to the orbital and radial dynamics of a particle in an N-dimensional space (discussed briefly for $N = 3$ in Sec. I).

Let $\mathbb{H}$ denote the Hilbert space of the N-dimensional harmonic oscillator and let $\{c_\alpha^\dagger, c^\alpha; \alpha = 1, \ldots, N\}$ denote the boson creation and annihilation operators of harmonic-oscillator quanta. The Hilbert space $\mathbb{H}$ is then a direct sum

$$\mathbb{H} = \bigoplus_{n=0}^{\infty} \mathbb{H}^{(n)}, \tag{187}$$

where $\mathbb{H}^{(n)} \subset \mathbb{H}$ is the subspace of states of n quanta. Each $\mathbb{H}^{(n)}$ is invariant under the U(1) group, whose infinitesimal generator is the boson number operator

$$\hat{n} = c^\dagger \cdot c = \sum_\alpha c_\alpha^\dagger c^\alpha. \tag{188}$$

Each $\mathbb{H}^{(n)}$ is also invariant and irreducible under the group U(N) and hence under its Lie algebra, whose complex extension is spanned by $\{\hat{C}_{\alpha\beta} = c_\alpha^\dagger c^\beta\}$. Moreover, $\hat{n}$ commutes with all elements of U(N) and any state of $\mathbb{H}$ having n quanta belongs to an irrep $\{n\} \times \{n\}$ of U(1) × U(N). This is a simple application of the bosonic unitary-unitary duality relationship of the previous section. However, the O(N) subgroup of U(N) has a more interesting dual partner, namely, SU(1,1). Note, however, that when N is odd, the representation of SU(1,1) is projective.

The relevant O(N) group is the subgroup of U(N) transformations that leave the scalar product $c^\dagger \cdot c^\dagger = \sum_\alpha c_\alpha^\dagger c_\alpha^\dagger$ invariant. A basis for the Lie algebra of this O(N) group, identical to the Lie algebra so(N) of SO(N), is given by the operators

$$\hat{L}_{\alpha\beta} = -\mathrm{i}(c_\alpha^\dagger c^\beta - c_\beta^\dagger c^\alpha), \qquad \alpha < \beta, \tag{189}$$

which are the analogs, in N-dimensional space, of the standard angular-momentum operators. Thus, an O(N) irrep is labeled (not always uniquely, as we shall see) by its highest weight $[\upsilon_1, \upsilon_2, \ldots, \upsilon_r]$ relative to the ordered basis of the usual Cartan subalgebra of so(N),

$$\hat{h}_1 = \hat{L}_{12}, \quad \hat{h}_2 = \hat{L}_{34}, \quad \ldots, \quad h_r = \hat{L}_{2r-1,2r}, \tag{190}$$

where $N = 2r$ or $N = 2r + 1$.

Recall that in addition to the subgroup SO(N) ⊂ O(N), the group O(N) also contains the discrete inversion subgroup. Thus, an extra label is often required in addition to the so(N) highest weight to characterize the inversion properties

(i.e., parity) of an O(N) irrep. However, when $m = 1$ and $N > 2$, this extra label is not needed because the inversion properties of an O(N) irrep contained within the space of the N-dimensional harmonic oscillator are then uniquely defined by the highest weight of the irrep. For example, the parity of a single-particle state of the three-dimensional harmonic oscillator with SO(3) angular momentum l is equal to $(-1)^l$.

The one-dimensional space $\mathbb{H}^{(0)}$, spanned by the vacuum state $|0\rangle$ (the harmonic-oscillator ground state), carries the identity irrep [0] of O(N) as well as the identity irrep {0} of U(N). The space $\mathbb{H}^{(1)}$, which carries the defining N-dimensional irrep {1} of U(N), likewise carries the N-dimensional irrep [1] of O(N). The highest-weight state for the latter O(N) irrep is determined from the observation that

$$[\hat{h}_i, c_1^\dagger + \mathrm{i}c_2^\dagger] = \delta_{i,1}(c_1^\dagger + \mathrm{i}c_2^\dagger), \tag{191}$$

where $\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_r$ are the so(N) Cartan operators of Eq. (190). Thus, the state

$$|1\rangle = \frac{1}{\sqrt{2}}(c_1^\dagger + \mathrm{i}c_2^\dagger)|0\rangle \tag{192}$$

has O(N) weight $[1, 0, 0, \ldots, 0] \equiv [1]$, given by the eigenvalues of the Cartan operators, and is the highest-weight state for the O(N) irrep [1].

The two-quantum space $\mathbb{H}^{(2)}$, which carries an irrep {2} of the group U(N), is reducible as the carrier space for an O(N) representation. It contains an O(N) highest-weight state

$$|2\rangle = \frac{1}{\sqrt{8}}(c_1^\dagger + \mathrm{i}c_2^\dagger)^2|0\rangle \tag{193}$$

for an irrep [2]. However, it also contains the state $c^\dagger \cdot c^\dagger|0\rangle$ which, because $c^\dagger \cdot c^\dagger$ is O(N) invariant, spans a one-dimensional irrep [0] isomorphic to the irrep spanned by the state $|0\rangle$. Continuing the pattern, the space $\mathbb{H}^{(n)}$, which carries the U(N) irrep {n}, contains a unique (normalized) state of maximal O(N) highest weight given by

$$|n\rangle = \frac{1}{\sqrt{2^n n!}}(c_1^\dagger + \mathrm{i}c_2^\dagger)^n|0\rangle. \tag{194}$$

This state is an eigenstate of the so(N) Cartan operators,

$$\begin{aligned}\hat{h}_i|n\rangle &= \frac{1}{\sqrt{2^n n!}}[\hat{h}_i, (c_1^\dagger + \mathrm{i}c_2^\dagger)^n]|0\rangle \\ &= \delta_{i,1} n|n\rangle, \qquad i = 1, \ldots, r,\end{aligned} \tag{195}$$

and the highest-weight state for the O(N) irrep [n].

However, there are other O(N) irreps in $\mathbb{H}^{(n)}$ as indicated by the branching rule

$$\begin{aligned}\mathrm{U}(N) \downarrow \mathrm{O}(N) : \{n\} \downarrow [n] \oplus [n-2] \oplus [n-4] \oplus \\ \cdots \oplus [1] \quad \text{or} \quad [0].\end{aligned} \tag{196}$$

The highest-weight states of these O(N) irreps are given, to within norm factors, by

$$\begin{aligned}&|n\rangle, \qquad (c^\dagger \cdot c^\dagger)|n-2\rangle, \qquad (c^\dagger \cdot c^\dagger)^2|n-4\rangle, \\ &\ldots, \quad (c^\dagger \cdot c^\dagger)^k|n-2k\rangle, \quad \ldots,\end{aligned} \tag{197}$$

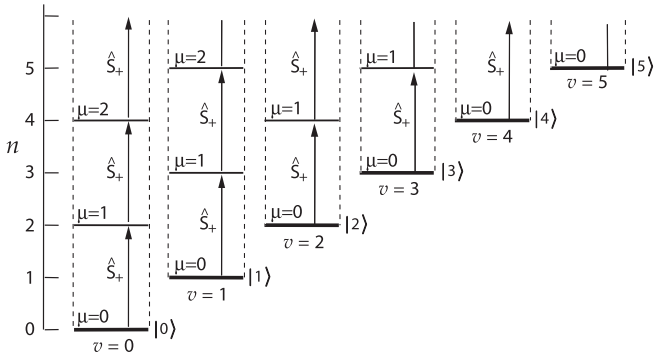


FIG. 1. Irreps of O(N) in the Hilbert space of the N-dimensional harmonic oscillator shown as horizontal lines. Equivalent O(N) irreps are placed in a common column and connected by the su(1,1) raising operator $\hat{S}_+$. Equivalent O(N) irreps are distinguished by the u(1) $\subset$ su(1, 1) quantum number μ. The set of O(N) irreps at a constant level n comprise a U(N) irrep {n} with $n = v + 2\mu$. The carrier space of this U(N) irrep {n} is $\mathbb{H}^{(n)}$, the space spanned by the states with n harmonic-oscillator quanta.

thus illustrating how the reduction of a U(N) irrep on restriction to O(N) is obtained by factoring out O(N) scalars.

The decomposition of the harmonic-oscillator space $\mathbb{H}$ is illustrated in Fig. 1. Each horizontal line corresponds to one of the subspaces $\mathbb{H}^{(n)}$ labeled by n on the vertical axis. The decomposition of $\mathbb{H}^{(n)}$ into irreducible O(N) subspaces is shown by the horizontal line segments, each of which represents an O(N) irrep [v], characterized by a value of v which, for each value of n, takes either even or odd integer values between n and zero in accordance with the branching rule (196). Equivalent O(N) irreps, i.e., irreps sharing a common value of v, are placed one above the other in a column labeled at the bottom by the value of v. It can be seen that the pattern of O(N) irreps obtained in this way is independent of N. For example, Fig. 1 gives the familiar spectrum of O(3) irreps of the three-dimensional harmonic oscillator.

Now observe that the operator $c^\dagger \cdot c^\dagger$ is the raising operator of an su(1,1) Lie algebra, whose complex extension is spanned by the O(N)-invariant operators

$$\begin{aligned}&\hat{S}_+ = \frac{1}{2}c^\dagger \cdot c^\dagger, \qquad \hat{S}_- = \frac{1}{2}c \cdot c, \\ &\hat{S}_0 = \frac{1}{4}(c^\dagger \cdot c + c \cdot c^\dagger) = \frac{1}{2}\left(\hat{n} + \frac{N}{2}\right),\end{aligned} \tag{198}$$

where $\hat{n}$ is the number operator for harmonic-oscillator quanta, as defined by Eq. (188). These su(1,1) operators satisfy the commutation relations

$$[\hat{S}_-, \hat{S}_+] = 2\hat{S}_0, \qquad [\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm. \tag{199}$$

Observe also that the states $\{|v\rangle; v = 0, 1, 2, \ldots\}$, defined (with $v = n$) by Eq. (194), are all annihilated by the su(1,1) lowering operator. Thus, the state $|v\rangle$ is simultaneously of highest U(N) weight {v}, of O(N) highest weight [v], and of SU(1,1) lowest weight $\langle\lambda(v)\rangle$, where $\lambda(v) = v + N/2$ is an eigenvalue of $2\hat{S}_0$, i.e., $|v\rangle$ satisfies

$$\hat{S}_-|v\rangle = 0, \quad \hat{S}_0|v\rangle = \tfrac{1}{2}\lambda(v)|v\rangle = \tfrac{1}{2}(v + N/2)|v\rangle. \tag{200}$$

By construction, the O(N) irrep with highest-weight state $|v\rangle$ lies lowest on a column of equivalent O(N) irreps (cf. Fig. 1). Moreover, it is seen that the highest-weight states for the successively higher O(N) irreps of a column are given, to within norm factors, by the states

$$|v\rangle, \quad (c^\dagger \cdot c^\dagger)|v\rangle, \quad (c^\dagger \cdot c^\dagger)^2|v\rangle, \\ \ldots, \ (c^\dagger \cdot c^\dagger)^\nu|v\rangle, \ \ldots . \tag{201}$$

Thus, the set of all O(N) highest-weight states in a column span an irrep $\langle\lambda(v)\rangle$ of SU(1,1) and all the states of a column span an irrep $\langle\lambda(v)\rangle \times [v]$ of the direct product group SU(1, 1) × O(N) labeled by v, with v taking the values 0, 1, 2, ... in successive columns.

Finally, observe that every irrep $\langle\lambda(v)\rangle \times [v]$ of SU(1, 1) × O(N) is multiplicity free and every irrep $\langle\lambda(v)\rangle$ of SU(1,1) is uniquely paired with an irrep $[v]$ of O(N). These are the properties required to demonstrate the duality of SU(1, 1) and O(N) representations on the Hilbert space $\mathbb{H}$ of the N-dimensional harmonic-oscillator space.

C. Applications of O(N) × SU(1, 1) duality

The duality of the representations of O(N) and SU(1, 1) on the Hilbert space $\mathbb{H}$ of the N-dimensional harmonic oscillator leads to many useful relationships in physics (Rowe, 2005). This is because $\mathbb{H}$, the space $\mathcal{L}^2(\mathbb{R}^N)$ of square integrable functions on the real N-dimensional Euclidean space $\mathbb{R}^N$, is also the Hilbert space for numerous other systems of interest.

1. Central-force problems

Suppose, for example, that $\hat{H}$ is an O(N)-invariant Hamiltonian on $\mathbb{H}$ that one wishes to diagonalize in a harmonic-oscillator basis. Such a basis is defined by the irrep labels of the subgroup chain

$$\begin{array}{ccccccccc} \mathrm{U}(N) & \supset & \mathrm{O}(N) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2), \\ n & & v & \rho & L & & M \end{array} \tag{202}$$

where we have assumed that $N \geq 3$, so that the standard rotation group SO(3) can be defined as a subgroup of O(N), and ρ is an additional quantum number to distinguish any multiplicity of SO(3) irreps that occur within a given irrep $[v]$ of O(N). As a result of the O(N) × SU(1, 1) duality, the basis states $\{|nv\rho LM\rangle\}$ also reduce the subgroup chain

$$\begin{array}{ccccccccc} \mathrm{SU}(1,1) \times \mathrm{O}(N) & \supset & \mathrm{U}(1) \times \mathrm{SO}(3) & \supset & \mathrm{SO}(2), \\ \lambda \quad v & \rho & \mu \quad L & & M \end{array} \tag{203}$$

where U(1) is the subgroup of SU(1,1) with infinitesimal generator $\hat{S}_0$. The above results show that the quantum numbers of the two chains are related by

$$\lambda = v + \tfrac{1}{2}N, \qquad n = v + 2\mu, \tag{204}$$

where v and μ run over all non-negative integer values.

The classification of states by the subgroup chain (203) is particularly useful because the direct product structure of the two commuting groups SU(1,1) and O(N) makes it natural to exploit the factorization of the wave functions for basis states into products of radial and orbital wave functions. Moreover, when the Hamiltonian is O(N) invariant, as it is for a generalized central-force problem, the orbital wave functions are simply SO(N) spherical harmonics (Rowe, Turner, and Repka, 2004), and the radial wave functions are eigenfunctions of a one-dimensional Schrödinger equation. In fact, as a consequence of the O(N) × SU(1, 1) duality, the spectral properties of many central-force Hamiltonians can be derived by algebraic methods using an su(1,1) Lie algebra as a spectrum generating algebra (Rowe, 2005). This is seen by expressing the harmonic oscillator raising and lowering operators in terms of Cartesian coordinates $\{x_i; i = 1, \ldots, N\}$ for $\mathbb{R}^N$:

$$c^i = \frac{1}{\sqrt{2}}\left(a x_i + \frac{1}{a}\frac{\partial}{\partial x_i}\right), \quad c_i^\dagger = \frac{1}{\sqrt{2}}\left(a x_i - \frac{1}{a}\frac{\partial}{\partial x_i}\right), \tag{205}$$

where a is an inverse unit of length. The su(1,1) operators are then obtained in the form

$$\hat{S}_\pm = \tfrac{1}{4}[a^{-2}\nabla^2 + a^2 r^2 \mp (\mathbf{r}\cdot\nabla + \nabla\cdot\mathbf{r})], \tag{206}$$

$$\hat{S}_0 = \tfrac{1}{4}[-a^{-2}\nabla^2 + a^2 r^2], \tag{207}$$

where $r^2 = \mathbf{r}\cdot\mathbf{r} = \sum_i x_i^2$ and $\nabla^2 = \sum_i \partial^2/\partial x_i^2$ is the Laplacian on $\mathcal{L}^2(\mathbb{R}^N)$.

The representations of SU(1,1) have been well studied; cf., for example, Wybourne (1974), Čížek and Paldus (1977), Rowe and Bahri (1998), and Rowe (2005). The harmonic series of SU(1,1) irreps are given by

$$\begin{aligned} \hat{S}_0|\lambda\mu\rangle &= \tfrac{1}{2}(\lambda + 2\mu)|\lambda\mu\rangle, \\ \hat{S}_+|\lambda\mu\rangle &= \sqrt{(\lambda+\mu)(\mu+1)}|\lambda, \mu+1\rangle, \\ \hat{S}_-|\lambda\mu\rangle &= \sqrt{(\lambda+\mu-1)\mu}|\lambda, \mu-1\rangle, \end{aligned} \tag{208}$$

with $\lambda = v + N/2$ and $v = 0, 1, 2, \ldots$. The operator $\hat{S}_0$ defined by Eq. (207) is proportional to a harmonic-oscillator Hamiltonian (160). Moreover, ∇^2 and r^2 are elements of the su(1,1) Lie algebra and expressible as linear combinations of $\hat{S}_\pm$ and $\hat{S}_0$ with known matrix elements in an su(1, 1) ⊃ u(1) coupled basis. Matrix elements of potential-energy functions of the form

$$V(r) = \sum_i P_i(r) e^{-\alpha_i r^2}, \tag{209}$$

where each P_i is an even polynomial, can also be computed with relative ease in such a basis. Thus, the dynamical group SU(1,1) very much simplifies the computation of matrix elements of a Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2M}\nabla^2 + V(r), \tag{210}$$

when the potential $V(r)$ is of the form (209).

Applications of SU(1,1) as a spectrum generating algebra for central-force problems have been considered; see e.g., Wybourne (1974), Čížek and Paldus (1977) or, for a pedagogical review, Cooke and Wood (2002). The fact that these applications are valid for any positive integer N and many discrete series irreps of SU(1,1) makes them particularly useful. The development of factorization methods and algebraic methods for the evaluation of matrix elements of SO(N) tensor operators makes it possible to extend the algebraic method to a much wider class of Hamiltonian (Rowe, 2005). By such means, SU(1, 1) × SO(5) has been used successfully as a dynamical

group in the development of an algebraic version of the Bohr model for collective quadrupole vibrations and rotations in nuclear physics (Rowe, Turner, and Repka, 2004; Rowe, 2004a; Rowe and Turner, 2005; Rowe, Welsh, and Caprio, 2009; Welsh and Rowe, 2012).

2. States for a six-dimensional harmonic oscillator

The Hilbert space of the six-dimensional harmonic oscillator $\mathcal{L}^2(\mathbb{R}^6)$ is a tensor product $\mathcal{L}^2(\mathbb{R}^5) \otimes \mathcal{L}^2(\mathbb{R})$ of Hilbert spaces for five-dimensional and one-dimensional harmonic oscillators. In the spirit of the interacting boson model (Iachello and Arima, 1987), to which this example applies, we may consider the boson creation operators of these harmonic oscillators as comprising five $\{d_m^\dagger; m = 0, \pm 1, \pm 2\}$ operators, which create harmonic-oscillator quanta of angular momentum $L = 2$ in $\mathcal{L}^2(\mathbb{R}^5)$, and one $s^\dagger$ operator, which creates $L = 0$ (angular-momentum-zero) quanta in $\mathcal{L}^2(\mathbb{R})$. We then have the following dual pairs of group representations:

$$\begin{array}{ccccccc} \mathcal{L}^2(\mathbb{R}^6) & & = & & \mathcal{L}^2(\mathbb{R}^5) & \otimes & \mathcal{L}^2(\mathbb{R}) \\ [\mathrm{U}(6) & \times & \mathrm{U}(1)_6] & & [\mathrm{U}(5) \times \mathrm{U}(1)_5] & & [\mathrm{U}(1)_1 \times \mathrm{U}(1)_1] \\ \cup & & \cap & & \cup \quad\quad \cap & & \cup \quad\quad \cap \\ [\mathrm{O}(6) & \times & \mathrm{SU}(1,1)_6] & & [\mathrm{O}(5) \times \mathrm{SU}(1,1)_5] & & [\mathrm{O}(1)_1 \times \mathrm{SU}(1,1)_1]. \end{array} \tag{211}$$

Note that, for logical consistency, we show a direct product of two copies of $\mathrm{U}(1)_1$ although, in fact, the representations of these two groups are dual to each other in a trivial way, i.e., they are identical. Note also that, because the irreps of one member of a dual pair are uniquely partnered with corresponding irreps of the other, we need only to specify the representations for one member of each pair. Thus, we label the paired irreps by common indices as follows:

$$\begin{array}{ccc} \mathrm{U}(6) \times \mathrm{U}(1)_6 & \mathrm{U}(5) \times \mathrm{U}(1)_5 & \mathrm{U}(1)_1 \times \mathrm{U}(1)_1 \\ N & n & N-n \\ \mathrm{O}(6) \times \mathrm{SU}(1,1)_6 & \mathrm{O}(5) \times \mathrm{SU}(1,1)_5 & \mathrm{O}(1)_1 \times \mathrm{SU}(1,1)_1 \\ \sigma & \upsilon & \pi. \end{array} \tag{212}$$

Note that there are only two irreps of $\mathrm{SU}(1,1)_1$ on the one-dimensional harmonic-oscillator Hilbert space: one carried by states of even numbers of oscillator quanta and one by states of odd numbers of quanta. The discrete group O(1) also has only two irreps of parity $\pi = \pm 1$. In $\mathcal{L}^2(\mathbb{R})$, the even and odd boson number $\mathrm{SU}(1,1)_1$ irreps are paired with the $\pi = 1$ and $\pi = -1$ irreps of $\mathrm{O}(1)_1$, respectively.

The interacting boson model has exactly solvable limits in which particular classes of Hamiltonians are diagonalized in bases that reduce corresponding subgroup chains. In particular, there are two exactly solvable classes of O(5)-invariant Hamiltonians: one that diagonalizes the chain

$$\begin{array}{ccccccccc} \mathrm{U}(6) & \supset & \mathrm{U}(5) \times \mathrm{O}(1)_1 & \supset & \mathrm{O}(5) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ N & & n \quad \pi & & \upsilon & & \rho \quad L & & M \end{array} \tag{213}$$

and another that diagonalizes the chain

$$\begin{array}{ccccccccc} \mathrm{U}(6) & \supset & \mathrm{O}(6) & \supset & \mathrm{O}(5) \times \mathrm{O}(1)_1 & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ N & & \sigma & & \upsilon \quad \pi & & \rho \quad L & & M \end{array}. \tag{214}$$

By using the duality relationships indicated above, we find that the basis states $\{|Nn\pi\upsilon\rho LM\rangle\}$ that reduce the subgroup chain (213) simultaneously reduce the chain

$$\begin{array}{ccccccccccccc} \mathrm{SO}(3) & \times & \mathrm{SU}(1,1)_1 & \times & \mathrm{SU}(1,1)_5 & \supset & \mathrm{SO}(2) & \times & \mathrm{U}(1)_1 & \times & \mathrm{U}(1)_5 & \supset & \mathrm{U}(1)_6 \\ L & & \pi & & \lambda = \upsilon + \frac{5}{2} & & M & & N-n & & n & & N \end{array}, \tag{215}$$

and the basis states $\{|N\sigma\upsilon\pi\rho LM\rangle\}$ that reduce the subgroup chain (214) simultaneously reduce the chain

$$\begin{array}{ccccccccccccc} \mathrm{SO}(3) & \times & \mathrm{SU}(1,1)_1 & \times & \mathrm{SU}(1,1)_5 & \supset & \mathrm{SO}(2) & \times & \mathrm{SU}(1,1)_6 & & \supset & & \mathrm{U}(1)_6 \\ L & & \pi & & \lambda = \upsilon + \frac{5}{2} & & M & & \kappa = \sigma + 3 & & & & N \end{array}, \tag{216}$$

where $\mathrm{SU}(1,1)_6$ is the subgroup of $\mathrm{SU}(1,1)_1 \times \mathrm{SU}(1,1)_5$ whose Lie algebra is spanned by the sums $\hat{S}_k^6 = \hat{S}_k^1 + \hat{S}_k^5$, where $\{\hat{S}_k^1\}$ and $\{\hat{S}_k^5\}$ are bases for $\mathrm{SU}(1,1)_1$ and $\mathrm{SU}(1,1)_5$, respectively.

The equivalence of alternative subgroup chains for classifying basis states means that one can choose whichever is the simplest for the purposes of diagonalizing a corresponding Hamiltonian. Thus, although the subgroup chains (213) and (214) are natural dynamical subgroup chains for diagonalizing an O(5)-invariant Hamiltonian on a U(6)-invariant subspace of the six-dimensional harmonic oscillator, it is generally much easier to use their SU(1,1) counterparts in Eqs. (215) and (216) (Rowe, 2004c).

It is also worth noting that one would sometimes like to know the unitary transformation between the U(5) and O(6) bases, i.e., the bases that, respectively, diagonalize the subgroups chains of Eqs. (213) and (214). The transformation coefficients are the ρLM-independent overlaps

$$\langle Nn\upsilon\pi\rho LM | N\sigma\upsilon\pi\rho LM\rangle = \langle Nn\upsilon | N\sigma\upsilon\rangle. \tag{217}$$

Knowledge of these coefficients immediately gives an algebraic expression for matrix elements of any mixture of Hamiltonians that are known in either the U(5) or O(6) bases. These coefficients can be computed with some effort by diagonalizing the O(6) Casimir operator in the U(5) basis as shown by Castaños *et al.* (1979). However, as the equivalent SU(1,1) chains reveal (Rowe, 2004c; Rowe and Thiamova, 2005), they are simply equal to already-known SU(1,1) Clebsch-Gordan coefficients: in the $(k_1, n_1; k_2, n_2|k, n)$ notation of Van der Jeugt (1997)

$$\langle Nn\upsilon|N\sigma\upsilon\rangle = \left(\frac{\upsilon+5/2}{2}, \frac{n-\upsilon}{2}; \frac{1}{4}, \frac{N-n}{2} \middle| \frac{\sigma+3}{2}, \frac{N-\sigma}{2}\right), \tag{218}$$

for $N-\upsilon$ even, and

$$\langle Nn\upsilon|N\sigma\upsilon\rangle = \left(\frac{\upsilon+5/2}{2}, \frac{n-\upsilon}{2}; \frac{3}{4}, \frac{N-n-1}{2} \middle| \frac{\sigma+3}{2}, \frac{N-\sigma}{2}\right), \tag{219}$$

for $N-\upsilon$ odd.

As considered further in Sec. X, this example is a prototype of many possible uses of dual subgroup chains to relate the basis states of one coupling scheme for a many-particle system to those of another. Thus, it can provide solutions to challenging problems, e.g., in the calculation of matrix elements of a Hamiltonian that contains mixtures of interactions that are diagonal in different coupling schemes.

An extension of the O(5)-invariant interacting boson model, that benefits even more from these duality relationship, is a model with mixed U(6) irreps, proposed by Lehmann and Jolie (1995). In this model, spherical neutron states mix with deformed states generated by the excitation of a proton pair into the active shell-model space from an otherwise inert closed subshell. The spherical states of this model are classified by the subgroup chains

$$\begin{array}{ccccc} \mathrm{U}(6) & \supset & \mathrm{U}(5) & \supset & \mathrm{O}(5) \\ N & & n_5 & & \upsilon \end{array}, \tag{220}$$

and the deformed states, with the addition of two excited protons, are classified by the chain

$$\begin{array}{ccccc} \mathrm{U}(6) & \supset & \mathrm{O}(6) & \supset & \mathrm{O}(5) \\ N+2 & & \sigma & & \upsilon \end{array}, \tag{221}$$

with the inclusion of multiplicity labels as needed. These states are equivalently classified by the dual subgroup chains

$$\begin{array}{ccccccccc} \mathrm{SU}(1,1)_1 & \times & \mathrm{SU}(1,1)_5 & \supset & \mathrm{U}(1)_1 & \times & \mathrm{U}(1)_5 & \supset & \mathrm{U}(1)_6 \\ \pi & & \lambda = \upsilon + \frac{5}{2} & & N - n_5 & & n_5 & & N \end{array}, \tag{222}$$

$$\begin{array}{ccccccccc} \mathrm{SU}(1,1)_1 & \times & \mathrm{SU}(1,1)_5 & \supset & \mathrm{SU}(1,1)_6 & \supset & \mathrm{U}(1)_6 \\ \pi & & \lambda = \upsilon + \frac{5}{2} & & \sigma & & N+2 \end{array}. \tag{223}$$

The mixing of these states is then simply described by an interaction of the form

$$\hat{V} = \alpha(\hat{S}^{(1)}_{+} + \hat{S}^{(1)}_{-}) + \beta(\hat{S}^{(5)}_{+} + \hat{S}^{(5)}_{-}), \tag{224}$$

where $\hat{S}^{(1)}_{\pm}$ and $\hat{S}^{(5)}_{\pm}$ are, respectively, raising and lowering operators for $\mathrm{SU}(1,1)_1$ and $\mathrm{SU}(1,1)_5$.

3. The vibron model and its q-deformed extension

A parallel application by Alvarez, Bonatsos, and Smirnov (1994) of the $\mathrm{O}(N) \times \mathrm{SU}(1,1)$ duality is to the classification of states of a four-dimensional harmonic oscillator used in the vibron model of the vibrations and rotations of diatomic molecules (Iachello, 1981). In this application, the Hilbert space of the four-dimensional harmonic oscillator is regarded as a tensor product $\mathcal{L}^2(\mathbb{R}^4) \simeq \mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{L}^2(\mathbb{R})$ of Hilbert spaces for three-dimensional and one-dimensional harmonic oscillators. Thus, it is determined that basis states for this model that reduce the subgroup chain

$$\mathrm{U}(4) \supset \mathrm{U}(3) \times \mathrm{U}(1) \supset \mathrm{O}(3) \times \mathrm{O}(1) \supset \mathrm{SO}(2) \tag{225}$$

simultaneously reduce the dual chain

$$\mathrm{SO}(2) \times \mathrm{SU}(1,1)_1 \times \mathrm{SU}(1,1)_3 \supset \mathrm{U}(1)_1 \times \mathrm{U}(1)_3 \supset \mathrm{U}(1)_4. \tag{226}$$

Similarly, basis states that reduce the subgroup chain

$$\mathrm{U}(4) \supset \mathrm{O}(4) \supset \mathrm{O}(3) \times \mathrm{O}(1) \supset \mathrm{SO}(2) \tag{227}$$

simultaneously reduce the dual chain

$$\mathrm{SO}(2) \times \mathrm{SU}(1,1)_1 \times \mathrm{SU}(1,1)_3 \supset \mathrm{SU}(1,1)_4 \supset \mathrm{U}(1)_4. \tag{228}$$

This formulation of the vibron model was used by Alvarez, Bonatsos, and Smirnov (1994) to show that, because the su(1,1) Lie algebra has a known q deformation (Kulish and Reshetikhin, 1983) to a so-called *quantum algebra*, the vibron model also has a q-deformed extension. A similar observation applies to the interacting boson model in its U(5) and O(6) dynamical symmetry limits.

D. $\mathrm{O}(N) \times \mathrm{Sp}(m, \mathbb{R})$ duality

We now consider applications of the general $\mathrm{O}(N) \times \mathrm{Sp}(m, \mathbb{R})$ duality relationship to the dynamics of a system of N particles in an m-dimensional configuration space.

Let $\{c^{\dagger}_{\alpha i}; \alpha = 1, \ldots, N; i = 1, \ldots, m\}$ and $\{c^{\alpha i}; \alpha = 1, \ldots, N; i = 1, \ldots, m\}$ denote boson creation and annihilation operators for an Nm-dimensional harmonic oscillator. The unitary groups $\mathrm{U}(N)$ and $\mathrm{U}(m)$ then have dual representations on the Hilbert space of this harmonic oscillator with infinitesimal generators defined, respectively, by

$$\hat{C}^{(N)}_{\alpha\beta} = \sum_{i=1}^{m} c^\dagger_{\alpha i} c^{\beta i}, \qquad \hat{C}^{(m)}_{ij} = \sum_{\alpha=1}^{N} c^\dagger_{\alpha i} c^{\alpha j}, \tag{229}$$

cf. Eq. (169). The group $O(N)$ is the subgroup of all real orthogonal transformations in $U(N)$ that leave the scalar products,

$$\hat{\mathcal{A}}_{ij} = c^\dagger_i \cdot c^\dagger_j = \sum_\alpha c^\dagger_{\alpha i} c^\dagger_{\alpha j}, \qquad i, j = 1, \ldots, m, \tag{230}$$

invariant. This group contains all $SO(N)$ rotations, for which infinitesimal generators are given by the generalized angular-momentum operators

$$\hat{L}_{\alpha\beta} = -\mathrm{i}(\hat{C}^{(N)}_{\alpha\beta} - \hat{C}^{(N)}_{\beta\alpha}), \qquad \alpha < \beta, \tag{231}$$

and an inversion operator that maps all the boson creation and annihilation operators to their negatives. The group $Sp(m, \mathbb{R})$, dual to $O(N)$, is a simple Lie group for which infinitesimal generators are given by Hermitian linear combinations of the $O(N)$ scalar operators

$$\hat{\mathcal{A}}_{ij} = \sum_\alpha c^\dagger_{\alpha i} c^\dagger_{\alpha j}, \qquad \hat{\mathcal{B}}_{ij} = \sum_\alpha c^{\alpha i} c^{\alpha j},$$
$$\hat{\mathcal{C}}_{ij} = \hat{C}^{(m)}_{ij} + \tfrac{1}{2} N \delta_{i,j}, \tag{232}$$

for $i, j = 1, \ldots, m$. Note that $\hat{\mathcal{C}}_{ij}$ and $\hat{C}^{(m)}_{ij}$, related as in Eq. (232), are infinitesimal generators of isomorphic $U(m)$ groups.

The Hilbert space of any Nm-dimensional harmonic oscillator carries a direct sum of $Sp(m, \mathbb{R})$ irreps known as a positive harmonic series, which are irreps with lowest but not highest weights. An $Sp(m, \mathbb{R})$ irrep on an Nm-dimensional harmonic-oscillator space is therefore conveniently characterized by a lowest-weight state $|\lambda\rangle$ that satisfies the following equations:

$$\hat{\mathcal{B}}_{ij}|\lambda\rangle = 0, \qquad 1 \le i \le j \le m, \tag{233}$$

$$\hat{\mathcal{C}}_{ij}|\lambda\rangle = 0, \qquad 1 \le i < j \le m, \tag{234}$$

$$\hat{\mathcal{C}}_{ii}|\lambda\rangle = (\lambda_i + \tfrac{1}{2}N)|\lambda\rangle, \qquad i = 1, \ldots, m. \tag{235}$$

To simplify the notation, we denote such an irrep with lowest weight $(\lambda_1 + \frac{1}{2}N, \lambda_2 + \frac{1}{2}N, \ldots, \lambda_m + \frac{1}{2}N)$ by $\langle \frac{1}{2}N(\lambda)\rangle$. Note that the $Sp(m, \mathbb{R})$ lowest-weight state $|\lambda\rangle$, defined in this way, is also the highest-weight state for an irrep $\{\lambda\}$ of the $U(m)$ group defined by Eq. (229).

Several copies of the $Sp(m, \mathbb{R})$ irrep $\langle \frac{1}{2}N(\lambda)\rangle$ appear in the N-particle, m-dimensional harmonic-oscillator space. However, because of the unitary-unitary duality relationship, Theorem 3, the $U(m)$ highest-weight state $|\lambda\rangle$ can be made unique by requiring that, in addition to satisfying Eqs. (233)–(235), it is also a $U(N)$ highest-weight state, i.e., it satisfies the following equations:

$$\hat{C}^{(N)}_{\alpha\beta}|\lambda\rangle = 0, \qquad \alpha < \beta, \tag{236}$$

$$\hat{C}^{(N)}_{\alpha\alpha}|\lambda\rangle = \lambda_\alpha|\lambda\rangle, \qquad \alpha = 1, \ldots, N, \tag{237}$$

where $\lambda_\alpha = 0$ for $\alpha > m$.

There is no duality relationship between the irreps of $Sp(m, \mathbb{R})$ and those of $U(N)$ because the irrep of $U(N)$ with highest weight λ also contains states that are not of $Sp(m, \mathbb{R})$ lowest weight, i.e., states that are not annihilated by the $\hat{\mathcal{B}}_{ij}$ lowering operators. However, because the $\hat{\mathcal{B}}_{ij}$ lowering operators are $O(N)$ invariant, the subset of states of the $U(N)$ irrep of highest weight λ that are also of $Sp(m, \mathbb{R})$ lowest weight carries a representation of $O(N)$. Moreover, by Theorem 4, such an $O(N)$ representation is irreducible. Thus, the state $|\lambda\rangle$ is the highest-weight state for an $O(N)$ irrep and, simultaneously, a lowest-weight state for a dual $Sp(m, \mathbb{R})$ irrep. This can be seen, for $m = 1$, in Fig. 1 which shows that only the rightmost $O(N)$ irrep belonging to a single $U(N)$ irrep lies at the bottom of a column of equivalent $O(N)$ irreps that together span an $Sp(1, \mathbb{R})$ irrep.

E. Applications of $O(N) \times Sp(m, \mathbb{R})$ duality

1. Relationships between branching rules

The following example shows how the seesaw relationship between the unitary-unitary and orthogonal-symplectic dual pairs

$$\begin{array}{ccc} U(N) & \times & U(m) \\ \cup & & \cap \\ O(N) & \times & Sp(m, \mathbb{R}) \end{array} \tag{238}$$

is used to determine $Sp(m, \mathbb{R}) \downarrow U(m)$ branching rules from known $U(N) \downarrow O(N)$ branching rules. This is important for two reasons: one is that a knowledge of the $Sp(m, \mathbb{R}) \downarrow U(m)$ branching rules is needed for nuclear shell-model calculations in an $Sp(3, \mathbb{R}) \supset SU(3) \supset SO(3)$ coupling scheme, appropriate for the microscopic theory of nuclear collective states; a second is that it serves as a prototype of ways to infer branching rules for a noncompact group from those of a compact group.

Note that the $U(m) \subset Sp(m, \mathbb{R})$ subgroup, Eq. (235), has infinitesimal generators $\{\hat{\mathcal{C}}_{ij} = \hat{C}^{(m)}_{ij} + \frac{1}{2}N\delta_{i,j}\}$ that differ from the infinitesimal generators $\{\hat{C}^{(m)}_{ij}\}$ of the $U(m)$ group defined by Eq. (229). The commutation relations of the $\{\hat{\mathcal{C}}_{ij}\}$ and $\{\hat{C}^{(m)}_{ij}\}$ operators are exactly the same. But they generate different, although simply related, representations when acting on the same states. Thus, when acting on a $U(m)$ highest-weight state $|\lambda\rangle$, defined by the equations

$$\hat{C}^{(m)}_{ij}|\lambda\rangle = 0, \qquad 1 \le i < j \le m, \tag{239}$$

$$\hat{C}^{(m)}_{ii}|\lambda\rangle = \lambda_i|\lambda\rangle, \qquad i = 1, \ldots, m, \tag{240}$$

the $U(m) \subset Sp(m, \mathbb{R})$ operators satisfy Eqs. (234) and (235). Thus, they generate a $U(m)$ irrep with a *shifted* highest weight $\lambda^{(N)}$ having components $\lambda^{(N)}_i = \lambda_i + \frac{1}{2}N$.

According to the unitary-unitary duality theorem, the Hilbert space $\mathbb{H}$ of the Nm-dimensional harmonic oscillator carries a direct sum $\bigoplus_\lambda \{\lambda\} \times \{\lambda^{(N)}\}$ of $U(N) \times U(m)$ irreps. From orthogonal-symplectic duality, $\mathbb{H}$ also carries a direct sum $\bigoplus_\lambda [\lambda] \times \langle \frac{1}{2}N(\lambda)\rangle$ of $O(N) \times Sp(m, \mathbb{R})$ irreps. To relate the $U(N) \downarrow O(N)$ and $Sp(m, \mathbb{R}) \downarrow U(m)$ branching rules, we express them in the form

$$\mathrm{U}(N)\downarrow \mathrm{O}(N) : \{\lambda\}\downarrow \bigoplus_{\kappa} P_{\lambda\kappa}[\kappa], \tag{241}$$

$$\mathrm{Sp}(m,\mathbb{R})\downarrow \mathrm{U}(m) : \langle \tfrac{1}{2}N(\kappa)\rangle \downarrow \bigoplus_{\lambda} \mathcal{P}_{\kappa\lambda}\{\lambda^{(N)}\}. \tag{242}$$

The $\mathrm{O}(N)\times \mathrm{U}(m)$ representation carried by $\mathbb{H}$ can now be expressed as a direct sum of irreps in two ways. One branching rule gives

$$\mathrm{U}(N)\times \mathrm{U}(m)\downarrow \mathrm{O}(N)\times \mathrm{U}(m):$$
$$\bigoplus_{\lambda}\{\lambda\}\times\{\lambda^{(N)}\}\downarrow \bigoplus_{\lambda,\kappa} P_{\lambda\kappa}[\kappa]\times\{\lambda^{(N)}\}. \tag{243}$$

The other gives

$$\mathrm{O}(N)\times \mathrm{Sp}(m,\mathbb{R})\downarrow \mathrm{O}(N)\times \mathrm{U}(m):$$
$$\bigoplus_{\kappa}[\kappa]\times\langle \tfrac{1}{2}N(\kappa)\rangle \downarrow \bigoplus_{\lambda,\kappa} \mathcal{P}_{\kappa\lambda}[\kappa]\times\{\lambda^{(N)}\}. \tag{244}$$

Thus, comparison of these two results reveals that

$$\mathcal{P}_{\kappa\lambda} = P_{\lambda\kappa}. \tag{245}$$

In this way, the $\mathrm{Sp}(m,\mathbb{R})\downarrow \mathrm{U}(m)$ branching rules were determined (Rowe, Wybourne, and Butler, 1985) from the $\mathrm{U}(N)\downarrow \mathrm{O}(N)$ branching rules of King (1975).

Cases for which $N\geq 2m$ turn out to be particularly simple. For example, in the symplectic shell-model theory of nuclear collective motion, $m=3$ is the dimension of ordinary 3-space and N is the nucleon number of the nucleus. Thus, for medium to heavy nuclei, for which the theory is most relevant, N is large compared to $2m=6$. When $N\geq 2m$, the $\mathrm{U}(N)\times \mathrm{U}(m)$ duality relationship implies that any $\mathrm{U}(N)$ irrep $\{\lambda\}$ carried by a subspace of the Hilbert space of the Nm-dimensional harmonic oscillator is labeled by a partition λ having at most $m\leq N/2$ parts. For such an irrep, the $\mathrm{U}(N)\downarrow \mathrm{O}(N)$ branching rule has a particularly simple expression (King, 1975). The dual $\mathrm{Sp}(m,\mathbb{R})\downarrow \mathrm{U}(m)$ branching rule is then equally simple and given by

$$\mathrm{Sp}(m,\mathbb{R})\downarrow \mathrm{U}(m) : \langle \tfrac{1}{2}N(\kappa)\rangle \downarrow \{\lambda^{(N)}\}\otimes\{D_m\}, \tag{246}$$

where $\{D_m\}=\sum_{n=0}^{\infty}\{2\}\,\textcircled{P}\,\{2n\}$ is the direct sum of the infinite sequence of $\mathrm{U}(m)$ irreps given by partitions whose parts are all even non-negative integers, i.e.,

$$\{D_m\} = \{0\}\oplus\{2\}\oplus\{4\}\oplus\{2^2\}\oplus\{6\}$$
$$\oplus\{42\}\oplus\{2^3\}\oplus\{8\}\oplus\{64\}\oplus\cdots, \tag{247}$$

with the understanding that the number of parts must not exceed m.[3]

2. Model spaces

A model space for a Lie group G, and/or its Lie algebra, is defined (Bernshtein, Gel'fand, and Gel'fand, 1975) as a Hilbert space that carries precisely one copy from every equivalence class of a specified set of irreps of G. For example, a model space for SU(3) is obtained as a subspace of all states of the Hilbert space $\mathbb{H}$ for the six-dimensional harmonic oscillator that are annihilated by the raising operator of a dual U(2) group. This follows because $\mathbb{H}=\bigoplus_{\lambda}\mathbb{H}^{\{\lambda\}}$ is the Hilbert space for a direct sum $\bigoplus_{\lambda}\{\lambda\}\times\{\lambda\}$ of all irreps of $\mathrm{U}(2)\times\mathrm{U}(3)$ that are labeled by partitions $\{\lambda\}=\{\lambda_1\lambda_2\}$ with no more than two integer parts. The subspace of $\mathbb{H}$ that is annihilated by U(2) raising operators is a model space for SU(3) because the branching rule for the restriction of U(3) to its SU(3) subgroup,

$$\mathrm{U}(3)\downarrow \mathrm{SU}(3) : \{\lambda_1\lambda_2\lambda_3\}\downarrow\{\lambda_1-\lambda_2,\lambda_2-\lambda_3\} \tag{248}$$

implies that the set of U(3) irreps with $\lambda_3=0$ restricts to a complete set of SU(3) irreps. Such a model is useful for the calculation of the subset of Clebsch-Gordan coefficients for the U(3) couplings

$$\{\kappa_1\kappa_2\}\otimes\{\lambda_1\lambda_2\} = \bigoplus_{\mu}\{\mu_1\mu_2\mu_3\} \tag{249}$$

for which $\mu_3=0$ (see further comments in Sec. X).

A similar example is given by the Hilbert space $\mathbb{H}$ of the $2m^2$-dimensional harmonic oscillator on which all the holomorphic discrete series irreps of $\mathrm{Sp}(m,\mathbb{R})$ are realized (see footnote 3). The subspace of states in $\mathbb{H}$ that are annihilated by the $\mathrm{Sp}(m,\mathbb{R})$ lowering operators is a model space for $\mathrm{O}(2m)$ and, conversely, the subspace of all states of $\mathbb{H}$ that are annihilated by the $\mathrm{O}(2m)$ raising operators is a model space for the holomorphic discrete series irreps of $\mathrm{Sp}(m,\mathbb{R})$. These model spaces were used in a study by Gelbart (1973).

3. The microscopic theory of nuclear collective dynamics

The $\mathrm{O}(A)\times\mathrm{Sp}(3,\mathbb{R})$ duality is central to the microscopic theory of nuclear collective dynamics in (Rowe, 1985) which $\mathrm{Sp}(3,\mathbb{R})$ is a dynamical group for an A-nucleon collective model Hamiltonian and $\mathrm{O}(A)$ is a symmetry group.

A group G of canonical transformations of a classical many-particle phase space is said to generate *collective motions* if it transforms the phase-space coordinates of all particles in the same way. Thus, if an element $g\in G$ maps a set of phase-space coordinates $\xi=(x,y,z,p_x,p_y,p_z)$ for a particle to a new set, denoted by $g\cdot\xi$, the corresponding collective transformation of an A-particle system is given by

$$(\xi_1,\xi_2,\ldots,\xi_A)\rightarrow(g\cdot\xi_1,g\cdot\xi_2,\ldots,g\cdot\xi_A). \tag{250}$$

Thus, by definition, a group of collective transformations of a many-particle system is a representation of a group of transformations of a single-particle phase space.

For a classical dynamical system described by Hamilton equations of motion, the possible motions are generated by groups of canonical (i.e., symplectic) transformations. Such dynamics satisfy Liouville's theorem, i.e., they preserve volumes in phase space, and are said to be Hamiltonian. Thus, the $\mathrm{Sp}(3,\mathbb{R})$ symplectic group, defined as the set of all linear canonical transformations of the phase space of a single particle in 3-space, is fundamental to the theory of collective structure.

[3]These $\mathrm{Sp}(m,\mathbb{R})$ irreps are the subset of positive harmonic series irreps that belong to the discrete series. It also follows from these results that all of the positive holomorphic discrete series of $\mathrm{Sp}(m,\mathbb{R})$ irreps (not including its double-valued metaplectic irreps) are realized within the Hilbert space of $N=2m$ particles in an m-dimensional harmonic oscillator [cf. Gelbart (1973)].

Infinitesimal generators of $\mathrm{Sp}(m,\mathbb{R})$ are defined by Eq. (232) in terms of harmonic oscillator raising and lowering operators. However, their physical significance is more apparent when expressed in terms of particle position and momentum coordinates. Thus, if $\{x_i; i=1,2,3\}$ are Cartesian coordinates for a single particle in $\mathbb{R}^3$ and $\{p_i; i=1,2,3\}$ are corresponding momentum coordinates, the basis for a unitary representation of the Lie algebra $\mathrm{sp}(3,\mathbb{R})$ is given by the operators

$$\hat{K}_{ij}=\hat{p}_i\hat{p}_j,\quad \hat{Q}_{ij}=\hat{x}_i\hat{x}_j,\quad \hat{T}_{ij}=\hat{x}_i\hat{p}_j+\hat{p}_j\hat{x}_i, \tag{251}$$

on the single-particle Hilbert space $\mathbb{H}_L=\mathcal{L}^2(\mathbb{R}^3)$, where $\hat{x}_i$ and $\hat{p}_i$ satisfy the standard commutation relations

$$[\hat{x}_i,\hat{x}_j]=[\hat{p}_i,\hat{p}_j]=0,\qquad [\hat{x}_i,\hat{p}_j]=i\hbar\delta_{i,j}. \tag{252}$$

Now, if $\{x_{ni}; n=1,\ldots,A; i=1,2,3\}$ are Cartesian coordinates for A particles and $\{p_{ni}; n=1,\ldots,A; i=1,2,3\}$ are corresponding momentum coordinates, infinitesimal generators for an A-particle representation of $\mathrm{Sp}(3,\mathbb{R})$ are given by the $\mathrm{O}(A)$ scalar operators on $\mathcal{L}^2(\mathbb{R}^{3A})$

$$\begin{aligned}&\hat{K}_{ij}=\sum_n \hat{p}_{ni}\hat{p}_{nj},\qquad \hat{Q}_{ij}=\sum_n \hat{x}_{ni}\hat{x}_{nj},\\ &\hat{T}_{ij}=\sum_n(\hat{x}_{ni}\hat{p}_{nj}+\hat{p}_{nj}\hat{x}_{ni}).\end{aligned} \tag{253}$$

The group $\mathrm{Sp}(3,\mathbb{R})$ proves to be just what is needed for a practical microscopic theory of nuclear collective motion. It has the particularly valuable property that the full many-particle kinetic energy $(1/2M)\sum_{ni}\hat{p}_{ni}^2$ is an element of its Lie algebra. Potential-energy functions of the nuclear quadrupole moments $\{Q_{ij}\}$ can then be added to this kinetic energy to form collective model Hamiltonians. The $\mathrm{sp}(3,\mathbb{R})$ Lie algebra also contains the Hamiltonian of the spherical harmonic oscillator

$$\hat{H}_{\mathrm{HO}}=\frac{1}{2M}\sum_{ni}\hat{p}_{ni}^2+\frac{1}{2}M\omega^2\sum_{ni}\hat{x}_{ni}^2, \tag{254}$$

which means that it provides a natural unification of the collective model with the harmonic-oscillator shell model. Combined with the fact that $\mathrm{Sp}(3,\mathbb{R})$ is a simple Lie group and that the representations and coupling coefficients of its $\mathrm{SO}(3)\subset\mathrm{U}(3)$ subgroups are already well known, these properties mean that it is straightforward to compute the matrix elements for irreps of the $\mathrm{sp}(3,\mathbb{R})$ Lie algebra in the harmonic-oscillator representations of $\mathrm{Sp}(3,\mathbb{R})$; they are most simply computed by so-called *vector-coherent-state* methods (Rowe, 1984; Rowe, Rosensteel, and Carr, 1984) as outlined by Rowe (1985) (see Sec. IX.B of this review).

We now show that the $\mathrm{O}(A)\times\mathrm{Sp}(3,\mathbb{R})$ duality on the Hilbert space $\mathcal{L}^2(\mathbb{R}^{3A})$ of spatial wave functions for A particles facilitates the construction of a shell-model coupling scheme with basis states that are products of center-of-mass states and antisymmetric combinations of spin, isospin, and spatial states in an $\mathrm{Sp}(3,\mathbb{R})$ basis.

Separation of center-of-mass states is accomplished by the factorization $\mathcal{L}^2(\mathbb{R}^{3A})=\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{L}^2(\mathbb{R}^{3(A-1)})$, where $\mathcal{L}^2(\mathbb{R}^3)$ is the Hilbert space of center-of-mass states and $\mathcal{L}^2(\mathbb{R}^{3(A-1)})$ is the complementary space for A nucleons relative to their center of mass. It remains to characterize the $\mathrm{Sp}(3,\mathbb{R})$ irreps in $\mathcal{L}^2(\mathbb{R}^{3(A-1)})$ by their S_A symmetries so that they may be combined with spin-isospin irreps of conjugate symmetry to form totally antisymmetric states. This is achieved by standard shell-model techniques for $\mathrm{Sp}(3,\mathbb{R})$ irreps for which the center of mass is in its harmonic-oscillator ground state. More generally, it is made possible by the $\mathrm{O}(A-1)\times\mathrm{Sp}(3,\mathbb{R})$ duality relationship.

By duality, $\mathcal{L}^2(\mathbb{R}^{3(A-1)})$ carries a multiplicity-free direct sum of irreps of the group $\mathrm{O}(A-1)\times\mathrm{Sp}(3,\mathbb{R})$. Because the symmetric group S_A is a subgroup of $\mathrm{O}(A-1)$, it is then possible to construct basis states for $\mathcal{L}^2(\mathbb{R}^{3(A-1)})$ that reduce the subgroup chain

$$\begin{array}{ccccccc}[& \mathrm{O}(A-1) & \supset & \mathrm{S}_A &] & \times & \mathrm{Sp}(3,\mathbb{R})\\ & [\kappa] & & (\lambda) & & & \langle\frac{1}{2}(A-1)(\kappa)\rangle\end{array} \tag{255}$$

and carry S_A irreps corresponding to partitions $\lambda\vdash A$. Thus, $\mathrm{Sp}(3,\mathbb{R})$ irreps are determined with well-defined S_A symmetry and can be coupled to spin-isospin irreps of conjugate S_A symmetry. The $\mathrm{O}(A-1)\downarrow S_A$ branching rules needed for this purpose have been given by Butler and King (1973), Dehuai and Wybourne (1981), and Carvalho (1990).

VIII. DUAL REPRESENTATIONS ON FERMION SPACES

We now consider duality relationships that are specific to fermions. A primary difference between the dual representations expressed in terms of boson operators and those expressed in terms of fermion operators is that the latter usually involve finite-dimensional representations. Thus, the fermionic counterparts of the bosonic duality relationships on harmonic-oscillator spaces involve groups with dual representations on a finite fermion Fock space $\mathbb{F}^{(wN)}$ spanned by multifermion states

$$|0\rangle,\ a_\mu^\dagger|0\rangle,\ a_\nu^\dagger a_\mu^\dagger|0\rangle,\ \ldots,\ a_1^\dagger a_2^\dagger\cdots a_{wN}^\dagger|0\rangle. \tag{256}$$

Pairs of groups with dual representations of relevance to fermion systems are found among the chains of subgroups

$$\begin{array}{ccc}\mathrm{SO}(2wN) & \times & \mathrm{O}(1)\\ \cup & & \cap\\ \mathrm{USp}(2w) & \times & \mathrm{USp}(N)\\ \cup & & \cap\\ \mathrm{U}(w) & \times & \mathrm{U}(N)\\ \cup & & \cap\\ \mathrm{U}(1) & \times & \mathrm{U}(wN),\end{array} \tag{257}$$

for N even, and

$$\begin{array}{ccc}\mathrm{SO}(2wN) & \times & \mathrm{O}(1)\\ \cup & & \cap\\ \mathrm{SO}(2w) & \times & \mathrm{O}(N)\\ \cup & & \cap\\ \mathrm{U}(w) & \times & \mathrm{U}(N)\\ \cup & & \cap\\ \mathrm{U}(1) & \times & \mathrm{U}(wN),\end{array} \tag{258}$$

for N even or odd. The groups $SO(2wN) \subset O(2wN)$ are defined below as groups of Bogoljubov-Valatin transformations and the group $U(wN)$ has a Lie algebra whose complex extension is spanned by the operators $\{a^\dagger_\mu a^\nu\}$.

The duality of $U(w)$ and $U(N)$ representations on $\mathbb{F}^{(wN)}$ follows once again from Theorem 3. The dualities between representations of $USp(2w)$ and $USp(N)$ for N even, and between representations of $SO(2w)$ and $O(N)$ for N even or odd, are established, respectively, in the following two theorems whose proofs are given elsewhere (Helmers, 1961; Rowe, Repka, and Carvalho, 2011).

Theorem 5 (Helmers): The groups $USp(2w)$ and $USp(N)$ have dual representations on $\mathbb{F}^{(wN)}$ for N even.

Theorem 6: The groups $SO(2w)$ and $O(N)$ have dual representations on $\mathbb{F}^{(wN)}$ for N even or odd.

Before discussing the relevance of these dual pairs in physical applications, we first define the group of Bogoljubov-Valatin transformations, $O(2wN)$.

A. The group of Bogoljubov-Valatin transformations

Let $\{a^\dagger_\nu, a^\nu; \nu = 1, \ldots, wN\}$ denote a set of fermion creation and annihilation operators that satisfy the anticommutation relations

$$\{a^\mu, a^\dagger_\nu\} = \delta^\mu_\nu, \qquad \{a^\dagger_\mu, a^\dagger_\nu\} = \{a^\mu, a^\nu\} = 0, \tag{259}$$

and the Hermiticity relations

$$(a^\nu)^\dagger = a^\dagger_\nu. \tag{260}$$

The group of Bogoljubov-Valatin transformations G is then the subset of complex-linear transformations

$$\begin{aligned} a^\dagger_\nu &\to \sum_\mu (a^\dagger_\mu u_{\mu\nu} + a^\mu v_{\mu\nu}), \\ a^\nu &\to \sum_\mu (a^\dagger_\mu v^*_{\mu\nu} + a^\mu u^*_{\mu\nu}), \end{aligned} \tag{261}$$

that preserve the fermion anticommutation relations (259) and the Hermiticity relationship.

To identify this group, consider its application to the Hermitian operators

$$\hat{\mathcal{Q}}_\nu = \frac{1}{\sqrt{2}}(a^\dagger_\nu + a^\nu), \qquad \hat{\mathcal{P}}_\nu = \frac{\mathrm{i}}{\sqrt{2}}(a^\dagger_\nu - a^\nu), \tag{262}$$

which satisfy the anticommutation relations

$$\{\hat{\mathcal{Q}}_\mu, \hat{\mathcal{Q}}_\nu\} = \{\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu\} = \delta_{\mu\nu}, \quad \{\hat{\mathcal{Q}}_\mu, \hat{\mathcal{P}}_\nu\} = 0. \tag{263}$$

If $\mathbb{R}^{2wN}$ is the real vector space spanned by the operators $\{\hat{Z}_i\} = \{\hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \ldots, \hat{\mathcal{P}}_1, \hat{\mathcal{P}}_2, \ldots\}$, then G is the set of linear transformations of $\mathbb{R}^{2wN}$ that preserve the anticommutation relations $\{\hat{Z}_i, \hat{Z}_j\} = \delta_{i,j}$. These transformations must be real; otherwise they would not preserve the Hermiticity of the $\{\hat{\mathcal{Q}}_\nu\}$ and $\{\hat{\mathcal{P}}_\nu\}$ operators. It follows that G is the subgroup of real linear transformations, $\hat{Z} \mapsto \hat{Z}g$, that satisfy the condition

$$\left\{\sum_i \hat{Z}_i g_{ij}, \sum_k \hat{Z}_k g_{kl}\right\} = \sum_i g_{ij} g_{il} = \delta_{j,l}. \tag{264}$$

This is the real orthogonal group $O(2wN)$.

A representation of this $O(2wN)$ group, of relevance to the quantum mechanics of many-fermion systems, is carried by the Fock space $\mathbb{F}^{(wN)}$. In this representation, the Lie algebra $so(2wN)$ of the group $O(2wN)$, known as the *fermion-pair* algebra, is spanned by the Hermitian linear combinations of the operators

$$a^\dagger_\mu a^\dagger_\nu, \qquad a^\mu a^\nu, \qquad a^\dagger_\mu a^\nu - a^\nu a^\dagger_\mu. \tag{265}$$

The Bogoljubov-Valatin group can be extended to a full dynamical group $O(2wN + 1)$ for the fermion system that includes both the even and odd fermion states of the Fock space $\mathbb{F}^{(wN)}$, by the addition of the operators $\{\hat{\mathcal{Q}}_\nu\}$ and $\{\hat{\mathcal{P}}_\nu\}$ to its Lie algebra. Such an addition was proposed by Fukutome, Yamamura, and Nishiyama (1977) based on the observation that the commutator $[a^\dagger_\mu, a^\nu] = a^\dagger_\mu a^\nu - a^\nu a^\dagger_\mu$ of a fermion creation and a fermion annihilation operator is in the complex extension of $so(2wN)$.

Note that if we replace the index ν, that labels a single-fermion state, by a double index $\nu \to (\tau m)$, where τ and m take w and N values, respectively, the space $\mathbb{F}^{(wN)}$ is seen as a tensor product. For example, τ might index the isospin states, $\tau = \pm 1/2$, of an isospin $T = 1/2$ nucleon and m might index a nucleon's angular-momentum states, $m = -j, j+1, \ldots, +j$ with j a half-odd positive integer. In this case $w = 2T + 1 = 2$, and $N = 2j + 1$ is an even integer. In another example, τ might index the four spin-isospin states of a nucleon (with spin and isospin $S = 1/2$, $T = 1/2$) and m might index the orbital angular-momentum states with labels $-l, \ldots, +l$, where l is a non-negative integer. In this case $w = 4$, and $N = 2l + 1$ is an odd integer.

B. Pair-coupling schemes for fermions of a single species

The special case of a $USp(2w) \times USp(N)$ duality, with $w = 1$ and $N = 2j + 1$, where j (a half-odd positive integer) is the angular momentum of a single fermion (a neutron, proton, or electron), is of historical significance because, as far as we know, it was the first duality relationship, independent of Schur-Weyl duality, to be identified (Helmers, 1961). Because of its simplicity, the $USp(2j + 1) \subset U(2j + 1)$ subgroup chain defines the coupling scheme most commonly used in the atomic shell model (Racah, 1949) and in the nuclear shell model for nuclei with either neutron or proton closed shells (Flowers, 1952a, 1952b, 1952c; French, 1960; Talmi, 1993). It also plays a central role in models of pairing and superconductivity in atomic and nuclear physics in which $USp(2j + 1)$ is dual to a so-called $SU(2)_{qs} \simeq USp(2)$ quasispin group (Anderson, 1958; Kerman, 1961; Kerman, Lawson, and Macfarlane, 1961). The Lie algebra $su(2)_{qs}$ is important for understanding situations in which the coupling of fermions to form angular-momentum-zero (Cooper) pairs is energetically favored over other couplings.

A system of n identical fermions of angular momentum j carries a fully antisymmetric irrep $\{1^n\}$ of $U(2j + 1)$ with n restricted to the range $0 \le n \le 2j + 1$. Thus, if the Hamiltonian is rotationally invariant, we seek basis states for this irrep that reduce the rotation subgroup $SU(2)_J \subset U(2j + 1)$ and have conserved angular-momentum quantum numbers. [Note that the rotation group for fermions is SU(2) rather than SO(3).] As observed by Racah (1949), additional

quantum numbers are supplied by inclusion of the group $\mathrm{USp}(2j+1)$ in the subgroup chain

$$\begin{array}{ccccccc} \mathrm{U}(2j+1) & \supset & \mathrm{USp}(2j+1) & \supset & \mathrm{SU}(2)_J & \supset & \mathrm{U}(1)_J \\ n & & \upsilon & & J & & M \end{array}. \tag{266}$$

Now from the paired subgroups of $\mathrm{O}(2(2j+1))$ with dual representations on the Fock space $\mathbb{F}^{(2j+1)}$ shown in Eq. (257), it is seen that $\mathrm{USp}(2j+1)$ has dual representations with an $\mathrm{SU}(2)_{\mathrm{qs}} \simeq \mathrm{USp}(2)$ quasispin group and that $\mathrm{U}(2j+1)$ has dual representations with the subgroup $\mathrm{U}(1)_{\mathrm{qs}} \subset \mathrm{SU}(2)_{\mathrm{qs}}$. It follows that the basis states of the coupling scheme (266) are identical with basis states of the Fock space $\mathbb{F}^{(2j+1)}$ that reduce the dual subgroup chain

$$\begin{array}{ccccccc} \mathrm{SU}(2)_{\mathrm{qs}} & \times & \mathrm{SU}(2)_J & \supset & \mathrm{U}(1)_{\mathrm{qs}} & \times & \mathrm{U}(1)_J \\ s & & J & & s_0 & & M \end{array}. \tag{267}$$

We show in the following that the quantum numbers for these two chains are related by

$$s(\upsilon) = \tfrac{1}{2}(j + \tfrac{1}{2} - \upsilon), \qquad s_0(n) = \tfrac{1}{2}(n - j - \tfrac{1}{2}). \tag{268}$$

[Note that the lowest-weight $\mathrm{SU}(2)_{\mathrm{qs}}$ state for the $\upsilon = 0$ irrep is the $n = 0$ state for which $s_0 = -s$.] We also show that the $\mathrm{USp}(2j+1)$ group in the chain (266) has the physical significance of being the subgroup of $\mathrm{U}(2j+1)$ transformations that leave the creation operator for a Cooper pair (i.e., an angular-momentum-zero fermion pair) invariant. Moreover, this pair-creation operator and the corresponding annihilation operator are shown to generate the Lie algebra $\mathrm{su}(2)_{\mathrm{qs}}$ of the group $\mathrm{SU}(2)_{\mathrm{qs}}$ that commutes with $\mathrm{USp}(2j+1)$. It follows that $\mathrm{USp}(2j+1)$ is a symmetry group of any Hamiltonian that is defined in terms of the $\mathrm{su}(2)_{\mathrm{qs}}$ pair operators. Thus, we have two groups, $\mathrm{SU}(2)_{\mathrm{qs}}$ and $\mathrm{USp}(2j+1)$, with commuting actions on the Fock space $\mathbb{F}^{(2j+1)}$, of which $\mathrm{SU}(2)_{\mathrm{qs}}$ is a dynamical group and $\mathrm{USp}(2j+1)$ is a symmetry group for a class of pairing model Hamiltonians.

1. The $\mathrm{su}(2)_{\mathrm{qs}}$ and $\mathrm{usp}(2j+1)$ Lie algebras

The fermion-pair creation operator used to define the $\mathrm{USp}(2j+1)$ group is the operator

$$\begin{aligned} \hat{\mathcal{A}} &= \sqrt{2j+1}\sum_m (j, -m; j, m|00) a_m^\dagger a_{\bar{m}}^\dagger \\ &= \sum_m (-1)^{j+m} a_m^\dagger a_{\bar{m}}^\dagger, \end{aligned} \tag{269}$$

where $\{a_m^\dagger; m = -j, \ldots, +j\}$ is a set of $2j+1$ creation operators for fermions of angular momentum j and $(j, -m; j, m|00) = (-1)^{j+m}$ is an SU(2) Clebsch-Gordan coefficient. The operator $\hat{\mathcal{A}}$ is the raising operator of the $\mathrm{su}(2)_{\mathrm{qs}}$ quasispin Lie algebra spanned by the operators

$$\begin{aligned} \hat{S}_+ &= \frac{1}{2}\hat{\mathcal{A}} = \sum_{m>0} a_m^\dagger a_{\bar{m}}^\dagger, \qquad \hat{S}_- = \sum_{m>0} a^{\bar{m}} a^m, \\ 2\hat{S}_0 &= \sum_{m>0} (a_m^\dagger a^m - a^{\bar{m}} a_{\bar{m}}^\dagger) = \hat{n} - \frac{1}{2}(2j+1)\hat{I}, \end{aligned} \tag{270}$$

where

$$a_{\bar{m}}^\dagger \equiv (-1)^{j+m} a_{-m}^\dagger, \qquad a^{\bar{m}} \equiv (-1)^{j+m} a^{-m}, \tag{271}$$

$\hat{n} = \sum_{m=-j}^{j} a_m^\dagger a^m$ is the fermion number operator, and $\hat{I}$ is the identity operator.[4] The quasispin operators satisfy the commutation relations

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_0, \qquad [\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm. \tag{272}$$

We now identify the operators of the $\mathrm{usp}(2j+1)$ Lie algebra. First observe that for each m the fermion creation and annihilation operators $a_m^\dagger$ and $a^{\bar{m}}$ are $\pm\frac{1}{2}$ components of a quasi-spin-$\frac{1}{2}$ tensor $\hat{\xi}_m = (a_m^\dagger, a^{\bar{m}})$ as can be seen from the commutation relations

$$\begin{aligned} &[\hat{S}_+, a^{\bar{m}}] = a_m^\dagger, \quad [\hat{S}_-, a_m^\dagger] = a^{\bar{m}}, \quad [\hat{S}_0, a^{\bar{m}}] = -\tfrac{1}{2}a^{\bar{m}}, \\ &[\hat{S}_0, a_m^\dagger] = \tfrac{1}{2}a_m^\dagger, \quad [\hat{S}_+, a_m^\dagger] = 0, \quad [\hat{S}_-, a^{\bar{m}}] = 0, \end{aligned} \tag{273}$$

which apply for any $-j \le m \le +j$. Next observe that pairs of these quasi-spin-$\frac{1}{2}$ tensors can be coupled with SU(2) Clebsch-Gordan coefficients to quasi-spin-scalar operators, i.e., $[\hat{\xi}_m \otimes \hat{\xi}_p]_0 = (a_m^\dagger a^{\bar{p}} - a^{\bar{m}} a_p^\dagger)/\sqrt{2}$. Moreover, to within numerical constants given, for example, by the anticommutators $\{a_m^\dagger, a^m\} = 1$, these are the only bilinear combinations of the fermion creation and annihilation operators that can commute with the $\mathrm{su}(2)_{\mathrm{qs}}$ operators. Thus, in the present context, the $\mathrm{usp}(2j+1)$ Lie algebra is spanned by the Hermitian linear combinations of the quasispin scalar operators

$$\hat{A}_{mp} = a_m^\dagger a^{\bar{p}} + a_p^\dagger a^{\bar{m}}, \qquad m \ge p > 0, \tag{274}$$

$$\hat{C}_{mp} = a_m^\dagger a^p - a_{\bar{p}}^\dagger a^{\bar{m}}, \qquad m, p > 0 \tag{275}$$

$$\hat{B}_{mp} = a_{\bar{m}}^\dagger a^p + a_{\bar{p}}^\dagger a^m, \qquad m \ge p > 0. \tag{276}$$

A basis for the Cartan subalgebra for this realization of $\mathrm{usp}(2j+1)$ is given by the operators

$$\hat{C}_{mm} = a_m^\dagger a^m - a_{\bar{m}}^\dagger a^{\bar{m}}, \qquad m > 0. \tag{277}$$

2. Labels for $\mathrm{SU}(2)_{\mathrm{qs}}$ and $\mathrm{USp}(2j+1)$ irreps

Quantum numbers for basis states defined by subgroup chains are given as usual by the labels for the irreps of the groups in the chain. Irreps of $\mathrm{SU}(2)_{\mathrm{qs}}$ and $\mathrm{USp}(2j+1)$ are conveniently labeled by their lowest and highest weights, respectively. A complete set of states in the Fock space $\mathbb{F}^{(2j+1)}$ that are simultaneously of lowest $\mathrm{SU}(2)_{\mathrm{qs}}$ and highest $\mathrm{USp}(2j+1)$ weight are the states

[4] The bar operation $a_m^\dagger \to a_{\bar{m}}^\dagger$ is equivalent to a rotation through angle π. Therefore, because $2j$ is odd, applying it twice changes the sign of a fermion operator, i.e., $a_{\bar{m}}^\dagger = -a_m^\dagger$ and $a^{\bar{m}} = -a^m$.

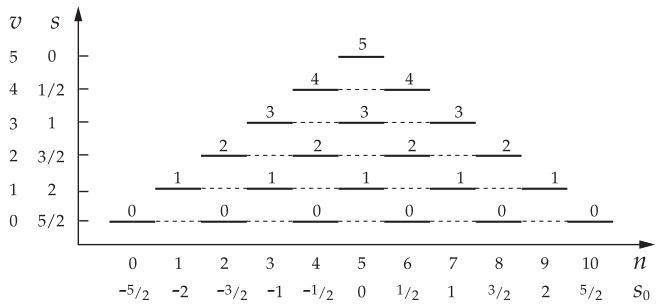


FIG. 2. The spectrum of USp(10) irreps in the $j = 9/2$ shell labeled by values of the seniority quantum number v. The combined states of all USp(10) irreps having a common value of the particle number n span the U(10) irrep $\{1^n\}$. The combined USp(10) irreps having a common value of the seniority v, connected by dotted lines, span an $SU(2)_{qs} \times USp(10)$ irrep.

$$|0\rangle, \qquad |1\rangle = a_j^\dagger|0\rangle, \qquad |2\rangle = a_j^\dagger a_{j-1}^\dagger|0\rangle, \quad \ldots,$$
$$|v\rangle = a_j^\dagger a_{j-1}^\dagger \cdots a_{j-v+1}^\dagger|0\rangle, \quad \ldots, \quad |j+\tfrac{1}{2}\rangle, \tag{278}$$

for v an integer in the range $0 \le v \le j + \frac{1}{2}$, which we describe as *extremal states*.

A USp$(2j + 1)$ irrep with highest-weight state $|v\rangle$ has highest weight defined by

$$\langle v|\hat{C}_{mm}|v\rangle = \begin{cases} 1 & \text{for } j \ge m \ge j - v + 1, \\ 0 & \text{for } \frac{1}{2} \le m < j - v + 1. \end{cases} \tag{279}$$

Thus, it has highest weight $\langle 1^v\rangle$ and states of this irrep are labeled by the quantum number v. On the other hand, an $SU(2)_{qs}$ irrep with lowest-weight state $|v\rangle$ has highest weight defined by the quasispin

$$s(v) = -\langle v|\hat{S}_0|v\rangle = \tfrac{1}{2}(j + \tfrac{1}{2} - v), \tag{280}$$

consistent with Eq. (268). The quantum number of the $U(1)_{qs} \subset SU(2)_{qs}$ subgroup with infinitesimal generator $\hat{S}_0$ is similarly given, by Eq. (268), i.e., $s_0(n) = \frac{1}{2}(n - j - \frac{1}{2})$.

The integer v, which labels the extremal states of Eq. (278) and both the $SU(2)_{qs}$ and USp$(2j + 1)$ irreps, is known as the *seniority* quantum number; it takes the values $v = 0, \ldots, j + \frac{1}{2}$ and has a physical interpretation as the number of unpaired particles in any state of an $SU(2)_{qs} \times USp(2j + 1)$ irrep; it is equal to the number of particles in an $SU(2)_{qs}$ lowest-weight state. This can be seen in Fig. 2.

3. A specific example and some general results

Suppose, for example, that $j = 9/2$. The states available to a many-fermion system (either protons, neutrons, or electrons) occupying the single-particle states of a $j = 9/2$ nuclear or atomic shell are shown in Fig. 2, by short lines, as subsets of states that span irreps of USp(10). The subsets belonging to equivalent USp(10) irreps are linked by dashed lines and together span irreps of $USp(10) \times SU(2)_{qs}$. By $U(1)_{qs} \times U(10)$ duality, all states belonging to a column of USp(10) irreps of particle number n belong to the single U(10) irrep $\{1^n\}$ that is determined by the $U(1)_{qs}$ irrep $\{n\}$. Thus, the figure shows that a subset of $U(10) \downarrow USp(10)$ branching rules are given, for $n \le 5$, by

$$\{1^n\} \downarrow \langle 1^n\rangle \oplus \langle 1^{n-2}\rangle \oplus \cdots \oplus \langle 1\rangle \text{ or } \langle 0\rangle, \tag{281}$$

and for $n > 5$ by

$$\{1^n\} \downarrow \langle 1^{10-n}\rangle \oplus \langle 1^{8-n}\rangle \oplus \cdots \oplus \langle 1\rangle \text{ or } \langle 0\rangle. \tag{282}$$

In fact, the duality relationships between the subgroup chains $SU(2)_{qs} \supset U(1)_{qs}$ and $U(2j + 1) \supset USp(2j + 1)$ imply that the coefficients in the two sets of branching rules

$$SU(2)_{qs} \downarrow U(1)_{qs} : (s(v)) \downarrow \bigoplus_n k_{vn}\{s_0(n)\}, \tag{283}$$

$$U(2j + 1) \downarrow USp(2j + 1) : \{1^n\} \downarrow \bigoplus_v k_{vn}\langle 1^v\rangle \tag{284}$$

are identical for any value of j. Thus, from the known coefficients for the $SU(2)_{qs} \supset U(1)_{qs}$ branching rules it is determined that a subset of $U(N) \downarrow USp(N)$ branching rules is given, for any positive integer N, by

$$\{1^n\} \downarrow \begin{cases} \langle 1^n\rangle \oplus \langle 1^{n-2}\rangle \oplus \cdots \oplus \langle 1\rangle \text{ or } \langle 0\rangle, & \text{for } n \le N/2, \\ \langle 1^{N-n}\rangle \oplus \cdots \oplus \langle 1\rangle \text{ or } \langle 0\rangle, & \text{for } n > N/2. \end{cases} \tag{285}$$

Branching rules such as these provide powerful tools for deriving many needed results in shell-model and other applications. For example, they make it possible to infer the angular-momentum states contained within a USp$(2j + 1)$ irrep $\langle 1^n\rangle$ from a knowledge of the $U(2j + 1) \downarrow SU(2)_J$ branching rules. Suppose these branching rules are expressed for $n \le j + \frac{1}{2}$ by

$$U(2j + 1) \downarrow SU(2)_J : \{1^n\} \downarrow \bigoplus_J F_J^n(J), \tag{286}$$

where we use the symbol (J) to denote an $SU(2)_J$ irrep of angular momentum J [corresponding to the U(2) irrep $\{2J\}$]. It follows from Eq. (285) that, for $n \le j + \frac{1}{2}$,

$$\begin{aligned} USp(2j+1)\downarrow SU(2)_J : &\langle 0\rangle \downarrow (0) \\ &: \langle 1\rangle \downarrow (j) \\ &: \langle 1^n\rangle \downarrow \bigoplus_J (F_J^n - F_J^{n-2})(J). \end{aligned} \tag{287}$$

This relationship is confirmed for $j = 9/2$ from Fig. 2.

The F_J^n coefficients can be evaluated by means of a plethysm. First observe that the single-particle $n = 1$ states all have angular momentum $J = j$. This means that they span a U(2) irrep labeled by the partition $\{2j\}$ and that the restriction of the fundamental U$(2j + 1)$ irrep $\{1\}$ to U(2) satisfies the branching rule

TABLE II. The spectrum of angular-momentum states contained in the USp(10) irreps of the $j = 9/2$ shell.

v	s	J
0	$\frac{5}{2}$	0
1	2	$\frac{9}{2}$
2	$\frac{3}{2}$	2, 4, 6, 8
3	1	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2}, \frac{21}{2}$
4	$\frac{1}{2}$	0, 2, 3, 4^2, 5, 6^2, 7, 8, 9, 10, 12
5	0	$\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2}, \frac{19}{2}, \frac{25}{2}$

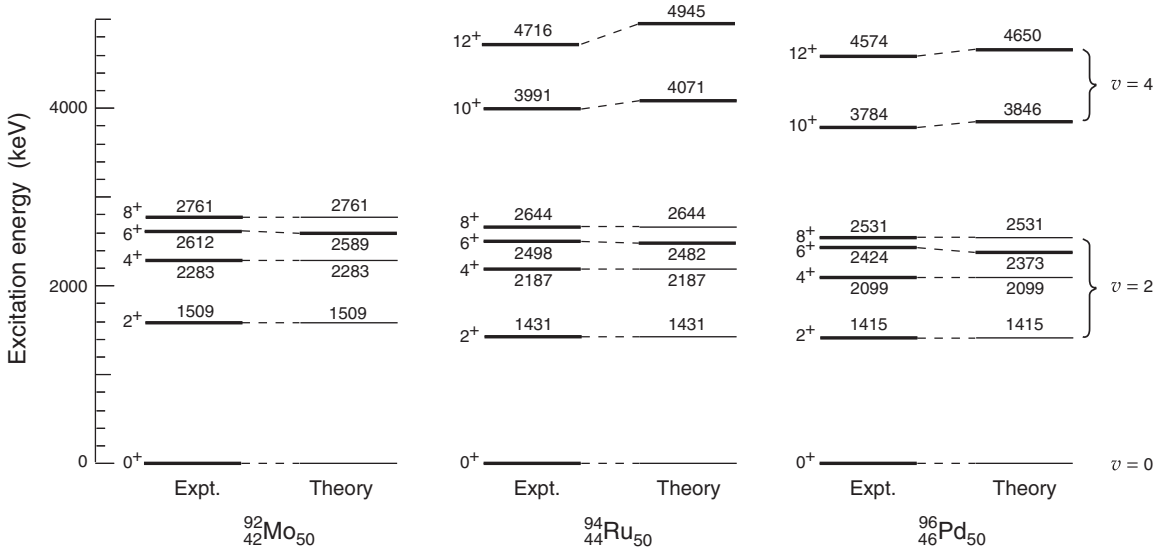


FIG. 3. Low-lying energy levels of some $N = 50$ isotones. The four parameters of a seniority-conserving interaction between protons in a $g_{9/2}$ shell were chosen to fit the energies of the lowest $J = 0$, 2, 4, and 8 states shown by light (theory) lines for each nucleus. The remaining energy levels shown by heavy (theory) lines were then predicted for these interactions. From Rosensteel and Rowe, 2003.

$$\mathrm{U}(2j+1) \downarrow \mathrm{U}(2) : \{1\} \downarrow \{2j\}. \tag{288}$$

It follows that n-particle irreps of $\mathrm{U}(2j+1)$ contain the angular-momentum states given by the plethysm

$$\mathrm{U}(2j+1) \downarrow \mathrm{U}(2) : \{1^n\} \downarrow \{2j\} \circledcirc \{1^n\}. \tag{289}$$

Each U(2) irrep $\{\lambda_1 \lambda_2\}$ then restricts to an $\mathrm{SU}(2)_J$ irrep of angular momentum $J = \frac{1}{2}(\lambda_1 - \lambda_2)$.

For example, using the plethysm code of Carvalho and D'Agostino (2001a), it is determined that

$$\mathrm{U}(10) \downarrow \mathrm{U}(2) : \{1^2\} \downarrow \{17,1\} \oplus \{15,3\} \oplus \{13,5\} \oplus \{11,7\} \oplus \{9,9\}, \tag{290}$$

and, hence, that

$$\mathrm{U}(10) \downarrow \mathrm{SU}(2)_J : \{1^2\} \downarrow (8) \oplus (6) \oplus (4) \oplus (2) \oplus (0). \tag{291}$$

It follows from Eq. (287) that

$$\mathrm{USp}(10) \downarrow \mathrm{SU}(2)_J : \langle 1^2 \rangle \downarrow (8) \oplus (6) \oplus (4) \oplus (2). \tag{292}$$

Repeating this process for other values of n and v, we obtain the spectrum of angular-momentum states contained in the USp(10) irreps of the $j = 9/2$ shell shown in Table II.

The above results show the duality relationship between the irreps of $\mathrm{USp}(2j+1)$ and $\mathrm{SU}(2)_{\mathrm{qs}}$ to be effective at giving a simple pairing model a microscopic expression within the framework of the many-nucleon shell model. They also reveal the more general circumstances under which the $\mathrm{USp}(2j+1)$ symmetry of a pair-coupling model is preserved. In particular, a number-conserving interaction that is expressible as a polynomial in the $\mathrm{usp}(2j+1)$ and $\mathrm{su}(2)_{\mathrm{qs}}$ Lie algebras cannot mix states belonging to different $\mathrm{USp}(2j+1)$ irreps. Such an interaction therefore conserves the seniority quantum number (French, 1960; Talmi, 1993; Rosensteel and Rowe, 2003). A remarkably large number of nuclear interactions have this property. For example, Fig. 3 shows the low-lying states of three isotones of neutron number $N = 50$ with their energy levels modeled in terms of 2, 4, and 6 protons in $j = 9/2$ single-particle states, outside of an $N = 50$, $Z = 40$ closed-shell core, and with states described by $\mathrm{U}(10) \supset \mathrm{USp}(10) \supset \mathrm{SU}(2)_J \supset \mathrm{U}(1)_J$ quantum numbers. The Hamiltonian used to fit the energy levels was chosen to comprise rotationally invariant and seniority-conserving two-body interactions that fit the lowest $J = 0$, 2, 4, and 8 states. Figure 3 shows the success of such a Hamiltonian in predicting the excitation energies of the other observed low-lying energy levels.

C. Dual groups for a multishell BCS Hamiltonian

The above coupling scheme for a single j shell has an interesting extension to a multishell (also called multilevel) scheme. Multilevel pair-coupling models have long been of interest in both nuclear and condensed matter physics as models of superconductivity. They are typically solved in the BCS approximation (Bardeen, Cooper, and Schrieffer, 1957) which is an approximation that violates particle-number conservation and other symmetries of the model. However, it was recently rediscovered that a method proposed by Richardson and Sherman (1964) and Richardson (1965) and pursued from a different perspective by Gaudin (1976) and Cambiaggio, Rivas, and Saraceno (1997) shows that a class of BCS Hamiltonians is integrable and has formally exact solutions.[5] The Richardson-Gaudin (RG) method and its many applications have been reviewed by Dukelsky, Pittel, and Sierra (2004).

A general BCS pairing Hamiltonian is of the form

$$\hat{H} = \sum_{k=1}^{p} \varepsilon_k \hat{S}_0^k - \sum_{i,k=1}^{p} g_{ik} \hat{S}_+^i \hat{S}_-^k, \tag{293}$$

[5]Solutions to the RG equations are described as "formally exact" because the equations can be solved only numerically.

where $\hat{S}_0^k$ and $\hat{S}_\pm^k$ span the complex extension of a quasispin algebra $\mathrm{SU}(2)_k$ and k indexes the levels of the system. Such a Hamiltonian has a dynamical group

$$G_2 = \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \cdots \times \mathrm{SU}(2)_p. \tag{294}$$

With the quasispin operators $\hat{S}_\pm^k$ interpreted as creation and annihilation operators for a pair of particles in level k, as defined by Eq. (270), $\hat{H}$ conserves the total particle number of the system. Thus, the $\mathrm{U}(1) \subset G_2$ subgroup, with infinitesimal generator $\hat{S}_0 = \sum_k \hat{S}_0^k$, is a symmetry group of $\hat{H}$. However, the Hamiltonian $\hat{H}$ has a much larger symmetry group

$$G_1 = \mathrm{USp}(2j_1 + 1) \times \cdots \times \mathrm{USp}(2j_p + 1), \tag{295}$$

where j_k is the angular momentum of a particle in level k that is dual to the dynamical group G_2. Thus, because the groups $\mathrm{U}(1)$ and $\mathrm{U}(\sum_k (2j_k + 1))$ form a dual pair, it is determined that a BCS Hamiltonian is diagonal in a basis that simultaneously reduces the subgroup chains $G_2 \supset \mathrm{U}(1)$ and $U(\sum_k (2j_k + 1)) \supset G_1$.

These observations indicate ways to extend single-j subshell coupling schemes to multi-j subshell schemes. For weak pairing correlations, i.e., when the off-diagonal elements g_{ik} are negligible for $i \neq k$, the eigenstates of $\hat{H}$ diagonalize the dynamical subgroup chain

$$G_2 \supset \mathrm{U}(1)_1 \times \cdots \times \mathrm{U}(1)_p \supset \mathrm{U}(1). \tag{296}$$

The dual chain that defines a corresponding shell-model coupling scheme is then the subgroup chain

$$\mathrm{U}\Big(\sum_k (2j_k + 1)\Big) \supset \prod_k \mathrm{U}(2j_k + 1) \supset G_1. \tag{297}$$

This is a standard shell-model coupling scheme. However, in the strong-coupling limit when ε_k and g_{ik} take i- and k-independent values, the eigenstates of $\hat{H}$ diagonalize the dynamical subgroup chain

$$G_2 \supset \mathrm{SU}(2) \supset \mathrm{U}(1), \tag{298}$$

where $\mathrm{SU}(2) \subset G_2$ is the subgroup with infinitesimal generators $\hat{S}_i = \sum_k \hat{S}_i^k$. They also simultaneously diagonalize the dual subgroup chain of the shell-model coupling scheme defined by

$$\mathrm{U}\Big(\sum_k (2j_k + 1)\Big) \supset \mathrm{USp}\Big(\sum_k (2j_k + 1)\Big) \supset G_1. \tag{299}$$

Other possible coupling schemes are available when more than two j subshells are involved in which some subshells are coupled strongly and others weakly. These several coupling schemes enable the selection of basis states for shell-model calculations that can be effectively truncated to include the dominant states coupled by an interaction with strong pairing components.

D. Isospin-invariant pair coupling in nuclei

In this section, we extend the duality relationships relevant for a valence shell of neutrons, protons, or electrons to a system of neutrons and protons. We show that such a system, with $w = 2$ and $N = (2j + 1)$, provides a $\mathrm{USp}(4) \times \mathrm{USp}(2j + 1)$ duality on a nuclear Fock space $\mathbb{F}^{(2(2j+1))}$. The group $\mathrm{USp}(4) \cong \mathrm{SO}(5)$ is a straightforward extension of the $\mathrm{USp}(2) \cong \mathrm{SU}(2)$ quasispin group for pair coupling of a single-nucleon species to a dynamical group for an isospin-invariant neutron-proton pairing model. The duality of its representations with those of $\mathrm{USp}(2j + 1)$ provides a practical interpretation and useful relationships for the application of the standard shell model in the jj- and isospin-coupled basis of Flowers (1952a, 1952b, 1952c) and French (1960).

To track the symmetries of a system of neutrons and protons, it is appropriate to regard neutrons and protons as different states of nucleons of isospin $T = 1/2$, labeled by $T_0 = \pm 1/2$, respectively. A basis for the Fock space $\mathbb{F}^{(2(2j+1))}$ is then one that reduces the subgroup chain

$$\begin{array}{ccccccccccc} \mathrm{U}(2(2j+1)) & \supset & \mathrm{U}(2j+1) & \times & \mathrm{U}(2)_T & \supset & \mathrm{USp}(2j+1) & \times & \mathrm{SU}(2)_T & \supset & \mathrm{SU}(2)_J \\ n & & \lambda & & \tilde{\lambda} & & \kappa & & TT_0 & & JM \end{array} \tag{300}$$

with quantum numbers defined by the representation labels shown. The group $\mathrm{U}(2)_T$ in this chain is the group of unitary transformations in two-dimensional isospin space.

From the paired subgroups of $\mathrm{O}(4(2j + 1))$ with dual representations on the Fock space $\mathbb{F}^{(2(2j+1))}$ shown in Eq. (257), it is now seen that $\mathrm{USp}(2j + 1)$ and the $\mathrm{USp}(4) \simeq \mathrm{SO}(5)$ quasispin group have dual representations as do $\mathrm{U}(2j + 1)$ and the isospin subgroup $\mathrm{U}(2)_T \subset \mathrm{SO}(5)$. It follows that the basis states of the coupling scheme (300) are identical to basis states of the Fock space $\mathbb{F}^{(2(2j+1))}$ that reduce the dual subgroup chain

$$\begin{array}{ccccccccc} \mathrm{SO}(5) & \times & \mathrm{SU}(2)_J & \supset & \mathrm{U}(2)_T & \supset & \mathrm{SU}(2)_T & \times & \mathrm{U}(1) \\ (v,t) & & JM & & \tilde{\lambda} & & TT_0 & & n \end{array}, \tag{301}$$

where v and t are related, as shown in the following, to the elements of the partition κ.

1. The Lie algebras of SO(5) and USp$(2j + 1)$ and their irreps

The so(5) Lie algebra is the natural extension of $\mathrm{su}(2)_{\mathrm{qs}}$ to include all $J = 0$ pair-creation operators that become possible when there are two kinds of fermions: neutrons and protons. Thus, its complex extension is spanned by the angular-momentum $J = 0$ operators

$$\hat{\mathcal{A}}_{\sigma\tau} = \sum_{m>0} (a^\dagger_{\sigma m} a^\dagger_{\tau \bar{m}} + a^\dagger_{\tau m} a^\dagger_{\sigma \bar{m}}), \tag{302}$$

$$\hat{\mathcal{B}}_{\sigma\tau} = \sum_{m>0} (a^{\tau \bar{m}} a^{\sigma m} + a^{\sigma \bar{m}} a^{\tau m}), \tag{303}$$

$$\hat{\mathcal{C}}_{\sigma\tau} = \sum_{m>0} (a^\dagger_{\sigma m} a^{\tau m} - a^{\tau \bar{m}} a^\dagger_{\sigma \bar{m}}) = \sum_m a^\dagger_{\sigma m} a^{\tau m} - \frac{1}{2}(2j + 1)\delta_{\sigma,\tau}, \tag{304}$$

where

$$a^\dagger_{\sigma\bar{m}} \equiv (-1)^{j+m} a^\dagger_{\sigma,-m}, \quad a^{\sigma\bar{m}} \equiv (-1)^{j+m} a^{\sigma,-m}, \\ \sigma,\tau = 1,2, \quad m = -j,\ldots,+j. \tag{305}$$

The $u(2)_T \subset so(5)$ subalgebra is spanned by the Hermitian linear combinations of the $\{\hat{C}_{\sigma\tau}\}$ operators.

The operators of the $usp(2j+1)$ Lie algebra are now realized as the subset of $u(2j+1)$ operators that commute with these so(5) operators. They are simply obtained by adding an infinitesimal generator of a neutron realization of $usp(2j+1)$ to the corresponding infinitesimal generator of a proton realization. For example, a basis for the Cartan subalgebra of a combined neutron-proton realization of $usp(2j+1)$ is given by

$$\hat{C}_{mm} = \sum_\tau (a^\dagger_{\tau m} a^{\tau m} - a^\dagger_{\tau\bar{m}} a^{\tau\bar{m}}), \qquad m > 0. \tag{306}$$

A dual pair of SO(5) and $USp(2j+1)$ irreps on the Fock space $\mathbb{F}^{(2(2j+1))}$, labeled by a $USp(2j+1)$ highest weight κ, is defined by a state

$$|\kappa\rangle = a^\dagger_{1,j} a^\dagger_{1,j-1} \cdots a^\dagger_{1,j+1-\tilde{\kappa}_1} |0\rangle, \tag{307}$$

when $\tilde{\kappa}_2 = 0$, and by

$$|\kappa\rangle = (a^\dagger_{1,j} a^\dagger_{1,j-1} \cdots a^\dagger_{1,j+1-\tilde{\kappa}_1})(a^\dagger_{2,j} a^\dagger_{2,j-1} \cdots a^\dagger_{2,j+1-\tilde{\kappa}_2})|0\rangle, \tag{308}$$

when $\tilde{\kappa}_2 \geq 1$, where $\tilde{\kappa}$ is the two-row partition conjugate to κ. Such a state is simultaneously of lowest SO(5) weight and highest $USp(2j+1)$ weight. Its SO(5) lowest weight is given by the nonzero expectation values of the $u(2)_T$ operators

$$\langle\kappa|\hat{C}_{\tau\tau}|\kappa\rangle = \langle\kappa|\sum_m a^\dagger_{\tau m} a^{\tau m} - \frac{1}{2}(2j+1)|\kappa\rangle \\ = \tilde{\kappa}_\tau - \frac{1}{2}(2j+1), \quad \text{for } \tau = 1,2, \tag{309}$$

and its $USp(2j+1)$ highest weight is given by the expectation values of

$$\kappa_k = \langle\kappa|\hat{C}_{j+1-k,j+1-k}|\kappa\rangle, \qquad k = 1,\ldots,j+\tfrac{1}{2}. \tag{310}$$

Thus, it is ascertained that

$$\kappa_k = \begin{cases} 2 & \text{for } 1 \leq k \leq \tilde{\kappa}_2, \\ 1 & \text{for } \tilde{\kappa}_2 < k \leq \tilde{\kappa}_1, \\ 0 & \text{for } k > \tilde{\kappa}_1. \end{cases} \tag{311}$$

The equivalent seniority and reduced isospin labels $v = \tilde{\kappa}_1 + \tilde{\kappa}_2$ and $t = \frac{1}{2}(\tilde{\kappa}_1 - \tilde{\kappa}_2)$ are understood physically as the particle number and isospin of the extremal state $|\kappa\rangle$.

2. Tabulation of basis states in jj coupling

The quantum numbers of basis states defined by the subgroup chain (300) signify irreps of the corresponding groups in the chains as follows. The n-nucleon Hilbert $\mathbb{H}^{(n)}$ carries an irrep $\{1^n\}$ of $U(2(2j+1))$. According to Theorem 3, it also carries a direct sum of irreps of the direct product group $U(2j+1) \times U(2)_T$ given by the branching rules

$$U(2(2j+1)) \downarrow U(2j+1) \times U(2)_T : \{1^n\} \downarrow \bigoplus_{\lambda \vdash n} \{\lambda\} \times \{\tilde{\lambda}\}, \tag{312}$$

with λ and $\tilde{\lambda}$ restricted to partitions having no more than $2j+1$ and 2 parts, respectively. The range of values of the $SU(2)_T$ isospin quantum number T is given by the branching rule

$$U(2)_T \downarrow SU(2)_T : \{\tilde{\lambda}\} \downarrow T = \tfrac{1}{2}(\tilde{\lambda}_1 - \tilde{\lambda}_2). \tag{313}$$

The $U(2j+1)$ irreps and isospins for $j = 3/2$ and $n \leq 4$ are shown, for example, in the first column of Table III.

To determine the $USp(2j+1)$ irreps $\langle\kappa\rangle$ in a given $U(2j+1)$ irrep $\{\lambda\}$, one needs the coefficients in the

$$U(2j+1) \downarrow USp(2j+1) : \{\lambda\} \downarrow \bigoplus_\kappa F_{\kappa\lambda} \langle\kappa\rangle \tag{314}$$

branching rule. These can be obtained from the corresponding $SO(5) \simeq USp(4) \downarrow U(2)_T$ branching rules or, more simply, from an algorithm given by King (1975) [and summarized by Rowe and Wood (2010)] which then determines the $SO(5) \downarrow U(2)_T$ branching rules. The values of v and t for each $USp(2j+1)$ irrep equal the values of n and T, respectively, for the lowest value of n for which the $USp(2j+1)$ irrep $\langle\kappa\rangle$ occurs. This irrep contains an $SO(5) \times USp(2j+1)$ extremal state and is noted in the last column of Table III by an asterisk.

The $SU(2)_J$ irreps contained in an $USp(2j+1)$ irrep can be obtained recursively from the $U(2j+1) \downarrow USp(2j+1)$ and $U(2j+1) \downarrow U(2)_J$ branching rules, where the latter is given by the plethysm

$$U(2j+1) \downarrow U(2)_J : \{1\} \downarrow \{2j\} \tag{315}$$

$$: \{\lambda\} \downarrow \{2j\} \,\textcircled{P}\, \{\lambda\}. \tag{316}$$

TABLE III. Irreps of the subgroups in the chains (300) and (301) to which the states of nucleon number $n = 1,\ldots,4$ and $j = 3/2$ belong. For $n = 5,\ldots,8$, the states are mirror images of the $n = 4,\ldots,1$ states, as can be seen in Fig. 4. Lowest-n multiplets of states of an $SO(5) \times USp(4)$ irrep are denoted by an asterisk in the last column. These are the lowest-n multiplets of states among sets of states linked by dotted lines in Fig. 4. These lowest-n multiplets of states contain a lowest-weight state for an $SO(5) \times USp(4)$ irrep.

n	$\{\lambda\}$	T	$\langle\kappa\rangle$	v	t	J	l.wt.
0	$\{00\}$	0	$\langle 00\rangle$	0	0	0	*
1	$\{10\}$	$\frac{1}{2}$	$\langle 10\rangle$	1	$\frac{1}{2}$	$\frac{3}{2}$	*
2	$\{20\}$	0	$\langle 20\rangle$	2	0	1, 3	*
2	$\{1^2\}$	1	$\langle 00\rangle$	0	0	0	
2	$\{1^2\}$	1	$\langle 11\rangle$	2	1	2	*
3	$\{21\}$	$\frac{1}{2}$	$\langle 10\rangle$	1	$\frac{1}{2}$	$\frac{3}{2}$	
3	$\{21\}$	$\frac{1}{2}$	$\langle 21\rangle$	3	$\frac{1}{2}$	$\frac{1}{2}, \frac{5}{2}, \frac{7}{2}$	*
3	$\{1^3\}$	$\frac{3}{2}$	$\langle 10\rangle$	1	$\frac{1}{2}$	$\frac{3}{2}$	
4	$\{2^2\}$	0	$\langle 00\rangle$	0	0	0	
4	$\{2^2\}$	0	$\langle 11\rangle$	2	1	2	
4	$\{2^2\}$	0	$\langle 22\rangle$	4	0	2, 4	*
4	$\{21^2\}$	1	$\langle 20\rangle$	2	0	1, 3	
4	$\{21^2\}$	1	$\langle 11\rangle$	2	1	2	
4	$\{1^4\}$	2	$\langle 00\rangle$	0	0	0	

Suppose, for example, that we have already determined the $USp(2j+1) \downarrow SU(2)_J$ branching rules for the $USp(2j+1)$ irreps that occur for $n < 4$ and we now wish to determine the rules for $n = 4$. The $U(2j+1) \downarrow USp(2j+1)$ branching rule for the $\{2^2\}$ irrep is given by

$$\{2^2\} \downarrow \langle 2^2\rangle \oplus \langle 1^2\rangle \oplus \langle 0\rangle \tag{317}$$

and by the plethysm of Eq. (316), we derive

$$U(2j+1) \downarrow SU(2)_J : \{2^2\} \downarrow (0) \oplus 2(2) \oplus (4). \tag{318}$$

Thus, knowing the branching rules for $n = 0$ and 2

$$USp(2j+1) \downarrow SU(2)_J : \langle 1^2\rangle \downarrow (2), \tag{319}$$

$$: \langle 0\rangle \downarrow (0), \tag{320}$$

it follows that

$$USp(2j+1) \downarrow SU(2)_J : \langle 2^2\rangle \downarrow (2) \oplus (4). \tag{321}$$

Note that for any n only one new $USp(2j+1)$ irrep ever occurs in any given $U(2j+1)$ irrep.

By the above means, we obtain a complete classification of shell-model states in any j^n configuration. For example, each $USp(2j+1)$ irrep $\langle\kappa\rangle$, listed for $j = 3/2$ in Table III, is shown as a short line in Fig. 4 with equivalent irreps linked by dotted lines. In combination, the states of each set of equivalent $USp(2j+1)$ irreps span an $SO(5) \times USp(2j+1)$ irrep. The seniority v, reduced isospin t, and angular-momentum values are shown for these $USp(2j+1)$ irreps and the extremal state is given explicitly for each $SO(5) \times USp(2j+1)$ irrep. The weight diagrams for the SO(5) irreps defined by the horizontal rows of Fig. 4 are shown in Fig. 5.

3. A simple SO(5) model

The subgroup chain (301) diagonalizes a simple isospin-invariant pairing model with Hamiltonian

$$\hat{H} = \varepsilon\hat{n} - \chi\sum_{\sigma\tau}\mathcal{A}_{\sigma\tau}\mathcal{B}_{\sigma\tau}, \tag{322}$$

where χ is a coupling constant; $\hat{H}$ has eigenvalues (Rowe and Wood, 2010) given in terms of the quantum numbers of the chain (301) by

$$E_{vtnJT} = \varepsilon n - \chi[t(t+1) - T(T+1) + \tfrac{3}{2}(n-v) - \tfrac{1}{4}(n-v)(n+v-4j-2)]. \tag{323}$$

This model may provide an acceptable description of some doubly open-shell nuclei in which seniority and isospin are expected to be approximately conserved. However, its primary value is to give a physical interpretation of the kind of Hamiltonian that is diagonalized by the classification of nuclear shell-model states in the jj-coupling scheme (Flowers, 1952a, 1952b, 1952c; French, 1960) defined by the subgroup chain (300). This coupling scheme is most commonly used in the nuclear shell model because it is the simplest and is one for which fractional parentage coefficients are readily available. Moreover, the SO(5) model gives a direct indication of the kinds of shell-model configurations needed to describe pairing correlations in doubly open-shell nuclei, which is important to know about even when these correlations are not dominant.

As shown for the SU(2) quasispin pairing model in Sec. VIII.C, the above SO(5) model has a multishell extension and corresponding shell-model coupling schemes. Interest in such extensions is stimulated by the recent observation that some such SO(5) models are integrable (Links *et al.*, 2002; Dukelsky *et al.*, 2006).

E. Pairing in *LST* coupling

In *LST* coupling, a many-nucleon wave function is a combination of spatial, spin, and isospin wave functions. There is more than one *LST* coupling scheme, as discussed in the following section. Here we consider the coupling scheme for nucleons occupying a single subshell of fixed orbital angular momentum l.

The primary objective is to construct wave functions that are totally antisymmetric and at the same time have good total angular momentum and isospin. This is achieved by use of

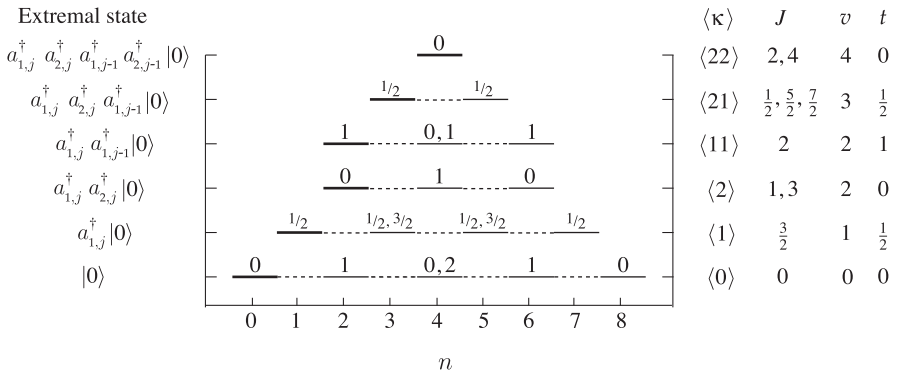


FIG. 4. The spectrum of $USp(2j+1)$ irreps, shown as lines, for $j = 3/2$. Each $USp(2j+1)$ irrep, labeled by $\langle\kappa\rangle$, contains states with the angular-momentum values J as shown on the right side of the figure. These irreps are also labeled by the SO(5) quantum numbers: seniority v, and reduced isospin t. The basis states for each SO(5) irrep are labeled by isospin T (shown above each level), and nucleon number n. Note that all the USp(4) irreps corresponding to lines at the same horizontal level and connected by dotted lines share common values of v and t, and common distributions of J values. They combine to span an $SO(5) \times USp(2j+1)$ irrep with $j = 3/2$. SO(5) lowest-weight states lie in the leftmost USp(4) irreps (shown as thick lines). Thus, it is shown that the many-nucleon states of a $j = 3/2$ shell fall into six distinct $SO(5) \times USp(2j+1)$ irreps.

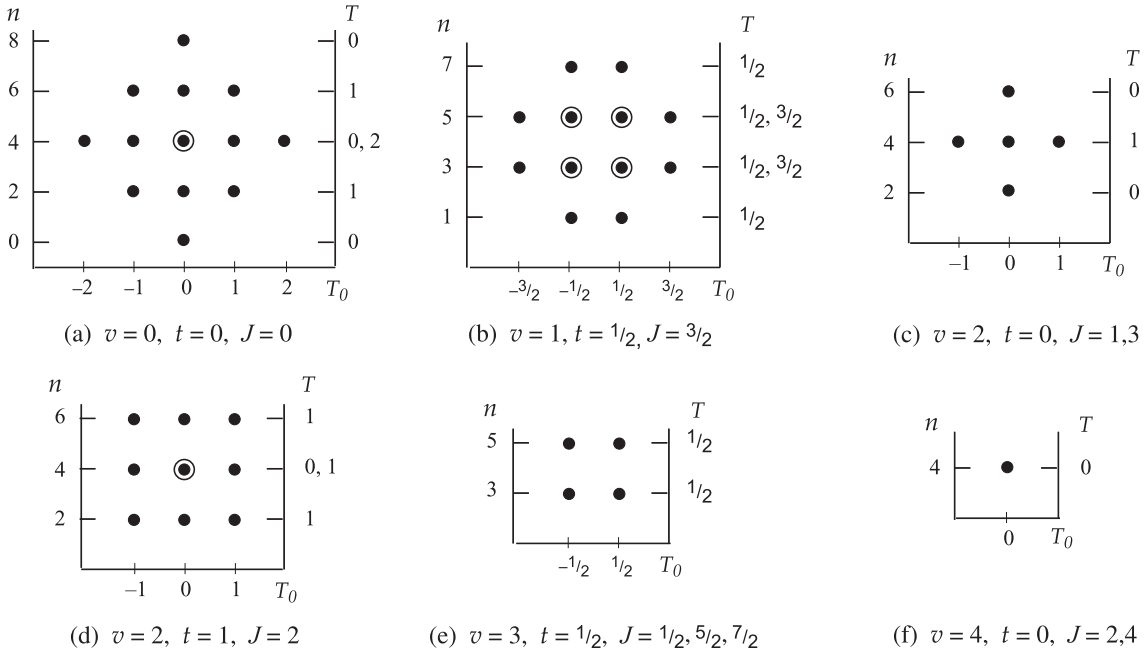


FIG. 5. Weight diagrams for the states of SO(5) ≃ USp(4) irreps whose lowest-weight states have no more than four particles. States are characterized by dots and labeled by their $U(2)_T \supset U(1)$ quantum numbers, i.e., the nucleon number n, isospin T, and component of isospin T_0. Weights with a multiplicity of two are denoted by dots with circles around them. For example, the $n = 4$ states of the $\langle 00 \rangle$ irrep contain two states of $n = 4$ and $T_0 = 0$; one has isospin $T = 0$ and the other $T = 2$.

Wigner's supermultiplet theory in concert with duality relationships as follows.

For nucleons of spin $s = \frac{1}{2}$ and isospin $t = \frac{1}{2}$, so that $(2s+1)(2t+1) = 4$, the space of single-particle wave functions of orbital angular momentum l is of dimension $4(2l+1)$. Thus, whereas in jj coupling the wave functions for n nucleons of angular momentum j transform according to an irrep $\{1^n\}$ of the unitary group $U(2(2j+1))$, the LST-coupled wave functions that we now consider transform according to an irrep $\{1^n\}$ of the unitary group $U(4(2l+1))$. A desirable basis for the corresponding Fock space $\mathbb{F}^{(4(2l+1))}$ is one that reduces the subgroup chain

$$\begin{array}{ccccccccccc} U(4(2l+1)) & \supset & U(2l+1) & \times & U(4) & \supset & O(2l+1) & \times & SU(2)_S & \times & SU(2)_T \\ n & & \lambda & & \tilde{\lambda} & & \kappa & & S & & T \end{array} \supset \begin{array}{c} O(3)_L \\ \pi LM \end{array}, \tag{324}$$

where U(4) is Wigner's supermultiplet group (Wigner, 1937; Hecht and Pang, 1969), i.e., the group of unitary transformations of the spin-isospin states of a nucleon. This subgroup chain was proposed for the definition of an LST coupling scheme by Bayman (1960). The orbital L and spin S angular momenta can now be coupled to good total angular momentum J in the knowledge that the antisymmetric requirement is looked after.

The dynamical content of this coupling scheme can be understood from a consideration of the dual subgroup chain. From the paired subgroups of $O(8(2l+1))$ with dual representations on the Fock space $\mathbb{F}^{(4(2l+1))}$ shown for $w = 4$ in Eq. (258), it is seen that $O(2l+1)$ and an SO(8) group have dual representations and that $U(2l+1)$ and the U(4) supermultiplet group have dual representations. It follows that the basis states of the coupling scheme (324) are identical to basis states for the Fock space $\mathbb{F}^{(4(2l+1))}$ that reduce the subgroup chain

$$\begin{array}{ccccccccc} O(3)_L & \times & SO(8) & \supset & U(4) & \supset & U(1) & \times & SU(2)_S & \times & SU(2)_T \\ \pi LM & & \frac{1}{2}d(\tilde{\kappa}) & & \frac{1}{2}d(\tilde{\lambda}) & & n & & S & & T \end{array}, \tag{325}$$

where $\frac{1}{2}d(\tilde{\kappa})$ and $\frac{1}{2}d(\tilde{\lambda})$, with $d = 2l+1$ label highest weights for SO(8) and U(4) irreps, respectively, as described below.

As mentioned in Sec. II, the discovery that the two subgroup chains (324) and (325) define common basis states followed the formulation of an SO(8) pairing model by Flowers and Szpikowski (1964) who subsequently learned of Bayman's LST coupling scheme.

1. The Lie algebras of SO(8) and $O(2l+1)$ and their irreps

Let $\{a^\dagger_{\sigma m}; \sigma = 1, \ldots, 4; m = -l, \ldots, +l\}$ denote nucleon creation operators, where σ indexes the spin-isospin state of a nucleon and m denotes the projection of its orbital angular momentum l onto the axis of quantization. The group $O(2l+1)$ is then the subgroup of $U(2l+1)$ transformations that leave invariant the $L = 0$ pair-creation operators

$$\hat{\mathcal{A}}_{\sigma\tau} = \sum_m (-1)^{l+m} a^\dagger_{\sigma m} a^\dagger_{\tau,-m} = -\hat{\mathcal{A}}_{\tau\sigma}, \quad \sigma,\tau = 1,\ldots,4. \tag{326}$$

In parallel with all the coupling schemes considered to this point, these $L = 0$ pair-creation operators and the corresponding pair-annihilation operators $\hat{\mathcal{B}}_{\sigma\tau}$, given by their Hermitian adjoints, generate the Lie algebra of a group that commutes with the group O($2l + 1$). They are in fact the raising and lowering operators of an so(8) Lie algebra, the complex extension of which has a basis given by the $L = 0$ angular-momentum-coupled operators

$$\hat{\mathcal{A}}_{\sigma\tau} = \sum_m a^\dagger_{\sigma m} a^\dagger_{\tau\bar{m}}, \tag{327}$$

$$\hat{\mathcal{B}}_{\sigma\tau} = \sum_m a^{\tau\bar{m}} a^{\sigma m}, \tag{328}$$

$$\hat{\mathcal{C}}_{\sigma\tau} = \sum_m a^\dagger_{\sigma m} a^{\tau m} - \frac{1}{2}(2l+1)\delta_{\sigma,\tau}, \tag{329}$$

where

$$a^\dagger_{\sigma\bar{m}} = (-1)^{l+m} a^\dagger_{\sigma,-m}, \quad a^{\sigma\bar{m}} = (-1)^{l+m} a^{\sigma,-m}. \tag{330}$$

The Lie algebra so($2l + 1$) of the group O($2l + 1$) is the subalgebra of u($2l + 1$) elements that commute with the $\hat{\mathcal{A}}_{\sigma\tau}$ operators of Eq. (327) and, hence, with all operators of the so(8) Lie algebra. In parallel with previous examples [see Eq. (277)], the Cartan subalgebra of so($2l + 1$) is spanned by the subset of u($2l + 1$) operators,

$$\hat{C}_{mm} = \sum_\sigma (a^\dagger_{\sigma m} a^{\sigma m} - a^\dagger_{\sigma\bar{m}} a^{\sigma\bar{m}}), \quad m = 1,\ldots,l. \tag{331}$$

If an so($2l + 1$) irrep has highest weight $\kappa = (\kappa_1,\ldots,\kappa_l)$, where κ_m is the eigenvalue of the Cartan operator $\hat{C}_{mm}$ on the highest-weight state for the irrep, the irrep of so($2l + 1$) and of its Lie group SO($2l + 1$) is denoted by $[\kappa]$. In addition to elements of its SO($2l + 1$) subgroup, the group O($2l + 1$) also contains reflections and inversions whose matrices have negative determinants. Thus, if we denote by "det" the one-dimensional irrep of O($2l + 1$), in which an element $g \in$ O($2l + 1$) is simply mapped to det(g), then the irreps of O($2l + 1$) occur in *associated* pairs, $[\kappa]$ and $[\kappa]^*$, that are related by

$$[\kappa]^* = \det \otimes [\kappa]. \tag{332}$$

In the more convenient notation of Littlewood (1950), if an irrep of O(N) is denoted by $[\kappa]$, the associated irrep $[\kappa]^*$ is denoted by the partition $[\kappa']$, whose conjugate $\tilde{\kappa}'$ is defined by

$$\tilde{\kappa}'_1 = N - \tilde{\kappa}_1, \qquad \tilde{\kappa}'_i = \tilde{\kappa}_i, \qquad \text{for } i \neq 1. \tag{333}$$

Note that in replacing $[\kappa]^*$ by $[\kappa']$ in this way, the length $l(\kappa') = \tilde{\kappa}'_1$ of the partition κ' will generally exceed $N/2$, which is the maximum length of an SO(N) highest weight. Thus, in the Littlewood convention, the restriction $\tilde{\kappa}_i \leq N/2$ on an O(N) weight is extended to allow all weights for which

$$\tilde{\kappa}'_1 + \tilde{\kappa}'_2 \leq N. \tag{334}$$

The branching rules for the restriction of the irreps of U(N) to O(N), compatible with the Littlewood convention, have been given, for example, by King (1975).

A dual pair of SO(8) and O($2l + 1$) irreps on the Fock space $\mathbb{F}^{[4(2l+1)]}$, labeled by an O($2l + 1$) highest weight κ, is defined generically by a state

$$|\kappa\rangle = (a^\dagger_{1,l} a^\dagger_{1,l-1} \cdots a^\dagger_{1,l+1-\tilde{\kappa}_1}) \cdots (a^\dagger_{4,l} a^\dagger_{4,l-1} \cdots a^\dagger_{4,l+1-\tilde{\kappa}_4})|0\rangle, \tag{335}$$

where $\tilde{\kappa}$ is the four-row partition conjugate to κ. Such a state is of highest weight for an O($2l + 1$) irrep, denoted in the Littlewood convention by $[\kappa]$, and of lowest weight for an SO(8) irrep of weight given by

$$\langle\kappa|\hat{C}_{\sigma\sigma}|\kappa\rangle = \tilde{\kappa}_\sigma - \tfrac{1}{2}(2l+1). \tag{336}$$

Thus, if we denote the SO(8) irrep with this lowest weight by $[\frac{1}{2}d(\tilde{\kappa})]$ with $d = 2l + 1$, it is seen that each irrep of O($2l + 1$) $\times$ SO(8) on a subspace of $\mathbb{F}^{[4(2l+1)]}$ is defined by a unique partition κ.

2. Tabulation of basis states in *LST* coupling

The quantum numbers of basis states defined by the subgroup chain (325) signify irreps of the corresponding groups in the chain as follows. For a given nucleon number n, the labels λ and $\tilde{\lambda}$ of the U($2l + 1$) $\times$ U(4) irreps are given by the branching rules

$$\mathrm{U}[4(2l+1)] \downarrow \mathrm{U}(2l+1) \times \mathrm{U}(4)\text{: } \{1^n\} \downarrow \bigoplus_{\lambda\vdash n} \{\lambda\} \times \{\tilde{\lambda}\}, \tag{337}$$

with λ and $\tilde{\lambda}$ restricted to partitions having no more than $2l + 1$ and 4 parts, respectively.

To determine the O($2l + 1$) irreps $[\kappa]$ in a given U($2l + 1$) irrep $\{\lambda\}$, one needs the coefficients in the known

$$\mathrm{U}(2l+1) \downarrow \mathrm{O}(2l+1) : \{\lambda\} \downarrow \bigoplus_\kappa F'_{\kappa\lambda}[\kappa] \tag{338}$$

branching rules (King, 1975; Rowe and Wood, 2010). Because of the duality relationships, the same coefficients appear in the

$$\mathrm{SO}(8) \downarrow \mathrm{U}(4) : \langle \tfrac{1}{2}d(\tilde{\kappa})\rangle \downarrow \bigoplus_\lambda F'_{\kappa\lambda}\{\tfrac{1}{2}d(\tilde{\lambda})\} \tag{339}$$

branching rules with $d = 2l + 1$.

The $\mathrm{U}(4) \downarrow \mathrm{SU}(2)_S \times \mathrm{SU}(2)_T$ branching rules are determined sequentially from the known branching rules for

$$\mathrm{U}(4) \downarrow \mathrm{O}(4) \downarrow \mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{340}$$

For example, if an SU(2) $\times$ SU(2) irrep is denoted by (S, T), the O(4) $\downarrow$ SU(2) $\times$ SU(2) reduction is given by

$$[\tilde{\lambda}_1\tilde{\lambda}_2] \downarrow \left(\frac{\tilde{\lambda}_1+\tilde{\lambda}_2}{2}, \frac{\tilde{\lambda}_1-\tilde{\lambda}_2}{2}\right) \oplus \left(\frac{\tilde{\lambda}_1-\tilde{\lambda}_2}{2}, \frac{\tilde{\lambda}_1+\tilde{\lambda}_2}{2}\right) \tag{341}$$

for $\tilde{\lambda}_2 \neq 0$ and

$$[\tilde{\lambda}_1, 0] \downarrow (\tfrac{1}{2}\tilde{\lambda}_1, \tfrac{1}{2}\tilde{\lambda}_1); \tag{342}$$

the reduction for a $[\tilde{\lambda}_1\tilde{\lambda}_2]^*$ irrep is identical, e.g.,

TABLE IV. Spectrum of nucleon states in the $l = 1$ shell classified by nucleon number n, for $n = 0, \ldots, 6$, $U(2l + 1) = U(3)$ symmetry $\{\lambda\}$, orbital angular momentum L, parity $\pi = (-1)^n$, U(4) symmetry $\{\tilde{\lambda}\}$, spin S, and isospin T.

n	$\{\lambda\}$	L	π	$\{\tilde{\lambda}\}$	(S, T)
0	$\{0\}$	0	+	$\{0\}$	(0, 0)
1	$\{1\}$	1	−	$\{1\}$	$(\frac{1}{2}, \frac{1}{2})$
2	$\{2\}$	0, 2	+	$\{1^2\}$	(1, 0), (0, 1)
	$\{1^2\}$	1	+	$\{2\}$	(0, 0), (1, 1)
3	$\{3\}$	1, 3	−	$\{1^3\}$	$(\frac{1}{2}, \frac{1}{2})$
	$\{21\}$	1, 2	−	$\{21\}$	$(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$
	$\{1^3\}$	0	−	$\{3\}$	$(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$
4	$\{4\}$	0, 2, 4	+	$\{1^4\}$	(0, 0)
	$\{31\}$	1, 2, 3	+	$\{21^2\}$	(0, 1), (1, 0), (1, 1)
	$\{2^2\}$	0, 2	+	$\{2^2\}$	(0, 0), (0, 2), (2, 0), (1, 1)
	$\{21^2\}$	1	+	$\{31\}$	(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)
5	$\{41\}$	1, 2, 3, 4	−	$\{21^3\}$	$(\frac{1}{2}, \frac{1}{2})$
	$\{32\}$	1, 2, 3	−	$\{2^21\}$	$(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$
	$\{31^2\}$	0, 2	−	$\{31^2\}$	$(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$
	$\{2^21\}$	1	−	$\{32\}$	$(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{5}{2})$, $(\frac{5}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$
6	$\{42\}$	0, 2^2, 3, 4	+	$\{2^21^2\}$	(1, 0), (0, 1)
	$\{41^2\}$	1, 3	+	$\{31^3\}$	(0, 0), (1, 1)
	$\{33\}$	1, 3	+	$\{2^3\}$	(0, 0), (1, 1)
	$\{321\}$	1, 2	+	$\{321\}$	(0, 1), (1, 0), (0, 2), (2, 0), 2(1, 1), (1, 2), (2, 1)
	$\{2^3\}$	0	+	$\{3^2\}$	(0, 1), (1, 0), (0, 3), (3, 0), (1, 2), (2, 1)

$$U(4) \downarrow O(4) : \{31^2\} \downarrow [3]^* \oplus [21] \oplus [1]^*, \tag{343}$$

and

$$\begin{aligned} O(4) \downarrow SU(2) \times SU(2) &: [3] \downarrow (\tfrac{3}{2}, \tfrac{3}{2}) \\ &: [21] \downarrow (\tfrac{3}{2}, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, \tfrac{3}{2}) \\ &: [1] \downarrow (\tfrac{1}{2}, \tfrac{1}{2}) \end{aligned} \tag{344}$$

give

$$\begin{aligned} &U(4) \downarrow SU(2) \times SU(2) : \\ &\{31^2\} \downarrow (\tfrac{3}{2}, \tfrac{3}{2}) \oplus (\tfrac{3}{2}, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, \tfrac{3}{2}) \oplus (\tfrac{1}{2}, \tfrac{1}{2}). \end{aligned} \tag{345}$$

As an illustration, the classification of states for $l = 1$ by the quantum numbers defined for the subgroup chains (324) and (325) is given in Table IV. From this table, the set of U(4) irreps that comprise each SO(8) irrep is determined by listing all the U(4) irreps $\{\tilde{\lambda}\}$ that occur in combination with a given $O(2l + 1)$ irrep [labeled for $O(2l + 1) = O(3)$ by L^π] as shown in Table V.

3. A simple SO(8) model

The dynamical subgroup chain (325) enables the construction of simply solvable spin- and isospin-invariant pairing models. Consider, for example, the Hamiltonian

$$\hat{H} = \varepsilon \hat{n} - \frac{1}{4}\chi \sum_{\sigma,\tau=1}^{2} \hat{\mathcal{A}}_{\sigma\tau}\hat{\mathcal{B}}_{\sigma\tau}, \tag{346}$$

where $\hat{\mathcal{A}}_{\sigma\tau}$ and $\hat{\mathcal{B}}_{\sigma\tau}$ are, respectively, so(8) raising and lowering operators. $\hat{H}$ could also include terms in the Casimir invariant of the SU(2) subalgebras of U(4). Such a Hamiltonian can be expressed in terms of Casimir operators of SO(8) and those of its subgroups. Its spectrum is then

TABLE V. The partitions $\{\tilde{\lambda}\}$ defining the $U(4) \subset SO(8)$ subrepresentations $\{\frac{3}{2}(\tilde{\lambda})\}$ contained in the $SO(8) \times O(2l + 1)$ irreps for $l = 1$ and even values of n. The $O(2l + 1) = O(3)_L$ irreps are labeled by $[\kappa]$ and equivalently by L^π. SO(8) irreps are labeled by their highest-weight U(4) subirreps, $[\frac{3}{2}(\tilde{\lambda})]_{\text{highest}}$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0	$\{0\}$				
2	$\{1^2\}$	$\{2\}$	$\{1^2\}$		
4	$\{1^4\}$, $\{2^2\}$	$\{21^2\}$, $\{31\}$	$\{1^4\}$, $\{21^2\}$, $\{2^2\}$	$\{21^2\}$	$\{1^4\}$
6	$\{2^21^2\}$, $\{3^2\}$	$\{31^3\}$, $\{2^3\}$, $\{321\}$	$2\{2^21^2\}$, $\{321\}$	$\{2^21^2\}$, $\{31^3\}$, $\{2^3\}$	$\{2^21^2\}$
8	$\{2^4\}$, $\{3^21^2\}$	$\{32^21\}$, $\{3^22\}$	$\{2^4\}$, $\{32^21\}$, $\{3^21^2\}$	$\{32^21\}$	$\{2^4\}$
10	$\{3^22^2\}$	$\{3^31\}$	$\{3^22^2\}$		
12	$\{3^4\}$				
$[\kappa]$	[0]	$[1^2] \equiv [1]^*$	[2]	$[31] \equiv [3]^*$	[4]
L^π	0^+	1^+	2^+	3^+	4^+
$[\frac{3}{2}(\tilde{\lambda})]_{\text{highest}}$	$[\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}]$	$[\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}]$	$[\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}]$	$[\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

immediately determined. As shown by Rowe and Carvalho (2007) [see also Rowe and Wood (2010)], the eigenvalues of the Hamiltonian (346) are given explicitly in terms of their SO(8) and U(4) labels by

$$E_{\kappa\lambda} = \varepsilon n - \frac{1}{4}\chi\sum_{\sigma}\left[\left(\tilde{\kappa}_\sigma - \frac{1}{2}d\right)\left(\tilde{\kappa}_\sigma - \frac{1}{2}d + 2 - 2\sigma\right) - \left(\tilde{\lambda}_\sigma - \frac{1}{2}d\right)\left(\tilde{\lambda}_\sigma - \frac{1}{2}d + 2 - 2\sigma\right)\right]. \quad (347)$$

A primary value of the SO(8) pair-coupling model is to give a physical interpretation of the kind of Hamiltonian that is diagonalized by the classification of nuclear shell-model states in the *LS*-coupling scheme, defined by the subgroup chain (324). In parallel with the SU(2) quasispin and SO(5) pairing models, the SO(8) model has an extension to a multishell pairing model together with corresponding shell-model coupling schemes. Such extensions are important for several reasons: one reason is that some such models are integrable [see Lerma *et al.* (2007)]; another, as we now discuss, is that they raise the possibility of exploring the competition between pairing and deformation correlations in nuclei.

F. The SU(3) and Sp(3, ℝ) *LST*-coupling models

In addition to the $U(2l+1)\times U(4) \supset O(2l+1)$ coupling scheme, two other *LST*-coupling schemes are of special interest. The first is based on the subgroup chain

$$U(N)\times U(4) \supset SU(3)\times O(4) \supset SO(3)\times SU(2)_S \times SU(2)_T, \quad (348)$$

where $N = \sum_l (2l+1)$ is a sum over the l values that occur in a single harmonic-oscillator shell, e.g., $l = 0$ and 2 for the $(2s1d)$ shell and $l = 1$ and 3 for the $(2p1f)$ shell. The second is based on the subgroup chain

$$Sp(3,\mathbb{R})\times U(4) \supset SU(3). \quad (349)$$

These coupling schemes are important for the microscopic description of nuclear collective states. The latter was discussed in Sec. VII.E.3. The former coupling scheme corresponds to Elliott's SU(3) model of rotational states in light nuclei.

It is now apparent that the two coupling schemes, given by Eq. (348) and by

$$U(N)\times U(4) \supset O(N)\times O(4) \supset SO(3)\times SU(2)_S \times SU(2)_T, \quad (350)$$

define shell-model basis states that diagonalize Hamiltonians for nuclei with deformation and pairing correlations, respectively. Thus, they provide useful bases for a study of the competition between these correlations.

A preliminary study of this competition (Rosensteel and Rowe, 2007) determined the spectra of $(2s1d)$-shell nuclei, for which $N = 6$, for a Hamiltonian

$$\hat{H} = -\alpha\hat{\mathfrak{C}}_{su(3)} - (1-\alpha)\hat{\mathfrak{C}}_{so(6)} + \beta\hat{\mathfrak{C}}_{so(3)}, \quad (351)$$

where $\hat{\mathfrak{C}}_{su(3)}$, $\hat{\mathfrak{C}}_{so(6)}$, and $\hat{\mathfrak{C}}_{so(3)}$ are Casimir operators for the respective subalgebras of u(6) and $0 \le \alpha \le 1$. A remarkable result was found; for $\alpha \lesssim 0.4$ the spectrum of low-energy states and their properties were characteristic of an O(6) phase and for $\alpha \gtrsim 0.6$ they became characteristic of an SU(3) phase. Such behavior has been seen in numerous similar studies and has been termed quasidynamical symmetry (see Sec. IX.C).

IX. OTHER DEVELOPMENTS IN THE APPLICATION OF SYMMETRY METHODS IN PHYSICS

From among the many developments in symmetry methods of importance in physics, we mention a few of particular relevance to the topic of this review.

A. Boson mappings

In low-density situations, systems of even fermion number often behave like bosons. For example, alpha particles are meaningfully approximated as bosons in the interpretation of superfluidity and Bose-Enstein condensation (Griffin, Stoke, and Stringari, 1995; Griffin, Nikuni, and Zaremba, 2009). Such *quasiboson approximations* to algebraic models can be obtained by group contraction methods (İnönü and Wigner, 1953) They are widely used in many-body theory (Sawada, 1957); early approaches in nuclear physics were described, for example, by Rowe (1970) and Ring and Schuck (1980).

Thus, it is natural to seek corrections to these approximations in terms of exact boson mappings as given by the Holstein and Primakoff (1940) representation of su(2)

$$\hat{J}_0 = -j + c^\dagger c, \qquad \hat{J}_+ = c^\dagger\sqrt{(2j - c^\dagger c)}, \qquad \hat{J}_- = \sqrt{(2j - c^\dagger c)}c, \quad (352)$$

for an arbitrary spin j, where c and $c^\dagger$ satisfy the boson commutation relations $[c, c^\dagger] = 1$. A comprehensive review of the many approaches to boson realizations of Lie algebras, initiated by Belyaev and Zelevinsky (1962), was given by Klein and Marshalek (1991).

Boson realizations are synonymous with coherent-state representations. This is apparent from the *Bargmann representation* of the Heisenberg-Weyl algebra (Bargmann, 1961) in which boson operators are represented in terms of a complex variable z by

$$c^\dagger \mapsto z, \qquad c \mapsto d/dz. \quad (353)$$

Coherent-state representations are defined for many Lie algebras. For example, according to Perelomov (1972, 1986), SU(2) coherent states are defined by $|z\rangle = \exp(z^*\hat{J}_+)|j,-j\rangle$, where z is a complex variable, and a coherent-state wave function for a state $|\psi\rangle \in \mathbb{H}$ is defined as the overlap function

$$\psi(z) = \langle z|\psi\rangle = \langle j,-j|e^{z\hat{J}_-}|\psi\rangle. \quad (354)$$

A component of SU(2) angular momentum J_k then has coherent-state representation defined by

$$\Gamma(J_k)\psi(z) = \langle j,-j|e^{z\hat{J}_-}\hat{J}_k|\psi\rangle = \langle j,-j|(e^{z\hat{J}_-}\hat{J}_k e^{-z\hat{J}_-})e^{z\hat{J}_-}|\psi\rangle. \quad (355)$$

Thus, an expansion of $e^{z\hat{J}_-}\hat{J}_k e^{-z\hat{J}_-}$ in terms of $\hat{J}_0$ and $\hat{J}_\pm$, and the identities

$$\langle j,-j|\hat{J}_0 e^{z\hat{J}_-}|\psi\rangle = -j\psi(z), \tag{356}$$

$$\langle j,-j|\hat{J}_- e^{z\hat{J}_-}|\psi\rangle = \frac{\partial}{\partial z}\psi(z), \tag{357}$$

$$\langle j,-j|\hat{J}_+ e^{z\hat{J}_-}|\psi\rangle = 0, \tag{358}$$

leads to the su(2) representation

$$\Gamma(J_0) = -j + zd/dz, \qquad \Gamma(J_-) = d/dz, \qquad \Gamma(J_+) = z(2j - d/dz). \tag{359}$$

This representation is now transformed into the Bargmann form of the Holstein-Primakoff representation

$$\Gamma(J_0) = -j + zd/dz, \qquad \Gamma(J_-) = \sqrt{(2j - d/dz)}d/dz, \qquad \Gamma(J_+) = z\sqrt{(2j - d/dz)}, \tag{360}$$

by a similarity transformation.

More generally, if an irrep of a Lie algebra on a Hilbert space $\mathbb{H}$ has a lowest weight that is uniquely defined for the vacuum of a commuting set of lowering operators $\{\hat{B}_k\}$ and the eigenvalues of a set of weight operator $\{\hat{C}_i\}$, i.e., by the formulas

$$\hat{B}_k|0\rangle = 0, \qquad \hat{C}_i|0\rangle = \lambda_i|0\rangle, \tag{361}$$

then coherent-state wave functions are defined in terms of a set $z = \{z_k\}$ of complex variables, for each state $|\psi\rangle \in \mathbb{H}$, by

$$\psi(z) = \langle 0|\exp\Big(\sum_k z_k \hat{B}_k\Big)|\psi\rangle. \tag{362}$$

Corresponding coherent-state representations of the Lie algebra are defined as illustrated above for su(2). Such coherent-state representations were used to determine exact boson mappings for a large variety of semisimple Lie algebras by Dobaczewski (1981a, 1981b, 1982).

The standard theory of coherent states and coherent-state representations has been reviewed by Klauder and Skagerstam (1985) and by Perelomov (1986).

B. More general scalar and vector-coherent-state representations

Coherent-state methods are extraordinarily powerful. In their most general forms (Rowe and Repka, 1991), they provide simple and versatile constructions of Lie group and Lie algebra representations, which include the methods of *induced representations* of Mackey (1968). They also provide an interface between classical and quantum mechanics (Bartlett, Rowe, and Repka, 2002a, 2002b; Bartlett and Rowe, 2003; Gazeau, 2009; Rowe, 2012.

However, the standard construction of coherent-state representations is limited to representations with lowest- and/or highest-weight states that are uniquely defined by sets of commuting raising operators.

Two extensions overcame this limitation. The first extension is from scalar to vector-coherent-state (VCS) irreps. Let $\{\hat{B}_k\}$ be a set of commuting lowering operators for an irrep of a semisimple Lie algebra $\mathfrak{g}$ on Hilbert space $\mathbb{H}$ and let $\{|\nu\rangle\}$ be an orthonormal basis for the subspace $\mathbb{H}_0 \subset \mathbb{H}$ of states that are annihilated by the lowering operators. Subject to certain conditions, a vector-valued coherent-state wave function,

$$\Psi(z) = \sum_\nu |\nu\rangle\langle\nu| \exp\Big(\sum_\nu z_k \hat{B}_k\Big)|\psi\rangle, \tag{363}$$

can then be defined for a state $|\psi\rangle \in \mathbb{H}$ and a corresponding construction of a coherent-state representation of an element $X \in \mathfrak{g}$ is defined by

$$\Gamma(X)\Psi(z) = \sum_\nu |\nu\rangle\langle\nu| \exp\Big(\sum_\nu z_k \hat{B}_k\Big)\hat{X}|\psi\rangle. \tag{364}$$

The construction is useful if the subset of commuting lowering operators is such that the subspace $\mathbb{H}_0$ carries a finite-dimensional unitary irrep of a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.

Such VCS irreps were introduced (Rowe, 1984; Rowe, Rosensteel, and Carr, 1984; Rowe, Rosensteel, and Gilmore, 1985) for the purpose of calculating matrix elements of the noncompact symplectic algebra sp(3, $\mathbb{R}$) in an SU(3) $\supset$ SO(3) basis as needed in the nuclear symplectic model. A *partial coherent-state theory*, which went some way toward solving this problem, was also proposed for this purpose (Deenen and Quesne, 1984, 1985). The VCS construction was then applied to calculate the explicit matrices for the irreps of numerous Lie algebras and even some superalgebras, as reviewed by Hecht (1987). It was also shown (Rowe and Repka, 1991) that VCS irreps are induced representations (Mackey, 1968) in which an irrep of $\mathfrak{g}$ is induced from an irrep of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

A second extension makes use of other kinds of coherent states besides those generated by exponentiating lowering (or raising) operators. It was introduced because a standard coherent-state irrep of su(3) enables its matrix elements to be computed in an SU(2) basis whereas, in applications with a rotationally invariant Hamiltonian, one needs a basis that reduces the SO(3) $\subset$ SU(3) subgroup. However, as shown by Elliott (1958a, 1958b), the rotated states $\{|\Omega\rangle = \hat{R}(\Omega)|\lambda\mu\rangle, \Omega \in \mathrm{SO}(3)\}$, where $|\lambda\mu\rangle$ is a highest-weight state for a generic su(3) irrep, span the Hilbert space for that irrep. Moreover, they are generalized coherent states, as defined by Perelomov (1986). Thus, an arbitrary $|\psi\rangle \in \mathbb{H}$ has a scalar coherent-state wave function defined by the overlap function

$$\psi(\Omega) = \langle\lambda\mu|\hat{R}(\Omega)|\psi\rangle. \tag{365}$$

The corresponding coherent-state irrep of the su(3) Lie algebra is then defined as usual by

$$\Gamma(X)\psi(\Omega) = \langle\lambda\mu|\hat{R}(\Omega)\hat{X}|\psi\rangle. \tag{366}$$

It is interesting to note that, whereas a standard coherent-state representation gives a boson realization of a $(\lambda, 0)$ irrep of SU(3) in an SU(2) basis, the new construction leads to a rotor realization of a generic $(\lambda\mu)$ irrep in an SO(3) basis. Thus quasiboson and quasirotor approximations are obtained, respectively, in large λ and/or μ contraction limits. The latter extension can be applied with other groups besides SO(3) and also within the framework of the VCS extension so that there is now a wealth of possibilities for handling a large variety of situations.

C. Quasidynamical symmetry

Although dynamical symmetry is of immense significance, as illustrated by its many applications mentioned in this review, it is in fact an idealization that is only achieved to some level of approximation in realistic situtations. What is remarkable is the extent to which models based on assumed dynamical symmetries are successful. Thus, as suggested by Hess *et al.* (2002), they should really be regarded as effective dynamical symmetries. Quasidynamical symmetry was introduced (Rochford and Rowe, 1988) as a mechanism for understanding the nature of these effective symmetries.

Quasidynamical symmetry is an approximate realization of the precise mathematical concept of an *embedded representation* (Rowe, Rochford, and Repka, 1988) loosely defined as follows. Let $\mathbb{H}$ be a Hilbert space for a representation T that is a direct sum of irreps of a Lie algebra $\mathfrak{g}$ and let $\mathbb{H}_0 \subset \mathbb{H}$ be a subspace. If the matrix elements $\langle\psi|\hat{X}|\psi'\rangle$ of all $\hat{X} = T(X)$, with $X \in \mathfrak{g}$, and all $|\psi\rangle$ and $|\psi'\rangle$ in $\mathbb{H}_0$ should happen to be equal to those of a representation of $\mathfrak{g}$, we say that this representation is an embedded representation of $\mathfrak{g}$.

Subrepresentations and linear combinations of equivalent representations are trivial examples of embedded representations. However, there are nontrivial examples for Lie algebras with irreps that are scale related such as rotor-model algebras. Consider, for example, a set of irreps of some Lie algebra $\mathfrak{g}$ labeled by $\{\lambda\}$ with basis states $\{|\lambda LM\rangle\}$, where L and M are SO(3) angular-momentum quantum numbers and suppose that elements $\{X_i\}$ of $\mathfrak{g}$ have matrix elements in this basis that scale in a manner given by the equation

$$\langle\lambda LM|\hat{X}_i|\lambda' L' M'\rangle = \delta_{\lambda,\lambda'} f^{\lambda}_{i\lambda_0} \langle\lambda_0 LM|\hat{X}_i|\lambda_0 L' M'\rangle, \tag{367}$$

where $f^{\lambda}_{i\lambda_0}$ is a real proportionality constant. Matrix elements between states given for each LM by

$$|LM\rangle = \sum_{\lambda} C_\lambda |\lambda LM\rangle, \tag{368}$$

where C_λ is an LM-independent set of coefficients, are then equal to those of an average irrep $\bar{\lambda}$ for which

$$f^{\bar{\lambda}}_{i\lambda_0} = \sum_{\lambda} |C_\lambda|^2 f^{\lambda}_{i\lambda_0}. \tag{369}$$

Only a limited number of Lie algebras have irreps that scale precisely in this way. However, most algebras of importance in physics have contraction limits with this property and so admit embedded representations approximately. Thus, quasidynamical symmetries as approximate embedded representations are common and particularly important for the interpretation of symmetry-related phases of physical systems and the transitions between them. Several examples were given by Rowe (2004b). Other applications and perspectives have been given by Hess *et al.* (2002), Yépez-Martínez, Cseh, and Hess (2006), Macek, Dobeš, and Cejnar (2009), and Bonatsos, McCutchan, and Casten (2010). A review of quantum phase transitions and the use of quasidynamical symmetry in understanding them has been given by Cejnar, Jolie, and Casten (2010).

D. Partial dynamical symmetry

In realistic situations, approximate dynamical symmetries may only be acceptable for a limited number of states of a system. In fact, mixed symmetry studies have shown that one symmetry may be dominant at low energies (in a quasidynamical symmetry sense) and another at higher energies (Caprio, Cejnar, and Iachello, 2008). Thus, a theory of partial dynamical symmetry (which also applies to quasidynamical symmetry) was introduced by Alhassid and Leviatan (1992). The occurrence of partial symmetry conservation in nuclear models has subsequently been considered by several authors and interesting examples have been discovered, e.g., by Zamick and colleagues (Escuderos and Zamick, 2006; Zamick and Isacker, 2008). Such examples and the development of efficient models of partial dynamical symmetry have been reviewed recently by Leviatan (2011).

X. DISCUSSION AND SUMMARY

Examples have been given in this review of many results, of importance in physics and mathematics, that follow from the duality of various group representations. Examples have been drawn primarily from applications in nuclear and atomic spectroscopy. However, many more applications of this extraordinarily powerful concept are known and undoubtedly more remain to be discovered.

We focused primarily on the subgroup chains of symmetry groups for sequences of Hamiltonians of increasing complexity which define coupling schemes for many-particle systems. Subgroup chains define basis states for Hilbert spaces that diagonalize Hamiltonians such as those given by combinations of the Casimir and other invariants of the groups in the chain. Moreover, they provide basis states for the description of more general Hamiltonians of interest. Thus, the study of such subgroup chains of potential symmetry groups for a system is an important step in understanding the range of possible dynamics that the system can exhibit. This approach was emphasized in the many studies of the interacting boson model (Iachello and Arima, 1987), where such chains of groups are said to define the dynamical symmetries of a model.

A primary motivation for studying dual subgroup chains is that they reveal associations of many phenomenological models of nuclear physics with shell-model coupling schemes. Thus, we showed that for many of the groups in a subgroup chain that define a coupling scheme, there are frequently other groups with dual representations on the same or an enlarged Hilbert space of the system. It then follows, as explained and illustrated in this review, that if one group is a dynamical group for a class of Hamiltonians its dual (if it has one) is a symmetry group for the same class of Hamiltonians. In this situation, the pair of groups with dual representations is of considerably greater value than either group separately. An even more useful situation arises when each of the several groups of a subgroup chain that defines a coupling scheme is partnered with a dual group belonging to another so-called *dual* subgroup chain. For then, if one subgroup chain defines a chain of symmetry groups for a sequence of Hamiltonians of decreasing symmetry, the dual

subgroup chain consists of dynamical groups for the same sequence of Hamiltonians, albeit in reversed order (i.e., in order of increasing dynamical symmetry). When such dual chains exist, they augment the tools available for studying the dynamical content of a system considerably. For example, it means that combinations of the elements of the Lie algebras of both a group and its dual will leave the irreps of both of the groups invariant. This property was used (Rosensteel and Rowe, 2003), for example, to identify subsets of two-body interactions that conserve seniority.

The relationship between algebraic models and shell-model coupling schemes is shown to be invaluable for embedding models, such as pairing models and collective models, into a more fundamental microscopic theory. The examples given were chosen to highlight the application of dual pairs of group and subgroup chain representations in the construction of simply solvable algebraic models and for providing useful basis states and coupling schemes for a general theory. In addition to providing a microscopic interpretation of successful phenomenological models, they also provide the means to identify the appropriate shell-model coupling scheme for a microscopic description of phenomena that have a simple model explanation.

We have given many examples of the use of duality relationships to infer the properties of one group from those of another. For example, Schur-Weyl duality relates the characters of unitary group irreps to irreps of symmetric groups. This relationship yields the extraordinarily valuable result that the characters of different $\mathrm{U}(n)$ groups are given by a common set of Schur functions which are defined by their S_N symmetries and take the same form, for each $\lambda \vdash N$, independent of the number n of variables. Parallel relationships exist between the characters of other dual pairs of group representations and can be used, for example, to determine many branching rules from simpler known rules. The branching rules for the representations of classical Lie groups were reviewed by King (1975), and many of them are related by duality relationships. For example, recall in Sec. VII.E that the duality of the groups in the chain $\mathrm{Sp}(N,\mathbb{R}) \supset \mathrm{U}(N)$ with those in the chain $\mathrm{O}(m) \subset \mathrm{U}(m)$ on the space of an Nm-dimensional harmonic oscillator enables one to calculate the branching rules for the restriction $\mathrm{Sp}(N,\mathbb{R}) \downarrow \mathrm{U}(N)$ from the known $\mathrm{U}(m) \downarrow \mathrm{O}(m)$ branching rules (Rowe, Wybourne, and Butler, 1985). This approach was extended by King and Wybourne (1985) to include branching rules for $\mathrm{U}(p,q)$ and $\mathrm{SO}^*(2n)$ to their respective $\mathrm{U}(p) \times \mathrm{U}(q)$ and $\mathrm{U}(n)$ subgroups. Such branching rules, which give the restrictions of the characters of a group to a unitary subgroup, can be more useful than explicit expressions for the characters of a noncompact group, such as $\mathrm{Sp}(N,\mathbb{R})$, whose unitary irreps are infinite dimensional. This is because one knows far more about the characters of the unitary and symmetric groups than about those of other groups. A review of many branching rules for dual reductive group and subgroup pairs was given by Howe, Tan, and Willenbring (2004) in terms of the Littlewood-Richardson coefficients for the tensor products of symmetric and unitary group characters (cf. Sec. IV.A). Examples are given in this review of how the tensor products of various groups of importance in many-particle spectroscopy can be derived as sums of irreps by methods that result from duality relationships.

It is known that the Clebsch-Gordan coefficients and more general Wigner-Racah algebras for one subgroup chain are related to those for a dual subgroup chain (Hecht, Le Blanc, and Rowe, 1987; Le Blanc, 1987; Le Blanc and Hecht, 1987). A prototype of such a relationship was given by Rowe (2004c) and Rowe and Thiamova (2005) and discussed in Sec. VII.C.2. The example showed that the transformation of basis states between a $\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{O}(5)$ coupling scheme for a six-dimensional harmonic oscillator and a $\mathrm{U}(6) \supset \mathrm{O}(6) \supset \mathrm{O}(5)$ coupling scheme is given simply by SU(1,1) Clebsch-Gordan coefficients which make the dual transformations from $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1) \supset \mathrm{SU}(1,1) \supset \mathrm{U}(1)$ coupled states to $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1) \supset \mathrm{U}(1) \times \mathrm{U}(1) \supset \mathrm{U}(1)$ coupled states. As observed, in an analysis of the phase transition from an O(6) to a U(5) coupling scheme with a change of a parameter in the interacting boson model (Rowe, 2004c), the availability of such transformation coefficients enables matrix elements that are diagonal in one coupling scheme to be expressed simply in another. The possibility of such relationships has the potential for relating the various coupling schemes in shell-model calculations. An example, given in Sec. VIII.C, is an extension of the single-shell $\mathrm{U}(2j+1) \supset \mathrm{USp}(2j+1)$ coupling scheme, relevant for Hamiltonians with strong pairing interactions, to multishell coupling schemes. Such extensions are similarly made for other shell-model coupling schemes.

It is mentioned, although it is not discussed in this review, that the Casimir operators of dual pairs of Lie groups are also simply related. This is to be expected because all multilinear combinations of the elements of a Lie algebra are invariants of a dual Lie algebra. In particular, the Casimir invariant of one Lie algebra is an invariant of the dual Lie algebra. Moreover, it was seen that the highest weights for dual irreps of a pair of semisimple or reductive Lie algebras are related. Thus, the eigenvalues of Casimir and other invariants for dual irreps of such groups, which are expressible in terms of related highest weights, are likewise related. For example, the Casimir invariant of $\mathrm{u}(n)$,

$$\hat{\mathfrak{C}}(u(n)) = \sum_{ij} \hat{C}_{ij}\hat{C}_{ji} = \sum_{i\le j}(2\hat{C}_{ji}\hat{C}_{ij} + \hat{C}_{ii} - \hat{C}_{jj}) + \sum_i \hat{C}_{ii}\hat{C}_{ii}, \quad (370)$$

is determined to have eigenvalue

$$\mathfrak{C}^{(n)}_{\{\lambda\}} = \sum_{i=1}^{n} \lambda_i(\lambda_i + n + 1 - 2i) \quad (371)$$

for an irrep with highest weight λ. Thus, a dual pair of $\mathrm{U}(n)$ and $\mathrm{U}(m)$ irreps with common highest weights, having the property that $l(\lambda)$ does not exceed either m or n, have Casimir invariants with distinct but closely related values. The Casimir invariants of semisimple Lie algebra are described and their values given in terms of highest weights by Wybourne (1974).

In spite of the simple origin of the relationships between Casimir invariants, the results can nevertheless be useful. For example, a simply solvable model with a Hamiltonian

expressed in terms of the Casimir invariants of chains of subgroups can also be expressed in terms of the Casimir invariants of a dual subgroup chain. This can be useful for the same reason that embedding a simple model in a much richer microscopic theory makes it possible to explore many more properties of the system being modeled.

Also not mentioned in this review is the duality between two copies of a single compact group G acting on $\mathcal{L}^2(G)$ by the left and right regular representations. The assertion of this duality is essentially the Peter-Weyl theorem. An explicit example is given by the regular representation of the rotation group SO(3). The Hilbert space $\mathcal{L}^2[\mathrm{SO}(3)]$ for this representation is spanned by the so-called Wigner $\mathcal{D}$ functions $\{\mathcal{D}^L_{KM}; L = 0, 1, 2, \ldots; K = -L, \ldots, +L; M = -L, \ldots, +L\}$, defined in terms of a basis $\{|LM\rangle; M = -L, \ldots, +L\}$ for the irrep of angular momentum L by

$$\hat{T}^{(L)}(\Omega)|LM\rangle = \sum_K |LK\rangle \mathcal{D}^L_{KM}(\Omega), \quad \Omega \in \mathrm{SO}(3). \tag{372}$$

The left and right regular representations of SO(3) are then defined, respectively, for $\omega \in \mathrm{SO}(3)$ by

$$\begin{aligned}\hat{L}_{\mathrm{Reg}}(\omega)\mathcal{D}^L_{KM}(\Omega) &= \mathcal{D}^L_{KM}(\omega^{-1}\Omega) \\ &= \sum_N \mathcal{D}^L_{NM}(\Omega)[\mathcal{D}^L_{NK}(\omega)]^*, \end{aligned} \tag{373}$$

$$\begin{aligned}\hat{R}_{\mathrm{Reg}}(\omega)\mathcal{D}^L_{KM}(\Omega) &= \mathcal{D}^L_{KM}(\Omega\omega) \\ &= \sum_N \mathcal{D}^L_{KN}(\Omega)\mathcal{D}^L_{NM}(\omega), \end{aligned} \tag{374}$$

where $[\mathcal{D}^L_{NK}(\omega)]^*$ is the complex conjugate of $\mathcal{D}^L_{NK}(\omega)$. It is seen that the Wigner $\mathcal{D}^L$ functions for any given value of L form a basis for an irrep $\mathcal{D}^{L*} \times \mathcal{D}^L$ of the direct product group $\mathrm{SO}(3) \times \mathrm{SO}(3)$ relative to the left and right actions, respectively, of these SO(3) groups. Thus, the two copies of SO(3) have dual representations on the Hilbert space of the SO(3) regular representation. This example has important applications for the representations of the rotor model in nuclear physics, in which the right representations of SO(3) correspond to rotations relative to a space-fixed reference frame and the left representations correspond to intrinsic rotations relative to a frame of reference fixed in the body of the rotor.

An aspect of dual group representations that merits further investigation is the related geometry of the systems to which they apply. It is known, for example, that the dynamics of a central-force Hamiltonian for a system with a Euclidean configuration space $\mathbb{R}^n$ can be regarded as a combination of rotations and radial motions. For such a system the symmetry group $\mathrm{O}(n)$ of rotations and inversions and the dynamical group SU(1,1) associated with the radial dynamics are determined to have dual representations on the Hilbert space $\mathcal{L}^2(\mathbb{R}^n)$ of the system. This dual pair of groups reflects the underlying geometrical structure of the Euclidean space as a product manifold of a radial line and a unit $(n-1)$ sphere. Thus, if $\{x_i, i = 1, \ldots, n\}$ is a set of Cartesian coordinates, the group $\mathrm{O}(n)$ is the set of all linear transformation of $\mathbb{R}^n$ that leave the squared radius of a point $r^2 = \sum_i x_i^2$ invariant. Moreover, the set of points generated by all $\mathrm{O}(n)$ transformations of a point in $\mathbb{R}^n$ at distance $r = 1$ from the origin is an $(n-1)$-dimensional unit sphere. Together, the radial coordinate r and a set of coordinates for the unit sphere define spherical polar coordinates for the points of $\mathbb{R}^n$.

A generalization of this geometric structure is observed for the collective dynamics of an N-particle system with a Euclidean configuration space $\mathbb{R}^{Nm}$ (Gelbart, 1973). For such a system, the symmetry group of the Hamiltonian is the group $\mathrm{O}(N)$ and the dynamical group is $\mathrm{Sp}(m, \mathbb{R})$. These groups have been shown to have dual representations on the Hilbert space $\mathcal{L}^2(\mathbb{R}^{Nm})$. If $\{x_{ni}; n = 1, \ldots, N, i = 1, \ldots, m\}$ is a set of Cartesian coordinates for $\mathbb{R}^{Nm}$, then $\mathrm{O}(N)$ is seen as the set of all linear transformations of $\mathbb{R}^{Nm}$ that leave the quadrupole moments $Q_{ij} = \sum_{n=1}^N x_{ni}x_{nj}$ invariant. The set of points generated by all $\mathrm{O}(N)$ transformations of a point in $\mathbb{R}^{Nm}$ of unit quadrupole moment is then the generalization of a unit sphere in $\mathbb{R}^N$ to a so-called Stieffel manifold

$$S^{N,m} = \left\{x \in \mathbb{R}^{Nm}; \sum_n x_{ni}x_{nj} = \delta_{i,j}\right\}. \tag{375}$$

Thus, a set of quadrupole moments (elements of $m \times m$ symmetric matrices) and coordinates for the Stieffel manifold provide a system of collective and intrinsic coordinates for $\mathbb{R}^{Nm}$. The geometrical structure underlying these coordinates introduced by Gelbart (1973) in a study of the representations of $\mathrm{Sp}(m, \mathbb{R})$ on $\mathcal{L}^2(\mathbb{R}^{Nm})$ proves to be of considerable significance for the development of the nuclear collective model (Rosensteel and Rowe, 1977, 1980; Rowe, 1985; Rowe and Repka, 1998). Moreover, it transpires that the intertwining of the representations of the symplectic and orthogonal groups on these spaces accounts for the centrifugal coupling of the dynamics on these two spaces. In particular, it leads to an understanding of the role of vorticity degrees of freedom.

The examples chosen in this review to illustrate the range of results that can be obtained from a consideration of dual group representations are primarily from nuclear and atomic physics. This is the area of physics most familiar to us and the one in which many of the known duality relationships have been discovered. However, they are far from complete. Indeed, we are optimistic that applications in other fields of physics will be brought to light by others. There are certainly potential applications in quantum optics. The fascinating concept of dual models in statistical mechanics (Girvin, 1996) also suggests interesting possibilities. While different systems with common algebraic structures can be expected to exhibit parallel properties, the concept of dual representations of different groups indicates a similarly close parallel between systems with dual algebraic structures. We are also aware that many other duality relations are known for which there are undoubtedly applications; cf., for example, Kashiwara and Vergne (1978), Gelbart (1979), Adams (1983), Howe (1985), and Leung and Ton-That (1994). Duality relationships are also known to exist for quantum groups and supersymmetric groups. For example, Lu and Howe (2010) recently explored an application of a duality relationship between O(3,1) and the orthosymplectic superalgebra osp(2,2) to Maxwell's equations.

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REFERENCES

Adams, J. D., 1983, Invent. Math. **74**, 449.
Alhassid, Y., and A. Leviatan, 1992, J. Phys. A **25**, L1265.
Alvarez, R. N., D. Bonatsos, and Y. F. Smirnov, 1994, Phys. Rev. A **50**, 1088.
Anderson, P. W., 1958, Phys. Rev. **112**, 1900.
Bardeen, J., L. N. Cooper, and J. R. Schrieffer, 1957, Phys. Rev. **108**, 1175.
Bargmann, V., 1961, Commun. Pure Appl. Math. **14**, 187.
Bartlett, S. D., and D. J. Rowe, 2003, J. Phys. A **36**, 1683.
Bartlett, S. D., D. J. Rowe, and J. Repka, 2002a, J. Phys. A **35**, 5599.
Bartlett, S. D., D. J. Rowe, and J. Repka, 2002b, J. Phys. A **35**, 5625.
Bayman, B. F., 1960, *Some Lectures on Groups and Their Applications to Spectroscopy* (Nordisk Institut for Teoretisk Atomfysik and NORDITA, Copenhagen).
Belyaev, S. T., and V. G. Zelevinsky, 1962, Nucl. Phys. **39**, 582.
Benkart, G., 1996, *Lie Algebras and Their Representation* (American Mathematical Society, Providence, RI), p. 1.
Bernshtein, I. N., I. M. Gel'fand, and S. I. Gel'fand, 1975, Funct. Anal. Appl. **9**, 322.
Black, G. R. E., R. C. King, and B. G. Wybourne, 1983, J. Phys. A **16**, 1555.
Black, G. R. E., and B. G. Wybourne, 1983, J. Phys. A **16**, 2405.
Bohm, A., Y. Ne'eman, and A. O. Barut, 1988, Eds., *Dynamical Groups and Spectrum Generating Algebras. Volumes 1 and 2* (World Scientific, Singapore).
Bonatsos, D., E. A. McCutchan, and R. F. Casten, 2010, Phys. Rev. Lett. **104**, 022502.
Born, M., W. Heisenberg, and P. Jordan, 1926, Z. Phys. A **35**, 557 [http://web.ihep.su/dbserv/compas/src/born26/ eng.pdf].
Borodin, A. M., and N. A. Rozhkovskaya, 1995, Erwin Schrödinger International Institute for Mathematical Physics, ESI preprint No. 246.
Brauer, R., 1937, Ann. Math. **38**, 857.
Butler, P. H., and R. C. King, 1973, J. Math. Phys. (N.Y.) **14**, 1176.
Cambiaggio, M. C., A. M. F. Rivas, and M. Saraceno, 1997, Nucl. Phys. **A624**, 157.
Caprio, M. A., P. Cejnar, and F. Iachello, 2008, Ann. Phys. (N.Y.) **323**, 1106.
Carvalho, M. J., 1990, J. Phys. A **23**, 1909.
Carvalho, M. J., and S. D'Agostino, 2001a, Comput. Phys. Commun. **141**, 282.
Carvalho, M. J., and S. D'Agostino, 2001b, J. Phys. A **34**, 1375.
Castaños, O., E. Chacòn, A. Frank, and M. Moshinsky, 1979, J. Math. Phys. (N.Y.) **20**, 35.
Cejnar, P., J. Jolie, and R. F. Casten, 2010, Rev. Mod. Phys. **82**, 2155.
Chacón, E., 1969, "The Orthogonal Group in n Dimensions and Nuclear Structure," (in Spanish), Ph.D. thesis, University of Mexico.
Cheng, S.-J., and W. Wang, 2000, Lett. Math. Phys. **52**, 143.
Cheng, S.-J., and W. Wang, 2001, Compos. Math. **128**, 55.
Cheng, S.-J., and R. B. Zhang, 2004, Adv. Math. **182**, 124.
Čížek, J., and J. Paldus, 1977, Int. J. Quantum Chem. **12**, 875.
Cooke, T. H., and J. L. Wood, 2002, Am. J. Phys. **70**, 945.
D'Agostino, S., 2005, "Group Duality and its Applications in Nuclear Physics," Ph.D. thesis, University of Toronto.
Dean, D. J., and M. Hjorth-Jensen, 2003, Rev. Mod. Phys. **75**, 607.
Deenen, J., and C. Quesne, 1984, J. Math. Phys. (N.Y.) **25**, 2354.
Deenen, J., and C. Quesne, 1985, J. Math. Phys. (N.Y.) **26**, 2705.
Dehuai, L., and B. G. Wybourne, 1981, J. Phys. A **14**, 327.
D'Hoker, E., 1984, J. Math. Phys. (N.Y.) **25**, 1.
Dirac, P. A. M., 1928a, Proc. R. Soc. A **117**, 610.
Dirac, P. A. M., 1928b, Proc. R. Soc. A **118**, 351.
Dobaczewski, J., 1981a, Nucl. Phys. **A369**, 213.
Dobaczewski, J., 1981b, Nucl. Phys. **A369**, 237.
Dobaczewski, J., 1982, Nucl. Phys. **A380**, 1.
Dukelsky, J., V. G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea, and H. S. Lerma, 2006, Phys. Rev. Lett. **96**, 072503.
Dukelsky, J., S. Pittel, and G. Sierra, 2004, Rev. Mod. Phys. **76**, 643.
Dvorsky, A., and S. Sahi, 1998, Sel. Math., New Ser. **4**, 11.
Elliott, J. P., 1958a, Proc. R. Soc. A **245**, 128.
Elliott, J. P., 1958b, Proc. R. Soc. A **245**, 562.
Engel, J., S. Pittel, M. Stoitsov, P. Vogel, and J. Dukelsky, 1997, Phys. Rev. C **55**, 1781.
Escuderos, A., and L. Zamick, 2006, Phys. Rev. C **73**, 044302.
Flowers, B. H., 1952a, Proc. R. Soc. A **212**, 248.
Flowers, B. H., 1952b, Proc. R. Soc. A **215**, 398.
Flowers, B. H., 1952c, Proc. R. Soc. A **210**, 497.
Flowers, B. H., and S. Szpikowski, 1964, Proc. Phys. Soc. London **84**, 673.
French, J. B., 1960, Nucl. Phys. **15**, 393.
Frobenius, F. G., 1900, *Über die charactere der symetrischen gruppe*, S'ber. Akad. Wiss. Berlin, p. 516 (reprinted in Gesammelte Abhandlungen. Band III, 1968) (Springer-Verlag, Berlin), English translation by J.-P. Serre, p. 148.
Fukutome, H., M. Yamamura, and S. Nishiyama, 1977, Prog. Theor. Phys. **57**, 1554.
Fulton, W., and J. Harris, 1991, *Representation Theory: A First Course* (Springer-Verlag, New York).
Gaudin, M., 1976, J. Phys. (France) **37**, 1087.
Gazeau, J. -P., 2009, *Coherent States in Quantum Physics* (Wiley-VCH, Weinheim).
Gelbart, S., 1973, Invent. Math. **19**, 49.
Gelbart, S., 1979, Proc. Symp. Pure Math. **33**, Pt. 1, 287 [http://www.ams.org/mathscinetgetitem?mr=81f:22035].
Girvin, S. M., 1996, Science **274**, 524.
Goodman, R., 2004, in *Representations of Real and P-Adic Groups*, edited by E.-C. Tan, and C.-B. Zhu, Lecture notes series Vol. 2 (World Scientific, Institute for Mathematical Sciences, National University of Singapore, Singapore), p. 305.
Green, G., 1999, J. Math. Phys. (N.Y.) **40**, 1074.
Griffin, A., T. Nikuni, and E. Zaremba, 2009, *Bose-Condensed Gases at Finite Temperatures* (Cambridge University Press, Cambridge, England).
Griffin, A., D. W. Stoke, and S. Stringari, 1995, (Eds.), *Bose-Einstein Condensation* (Cambridge University Press, Cambridge, England).
Grudzinski, K., and B. G. Wybourne, 1996, J. Phys. A **29**, 6631.
Haase, R. W., and P. H. Butler, 1984a, J. Phys. A **17**, 47.
Haase, R. W., and P. H. Butler, 1984b, J. Phys. A **17**, 61.
Hamermesh, M., 1962, *Group Theory and its Application to Physical Problems.* (Addison-Wesley, Reading, MA).
Hecht, K. T., 1987, *The Vector Coherent State Method and Its Application to Problems of Higher Symmetries*, Lecture Notes in Physics Vol. 290 (Springer-Verlag, Berlin).
Hecht, K. T., R. Le Blanc, and D. J. Rowe, 1987, J. Phys. A **20**, 2241.
Hecht, K. T., and S. C. Pang, 1969, J. Math. Phys. (N.Y.) **10**, 1571.

Helmers, K., 1961, Nucl. Phys. **23**, 594.
Hess, P., A. Algora, N. Hunyadi, and J. Cseh, 2002, Eur. Phys. J. A **15**, 449.
Holstein, T., and H. Primakoff, 1940, Phys. Rev. **58**, 1098.
Howe, R., 1985, in *Applications of Group Theory in Physics and Mathematical Physics*, edited by M. Flato, P. Sully, and G. Zuckermann, Lectures in Applied Mathematics Vol. 21 (American Mathematical Society, Providence), p. 179.
Howe, R., 1989, Trans. Am. Math. Soc. **313**, 539.
Howe, R., 1995, in *The Schur Lectures (1992)*, edited by I. Piatestski-Shapiro and S. Gelbart (Bar-Ilan University, Ramat-Gan), p. 3.
Howe, R., E.-C. Tan, and J. Willenbring, 2005, Trans. Am. Math. Soc. **357**, 1601 [http://www.ams.org/jourcgi/jrnl_toolbar_nav/tran_all}.
Iachello, F., 1981, Chem. Phys. Lett. **78**, 581.
Iachello, F., and A. Arima, 1987, *The Interacting Boson Model* (Cambridge University Press, Cambridge, England).
Ibrahim, E. M., 1951, "Royal Society depository for unpublished mathematical tables (1)."
Ibrahim, E. M., 1952, Quart. J. Math. (Oxford Series) **3**, 50.
İnönü, E., and E. P. Wigner, 1953, Proc. Natl. Acad. Sci. U.S.A. **39**, 510.
Isacker, P. V., 1999, Rep. Prog. Phys. **62**, 1661.
Isacker, P. V., K. Heyde, J. Jolie, and A. Sevrin, 1986, Ann. Phys. (N.Y.) **171**, 253.
Jordan, P., and E. Wigner, 1928, Z. Phys. A **47**, 631.
Junge, M., P. T. Kim, and D. W. Kribs, 2005, J. Math. Phys. (N.Y.) **46**, 022102.
Kashiwara, M., and M. Vergne, 1978, Invent. Math. **44**, 1.
Kerman, A. K., 1961, Ann. Phys. (N.Y.) **12**, 300.
Kerman, A. K., R. D. Lawson, and M. H. Macfarlane, 1961, Phys. Rev. **124**, 162.
King, R. C., 1970, J. Math. Phys. (N.Y.) **11**, 280.
King, R. C., 1971, J. Math. Phys. (N.Y.) **12**, 1588.
King, R. C., 1975, J. Phys. A **8**, 429.
King, R. C., 1993, in *Symmetries and Structural Properties of Condensed Matter*, edited by W. Florek, D. Lipinski, and T. Lulek (World Scientific, Singapore), p. 57.
King, R. C., and B. G. Wybourne, 1985, J. Phys. A **18**, 3113.
King, R. C., and B. G. Wybourne, 1998, J. Phys. A **31**, 6669.
King, R. C., and B. G. Wybourne, 2000a, J. Math. Phys. (N.Y.) **41**, 5002.
King, R. C., and B. G. Wybourne, 2000b, J. Math. Phys. (N.Y.) **41**, 5656.
Klauder, J. R., and B. -S. Skagerstam, 1985, *Coherent States; Applications in Physics and Mathematical Physics* (World Scientific, Singapore).
Klein, A., and E. R. Marshalek, 1991, Rev. Mod. Phys. **63**, 375.
Koike, K., and I. Terada, 1987, J. Algebra **107**, 466.
Kota, V. K. B., and J. A. C. Alcarás, 2006, Nucl. Phys. **A764**, 181.
Kulish, P. P., and N. Y. Reshetikhin, 1983, J. Sov. Math. **23**, 2435.
Landsman, N. P., 1999, J. Nonlinear Math. Phys. **6**, 161.
Le Blanc, R., 1987, J. Phys. A **20**, 5015.
Le Blanc, R., and K. T. Hecht, 1987, J. Phys. A **20**, 4613.
Ledermann, W., 1987, *Introduction to Group Characters* (Cambridge University Press, Cambridge, England), 2nd ed.
Lehmann, H., and J. Jolie, 1995, Nucl. Phys. **A588**, 623.
Lerma, H. S., B. Errea, J. Dukelsky, and W. Satuła, 2007, Phys. Rev. Lett. **99**, 032501.
Leung, E. Y., and T. Ton-That, 1994, Proc. Am. Math. Soc. **120**, 53 [http://www.ams.org/journals/proc/1994-120-01/S0002-9939-1994-1165060-9/home.html].
Leung, E. Y., and T. Ton-That, 1995, Proc. Am. Math. Soc. **123**, 1281 [http://www.ams.org/journals/proc/1995-123-04/S0002-9939-1995-1227520-2/home.html].
Leviatan, A., 2011, Prog. Part. Nucl. Phys. **66**, 93.
Li, J.-S., 2000, in *Representation Theory of Lie Groups*, edited by J. Adams, and D. Vogan (American Mathematical Society, IAS/Park City), Vol. 8, p. 293.
Links, J., H.-Q. Zhou, M. D. Gould, and R. H. McKenzie, 2002, J. Phys. A **35**, 6459.
Lipkin, H. J., 2002, *Lie Groups for Pedestrians* (Dover, Mineola, NY), 2nd ed. (originally published in 1966 by North-Holland, Amsterdam).
Littlewood, D., 1944a, Phil. Trans. R. Soc. A **239**, 387.
Littlewood, D. E., 1936, J. Lond. Math. Soc. **s1-11**, 49.
Littlewood, D. E., 1944b, Phil. Trans. R. Soc. A **239**, 305.
Littlewood, D. E., 1950, *The Theory of Group Characters and Matrix Representations of Groups* (Oxford University Press, Oxford), 2nd ed.
Lorazo, B., 1970, Nucl. Phys. **A153**, 255.
Lu, D., and R. Howe, 2010, in *Casimir Force, Casimir Operators and the Riemann Hypothesis*, edited by v. Dijk, G, and M. Wakayama, de Gruyter Proceedings in Mathematics (Walter de Gruyter, Berlin/New York), p. 105 [http://www.degruyter.com/view/product/177072].
Macdonald, I. G., 1995, *Symmetric Functions and Hall Polynomials* (Clarendon Press, Oxford), 2nd ed.
Macdonald, I. G., 1998, *Symmetric Functions and Orthogonal Polynomials* (American Mathematical Society, Providence, RI).
Macek, M., J. Dobeš, and P. Cejnar, 2009, Phys. Rev. C **80**, 014319.
Mackey, G. W., 1968, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New York).
Moshinsky, M., and C. Quesne, 1969, Phys. Lett. **29B**, 482.
Moshinsky, M., and C. Quesne, 1970, J. Math. Phys. (N.Y.) **11**, 1631.
Palchikov, Y. V., J. Dobeš, and R. V. Jolos, 2001, Phys. Rev. C **63**, 034320.
Parikh, J. C., 1978, *Group Symmetries in Nuclear Structure* (Plenum Press, New York/London).
Perelomov, A. M., 1972, Commun. Math. Phys. **26**, 222.
Perelomov, A. M., 1986, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin).
Racah, G., 1943, Phys. Rev. **63**, 367.
Racah, G., 1949, Phys. Rev. **76**, 1352.
Richardson, R. W., 1965, J. Math. Phys. (N.Y.) **6**, 1034.
Richardson, R. W., and N. Sherman, 1964, Nucl. Phys. **52**, 221.
Ring, P., and P. Schuck, 1980, *The Nuclear Many-Body Problem* (Springer-Verlag, New York).
Robinson, G. d. B., 1977, Ed., *The Collected Papers of Alfred Young*, Mathematical Expositions Vol. 21 (University of Toronto Press, Toronto).
Rochford, P., and D. J. Rowe, 1988, Phys. Lett. B **210**, 5.
Rosensteel, G., and D. J. Rowe, 1977, Phys. Rev. Lett. **38**, 10.
Rosensteel, G., and D. J. Rowe, 1980, Ann. Phys. (N.Y.) **126**, 343.
Rosensteel, G., and D. J. Rowe, 2003, Phys. Rev. C **67**, 014303.
Rosensteel, G., and D. J. Rowe, 2005, Nucl. Phys. **A759**, 92.
Rosensteel, G., and D. J. Rowe, 2007, Nucl. Phys. **A797**, 94.
Rowe, D. J., 1970, *Nuclear Collective Motion: Models and Theory* (Methuen, London) (reprinted in 2010 by World Scientific, Singapore).
Rowe, D. J., 1984, J. Math. Phys. (N.Y.) **25**, 2662.
Rowe, D. J., 1985, Rep. Prog. Phys. **48**, 1419.
Rowe, D. J., 2004a, Nucl. Phys. **A735**, 372.
Rowe, D. J., 2004b, in *Computational and Group-Theoretical Methods in Nuclear Physics*, edited by J. Escher, O. Castaños,

J. G. Hirsch, S. Pittel, and G. Stoitcheva (World Scientific, Singapore), p. 165.
Rowe, D. J., 2004c, Nucl. Phys. **A745**, 47.
Rowe, D. J., 2005, J. Phys. A **38**, 10181.
Rowe, D. J., and C. Bahri, 1998, J. Phys. A **31**, 4947.
Rowe, D. J., and M. J. Carvalho, 2007, J. Phys. A **40**, 471.
Rowe, D. J., 2012, "Vector Coherent State Representations and Their Inner Product" J. Phys. A (to be published).
Rowe, D. J., and J. Repka, 1991, J. Math. Phys. (N.Y.) **32**, 2614.
Rowe, D. J., and J. Repka, 1998, J. Math. Phys. (N.Y.) **39**, 6214.
Rowe, D. J., J. Repka, and M. J. Carvalho, 2011, J. Math. Phys. (N.Y.) **52**, 013507.
Rowe, D. J., P. Rochford, and J. Repka, 1988, J. Math. Phys. (N.Y.) **29**, 572.
Rowe, D. J., and G. Rosensteel, 2001, Phys. Rev. Lett. **87**, 172501.
Rowe, D. J., G. Rosensteel, and R. Carr, 1984, J. Phys. A **17**, L399.
Rowe, D. J., G. Rosensteel, and R. Gilmore, 1985, J. Math. Phys. (N.Y.) **26**, 2787.
Rowe, D. J., and G. Thiamova, 2005, Nucl. Phys. **A760**, 59.
Rowe, D. J., and P. S. Turner, 2005, Nucl. Phys. **A753**, 94.
Rowe, D. J., P. S. Turner, and J. Repka, 2004, J. Math. Phys. (N.Y.) **45**, 2761.
Rowe, D. J., T. A. Welsh, and M. A. Caprio, 2009, Phys. Rev. C **79**, 054304.
Rowe, D. J., and J. L. Wood, 2010, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore).
Rowe, D. J., B. G. Wybourne, and P. H. Butler, 1985, J. Phys. A **18**, 939.
Sawada, K., 1957, Phys. Rev. **106**, 372.
Schmidt, M., 1999, J. Geom. Phys. **29**, 283.
Schur, I., 1901, Ph.D. thesis, Berlin University (reprinted in Schur's collected papers: I. Schur: Gesammelte Abhandlungen. Band I, English translation by Alfred Brauer und Hans Rohrbach, Springer-Verlag, Berlin, New York, 1973).
Schur, I., 1927, S'ber. Akad. Wiss., Berlin, p. 58 (reprinted in Schur's collected papers: I. Schur: Gesammelte Abhandlungen. Band III, English translation by Alfred Brauer und Hans Rohrbach, Springer-Verlag, Berlin, New York, 1973).
Segal, E., 1959, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **31**, No. 12, 1.
Shale, D., 1962, Trans. Am. Math. Soc. **103**, 149.
Smith, P. R., and B. G. Wybourne, 1967, J. Math. Phys. (N.Y.) **8**, 2434.
Smith, P. R., and B. G. Wybourne, 1968, J. Math. Phys. (N.Y.) **9**, 1040.
Sternberg, S., 1994, *Group Theory and Physics* (Cambridge University Press Cambridge, England).
Sviratcheva, K. D., A. I. Georgieva, and J. P. Draayer, 2003, J. Phys. G **29**, 1281.
Talmi, I., 1993, *Simple Models of Complex Nuclei: The Shell Model and Interacting Boson Model* (Harwood Academic Publishers, Chur, Switzerland).
Thibon, J.-Y., F. Toumazet, and B. G. Wybourne, 1997, J. Phys. A **30**, 4851.
Van der Jeugt, J., 1997, J. Math. Phys. (N.Y.) **38**, 2728.
Van Isacker, P., 1999, Rep. Prog. Phys. **62**, 1661.
Weil, A., 1964, Acta Math. **111**, 143.
Welsh, T. A., and D. J. Rowe, 2012 (unpublished).
Weyl, H., 1946, *The Classical Groups, Their Invariants and Representations* (Princeton University Press, Princeton, NJ), 2nd ed.
Weyl, H., 1950, *The Theory of Groups and Quantum Mechanics* (Dover, New York, originally published in 1928 in German), 2nd ed. (English edition 1, translated by H. P. Robertson, published by Methuen in 1931).
Whippman, M. L., 1965, J. Math. Phys. (N.Y.) **6**, 1534.
Wigner, E., 1937, Phys. Rev. **51**, 106.
Wulfman, C. E., 2011, *Dynamical Symmetry* (World Scientific, Singapore).
Wybourne, B. G., 1970, *Symmetry Principles and Atomic Spectroscopy* (Wiley, New York).
Wybourne, B. G., 1974, *Classical Groups for Physicists* (Wiley, New York).
Wybourne, B. G., 1991, in *Symmetry and Structural Properties of Condensed Matter*, edited by W. Florek, T. Lulek, and M. Mucha (World Scientific, Singapore), p. 187.
Wybourne, B. G., 1993, in *Symmetry and Structural Properties of Condensed Matter*, edited by W. Florek, D. Lipinski, and T. Lulek, The Second International School of Theoretical Physics, Poznan, Poland (World Scientific, Singapore), p. 79.
Yépez-Martínez, H., J. Cseh, and P. O. Hess, 2006, Phys. Rev. C **74**, 024319.
Zamick, L., and P. V. Isacker, 2008, Phys. Rev. C **78**, 044327.