# Theoretical aspects of massive gravity

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Massive gravity has seen a resurgence of interest due to recent progress which has overcome its traditional problems, yielding an avenue for addressing important open questions such as the cosmological constant naturalness problem. The possibility of a massive graviton has been studied on and off for the past 70 years. During this time, curiosities such as the van Dam, Veltman, and Zakharov (vDVZ) discontinuity and the Boulware-Deser ghost were uncovered. These results are rederived in a pedagogical manner and the Stückelberg formalism to discuss them from the modern effective field theory viewpoint is developed. Recent progress of the last decade is reviewed, including the dissolution of the vDVZ discontinuity via the Vainshtein screening mechanism, the existence of a consistent effective field theory with a stable hierarchy between the graviton mass and the cutoff, and the existence of particular interactions which raise the maximal effective field theory cutoff and remove the ghosts. In addition, some peculiarities of massive gravitons on curved space, novel theories in three dimensions, and examples of the emergence of a massive graviton from extra dimensions and brane worlds are reviewed.

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## I. INTRODUCTION

Our goal is to explore what happens when one tries to give the graviton a mass. This is a modification of gravity, so we first discuss what gravity is and what it means to modify it.

### A. General relativity is massless spin 2

General relativity (GR) (Einstein, 1916) is by now widely accepted as the correct theory of gravity at low energies or large distances. The discovery of GR was in many ways ahead of its time. It was a leap of insight, from the equivalence principle and general coordinate invariance, to a fully nonlinear theory governing the dynamics of spacetime itself. It provided a solution, one more elaborate than necessary, to the problem of reconciling the insights of special relativity with the nonrelativistic action at a distance of Newtonian gravity.

Had it not been for Einstein's intuition and years of hard work, general relativity would likely have been discovered anyway, but its discovery may have had to wait several more decades, until developments in field theory in the 1940s and 1950s primed the culture. But in this hypothetical world without Einstein, the path of discovery would likely have been very different and in many ways more logical.

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This logical path starts with the approach to field theory espoused in the first volume of Weinberg's field theory text (Weinberg, 1995). Degrees of freedom in flat four-dimensional spacetime are particles, classified by their spin. These degrees of freedom are carried by fields. If we wish to describe long-range macroscopic forces, only bosonic fields will do, since fermionic fields cannot build up classical coherent states. By the spin statistics theorem, these bosonic fields must be of integer spin $s = 0, 1, 2, 3$, etc. A field $\psi$, which carries a particle of mass $m$, will satisfy the Klein-Gordon equation $(\Box - m^2)\psi = 0$, whose solution a distance $r$ from a localized source is similar to $\sim r^{-1}e^{-mr}$. Long-range forces, those without exponential suppression, must therefore be described by massless fields $m = 0$.

Massless particles are characterized by how they transform under rotations transverse to their direction of motion. The transformation rule for bosons is characterized by an integer $h \geq 0$, which we call the helicity. For $h = 0$, such massless particles can be carried most simply by a scalar field $\phi$. For a scalar field, any sort of interaction terms consistent with Lorentz invariance can be added, and so there are a plethora of possible self-consistent interacting theories of spin 0 particles.

For helicities $s \geq 1$, the field must carry a gauge symmetry if we are to write interactions with manifest Lorentz symmetry and locality. For helicity 1, if we choose a vector field $A_\mu$ to carry the particle, its action is fixed to be the Maxwell action, so even without Maxwell, we could have discovered electromagnetism via these arguments. If we now ask for consistent self-interactions of such massless particles, we are led to the problem of deforming the action (and possibly the form of the gauge transformations), in such a way that the linear form of the gauge transformations is preserved. These requirements are enough to lead us essentially uniquely to the non-Abelian gauge theories, two of which describe the strong and weak forces (Henneaux, 1998).

Moving on to helicity 2, the required gauge symmetry is linearized general coordinate invariance. Asking for consistent self-interactions leads essentially uniquely to GR and full general coordinate invariance (Gupta, 1954; Kraichnan, 1955; Weinberg, 1965; Deser, 1970; Boulware and Deser, 1975; Fang and Fronsdal, 1979; Wald, 1986) [see also Chapter 13 of Weinberg (1995), which shows how helicity 2 implies the equivalence principle]. For helicity $\geq 3$, the story ends, because there are no self-interactions that can be written (Berends, Burgers, and van Dam, 1984) [see also Chapter 13 of Weinberg (1995), which shows that the scattering amplitudes for helicity $\geq 3$ particles vanish].

This path is straightforward, starting from the principles of special relativity (Lorentz invariance) to the classification of particles and fields that describe them, and finally to their possible interactions. The path Einstein followed, on the other hand, is a leap of insight and has logical gaps; the equivalence principle and general coordinate invariance, although they suggest GR, do not lead uniquely to GR.

General coordinate invariance is a gauge symmetry, and gauge symmetries are redundancies of description, not fundamental properties. In any system with gauge symmetry, one can always fix the gauge and eliminate the gauge symmetry, without breaking the physical global symmetries (such as Lorentz invariance) or changing the physics of the system in any way. One often hears that gauge symmetry is fundamental, in electromagnetism, for example, but the more correct statement is that gauge symmetry in electromagnetism is necessary only if one demands the convenience of linearly realized Lorentz symmetry and locality. Fixing a gauge will not change the physics, but the price paid is that the Lorentz symmetries and locality are not manifest.

On the other hand, starting from a system without gauge invariance, it is always possible to introduce gauge symmetry by putting in redundant variables. Often this can be very useful for studying a system and can elucidate properties which are otherwise difficult to see. This is the essence of the Stückelberg trick, which we make use of extensively in our study of massive gravity. In fact, as we will see, this trick can be used to make any Lagrangian invariant under general coordinate diffeomorhpisms, the same group under which GR is invariant. Thus general coordinate invariance cannot be the defining feature of GR.

Similarly, the principle of equivalence, which demands that all mass and energy gravitate with the same strength, is not unique to GR. It can be satisfied even in scalar field theories, if one chooses the interactions properly. For example, this can be achieved by iteratively coupling a canonical massless scalar to its own energy momentum tensor. Such a theory in fact solves all the problems Einstein set out to solve; it provides a universally attractive force which conforms to the principles of special relativity, reduces to Newtonian gravity in the nonrelativistic limit, and satisfies the equivalence principle.[1] By introducing diffeomorphism invariance via the Stückelberg trick, it can even be made to satisfy the principle of general coordinate invariance.

The real underlying principle of GR has nothing to do with coordinate invariance or equivalence principles or geometry, rather it is the statement: General relativity is the theory of a nontrivially interacting massless helicity 2 particle. The other properties are consequences of this statement, and the implication cannot be reversed.

As a quantum theory, GR is not UV complete. It must be treated as an effective field theory valid at energies up to a cutoff at the Planck mass $M_P$, beyond which unknown high energy effects will correct the Einstein-Hilbert action. For a given background such as the spherical solution around a heavy source of mass $M$ such as the Sun, GR has three distinct regimes. There is a classical linear regime, where both nonlinear effects and quantum effects can be ignored. This is the regime in which $r$ is greater than the Schwarzschild radius $r > r_S \sim M/M_P^2$. For $M$ the mass of the Sun, we have $r_S \sim 1$ km, so the classical linear approximation is good nearly everywhere in the Solar System. There is the quantum regime $r < 1/M_P$, very near the singularity of the black hole, where the effective field theory description breaks down. Most importantly, there is a well-separated

[1]This theory is sometimes known as the Einstein-Fokker theory, first introduced in 1913 by Nordström (1913a, 1913b), and later in a different form (Freund and Nambu, 1968; Deser and Halpern, 1970). It was even studied by Einstein when he was searching for a relativistic theory of gravity that embodied the equivalence principle (Einstein and Fokker, 1914).

middle ground, a classical nonlinear regime $1/M_P < r < r_S$, where nonlinearities can be summed up without worrying about quantum corrections, the regime which can be used to make controlled statements about what is going on inside a black hole. One of the challenges of adding a mass to the graviton, or any modification of gravity, is to retain calculable yet interesting regimes such as this.

### B. Modifying general relativity

A theory of massive gravity is a theory which propagates a massive spin 2 particle. The most straightforward way to construct such a theory is to simply add a mass term to the Einstein-Hilbert action, giving the graviton a mass $m$ in such a way that GR is recovered as $m \to 0$. This is a modification of gravity, a deformation away from the elegant theory of Einstein. Since GR is the essentially unique theory of a massless spin 2 degree of freedom, it should be remembered that modifying gravity means changing its degrees of freedom.

Despite the universal consensus that GR is a beautiful and accurate theory, there has in recent years arisen a small industry of physicists working to modify it and test these modifications. When asked to cite their motivation, they more often than not point to supernova data (Riess *et al.*, 1998; Perlmutter *et al.*, 1999) which show that the Universe has recently started accelerating in its expansion. If GR is correct, there must exist some dark energy density $\rho \sim 10^{-29}$ g/cm$^3$. The simplest interpretation is that there is a constant term $\Lambda$ in the Einstein-Hilbert action, which would give $\rho \sim M_P^2\Lambda$. To give the correct vacuum energy, this constant has to take the small value $\Lambda/M_P^2 \sim 10^{-65}$, whereas arguments from quantum field theory suggest a value much larger, up to the order of unity (Weinberg, 1989). It is therefore tempting to speculate that perhaps GR is wrong, and instead of a dark energy component, gravity is modified in the infrared (Deffayet, 2001; Deffayet, Dvali, and Gabadadze, 2002), in such a way as to produce an accelerating universe from nothing. Indeed many modifications can be cooked up which produce these so-called self-accelerating solutions. For example, one well-studied modification is to replace the Einstein-Hilbert Lagrangian with $F(R)$, a general function of the Ricci scalar (Sotiriou and Faraoni, 2008; De Felice and Tsujikawa, 2010), which can lead to self-accelerating solutions (Carroll *et al.*, 2004, 2005). This modification is equivalent to adding an additional scalar degree of freedom.

These cosmological reasons for studying modifications to gravity are often criticized on the grounds that they can take us only so far; the small value of the cosmological acceleration relative to the Planck mass must come from somewhere, and the best these modifications can do is to shift the fine-tuning into other parameters [see Batra *et al.* (2008) for an illustration in the $F(R)$ scalar-tensor case].

While it is true the small number must come from somewhere, there remains hope that it can be put somewhere which is technically natural, i.e., stable to quantum corrections. Some small parameters, such as the ratio of the Higgs mass to the Planck mass in the standard model, are not technically natural, whereas others, such as small fermion masses, are technically natural, because their small values are stable under quantum corrections. A rule of thumb is that a small parameter is technically natural if there is a symmetry that appears as the small parameter is set to zero. When this is the case, symmetry protects a zero value of the small parameter from quantum corrections. This means corrections due to the small parameter must be proportional to the parameter itself. In the case of small fermion masses, it is chiral symmetry that appears, whereas in the case of the Higgs mass and the cosmological constant, there is no obvious symmetry that appears.

Of course, there is no logical inconsistency with having small parameters, technically natural or not, and nature may explain them anthropically (Barrow and Tipler, 1988), or may just employ them without reason. But as practical working physicists, we hope that it is the case that a small parameter is technically natural, because then there is hope that perhaps some classical mechanism can be found that drives the parameter toward zero, or otherwise explains its small value. If it is not technically natural, any such mechanism will be much harder to find because it must know about the quantum corrections in order to compensate them.

One does not need a cosmological constant problem, however, to justify studying modifications to GR. There are few better ways to learn about a structure, whether it is a car, a computer program, or a theory, than to attempt to modify it. With a rigid theory such as GR, there is a level of appreciation that can be achieved only by witnessing how easily things can go badly with the slightest modification. In addition, deforming a known structure is one of the best ways to go about discovering new structures, structures which may have unforeseen applications.

One principle that comes into play is the continuity of physical predictions of a theory in the parameters of the theory. Surely, we should not be able to say experimentally, given our finite experimental precision, that a parameter of nature is exactly mathematically zero and not just very small. If we deform GR by a small parameter, the predictions of the deformed theory should be very close to GR, to the extent that the deformation parameter is small. It follows that any undesirable pathologies associated with the deformation should cure themselves as the parameter is set to zero. Thus, we uncover a mechanism by which such pathologies can be cured, a mechanism which may have applications in other areas.

Massive gravity is a well-developed case study in the infrared modification of gravity, where all of these points are nicely illustrated. Purely from the consideration of degrees of freedom, it is a natural modification to consider, since it amounts to simply giving a mass to the particle which is already present in GR. In another sense, it is less minimal than $F(R)$ or scalar-tensor theory, which adds a single scalar degree of freedom, because to reach the five polarizations of the massive graviton we must add at least 3 degrees of freedom beyond the 2 of the massless graviton.

With regard to the cosmological constant problem, there is the possibility of a technically natural explanation. The deformation parameter is $m$, the graviton mass, and GR should be restored as $m \to 0$. The force mediated by a massive graviton has a Yukawa profile $\sim r^{-1}e^{-mr}$, which drops off from that of a massless graviton at distances $r \gtrsim 1/m$, so one hopes to explain the acceleration of the Universe without dark energy by choosing the graviton mass to be of the order of the Hubble constant $m \sim H$. Of course, this does not eliminate the small cosmological constant, which reappears as the ratio $m/M_P$. But there is now hope that this is a technically natural choice, because deformation by a mass term breaks the gauge symmetry of GR, which is restored in the limit $m \rightarrow 0$. As we will see, a small $m$ is indeed protected from quantum corrections (although there are other issues that prevent this, at our current stage of understanding, from being a completely satisfactory realization of a technically natural cosmological constant).

There are also interesting lessons to be learned regarding the continuity of physical predictions. The addition of a mass term is a brutality upon the structure of GR and does not go unpunished. Various pathologies appear, which are representative of common pathologies associated with any infrared modification of gravity. These include strong classical nonlinearities, ghostlike instabilities, and a very low cutoff, or region of trustability, for the resulting quantum effective theory. In short, modifying the infrared often messes up the UV. New mechanisms also come into play, because the extra degrees of freedom carried by the massive graviton must somehow decouple themselves as $m \rightarrow 0$ to restore the physics of GR.

The study of the massless limit leads to the discovery of the *Vainshtein mechanism*, by which these extra degrees of freedom hide themselves at short distances using nonlinearities. This mechanism has already proven useful for model builders who have long-range scalars, such as moduli from the extra dimensions of string theory, that they want to shield from local experiments that would otherwise rule them out.

### C. History and outline

The possibility of a graviton mass has been studied off and on since 1939, when Fierz and Pauli (1939) first wrote the action describing a free massive graviton. Following this, not much occurred until the early 1970s, when there was a flurry of renewed interest in quantum field theory. The linear theory coupled to a source was studied by van Dam and Veltman (1970) and Zakharov (1970) (vDVZ), who discovered the curious fact that the theory makes predictions different from those of linear GR even in the limit as the graviton mass goes to zero. For example, massive gravity in the $m \rightarrow 0$ limit gives a prediction for light bending that is off by 25% from the GR prediction. The linear theory violates the principle of continuity of the physics in the parameters of the theory. This is known as the *vDVZ discontinuity*. The discontinuity was soon traced to the fact that not all of the degrees of freedom introduced by the graviton mass decouple as the mass goes to zero. The massive graviton has five spin states, which in the massless limit become the two helicity states of a massless graviton, two helicity states of a massless vector, and a single massless scalar. The scalar is essentially the longitudinal graviton, and it maintains a finite coupling to the trace of the source stress tensor even in the massless limit. In other words, the massless limit of a massive graviton is not a massless graviton, but rather a massless graviton plus a coupled scalar, and the scalar is responsible for the vDVZ discontinuity.

If the linear theory is accurate, then the vDVZ discontinuity represents a true physical discontinuity in predictions, violating our intuition that physics should be continuous in the parameters. Measuring the light bending in this theory would be a way to show that the graviton mass is mathematically zero rather than just very small. However, the linear theory is only the start of a complete nonlinear theory, coupled to all the particles of the standard model. The possible nonlinearities of a real theory were studied several years later by Vainshtein (1972), who found that the nonlinearities of the theory become stronger and stronger as the mass of the graviton shrinks. What he found was that around any massive source of mass $M$, such as the Sun, there is a new length scale known as the *Vainshtein radius* $r_V \sim (M/m^4 M_P^2)^{1/5}$. At distances $r \lesssim r_V$, nonlinearities begin to dominate and the predictions of the linear theory cannot be trusted. The Vainshtein radius goes to infinity as $m \rightarrow 0$, so there is no radius at which the linear approximation tells us something trustworthy about the massless limit. This opens the possibility that the nonlinear effects cure the discontinuity. To have some values in mind, if we take $M$ the mass of the Sun and $m$ a very small value, say the Hubble constant $m \sim 10^{-33}$ eV, the scale at which we might want to modify gravity to explain the cosmological constant, we have $r_V \sim 10^{18}$ km, about the size of the Milky Way.

Later the same year, Boulware and Deser (1972) studied some specific fully nonlinear massive gravity theories and showed that they possess a ghostlike instability. Whereas the linear theory has 5 degrees of freedom, the nonlinear theories they studied turned out to have 6, and the extra degree of freedom manifests itself around nontrivial backgrounds as a scalar field with a wrong sign kinetic term, known as the *Boulware-Deser ghost*.

Meanwhile, the ideas of effective field theory were being developed, and it was realized that a nonrenormalizable theory, even one with apparent instabilities such as massive gravity, can be made sense of as an effective field theory, valid only at energies below some ultraviolet cutoff scale $\Lambda$. In 2003, Arkani-Hamed, Georgi, and Schwartz (2003) brought to attention a method of restoring gauge invariance to massive gravity in a way that makes it very simple to see what the effective field theory properties are. They showed that massive gravity generically has a maximum UV cutoff of $\Lambda_5 = (M_P m^4)^{1/5}$. For Hubble scale graviton mass, this is a length scale $\Lambda_5^{-1} \sim 10^{11}$ km. This is a very small cutoff, parametrically smaller than the Planck mass, and goes to zero as $m \rightarrow 0$. Around a massive source, the quantum effects become important at the radius $r_Q = (M/M_{\rm Pl})^{1/3}(1/\Lambda_5)$, which is parametrically larger than the Vainshtein radius at which nonlinearities enter. For the Sun, $r_Q \sim 10^{24}$ km. Without finding a UV completion or some other resummation, there is no sense in which we can trust the solution inside this radius, and the usefulness of massive gravity is limited. In particular, since the whole nonlinear regime is below this radius, there is no hope to examine the continuity of physical quantities in $m$ and explore the Vainshtein mechanism in a controlled way. On the other hand, it can be seen that the mass of the Boulware-Deser ghost drops below the cutoff only when $r \lesssim r_Q$, so the ghost is not really in the effective theory at all and can be consistently excluded.

Putting aside the issue of quantum corrections, there has been continued study of the Vainshtein mechanism in a purely classical context. It has been shown that classical nonlinearities do indeed restore continuity with GR in certain circumstances. In fact, the ghost degree of freedom can play an essential role in this, by providing a repulsive force in the nonlinear region to counteract the attractive force of the longitudinal scalar mode.

By adding higher order graviton self-interactions with appropriately tuned coefficients, it is in fact possible to raise the UV cutoff of the theory to $\Lambda_3 = (M_P m^2)^{1/3}$, corresponding to roughly $\Lambda_3^{-1} \sim 10^3$ km. In 2010, the complete action of this theory in a certain decoupling limit was worked out by de Rham and Gabadadze (2010a), and they show that, remarkably, it is free of the Boulware-Deser ghost. Recently, it was shown that the complete theory is free of the Boulware-Deser ghost. This $\Lambda_3$ theory is the best hope of realizing a useful and interesting massive gravity theory.

The subject of massive gravity also naturally arises in extra-dimensional setups. In a Kaluza-Klein scenario such as GR in $5d$ compactified on a circle, the higher Kaluza-Klein modes are massive gravitons. Brane world setups such as the Dvali-Gabadadze-Porrati (DGP) model (Dvali, Gabadadze, and Porrati, 2000a) give more intricate gravitons with resonance masses. The study of such models has complemented the study of pure $4d$ massive gravity and has pointed toward new research directions.

The major outstanding question is whether it is possible to UV extend the effective field theory of massive gravity to the Planck scale and what this UV extension may look like. This would provide a solution to the problem of making the small cosmological constant technically natural and is bound to be an interesting theory in its own right (the analogous question applied to massive vector bosons leads to the discovery of the Higgs mechanism and spontaneous symmetry breaking). In the case of massive gravity, there are indications that a UV completion may not have a local Lorentz invariant form, although the issue is not settled. Another long shot, if UV completion can be found, would be to take the $m \to 0$ limit of the completion and hope to obtain a UV completion to ordinary GR.

As this review is focused on the theoretical aspects of Lorentz invariant massive gravity, we do not have much to say about the large literature on Lorentz-violating massive gravity. We also do not say much about the experimental search for a graviton mass, or what the most likely signals and search modes would be. There has been much work in these areas, and each could be the topic of a separate review.

*Conventions*: Often we work in an arbitrary number of dimensions, just because it is easy to do so. In this case, $D$ signifies the number of spacetime dimension and we stick to $D \geq 3$. $d$ signifies the number of space dimensions $d = D - 1$. We use the mostly plus metric signature convention $\eta_{\mu\nu} = (-, +, +, +, \ldots)$. Tensors are symmetrized and antisymmetrized with unit weight, i.e., $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$, $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$. The reduced $4d$ Planck mass is $M_P = 1/(8\pi G)^{1/2} \approx 2.43 \times 10^{18}$ GeV. Conventions for the curvature tensors, covariant derivatives, and Lie derivatives are those of Carroll (2004).

## II. THE FREE FIERZ-PAULI ACTION

We start by displaying an action for a single massive spin 2 particle in flat space, carried by a symmetric tensor field $h_{\mu\nu}$,

$$S = \int d^D x -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \quad (2.1)$$

This is known as the *Fierz-Pauli action* (Fierz and Pauli, 1939). Our point of view is to take this action as given and then show that it describes a massive spin 2. There are, however, some (less than thorough) ways of motivating this action. To start with, the action above contains all possible contractions of two powers of $h$, with up to two derivatives. The two derivative terms, those which survive when $m = 0$, are chosen to exactly match those obtained by linearizing the Einstein-Hilbert action. The $m = 0$ terms describe a massless helicity 2 graviton and have the gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (2.2)$$

for a spacetime dependent gauge parameter $\xi_\mu(x)$. This symmetry fixes all the coefficients of the two-derivative part of Eq. (2.1), up to an overall coefficient. The mass term, however, violates this gauge symmetry. The relative coefficient of $-1$ between the $h^2$ and $h_{\mu\nu}h^{\mu\nu}$ contractions is called the *Fierz-Pauli tuning*, and it is not enforced by any known symmetry.

However, the only thing that needs to be said about this action is that it describes a single massive spin 2 degree of freedom of mass $m$. We show this explicitly in what follows. Any deviation from the form (2.1) and the action will no longer describe a single massive spin 2. For example, violating the Fierz-Pauli tuning in the mass term by changing to $-\frac{1}{2}m^2[h_{\mu\nu}h^{\mu\nu} - (1-a)h^2]$ for $a \neq 0$ gives an action describing a scalar ghost (a scalar with negative kinetic energy) of mass $m_g^2 = [(3-4a)/2a]m^2$, in addition to the massive spin 2. For small $a$, the ghost mass squared goes like $\sim 1/a$. It goes to infinity as the Fierz-Pauli tuning is approached, rendering it nondynamical. Violating the tuning in the kinetic terms similarly alters the number of degrees of freedom; see van Nieuwenhuizen (1973a) for a general analysis.

There is a method of constructing Lagrangians such as Eq. (2.1) to describe any given spin. See, for example, the first few chapters of Weinberg (1995), the classic papers on higher spin Lagrangians by Singh and Hagen (1974) and Fronsdal (1978), and the reviews by Bouatta, Compere, and Sagnotti (2004) and Sorokin (2005).

### A. Hamiltonian and degree of freedom count

We begin our study of the Fierz-Pauli action (2.1) by casting it into Hamiltonian form and counting the number of degrees of freedom. We show that it propagates $D(D-1)/2 - 1$ degrees of freedom in $D$ dimensions (5 degrees of freedom for $D = 4$), the right number for a massive spin 2 particle.

We start by Legendre transforming Eq. (2.1) only with respect to the spatial components $h_{ij}$. The canonical momenta are[2]

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij}. \quad (2.3)$$

Inverting for the velocities, we have

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}. \quad (2.4)$$

In terms of these Hamiltonian variables, the Fierz-Pauli action (2.1) becomes

$$S = \int d^D x \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}(\partial_j \pi_{ij}) + m^2 h_{0i}^2 + h_{00}(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}), \quad (2.5)$$

where

$$\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2). \quad (2.6)$$

First consider the case $m = 0$. The timelike components $h_{0i}$ and $h_{00}$ appear linearly multiplied by terms with no time derivatives. We interpret them as Lagrange multipliers enforcing the constraints $\partial_j \pi_{ij} = 0$ and $\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} = 0$. It is straightforward to check that these are first class constraints, and that the Hamiltonian (2.6) is first class. Thus Eq. (2.5) is a first class gauge system. For $D = 4$, the $h_{ij}$ and $\pi_{ij}$ each have six components, because they are $3 \times 3$ symmetric tensors, so together they span a 12-dimensional (for each space point) phase space. We have four constraints (at each space point), leaving an eight-dimensional constraint surface. The constraints then generate four gauge invariances, so the gauge orbits are four dimensional, and the gauge invariant quotient by the orbits is four dimensional [see Henneaux and Teitelboim (1992) for an introduction to constrained Hamiltonian systems, gauge theories, and the terminology used here]. These are the two polarizations of the massless graviton, along with their conjugate momenta.

In the case $m \neq 0$, the $h_{0i}$ are no longer Lagrange multipliers. Instead, they appear quadratically and are auxiliary variables. Their equations of motion yield

$$h_{0i} = -\frac{1}{m^2}\partial_j \pi_{ij}, \quad (2.7)$$

which can be plugged back into the action (2.5) to give

$$S = \int d^D x \pi_{ij}\dot{h}_{ij} - \mathcal{H} + h_{00}(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}), \quad (2.8)$$

where

$$\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2) + \frac{1}{m^2}(\partial_j \pi_{ij})^2. \quad (2.9)$$

The component $h_{00}$ remains a Lagrange multiplier enforcing a single constraint $\mathcal{C} = -\vec{\nabla}^2 h_{ii} + \partial_i\partial_j h_{ij} + m^2 h_{ii} = 0$, but the Hamiltonian is no longer first class. One secondary constraint arises from the Poisson bracket with the Hamiltonian $H = \int d^d x \mathcal{H}$, namely, $\{H, \mathcal{C}\}_{\rm PB} = [1/(D-2)] \times m^2 \pi_{ii} + \partial_i\partial_j \pi_{ij}$. The resulting set of two constraints is second class, so there is no longer any gauge freedom. For $D = 4$ the 12-dimensional phase space has two constraints for a total of 10 degrees of freedom, corresponding to the five polarizations of the massive graviton and their conjugate momenta.

Note that the Fierz-Pauli tuning is crucial to the appearance of $h_{00}$ as a Lagrange multiplier. If the tuning is violated, then $h_{00}$ appears quadratically and is an auxiliary variable and no longer enforces a constraint. There are then no constraints, and the full 12 degrees of freedom in the phase space are active. The extra 2 degrees of freedom are the scalar ghost and its conjugate momentum.

### B. Free solutions and graviton mode functions

We now proceed to find the space of solutions of Eq. (2.1) and show that it transforms as a massive spin 2 representation of the Lorentz group, showing that the action propagates precisely one massive graviton. The equations of motion from Eq. (2.1) are

$$\frac{\delta S}{\delta h^{\mu\nu}} = \Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0. \quad (2.10)$$

Acting on Eq. (2.10) with $\partial^\mu$, we find, assuming $m^2 \neq 0$, the constraint $\partial^\mu h_{\mu\nu} - \partial_\nu h$. Plugging this back into the equations of motion, we find $\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0$. Taking the trace of this we find $h = 0$, which in turn implies $\partial^\mu h_{\mu\nu} = 0$. This, along with $h = 0$ applied to the equation of motion (2.10), gives $(\Box - m^2)h_{\mu\nu} = 0$.

Thus the equation of motion (2.10) implies the three equations

$$(\Box - m^2)h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \quad h = 0. \quad (2.11)$$

Conversely, it is straightforward to see that these three equations imply the equation of motion (2.10), so Eqs. (2.10) and (2.11) are equivalent. The form (2.11) makes it easy to count the degrees of freedom as well. For $D = 4$, the first of Eq. (2.11) is an evolution equation for the ten components of the symmetric tensor $h_{\mu\nu}$, and the last two are constraint equations on the initial conditions and velocities of $h_{\mu\nu}$. The last determines the trace completely, killing 1 real space degree of freedom. The second gives four initial value constraints, and the vanishing of its time derivative,

[2]Note that canonical momenta can change under integrations by parts of the time derivatives. We fixed this ambiguity by integrating by parts such as to remove all derivatives from $h_{0i}$ and $h_{00}$.

i.e., demanding that it be preserved in time, implies four more initial value constraints, thus killing 4 real space degrees of freedom. In total, we are left with the 5 real space degrees of freedom of a four-dimensional spin 2 particle, in agreement with the Hamiltonian analysis of Sec. II.A.

The first equation in Eq. (2.11) is the standard Klein-Gordon equation, with the general solution

$$h^{\mu\nu}(x) = \int \frac{d^d \mathbf{p}}{\sqrt{(2\pi)^d 2\omega_{\mathbf{p}}}} [h^{\mu\nu}(\mathbf{p}) e^{ip\cdot x} + h^{\mu\nu *}(\mathbf{p}) e^{-ip\cdot x}]. \tag{2.12}$$

Here $\mathbf{p}$ are the spatial momenta, $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, and the $D$ momenta $p^\mu$ are on shell so that $p^\mu = (\omega_{\mathbf{p}}, \mathbf{p})$.

Next we expand the Fourier coefficients $h^{\mu\nu}(\mathbf{p})$ over some basis of symmetric tensors, indexed by $\lambda$,

$$h^{\mu\nu}(\mathbf{p}) = a_{\mathbf{p},\lambda} \bar{\epsilon}^{\mu\nu}(\mathbf{p}, \lambda). \tag{2.13}$$

We fix the momentum dependence of the basis elements $\bar{\epsilon}^{\mu\nu}(\mathbf{p}, \lambda)$ by choosing some basis $\bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda)$ at the standard momentum $k^\mu = (m, 0, 0, 0, \ldots)$ and then acting with some standard boost[3] $L(p)$, which takes $k$ into $p$, $p^\mu = L(p)^\mu{}_\nu k^\nu$. This standard boost chooses for us the basis at $\mathbf{p}$, relative to that at $\mathbf{k}$. Thus we have

$$\bar{\epsilon}^{\mu\nu}(\mathbf{p}, \lambda) = L(p)^\mu{}_\alpha L(p)^\nu{}_\beta \bar{\epsilon}^{\alpha\beta}(\mathbf{k}, \lambda). \tag{2.15}$$

Imposing the conditions $\partial_\mu h^{\mu\nu} = 0$ and $h = 0$ on Eq. (2.12) then reduces to imposing

$$k_\mu \bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda) = 0, \qquad \eta_{\mu\nu} \bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda) = 0. \tag{2.16}$$

The first says that $\bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda)$ is purely spatial, i.e., $\bar{\epsilon}^{0\mu}(\mathbf{k}, \lambda) = 0$. The second says that it is traceless, so that $\bar{\epsilon}^i_i(\mathbf{k}, \lambda) = 0$ also. Thus the basis need only be a basis of symmetric traceless spatial tensors, $\lambda = 1, \ldots, d(d+1)/2 - 1$. We demand that the basis be orthonormal,

$$\bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda) \bar{\epsilon}^*_{\mu\nu}(\mathbf{k}, \lambda') = \delta_{\lambda\lambda'}. \tag{2.17}$$

This basis forms the symmetric traceless representation of the rotation group SO($d$), which is the little group for a massive particle in $D$ dimensions. If $R^\mu{}_\nu$ is a spatial rotation, we have

$$R^\mu{}_{\mu'} R^\nu{}_{\nu'} \bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda') = R^{\lambda'}{}_\lambda \bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda'), \tag{2.18}$$

where $R^{\lambda'}{}_\lambda$ is the symmetric traceless tensor representation of $R^\mu{}_{\mu'}$. We are free to use any other basis $\epsilon^{\mu\nu}(\mathbf{k}, \lambda)$, related to the $\bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda)$ by

$$\epsilon^{\mu\nu}(\mathbf{k}, \lambda) = B^{\lambda'}{}_\lambda \bar{\epsilon}^{\mu\nu}(\mathbf{k}, \lambda'), \tag{2.19}$$

where $B$ is any unitary matrix.

Given a particular spatial direction, with unit vector $\hat{k}^i$, there is an SO($d-1$) subgroup of the little group SO($d$) which leaves $\hat{k}^i$ invariant, and the symmetric traceless representation (rep) of SO($d$) breaks up into three reps of SO($d-1$), a scalar, a vector, and a symmetric traceless tensor. The scalar mode is called the longitudinal graviton and has spatial components

$$\epsilon_L^{ij} = \sqrt{\frac{d}{d-1}} \left( \hat{k}^i \hat{k}^j - \frac{1}{d} \delta^{ij} \right). \tag{2.20}$$

After a large boost in the $\hat{k}^i$ direction, it goes similar to $\epsilon_L \sim p^2/m^2$. As we see later, in the massless limit, or large boost limit, this mode is carried by a scalar field, which generally becomes strongly coupled once interactions are taken into account. The vector modes have spatial components

$$\epsilon_{V,k}^{ij} = \sqrt{2} \hat{k}^{(i} \delta_k^{j)}, \tag{2.21}$$

and after a large boost in the $\hat{k}^i$ direction, they go similar to $\epsilon_L \sim p/m$. In the massless limit, these modes are carried by a vector field, which decouples from conserved sources. The remaining linearly independent modes are symmetric traceless tensors with no components in the $\hat{k}^i$ directions, and they form the symmetric traceless mode of SO($d-1$). They are invariant under a boost in the $\hat{k}^i$ direction, and in the massless limit, they are carried by a massless graviton. In the massless limit, we therefore expect that the extra degrees of freedom of the massive graviton organize themselves into a massless vector and a massless scalar. We see later explicitly how this comes about at the Lagrangian level.

Upon boosting to $\mathbf{p}$, the polarization tensors satisfy the following properties: they are transverse to $p^\mu$ and traceless,

$$p_\mu \epsilon^{\mu\nu}(\mathbf{p}, \lambda) = 0, \qquad \eta_{\mu\nu} \epsilon^{\mu\nu}(\mathbf{p}, \lambda) = 0, \tag{2.22}$$

and they satisfy orthogonality and completeness relations

$$\epsilon^{\mu\nu}(\mathbf{p}, \lambda) \epsilon^*_{\mu\nu}(\mathbf{p}, \lambda') = \delta_{\lambda\lambda'}, \tag{2.23}$$

$$\sum_\lambda \epsilon^{\mu\nu}(\mathbf{p}, \lambda) \epsilon^{*\alpha\beta}(\mathbf{p}, \lambda) = \frac{1}{2}(P^{\mu\alpha} P^{\nu\beta} + P^{\mu\beta} P^{\nu\alpha}) - \frac{1}{D-1} P^{\mu\nu} P^{\alpha\beta}, \tag{2.24}$$

where $P^{\alpha\beta} \equiv \eta^{\alpha\beta} + p^\alpha p^\beta / m^2$. The right-hand side of the completeness relation (2.24) is the projector onto the symmetric and transverse traceless subspace of tensors, i.e., the identity on this space. We also have the following symmetry properties in $\mathbf{p}$, which can be deduced from the form of the standard boost (2.14):

$$\epsilon^{ij}(-\mathbf{p}, \lambda) = \epsilon^{ij}(\mathbf{p}, \lambda), \qquad i, j = 1, 2, \ldots, d, \tag{2.25}$$

$$\epsilon^{0i}(-\mathbf{p}, \lambda) = -\epsilon^{0i}(\mathbf{p}, \lambda), \qquad i = 1, 2, \ldots, d, \tag{2.26}$$

$$\epsilon^{00}(-\mathbf{p}, \lambda) = \epsilon^{00}(\mathbf{p}, \lambda). \tag{2.27}$$

[3] We choose the standard boost to be

$$L^i{}_j(p) = \delta_{ij} + \frac{1}{|\mathbf{p}|^2}(\gamma - 1)\mathbf{p}^i \mathbf{p}^j, \qquad L^i{}_0(p) = L^0{}_i(\mathbf{p}) = \frac{\mathbf{p}^i}{|\mathbf{p}|}\sqrt{\gamma^2 - 1}, \qquad L^0{}_0(p) = \gamma, \tag{2.14}$$

where

$$\gamma = p^0/m = \sqrt{|\mathbf{p}|^2 + m^2}/m$$

is the usual relativistic $\gamma$. See Chapter 2 of Weinberg (1995) for discussions of this standard boost and general representation theory of the Poincaré group.

The general solution to Eq. (2.10) thus reads

$$h^{\mu\nu}(x) = \int \frac{d^d\mathbf{p}}{\sqrt{(2\pi)^d 2\omega_{\mathbf{p}}}} \sum_\lambda a_{\mathbf{p},\lambda}\epsilon^{\mu\nu}(\mathbf{p},\lambda)e^{ip\cdot x} + a^*_{\mathbf{p},\lambda}\epsilon^{*\mu\nu}(\mathbf{p},\lambda)e^{-ip\cdot x}. \tag{2.28}$$

The solution is a general linear combination of the following mode functions and their conjugates

$$u^{\mu\nu}_{\mathbf{p},\lambda}(x) \equiv \frac{1}{\sqrt{(2\pi)^d 2\omega_{\mathbf{p}}}}\epsilon^{\mu\nu}(\mathbf{p},\lambda)e^{ip\cdot x}, \quad \lambda = 1, 2, \ldots, d. \tag{2.29}$$

These are the solutions representing gravitons, and they have the following Poincaré transformation properties:

$$u^{\mu\nu}_{\mathbf{p},\lambda}(x-a) = u^{\mu\nu}_{\mathbf{p},\lambda}(x)e^{-ip\cdot a}, \tag{2.30}$$

$$\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}u^{\mu'\nu'}_{\mathbf{p},\lambda}(\Lambda^{-1}x) = \sqrt{\frac{\omega_{\Lambda\mathbf{p}}}{\omega_{\mathbf{p}}}}W(\Lambda,p)_{\lambda'\lambda}u^{\mu\nu}_{\Lambda\mathbf{p},\lambda'}(x), \tag{2.31}$$

where $W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)$ is the Wigner rotation, and $W(\Lambda,p)_{\lambda'\lambda}$ is its spin 2 rep, $R^\mu{}_\nu \to (B^{-1}RB)_{\lambda'\lambda}$.[4] Thus the gravitons are spin 2 solutions.

In terms of the modes, the general solution reads

$$h^{\mu\nu}(x) = \int d^d\mathbf{p}\sum_\lambda [a_{\mathbf{p},\lambda}u^{\mu\nu}_{\mathbf{p},\lambda}(x) + a^*_{\mathbf{p},\lambda}u^{\mu\nu *}_{\mathbf{p},\lambda}(x)]. \tag{2.32}$$

The inner (symplectic) product on the space of solutions to the equations of motion is

$$(h, h') = i\int d^d\mathbf{x}\, h^{\mu\nu *}(x)\overleftrightarrow{\partial}_0 h'_{\mu\nu}(x)|_{t=0}. \tag{2.33}$$

The $u$ functions are orthonormal with respect to this product,

$$(u_{\mathbf{p},\lambda}, u_{\mathbf{p}',\lambda'}) = \delta^d(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}, \tag{2.34}$$

$$(u^*_{\mathbf{p},\lambda}, u^*_{\mathbf{p}',\lambda'}) = -\delta^d(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}, \tag{2.35}$$

$$(u_{\mathbf{p},\lambda}, u^*_{\mathbf{p}',\lambda'}) = 0, \tag{2.36}$$

and we can use the product to extract the $a$ and $a^*$ coefficients from any solution $h_{\mu\nu}(x)$,

$$a_{\mathbf{p},\lambda} = (u_{\mathbf{p},\lambda}, h), \tag{2.37}$$

$$a^*_{\mathbf{p},\lambda} = -(u^*_{\mathbf{p},\lambda}, h). \tag{2.38}$$

In the quantum theory, the $a$ and $a^*$ become creation and annihilation operators which satisfy the usual commutation relations and produce massive spin 2 states. The fields $h_{ij}$ and their canonical momenta $\pi_{ij}$, constructed from the $a$ and $a^*$, will then automatically satisfy the Dirac algebra and constraints of the Hamiltonian analysis of Sec. II.A, providing a quantization of the system. Once interactions are taken into account, external lines of Feynman diagrams will get a factor of the mode functions (2.29).

### C. Propagator

Integrating by parts, we can rewrite the Fierz-Pauli action (2.1) as

$$S = \int d^D x \frac{1}{2}h_{\mu\nu}\mathcal{O}^{\mu\nu,\alpha\beta}h_{\alpha\beta}, \tag{2.39}$$

where

$$\mathcal{O}^{\mu\nu}{}_{\alpha\beta} = (\eta^{(\mu}{}_\alpha\eta^{\nu)}{}_\beta - \eta^{\mu\nu}\eta_{\alpha\beta})(\Box - m^2) - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} + \partial^\mu\partial^\nu\eta_{\alpha\beta} + \partial_\alpha\partial_\beta\eta^{\mu\nu}, \tag{2.40}$$

is a second order differential operator satisfying

$$\mathcal{O}^{\mu\nu,\alpha\beta} = \mathcal{O}^{\nu\mu,\alpha\beta} = \mathcal{O}^{\mu\nu,\beta\alpha} = \mathcal{O}^{\alpha\beta,\mu\nu}. \tag{2.41}$$

In terms of this operator, the equation of motion (2.10) can be written simply as $\delta S/\delta h_{\mu\nu} = \mathcal{O}^{\mu\nu,\alpha\beta}h_{\alpha\beta} = 0$.

To derive the propagator, we go to momentum space,

$$\mathcal{O}^{\mu\nu}{}_{\alpha\beta}(\partial \to ip) = -(\eta^{(\mu}{}_\alpha\eta^{\nu)}{}_\beta - \eta^{\mu\nu}\eta_{\alpha\beta})(p^2 + m^2) + 2p^{(\mu}p_{(\alpha}\eta^{\nu)}{}_{\beta)} - p^\mu p^\nu\eta_{\alpha\beta} - p_\alpha p_\beta\eta^{\mu\nu}. \tag{2.42}$$

The propagator is the operator $\mathcal{D}_{\alpha\beta,\sigma\lambda}$ with the same symmetries Eq. (2.41) which satisfies

$$\mathcal{O}^{\mu\nu,\alpha\beta}\mathcal{D}_{\alpha\beta,\sigma\lambda} = \frac{i}{2}(\delta^\mu_\sigma\delta^\nu_\lambda + \delta^\nu_\sigma\delta^\mu_\lambda). \tag{2.43}$$

[4] We show the Lorentz transformation property as follows:

$$\begin{aligned}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\epsilon^{\mu'\nu'}(p,\lambda)e^{ip\cdot\Lambda^{-1}x} &= [\Lambda L(p)]^\mu{}_{\mu'}[\Lambda L(p)]^\nu{}_{\nu'}\epsilon^{\mu'\nu'}(k,\lambda)e^{i\Lambda p\cdot x} \\ &= [L(\Lambda p)(L^{-1}(\Lambda p)\Lambda L(p))]^\mu{}_{\mu'}[L(\Lambda p)(L^{-1}(\Lambda p)\Lambda L(p))]^\nu{}_{\nu'}\epsilon(k,\lambda)^{\mu'\nu'}e^{i\Lambda p\cdot x} \\ &= [L(\Lambda p)W(\Lambda,p)]^\mu{}_{\mu'}[L(\Lambda p)W(\Lambda,p)]^\nu{}_{\nu'}\epsilon(k,\lambda)^{\mu'\nu'}e^{i\Lambda p\cdot x}.\end{aligned}$$

The little group element is a spatial rotation. For any spatial rotation $R^\mu{}_\nu$, we have

$$R^\mu{}_{\mu'}R^\nu{}_{\nu'}\epsilon^{\mu'\nu'}(k,\lambda) = R^\mu{}_{\mu'}R^\nu{}_{\nu'}B^{\lambda'}{}_\lambda\bar\epsilon^{\mu'\nu'}(\mathbf{k},\lambda') = B^{\lambda'}{}_\lambda R^{\lambda''}{}_{\lambda'}\bar\epsilon^{\mu\nu}(k,\lambda'') = (B^{-1}RB)^{\lambda'}{}_\lambda\epsilon^{\mu\nu}(k,\lambda').$$

Plugging back into the above,

$$\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\epsilon^{\mu'\nu'}(p,\lambda)e^{ip\cdot\Lambda^{-1}x} = L(\Lambda p)^\mu{}_{\mu'}L(\Lambda p)^\nu{}_{\nu'}W(\Lambda,p)^{\lambda'}{}_\lambda\epsilon^{\mu'\nu'}(k,\lambda')e^{i\Lambda p\cdot x} = W(\Lambda,p)^{\lambda'}{}_\lambda\epsilon^{\mu\nu}(\Lambda p,\lambda')e^{i\Lambda p\cdot x},$$

where $W$ is the spin 2 representation of the little group in a basis rotated by $B$, $W = B^{-1}RB$.

The right side is the identity operator on the space of symmetric tensors.

Solving Eq. (2.43), we find

$$\mathcal{D}_{\alpha\beta,\sigma\lambda} = \frac{-i}{p^2+m^2}\left[\frac{1}{2}(P_{\alpha\sigma}P_{\beta\lambda}+P_{\alpha\lambda}P_{\beta\sigma}) - \frac{1}{D-1}P_{\alpha\beta}P_{\sigma\lambda}\right], \tag{2.44}$$

where $P_{\alpha\beta} \equiv \eta_{\alpha\beta} + p_\alpha p_\beta/m^2$.

In the interacting quantum theory, internal lines with momentum $p$ will be assigned this propagator, which for large $p$ behaves as $\sim p^2/m^4$. This growth with $p$ means we cannot apply standard power counting arguments [such as those of Chapter 12 of Weinberg (1995)] to deduce the renormalizability properties or strong coupling scales of a theory. We see later how to overcome this difficulty by rewriting the theory in a way in which all propagators go similar to $\sim 1/p^2$ at high energy.

The massive graviton propagator (2.44) can be compared to the propagator for the case $m = 0$. For $m = 0$, the action becomes

$$S_{m=0} = \int d^Dx \frac{1}{2} h_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h_{\alpha\beta}, \tag{2.45}$$

where the kinetic operator is

$$\begin{aligned}\mathcal{E}^{\mu\nu}{}_{\alpha\beta} &= \mathcal{O}^{\mu\nu}{}_{\alpha\beta}|_{m=0} \\ &= (\eta^{(\mu}{}_\alpha\eta^{\nu)}{}_\beta - \eta^{\mu\nu}\eta_{\alpha\beta})\Box - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} \\ &\quad + \partial^\mu\partial^\nu\eta_{\alpha\beta} + \partial_\alpha\partial_\beta\eta^{\mu\nu}.\end{aligned} \tag{2.46}$$

This operator has the symmetries (2.41). Acting on a symmetric tensor $Z_{\mu\nu}$ it reads

$$\begin{aligned}\epsilon^{\mu\nu,\alpha\beta}Z_{\alpha\beta} &= \Box Z^{\mu\nu} - \eta^{\mu\nu}\Box Z - 2\partial^{(\mu}\partial_\alpha Z^{\nu)\alpha} + \partial^\mu\partial^\nu Z \\ &\quad + \eta^{\mu\nu}\partial_\alpha\partial_\beta Z^{\alpha\beta}.\end{aligned} \tag{2.47}$$

The $m = 0$ action has the gauge symmetry (2.2), and the operator (2.46) is not invertible. Acting with it results in a tensor which is automatically transverse, and it annihilates anything which is pure gauge

$$\partial_\mu(\epsilon^{\mu\nu,\alpha\beta}Z_{\alpha\beta}) = 0, \quad \epsilon^{\mu\nu,\alpha\beta}(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha) = 0. \tag{2.48}$$

To find a propagator, we must fix the gauge freedom. We choose the Lorenz gauge (also called harmonic, or de Donder gauge),

$$\partial^\mu h_{\mu\nu} - \tfrac{1}{2}\partial_\nu h = 0. \tag{2.49}$$

We reach this gauge by making a gauge transformation with $\xi_\mu$ chosen to satisfy $\Box\xi_\mu = -(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h)$. This condition fixes the gauge only up to gauge transformations with parameter $\xi_\mu$ satisfying $\Box\xi_\mu = 0$. In this gauge, the equations of motion simplify to

$$\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = 0. \tag{2.50}$$

The solutions to this equation which also satisfy the gauge condition (2.49) are the Lorenz gauge solutions to the original equations of motion.

To the Lagrangian of Eq. (2.45) we add the following gauge fixing term:

$$\mathcal{L}_{\rm GF} = -(\partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h)^2. \tag{2.51}$$

Quantum mechanically, this results from the Fadeev-Popov gauge fixing procedure. We have

$$\mathcal{L} + \mathcal{L}_{\rm GF} = \tfrac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - \tfrac{1}{4}h\Box h, \tag{2.52}$$

whose equations of motion are (2.50). Note, however, that the classical gauge condition we have been using is not obtained as an equation of motion and must be imposed separately if solutions are to be compared.

We can write the gauge fixed Lagrangian as $\mathcal{L} + \mathcal{L}_{\rm GF} = \frac{1}{2}h_{\mu\nu}\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta}h_{\alpha\beta}$, where

$$\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta} = \Box[\tfrac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}]. \tag{2.53}$$

Going to momentum space and inverting, we obtain the propagator

$$\mathcal{D}_{\alpha\beta,\sigma\lambda} = \frac{-i}{p^2}\left[\frac{1}{2}(\eta_{\alpha\sigma}\eta_{\beta\lambda}+\eta_{\alpha\lambda}\eta_{\beta\sigma}) - \frac{1}{D-2}\eta_{\alpha\beta}\eta_{\sigma\lambda}\right], \tag{2.54}$$

which satisfies the Eq. (2.43) with $\tilde{\mathcal{O}}$ in place of $\mathcal{O}$. This propagator grows similar to $\sim 1/p^2$ at high energy. Comparing the massive and massless propagators, Eqs. (2.44) and (2.54), and ignoring for a second the terms in Eq. (2.44) which are singular as $m \to 0$, there is a difference in coefficient for the last term, even as $m \to 0$. For $D = 4$, it is 1/2 vs 1/3. This is the first sign of a discontinuity in the $m \to 0$ limit.

## III. LINEAR RESPONSE TO SOURCES

We now add a fixed external symmetric source $T^{\mu\nu}(x)$ to the action (2.1),

$$\begin{aligned}S = \int d^Dx &- \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h \\ &+ \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu}.\end{aligned} \tag{3.1}$$

Here $\kappa = M_P^{-(D-2)/2}$ is the coupling strength to the source.[5]

The equations of motion are now sourced by $T_{\mu\nu}$,

$$\begin{aligned}&\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} \\ &\quad + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = -\kappa T_{\mu\nu}.\end{aligned} \tag{3.2}$$

In the case $m = 0$, acting on the left with $\partial^\mu$ gives identically zero, so we must have the conservation condition $\partial^\mu T_{\mu\nu} = 0$ if there is to be a solution. For $m \neq 0$, there is no such condition.

[5]The normalizations chosen here are in accord with the general relativity definition $T^{\mu\nu} = (2/\sqrt{-g})\delta\mathcal{L}/\delta g_{\mu\nu}$, as well as the normalization $\delta g_{\mu\nu} = 2\kappa h_{\mu\nu}$.

### A. General solution to the sourced equations

We now find the retarded solution of Eq. (3.2), to which the homogeneous solutions of Eq. (2.2) can be added to obtain the general solution. Acting on the equation of motion (3.2) with $\partial^\mu$, we find

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu}. \tag{3.3}$$

Plugging this back into Eq. (3.2), we find

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = -\kappa T_{\mu\nu} + \frac{\kappa}{m^2}[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \eta_{\mu\nu}\partial\partial T],$$

where $\partial\partial T$ is short for the double divergence $\partial_\mu\partial_\nu T^{\mu\nu}$. Taking the trace of this we find

$$h = -\frac{\kappa}{m^2(D-1)}T - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T. \tag{3.4}$$

Applying this to Eq. (3.3), we find

$$\partial^\mu h_{\mu\nu} = -\frac{\kappa}{m^2(D-1)}\partial_\nu T + \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial_\nu\partial\partial T, \tag{3.5}$$

which when applied along with Eq. (3.4) to the equations of motion gives

$$(\partial^2 - m^2)h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right] + \frac{\kappa}{m^2}\left[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \frac{1}{D-1}\left(\eta_{\mu\nu} + (D-2)\frac{\partial_\mu\partial_\nu}{m^2}\right)\partial\partial T\right]. \tag{3.6}$$

Thus we have seen that the equation of motion (3.2) implies the following three equations:

$$\begin{aligned}
(\Box - m^2)h_{\mu\nu} &= -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right] + \frac{\kappa}{m^2}\left[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \frac{1}{D-1}\left(\eta_{\mu\nu} + (D-2)\frac{\partial_\mu\partial_\nu}{m^2}\right)\partial\partial T\right],\\
\partial^\mu h_{\mu\nu} &= -\frac{\kappa}{m^2(D-1)}\partial_\nu T + \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial_\nu\partial\partial T,\\
h &= -\frac{\kappa}{m^2(D-1)}T - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T.
\end{aligned} \tag{3.7}$$

Conversely, it is straightforward to see that these three equations imply the equation of motion (3.2).

Taking the first equation of (3.7) and tracing, we find

$$(\Box - m^2)\left[h + \frac{\kappa}{m^2(D-1)}T + \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T\right] = 0.$$

Under the assumption that $(\partial^2 - m^2)f = 0 \Rightarrow f = 0$ for any function $f$, the third equation is implied. This will be the case with good boundary conditions, such as the retarded boundary conditions we impose when we are interested in the classical response to sources. The second equation of (3.7) can also be shown to follow under this assumption, so that we may obtain the solution by Fourier transforming only the first equation of (3.7). This solution can also be obtained by applying the propagator (2.44) to the Fourier transform of the source.

Despite the absence of gauge symmetry, we will often be interested in sources which are conserved anyway, $\partial_\mu T^{\mu\nu} = 0$. When the source is conserved, and under the assumptions in the paragraph above, we are left with just the equation

$$(\partial^2 - m^2)h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right]. \tag{3.8}$$

The general solution for a conserved source is then

$$h_{\mu\nu}(x) = \kappa\int\frac{d^Dp}{(2\pi)^D}e^{ipx}\frac{1}{p^2+m^2} \times\left[T_{\mu\nu}(p) - \frac{1}{D-1}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)T(p)\right], \tag{3.9}$$

where $T^{\mu\nu}(p)$ is the Fourier transform of the source, $T^{\mu\nu}(p) = \int d^Dx e^{-ipx}T^{\mu\nu}(x)$. To get the retarded field, we integrate above the poles in the $p^0$ plane.

### B. Solution for a point source

We now specialize to four dimensions so that $\kappa = 1/M_P$, and we consider as a source the stress tensor of a mass $M$ point particle at rest at the origin

$$T^{\mu\nu}(x) = M\delta_0^\mu\delta_0^\nu\delta^3(\mathbf{x}), \quad T^{\mu\nu}(p) = 2\pi M\delta_0^\mu\delta_0^\nu\delta(p^0). \tag{3.10}$$

Note that this source is conserved. For this source, the general solution (3.9) gives

$$\begin{aligned}
h_{00}(x) &= \frac{2M}{3M_P}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\mathbf{p}^2+m^2},\\
h_{0i}(x) &= 0,\\
h_{ij}(x) &= \frac{M}{3M_P}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\mathbf{p}^2+m^2}\left(\delta_{ij} + \frac{p_ip_j}{m^2}\right).
\end{aligned} \tag{3.11}$$

Using the formulas

$$\begin{aligned}
\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\mathbf{p}^2+m^2} &= \frac{1}{4\pi}\frac{e^{-mr}}{r},\\
\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{p_ip_j}{\mathbf{p}^2+m^2} &= -\partial_i\partial_j\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\mathbf{p}^2+m^2}\\
&= \frac{1}{4\pi}\frac{e^{-mr}}{r}\left[\frac{1}{r^2}(1+mr)\delta_{ij} - \frac{1}{r^4}(3+3mr+m^2r^2)x_ix_j\right],
\end{aligned} \tag{3.12}$$

where $r \equiv \sqrt{x_i x_i}$, we have

$$h_{00}(x) = \frac{2M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \qquad h_{0i}(x) = 0,$$
$$h_{ij}(x) = \frac{M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r}\left[\frac{1+mr+m^2r^2}{m^2r^2}\delta_{ij} - \frac{1}{m^2r^4}(3+3mr+m^2r^2)x_ix_j\right]. \tag{3.13}$$

Note the Yukawa suppression factors $e^{-mr}$ characteristic of a massive field.

For future reference, it is convenient to record these expressions in spherical coordinates for the spatial variables. Using the formula $[F(r)\delta_{ij} + G(r)x_ix_j]dx^idx^j = [F(r) + r^2G(r)]dr^2 + F(r)r^2d\Omega^2$ to get to spherical coordinates we find

$$h_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2d\Omega^2, \tag{3.14}$$

where

$$B(r) = -\frac{2M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r},$$
$$C(r) = -\frac{2M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2}, \tag{3.15}$$
$$A(r) = \frac{M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2}.$$

In the limit $r \ll 1/m$ these reduce to

$$B(r) = -\frac{2M}{3M_P}\frac{1}{4\pi r},$$
$$C(r) = -\frac{2M}{3M_P}\frac{1}{4\pi m^2 r^3}, \tag{3.16}$$
$$A(r) = \frac{M}{3M_P}\frac{1}{4\pi m^2 r^3}.$$

Corrections are suppressed by powers of $mr$.

For comparison, we compute the point source solution for the massless case as well. We choose the Lorenz gauge (2.49). In this gauge, the equations of motion simplify to

$$\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu}. \tag{3.17}$$

Taking the trace, we find $\Box h = [2/(D-2)]\kappa T$, and upon substituting back, we get

$$\Box h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T\right]. \tag{3.18}$$

This equation, along with the Lorenz gauge condition (2.49), is equivalent to the original equation of motion in the Lorenz gauge.

Taking $\partial^\mu$ on Eq. (3.17) and on its trace, using conservation of $T_{\mu\nu}$ and comparing, we have $\Box(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h) = 0$, so that the Lorentz condition is automatically satisfied when boundary conditions are satisfied with the property that $\Box f = 0 \Rightarrow f = 0$ for any function $f$, as is the case when we impose retarded boundary conditions. We can then solve Eq. (3.17) by Fourier transforming.

$$h_{\mu\nu}(x) = \kappa\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\frac{1}{p^2}\left[T_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}T(p)\right], \tag{3.19}$$

where $T^{\mu\nu}(p) = \int d^Dx e^{-ip\cdot x}T^{\mu\nu}(x)$ is the Fourier transform of the source. To get the retarded field, we integrate above the poles in the $p^0$ plane.

Now we specialize to $D = 4$, and we consider as a source the point particle of mass $M$ at the origin (3.10). For this source, the general solution (3.19) gives

$$h_{00}(x) = \frac{M}{2M_P}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2} = \frac{M}{2M_P}\frac{1}{4\pi r},$$
$$h_{0i}(x) = 0, \tag{3.20}$$
$$h_{ij}(x) = \frac{M}{2M_P}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2}\delta_{ij} = \frac{M}{2M_P}\frac{1}{4\pi r}\delta_{ij}.$$

For later reference, we record this result in spherical spatial coordinates as well. Using the formula $[F(r)\delta_{ij} + G(r)x_ix_j]dx^idx^j = [F(r) + r^2G(r)]dr^2 + F(r)r^2d\Omega^2$ to get to spherical coordinates we find a metric of the form (3.14) with

$$B(r) = -\frac{M}{2M_P}\frac{1}{4\pi r}, \qquad C(r) = \frac{M}{2M_P}\frac{1}{4\pi r},$$
$$A(r) = \frac{M}{2M_P}\frac{1}{4\pi r}. \tag{3.21}$$

### C. The vDVZ discontinuity

We now extract some physical predictions from the point source solution. Assume we have a test particle moving in this field, and that this test particle responds to $h_{\mu\nu}$ in the same way that a test particle in general relativity responds to the metric deviation $\delta g_{\mu\nu} = (2/M_P)h_{\mu\nu}$. We know from the textbooks [see, for example, Chapter 7 of Carroll (2004)] that if $h_{\mu\nu}$ takes the form $2h_{00}/M_P = -2\phi$, $2h_{ij}/M_P = -2\psi\delta_{ij}$, $h_{0i} = 0$ for some functions $\phi(r)$ and $\psi(r)$, then the Newtonian potential experienced by the particle is given by $\phi(r)$. Furthermore, if $\psi(r) = \gamma\phi(r)$ for some constant $\gamma$, called the parametrized post-Newtonian (PPN) parameter, and if $\phi(r) = -k/r$ for some constant $k$, then the angle for the bending of light at impact parameter $b$ around the heavy source is given by $\alpha = 2(1+\gamma)/b$. Looking at Eq. (3.20), the massless graviton gives us the values

$$\phi = -\frac{GM}{r}, \quad \psi = -\frac{GM}{r}, \quad \text{massless graviton}, \tag{3.22}$$

using $1/M_P^2 = 8\pi G$. The PPN parameter is therefore $\gamma = 1$ and the magnitude of the light bending angle for light incident at impact parameter $b$ is

$$\alpha = \frac{4GM}{b}, \quad \text{massless graviton}. \tag{3.23}$$

For the massive case, the metric (3.13) is not quite in the right form to read off the Newtonian potential and light bending. To simplify things, we notice that while the massive gravity action is not gauge invariant, we assumed that the coupling to the test particle is that of GR, so this coupling is gauge invariant. Thus we are free to make a gauge transformation on the solution $h_{\mu\nu}$, and there will be no effect on the test particle. To simplify the metric (3.13), we go back to Eq. (3.11) and notice that the $p_i p_j/m^2$ term in $h_{ij}$ is pure gauge, so we can ignore this term. Thus our metric is gauge equivalent to the metric

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{3M_P}\frac{1}{4\pi}\frac{e^{-mr}}{r}\delta_{ij}. \end{aligned} \tag{3.24}$$

We then have, in the small mass limit,

$$\begin{aligned} \phi &= -\frac{4}{3}\frac{GM}{r}, \\ \psi &= -\frac{2}{3}\frac{GM}{r}\delta_{ij}, \quad \text{massive graviton.} \end{aligned} \tag{3.25}$$

These are the same values as obtained for the $\omega = 0$ Brans-Dicke theory. The Newtonian potential is larger than for the massless case. The PPN parameter is $\gamma = \frac{1}{2}$, and the magnitude of the light bending angle for light incident at impact parameter $b$ is the same as in the massless case,

$$\alpha = \frac{4GM}{b}, \quad \text{massive graviton.} \tag{3.26}$$

If we want, we can make the Newtonian potential agree with GR by scaling $G \to \frac{3}{4}G$. Then the light bending would change to $\alpha = 3GM/b$, off by 25% from GR.

What this all means is that linearized massive gravity, even in the limit of zero mass, gives predictions which are order 1 different from linearized GR. If nature were described by either one or the other of these theories, we would, by making a finite measurement, be able to tell whether the graviton mass is mathematically zero or not, in violation of our intuition that the physics of nature should be continuous in its parameters. This is the vDVZ discontinuity (van Dam and Veltman, 1970; Zakharov, 1970) [see also Iwasaki (1970) and Carrera and Giulini (2001)]. It is present in other physical predictions as well, such as the emission of gravitational radiation (van Nieuwenhuizen, 1973b).

## IV. THE STÜCKELBERG TRICK

We have seen that there is a discontinuity in the physical predictions of linear massless gravity and the massless limit of linear massive gravity, known as the vDVZ discontinuity. In this section, we expose the origin of this discontinuity. We see explicitly that the correct massless limit of massive gravity is not massless gravity, but rather massless gravity plus extra degrees of freedom, as expected since the gauge symmetry which kills the extra degrees of freedom appears only when the mass is strictly zero. The extra degrees of freedom are a massless vector and a massless scalar which couples to the trace of the energy momentum tensor. This extra scalar coupling is responsible for the vDVZ discontinuity.

Taking $m \to 0$ straight away in the Lagrangian (3.1) does not yield a smooth limit, because degrees of freedom are lost. To find the correct limit, the trick is to introduce new fields and gauge symmetries into the massive theory in a way that does not alter the theory. This is the Stückelberg trick. Once this is done, a limit can be found in which no degrees of freedom are gained or lost.

### A. Vector example

To introduce the idea, we consider a simpler case, the theory of a massive photon $A_\mu$ coupled to a (not necessarily conserved) source $J_\mu$,

$$S = \int d^D x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{4.1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The mass term breaks the would-be gauge invariance $\delta A_\mu = \partial_\mu \Lambda$, and for $D = 4$ this theory describes the 3 degrees of freedom of a massive spin 1 particle. Recall that the propagator for a massive vector is $[-i/(p^2 + m^2)](\eta_{\mu\nu} + p_\mu p_\nu/m^2)$, which is similar to $\sim 1/m^2$ for large momenta, invalidating the usual power counting arguments.

As it stands, the limit $m \to 0$ of the Lagrangian (4.1) is not a smooth limit because we lose a degree of freedom; for $m = 0$ we have Maxwell electromagnetism which in $D = 4$ propagates only 2 degrees of freedom, the two polarizations of a massless helicity 1 particle. Also, the limit does not exist unless the source is conserved, as this is a consistency requirement in the massless case.

The Stückelberg trick consists of introducing a new scalar field $\phi$, in such a way that the new action has gauge symmetry but is still dynamically equivalent to the original action. It will expose a different $m \to 0$ limit which is smooth, in that no degrees of freedom are gained or lost. We introduce a field $\phi$ by making the replacement

$$A_\mu \to A_\mu + \partial_\mu \phi, \tag{4.2}$$

following the pattern of the gauge symmetry we want to introduce (Stückelberg, 1957). This is emphatically not a change of field variables. It is not a decomposition of $A_\mu$ into transverse and longitudinal parts ($A_\mu$ is not meant to identically satisfy $\partial_\mu A^\mu = 0$ after the replacement), and it is not a gauge transformation [the Lagrangian (4.1) is not gauge invariant]. Rather, this is creating a new Lagrangian from the old one, by the addition of a new field $\phi$. $F_{\mu\nu}$ is invariant under this replacement, since the replacement looks similar to a gauge transformation and $F_{\mu\nu}$ is gauge invariant. The only thing that changes is the mass term and the coupling to the source,

$$\begin{aligned} S = \int d^D x &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 (A_\mu + \partial_\mu \phi)^2 \\ &+ A_\mu J^\mu - \phi \partial_\mu J^\mu. \end{aligned} \tag{4.3}$$

We have integrated by parts in the coupling to the source. The new action now has the gauge symmetry

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = -\Lambda. \tag{4.4}$$

By fixing the gauge $\phi = 0$, called the unitary gauge (a gauge condition for which it is permissible to substitute back into the action, because the potentially lost $\phi$ equation is implied by the divergence of the $A_\mu$ equation), we recover the original massive Lagrangian (4.1), which means Eqs. (4.1) and (4.3) are equivalent theories. They both describe the 3 degrees of freedom of a massive spin 1 in $D = 4$. The new Lagrangian (4.3) does the job using more fields and gauge symmetry.

The Stückelberg trick is a terrific illustration of the fact that gauge symmetry is a complete sham. It represents nothing more than a redundancy of description. We can take any theory and make it a gauge theory by introducing redundant variables. Conversely, given any gauge theory, we can always eliminate the gauge symmetry by eliminating the redundant degrees of freedom. The catch is that removing the redundancy is not always a smart thing to do. For example, in Maxwell electromagnetism it is impossible to remove the redundancy and at the same time preserve manifest Lorentz invariance and locality. Of course, electromagnetism with gauge redundancy removed is still electromagnetism, so it is still Lorentz invariant and local, it is just not manifestly so. With the Stückelberg trick presented here, on the other hand, we are adding and removing extra gauge symmetry in a rather simple way, which does not mess with the manifest Lorentz invariance and locality.

We see from Eq. (4.3) that $\phi$ has a kinetic term, in addition to cross terms. Rescaling $\phi \to m^{-1}\phi$ in order to normalize the kinetic term, we have

$$S = \int d^D x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - m A_\mu \partial^\mu \phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu - \frac{1}{m}\phi\partial_\mu J^\mu, \tag{4.5}$$

and the gauge symmetry reads

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = -m\Lambda. \tag{4.6}$$

Consider now the $m \to 0$ limit. Note that if the current is not conserved [or its divergence does not go to zero with at least a power of $m$ (Fronsdal, 1980)], then the scalar becomes strongly coupled to the divergence of the source and the limit does not exist. Assuming a conserved source, the Lagrangian becomes in the limit

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu, \tag{4.7}$$

and the gauge symmetry is

$$\delta A_\mu = \partial_\mu\Lambda, \qquad \delta\phi = 0. \tag{4.8}$$

It is now clear that the number of degrees of freedom is preserved in the limit. For $D = 4$ two of the 3 degrees of freedom go into the massless vector, and one goes into the scalar.

In the limit the vector decouples from the scalar, and we are left with a massless gauge vector interacting with the source, as well as a completely decoupled free scalar. This $m \to 0$ limit is a different limit than the nonsmooth limit we would have obtained by taking $m \to 0$ straight away in Eq. (4.1). We have scaled $\phi \to m^{-1}\phi$ in order to canonically normalize the scalar kinetic term, so we are actually using a new scalar $\phi_{\text{new}} = m\phi_{\text{old}}$ which does not scale with $m$, so the smooth limit we are taking is to scale the old scalar degree of freedom up as we scale $m$ down, in such a way that the new scalar degree of freedom remains preserved.

Rather than unitary gauge, we can instead fix a Lorentz-like gauge for the action (4.3),

$$\partial_\mu A^\mu + m\phi = 0. \tag{4.9}$$

This gauge fixes the gauge freedom up to a residual gauge parameter satisfying $(\Box - m^2)\Lambda = 0$. We can add the gauge fixing term

$$S_{\text{GF}} = \int d^D x - \frac{1}{2}(\partial_\mu A^\mu + m\phi)^2. \tag{4.10}$$

As in the massless case, quantum mechanically this term results from the Fadeev-Popov gauge fixing procedure. Adding the gauge fixing term diagonalizes the Lagrangian,

$$S + S_{\text{GF}} = \int d^D x \frac{1}{2}A_\mu(\Box - m^2)A^\mu + \frac{1}{2}\phi(\Box - m^2)\phi + A_\mu J^\mu - \frac{1}{m}\phi\partial_\mu J^\mu, \tag{4.11}$$

and the propagators for $A_\mu$ and $\phi$ are, respectively,

$$\frac{-i\eta_{\mu\nu}}{p^2 + m^2}, \qquad \frac{-i}{p^2 + m^2}, \tag{4.12}$$

which are similar to $\sim 1/p^2$ at high momenta. Thus we have managed to restore the good high energy behavior of the propagators.

It is possible to find the gauge invariant mode functions for $A_\mu$ and $\phi$, which can then be compared to the unitary gauge mode functions of the massive photon. In the massless limit, there is a direct correspondence; $\phi$ is gauge invariant and becomes the longitudinal photon, $A_\mu$ has the usual Maxwell gauge symmetry and its gauge invariant transverse modes are exactly the transverse modes of the massive photon.

### B. Graviton Stückelberg and origin of the vDVZ discontinuity

Now consider massive gravity,

$$S = \int d^D x \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu}, \tag{4.13}$$

where $\mathcal{L}_{m=0}$ is the Lagrangian of the massless graviton. We want to preserve the gauge symmetry $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ present in the $m = 0$ case, so we introduce a Stückelberg field $A_\mu$ patterned after the gauge symmetry,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu. \tag{4.14}$$

The $\mathcal{L}_{m=0}$ term remains invariant because it is gauge invariant and Eq. (4.14) looks like a gauge transformation, so all that changes is the mass term,

$$S = \int d^D x \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} - 2m^2(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) + \kappa h_{\mu\nu}T^{\mu\nu} - 2\kappa A_\mu\partial_\nu T^{\mu\nu}, \tag{4.15}$$

where we have integrated by parts in the last term, and where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

There is now a gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta A_\mu = -\xi_\mu, \tag{4.16}$$

and fixing the gauge $A_\mu = 0$ recovers the original massive gravity action (as in the vector case, this is a gauge condition for which it is permissible to substitute back into the action, because the potentially lost $A_\mu$ equation is implied by the divergence of the $h_{\mu\nu}$ equation). At this point, we might consider scaling $A_\mu \to m^{-1} A_\mu$ to normalize the vector kinetic term, and then take the $m \to 0$ limit. In this limit, we end up with a massless graviton and a massless photon, for a total of 4 degrees of freedom (in four dimensions). So at this point, $m \to 0$ is still not a smooth limit, since we would be losing one of the original 5 degrees of freedom.

We go one step further and introduce a scalar gauge symmetry, by introducing another Stückelberg field $\phi$,

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{4.17}$$

The action (4.15) now becomes

$$\begin{aligned} S = \int d^D x \mathcal{L}_{m=0} &- \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \\ &- \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} - 2m^2 (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) \\ &- 2m^2 (h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \partial^2 \phi) + \kappa h_{\mu\nu} T^{\mu\nu} \\ &- 2\kappa A_\mu \partial_\nu T^{\mu\nu} + 2\kappa \phi \partial\partial T, \end{aligned} \tag{4.18}$$

where $\partial\partial T \equiv \partial_\mu \partial_\nu T^{\mu\nu}$ and we have integrated by parts in the last term.

There are now two gauge symmetries

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta A_\mu = -\xi_\mu, \tag{4.19}$$

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = -\Lambda. \tag{4.20}$$

By fixing the gauge $\phi = 0$ we recover the Lagrangian (4.15).

Suppose we now rescale $A_\mu \to (1/m) A_\mu$, $\phi \to (1/m^2)\phi$, under which the action becomes

$$\begin{aligned} S = \int d^D x \mathcal{L}_{m=0} &- \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \\ &- 2m (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) \\ &- 2 (h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \partial^2 \phi) + \kappa h_{\mu\nu} T^{\mu\nu} \\ &- \frac{2}{m} \kappa A_\mu \partial_\nu T^{\mu\nu} + \frac{2}{m^2} \kappa \phi \partial\partial T, \end{aligned} \tag{4.21}$$

and the gauge transformations become

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta A_\mu = -m \xi_\mu, \\ \delta A_\mu &= \partial_\mu \Lambda, \qquad \delta\phi = -m\Lambda, \end{aligned} \tag{4.22}$$

where we have absorbed one factor on $m$ into the gauge parameter $\Lambda$.

Now take the $m \to 0$ limit. [If the source is not conserved and the divergences do not go to zero fast enough with $m$ (Fronsdal, 1980), then $\phi$ and $A_\mu$ become strongly coupled to the divergence of the source, so we now assume the source is conserved.] In this limit, the theory now takes the form

$$\begin{aligned} S = \int d^D x \mathcal{L}_{m=0} &- \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2 (h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \partial^2 \phi) \\ &+ \kappa h_{\mu\nu} T^{\mu\nu}. \end{aligned} \tag{4.23}$$

We see that this has all 5 degrees of freedom: a scalar-tensor vector theory where the vector is completely decoupled but the scalar is kinetically mixed with the tensor.

To see this, we unmix the scalar and tensor, at the expense of the minimal coupling to $T^{\mu\nu}$, by a field redefinition. Consider the change

$$h_{\mu\nu} = h'_{\mu\nu} + \pi \eta_{\mu\nu}, \tag{4.24}$$

where $\pi$ is any scalar. This is the linearization of a conformal transformation. The change in the massless spin 2 part is (no integration by parts here)

$$\begin{aligned} \mathcal{L}_{m=0}(h) = \mathcal{L}_{m=0}(h') &+ (D-2)[\partial_\mu \pi \partial^\mu h' \\ &- \partial_\mu \pi \partial_\nu h'^{\mu\nu} + \tfrac{1}{2}(D-1) \partial_\mu \pi \partial^\mu \pi]. \end{aligned} \tag{4.25}$$

This is simply the linearization of the effect of a conformal transformation on the Einstein-Hilbert action.

By taking $\pi = [2/(D-2)]\phi$ in the transformation (4.24), we can arrange to cancel all the off-diagonal $h\phi$ terms in the Lagrangian (4.23), trading them in for a $\phi$ kinetic term. The Lagrangian (4.23) now takes the form

$$\begin{aligned} S = \int d^D x \mathcal{L}_{m=0}(h') &- \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2 \frac{D-1}{D-2} \partial_\mu \phi \partial^\mu \phi \\ &+ \kappa h'_{\mu\nu} T^{\mu\nu} + \frac{2}{D-2} \kappa \phi T, \end{aligned} \tag{4.26}$$

and the gauge transformations read

$$\delta h'_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta A_\mu = 0, \tag{4.27}$$

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = 0. \tag{4.28}$$

There are now (for $D = 4$) manifestly 5 degrees of freedom, two in a canonical massless graviton, two in a canonical massless vector, and one in a canonical massless scalar.[6]

Note, however, that the coupling of the scalar to the trace of the stress tensor survives the $m = 0$ limit. We exposed the origin of the vDVZ discontinuity. The extra scalar degree of freedom, since it couples to the trace of the stress tensor, does not affect the bending of light (for which $T = 0$), but it does affect the Newtonian potential. This extra scalar potential exactly accounts for the discrepancy between the massless limit of massive gravity and massless gravity.

As a side note, one can see from this Stückelberg trick that violating the Fierz-Pauli tuning for the mass term leads to a ghost. Any deviation from this form, and the Stückelberg scalar will acquire a kinetic term with four derivatives $\sim(\Box\phi)^2$, indicating that it carries 2 degrees of freedom, one of which is ghostlike (de Urries and Julve, 1995, 1998). The Fierz-Pauli tuning is required to exactly cancel these terms, up to total derivative.

[6]Ordinarily the Maxwell term would come with a $\frac{1}{4}$ and the scalar kinetic term with a $\frac{1}{2}$, but we leave different factors here just to avoid unwieldiness.

Returning to the action for $m \neq 0$ (and a not necessarily conserved source), we now know to apply the transformation

$$h_{\mu\nu} = h'_{\mu\nu} + \frac{2}{D-2}\phi\eta_{\mu\nu},$$

which yields

$$\begin{aligned} S = \int d^D x \mathcal{L}_{m=0}(h') &- \frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) \\ &- \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2\frac{D-1}{D-2}\phi\left(\Box + \frac{D}{D-2}m^2\right)\phi \\ &- 2m(h'_{\mu\nu}\partial^\mu A^\nu - h'\partial_\mu A^\mu) \\ &+ 2\frac{D-1}{D-2}(m^2 h'\phi + 2m\phi\partial_\mu A^\mu) + \kappa h'_{\mu\nu}T^{\mu\nu} \\ &+ \frac{2}{D-2}\kappa\phi T - \frac{2}{m}\kappa A_\mu \partial_\nu T^{\mu\nu} + \frac{2}{m^2}\kappa\phi\partial\partial T. \end{aligned} \tag{4.29}$$

The gauge symmetry reads

$$\begin{aligned} \delta h'_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \frac{2}{D-2}m\Lambda\eta_{\mu\nu}, \\ \delta A_\mu &= -m\xi_\mu, \end{aligned} \tag{4.30}$$

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = -m\Lambda. \tag{4.31}$$

We can go to a Lorentz-like gauge, by imposing the gauge conditions (Huang and Parker, 2007; Nibbelink, Peloso, and Sexton, 2007)

$$\partial^\nu h'_{\mu\nu} - \tfrac{1}{2}\partial_\mu h' + mA_\mu = 0, \tag{4.32}$$

$$\partial_\mu A^\mu + m\left(\frac{1}{2}h' + 2\frac{D-1}{D-2}\phi\right) = 0. \tag{4.33}$$

The first condition fixes the $\xi_\mu$ symmetry up to a residual transformation satisfying $(\Box - m^2)\xi_\mu = 0$. It is invariant under $\Lambda$ transformations, so it fixes none of this symmetry. The second condition fixes the $\Lambda$ symmetry up to a residual transformation satisfying $(\Box - m^2)\Lambda = 0$. It is invariant under $\xi_\mu$ transformations, so it fixes none of this symmetry. We add two corresponding gauge fixing terms to the action, resulting from either Fadeev-Popov gauge fixing or classical gauge fixing,

$$S_{\rm GF1} = \int d^D x - \left(\partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + mA_\mu\right)^2, \tag{4.34}$$

$$S_{\rm GF2} = \int d^D x - \left[\partial_\mu A^\mu + m\left(\frac{1}{2}h' + 2\frac{D-1}{D-2}\phi\right)\right]^2. \tag{4.35}$$

These have the effect of diagonalizing the action,

$$\begin{aligned} S + S_{\rm GF1} + S_{\rm GF2} &= \int d^D x \frac{1}{2}h'_{\mu\nu}(\Box - m^2)h'^{\mu\nu} \\ &- \frac{1}{4}h'(\Box - m^2)h' + A_\mu(\Box - m^2)A^\mu \\ &+ 2\frac{D-1}{D-2}\phi(\Box - m^2)\phi + \kappa h'_{\mu\nu}T^{\mu\nu} \\ &+ \frac{2}{D-2}\kappa\phi T - \frac{2}{m}\kappa A_\mu\partial_\nu T^{\mu\nu} + \frac{2}{m^2}\kappa\phi\partial\partial T. \end{aligned} \tag{4.36}$$

The propagators of $h'_{\mu\nu}$, $A_\mu$, and $\phi$ are now, respectively,

$$\begin{aligned} &\frac{-i}{p^2+m^2}\left[\frac{1}{2}(\eta_{\alpha\sigma}\eta_{\beta\lambda} + \eta_{\alpha\lambda}\eta_{\beta\sigma}) - \frac{1}{D-2}\eta_{\alpha\beta}\eta_{\sigma\lambda}\right], \\ &\frac{1}{2}\frac{-i\eta_{\mu\nu}}{p^2+m^2}, \qquad \frac{D-2}{4(D-1)}\frac{-i}{p^2+m^2}, \end{aligned} \tag{4.37}$$

which all behave as $\sim 1/p^2$ for high momenta, so we may now apply standard power counting arguments.

With some amount of work, it is possible to find the gauge invariant mode functions for $h'_{\mu\nu}$, $A_\mu$, and $\phi$, which can then be compared to the unitary gauge mode functions of Sec. II.B. In the massless limit, there is a direct correspondence; $\phi$ is gauge invariant and its 1 degree of freedom is exactly the longitudinal mode (2.20), the $A_\mu$ has the usual Maxwell gauge symmetry and its gauge invariant transverse modes are exactly the vector modes (2.21), and finally the $h'_{\mu\nu}$ has the usual massless gravity gauge symmetry and its gauge invariant transverse modes are exactly the transverse modes of the massive graviton.

## V. NONLINEAR INTERACTIONS

Up to this point, we have studied only the linear theory of massive gravity, which is determined by the requirement that it propagates only one massive spin 2 degree of freedom. We now turn to the study of the possible interactions and nonlinearities for massive gravity.

### A. Massive general relativity

What we want in a full theory of massive gravity is some nonlinear theory whose linear expansion around some background is the massive Fierz-Pauli theory (2.1). Unlike in GR, where the gauge invariance constrains the full theory to be Einstein gravity, the extension for massive gravity is not unique. In fact, there is no obvious symmetry to preserve, so any interaction terms whatsoever are allowed.

The first extension we consider would be to deform GR by simply adding the Fierz-Pauli term to the full nonlinear GR action, that is, choosing the only nonlinear interactions to be those of GR,

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^D x\Big[&(\sqrt{-g}R) \\ &- \sqrt{-g^0}\frac{1}{4}m^2 g^{(0)\mu\alpha}g^{(0)\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\Big]. \end{aligned} \tag{5.1}$$

Here there are several subtleties. Unlike GR, the Lagrangian now explicitly depends on a fixed metric $g^{(0)}_{\mu\nu}$, which we call the absolute metric, on which the linear massive graviton propagates. We have $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$ as before. The mass term is unchanged from its linear version, so the indices on $h_{\mu\nu}$ are raised and traced with the absolute metric. The presence of this absolute metric in the mass term breaks the diffeomorphism invariance of the Einstein-Hilbert term. Note that there is no way to introduce a mass term using only the full metric $g_{\mu\nu}$, since tracing it with itself just gives a constant, so the nondynamical absolute metric is required to create the traces and contractions.

Varying with respect to $g_{\mu\nu}$ we obtain the equations of motion

$$\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) + \sqrt{-g^{(0)}}\frac{m^2}{2}(g^{(0)\mu\alpha}g^{(0)\nu\beta}h_{\alpha\beta} - g^{(0)\alpha\beta}h_{\alpha\beta}g^{(0)\mu\nu}) = 0. \quad (5.2)$$

Indices on $R_{\mu\nu}$ are raised with the full metric, and those on $h_{\mu\nu}$ with the absolute metric. We see that if the absolute metric $g^{(0)}_{\mu\nu}$ satisfies the Einstein equations, then $g_{\mu\nu} = g^{(0)}_{\mu\nu}$, i.e., $h_{\mu\nu} = 0$, is a solution. When dealing with massive gravity and more complicated nonlinear solutions thereof, there can be at times two different background structures. On the one hand, there is the absolute metric, the structure which breaks explicitly the diffeomorphism invariance. On the other hand, there is the background metric, which is a solution to the full nonlinear equations, about which we may expand the action. Often the solution metric we are expanding around will be the same as the absolute metric, but if we were expanding around a different solution, say a black hole, there would be two distinct structures, the black hole solution metric and the absolute metric.

If we add matter to the theory and agree to use only minimal coupling to the metric $g_{\mu\nu}$, then the absolute metric does not directly influence the matter. It is the geodesics and lengths as measured by the full metric (i.e., the solution of the field equations) that we care about. In massive gravity, unlike in GR, if we have a solution, we cannot perform a diffeomorphism on it to obtain a second solution to the same theory. What we obtain instead is a solution to a different massive gravity theory, one in which the absolute metric is related to the original absolute metric by the same diffeomorphism.

Going to more general interactions beyond Eq. (5.1), our main interest will be in adding interaction terms with no derivatives, since these are most important at low energies. The most general such potential which reduces to Fierz-Pauli at quadratic order involves adding terms cubic and higher in $h_{\mu\nu}$ in all possible ways

$$S = \frac{1}{2\kappa^2}\int d^D x\left[(\sqrt{-g}R) - \sqrt{-g^0}\frac{1}{4}m^2 U(g^{(0)}, h)\right], \quad (5.3)$$

where the interaction potential $U$ is the most general one that reduces to Fierz-Pauli at linear order,

$$U(g^{(0)}, h) = U_2(g^{(0)}, h) + U_3(g^{(0)}, h) + U_4(g^{(0)}, h) + U_5(g^{(0)}, h) + \cdots, \quad (5.4)$$

$$U_2(g^{(0)}, h) = [h^2] - [h]^2, \quad (5.5)$$

$$U_3(g^{(0)}, h) = +C_1[h^3] + C_2[h^2][h] + C_3[h]^3, \quad (5.6)$$

$$U_4(g^{(0)}, h) = +D_1[h^4] + D_2[h^3][h] + D_3[h^2]^2 + D_4[h^2][h]^2 + D_5[h]^4, \quad (5.7)$$

$$U_5(g^{(0)}, h) = +F_1[h^5] + F_2[h^4][h] + F_3[h^3][h]^2 + F_4[h^3][h^2] + F_5[h^2]^2[h] + F_6[h^2][h]^3 + F_7[h]^5, \quad \vdots \quad (5.8)$$

The square bracket indicates a trace, with indices raised with $g^{(0),\mu\nu}$, i.e., $[h] = g^{(0)\mu\nu}h_{\mu\nu}$, $[h^2] = g^{(0)\mu\alpha}h_{\alpha\beta}g^{(0)\beta\nu}h_{\nu\mu}$, etc. The coefficients $C_1$, $C_2$, etc. are generic coefficients. Note that the coefficients in $U_n(g^{(0)}, h)$ for $n > D$ are redundant by 1, because there is a combination of the various contractions, the characteristic polynomial $\mathcal{L}^{\rm TD}_n(h)$ (see the Appendix), which vanishes identically. Thus one of the coefficients in $U_n(g^{(0)}, h)$ for $n > D$ (or any one linear combination) can be set to zero.

If we want, we can reorganize the terms in the potential by raising and lowering with the full metric $g^{\mu\nu}$ rather than the absolute metric $g^{(0)\mu\nu}$,

$$S = \frac{1}{2\kappa^2}\int d^D x\left[(\sqrt{-g}R) - \sqrt{-g}\frac{1}{4}m^2 V(g, h)\right], \quad (5.9)$$

where

$$V(g, h) = V_2(g, h) + V_3(g, h) + V_4(g, h) + V_5(g, h) + \cdots, \quad (5.10)$$

$$V_2(g, h) = \langle h^2\rangle - \langle h\rangle^2, \quad (5.11)$$

$$V_3(g, h) = +c_1\langle h^3\rangle + c_2\langle h^2\rangle\langle h\rangle + c_3\langle h\rangle^3, \quad (5.12)$$

$$V_4(g, h) = +d_1\langle h^4\rangle + d_2\langle h^3\rangle\langle h\rangle + d_3\langle h^2\rangle^2 + d_4\langle h^2\rangle\langle h\rangle^2 + d_5\langle h\rangle^4, \quad (5.13)$$

$$V_5(g, h) = +f_1\langle h^5\rangle + f_2\langle h^4\rangle\langle h\rangle + f_3\langle h^3\rangle\langle h\rangle^2 + f_4\langle h^3\rangle\langle h^2\rangle + f_5\langle h^2\rangle^2\langle h\rangle + f_6\langle h^2\rangle\langle h\rangle^3 + f_7\langle h\rangle^5, \quad \vdots \quad (5.14)$$

where the angled brackets are traces with the indices raised with respect to $g^{\mu\nu}$. It does not matter whether we use potential (5.3) with indices raised by $g^{(0)\mu\nu}$, or the potential (5.9) with indices raised by $g^{\mu\nu}$. The two carry the same information and we can easily relate the coefficients of the two by expanding the inverse full metric and the full determinant in powers of $h_{\mu\nu}$ raised with the absolute metric,

$$g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_{\lambda}{}^{\nu} - h^{\mu\lambda}h_{\lambda}{}^{\sigma}h_{\sigma}{}^{\nu} + \cdots, \tag{5.15}$$

$$\sqrt{-g} = \sqrt{-g^{(0)}}\left[1 + \frac{1}{2}h - \frac{1}{4}\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right) + \cdots\right]. \tag{5.16}$$

The following is useful for this purpose:

$$\langle h^n \rangle = \sum_{l=0}^{\infty}(-1)^l \binom{l+n-1}{l}[h^{l+n}]. \tag{5.17}$$

While the zero derivative interaction terms we have written in Eq. (5.3) are general, the two derivative terms are not, since we have demanded they sum up to the Einstein-Hilbert form. The potential has broken the diffeomorphism invariance, so there is no symmetry reason for the two derivative interaction terms to take the Einstein-Hilbert form. We could deviate from it if we wanted, but we will see later that there are good reasons why it is better not to. We may also conceivably add general interactions with more than two derivatives, but we omit these for the same reasons we omit them in GR, because they are associated with higher order effective field theory effects which we hope will be small in suitable regimes.

### B. Spherical solutions and the Vainshtein radius

We now look at static spherical solutions. We specialize to four dimensions, and for definiteness we pick the action (5.1) with the minimal mass term. We attempt to find spherically symmetric solutions to the equation of motion (5.2), in the case where the absolute metric is flat Minkowski in spherical coordinates,

$$g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\Omega^2.$$

We consider a spherically symmetric static ansatz for the dynamical metric[7]

$$g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2. \tag{5.18}$$

Plugging this ansatz into the equations of motion, we get the following from the $tt$ equation, $rr$ equation, and $\theta\theta$ equation (which is the same as the $\phi\phi$ equation by spherical symmetry), respectively,

$$\begin{aligned} &4BC^2m^2r^2A^3 + [2B(C-3)C^2m^2r^2 \\ &\quad - 4\sqrt{A^2BC}(C - rC')]A^2 \\ &\quad + 2\sqrt{A^2BC}[2C^2 - 2r(3A' + rA'')C + r^2A'C']A \\ &\quad + C\sqrt{A^2BC}r^2(A')^2 = 0, \end{aligned} \tag{5.19}$$

$$\begin{aligned} &\frac{4(B + rB')A^2 + [2r^2A'B' - 4B(C - rA')]A + Br^2(A')^2}{A^2BC^2r^2} \\ &\quad - \frac{2(2A + B - 3)m^2}{\sqrt{A^2BC}} = 0, \end{aligned} \tag{5.20}$$

$$\begin{aligned} &-2B^2C^2m^2rA^4 - 2B^2C^2(B + C - 3)m^2rA^3 \\ &- \sqrt{A^2BC}\{2C'B^2 + [rB'C' - 2C(B' + rB'')]B \\ &+ Cr(B')^2\}A^2 \\ &+ B\sqrt{A^2BC}[CrA'B' + B(4CA' - rC'A' + 2CrA'')]A \\ &- B^2C\sqrt{A^2BC}r(A')^2 = 0. \end{aligned} \tag{5.21}$$

In the massless case, $A(r)$ could be removed by a coordinate gauge transformation, and the last equation was redundant; it was a consequence of the first two. With nonzero $m$, there is no diffeomorphism invariance, so no such coordinate change can be made, and the last equation is independent.

We expand these equations around the flat space solution

$$B_0(r) = 1, \qquad C_0(r) = 1, \qquad A_0(r) = 1. \tag{5.22}$$

We introduce the expansion

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots, \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots, \\ A(r) &= A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots. \end{aligned} \tag{5.23}$$

Plugging into the equations of motion and collecting like powers of $\epsilon$, the $\mathcal{O}(0)$ part gives $0 = 0$ because $B_0$, $C_0$, and $A_0$ are solutions to the full nonlinear equations. At each higher order in $\epsilon$ we obtain a linear equation that lets us solve for the next term. At $\mathcal{O}(\epsilon)$ we obtain

$$\begin{aligned} &2(m^2r^2 - 1)A_1 + (m^2r^2 + 2)C_1 \\ &\quad + 2r(-3A_1' + C_1' - rA_1'') = 0, \end{aligned} \tag{5.24}$$

$$-\frac{1}{2}B_1m^2 + \left(\frac{1}{r^2} - m^2\right)A_1 + \frac{r(A_1' + B_1') - C_1}{r^2} = 0, \tag{5.25}$$

$$\begin{aligned} &rA_1m^2 + rB_1m^2 + rC_1m^2 - 2A_1' - B_1' + C_1' \\ &\quad - rA_1'' - rB_1'' = 0. \end{aligned} \tag{5.26}$$

One way to solve these equations is as follows. Algebraically solve them simultaneously for $A_1$, $A_1'$, and $A_1''$ in terms of $B_1$'s and $C_1$'s and their derivatives. Then write the equations $(d/dr)A_1(B, C) = A_1'(B, C)$ and $(d/dr)A'(B, C) = A''(B, C)$. Solve these two equations for $C_1$ and $C_1'$ in terms of $B_1$'s derivatives. Then write $(d/dr)C_1(B) = C_1'(B)$. What is left is

$$-3rB_1m^2 + 6B_1' + 3rB_1'' = 0. \tag{5.27}$$

[7]In general, when there are two metrics staticity and spherical symmetry are not enough to put both in diagonal form. An $r$ dependent off-diagonal $drdt$ term can remain in one of them. We will not seek such off-diagonal metrics and will limit ourselves to the diagonal ansatz.

There are two integration constants in the solution to Eq. (5.27); one is left arbitrary and the other must be sent to zero to prevent the solutions from blowing up at infinity. We then recursively determine $C_1$ and $A_1$. Thus the whole solution is determined by two pieces of initial data.[8]

The solution is

$$B_1(r) = -\frac{8GM}{3}\frac{e^{-mr}}{r}, \tag{5.28}$$

$$C_1(r) = -\frac{8GM}{3}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2}, \tag{5.29}$$

$$A_1(r) = \frac{4GM}{3}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2}, \tag{5.30}$$

where we have chosen the integration constant so that we agree with the solution (3.15) obtained from the Green's function.

We can now proceed to $\mathcal{O}(\epsilon^2)$. Going through the same procedure, we find for the solution, when $mr \ll 1$,

$$B(r) - 1 = -\frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4r^5} + \cdots\right), \tag{5.31}$$

$$C(r) - 1 = -\frac{8}{3}\frac{GM}{m^2r^3}\left(1 - 14\frac{GM}{m^4r^5} + \cdots\right), \tag{5.32}$$

$$A(r) - 1 = \frac{4}{3}\frac{GM}{4\pi m^2r^3}\left(1 - 4\frac{GM}{m^4r^5} + \cdots\right). \tag{5.33}$$

The dots represent higher powers in the nonlinearity parameter $\epsilon$. We see that the nonlinearity expansion is an expansion in the parameter $r_V/r$, where

$$r_V \equiv \left(\frac{GM}{m^4}\right)^{1/5} \tag{5.34}$$

is known as the Vainshtein radius. As the mass $m$ approaches 0, $r_V$ grows, and hence the radius beyond which the solution can be trusted gets pushed out to infinity. As argued by Vainshtein (1972), this perturbation expansion breaks down and says nothing about the true nonlinear behavior of massive gravity in the massless limit. Thus there was reason to hope that the vDVZ discontinuity was merely an artifact of linear perturbation theory, and that the true nonlinear solutions showed a smooth limit (Vainshtein, 1972; Deffayet *et al.*, 2002; Porrati, 2002; Gruzinov, 2005).

One might hope that a smooth limit could be seen by setting up an alternative expansion in the mass $m^2$. We take a solution to the massless equations, the ordinary Schwarzschild solution, with metric coefficients $B_0$, $C_0$, and $A_0$, and then plug an expansion,

$$\begin{aligned} B(r) &= B_0(r) + m^2B_1(r) + m^4B_2(r) + \cdots, \\ C(r) &= C_0(r) + m^2C_1(r) + m^4C_2(r) + \cdots, \\ A(r) &= A_0(r) + m^2A_1(r) + m^4A_2(r) + \cdots, \end{aligned} \tag{5.35}$$

into the equations of motion. Collecting powers of $m$ yields a new perturbation equation at each order, but in this case the equations are generally nonlinear. Even the equation we obtain at $\mathcal{O}(m^2)$ for the first correction to Schwarzschild is nonlinear, so working with this expansion is much more difficult than working with the linearized expansion.

The linearity expansion is valid is the region $r \gg r_V$. If general relativity is restored at distances near the source, the mass expansion should be valid in the opposite regime $r \ll r_V$, and the full solutions should interpolate between the two expansions. There have been several extensive numerical studies of the full nonlinear solutions in Damour, Kogan, and Papazoglou (2003), in the decoupling limit in Babichev, Deffayet, and Ziour (2009b), and more extensively in the full theory in Deffayet (2008), Babichev, Deffayet, and Ziour (2009a, 2010), with the final result being that the nonlinearities can, in fact, work to restore continuity with GR. We see later the mechanism by which this occurs. Some analytic solutions in various cases are given by Berezhiani *et al.* (2008), Comelli *et al.* (2011), Koyama, Niz, and Tasinato (2011a, 2011b), and Nieuwenhuizen (2011).

### C. Nonlinear Hamiltonian and the Boulware-Deser mode

We now study the Hamiltonian of the nonlinear massive gravity action (5.1) with flat absolute metric $\eta^{\mu\nu}$,

$$S = \frac{1}{2\kappa^2}\int d^Dx\Big[(\sqrt{-g}R) - \frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\Big]. \tag{5.36}$$

We saw in Sec. II.A that the free theory carries 5 degrees of freedom in $D = 4$, due to the fact that the time components $h_{00}$ appeared as a Lagrange multiplier in the action. We see that this no longer remains true once the nonlinearities of Eq. (5.36) are taken into account, so there is now an extra degree of freedom.

A particularly nice way to study gravity Hamiltonians is through the Arnowitt-Deser-Misner (ADM) formalism (Arnowitt, Deser, and Misner, 1960, 1962). A spacelike slicing of spacetime by hypersurfaces $\Sigma_t$ is chosen, and we change variables from components of the metric $g_{\mu\nu}$ to the spatial metric $g_{ij}$, the lapse $N_i$, and the shift $N$, according to

$$g_{00} = -N^2 + g^{ij}N_iN_j, \tag{5.37}$$

$$g_{0i} = N_i, \tag{5.38}$$

$$g_{ij} = g_{ij}. \tag{5.39}$$

Here $i, j, \ldots$ are spatial indices, and $g^{ij}$ is the inverse of the spatial metric $g_{ij}$ (not the $ij$ components of inverse metric $g^{\mu\nu}$).

The Einstein-Hilbert part of the action in these variables reads [see Poisson (2004) and Dyer and Hinterbichler (2009) for detailed derivations and formulas]

$$\frac{1}{2\kappa^2}\int d^Dx\sqrt{g}N[{}^{(d)}R - K^2 + K^{ij}K_{ij}], \tag{5.40}$$

[8] Naively, it is a second order equation in $A_1$ and $B_1$, first order in $C_1$, and we think this requires five initial conditions, but, in fact, it is a degenerate system, and there are second class constraints bringing the required boundary data to 2.

where ${}^{(d)}R$ is the curvature of the spatial metric $g_{ij}$. The quantity $K_{ij}$ is the extrinsic curvature of the spatial hypersurfaces, defined as

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{5.41}$$

where the dot means a time derivative, and the covariant derivatives are with respect to the spatial metric $g_{ij}$. We then Legendre transform the spatial variables $g_{ij}$, defining the canonical momenta

$$p^{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \frac{1}{2\kappa^2}\sqrt{g}(K^{ij} - Kg^{ij}), \tag{5.42}$$

and writing the action in Hamiltonian form

$$2\kappa^2 L = \left(\int_{\Sigma_t} d^d x p^{ij}\dot{g}_{ij}\right) - H, \tag{5.43}$$

where the Hamiltonian $H$ is defined by

$$H = \left(\int_{\Sigma_t} d^d x p^{ab}\dot{g}_{ab}\right) - L = \int_{\Sigma_t} d^d x N\mathcal{C} + N_i\mathcal{C}^i, \tag{5.44}$$

and the quantities $\mathcal{C}$ and $\mathcal{C}_i$ are

$$\mathcal{C} = \sqrt{g}[{}^{(d)}R + K^2 - K^{ij}K_{ij}],$$
$$\mathcal{C}^i = 2\sqrt{g}\nabla_j(K^{ij} - Kg^{ij}), \tag{5.45}$$

and here $K_{ij}$ should be thought of as a function of $p^{ij}$ and $g_{ij}$, obtained by inverting Eq. (5.42) for $\dot{g}_{ij}$ and plugging into Eq. (5.41),

$$K_{ij} = \frac{2\kappa^2}{\sqrt{g}}\left(p_{ij} - \frac{1}{D-2}pg_{ij}\right). \tag{5.46}$$

All traces and index manipulations are performed with $g_{ij}$ and its inverse.

For $m = 0$, the action is pure constraint, and the Hamiltonian vanishes, a characteristic of diffeomorphism invariance. The shift $N$ and lapse $N_i$ appear as Lagrange multipliers, enforcing the Hamiltonian constraint $\mathcal{C} = 0$ and momentum constraints $\mathcal{C}_i = 0$. It can be checked that these are first class constraints, generating the $D$ diffeomorphism symmetries of the action. In $D = 4$, we have 12 phase space metric components, minus four constraints, minus four gauge symmetries, leaves 4 phase space degrees of freedom, the same counting as in the linear theory. The nonlinear theory contains the same number of degrees of freedom as the linearized theory.

Now looking at the mass term, in ADM variables we have

$$\begin{aligned}\eta^{\mu\alpha}\eta^{\mu\beta}&(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\mu\beta}) \\ &= \delta^{ik}\delta^{jl}(h_{ij}h_{kl} - h_{ik}h_{jl}) + 2\delta^{ij}h_{ij} \\ &\quad - 2N^2\delta^{ij}h_{ij} + 2N_i(g^{ij} - \delta^{ij})N_i,\end{aligned} \tag{5.47}$$

where $h_{ij} \equiv g_{ij} - \delta_{ij}$. The action becomes

$$\begin{aligned}S = \frac{1}{2\kappa^2}\int d^D x p^{ij}\dot{g}_{ij} &- N\mathcal{C} - N_i\mathcal{C}^i \\ &- \frac{m^2}{4}[\delta^{ik}\delta^{jl}(h_{ij}h_{kl} - h_{ik}h_{jl}) + 2\delta^{ij}h_{ij} \\ &- 2N^2\delta^{ij}h_{ij} + 2N_i(g^{ij} - \delta^{ij})N_i].\end{aligned} \tag{5.48}$$

In the $m \neq 0$ case, the Fierz-Pauli term brings in contributions to the action that are quadratic in the lapse and shift (but still free of time derivatives). Thus the lapse and shift no longer serve as Lagrange multipliers, but rather as auxiliary fields, because their equations of motion can be algebraically solved to determine their values,

$$N = \frac{\mathcal{C}}{m^2\delta^{ij}h_{ij}}, \qquad N_i = \frac{1}{m^2}(g^{ij} - \delta^{ij})^{-1}\mathcal{C}^j. \tag{5.49}$$

When these values are plugged back into Eq. (5.48), we have an action with no constraints or gauge symmetries at all, so all the phase space degrees of freedom are active. The resulting Hamiltonian is

$$\begin{aligned}H = \frac{1}{2\kappa^2}\int d^d x \frac{1}{2m^2}\frac{\mathcal{C}^2}{\delta^{ij}h_{ij}} &+ \frac{1}{2m^2}\mathcal{C}^i(g^{ij} - \delta^{ij})^{-1}\mathcal{C}^j \\ &+ \frac{m^2}{4}[\delta^{ik}\delta^{jl}(h_{ij}h_{kl} - h_{ik}h_{jl}) + 2\delta^{ij}h_{ij}],\end{aligned} \tag{5.50}$$

which is nonvanishing, unlike in GR. In four dimensions, we thus have 12 phase space degrees of freedom, or 6 real degrees of freedom. The linearized theory had only 5 degrees of freedom, and we have here a case where the nonlinear theory contains more degrees of freedom than the linear theory. It should not necessarily be surprising that this can happen, because there is no reason nonlinearities cannot change the constraint structure of a theory or that kinetic terms cannot appear at higher order.

As argued by Boulware and Deser (1972), the Hamiltonian (5.50) is not bounded, and since the system is nonlinear, it is not surprising that it has instabilities (Gabadadze and Gruzinov, 2005). The nature of the instability, i.e., whether it is a ghost of a tachyon, what backgrounds it appears around, and its severity, is hard to see in the Hamiltonian formalism. But in Sec. VII.B we see that this instability is a ghost, a scalar with a negative kinetic term, and that its mass around a given background can be determined. It turns out that around flat space, the ghost degree of freedom is not excited because its mass is infinite, but around nontrivial backgrounds its mass becomes finite. This ghostly extra degree of freedom is referred to as the Boulware-Deser ghost (Boulware and Deser, 1972).

There is still the possibility that adding higher order interaction terms such as $h^3$ terms and higher can remove the ghostly sixth degree of freedom. Boulware and Deser analyzed a large class of various mass terms, showing that the sixth degree of freedom remained (Boulware and Deser, 1972), but they did not consider the most general possible potential. This was addressed by Creminelli *et al.* (2005), where the analysis was done perturbatively in powers of $h$. The lapse is expanded around its flat space values $N = 1 + \delta N$. In this case, $\delta N$ plays the role of the Lagrange multiplier, and it is shown that at fourth order, interaction terms involving higher powers of $\delta N$ cannot be removed. It was concluded by Creminelli *et al.* (2005) that the Boulware-Deser ghost is unavoidable, but this conclusion is too quick. It may be possible that there are field redefinitions under which the lapse is made to appear linearly. Alternatively, it may be possible that after one solves for the shift using its equation of motion, then replaces into the action, the resulting action is linear in the lapse, even though it contained higher powers of the lapse before integrating out the shift. It is also possible that the lapse appears linearly in the full nonlinear action, even though at any finite order the action contains higher powers of the lapse. [For discussions and examples of these points, see de Rham and Gabadadze (2010a) and de Rham, Gabadadze, and Tolley (2010)).]

As it turns out, it is, in fact, possible to add appropriate interactions that eliminate the ghost (Hassan and Rosen, 2011a, 2011c). In $D$ dimensions, there is a $D-2$ parameter family of such interactions. We study these in Sec. VIII, where we see that they also have the effect of raising the maximum energy cutoff at which massive gravity is valid as an effective field theory.[9] This class of theories solves the problem of the Boulware-Deser ghost.[10].

## VI. THE NONLINEAR STÜCKELBERG FORMALISM

In this section we extend the Stückelberg trick to full nonlinear order. This is a powerful tool with which to elucidate the nonlinear dynamics of massive gravity. It allows us to trace the breakdown in the linear expansion to strong coupling of the longitudinal mode. It also tells us about quantum corrections, the scale of the effective field theory and where it breaks down, as well as the nature of the Boulware-Deser ghost and whether it lies within the effective theory or can be consistently ignored.

### A. Stückelberg for gravity and the restoration of diffeomorphism invariance

We now construct the full nonlinear gravitational Stückelberg. This method was brought to our attention by Arkani-Hamed, Georgi, and Schwartz (2003) and Schwartz (2003), but was, in fact, known previously from work in string theory (Green and Thorn, 1991; Siegel, 1994).

The full finite gauge transformation for gravity is

$$g_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}(f(x)), \tag{6.1}$$

where $f(x)$ is the arbitrary gauge function, which must be a diffeomorphism. In massive gravity this gauge invariance is broken only by the mass term. To restore it, we introduce a Stückelberg field $Y^\mu(x)$, patterned after the gauge symmetry (6.1), and we apply it to the metric $g_{\mu\nu}$,

$$g_{\mu\nu}(x) \to G_{\mu\nu} = \frac{\partial Y^\alpha}{\partial x^\mu}\frac{\partial Y^\beta}{\partial x^\nu} g_{\alpha\beta}(Y(x)). \tag{6.2}$$

The Einstein-Hilbert term $\sqrt{-g}R$ will not change under this substitution, because it is gauge invariant, and the substitution looks similar to a gauge transformation with gauge parameter $Y^\mu(x)$, so no $Y$ fields are introduced into the Einstein-Hilbert part of the action.

The graviton mass term, however, will pick up dependence on $Y$'s in such a way that it will now be invariant under the following gauge transformation:

$$\begin{aligned} g_{\mu\nu}(x) &\to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}(f(x)), \\ Y^\mu(x) &\to f^{-1}(Y(x))^\mu \end{aligned} \tag{6.3}$$

with $f(x)$ the gauge function. This is because the combination $G_{\mu\nu}$ is gauge *invariant* (not *covariant*). To see this, first transform[11] $g_{\mu\nu}$,

$$\begin{aligned} &\partial_\mu Y^\alpha \partial_\nu Y^\beta g_{\alpha\beta}(Y(x)) \\ &\quad \to \partial_\mu Y^\alpha \partial_\nu Y^\beta \partial_\alpha f^\lambda|_Y \partial_\beta f^\sigma|_Y g_{\lambda\sigma}(f(Y(x))) \end{aligned} \tag{6.7}$$

[here $|_Y$ means the function is evaluated at $Y(x)$] and then transform $Y$,

[9]Note that merely finding a ghost free interacting Lorentz invariant massive gravity theory is not hard totake; for instance, $U(\eta, h) = -2[\det(\delta_\mu{}^\nu + h_\mu{}^\nu) - h]$ in Eq. (5.3), while letting the kinetic interactions be those of the linear graviton only. A Hamiltonian analysis just like that of Sec. II.A shows that $h_{00}$ and $h_{0i}$ both remain Lagrange multipliers. The problem is that this theory does not go to GR in the $m \to 0$ limit, it goes to massless gravity. The real challenge is to construct a ghost free Lorentz invariant massive gravity that reduces to GR.

[10]The objections of Alberte, Chamseddine, and Mukhanov (2011), Folkerts, Pritzel, and Wintergerst (2011), and Kluson (2011) are addressed by Hassan and Rosen (2011a), de Rham, Gabadadze, and Tolley (2011a, 2011c), respectively.

[11]The transformation of fields that depend on other fields is potentially tricky. To get it right, it is sometimes convenient to tease out the dependencies using delta functions. For example, suppose we have a scalar field $\phi(x)$, which we know transforms according to $\phi(x) \to \phi(f(x))$. How should $\phi(Y(x))$ transform? To make it clear, write

$$\phi(Y(x)) = \int dy \phi(y) \delta(y - Y(x)). \tag{6.4}$$

Now the field $\phi$ appears with coordinate dependence, which we know how to deal with,

$$\to \int dy \phi(f(y)) \delta(y - Y(x)) = \phi(f(Y(x))). \tag{6.5}$$

Going through an identical trick for the metric, which we know transforms as

$$g_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}(f(x)),$$

we find

$$g_{\alpha\beta}(Y(x)) \to \partial_\alpha f^\lambda|_Y \partial_\beta f^\sigma|_Y g_{\lambda\sigma}(f(Y(x))). \tag{6.6}$$

$$\begin{aligned}
&\to \partial_\mu[f^{-1}(Y)]^\alpha \partial_\nu[f^{-1}(Y)]^\beta \partial_\alpha f^\lambda|_{f^{-1}(Y)} \\
&\quad\times \partial_\beta f^\sigma|_{f^{-1}(Y)} g_{\lambda\sigma}(Y(x)) \\
&= \partial_\rho[f^{-1}]^\alpha|_Y \partial_\mu Y^\rho \partial_\tau[f^{-1}]^\beta|_Y \\
&\quad\times \partial_\nu Y^\tau \partial_\alpha f^\lambda|_{f^{-1}(Y)} \partial_\beta f^\sigma|_{f^{-1}(Y)} g_{\lambda\sigma}(Y(x)) \\
&= \delta^\lambda_\rho \delta^\sigma_\tau \partial_\mu Y^\rho \partial_\nu Y^\tau g_{\lambda\sigma}(Y(x)) \\
&= \partial_\mu Y^\lambda \partial_\nu Y^\sigma g_{\lambda\sigma}(Y(x)). \qquad (6.8)
\end{aligned}$$

We now expand $Y$ about the identity,

$$Y^\alpha(x) = x^\alpha + A^\alpha(x). \qquad (6.9)$$

The quantity $G_{\mu\nu}$ is expanded as

$$\begin{aligned}
G_{\mu\nu} &= \frac{\partial Y^\alpha(x)}{\partial x^\mu}\frac{\partial Y^\beta(x)}{\partial x^\nu} g_{\alpha\beta}(Y(x)) \\
&= \frac{\partial(x^\alpha + A^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta + A^\beta)}{\partial x^\nu} g_{\alpha\beta}(x + A) \\
&= (\delta^\alpha_\mu + \partial_\mu A^\alpha)(\delta^\beta_\nu + \partial_\nu A^\beta)\Big(g_{\alpha\beta} + A^\mu \partial_\mu g_{\alpha\beta} \\
&\quad + \frac{1}{2} A^\mu A^\nu \partial_\mu \partial_\nu g_{\alpha\beta} + \cdots\Big) \\
&= g_{\mu\nu} + A^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu A^\alpha g_{\alpha\nu} + \partial_\nu A^\alpha g_{\alpha\mu} \\
&\quad + \frac{1}{2} A^\alpha A^\beta \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu A^\alpha \partial_\nu A^\beta g_{\alpha\beta} \\
&\quad + \partial_\mu A^\alpha A^\beta \partial_\beta g_{\alpha\nu} + \partial_\nu A^\alpha A^\beta \partial_\beta g_{\mu\alpha} + \cdots. \qquad (6.10)
\end{aligned}$$

We now look at the infinitesimal transformation properties of $g$, $Y$, $G$, and $Y$, under infinitesimal general coordinate transformations generated by $f(x) = x + \xi(x)$. The metric transforms in the usual way,

$$\delta g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda}. \qquad (6.11)$$

The transformation law for the $A$'s comes from the transformation of $Y$,

$$\begin{aligned}
&Y^\mu(x) \to f^{-1}(Y(x))^\mu \approx Y^\mu(x) - \xi^\mu(Y(x)), \\
&\delta Y^\mu = -\xi^\mu(Y), \\
&\delta A^\mu = -\xi^\mu(x + A) \\
&\qquad = -\xi^\mu - A^\alpha \partial_\alpha \xi^\mu - \tfrac{1}{2} A^\alpha A^\beta \partial_\alpha \partial_\beta \xi^\mu - \cdots. \qquad (6.12)
\end{aligned}$$

The $A^\mu$ are the Goldstone bosons that nonlinearly carry the broken diffeomorphism invariance in massive gravity. The combination $G_{\mu\nu}$, as we noted before, is gauge invariant

$$\delta G_{\mu\nu} = 0. \qquad (6.13)$$

We now have a recipe for Stückelberg-ing the general massive gravity action of the form (5.3). We leave the Einstein-Hilbert term alone. In the mass term, we write all the $h_{\mu\nu}$'s with lowered indices to get rid of the dependence on the absolute metric, and then we replace all occurrences of $h_{\mu\nu}$ with

$$H_{\mu\nu}(x) = G_{\mu\nu}(x) - g^{(0)}_{\mu\nu}(x), \qquad (6.14)$$

where $g^{(0)}_{\mu\nu}(x)$ is the absolute metric, which here is also the background metric. We then expand $G_{\mu\nu}$ as in Eq. (6.10), and $Y^\mu$ as in Eq. (6.9). To linear order in $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$ and $A_\mu$, the expansion reads

$$H_{\mu\nu} = h_{\mu\nu} + \nabla^{(0)}_\mu A_\nu + \nabla^{(0)}_\nu A_\mu, \qquad (6.15)$$

where indices on $A$ are lowered with the background metric. This is exactly the Stückelberg substitution we made in the linear case.

In the case where the absolute metric is flat, $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, we have from Eq. (6.10),

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + \partial_\mu A^\alpha \partial_\nu A_\alpha + \cdots. \qquad (6.16)$$

Here indices on $A^\mu$ are lowered with $\eta_{\mu\nu}$ and the ellipsis are terms quadratic and higher in the fields and containing at least one power of $h$. This takes into account the full nonlinear gauge transformation.

As in the linear case, we usually want to do another scalar Stückelberg replacement to introduce a U(1) gauge symmetry,

$$A_\mu \to A_\mu + \partial_\mu \phi. \qquad (6.17)$$

Then the expansion for the flat absolute metric takes the form

$$\begin{aligned}
H_{\mu\nu} &= h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu \partial_\nu \phi + \partial_\mu A^\alpha \partial_\nu A_\alpha \\
&\quad + \partial_\mu A^\alpha \partial_\nu \partial_\alpha \phi + \partial_\mu \partial^\alpha \phi \partial_\nu A_\alpha \\
&\quad + \partial_\mu \partial^\alpha \phi \partial_\nu \partial_\alpha \phi + \cdots, \qquad (6.18)
\end{aligned}$$

where again the ellipsis are terms quadratic and higher in the fields and containing at least one power of $h$. The gauge transformation laws are

$$\begin{aligned}
\delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{L}_\xi h_{\mu\nu}, \\
\delta A_\mu &= \partial_\mu \Lambda - \xi_\mu - A^\alpha \partial_\alpha \xi_\mu - \tfrac{1}{2} A^\alpha A^\beta \partial_\alpha \partial_\beta \xi_\mu - \cdots, \\
\delta \phi &= -\Lambda. \qquad (6.19)
\end{aligned}$$

This method of Stückelberging can be extended to any number of gravitons and general coordinate invariances, as done by Arkani-Hamed, Georgi, and Schwartz (2003) and Arkani-Hamed and Schwartz (2004), in analogy with the gauge theory little Higgs models and dimensional deconstruction (Arkani-Hamed, Cohen, and Georgi, 2001a, 2001b). When multiple gravitons are present, all but one must become massive, since there are no nontrivial interactions between multiple massless gravitons (Boulanger *et al.*, 2001) [see Bachas and Petropoulos (1993) and Kiritsis (2006) for string theory and holographic proofs of this], and these gravitons mimic the Kaluza-Klein spectrum of a discrete extra dimension. Other work in this area, including applications to bigravity and multigravity models, can be found in Jejjala, Leigh, and Minic (2003), Kan and Shiraishi (2003), Deffayet and Mourad (2004), Groot Nibbelink and Peloso (2005), Nibbelink, Peloso, and Sexton (2007), and Deffayet and Randjbar-Daemi (2011).

### B. Another way to Stückelberg

In the last section, we introduced gauge invariance and the Stückelberg fields by replacing the metric $g_{\mu\nu}$ with the gauge invariant object $G_{\mu\nu}$. This is well suited to the case where we have a potential arranged in the form (5.3), because all the background $g^{(0)\mu\nu}$'s appearing in the contractions and determinant of the mass term do not need replacing. The drawback is that the Stückelberg expansion involves an infinite number of terms higher order in $h_{\mu\nu}$. If we wish to keep track of the $h_{\mu\nu}$'s, this is not very convenient.

Instead, we develop another method, which is to introduce the Stückelberg fields through the background metric $g^{(0)}_{\mu\nu}$, and then allow $g_{\mu\nu}$ to transform covariantly. This method will be better suited to a potential arranged in the form (5.9) and will have the advantage that the Stückelberg expansion contains no higher powers of $h_{\mu\nu}$.

We make the replacement

$$g^{(0)}_{\mu\nu} \to g^{(0)}_{\alpha\beta}\partial_\mu Y^\alpha \partial_\nu Y^\beta. \tag{6.20}$$

The $Y^\alpha(x)$ that are introduced are four fields, which despite the index $\alpha$ are to transform as *scalars* under diffeomorphisms

$$Y^\alpha(x) \to Y^\alpha(f(x)), \tag{6.21}$$

or infinitesimally,

$$\delta Y^\alpha = \xi^\nu \partial_\nu Y^\alpha. \tag{6.22}$$

This is to be contrasted with the transformation rule $\delta Y^\alpha = \xi^\nu\partial_\nu Y^\alpha - (\partial_\nu \xi^\alpha)Y^\nu$ which would hold if $Y^\mu$ were a vector. Given this scalar transformation rule for $Y^\alpha$, the replaced $g^{(0)}_{\mu\nu}$ now transforms similar to a metric tensor. If we now assign the usual diffeomorpshim transformation law to the metric $g_{\mu\nu}$ (so that it is now covariant), quantities such as $g^{(0)}_{\mu\nu}g^{\mu\nu}$ and other contractions will transform as diffeomorphism scalars. We can take any action which is a scalar function of $g^{(0)}_{\mu\nu}$ and $g_{\mu\nu}$, and introduce gauge invariance in this way.[12].

This is convenient when we have a potential of the form (5.9). First we lower all indices on the $h_{\mu\nu}$'s in the potential. Now the background metric $g^{(0)}_{\mu\nu}$ appears only through $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$, so we replace all occurrences of $h_{\mu\nu}$ with

$$H_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\alpha\beta}\partial_\mu Y^\alpha \partial_\nu Y^\beta. \tag{6.23}$$

Note that we do not need to make a replacement on the $g^{\mu\nu}$'s used to contract the indices, nor on the $\sqrt{-g}$ out front of the potential in Eq. (5.9).

Expanding

$$Y^\alpha = x^\alpha - A^\alpha, \tag{6.24}$$

and using $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, we have

$$H_{\mu\nu} = h_{\mu\nu} + g^{(0)}_{\nu\alpha}\partial_\mu A^\alpha + g^{(0)}_{\mu\alpha}\partial_\nu A^\alpha - g^{(0)}_{\alpha\beta}\partial_\mu A^\alpha \partial_\nu A^\beta. \tag{6.25}$$

Note the difference in sign for the term quadratic in $A^\mu$ compared with Eq. (6.16).

Under infinitesimal gauge transformations we have

$$\delta A^\alpha = -\xi^\alpha + \xi^\nu \partial_\nu A^\alpha, \tag{6.26}$$

$$\delta h_{\mu\nu} = \nabla^{(0)}_\mu \xi_\nu + \nabla^{(0)}_\nu \xi_\mu + \mathcal{L}_\xi h_{\mu\nu}, \tag{6.27}$$

where the covariant derivatives are with respect to $g^{(0)}_{\mu\nu}$ and the indices on $\xi^\mu$ are lowered using $g^{(0)}_{\mu\nu}$. To linear order, the transformations are

$$\delta A^\alpha = -\xi^\alpha, \tag{6.28}$$

$$\delta h_{\mu\nu} = \nabla^{(0)}_\mu \xi_\nu + \nabla^{(0)}_\nu \xi_\mu, \tag{6.29}$$

which reproduces the linear Stückelberg expansion.

In the case of a flat background $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, the replacement is

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu - \partial_\mu A^\alpha \partial_\nu A_\alpha, \tag{6.30}$$

with indices on $A^\alpha$ lowered by $\eta_{\mu\nu}$. Notice that this is the complete expression; there are no higher powers of $h_{\mu\nu}$, unlike Eq. (6.16).

We often follow this with the replacement $A_\mu \to A_\mu + \partial_\mu\phi$ to extract the helicity 0 mode. The full expansion thus reads

$$\begin{aligned} H_{\mu\nu} = {}& h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\phi + \partial_\mu A^\alpha \partial_\nu A_\alpha \\ & + \partial_\mu A^\alpha \partial_\nu\partial_\alpha\phi + \partial_\mu\partial^\alpha\phi\partial_\nu A_\alpha + \partial_\mu\partial^\alpha\phi\partial_\nu\partial_\alpha\phi. \end{aligned} \tag{6.31}$$

Under infinitesimal gauge transformations,

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathcal{L}_\xi h_{\mu\nu}, \tag{6.32}$$

$$\delta A_\mu = \partial_\mu\Lambda - \xi_\mu + \xi^\nu\partial_\nu A_\mu, \tag{6.33}$$

$$\delta\phi = -\Lambda. \tag{6.34}$$

Yet another way to introduce Stückelberg fields is advocated by Alberte, Chamseddine, and Mukhanov (2010, 2011) and Chamseddine and Mukhanov (2010), in which they make the inverse metric $g^{\mu\nu}$ covariant through the introduction of scalars $g^{\mu\nu} \to g^{\alpha\beta}\partial_\alpha Y^\mu \partial_\beta Y^\nu$. There have also been many studies, initiated by 't Hooft, of the so-called gravitational Higgs mechanism, which is also essentially a Stückelberging of different forms of massive gravity (Kirsch, 2005; Leclerc, 2006; 't Hooft, 2007; Kakushadze, 2008a, 2008b; Demir and Pak, 2009; Kluson, 2010; Oda, 2010a, 2010b). All of these

[12]This is essentially the technique of spurion analysis, where a coupling constant is made to transform as a field. A quantity which is normally a background quantity, a coupling constant in the case of spurions, or the background $g^{(0)}_{\mu\nu}$ in this case, is made to transform in some way that gives the action more symmetries. Note that this method of introducing gauge invariance can be carried out on any Lorentz invariant action, even one that does not contain a dynamical metric $g_{\mu\nu}$. For example, a plain old scalar field in flat space can be made diffeomorphism invariant in this way. This highlights the fact that general coordinate invariance is not the critical ingredient that leads one to a theory of gravity, since it can be made to hold in any theory.

are equivalent to the theories we study, as can be seen simply by going to unitary gauge (Berezhiani and Mirbabayi, 2010). At the end of the day, Eq. (5.3) is the most general Lorentz invariant graviton potential, and any Lorentz invariant massive gravity theory will have a unitary gauge with a potential which is equivalent to it for some choice of the coefficients $C_1$, $C_2$, etc.

## VII. STÜCKELBERG ANALYSIS OF INTERACTING MASSIVE GRAVITY

In this section, we set $D = 4$ and apply the Stückelberg analysis to the massive GR action (5.1) in the case of a flat absolute metric. The mass term reads

$$S_{\rm mass} = -\frac{M_P^2}{2}\frac{m^2}{4}\int d^4x \eta^{\mu\nu}\eta^{\alpha\beta}(h_{\mu\alpha}h_{\nu\beta} - h_{\mu\nu}h_{\alpha\beta}). \tag{7.1}$$

The Stückelberg analysis instructs us to make the replacement (6.18),

$$\begin{aligned} h_{\mu\nu} \to H_{\mu\nu} &= h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + \partial_\mu A^\alpha \partial_\nu A_\alpha + 2\partial_\mu\partial_\nu\phi \\ &\quad + \partial_\mu\partial^\alpha\phi\partial_\nu\partial_\alpha\phi \cdots. \end{aligned} \tag{7.2}$$

The extra terms with $h$ in the ellipsis will not be important for this theory, as we will see.

At the linear level, this replacement is exactly the linear Stückelberg expansion of Sec. IV. We have to canonically normalize the fields here to match the fields of the linear analysis. Using a hat to signify the canonically normalized fields with the same coefficients as used in Sec. IV (although there we omitted the hats), we have

$$\hat{h} = \tfrac{1}{2}M_P h, \qquad \hat{A} = \tfrac{1}{2}mM_P A, \qquad \hat{\phi} = \tfrac{1}{2}m^2 M_P \phi. \tag{7.3}$$

We also get a whole slew of interaction terms, third order and higher in the fields, suppressed by various scales. We always assume $m < M_P$. $\phi$ always appears with two derivatives, $A$ always appears with one derivative, and $h$ always appears with none, so a generic term, with $n_h$ powers of $h_{\mu\nu}$, $n_A$ powers of $A_\mu$, and $n_\phi$ powers of $\phi$, reads

$$\begin{aligned} &\sim m^2 M_P^2 h^{n_h}(\partial A)^{n_A}(\partial^2\phi)^{n_\phi} \\ &\sim \Lambda_\lambda^{4-n_h-2n_A-3n_\phi}\hat{h}^{n_h}(\partial\hat{A})^{n_A}(\partial^2\hat{\phi})^{n_\phi}, \end{aligned} \tag{7.4}$$

where the scale suppressing the term is

$$\Lambda_\lambda = (M_P m^{\lambda-1})^{1/\lambda}, \qquad \lambda = \frac{3n_\phi + 2n_A + n_h - 4}{n_\phi + n_A + n_h - 2}. \tag{7.5}$$

The larger $\lambda$, the smaller the scale, since $m < M_P$. We have $n_\phi + n_A + n_h \geq 3$, since we are only considering interaction terms. The term suppressed by the smallest scale is the cubic scalar term $n_\phi = 3$, $n_A = n_h = 0$, which is suppressed by the scale $\Lambda_5 = (M_P m^4)^{1/5}$,

$$\sim \frac{(\partial^2\hat{\phi})^3}{\Lambda_5^5}, \qquad \Lambda_5 = (M_P m^4)^{1/5}. \tag{7.6}$$

In terms of the canonically normalized fields (7.3), the gauge symmetries (6.19) read

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu\hat{\xi}_\nu + \partial_\nu\hat{\xi}_\mu + \frac{2}{M_P}\mathcal{L}_{\hat{\xi}}\hat{h}_{\mu\nu}, \\ \delta\hat{A}_\mu &= \partial_\mu\hat{\Lambda} - m\hat{\xi}_\mu + \frac{2}{M_P}\hat{\xi}^\nu\partial_\nu\hat{A}_\mu \\ &\quad - \frac{2}{mM_P^2}\hat{A}^\alpha\hat{A}^\beta\partial_\alpha\partial_\beta\hat{\xi}_\mu - \cdots, \\ \delta\phi &= -m\hat{\Lambda}, \end{aligned} \tag{7.7}$$

where we rescaled $\hat{\Lambda} = (mM_P/2)\Lambda$ and $\hat{\xi}^\mu = (M_P/2)\xi^\mu$.

Finally, note that since the scalar field $\phi$ always appears with at least two derivatives in the Stückelberg replacement (7.2), the resulting action is automatically invariant under the global *Galilean symmetry*

$$\phi(x) \to c + b_\mu x^\mu, \tag{7.8}$$

where $c$ and $b_\mu$ are constants. In addition, the action is automatically invariant under global shifts in $A_\mu \to A_\mu + c_\mu$ for constant $c_\mu$. It will persist even in limits where the gauge symmetries on $A_\mu$ and $\phi$ no longer act.

### A. Decoupling limit and breakdown of linearity

As seen in Sec. IV.B, the propagators have all been made to go as $\sim 1/p^2$, so normal power counting applies, and the lowest scale $\Lambda_5$ is the cutoff of the effective field theory. To focus in on the cutoff scale, we take the decoupling limit

$$m \to 0, \quad M_P \to \infty, \quad T \to \infty, \qquad \Lambda_5, \quad \frac{T}{M_P} \text{ fixed.} \tag{7.9}$$

All interaction terms go to zero, except for the scalar cubic term (7.6) responsible for the strong coupling, which we calculate using the replacement $H_{\mu\nu} = 2\partial_\mu\partial_\nu\phi + \partial_\mu\partial^\alpha\phi\partial_\nu\partial_\alpha\phi$ since we do not need the vector and tensor terms. As discussed in Sec. IV.B, we must also do the conformal transformation $h_{\mu\nu} = h'_{\mu\nu} + m^2\phi\eta_{\mu\nu}$. This will diagonalize all the kinetic terms (except for various cross terms proportional to $m$ which are eliminated with appropriate gauge fixing terms, as discussed in Sec. IV.B, and which go to zero anyway in the decoupling limit).

After all this, the Lagrangian for the scalar reads, up to a total derivative,

$$\begin{aligned} S_\phi = \int d^4x - 3(\partial\hat{\phi})^2 + \frac{2}{\Lambda_5^5}[(\Box\hat{\phi})^3 - (\Box\hat{\phi})(\partial_\mu\partial_\nu\hat{\phi})^2] \\ + \frac{1}{M_P}\hat{\phi}T. \end{aligned} \tag{7.10}$$

The free graviton coupled to the source via $(1/M_P)\hat{h}'_{\mu\nu}T^{\mu\nu}$ also survives the limit, as does the free decoupled vector.

We can now understand the origin of the Vainshtein radius at which the linear expansion breaks down around heavy point sources. The scalar couples to the source through the trace, $(1/M_P)\hat{\phi}T$. To linear order around a central source of mass $M$, we have

$$\hat{\phi} \sim \frac{M}{M_P}\frac{1}{r}. \tag{7.11}$$

The nonlinear term is suppressed relative to the linear term by the factor

$$\frac{\partial^4\hat{\phi}}{\Lambda_5^5} \sim \frac{M}{M_P}\frac{1}{\Lambda_5^5 r^5}. \tag{7.12}$$

Nonlinearities become important when this factor becomes of order 1, which happens at the radius

$$r_V \sim \left(\frac{M}{M_P}\right)^{1/5}\frac{1}{\Lambda_5} \sim \left(\frac{GM}{m^4}\right)^{1/5}. \tag{7.13}$$

When $r \lesssim r_V$, linear perturbation theory breaks down and nonlinear effects become important. This is exactly the Vainshtein radius found in Sec. V.B by directly calculating the second order correction to spherical solutions.

In the decoupling limit, the gauge symmetries (7.7) reduce to their linear forms,

$$\delta h_{\mu\nu} = \partial_\mu\hat{\xi}_\nu + \partial_\nu\hat{\xi}_\mu, \qquad \delta\hat{A}_\mu = \partial_\mu\hat{\Lambda}, \qquad \delta\phi = 0. \tag{7.14}$$

Even though $\phi$ is gauge invariant in the decoupling limit, the fact that it always comes with two derivatives means that the global Galileon symmetry (7.8) is still present, as is the shift symmetry on $A_\mu$.

### B. Ghosts

Note that the Lagrangian (7.10) is a higher derivative action, and its equations of motion are fourth order. This means that this Lagrangian actually propagates two Lagrangian degrees of freedom rather than one, since we need to specify twice as many initial conditions to uniquely solve the fourth order equations of motion (de Urries and Julve, 1998), and by Ostrogradski's theorem (Ostrogradski, 1850; Woodard, 2007), one of these degrees of freedom is a ghost. The decoupling limit contains 6 degrees of freedom: two in the massless tensor, two in the free vector, and two in the scalar. This matches the number of degrees of freedom in the full theory as determined in Sec. V.C, so the decoupling limit we have taken is smooth. The extra ghostly scalar degree of freedom is the Boulware-Deser ghost. Note that at linear order, the higher derivative scalar terms for the scalar are not visible, so the linear theory has only 5 degrees of freedom.

Following Creminelli *et al.* (2005), let us consider the stability of the classical solutions to Eq. (7.10) around a massive point source. We have a classical background $\Phi(r)$, which is a solution of the $\hat{\phi}$ equation of motion, and we expand the Lagrangian of Eq. (7.10) to quadratic order in the fluctuation $\varphi \equiv \hat{\phi} - \Phi$. The result is schematically

$$\mathcal{L}_\varphi \sim -(\partial\varphi)^2 + \frac{(\partial^2\Phi)}{\Lambda_5^5}(\partial^2\varphi)^2. \tag{7.15}$$

There is a four-derivative contribution to the $\varphi$ kinetic term, signaling that this theory propagates 2 linear degrees of freedom. As shown in Sec. 2 of Creminelli *et al.* (2005), one is stable and massless, and the other is a ghost with a mass of the order of the scale appearing in front of the higher derivative terms. So in this case the ghost has an $r$-dependent mass

$$m_{\rm ghost}^2(r) \sim \frac{\Lambda_5^5}{\partial^2\Phi(r)}. \tag{7.16}$$

This shows that around a flat background, or far from the source, the ghost mass goes to infinity and the ghost freezes, explaining why it was not seen in the linear theory. It is only around nontrivial backgrounds that it becomes active. Notice, however, that the backgrounds around which the ghost becomes active are perfectly nice, asymptotically flat configurations sourced by compact objects such as the Sun, and not disconnected in any way in field space (this is in contrast to the ghost in DGP, which occurs around only asymptotically de Sitter solutions).

We are working in an effective field theory with a UV cutoff $\Lambda_5$; therefore we should not worry about instabilities until the mass of the ghost drops below $\Lambda_5$. This happens at the distance $r_{\rm ghost}$ where $\partial^2\Phi^c \sim \Lambda_5^3$. For a source of mass $M$, at distances $r \gg r_V$ the background field is similar to $\Phi(r) \sim (M/M_P)(1/r)$, so

$$r_{\rm ghost} \sim \left(\frac{M}{M_P}\right)^{1/3}\frac{1}{\Lambda_5} \gg r_V \sim \left(\frac{M}{M_P}\right)^{1/5}\frac{1}{\Lambda_5}. \tag{7.17}$$

$r_{\rm ghost}$ is parametrically larger than the Vainshtein radius $r_V$.

As we see in Sec. VII.D, the distance $r_{\rm ghost}$ is the same distance at which quantum effects become important. Whatever UV completion takes over should cure the ghost instabilities that become present at this scale, so we will be able to consistently ignore the ghost. We see already that we cannot trust the classical solution even in regions parametrically farther than the Vainshtein radius. The best we can do is make predictions outside $r_{\rm ghost}$, and we have more to say about this later.

### C. Resolution of the vDVZ discontinuity and the Vainshtein mechanism

We are now in a position to see the mechanism by which nonlinearities can resolve the vDVZ discontinuity. This is known as the Vainshtein mechanism. It turns out to involve the ghost in a critical role.

Far outside the Vainshtein radius, where the linear term of Eq. (7.10) dominates, the field has the usual Coulombic $1/r$ form. But inside the Vainshtein radius, where the cubic term dominates, it is easy to see by power counting that the field gets an $r^{3/2}$ profile,

$$\hat{\phi} \sim \begin{cases} \frac{M}{M_P}\frac{1}{r}, & r \gg r_V, \\ \left(\frac{M}{M_P}\right)^{1/2}\Lambda_5^{5/2} r^{3/2}, & r \ll r_V. \end{cases} \tag{7.18}$$

At distances much below the Vainshtein radius, the ghost mass (7.16) becomes very small, and the ghost starts to mediate a long-range force. Usually a scalar field mediates an attractive force, but due to the ghost's wrong sign kinetic term, the force mediated by it is repulsive. In fact, it cancels the attractive force due to the longitudinal mode, the force responsible for the vDVZ discontinuity, and so general relativity is restored inside the Vainshtein radius.

We now see this more explicitly. Following Deffayet and Rombouts (2005), some field redefinitions can be done on the scalar action (7.10), and the result is an action schematically of the form

$$\mathcal{L} = -(\partial\tilde{\phi})^2 + (\partial\psi)^2 + \Lambda_5^{5/2}\psi^{3/2} + \frac{1}{M_P}\tilde{\phi}T + \frac{1}{M_P}\psi T.$$

Here $\tilde{\phi}$ is the healthy longitudinal mode, $\psi$ is the ghost mode, and the original scalar can be found from $\hat{\phi} = \tilde{\phi} - \psi$. Both are coupled gravitationally to the stress tensor. Note that the self-interactions appear in these variables as a peculiar nonanalytic $\psi^{3/2}$ term (we can also see that the ghost mass around a background $\langle\psi\rangle$ will be $\Lambda_5^{5/2}/\langle\psi\rangle^{1/2}$). The $\tilde{\phi}$ field is free and has the profile $\tilde{\phi} \sim (M/M_P)(1/r)$ everywhere, mediating an attractive force.

The $\psi$ field, however, has two competing terms, which become comparable at the Vainshtein radius. The linear term dominates at radii *smaller* than the Vainshtein radius, so $\psi \sim (M/M_P)(1/r)$ for $r \ll r_V$. This profile generates a repulsive Coulomb force that exactly cancels the attractive force mediated by $\tilde{\phi}$, so in sum there are no extra forces beyond gravity in this region. [The leading correction to the profile is found by treating the $\psi^{3/2}$ term as a perturbation, $\psi \sim \psi_0 + \psi_{(1)} + \cdots$, with $\psi_0 \sim (M/M_P)(1/r)$, plugging in the equation of motion $\partial^2\psi_{(1)} + \Lambda_5^{5/2}\psi_{(0)}^{1/2} = 0$ obtaining $\psi_{(1)} \sim (M/M_P)^{1/2}\Lambda_5^{5/2}r^{3/2}$, in agreement with Eq. (7.18).] The funny nonlinear term dominates at radii larger than the Vainshtein radius, so $\psi \sim (M/M_P)^2 1/\Lambda_5^5 r^6$ for $r \gg r_V$, and so the ghost profile is negligible in this region compared to the $\tilde{\phi}$ profile. Thus the ghost ceases to be active beyond the Vainshtein radius, and the longitudinal mode generates a fifth force. This is known as a *screening mechanism*, a mechanism for rendering a light scalar inactive at short distances through nonlinearities [see the Introduction and references in Hinterbichler and Khoury (2010) and Hinterbichler, Khoury, and Nastase (2010), and in a different context (Gabadadze and Iglesias, 2008)].

One can think of this as a kind of classical version of a weakly coupled UV completion via a Higgs. Above the Vainshtein radius (low energies), there is only the long distance scalar, which starts to become nonlinear (strongly coupled) around the Vainshtein radius, so one can think of this regime in terms of an effective field theory with cutoff of the Vainshtein radius. Below the Vainshtein radius (high energies), a new degree of freedom, the ghost (analogous to the physical Higgs in the standard model), kicks in. Much below the Vainshtein radius, everything is again linear and weakly coupled, with the difference that there are now 2 active degrees of freedom, so one can think of this as a classical UV completion of the effective theory.

Of course, this ghostly mechanism for restoring continuity with GR relies on an instability, which would become apparent were we to investigate small fluctuations beyond the gross-scale features described here. Furthermore, as we see in the next section, the ghost issue is moot, since the classical mechanism described in this section occurs outside the regime of validity of the quantum effective theory and is swamped by unknown quantum corrections.

### D. Quantum corrections and the effective theory

Quantum mechanically, massive gravity is an effective field theory, since there are nonrenormalizable operators suppressed by the mass scale $\Lambda_5$. The amplitude for $\pi\pi \to \pi\pi$ scattering at energy $E$, coming from the cubic coupling in Eq. (7.10), is similar to $\mathcal{A} \sim (E/\Lambda_5)^{10}$. This amplitude should correspond to the scattering of longitudinal gravitons. The wave function of the longitudinal graviton (2.20) for a large boost is proportional to $m^{-2}$, while the largest term at high momentum in the graviton propagator (2.44) is proportional to $m^{-4}$, so naive power counting suggests that the amplitude at energies much larger than $m$ is similar to $\mathcal{A} \sim E^{14}/M_P^2 m^{12}$. However, as recognized by Arkani-Hamed, Georgi, and Schwartz (2003) and calculated explicitly by Aubert (2004), there is a cancellation in the diagrams so that the result agrees with the result of the Stückelberg description. We encounter these kinds of cancellations again in loops, and part of the usefulness of the Stückelberg description is that they are made manifest.

The amplitude becomes of the order of 1 and hence strongly coupled when $E \sim \Lambda_5$. Thus $\Lambda_5$ is the maximal cutoff of the theory. We expect to generate all operators compatible with the symmetries, suppressed by appropriate powers of the cutoff. In the unitary gauge, there are no symmetries, so we generate all operators of the form

$$c_{p,q}\partial^q h^p. \tag{7.19}$$

We determine the scales in the coefficient $c_{p,q}$.

After Stückelberging, the decoupling limit theory contains only the scalar $\hat{\phi}$ and the single coupling scale $\Lambda_5$. In addition, there is the Galileon symmetry $\hat{\phi} \to \hat{\phi} + c + c_\mu x^\mu$. Quantum mechanically, we expect to generate in the quantum effective action all possible operators with this symmetry, suppressed by the appropriate power of the cutoff $\Lambda_5$. The Galileon symmetry forces each $\hat{\phi}$ to carry at least two derivatives,[13] so the general term we can have is

$$\sim \frac{\partial^q(\partial^2\hat{\phi})^p}{\Lambda_5^{3p+q-4}}. \tag{7.20}$$

[13]Actually, there are a finite number of terms which have fewer than two derivatives per field, the so-called Galileon terms (Nicolis, Rattazzi, and Trincherini, 2008) which change by a total derivative under the Galileon symmetry (Nicolis, Rattazzi, and Trincherini, 2008). However, there is a nonrenormalization theorem that says these are not generated at any loop by quantum corrections (Hinterbichler, Trodden, and Wesley, 2010), so we need not include them. We encounter them later when we raise the cutoff to $\Lambda_3$.

To compare with Eq. (7.19), we go back to the original normalization for the fields by replacing $\hat{\phi} \sim m^2 M_P \phi$ and recall that $\partial_\mu \partial_\nu \phi$ comes from an $h_{\mu\nu}$ to find that in unitary gauge the coefficients $c_{p,q}$ are similar to

$$c_{p,q} \sim \Lambda_5^{-3p-q+4} M_P^p m^{2p} = (m^{16-4q-2p} M_P^{2p-q+4})^{1/5}. \tag{7.21}$$

This comparison is possible because the operations of taking the decoupling limit and computing quantum corrections should commute.

Notice that the term with $p = 2$, $q = 0$ is a mass term, $\sim (M_P^2 m^4/\Lambda_5^2) h^2$, corresponding to a mass correction $\delta m^2 = m^2(m^2/\Lambda_5^2)$. This is down by a factor of $m^2/\Lambda_5^2$ from the tree level mass term. Thus a small mass graviton $m \ll \Lambda_5$ is technically natural, and there is no quantum hierarchy problem associated with a small mass. This is in line with the general rule of thumb that a small term is technically natural if a symmetry emerges as the term is dialed to zero. In this case, it is the diffeomorphism symmetry of GR which is restored as the mass term goes to zero. The quantum mass correction will also generically ruin the Fierz-Pauli tuning, but its coefficient is small enough that ghosts and tachyons associated with the tuning violation are postponed to the cutoff; indeed the resulting ghost mass, using the relations in Sec. II, is $\sim \Lambda_5$.

It is important that there are no nonparametric modifications to the kinetic structure of the Einstein-Hilbert term, even though the lack of gauge symmetry suggests that we are free to make such modifications. Suppose we try to calculate the mass correction directly in unitary gauge. The graviton mass term contributes no vertices but alters the propagator so that its high energy behavior is $\sim k^2/m^4$ (the next leading terms are similar to $1/m^2$ and then $1/k^2$). At one loop, there are two one-particle irreducible diagrams correcting the mass: one containing two cubic vertices $(1/M_P)\partial^2 \hat{h}^3$ from the Einstein-Hilbert action and two propagators, and another containing a single quartic vertex $(1/M_P^2)\partial^2 \hat{h}^4$ from the Einstein-Hilbert action and a single propagator. Cutting off the loop at the momenta $k_{\max} \sim \Lambda_5$, the first diagram gives the largest naive correction $\delta m^2 \sim (1/M_P^2 m^8)\Lambda_5^{12} \sim \Lambda_5^2$. (The second diagram gives a smaller correction.) This is at the cutoff, dangerously higher than the small correction $\delta m^2 \sim m^2(m^2/\Lambda_5^2)$ we found in the Stückelberg formalism.

This means that there must be a nontrivial cancellation of this leading divergence in unitary gauge, so that we recover the Stückelberg result. This cancellation happens because the kinetic interactions of Einstein-Hilbert are gauge invariant, implying that the dangerous $k^\mu k^\nu k^\alpha k^\beta/m^4$ terms in the graviton propagator do not contribute. Without these terms, the propagator is similar to $1/m^2$ and the estimate for the first diagram is $\delta m^2 \sim (1/M_P^2 m^4)\Lambda_5^8 \sim m^2(m^2/\Lambda_5^2)$, in agreement with the Stückelberg prediction (again the second diagram again gives a smaller correction). Nonparametrically altering the coefficients in the kinetic structure would spoil this cancellation and the resulting technical naturalness of the small mass (although such alterations could be done without spoiling technical naturalness if the alterations to the kinetic terms are parametrically suppressed by appropriate powers of $m$). These kinds of cancellations can be seen explicitly in the calculations of Aubert (2004). Some loop calculations for massive gravity have been done by Park (2010a, 2010b).

In summary, in unitary gauge the theory (5.1) in $D = 4$ is a natural effective field theory with a cutoff parametrically larger than the graviton mass, with the effective action

$$S = \int d^4x \frac{M_P^2}{2}\left[\sqrt{-g}R - \frac{m^2}{4}(h_{\mu\nu}^2 - h^2)\right] + \sum_{p,q} c_{p,q} \partial^q h^p, \tag{7.22}$$

and a cutoff $\Lambda_5 = (m^4 M_P)^{1/5}$.

We take into account the effect that the unknown quantum operators have on the solution around a heavy source. Given that the linear field is similar to $\hat{\phi} \sim (M/M_P)(1/r)$, the radius $r_{p,q}$ at which the term (7.20) becomes comparable to the kinetic term $(\partial\hat{\phi})^2$ is

$$r_{p,q} \sim \left(\frac{M}{M_{\rm Pl}}\right)^{(p-2)/(3p+q-4)} \frac{1}{\Lambda_5}. \tag{7.23}$$

This distance increases with $p$ and asymptotes to its highest value

$$r_Q \sim \left(\frac{M}{M_{\rm Pl}}\right)^{1/3} \frac{1}{\Lambda_5}. \tag{7.24}$$

Thus we cannot trust the classical solution at distances below $r_Q$, since quantum operators become important there. This distance is parametrically larger than the Vainshtein radius, where classical nonlinearities become important. Unlike the case in GR, there is no intermediate regime where the linear approximation breaks down but quantum effects are still small, so there is no sense in which a nonlinear solution to massive gravity can be trusted for making real predictions in light of quantum mechanics.

In particular, the entire ghost screening mechanism of Sec. VII.C is in the nonlinear regime, and so it becomes swamped in quantum corrections. Thus there is no regime for which GR is a good approximation; the theory transitions directly from the linear classical regime with a long-range fifth force scalar, to the full quantum regime. Note that it is the higher dimension operators that become important first,

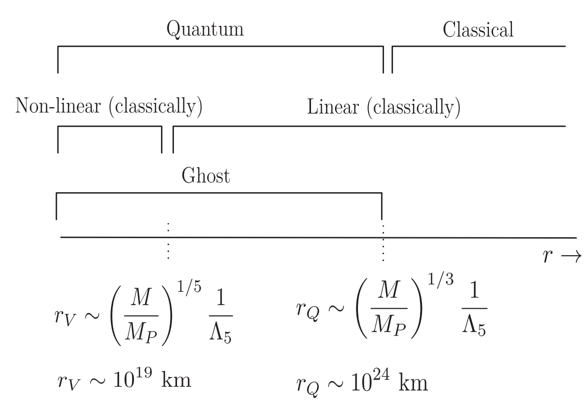


FIG. 1. Regimes for massive gravity with cutoff $\Lambda_5 = (M_P m^4)^{1/5}$, and some values within the Solar System (i.e., $M$ is the solar mass and $m$ is taken to be the Hubble scale), for which $\Lambda_5^{-1} \sim 10^{11}$ km. Note that $r_Q$ is a bit larger than the observable universe, i.e., this theory makes no observable predictions within its range of validity.

so there is no hope of finding the leading quantum corrections. Finally, the radius $r_Q$ is the same as the radius $r_{\rm ghost}$, where the ghost mass drops below the cutoff, so it is consistent to ignore the ghost since it lies beyond the reach of the quantum effective theory. The various regions are shown in Fig. 1. Note that in the decoupling limit we are working in, the Schwarzschild radius (and the radii associated to all scales larger than $\Lambda_5$) are sent to zero, while the scale $r \sim 1/m$ where Yukawa suppression takes hold is sent to infinity.

## VIII. THE $\Lambda_3$ THEORY

We have seen that the theory (5.1) containing only the linear graviton mass term has some undesirable features, including a ghost instability and quantum corrections that become important before classical nonlinearities can restore continuity with GR. In this section, we consider the higher order potential terms in Eq. (5.3) and ask whether they can alleviate these problems. It turns out that there is a special choice of potential that cures all these problems, at least in the decoupling limit.

This choice also has the advantage of raising the cutoff. With only the Fierz-Pauli mass term, the strong coupling cutoff was set by the cubic scalar self-coupling $\sim(\partial^2\hat{\phi})^3/\Lambda_5^5$. The cutoff $\Lambda_5 = (M_P m^4)^{1/5}$ is very low, and as we see generically any interaction term will have this cutoff. But by choosing this special tuning of the higher order interactions, we end up raising the cutoff to the higher scale $\Lambda_3 = (M_P m^2)^{1/3}$.

Arkani-Hamed, Georgi, and Schwartz (2003) already recognized that if the scalar self-interactions could be eliminated, the cutoff would be raised to $\Lambda_3$. This was studied more fully by Creminelli *et al.* (2005), where the cancellation was worked through and it was (mistakenly) concluded that ghosts would be unavoidable once the cutoff was raised. Motivated by constructions of massive gravity with auxiliary extra dimensions (Gabadadze, 2009; de Rham, 2010; de Rham and Gabadadze, 2010b), this was revisited by de Rham and Gabadadze (2010a) and de Rham, Gabadadze, and Tolley (2010), where the decoupling limit Lagrangian was calculated explicitly and was seen to be ghost free. The full theory was shown to be ghost free by Hassan and Rosen (2011a, 2011c).

### A. Tuning interactions to raise the cutoff

Looking back at the scales (7.5), the term suppressed by the smallest scale is the cubic scalar term, which is suppressed by the scale $\Lambda_5 = (M_P m^4)^{1/5}$,

$$\sim \frac{(\partial^2\hat{\phi})^3}{M_P m^4}. \tag{8.1}$$

The next highest scale is $\Lambda_4 = (M_P m^3)^{1/4}$, carried by a quartic scalar interaction, and a cubic term with a single vector and two scalars,

$$\sim \frac{(\partial^2\hat{\phi})^4}{M_P^2 m^6}, \qquad \sim \frac{\partial\hat{A}(\partial^2\hat{\phi})^2}{M_P m^3}. \tag{8.2}$$

The next highest is a quintic scalar, and so on. The only terms which carry a scale less than $\Lambda_3 = (M_P m^2)^{1/3}$ are terms with only scalars $(\partial^2\hat{\phi})^n$, and terms with one vector and the rest scalars $\partial\hat{A}(\partial^2\hat{\phi})^n$.

The scale $\Lambda_3$ is carried only by the following terms:

$$\sim \frac{\hat{h}(\partial^2\hat{\phi})^n}{M_P^{n-1} m^{2n-2}}, \qquad \sim \frac{(\partial\hat{A})^2(\partial^2\hat{\phi})^n}{M_P^n m^{2n}}. \tag{8.3}$$

All other terms carry scales higher than $\Lambda_3$.

It turns out that we can arrange to cancel all of the scalar self-couplings by appropriately choosing the coefficients of the higher order terms. We work with the form of the potential in Eq. (5.9) where indices are raised with the full metric, and the Stückelberg formalism of Sec. VI.B. We do so because we eventually want to keep track of powers of $h$, so the form of the Stückelberg replacement in Sec. VI.B is simpler. We are interested only in scalar self-interactions, so we may make the replacement (6.31) with the vector field set to zero,

$$H_{\mu\nu} \to 2\partial_\mu\partial_\nu\phi - \partial_\mu\partial_\alpha\phi\partial_\nu\partial^\alpha\phi. \tag{8.4}$$

The interaction terms are a function of the matrix of second derivatives $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\phi$. As reviewed in the Appendix, there is at each order in $\phi$ a single polynomial in $\Pi_{\mu\nu}$ which is a total derivative. By choosing the coefficients (5.9) correctly, we can arrange for the $\phi$ terms to appear in these total derivative combinations. The total derivative combinations have at each order in $\phi$ as many terms as there are terms in the potential of Eq. (5.9), so all the coefficients must be fixed, except for one at each order which becomes the overall coefficient of the total derivative combination.

The choice of coefficients in the potential (5.9) which removes the scalar self-interactions is, to fifth order (de Rham and Gabadadze, 2010a),

$$c_1 = 2c_3 + \tfrac{1}{2}, \qquad c_2 = -3c_3 - \tfrac{1}{2}, \tag{8.5}$$

$$\begin{aligned} d_1 &= -6d_5 + \tfrac{1}{16}(24c_3 + 5), \\ d_2 &= 8d_5 - \tfrac{1}{4}(6c_3 + 1), \\ d_3 &= 3d_5 - \tfrac{1}{16}(12c_3 + 1), \\ d_4 &= -6d_5 + \tfrac{3}{4}c_3, \end{aligned} \tag{8.6}$$

$$\begin{aligned} f_1 &= \tfrac{7}{32} + \tfrac{9}{8}c_3 - 6d_5 + 24f_7, \\ f_2 &= -\tfrac{5}{32} - \tfrac{15}{16}c_3 + 6d_5 - 30f_7, \\ f_3 &= \tfrac{3}{8}c_3 - 3d_5 + 20f_7, \\ f_4 &= -\tfrac{1}{16} - \tfrac{3}{4}c_3 + 5d_5 - 20f_7, \\ f_5 &= \tfrac{3}{16}c_3 - 3d_5 + 15f_7, \\ f_6 &= d_5 - 10f_7. \end{aligned} \tag{8.7}$$

At each order, there is a one-parameter family of choices that works to create a total derivative. Here $c_3$, $d_5$, and $f_7$ are chosen to carry that parameter at order 3, 4, and 5, respectively. Note, however, that at order 5 and above (or $D+1$ and above if we were doing this in $D$ dimensions), there is one linear combination of all the terms, the characteristic polynomial of $h$ mentioned below Eq. (5.8) that vanishes identically. This means that one of the coefficients is redundant, and we can, in fact, set $d_5$ and its higher counterparts to any value we like without changing the theory. Thus there is only a two parameter family ($D-2$ parameter in dimension $D$) of theories with no scalar self-interactions. This can be carried through at all orders, and at the end there will be no terms $\sim(\partial^2\phi)^n$.

The only terms with interaction scales lower than $\Lambda_3$ were the scalar self-interactions $(\partial^2\hat{\phi})^n$, and terms with one vector and the rest scalars $\partial\hat{A}(\partial^2\hat{\phi})^n$. We succeeded in eliminating the scalar self-interactions, but since these always came from combinations $(A_\mu+\partial_\mu)$ the terms $\partial\hat{A}(\partial^2\hat{\phi})^n$ are automatically of the form $\partial^\mu A^\nu X^{(n)}_{\mu\nu}$, where the $X^{(n)}_{\mu\nu}$ are the functions of $\partial_\mu\partial_\nu\phi$ described in the Appendix, which are identically conserved $\partial^\mu X^{(n)}_{\mu\nu}=0$. Thus, once the scalar self-interactions are eliminated, the $\partial\hat{A}(\partial^2\hat{\phi})^n$ terms are all total derivatives and are also eliminated.

Now the lowest interaction scale will be due to the terms in Eq. (8.3),

$$\sim\frac{\hat{h}(\partial^2\hat{\phi})^n}{M_P^{n+1}m^{2n+2}},\qquad \sim\frac{(\partial\hat{A})^2(\partial^2\hat{\phi})^n}{M_P^{n+2}m^{2n+4}},\tag{8.8}$$

which are suppressed by the scale $\Lambda_3=(M_Pm^2)^{1/3}$, so the cutoff has be raised to $\Lambda_3$, carried by the terms (8.8).

This theory can, in fact, be resummed in an interesting way, using an action involving square roots (de Rham, Gabadadze, and Tolley, 2010; Hassan and Rosen, 2011b).

The decoupling limit is now

$$m\to 0,\quad M_P\to\infty,\qquad \Lambda_3\ \text{fixed},\tag{8.9}$$

and the only terms which survive are those in Eq. (8.3). To find these terms we must now go back to the full Stückelberg replacement (6.31), and we must also expand the inverse metric and determinant in the potential of Eq. (5.9) in powers of $h$. The $h(\partial^2\phi)^n$ terms, up to quintic order in the decoupling limit, and up to total derivatives are (de Rham and Gabadadze, 2010a)

$$\begin{aligned}S=\int d^4x\frac{1}{2}\hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}\hat{h}_{\alpha\beta}-\frac{1}{2}\hat{h}^{\mu\nu}\Big[-4X^{(1)}_{\mu\nu}(\hat{\phi})\\ +\frac{4(6c_3-1)}{\Lambda_3^3}X^{(2)}_{\mu\nu}(\hat{\phi})+\frac{16(8d_5+c_3)}{\Lambda_3^6}X^{(3)}_{\mu\nu}(\hat{\phi})\Big]\\ +\frac{1}{M_P}\hat{h}_{\mu\nu}T^{\mu\nu}.\end{aligned}\tag{8.10}$$

Here the $X^{(n)}_{\mu\nu}$ are the identically conserved combinations of $\partial_\mu\partial_\nu\hat{\phi}$ described in the Appendix. The $(\partial A)^2(\partial^2\phi)$ terms are found to cubic order by de Rham and Gabadadze (2010b). The terms with $A$'s can in any case be consistently set to zero at the classical level, since they never appear linearly in the Lagrangian, so we focus only on the terms involving $h$ and $\phi$. de Rham, Gabadadze, and Tolley (2010) used a nice trick to show that the decoupling limit Lagrangian (8.10) is exact to all orders in the fields, that is, there are no further terms $h(\partial^2\phi)^n$ for $n\geq 4$. Properties of this Lagrangian, including its cosmological solutions, degravitation effects, and phenomenology are studied by de Rham *et al.* (2010). Spherical solutions are studied by Chkareuli and Pirtskhalava (2011). The cosmology of a covariantized version was studied by de Rham and Heisenberg (2011).

In terms of the canonically normalized fields (7.3), the gauge symmetries (6.34) of the full theory are

$$\delta\hat{A}_\mu=\partial_\mu\hat{\Lambda}-m\hat{\xi}_\mu+\frac{2}{M_P}\hat{\xi}^\nu\partial_\nu\hat{A}_\mu,\tag{8.11}$$

$$\delta h_{\mu\nu}=\partial_\mu\hat{\xi}_\nu+\partial_\nu\hat{\xi}_\mu+\frac{2}{M_P}\mathcal{L}_{\hat{\xi}}\hat{h}_{\mu\nu},\tag{8.12}$$

$$\delta\phi=-m\hat{\Lambda},\tag{8.13}$$

where we rescaled $\hat{\Lambda}=(mM_P/2)\Lambda$ and $\hat{\xi}^\mu=(M_P/2)\xi^\mu$. In the decoupling limit (8.9), this gauge symmetry reduces to its linear form

$$\delta\hat{A}_\mu=\partial_\mu\hat{\Lambda},\tag{8.14}$$

$$\delta h_{\mu\nu}=\partial_\mu\hat{\xi}_\nu+\partial_\nu\hat{\xi}_\mu,\tag{8.15}$$

$$\delta\phi=0.\tag{8.16}$$

The Lagrangian (8.10) should be invariant under the decoupling limit gauge symmetries (8.16). Indeed, the identity $\partial^\mu X^{(n)}_{\mu\nu}=0$ ensures that it is. The scalar $\phi$ is gauge invariant in the decoupling limit, but the fact that it always comes with two derivatives means that the global Galileon symmetry (7.8) is still present, as is the shift symmetry on $A_\mu$.

Note that for the specific choices $c_3=1/6$ and $d_5=-1/48$, all the interaction terms disappear. This could mean that the theory becomes strongly coupled at some scale larger than $\Lambda_3$, or there could be no lowest scale, since there are scales arbitrarily close to but above $\Lambda_3$. In the latter case, the theory would have no nonlinear behavior, and so no mechanism to recover continuity with GR, and it would therefore be ruled out observationally.

### B. The appearance of Galileons and the absence of ghosts

We partially diagonalize the interaction terms in Eq. (8.10) by using the properties (A.18). First we perform the conformal transformation needed to diagonalize the linear terms, $\hat{h}_{\mu\nu}\to\hat{h}_{\mu\nu}+\hat{\phi}\eta_{\mu\nu}$, after which the Lagrangian takes the form

$$\begin{aligned}S=\int d^4x\frac{1}{2}\hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}\hat{h}_{\alpha\beta}\\ -\frac{1}{2}\hat{h}^{\mu\nu}\Big[\frac{4(6c_3-1)}{\Lambda_3^3}\hat{X}^{(2)}_{\mu\nu}+\frac{16(8d_5+c_3)}{\Lambda_3^6}\hat{X}^{(3)}_{\mu\nu}\Big]\\ +\frac{1}{M_P}\hat{h}_{\mu\nu}T^{\mu\nu}-3(\partial\hat{\phi})^2+\frac{6(6c_3-1)}{\Lambda_3^3}(\partial\hat{\phi})^2\square\hat{\phi}\\ +\frac{16(8d_5+c_3)}{\Lambda_3^6}(\partial\hat{\phi})^2([\hat{\Pi}]^2-[\hat{\Pi}^2])+\frac{1}{M_P}\hat{\phi}T.\end{aligned}\tag{8.17}$$

Here the brackets are traces of $\hat{\Pi}_{\mu\nu}\equiv\partial_\mu\partial_\nu\hat{\phi}$ and its powers (the notation is explained at the end of the Introduction).

The cubic $h\phi\phi$ couplings can be eliminated with a field redefinition

$$\hat{h}_{\mu\nu}\to\hat{h}_{\mu\nu}+\frac{2(6c_3-1)}{\Lambda_3^3}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi},$$

after which the Lagrangian reads

$$\begin{aligned}S = \int d^4x \frac{1}{2}\hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}\hat{h}_{\alpha\beta} - \frac{8(8d_5+c_3)}{\Lambda_3^6}\hat{h}^{\mu\nu}\hat{X}^{(3)}_{\mu\nu} \\ + \frac{1}{M_P}\hat{h}_{\mu\nu}T^{\mu\nu} - 3(\partial\hat{\phi})^2 + \frac{6(6c_3-1)}{\Lambda_3^3}(\partial\hat{\phi})^2\Box\hat{\phi} \\ - 4\frac{(6c_3-1)^2 - 4(8d_5+c_3)}{\Lambda_3^6}(\partial\hat{\phi})^2([\hat{\Pi}]^2 - [\hat{\Pi}^2]) \\ - \frac{40(6c_3-1)(8d_5+c_3)}{\Lambda_3^9}(\partial\hat{\phi})^2([\hat{\Pi}]^3 - 3[\hat{\Pi}^2][\hat{\Pi}] \\ + 2[\hat{\Pi}^3]) + \frac{1}{M_P}\hat{\phi}T + \frac{2(6c_3-1)}{\Lambda_3^3 M_P}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}T^{\mu\nu}.\end{aligned} \quad (8.18)$$

There is no local field redefinition that can eliminate the $h\phi\phi\phi$ quartic mixing (there is a nonlocal redefinition that can do it), so this is as unmixed as the Lagrangian can get while staying local.

The scalar self-interactions in Eq. (8.18) are given by the following four Lagrangians:

$$\begin{aligned}&\mathcal{L}_2 = -\tfrac{1}{2}(\partial\phi)^2, \qquad \mathcal{L}_3 = -\tfrac{1}{2}(\partial\phi)^2[\Pi], \\ &\mathcal{L}_4 = -\tfrac{1}{2}(\partial\phi)^2([\Pi]^2 - [\Pi^2]), \\ &\mathcal{L}_5 = -\tfrac{1}{2}(\partial\phi)^2([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]).\end{aligned} \quad (8.19)$$

These are known as the *Galileon* terms (Nicolis, Rattazzi, and Trincherini, 2008) [see also Sec. II of Hinterbichler, Trodden, and Wesley (2010) for a summary of the Galileons]. They share two special properties: their equations of motion are purely second order (despite the appearance of higher derivative terms in the Lagrangians), and they are invariant up to a total derivative under the Galilean symmetry (7.8), $\phi(x) \to \phi(x) + c + b_\mu x^\mu$. As shown by Nicolis, Rattazzi, and Trincherini (2008), the terms (8.19) are the only polynomial terms in four dimensions with these properties.

The Galileon was first discovered in studies of the DGP brane world model (Dvali, Gabadadze, and Porrati, 2000a), for which the cubic Galileon $\mathcal{L}_3$ was found to describe the leading interactions of the brane bending mode (Luty, Porrati, and Rattazzi, 2003; Nicolis and Rattazzi, 2004). The rest of the Galileons were then discovered by Nicolis, Rattazzi, and Trincherini (2008), by abstracting the properties of the cubic term away from DGP. They have some other very interesting properties, such as a nonrenormalization theorem [see, e.g., Sec. VI of Hinterbichler, Trodden, and Wesley (2010)], and a connection to the Lovelock invariants through brane embedding (de Rham and Tolley, 2010). Because of these unexpected and interesting properties, they have since taken on a life of their own. They have been generalized in many directions (Deffayet, Deser, and Esposito-Farese, 2009, 2010; Deffayet, Esposito-Farese, and Vikman, 2009; Padilla, Saffin, and Zhou, 2010; Deffayet *et al.*, 2011; Goon, Hinterbichler, and Trodden, 2011; Khoury, Lehners, and Ovrut, 2011) and are the subject of much recent activity [see, for instance, the many papers citing Nicolis, Rattazzi, and Trincherini (2008)].

The fact that the equations are second order ensures that, unlike Eq. (7.10), no extra degrees of freedom propagate. In fact, as pointed out by de Rham and Gabadadze (2010a), the properties (A.17) of the tensors $X_{\mu\nu}$ guarantee that there are no ghosts in the Lagrangian (8.10) of the decoupling limit theory.[14] By going through a Hamiltonian analysis similar to that of Sec. II.A, we see that $h_{00}$ and $h_{0i}$ remain Lagrange multipliers enforcing first class constraints [as they should since the Lagrangian (8.10) is gauge invariant]. In addition, the equations of motion remain second order, so the decoupling limit Lagrangian (8.10) is free of the Boulware-Deser ghost and propagates 3 degrees of freedom around any background.

Once the 2 degrees of freedom of the vector $A_\mu$ are included, and if there are no ghosts in the vector part or its interactions, the total number of degrees of freedom goes to 5, the same as the linear massive graviton. The vector interactions were shown to be ghost free at cubic order by de Rham and Gabadadze (2010b). de Rham, Gabadadze, and Tolley (2010) showed that the full theory beyond the decoupling limit, including all the fields, is ghost free, up to quartic order in the fields. This guarantees that any ghost must carry a mass scale larger than $\Lambda_3$ and hence can be consistently excluded from the quantum theory. Finally, Hassan and Rosen (2011a, 2011c) showed, using the Hamiltonian formalism, that the full theory, including all modes and to all orders beyond the decoupling limit, carries 5 degrees of freedom. The $\Lambda_3$ theory is therefore free of the Boulware-Deser ghost, around any background. This can also been seen in the Stückelberg language (de Rham, Gabadadze, and Tolley, 2011a).

### C. The $\Lambda_3$ Vainshtein radius

We now derive the scale at which the linear expansion breaks down around heavy point sources in the $\Lambda_3$ theory. To linear order around a central source of mass $M$, the fields still have their usual Coulomb form

$$\hat{\phi}, \hat{h} \sim \frac{M}{M_P}\frac{1}{r}. \quad (8.20)$$

The nonlinear terms in Eqs. (8.10) or (8.18) are suppressed relative to the linear term by a different factor than in the $\Lambda_5$ theory,

$$\frac{\partial^2\hat{\phi}}{\Lambda_3^3} \sim \frac{M}{M_P}\frac{1}{\Lambda_3^3 r^3}. \quad (8.21)$$

Nonlinearities become important when this factor becomes of the order of 1, which happens at the radius

$$r_V^{(3)} \sim \left(\frac{M}{M_P}\right)\frac{1}{\Lambda_3} \sim \left(\frac{GM}{m^2}\right)^{1/3}. \quad (8.22)$$

This is parametrically larger than the Vainshtein radius found in the $\Lambda_5$ theory.

It is important that the decoupling limit Lagrangian is ghost free. To see what could go wrong if there were a ghost, expand around some spherical background $\hat{\phi} = \Phi(r) + \varphi$

[14]This is contrary to Creminelli *et al.* (2005) who claims that a ghost is still present at quartic order. As remarked, however, by de Rham and Gabadadze (2010a), they arrive at the incorrect decoupling limit Lagrangian, which can be traced to a minus sign mistake in their Eq. 5, which should be as in Eq. (8.4).

and similarly for $h_{\mu\nu}$. The cubic coupling and quartic couplings could possibly give fourth order kinetic contributions of the schematic form, respectively,

$$\frac{1}{\Lambda_3^3}\Phi(\partial^2\varphi)^2, \qquad \frac{1}{\Lambda_3^6}\Phi\partial^2\Phi(\partial^2\varphi)^2. \tag{8.23}$$

These would correspond to ghosts with $r$-dependent masses,

$$m^2_{\rm ghost}(r)\sim\frac{\Lambda_3^3}{\Phi}, \qquad \frac{\Lambda_3^6}{\Phi\partial^2\Phi}, \tag{8.24}$$

or, given that the background fields are similar to $\Phi\sim(M/M_P)(1/r)$,

$$m^2_{\rm ghost}(r)\sim\frac{M_P}{M}\Lambda_3^3 r, \qquad \left(\frac{M_P}{M}\right)^2\Lambda_3^6 r^4. \tag{8.25}$$

Thus the ghost mass sinks below the cutoff $\Lambda_3$ at the radius

$$r^{(3)}_{\rm ghost}\sim\left(\frac{M}{M_P}\right)\frac{1}{\Lambda_3}, \qquad \left(\frac{M}{M_P}\right)^{1/2}\frac{1}{\Lambda_3}. \tag{8.26}$$

As happened in the $\Lambda_5$ theory, these radii are parametrically larger than the Vainshtein radius. This is a fatal instability which renders the whole nonlinear region inaccessible, unless we lower the cutoff of the effective theory so that the ghost stays above it, in which case unknown quantum corrections would also kick in at $\sim r^{(3)}_{\rm ghost}$, swamping the entire nonlinear Vainshtein region.

### D. The Vainshtein mechanism in the $\Lambda_3$ theory

In the $\Lambda_5$ theory, the key to the resolution of the vDVZ discontinuity and recovery of GR was the activation of the Boulware-Deser ghost, which canceled the force due to the longitudinal mode. In the $\Lambda_3$ theory, there is no ghost (at least in the decoupling limit), so there must be some other method by which the scalar screens itself to restore continuity with general relativity. This method uses nonlinearities to enlarge the kinetic terms of the scalar, rendering its couplings small.

To see how this works, consider the Lagrangian in the form (8.17). Set $d_5=-c_3/8$, $c_3=5/36$ to simplify coefficients, and ignore for a second the cubic $h\phi\phi$ coupling, so that we have only a cubic $\phi$ self-interaction governed by the Galileon term $\mathcal{L}_3$,

$$S=\int d^4x-3(\partial\hat\phi)^2-\frac{1}{\Lambda^3}(\partial\hat\phi)^2\Box\hat\phi+\frac{1}{M_4}\hat\phi T. \tag{8.27}$$

This is the same Lagrangian studied by Nicolis and Rattazzi (2004) in the DGP context.

Consider the static spherically symmetric solution $\hat\phi(r)$ around a point source of mass $M$, $T\sim M\delta^3(r)$. The solution transitions, at the Vainshtein radius $r_V^{(3)}\equiv(M/M_{\rm Pl})^{1/3}(1/\Lambda_3)$, between a linear and nonlinear regime. For $r\gg r_V^{(3)}$ the kinetic term in Eq. (8.27) dominates over the cubic term, linearities are unimportant, and we get the usual $1/r$ Coulomb behavior. For $r\ll r_V^{(3)}$, the cubic term is dominant, and we get a nonlinear $\sqrt{r}$ potential,

$$\hat\phi(r)\sim\begin{cases}\Lambda_3^3 r_V^{(3)2}\left(\frac{r}{r_V^{(3)}}\right)^{1/2} & r\ll r_V^{(3)},\\ \Lambda_3^3 r_V^{(3)2}\left(\frac{r_V^{(3)}}{r}\right) & r\gg r_V^{(3)}.\end{cases} \tag{8.28}$$

We can see the Vainshtein mechanism at work already by calculating the ratio of the fifth force due to the scalar to the force from ordinary Newtonian gravity,

$$\frac{F_\phi}{F_{\rm Newton}}=\frac{\hat\phi'(r)/M_P}{M/M_P^2r^2}\sim\begin{cases}\left(\frac{r}{r_V^{(3)}}\right)^{3/2} & r\ll r_V^{(3)},\\ 1 & r\gg r_V^{(3)}.\end{cases} \tag{8.29}$$

There is a gravitational strength fifth force at distances much farther than the Vainshtein radius, but the force is suppressed at distances smaller than the Vainshtein radius.

This suppression extends to all scalar interactions in the presence of the source. To see how this comes about, we study perturbations around a given background solution $\Phi(x)$. Expanding

$$\hat\phi=\Phi+\varphi, \qquad T=T_0+\delta T, \tag{8.30}$$

we have after using the identity $(\partial^\mu\varphi)\Box\varphi=\partial_\nu[\partial^\nu\varphi\partial^\mu\varphi-\frac12\eta^{\mu\nu}(\partial\varphi)^2]$ on the quadratic parts and integrating by parts

$$S_\varphi=\int d^4x-3(\partial\varphi)^2+\frac{2}{\Lambda^3}(\partial_\mu\partial_\nu\Phi-\eta_{\mu\nu}\Box\Phi)\partial^\mu\varphi\partial^\nu\varphi \\ -\frac{1}{\Lambda^3}(\partial\varphi)^2\Box\varphi+\frac{1}{M_4}\varphi\delta T. \tag{8.31}$$

Note that expanding the cubic term yields new contributions to the kinetic terms, with coefficients that depend on the background. Unlike the $\Lambda_5$ Lagrangian (7.10), no higher derivative kinetic terms are generated, so no extra degrees of freedom are propagated on any background. This is a property shared by all the Galileon Lagrangians (8.19) (Endlich *et al.*, 2010).

Around the solution (8.28), the coefficient of the kinetic term in Eq. (8.31) is $\mathcal{O}(1)$ at distances $r\gg r_V^{(3)}$, but goes as $(r_V^{(3)}/r)^{3/2}$ for distances $r\ll r_V^{(3)}$. Thus the kinetic term is enhanced at distances below the Vainshtein radius, which means that after canonical normalization the couplings of the fluctuations to the source are reduced. The fluctuations $\varphi$ effectively decouple near a large source, so the scalar force between two small test particles in the presence of a large source is reduced, and continuity with GR is restored. A more careful study of the Vainshtein screening in the $\Lambda_3$ theory, including numerical solutions of the decoupling limit action, can be found in Chkareuli and Pirtskhalava (2011).

### E. Quantum corrections in the $\Lambda_3$ theory

As in Sec. VII.D, we expect quantum mechanically the presence of all operators with at least two derivatives per $\phi$, now suppressed by the cutoff $\Lambda_3$ (we ignore for simplicity the scalar-tensor interactions),

$$\sim\frac{\partial^q(\partial^2\hat\phi)^p}{\Lambda_3^{3p+q-4}}. \tag{8.32}$$

These are in addition to the classical Galileon terms in Eq. (8.18), which have fewer derivatives per $\phi$ and are of the form

$$\sim\frac{(\partial\hat\phi)^2(\partial^2\hat\phi)^p}{\Lambda_3^{3p}}. \tag{8.33}$$

An analysis similar to that of Sec. VII.D shows that the terms (8.32) become important relative to the kinetic term at the radius $r \sim (M/M_{Pl})^{1/3}(1/\Lambda_3)$. This is the same radius at which classical nonlinear effects due to Eq. (8.33) become important and alter the solution from its Coulomb form. Thus we must instead compare the terms (8.32) to the classical nonlinear Galileon terms (8.33). We see that the terms (8.32) are all suppressed relative to the Galileon terms (8.33) by powers of $\partial/\Lambda_3$, which is $\sim 1/\Lambda_3 r$ regardless of the nonlinear solution. Thus, quantum effects do not become important until the radius

$$r_Q \sim \frac{1}{\Lambda_3}, \tag{8.34}$$

which is parametrically smaller than the Vainshtein radius (8.22).

This behavior is much improved from that of the $\Lambda_5$ theory, in which the Vainshtein region was swamped by quantum correction. Here there is a parametrically large intermediate classical region in which nonlinearities are important but quantum effects are not, and in which the Vainshtein mechanism should screen the extra scalar. In this region, GR should be a good approximation; see Fig. 2.

As in the $\Lambda_5$ theory, quantum corrections are generically expected to ruin the various classical tunings for the coefficients, but the tunings are still technically natural because the corrections are parametrically small. For example, cutting off loops by $\Lambda_3$, we generate the operator $\sim(1/\Lambda_3^2)(\Box\hat{\phi})^2$, which corrects the mass term. The canonically normalized $\hat{\phi}$ is related to the original dimensionless metric by $h \sim (1/\Lambda_3^3)\partial\partial\hat{\phi}$, so the generated term corresponds in unitary gauge to $\Lambda_3^4 h^2 = M_p^2 m^2(\Lambda_3/M_p)h^2$, representing a mass correction $\delta m^2 \sim m^2(\Lambda_3/M_p)$. This mass correction is parametrically smaller than the mass itself and so the hierarchy $m \ll \Lambda_3$ is technically natural. This correction also ruins the Fierz-Pauli tuning, but the pathology associated with the detuning of Fierz-Pauli, the ghost mass, is $m_g^2 \sim m^2/(\delta m^2/m^2) \sim \Lambda_3^2$, safely at the cutoff.

We mention another potential issue with the $\Lambda_3$ theory. It was found by Nicolis, Rattazzi, and Trincherini (2008) that Lagrangians of the Galileon type inevitably have superluminal propagation around spherical background solutions. No matter what the choice of parameters in the Lagrangian, if the solution is stable, then superluminality is always present at distances far enough from the source [see also Osipov and Rubakov (2008)]. It was argued that such superluminality is a sign that the theory cannot be UV completed by a standard local Lorentz invariant theory (Adams et al., 2006), though this remains controversial and others have argued that this is not a problem (Babichev, Mukhanov, and Vikman, 2008). In addition, the analysis of Nicolis, Rattazzi, and Trincherini (2008) was for pure Galileons only, and the scalar-tensor couplings of the massive gravity Lagrangian can potentially change the story. These issues have been studied within massive gravity by Gruzinov (2011) and de Rham, Gabadadze, and Tolley (2011b).

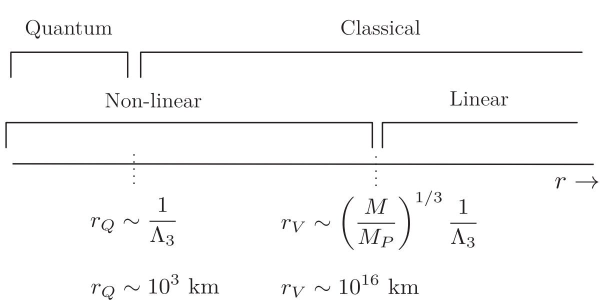


FIG. 2. Regimes for massive gravity with cutoff $\Lambda_3 = (M_P m^2)^{1/3}$ (i.e., $M$ is the solar mass and $m$ is taken to be the Hubble scale) and some values within the Solar System. The values are much more reasonable than those of the $\Lambda_5$ theory.

## IX. BRANE WORLDS AND THE RESONANCE GRAVITON

So far, we have stuck to the effective field theorist's philosophy. We have explored the possibility of a massive graviton by simply writing down the most general mass term a graviton can have, remaining agnostic as to its origin. However, it is important to ask whether such a mass term has a top down construction or embedding into a wider structure, one which would determine the coefficients of all the various interactions. This goes back to the question of whether it is possible to UV complete (or UV extend) the effective field theory of a massive graviton.

One way in which a massive graviton naturally arises is from higher dimensions. We now study one of these higher dimensional scenarios, the DGP brane world model, showing how massive gravitons emerge in a $4d$ description.

The DGP model (Dvali, Gabadadze, and Porrati, 2000a) is an extra-dimensional model which has spawned a great deal of interest [see the many papers citing Dvali, Gabadadze, and Porrati (2000a)]. It provides another, more novel realization of a graviton mass. Unlike the Kaluza-Klein scenario, in DGP the extra dimensions can be infinite in extent, although there must be a brane on which to confine standard model matter [see (Gabadadze (2003) for discussion on large extra dimensions]. By integrating out the extra dimensions, we can write an effective $4d$ action for this scenario which contains a momentum dependent mass term for the graviton. This provides an example of a graviton resonance, i.e., a continuum of massive gravitons.

Another model that has received a great deal of attention is the Randall-Sundrum brane world (Randall and Sundrum, 1999), in which there is a brane floating in large warped extra dimensions. This model is not as interesting from the point of view of massive gravity at low energies, since the $4d$ spectrum is similar to ordinary Kaluza-Klein theory, containing ordinary Einstein gravity as a zero mode, and then massive gravitons as higher Kaluza-Klein modes. See Langlois (2002), Brax, van de Bruck, and Davis (2004), Kiritsis (2005), and Maartens and Koyama (2010) for other reviews on aspects of brane world gravity and cosmology.

### A. The DGP action

DGP is the model of a $(3+1)$-dimensional brane (the 3-brane) floating in a $(4+1)$-dimensional bulk spacetime. Gravity is dynamical in the bulk and the brane position is dynamical as well, and the action contains both $4d$ and $5d$ parts,

$$S = \frac{M_5^3}{2}\int d^5X\sqrt{-G}R(G) + \frac{M_4^2}{2}\int d^4x\sqrt{-g}R(g) + S_M. \tag{9.1}$$

Here $X^A$, $A, B, \ldots = 0, 1, 2, 3$, and 5 are the $5d$ bulk coordinates, $G_{AB}(X)$ is the $5d$ metric, and $M_5$ is the $5d$ Planck mass. $x^\mu$, $\mu, \nu, \ldots = 0, 1, 2$, and 3 are the $4d$ brane coordinates, $g_{\mu\nu}(x)$ is the $4d$ metric which is given by inducing the $5d$ metric $G_{AB}$ onto the brane, and $M_4$ is the $4d$ Planck mass. $S_M$ is the matter action, which we imagine to be localized to the brane,

$$S_M = \int d^4x \mathcal{L}_M(g, \psi), \tag{9.2}$$

where $\psi(x)$ are the $4d$ matter fields. Because of the presence of a brane Einstein-Hilbert term, this scenario is also called *brane induced gravity* (Gabadadze, 2007) [see Kiritsis, Tetradis, and Tomaras (2001) and Antoniadis, Minasian, and Vanhove (2003) for attempts at string theory realizations].

The dynamical variables are the $5d$ metric depending on the $5d$ coordinates, the embedding $X^A(x)$ of the brane depending on the $4d$ coordinates, and the $4d$ matter fields depending on the $4d$ coordinates

$$G_{AB}(X), \quad X^A(x), \quad \psi(x). \tag{9.3}$$

The $4d$ metric is not independent, but is fixed to be the pullback of the $5d$ metric,

$$g_{\mu\nu}(x) = \partial_\mu X^A \partial_\nu X^B G_{AB}(X(x)). \tag{9.4}$$

Note that the dependence of the action on the $X^A$ enters only through the induced metric $g_{\mu\nu}$.

The action (9.1) has a lot of gauge symmetry. First, there are the reparametrizations of the brane given by infinitesimal vector fields $\xi^\mu(x)$, under which the $X^A$ transform as scalars and the matter fields transform as tensors (i.e., with a Lie derivative),

$$\delta_\xi X^A = \xi^\mu \partial_\mu X^A, \qquad \delta_\xi \psi = \mathcal{L}_\xi \psi. \tag{9.5}$$

Second, there are reparametrizations of the bulk given by infinitesimal vector fields $\Xi^A(X)$, under which $G_{AB}$ transforms as a tensor and the $X^A$ shift,

$$\delta_\Xi G_{AB} = \nabla_A \Xi_B + \nabla_B \Xi_A, \quad \delta_\Xi X^A = -\Xi^A(X). \tag{9.6}$$

The induced metric $g_{\mu\nu}$ transforms as a tensor under $\delta_\xi$, and is invariant under $\delta_\Xi$[15].

[15]To see invariance under $\delta_\Xi$, transform

$$\begin{aligned}\delta_\Xi G_{AB} &= \delta_\Xi(\partial_\mu X^A \partial_\nu X^B G_{AB}(X(x))) \\ &= -\partial_\mu \Xi^A \partial_\nu X^B G_{AB}(X(x)) - \partial_\mu X^A \partial_\nu \Xi^B G_{AB}(X(x)) \\ &\quad + \partial_\mu X^A \partial_\nu X^B \delta_\Xi G_{AB}(X(x)),\end{aligned} \tag{9.7}$$

then in transforming $G_{AB}$, remember that both the function and the argument are changing,

$$\delta_\Xi G_{AB}(X(x)) = \mathcal{L}_\Xi G_{AB}(X(x)) - \Xi^C \partial_C G_{AB}. \tag{9.8}$$

Putting all this together, we find $\delta_\Xi G_{AB} = 0$.

We first proceed to fix some of this gauge symmetry. In particular, we freeze the position of the brane. Note that the brane coordinate functions $X^A(x)$ are essentially Goldstone bosons since they shift under the bulk gauge symmetry, $X^A(x) \to X^A(x) - \Xi^A(X(x))$. We can thus reach a sort of unitary gauge where the $X^A$ are fixed to some specified values. We set values so that the brane is the surface $X^5 = 0$, and the brane coordinates $x^\mu$ coincide with the coordinates $X^\mu$; thus we set

$$X^\mu(x) = x^\mu, \qquad \mu = 0, 1, 2, 3, \tag{9.9}$$

$$X^5(x) = 0. \tag{9.10}$$

There are still residual gauge symmetries which leave this gauge choice invariant. Acting with the two gauge transformations $\delta_\xi$ and $\delta_\Xi$ on the gauge conditions and demanding that the change be zero, we find

$$\begin{aligned}&\delta_\Xi X^5(x) + \delta_\xi X^5(x) \\ &\quad = -\Xi^5(X(x)) + \xi^\mu \partial_\mu X^5 \underset{X^5(x)=0}{\longrightarrow} -\Xi^5(X(x)) \\ &\quad \Rightarrow \Xi^5(X(x)) = 0.\end{aligned} \tag{9.11}$$

$$\begin{aligned}&\delta_\Xi X^\mu(x) + \delta_\xi X^\mu(x) \\ &\quad = -\Xi^\mu(X(x)) + \xi^\nu \partial_\nu X^\mu(x) \\ &\quad \underset{X^\mu(x)=x^\mu}{\longrightarrow} -\Xi^\mu(X(x)) + \xi^\mu(x) \\ &\quad \Rightarrow \Xi^\mu(X(x)) = \xi^\mu(x).\end{aligned} \tag{9.12}$$

The residual gauge transformations are bulk gauge transformations that do not move points onto or off of the brane, but only move brane points to other brane points. Furthermore, the brane diffeomorphism invariance is no longer an independent invariance but is fixed to be the diffeomorphisms induced from the bulk.

We now fix this gauge in the action (9.10), which is permissible since no equations of motion are lost. This means that the induced metric is now

$$g_{\mu\nu}(x) = G_{\mu\nu}(x, X^5 = 0). \tag{9.13}$$

We split the action into two regions, region $L$ to the left of the brane, and region $R$ to the right of the brane, with outward pointing normals, as in Fig. 3. We call the fifth coordinate $X^5 \equiv y$. The brane is at $y = 0$:

$$S = \frac{M_5^3}{2}\left(\int_L + \int_R\right) d^4x dy \sqrt{-G}R(G) + \int d^4x \mathcal{L}_4, \tag{9.14}$$

where $\mathcal{L}_4 \equiv (M_4^2/2)\sqrt{-g}R(g) + \mathcal{L}_M(g, \psi)$ is the $4d$ part of the Lagrangian. To have a well-defined variational principle, we must have Gibbons-Hawking terms on both sides (Dyer and Hinterbichler, 2009), corresponding to the outward pointing normals. Adding these, the resulting action is

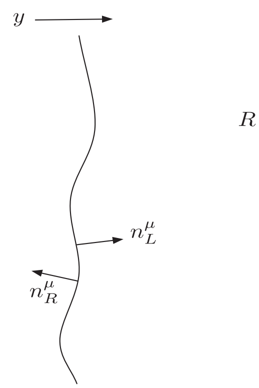


FIG. 3. Splitting the DGP action.

$$S = \frac{M_5^3}{2}\left[\left(\int_L + \int_R\right) d^4x dy \sqrt{-G} R(G) + 2\oint_L d^4x\sqrt{-g}K_L + 2\oint_R d^4x\sqrt{-g}K_R\right] + \int d^4x \mathcal{L}_4, \tag{9.15}$$

where $K_R$ and $K_L$ are the extrinsic curvatures relative to the normals $n_R$ and $n_L$, respectively.

We now go to spacelike ADM variables (Arnowitt, Deser, and Misner, 1960, 1962) adapted to the brane [see Poisson (2004) and Dyer and Hinterbichler (2009) for detailed derivations and formulas]. The lapse and shift relative to $y$ are $N^\mu(x, y)$ and $N(x, y)$, and the $4d$ metric is $g_{\mu\nu}(x, y)$. The $5d$ metric is

$$G_{AB} = \begin{pmatrix} N^2 + N^\mu N_\mu & N_\mu \\ N_\mu & g_{\mu\nu} \end{pmatrix}. \tag{9.16}$$

The $4d$ extrinsic curvature is taken with respect to the positive pointing normal $n_L$ and is given by

$$K_{\mu\nu} = \frac{1}{2N}(g'_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu), \tag{9.17}$$

where a prime means a derivative with respect to $y$. The action is now[16]

$$S = \frac{M_5^3}{2}\left(\int_L + \int_R\right) d^4x dy N\sqrt{-g}[R(g) + K^2 - K_{\mu\nu}K^{\mu\nu}] + \int d^4x \mathcal{L}_4. \tag{9.20}$$

It can be checked that a flat brane living in flat space is a solution to the equations of motion of this action. This is called the *normal* branch. There is another maximally symmetric solution with a flat $5d$ bulk, which contains a de Sitter brane with a $4d$ Hubble scale $H \sim M_5^3/M_4^2$. This is called the *self-accelerating* branch and has caused much interest because the solution exists even though the brane and bulk cosmological constants vanish.

[16]The Ricci scalar and metric determinant are

$$(5)R \overset{(4)}{=} R + (K^2 - K_{\mu\nu}K^{\mu\nu}) + 2\nabla_A(n^B\nabla_B n^A - n^A K), \tag{9.18}$$

$$\sqrt{-G} = N\sqrt{-g}. \tag{9.19}$$

The total derivatives coming from $2\nabla_A(n^B\nabla_B n^A - n^A K)$ in the Einstein-Hilbert part of the action exactly cancel the Gibbons-Hawking terms.

### B. Linear expansion

To see the particle content of DGP, we expand the action (9.20) to linear order around the flat space solution and then integrate out the bulk to obtain an effective $4d$ action. We start by expanding the $5d$ graviton about flat space

$$G_{AB} = \eta_{AB} + H_{AB}. \tag{9.21}$$

We use the lapse, shift, and $4d$ metric variables, with their expansions around flat space,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad N_\mu = n_\mu, \qquad N = 1 + n. \tag{9.22}$$

We have the relations, to linear order in $h_{\mu\nu}$, $n_\mu$, and $n$,

$$H_{\mu\nu} = h_{\mu\nu}, \qquad H_{\mu 5} = n_\mu, \qquad H_{55} = 2n. \tag{9.23}$$

We first expand the DGP action (9.20) to quadratic order in $h_{\mu\nu}$, $n_\mu$, and $n$. We then solve the $5d$ equations of motion, subject to arbitrary boundary values on the brane and going to zero at infinity. We then plug this solution back into the action to obtain an effective $4d$ theory for the arbitrary brane boundary values.

The $5d$ equations of motion away from the brane are simply the vacuum Einstein equations, which read, to linear order,

$$-2R_{AB}(G)_{\text{linear}} = \square^{(5)}H_{AB} + \partial_A\partial_B H - \partial^C\partial_A H_{BC} - \partial^C\partial_B H_{AC} = 0. \tag{9.24}$$

We solve Eq. (9.24) in the de Donder gauge,

$$\partial^B H_{AB} - \tfrac{1}{2}\partial_A H = 0. \tag{9.25}$$

With this, Eq. (9.24) is equivalent to

$$\square^{(5)}H_{AB} = 0, \tag{9.26}$$

along with the de Donder gauge condition (9.25). In terms of the ADM variables, Eq. (9.26) becomes

$$\square h_{\mu\nu} + \partial_y^2 h_{\mu\nu} = 0, \tag{9.27}$$

$$\square n_\mu + \partial_y^2 n_\mu = 0, \tag{9.28}$$

$$\square n + \partial_y^2 n = 0, \tag{9.29}$$

where $\square$ is the $4d$ Laplacian. These have the following solutions in terms of boundary values $h_{\mu\nu}(x)$, $n_\mu(x)$, and $n(x)$:

$$h_{\mu\nu}(x, y) = e^{-y\Delta}h_{\mu\nu}(x), \tag{9.30}$$

$$n_\mu(x, y) = e^{-y\Delta}n_\mu(x), \tag{9.31}$$

$$n(x,y) = e^{-y\Delta} n(x). \tag{9.32}$$

Here the operator $\Delta$ is the formal square root of the $4d$ Laplacian,

$$\Delta \equiv \sqrt{-\Box}. \tag{9.33}$$

The $A = \mu$ and $A = 5$ components of the gauge condition (9.25) are, respectively,

$$\partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h + \partial_y n_\mu - \partial_\mu n = 0, \tag{9.34}$$

$$\partial^\mu n_\mu - \tfrac{1}{2}\partial_y h + \partial_y n = 0. \tag{9.35}$$

For these to be satisfied everywhere, it is necessary and sufficient that the boundary fields satisfy the following at $y = 0$:

$$\partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h - \Delta n_\mu - \partial_\mu n = 0, \quad \partial^\mu n_\mu + \tfrac{1}{2}\Delta h - \Delta n = 0. \tag{9.36}$$

These should be thought of as constraints determining some of the boundary variables in terms of the others.[17] We at this point imagine that we have solved these constraints, and that the action is really a function of the independent variables.

The de Donder gauge is preserved by any $5d$ gauge transformation $\Xi^A$ satisfying

$$\Box^{(5)} \Xi^A = 0. \tag{9.38}$$

The component $\Xi^5$ must vanish at $y = 0$ because the position of the brane is fixed. Equation (9.38) then implies that $\Xi^5$ vanishes everywhere. The other components can have arbitrary values $\Xi^\mu(x,0) = \xi^\mu(x)$ on the brane, which are then extended into bulk in order to satisfy Eq. (9.38),

$$\Xi^\mu(x,y) = e^{-y\Delta}\xi(x). \tag{9.39}$$

The residual gauge transformations acting on the boundary fields are then

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\nu, \quad \delta n_\mu = -\Delta \xi_\mu, \quad \delta n = 0. \tag{9.40}$$

The constraints (9.36) are invariant under these gauge transformations. The $4d$ effective action must and will be invariant under Eq. (9.40).

The $5d$ part of the action reads

$$S_5 = \frac{M_5^3}{2}\int d^4x dy N\sqrt{-g}[R(g) + K^2 - K_{\mu\nu}K^{\mu\nu}]. \tag{9.41}$$

[17]Note that we cannot think of them as determining $n^\mu$, $n$ in terms of $h_{\mu\nu}$. Acting with $\partial_\mu$ on the first equation, $\Delta$ on the second, and then adding, we find the equation

$$\partial_\mu\partial_\nu h^{\mu\nu} - \Box h = 0, \tag{9.37}$$

which is precisely the statement that the $4d$ linearized curvature vanishes (which is, in turn, the linearized Hamiltonian constraint in general relativity). Thus, we must think of these constraints as determining some of the components of the metric.

We want to expand this to quadratic order in $h_{\mu\nu}$, $n_\mu$, and $n$ and then plug in our solution. We need the expansion of $K_{\mu\nu}$ to first order,

$$K_{\mu\nu} = \tfrac{1}{2}(\partial_y h_{\mu\nu} - \partial_\mu n_\nu - \partial_\nu n_\mu). \tag{9.42}$$

Expanding, we have (after much integration by parts in $4d$)

$$\begin{aligned}\frac{2}{M_5^3}S_5 = \int d^4x dy\, & n\partial_\mu\partial_\nu h^{\mu\nu} - n\Box h + \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\nu h^{\mu\lambda} \\ & - \frac{1}{2}\partial_\mu h \partial_\nu h^{\mu\nu} - \partial_y h \partial_\mu n^\mu + \frac{1}{2}(\partial_\mu n^\mu)^2 \\ & + \partial_y h_{\mu\nu}\partial^\mu n^\nu + \frac{1}{2} n_\mu \Box n^\mu \frac{1}{4} h_{\mu\nu}\Box h^{\mu\nu} \\ & - \frac{1}{4}\partial_y h_{\mu\nu}\partial_y h^{\mu\nu} - \frac{1}{4} h\Box h + \frac{1}{4}(\partial_y h)^2.\end{aligned}$$

Now, in the last line, integrate by parts in $y$, picking up a boundary term at $y = 0$, and use Eq. (9.26) to kill the bulk part,

$$\begin{aligned}\frac{2}{M_5^3}S_5 = \int d^4x dy\, & n\partial_\mu\partial_\nu h^{\mu\nu} - n\Box h + \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\nu h^{\mu\lambda} \\ & - \frac{1}{2}\partial_\mu h \partial_\nu h^{\mu\nu} - \partial_y h \partial_\mu n^\mu + \frac{1}{2}(\partial_\mu n^\mu)^2 \\ & + \partial_y h_{\mu\nu}\partial^\mu n^\nu + \frac{1}{2} n_\mu \Box n^\mu + \int d^4x - \frac{1}{4} h\partial h \\ & + \frac{1}{4} h_{\mu\nu}\partial_y h^{\mu\nu}.\end{aligned}$$

We now insert the following term into the action:

$$S_{\rm GF} = -\frac{M_5^3}{4}\int d^5X\left(\partial^B H_{AB} - \frac{1}{2}\partial_A H\right)^2. \tag{9.43}$$

The $5d$ equations of motion solve the de Donder gauge condition, so this term contributes 0 to the action (thought of as a function of the unconstrained variables) and we are free to add it. However, we write it in terms of the unconstrained $4d$ variables for now,

$$\frac{2}{M_5^3}S_{\rm GF} = \int d^4x dy$$

$$-\frac{1}{2}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h + \partial_y n_\mu - \partial_\mu n\right)^2 \tag{9.44}$$

$$-\frac{1}{2}\left(\partial_\mu n^\mu - \frac{1}{2}\partial_y h + \partial_y n\right)^2. \tag{9.45}$$

Adding this to the previous $5d$ term, we find that after using the $5d$ Laplace equations, the entire action can be reduced to a boundary term at $y = 0$,

$$\begin{aligned}\frac{2}{M_5^3}(S_5 + S_{GF}) = \int d^4x & - \frac{1}{4} h_{\mu\nu}\Delta h^{\mu\nu} + \frac{1}{8} h\Delta h \\ & - \frac{1}{2} n\Delta n - \frac{1}{2} n_\mu \Delta n^\mu + \frac{1}{2} h\Delta n \\ & + n^\mu\left(-\partial_\mu n - \frac{1}{2}\partial_\mu h + \partial^\nu h_{\mu\nu}\right).\end{aligned} \tag{9.46}$$

Now a crucial point. We have been imagining solving the constraints (9.36) for the independent variables. But now, consider the action (9.46) as a function of the original variables $h_{\mu\nu}$, $n^\mu$, and $n$. Varying with respect to $n^\mu$ and $n$, we recover precisely the constraints (9.36). Thus, we can reintroduce the solved variables as auxiliary fields, since the constraints are then implied. The action now becomes a function of $h_{\mu\nu}$, $n^\mu$, and $n$.

Now add in the $4d$ part of the action,

$$S = \frac{M_4^2}{2}\int d^4x\sqrt{-g}R(g) + S_M + 2(S_5 + S_{\rm GF}), \qquad (9.47)$$

where $S_M$ is the $4d$ matter action and the factor of 2 in front of the $5d$ parts results from taking into account both sides of the bulk (through boundary conditions at infinity we have thus implicitly imposed a $\mathbb{Z}_2$ symmetry).

Expanded to quadratic order,

$$\begin{aligned} S = \int d^4x \frac{M_4^2}{4}\frac{1}{2}h_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h_{\alpha\beta} \\ + \frac{M_4^2 m}{4}\Big[-\frac{1}{2}h_{\mu\nu}\Delta h^{\mu\nu} + \frac{1}{4}h\Delta h - n\Delta n - n_\mu\Delta n^\mu \\ + h\Delta n + n^\mu(-2\partial_\mu n - \partial_\mu h + 2\partial^\nu h_{\mu\nu})\Big] \\ + \frac{1}{2}h_{\mu\nu}T^{\mu\nu}, \end{aligned} \qquad (9.48)$$

where $\mathcal{E}^{\mu\nu,\alpha\beta}$ is the massless graviton kinetic operator (2.46), and

$$m \equiv \frac{2M_5^3}{M_4^2} \qquad (9.49)$$

is known as the DGP scale.

It is invariant under the gauge transformations (9.40), under which $n^\mu$ plays the role of the vector Stückelberg field. $n$ plays the role of a gauge invariant auxiliary field. To get this into Fierz-Pauli form, first eliminate $n$ as an auxiliary field by using its equation of motion. Then use Eq. (9.40) to fix the gauge $n^\mu = 0$. The resulting action is

$$\begin{aligned} S = \int d^4x \frac{M_4^2}{4}\Big[\frac{1}{2}h_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h_{\alpha\beta} \\ - \frac{1}{2}m(h_{\mu\nu}\Delta h^{\mu\nu} - h\Delta h)\Big] + \frac{1}{2}h_{\mu\nu}T^{\mu\nu}, \end{aligned} \qquad (9.50)$$

which is of the Fierz-Pauli form, with an operator dependent mass term $m\Delta$.

One can go on to study interaction terms for DGP, and the longitudinal mode turns out to be governed by interactions which include the cubic Galileon term $\sim(\partial\phi)^2\Box\phi$ (Luty, Porrati, and Rattazzi, 2003; Nicolis and Rattazzi, 2004; Gabadadze and Iglesias, 2006) and are suppressed by the scale $\Lambda_3 = (M_4 m^2)^{1/3}$ (in fact, this was where the Galileons were first uncovered). In this sense, DGP is analogous to the nicer $\Lambda_3$ theories of Sec. VIII. The theory is free of ghosts and instabilities around solutions connected to flat space (Nicolis and Rattazzi, 2004), but changing the asymptotics to the self-accelerating de Sitter brane solutions flips the sign of the kinetic term of the longitudinal mode, so there is a massless ghost around the self-accelerating branch (Koyama, 2007). This branch is completely unstable, which is bad news for doing cosmology on this branch. In addition to ghosts, there are other issues with other nontrivial branches, such as superluminal fluctuations (Hinterbichler, Nicolis, and Porrati, 2009), and uncontrolled singularities and tunneling (Gregory, 2008).

### C. Resonance gravitons

The operator dependent mass term in Eq. (9.50) is known as a *resonance* mass, or *soft* mass (Dvali, Gabadadze, and Porrati, 2000b; Dvali *et al.*, 2001a, 2001b; Gabadadze, 2004; Gabadadze and Shifman, 2004; Dvali, 2006; Dvali, Hofmann, and Khoury, 2007; Lopez Nacir and Mazzitelli, 2007; Gabadadze and Iglesias, 2008; Patil, 2010). To see the particle content of this theory, we decompose the propagator into a sum of massive gravity propagators. The linear Stückelberg analysis, leading to the propagators (4.37), goes through identically, with the replacement $m^2 \to m\Delta$. The momentum part of the propagators now reads

$$\frac{-i}{p^2 + m\sqrt{p^2}}. \qquad (9.51)$$

Setting $z = -p^2$, the propagator has a branch cut in the $z$ plane from $(0, \infty)$, with discontinuity

$$-\frac{2m}{\sqrt{z}(z+m^2)}. \qquad (9.52)$$

A branch cut can be thought of as a string of simple poles, in the limit where the spacing between the poles and their residues both go to zero. The function $f(z) = \int_{-\infty}^{\infty} d\lambda\rho(\lambda)/(z-\lambda)$ has a cut along the real axis everywhere that $\rho$ is nonzero, with discontinuity $-2i\pi\rho(z)$. We can see this by noting

$$\begin{aligned} \operatorname{disc} f(z) &= \int_{-\infty}^{\infty} d\lambda\rho(\lambda)\frac{1}{z-\lambda+i\epsilon} - \frac{1}{z-\lambda-i\epsilon} \\ &= \int_{-\infty}^{\infty} d\lambda\rho(\lambda)[-2\pi i\delta(z-\lambda)]. \end{aligned}$$

Using all this, and the fact that analytic functions are determined by their poles and cuts, we can write the propagator in the spectral form

$$\begin{aligned} \frac{-i}{p^2+m\sqrt{p^2}} &= \int_0^\infty ds \frac{-i}{p^2+s}\rho(s), \\ \rho(s) &= \frac{m}{\pi\sqrt{s}(s+m^2)} > 0. \end{aligned} \qquad (9.53)$$

The spectral function is greater than zero, so this theory contains a continuum of ordinary (nonghost, nontachyon) gravitons, with masses ranging from 0 to $\infty$. This is what would be expected from dimensionally reducing a noncompact fifth dimension. The Kaluza-Klein tower has collapsed down into a Kaluza-Klein continuum.

In the limit $m \to 0$, where the action becomes purely four dimensional, the spectral function reduces to a delta function,

$$\rho(s) \to 2\delta(s), \qquad (9.54)$$

and the propagator reduces to $-i/p^2$ representing a single massless graviton, vector, and scalar, as can be seen from

Eq. (4.37) (the extra factor of 2 is taken care of by noting that the integral is from 0 to $\infty$, so only half of the delta function actually gets counted). This theory therefore contains a vDVZ discontinuity.

The potential of a point source of mass $M$ sourced by this resonance graviton displays an interesting crossover behavior. Looking back at Eq. (3.11) with the momentum space replacement $m^2 \to m\sqrt{p^2}$, and using the relation $\phi = -h_{00}/M_4$ for the Newtonian potential, we have

$$\begin{aligned}\phi(r) &= \frac{-2M}{3M_4^2}\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{-1}{\mathbf{p}^2 + m|\mathbf{p}|} \\ &= \frac{2}{3}\frac{M}{M_4^2 2\pi^2 r}\left\{\sin\left(\frac{r}{r_0}\right)\mathrm{ci}\left(\frac{r}{r_0}\right) \right. \\ &\quad \left. + \frac{1}{2}\cos\left(\frac{r}{r_0}\right)\left[\pi - 2\,\mathrm{si}\left(\frac{r}{r_0}\right)\right]\right\}, \end{aligned} \tag{9.55}$$

where

$$\begin{aligned}\mathrm{si}(x) &\equiv \int_0^x \frac{dt}{t}\sin t, \\ \mathrm{ci}(x) &\equiv \gamma + \ln x + \int_0^x \frac{dt}{t}(\cos t - 1), \end{aligned} \tag{9.56}$$

$\gamma \approx 0.577\ldots$ is the Euler-Masceroni constant, and the length scale $r_0$ is

$$r_0 \equiv \frac{1}{m} = \frac{M_4^2}{2M_5^3}. \tag{9.57}$$

The potential interpolates between $4d \sim 1/r$ and $5d \sim 1/r^2$ behavior at the scale $r_0$,

$$V(r) = \begin{cases} \frac{2}{3}\frac{M}{M_4^2 4\pi r} + \frac{M}{3\pi^2 M_4^2}\frac{1}{r_0}[\gamma - 1 + \ln(\frac{r}{r_0})] + \mathcal{O}(r), & r \ll r_0, \\ \frac{2}{3}\frac{M}{M_5^3 4\pi^2 r^2} + \mathcal{O}(\frac{1}{r^3}), & r \gg r_0. \end{cases} \tag{9.58}$$

Physically, we think of gravity as being confined to the brane out to a distance $\sim r_0$, at which point it starts to weaken and leak off the brane, becoming five dimensional. This is the behavior that is morally responsible for the self-accelerated solutions seen in DGP (Deffayet, Dvali, and Gabadadze, 2002). It has been suggested that corrections to the Newtonian potential for $r \ll r_0$ may be observable in lunar laser ranging experiments (Dvali, Gruzinov, and Zaldarriaga, 2003; Lue and Starkman, 2003).

The resonance massive graviton can also be generalized away from DGP, by replacing the mass term with an arbitrary function of the Laplacian (Gabadadze and Shifman, 2004; Dvali, 2006; Dvali, Hofmann, and Khoury, 2007),

$$m^2 \to m^2(\Box). \tag{9.59}$$

[See Dvali, Pujolas, and Redi (2008) for even further generalizations.] At large distances, where we want modifications to occur, the mass term has a leading Taylor expansion

$$m^2(\Box) = L^{2(\alpha-1)}\Box^\alpha, \tag{9.60}$$

with $L$ being a length scale and $\alpha$ being a constant. In order to modify Newtonian dynamics at large scales, $\partial \ll 1/L$, the mass term should dominate over the two derivative kinetic terms, so we should have $\alpha < 1$. Additionally, there is the constraint that the spectral function (9.53) should be positive definite, so that there are no ghosts. This puts a lower bound $\alpha \geq 0$ (Dvali, 2006). It turns out that degravitation can be made to work only for $\alpha < 1/2$ (Dvali, Hofmann, and Khoury, 2007). DGP corresponds to $\alpha = 1/2$, and so it just barely fails to degravitate, but by extending the DGP idea to higher codimension (Kiritsis, Tetradis, and Tomaras, 2001; de Rham, 2008; Hassan, Hofmann, and von Strauss, 2011) or to multibrane cascading DGP models (de Rham *et al.*, 2008; de Rham, 2009; de Rham, Khoury, and Tolley, 2010), $\alpha < 1/2$ can be achieved and degravitation made to work (de Rham *et al.*, 2008). Some $N$-body simulations of degravitation and DGP have been done by Chan and Scoccimarro (2009), Khoury and Wyman (2009), and Schmidt (2009).

## X. CONCLUSIONS AND FUTURE DIRECTIONS

Massive gravity remains an active research area, one which may provide a viable solution to the cosmological constant naturalness problem. As seen, many interesting effects arise from the naive addition of a hard mass term to Einstein gravity. There is a well-defined effective field theory with a protected hierarchy between the cutoff and the graviton mass, and a screening mechanism which nonlinearly hides the new degrees of freedom and restores continuity with GR in the massless limit.

A massive graviton can screen a large cosmological constant, and a stable theory of massive gravity with a small protected mass offers a solution to the problem of quantum corrections to the cosmological constant. It is a remarkable fact that the $\Lambda_3$ theories of Sec. VIII exist and are ghost free, and that they are found simply by tuning some coefficients in the generic graviton potential.

There are many interesting outstanding issues. One is the nature of the $\Lambda_3$ theory. Is there a symmetry or a topological construction that explains the tunings of the coefficients necessary to achieve the $\Lambda_3$ cutoff? Is there some construction free of prior geometry that would contain this theory? Is there an extra-dimensional construction?

There are also many questions related to the quantum properties of these theories. Apart from the order of magnitude estimates presented in this review and a few sporadic calculations, the detailed quantum properties of this theory and others remain relatively unexplored. The same goes for nonperturbative quantum properties, such as how a massive graviton would modify black hole thermodynamics, Hawking radiation, or holography (Babak and Grishchuk, 2003; Aharony, Clark, and Karch, 2006; Katz, Lewandowski, and Schwartz, 2006; Kiritsis, 2006; Kiritsis and Niarchos, 2009; Niarchos, 2009; Capela and Tinyakov, 2011).

The cutoff $\Lambda_3$ is still rather low, however, so at best this theory in its current perturbative expansion can provide only a partial solution to the cosmological constant naturalness problem. Finding more natural constructions of these theories would go a long way toward solving the major issue, which is that of UV completion; is it possible to find a standard UV completion for a massive graviton, analogous to what the Higgs mechanism provides for a massive vector? Or is there some incontrovertible obstruction that forces any UV completion to violate Lorentz invariance, locality, or some other cherished property? If so, there may be a nonstandard UV completion, or it could be that massive gravity really is inconsistent, in the sense that there really is no way whatsoever to UV complete it. There has been work on holography of massive gravitons in anti–de Sitter/conformal field theory (AdS/CFT) (Aharony, Clark, and Karch, 2006; Kiritsis, 2006; Kiritsis and Niarchos, 2009; Niarchos, 2009), which provides a UV completion for theories in AdS space containing massive gravity.

Even a partial UV completion, one that raises the cutoff to $M_P$, would be extremely important, as this is all that is required to offer a solution to the cosmological constant naturalness problem. One possibility is that the scale $\Lambda_3$, while indicating a breakdown in perturbation theory, does not signal the activation of any new degrees of freedom, so that the theory is already self-complete up to $M_P$. Since there are multiple parameters in the theory, it is likely that there is some other expansion, such as a small $m$ expansion, which reorganizes the perturbation theory into one which yields perturbative access to scales above $\Lambda_3$. If this is true, it is important that the $\Lambda_3$ theory is ghost free beyond the decoupling limit.

It should also be noted that massive gravitons already exist in nature, in the form of tensor mesons which carry spin 2. There is a nonet of them, which at low energies can be described in chiral perturbation theory as a multiplet of massive gravitons (Chow and Rey, 1998). Here we know that these states find a UV completion in QCD, where they are simply excited states of bound quarks.

In this review we focused on theories with a vacuum that preserves Lorentz invariance, but there is a whole new world that opens up when one allows for Lorentz violation. There exist theories that explicitly break Lorentz invariance, and theories such as the ghost condensate (Arkani-Hamed *et al.*, 2004) which have Lorentz invariance spontaneously broken (Arkani-Hamed *et al.*, 2005) by some non-Lorentz invariant background. In the former case, a systematic study of the possible mass terms and their degrees of freedom, generalizing the Fierz-Pauli analysis to the case where the mass term preserves only rotation invariance, is performed by Dubovsky (2004). For examples of the latter case, see Berezhiani *et al.* (2007) and Blas, Deffayet, and Garriga (2007). See also Rubakov (2004), Gabadadze and Grisa (2005), Rubakov and Tinyakov (2008), Bebronne (2009b), and Mironov *et al.* (2010) for reviews.

There is still much to be learned about massive gravitons on curved spaces and cosmologies. For instance, does a generalization exist of the higher cutoff $\Lambda_3$ theory around curved backgrounds? Is there a consistent fully interacting theory of the partially massless theories on de Sitter space? Are there consistent theories with cosmological backgrounds, and, in particular, can they nonlinearly realize the screening of a large bare cosmological constant while maintaining consistency with Solar System constraints?

Finally, a topic worthy of a separate review is the observable signatures that would be characteristic of a massive graviton. What would be the signatures of a cosmological constant screened by a graviton mass? For some examples of various proposed signatures, see Dubovsky, Tinyakov, and Tkachev (2005), Bebronne (2009a), Bessada and Miranda (2009a, 2009b), Dubovsky *et al.* (2010), and Wyman (2011). We end this review by quoting the tantalizing current experimental limits on the mass of the graviton (under some hypotheses, of course) $m \lesssim 7 \times 10^{-32}$ eV (Goldhaber and Nieto, 2010; Nakamura *et al.*, 2010), about an order of magnitude above the Hubble scale, the value needed to theoretically explain the cosmological constant naturalness problem.

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## APPENDIX: TOTAL DERIVATIVE COMBINATIONS

Define the matrix of second derivatives

$$\Pi_{\mu\nu} = \partial_\mu \partial_\nu \phi. \tag{A.1}$$

At every order in $\phi$, there is a unique (up to overall constant) contraction of $\Pi$'s that reduces to a total derivative[18],

$$\mathcal{L}_1^{\mathrm{TD}}(\Pi) = [\Pi], \tag{A.2}$$

$$\mathcal{L}_2^{\mathrm{TD}}(\Pi) = [\Pi]^2 - [\Pi^2], \tag{A.3}$$

$$\mathcal{L}_3^{\mathrm{TD}}(\Pi) = [\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3], \tag{A.4}$$

$$\begin{aligned} \mathcal{L}_4^{\mathrm{TD}}(\Pi) = {}& [\Pi]^4, -6[\Pi^2][\Pi]^2 + 8[\Pi^3][\Pi] \\ & + 3[\Pi^2]^2 - 6[\Pi^4], \\ & \vdots, \end{aligned} \tag{A.5}$$

where the brackets are traces. $\mathcal{L}_2^{\mathrm{TD}}(h)$ is just the Fierz-Pauli term, and the others can be thought of as higher order generalizations of it. They are characteristic polynomials, terms in the expansion of the determinant in powers of $H$,

$$\begin{aligned} \det(1 + \Pi) = {}& 1 + \mathcal{L}_1^{\mathrm{TD}}(\Pi) + \frac{1}{2}\mathcal{L}_2^{\mathrm{TD}}(\Pi) + \frac{1}{3!}\mathcal{L}_3^{\mathrm{TD}}(\Pi) \\ & + \frac{1}{4!}\mathcal{L}_4^{\mathrm{TD}}(\Pi) + \cdots . \end{aligned} \tag{A.6}$$

The term $\mathcal{L}_n^{\mathrm{TD}}(\Pi)$ vanishes identically when $n > D$, with $D$ the spacetime dimension, so there are only $D$ nontrivial such combinations, those with $n = 1, \ldots, D$.

[18]The proof of this fact is the same as the proof showing the uniqueness of the Galileons in Nicolis, Rattazzi, and Trincherini (2008). See also Creminelli *et al.* (2005).

They can be written explicitly as

$$\mathcal{L}_n^{\rm TD}(\Pi) = \sum_p (-1)^p \eta^{\mu_1 p(\nu_1)}\eta^{\mu_2 p(\nu_2)}\cdots\eta^{\mu_n p(\nu_n)} \times (\Pi_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\cdots\Pi_{\mu_n\nu_n}). \quad (A.7)$$

The sum is over all permutations of the $\nu$ indices, with $(-1)^p$ the sign of the permutation.

They satisfy a recursion relation

$$\mathcal{L}_n^{\rm TD}(\Pi) = -\sum_{m=1}^{n}(-1)^m \frac{(n-1)!}{(n-m)!}[\Pi^m]\mathcal{L}_{n-m}^{\rm TD}(\Pi), \quad (A.8)$$

with $\mathcal{L}_0^{\rm TD}(\Pi) = 1$.

In addition, there are tensors $X_{\mu\nu}^{(n)}$ that we construct out of $\Pi_{\mu\nu}$ as follows[19]:

$$X_{\mu\nu}^{(n)} = \frac{1}{n+1}\frac{\delta}{\delta\Pi_{\mu\nu}}\mathcal{L}_{n+1}^{\rm TD}(\Pi). \quad (A.9)$$

The first few are

$$X_{\mu\nu}^{(0)} = \eta_{\mu\nu}, \quad (A.10)$$

$$X_{\mu\nu}^{(1)} = [\Pi]\eta_{\mu\nu} - \Pi_{\mu\nu}, \quad (A.11)$$

$$X_{\mu\nu}^{(2)} = ([\Pi]^2 - [\Pi^2])\eta_{\mu\nu} - 2[\Pi]\Pi_{\mu\nu} + 2\Pi_{\mu\nu}^2, \quad (A.12)$$

$$X_{\mu\nu}^{(3)} = ([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3])\eta_{\mu\nu} - 3([\Pi]^2 - [\Pi^2])\Pi_{\mu\nu} + 6[\Pi]\Pi_{\mu\nu}^2 - 6\Pi_{\mu\nu}^3, \quad \vdots \quad (A.13)$$

The following is an explicit expression:

$$X_{\mu\nu}^{(n)} = \sum_{m=0}^{n}(-1)^m \frac{n!}{(n-m)!}\Pi_{\mu\nu}^m\mathcal{L}_{n-m}^{\rm TD}(\Pi). \quad (A.14)$$

They satisfy the recursion relation

$$X_{\mu\nu}^{(n)} = -n\Pi_\mu{}^\alpha X_{\alpha\nu}^{(n-1)} + \Pi^{\alpha\beta}X_{\alpha\beta}^{(n-1)}\eta_{\mu\nu}. \quad (A.15)$$

Since $\mathcal{L}_n^{\rm TD}(\Pi)$ vanishes for $n > D$, $X_{\mu\nu}^{(n)}$ vanishes for $n \geq D$.

The $X_{\mu\nu}^{(n)}$ satisfy the following important properties.

- They are symmetric and identically conserved and are the only combinations of $\Pi_{\mu\nu}$ at each order with these properties:

$$\partial^\mu X_{\mu\nu}^{(n)} = 0. \quad (A.16)$$

- For spatial indices $i$, $j$ and time index 0,

$$\begin{aligned} &X_{ij}^{(n)} \text{ has at most two time derivatives,}\\ &X_{0i}^{(n)} \text{ has at most one time derivative,}\\ &X_{00}^{(n)} \text{ has no time derivatives.}\end{aligned} \quad (A.17)$$

Finally, we have the following relations involving the massless kinetic operator (2.46):

$$\begin{aligned}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\phi\eta_{\alpha\beta}) &= -(D-2)X_{\mu\nu}^{(1)},\\ \mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\partial_\alpha\phi\partial_\beta\phi) &= X_{\mu\nu}^{(2)}.\end{aligned} \quad (A.18)$$

[19]Note that our definition of the $X_{\mu\nu}^{(n)}$ used here differs by a factor of 2 from that of de Rham, Gabadadze, and Tolley (2010).

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