Colloquium: Nonequilibrium dynamics of closed interacting quantum systems

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This Colloquium gives an overview of recent theoretical and experimental progress in the area of nonequilibrium dynamics of isolated quantum systems. There is particularly a focus on quantum quenches: the temporal evolution following a sudden or slow change of the coupling constants of the system Hamiltonian. Several aspects of the slow dynamics in driven systems are discussed and the universality of such dynamics in gapless systems with specific focus on dynamics near continuous quantum phase transitions is emphasized. Recent progress on understanding thermalization in closed systems through the eigenstate thermalization hypothesis is also reviewed and relaxation in integrable systems is discussed. Finally key experiments probing quantum dynamics in cold atom systems are overviewed and put into the context of our current theoretical understanding.

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CONTENTS

I. INTRODUCTION

In the past two decades the outlook of condensed matter physics has been deeply and unexpectedly revolutionized by a few experimental breakthroughs in atomic physics, quantum optics, and nanoscience. In synthesis, crucial advances in these fields have made it possible to realize artificial systems (e.g., optical lattices, quantum dots, nanowires) that are described to a high degree of accuracy by models (e.g., Hubbard, Kondo, and Luttinger models) whose physics has been a subject of intense investigation in various contexts ranging from high temperature superconductivity to low temperature transport in metals. It is fair to say that this experimental progress has changed the way theory and experiment look at each other. In the past, effective models were largely devised to explain the low-energy physics of highly complex systems. The situation has now been reversed so that one can experimentally realize and simulate the physics of such models. On the one hand, the design and realization of interacting many-body systems could in principle be used to perform practical tasks, such as quantum information processing (Farhi *et al.*, 2001). On the other hand, direct simulations of simple models could help in resolving important problems in condensed matter physics. But most importantly, the availability of experimental controllable systems whose properties can be accurately described by simple models provides an unprecedented opportunity to explore several new frontiers of condensed matter physics including the nonequilibrium dynamics in closed interacting quantum systems.

Equilibrium systems can often be understood using a combination of mean field theory, renormalization group, and universality. This allows us to understand low temperature experimental data obtained in complex systems, such as interacting electrons in solids, in terms of simple effective models containing a few relevant parameters. Away from equilibrium the situation is much less clear. While some

progress was made in the past for classical systems (Schmittmann and Zia, 1995), there are no rigorously justified generalizations of any of these approaches to generic quantum nonequilibrium systems. It is thus not obvious that the theoretical study of the dynamics of simplified models would accurately describe experiments of more complex systems. In addition there are fewer available tools for analyzing dynamics of even simple interacting models. In this respect cold atomic gases and nanostructures make possible what would be arduous otherwise: a fruitful comparison between nonequilibrium theories based on simple models and carefully designed experiments with tunable system parameters.

Finding systematic ways to understand the nonequilibrium physics of interacting systems is not only of fundamental importance, but could also be crucial for future technologies. A quantum computer, for example, will definitely require the capability of performing real time manipulations of interacting quantum systems. Although large scale quantum computers have yet to appear on the horizon, it is evident that quantum coherent dynamics will be one of the focus points of various experimental systems and of future technologies.

Nonequilibrium dynamics is a potentially vast field: There are many ways to take a system out of equilibrium, such as applying a driving field or pumping energy or particles in the system through external reservoirs as in transport problems. It is thus of utter importance to focus on simple, yet fundamental protocols. In this Colloquium we concentrate on the simplest paradigm: the study of the nonequilibrium dynamics of closed interacting quantum systems following a change in one of the system parameters (quantum quench). Such a change, which could be either fast or slow, is particularly interesting when it takes the system through a quantum phase transition involving macroscopic changes in the state of the many-body system at the initial and final points. Seminal work in this direction includes groundbreaking experiments (Greiner *et al.*, 2002a, 2002b) showing both the feasibility of observing a quantum phase transition in cold atoms and the possibility of observing quantum coherent dynamics. Following this work, a number of different experiments explored the dynamics of cold atom systems driven across the BCS-BEC (Bose-Einstein condensate) crossover (Regal and Jin, 2006, Regal, 2006), polar and ferromagnetic phases of spinor condensates (Sadler *et al.*, 2006), insulating and superfluid phases of ultracold bosons (Tuchman *et al.*, 2006), and many others [see Bloch *et al.* (2008) for a review].

These experiments stimulated an active theoretical research in the relatively unexplored area of quantum dynamics in closed interacting systems. An interesting characteristic common to these systems is that despite the absence of energy exchange with an environment and of the consequent global relaxation, it is nevertheless frequently possible to look at the long time dynamics and characterize it in terms of an asymptotic state attained by physical (measurable) observables (Cramer *et al.*, 2008; Flesch *et al.*, 2008; Reimann, 2008; Rigol *et al.*, 2008; Linden *et al.*, 2009; Gogolin *et al.*, 2011). In connection with this, it is possible to categorize recent research on the subject of this Colloquium in two main questions:

- What is universal in the dynamics of a system following a quantum quench?
- What are the characteristics of the asymptotic, steady state reached after a quench? When is it thermal?

In this Colloquium we discuss both of these questions extensively. We outline our current level of understanding of these issues and chart out the outstanding open questions in the field. In Sec. II we focus on the first question and describe, from various points of view, the universal aspects of nearly adiabatic dynamics near quantum critical points as well as in generic gapped and gapless systems. We argue that the proximity to the adiabatic limit allows us to make specific universal predictions of scaling of various quantities such as the defect density and heat with the quench rate.

In Sec. III we discuss recent progress in understanding thermalization of a quantum system following a quench. In classical systems active interest in this topic was stimulated by the celebrated work of Fermi, Pasta, and Ulam (FPU) on the dynamics of a one-dimensional (1D) anharmonic chain (Fermi *et al.*, 1955; Campbell *et al.*, 2005; Porter *et al.*, 2009) which demonstrated the absence of such thermalization. It was realized much later that the nonlinearity of the interaction is not sufficient for thermalization which occurs, in this system, only if the initial amplitude of interaction exceeds a certain threshold (Izrailev and Chirikov, 1966). Below this threshold, the solution splits into solitons and retains its quasiperiodic nature (Zabusky and Kruskal, 1965) which is a consequence of the Kolmogorov-Arnold-Moser (KAM) theorem (Tabor, 1989). In quantum systems, the question of sufficient criteria for thermalization has remained largely unaddressed so far. Some experimental progress in this direction has been made by a recent experiment from Kinoshita *et al.* (2006) on nonthermalizing dynamics of 1D bosons with short range interactions. This experiment constitutes the first clear demonstration of the fact that a nearly integrable quantum interacting many-particle system does not thermalize for a long time. Currently, the question of extension of the KAM theorem to quantum systems is a subject of active theoretical debate [see, e.g., Olshanii and Yurovsky (2009)].

Finally, we note that many important topics concerning the physics of closed interacting systems did not find space in this Colloquium. Most important among these are the tools that are being developed to describe theoretically the physics of interacting systems out of equilibrium. Among such methods we mention the density matrix renormalization group (DMRG) and time-evolving block decimation (TEBD) for analyzing equilibrium and nonequilibrium 1D systems (White, 1992; Vidal, 2003, 2004; Schollwöck, 2005; Schollwöck and White, 2006) and higher dimensional ones (Verstraete *et al.*, 2008), the Keldysh technique (Kamenev and Levchenko, 2009), which is particularly helpful for deriving quantum kinetic equations, and closely related functional integral methods (Plimak *et al.*, 2001; Rey *et al.*, 2005; Gasenzer, 2009). Cold atom experiments also prompted rapid developments in phase space methods, where quantum dynamics is represented as an evolution in the classical phase space (Blakie *et al.*, 2008; Polkovnikov, 2010b). These methods were originally developed and applied to various problems in single-particle dynamics

(Hillery *et al.*, 1984; Zurek, 2003) and independently in quantum optics in the context of coherent states (Walls and Milburn, 1994; Gardiner and Zoller, 2004). There are other reviews available in the literature (see the references above) which specifically target these areas.

II. NEARLY ADIABATIC DYNAMICS IN QUANTUM SYSTEMS

A. Universality in a nutshell

Universality (or insensitivity to microscopic details) is one of the crucial concepts of modern condensed matter physics. It naturally emerged from one of the milestones of modern physics: the renormalization group. In condensed matter physics universality is a powerful tool for the understanding of continuous (second order) phase transitions, both classical (Landau and Lifshitz, 1980; Chaikin and Lubensky, 1995) and quantum (Sondhi *et al.*, 1997; Sachdev, 1999; Vojta, 2003). As a consequence of the divergence of the correlation length, a system undergoing such a continuous phase transition is typically scale invariant in the vicinity of the critical point and can be characterized by relatively simple massless field theories, which permit a classification of perturbations driving the system away from the critical point. Consequently, universality manifests itself in the scaling behavior of various quantities such as the order parameter, (free) energy, susceptibilities, and correlation functions near the critical point. In this Colloquium we focus mostly on quantum phase transitions occurring at zero temperature upon the variation of a control parameter λ through a critical point λ_c. A standard example of universality is the fact that the exponent ν characterizing the divergence of the correlation length $\xi \sim 1/|\lambda - \lambda_c|^{\nu}$ near the quantum critical point (QCP) is insensitive to the microscopic details of the system and depends only on the *universality class* of the transition, determined by the dimensionality, overall symmetries, and range of the interactions. For classical (thermal) phase transitions similar universality manifests in the divergence of the relaxation time $\tau_{\mathrm{rel}} \sim 1/|\lambda - \lambda_c|^{z\nu}$, where z is the dynamical critical exponent. For quantum phase transitions the exponent z can be associated with a vanishing energy scale $\Delta \sim |\lambda - \lambda_c|^{z\nu}$, which can be either a gap or a crossover scale where the spectrum changes qualitatively. By the uncertainty principle this energy scale corresponds to a divergent time scale, which typically describes the crossover in the scaling behavior of unequal time correlation functions. Phase transitions can also be characterized by singular susceptibilities, which are in turn connected through the fluctuation-dissipation theorem to the correlation functions of conjugate variables (e.g., the magnetic susceptibility is related to the correlation function of the magnetization). At critical points these correlation functions often have power law scaling behavior at long distances, e.g., $\langle m(x)m(x')\rangle \approx 1/|x - x'|^{2\alpha}$. The exponent α sets the scaling dimension of the corresponding operator $m(x)$: $\dim[m(x)] = \alpha$. Because similar correlation functions can enter different susceptibilities not all the scaling exponents are independent but must satisfy scaling relations (Chaikin and Lubensky, 1995; Vojta, 2003).

As mentioned in the Introduction, the idea of universality makes it possible to interpret experimental data obtained in real systems in terms of effective models with a few parameters. Universality can be ultimately understood using the renormalization group, which shows that as a system is coarse grained to lower energies and longer length scales, more and more parameters of its original, *ab initio* description become unimportant (irrelevant), while the remaining few (relevant) parameters define an effective low-energy model. A standard example of universality in this context is the scaling relation between energy and momentum of elementary excitations, $\epsilon \propto k^z$, controlled by the dynamical exponent z which depends on the symmetries of the system. In particular, $z = 1$ in most phases characterized by a continuous broken symmetry (crystals, superfluids, antiferromagnets), $z = 2$ in ferromagnets, where there is an additional conservation law of the order parameter.

Universality is well established and understood in equilibrium. It is, however, crucial for many experimentally relevant situations to understand the extent to which this concept can be extended to out-of-equilibrium physics. Can irrelevant interactions turn out to be important away from equilibrium? Since there are many ways to take a system out of equilibrium, for which specific protocols will universality emerge and which details of the protocol are potentially important? Below we focus on recent studies addressing these important issues in closed interacting quantum systems, and, in particular, on the dynamics of a system whose parameters are dynamically tuned either through a quantum critical point or in general within a gapless and/or gapped phase.

B. Universal dynamics near quantum critical points

We start by considering the simplest nonequilibrium protocol (Dziarmaga, 2005; Polkovnikov, 2005; Zurek *et al.*, 2005): The system is prepared in its ground state and is then driven through a QCP by changing an external parameter λ in time. As long as the rate change of the gap in the spectrum Δ[1] caused by changing λ is smaller than the square of the gap one can expect the system to approximately follow the ground state adiabatically (we revisit this statement in the next section). However, the vanishing of the gap at $\lambda = \lambda_c$ implies that the system will always violate adiabaticity close to the quantum critical point, no matter how slowly the parameters are changed. It is then natural to ask how many excitations will be generated while passing though the critical point and how their density as well as generated entropy and energy will depend on the rate of change of λ.

A similar question known under the name of the Kibble-Zurek (KZ) mechanism (Kibble, 1976; Zurek, 1985, 1996; Kibble, 2007) has been addressed in the past for classical phase transitions. In that case the excitation density of defects is described by a simple scaling argument (Zurek, 1985, 1996). Suppose that the tuning parameter, for example, external temperature T, slowly decreases in time across the critical value T_c: $T = T_c - vt$. The system will respond adiabatically (quasistatically if the system is not thermally

[1]We use the word "gap" for brevity. However, the system can be gapless on one or both sides of the transition (e.g., superfluid-insulator transition). Then Δ would denote a crossover energy scale vanishing at the QCP.

insulated) up to some close vicinity of the critical point, where adiabaticity will be violated as a result of the divergence of the relaxation time ($\tau_{\rm rel} \sim 1/|T - T_c|^{z\nu}$) and the dynamics will become diabatic (sudden). The adiabatic response is once again resumed after the system moves out of the vicinity of the critical point. Zurek suggested a simple criterion for separating such adiabatic and diabatic (impulse) regimes: The time to reach the critical point $t = |T - T_c|/v$ should be equal to the relaxation time. This immediately introduces the time and length scales characterizing the adiabatic to diabatic crossover: $t^\star \sim 1/|v|^{z\nu/(z\nu+1)}$, $\xi^\star \sim 1/|v|^{\nu/(z\nu+1)}$. The violation of adiabaticity implies that order cannot form on distances larger than $\xi^\star$ leading to the formation of a domain structure with a characteristic distance $\xi^\star$ between the domain boundaries.

In two- and three-dimensional systems, when the order parameter is characterized by a continuous broken symmetry, the points where several domains meet correspond to vortices or vortex rings. These are robust topological excitations with a long lifetime (see Fig. 1). Since $\xi^\star$ determines the average distance between the defects their density is given by a simple universal expression

$$n_{\rm ex} \sim (\xi^\star)^d \sim |v|^{d\nu/(z\nu+1)}. \tag{1}$$

The universality of the KZ prediction above is manifest in the appearance of the universal critical exponents z and ν in the scaling law. This scaling was confirmed in experiments in liquid crystals (Ducci *et al.*, 1999). Experiments in other systems [superconductors (Maniv *et al.*, 2003), arrays of Josephson junctions (Monaco *et al.*, 2006)] observed the production of topological defects with a power law scaling on the quench rate but gave a different exponent. The KZ scaling was also confirmed theoretically using stochastic dynamics (Ginzburg-Landau dynamics with Langevin noise or Glauber dynamics) where temperature changes in time (Laguna and Zurek, 1997; Yates and Zurek, 1998; Rivers, 2001; Krapivsky, 2010), although there are also works suggesting various modifications (Hindmarsh and Rajantie, 2000; Biroli *et al.*, 2010). One can also interpret Eq. (1) as a measure of nonadiabaticity near the critical point. It is naturally expected that other measures such as nonadiabatic energy production and entropy generation will display similar universality. These measures might be preferable over $n_{\rm ex}$ in situations where it is difficult to identify defects. Finding the scaling of these quantities remains an open question.

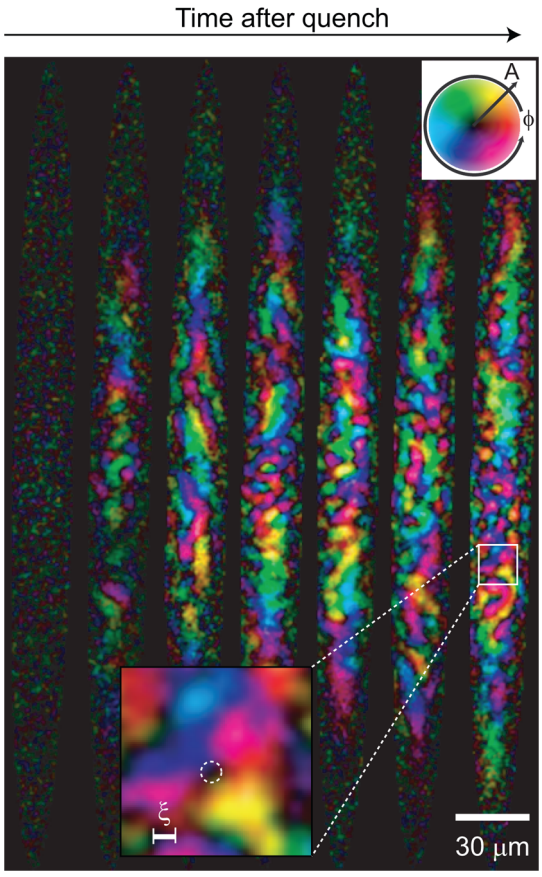

FIG. 1 (color online). Defect generation after a quench in a spinor condensate. These images show the transverse magnetization density of spinor condensates for variable evolution times after a quench to a ferromagnetic state, revealing a spatially inhomogeneous formation of ferromagnetic domains. The orientation ϕ and amplitude A are depicted by the color and brightness according to the color wheel shown. Inset: An instance of a spin vortex spontaneously created during the quench. For reference, the length scale corresponding to the characteristic healing length ξ is also shown (see Sec. IV for more details). Adapted from Sadler *et al.*, 2006.

The arguments above were recently generalized to the crossing of quantum phase transitions (Dziarmaga, 2005; Polkovnikov, 2005; Zurek *et al.*, 2005) [for reviews on this subject see Dziarmaga (2010), and Gritsev and Polkovnikov (2010)]. As discussed in the quantum case the parameter to be varied is not temperature T but rather the coupling λ tuning the system through the quantum critical point. In order to obtain the scaling for the number of excitations produced in the quantum case first recall the Landau-Zener (LZ) analysis of the crossover between adiabatic and nonadiabatic dynamics in a simple driven two-level system:

$$\mathcal{H}_{lz} = g(t)\sigma_z + \Delta\sigma_x, \tag{2}$$

where $g(t) = vt$. If the system was initially prepared in the ground state at $t \to -\infty$, the probability of transition to the excited state at $t \to +\infty$ is (Landau, 1932; Majorana, 1932; Stückelberg, 1932; Zener, 1932)

$$p_{\rm ex} = \exp[-\pi\gamma], \tag{3}$$

where we introduced the Landau-Zener parameter $\gamma = \Delta^2/v$. Note that the limit $\gamma \gg 1$ corresponds to the adiabatic limit with an exponentially suppressed transition probability while $\gamma \ll 1$ corresponds to the diabatic limit where the transition happens with probability close to unity. Hence, when the rate of change of the energy splitting between two levels becomes larger or comparable to the energy splitting squared, one observes a crossover from adiabatic to diabatic dynamics. An alternative qualitative explanation of this result was recently formulated (Damski and Zurek, 2006).

The Landau-Zener argument can be straightforwardly extended to the crossing of a QCP. The characteristic energy scale which changes in time is now the gap Δ. As discussed this gap universally depends on the tuning parameter λ: $\Delta(\lambda) \sim |\lambda - \lambda_c|^{z\nu} \sim |vt|^{z\nu}$, where we assumed that the

dependence $\lambda(t)$ can be linearized near the QCP, $\lambda(t) \approx \lambda_c + vt$. Comparing the rate of change of the gap with its square, i.e., solving the equation $d\Delta/dt \approx \Delta^2$, we find the energy scale at which adiabaticity breaks down is $\Delta^\star \sim |v|^{z\nu/(z\nu+1)}$. At this point the system is characterized by the length scale $\xi^\star \sim |v|^{-\nu/(z\nu+1)}$, which can be interpreted as the typical length scale of fluctuations of the order parameter. Beyond this point the adiabatic approximation breaks down and fluctuations at longer scales cannot adiabatically follow the ground state. This results in the creation of defects with typical distance ξ between them and density $n_{\rm ex} \sim |\xi^\star|^d \sim |v|^{d\nu/(z\nu+1)}$. This scaling is identical to the classical one, Eq. (1) with $\lambda \to T$, and was proposed independently by Polkovnikov (2005) and Zurek *et al.* (2005). There is a simple quasiparticle interpretation for this scaling: Assuming that the excitations in the system are characterized by isolated quasiparticles then their density can be found from $n_{\rm ex} \approx \int_0^{\Delta^\star} d\epsilon \rho(\epsilon)$, where $\rho(\epsilon)$ is the single-particle density of states near the QCP. In uniform d-dimensional systems $\rho(\epsilon) \sim \epsilon^{d/z-1}$, which again reproduces Eq. (1).

The scaling in Eq. (1) was verified in a series of exact solutions of the dynamics across the QCP in integrable systems whose dynamics can be mapped into a series of Landau-Zener transitions of a few quasiparticle modes. In particular, it was verified for various spin models in one and two dimensions which can be mapped to noninteracting fermions (Mukherjee *et al.*, 2007), for models where low-energy excitations near phase transitions can be described by bosonic Goldstone modes (Polkovnikov, 2005; Lamacraft, 2007; Dziarmaga *et al.*, 2008), the sine-Gordon model, where elementary excitations are solitons and breathers with fractional statistics (De Grandi *et al.*, 2008; De Grandi *et al.*, 2010a), graphene (Dóra *et al.*, 2010; Dóra and Moessner, 2010). This scaling was also extended to disordered systems, such as a disordered Ising spin chain, where it was found that $n_{\rm ex} \sim 1/\log^2(v)$ (Dziarmaga, 2006; Caneva *et al.*, 2007), as expected from Eq. (1) due to the divergence of the exponent z near the critical point (Fisher, 1995).

The scaling in Eq. (1) can be generalized to the case of nonlinear dependence of the tuning parameter on time $\lambda(t) \sim \lambda_c \pm v|t|^r$, where considerations similar to those leading to Eq. (1) give (Barankov and Polkovnikov, 2008; Sen *et al.*, 2008; De Grandi *et al.*, 2010b)

$$n_{\rm ex} \sim |v|^{d\nu/(z\nu r+1)}. \tag{4}$$

In all cases v in Eq. (4) plays the role of the adiabatic parameter: The limit $v \to 0$ corresponds to the adiabatic limit (this interpretation is valid even for instantaneous quenches $r = 0$, where v plays the role of the quench amplitude). This suggests the dynamics can be systematically analyzed using adiabatic perturbation theory (Polkovnikov, 2005; Rigolin *et al.*, 2008; De Grandi and Polkovnikov, 2010), i.e., expanding the transition amplitudes to the instantaneous eigenstates of the system in powers of v. By using such an analysis, De Grandi *et al.* (2010a, 2010b) showed that the scaling (4) can be derived from the scaling of the adiabatic fidelity, defined as the overlap of the time-dependent wave function with the instantaneous ground state:

$$F(t) = |\langle \psi(t) | \psi_{\rm gs}(t) \rangle|. \tag{5}$$

In particular, for $\lambda(t) = \lambda_c + vt^r/r!\theta(t)$, where $\theta(t)$ is a step function,

$$P_{\rm ex}(v) = 1 - F(t)^2 \approx L^d |v|^2 \chi_{2r+2}(\lambda_c), \tag{6}$$

where

$$\chi_{2r+2}(\lambda) = \frac{1}{L^d} \sum_{n \neq 0} \frac{|\langle n|V|0\rangle|^2}{(E_n - E_0)^{2r+2}} \tag{7}$$

is the adiabatic (fidelity) susceptibility of the order $2r+2$ [χ_2 is the conventional fidelity susceptibility (Gu and Lin, 2009)]. Here E_n are the eigenenergies and V is the operator coupled to the parameter λ: $V = \partial_\lambda H(\lambda)|_{\lambda=\lambda_c}$. If the perturbation is local and spatially uniform, i.e., $V = \int d^d x u(x)$, then the scaling dimension of the adiabatic fidelity susceptibility is obtained from a straightforward generalization of the result by Venuti and Zanardi (2007) and Gu and Lin (2009), i.e., $\dim[\chi_{2r+2}] = 2\Delta_u - 2z(r+1) - d$, where $\Delta_u \equiv \dim[u]$ is the scaling dimension of $u(x)$.

We now discuss from this general perspective the arguments leading to the generalized scaling relation, Eq. (4). If the scaling dimension of the susceptibility is negative, this implies that χ_{2r+2} diverges at the critical point. In this case from Eq. (6) we find that asymptotically at $v \to 0$

$$P_{\rm ex}(v) \sim |v|^2 L^{2d+2z(r+1)-2\Delta_u}. \tag{8}$$

From Eq. (8) we see that the probability of exciting the system becomes of order 1 when $L \sim 1/|v|^{1/(d+z(r+1)-\Delta_u)}$. This length scale can be interpreted as the typical distance between elementary excitations (defects) and thus we find that instead of Eq. (8) we get

$$n_{\rm ex} \sim |v|^{d/[d+z(r+1)-\Delta_u]}. \tag{9}$$

This expression reduces to Eq. (4) if $u(x)$ is a relevant operator driving the system to the new phase. Indeed, in this case $\lambda \int d^d x u(x)$ should have the same scaling dimension as the gap, i.e., z, which immediately implies that the scaling dimension of $u(x)$ is $\Delta_u = d + z - 1/\nu$ and that $\dim[\chi_{2r+2}] = d - 2zr - 2/\nu$ (Schwandt *et al.*, 2009; De Grandi *et al.*, 2010a). Finally note that if the scaling dimension of χ_{2r+2} is positive then the asymptotics in Eq. (8) gives a subleading correction to the regular analytic part $P_{\rm ex} \approx \text{const} \times L^d v^2$, coming from the high-energy (ultraviolet) contribution to the susceptibility. We discuss its importance in the next section.

Other possible generalizations of the scaling law Eq. (1) involve studies of defect production in systems where the dynamics describes the passage through quantum critical lines. A concrete example of such a situation occurs in the transverse-field XY model (Divakaran *et al.*, 2008; Mukherjee *et al.*, 2008). Here the quench takes one through a gapless line where the critical point occurs at the same momenta ($\vec{k} = 0$ for the present case) at each point on the line. A detailed analysis shows that in such cases, for critical lines with $z = \nu = 1$, the defect density still obeys a universal scaling law albeit with a different power: $n \sim v^{1/3}$ (Divakaran *et al.*, 2008; Mukherjee *et al.*, 2008; Mondal *et al.*, 2009). The second situation involves the 2D Kitaev model (Sengupta *et al.*, 2008), where a quench once again involves the passage through a gapless line with an energy

gap vanishing for different momenta at different points on the line. It can be shown that in such a case, the defect density scales as $\sqrt{v}$ for the 2D Kitaev model instead of the expected $n \sim v$ behavior for 2D systems with $z = \nu = 1$ (Sengupta *et al.*, 2008). A generalization of these results for linear quenches through critical lines with arbitrary z and ν has also been worked out (Mondal *et al.*, 2008; Mukherjee *et al.*, 2008; Sengupta *et al.*, 2008; Mondal *et al.*, 2009). Many other situations involving anisotropic phase transitions and quenching through multicritical points were analyzed in the literature leading to various deviations from the scaling equation (4) (Deng *et al.*, 2008; Bermudez *et al.*, 2009; Sen and Vishveshwara, 2010).

In order to detect experimentally the density of excitations generated by passing through a QCP, it is evident that one should distinguish between situations where such excitations are long-lived quasiparticles (as for nearly integrable systems) or decay after being created (as for nonintegrable systems). In the first case, the presence of excitations above the ground state could, for example, be detected by measuring correlation functions long after the quench. This has been shown for a quantum Ising chain linearly tuned through its quantum critical point (Cherng and Levitov, 2006). The presence of defects with respect to the ferromagnetic ground state leads to exponentially decaying correlations of the order parameter superimposed, for slow enough quenches, to characteristic oscillations with period scaling with the quench velocity. This second feature is observed for abrupt quenches as well (Sengupta *et al.*, 2004) and is a consequence of the integrability of the model (Rossini *et al.*, 2010). If in turn we consider a generic nonintegrable system, it is necessary to express deviations from adiabaticity in terms of quantities, such as the excess energy or the entropy generated by passing through the QCP, which are not sensitive to the decay of quasiparticles, but can still be related to the density of excitations created close to the quantum critical point. Energy can be unambiguously determined both experimentally and numerically for both integrable and nonintegrable systems and its scaling with the rate of quench can be used to differentiate between different nonadiabatic regimes (Polkovnikov and Gritsev, 2008; Eckstein and Kollar, 2010; Moeckel and Kehrein, 2010). The excess energy or equivalently heat (Q) (Polkovnikov, 2008b) generated during the quench process is in general universal if the process ends near the critical point. The scaling of Q is associated with the singularity of the susceptibility χ_{2r+1} at the critical point (De Grandi and Polkovnikov, 2010), which implies that for relevant perturbations in the thermodynamic limit

$$Q \sim |v|^{(d+z)\nu/(z\nu r+1)}. \tag{10}$$

Unless one considers cyclic processes, the drawback of using heat as a measure of nonadiabaticity is that it is hard to separate it from the adiabatic part of the energy change, corresponding to the limit $v \to 0$. Moreover, if the position of the QCP is not exactly known, the heat becomes sensitive to the nonuniversal details of the spectrum at the final point of the evolution. A way out could be to measure the higher moments of the energy or the whole distribution function of the energy, connected to the statistics of the work (Talkner *et al.*, 2007; Silva, 2008) in finite size systems. In particular, in the case of abrupt quenches close to critical points the statics of the work is characterized by sharp edge singularities (Silva, 2008; Paraan and Silva, 2009). A related natural measure of nonadiabaticity is obtained by focusing on the entropy since entropy is conserved only for slow (adiabatic) processes, while it is expected to increase as the system passes through the QCP. Moreover entropy production can be detected experimentally in certain systems, e.g., in cold atoms by driving the system to the weakly interacting regime, where the relation between entropy and energy is known (Luo *et al.*, 2007). Theoretically, the quantification of entropy production in a closed quantum system is rather subtle. Indeed, von Neumann's entropy of the entire system, being conserved throughout unitary evolution (Landau and Lifshitz, 1980), cannot be a good characterization of deviations from adiabatic dynamics. However, the concept of diagonal entropy (Polkovnikov, 2010a), defined as $S_d = -\sum_n \rho_{nn} \ln(\rho_{nn})$, where ρ_{nn} are the diagonal matrix elements of the density matrix in the instantaneous basis, avoids this difficulty. In stationary systems, the diagonal entropy is simply von Neumann's entropy of the time averaged density matrix, also called diagonal ensemble. It is clear that the diagonal entropy is generated due only to nonadiabatic transitions and thus satisfies the key requirement of the thermodynamic entropy: It is conserved for adiabatic processes and can only increase or stay constant in closed systems if the initial state is stationary (Polkovnikov, 2010a). For initial equilibrium states the diagonal entropy also satisfies the fundamental thermodynamic relation $dE = TdS - Fd\lambda$, where $F = -\langle \partial_\lambda H \rangle$ is the generalized force. For particular noninteracting models, the scaling of the diagonal entropy was found to be the same as that of the density of quasiparticles Eq. (4) (Mukherjee *et al.*, 2008; De Grandi *et al.*, 2010a, 2010b). It is also possible to analyze the entanglement entropy (Vidal *et al.*, 2003; Calabrese and Cardy, 2004; Refael and Moore, 2004; Calabrese and Cardy, 2005), i.e., von Neumann's entropy of the reduced density matrix of a part of the system, and, in particular, at its time evolution following a quench (Cincio *et al.*, 2007; Sengupta and Sen, 2009; Pollmann *et al.*, 2010). For specific 1D spin systems it was found that the entanglement entropy scales logarithmically with the quench time (Cincio *et al.*, 2007; Pollmann *et al.*, 2010). Note that currently it is unclear how one can measure entanglement in many-body systems and the entanglement entropy, in particular [see suggestions by Klich *et al.* (2006), and Klich and Levitov (2009)] and what its relation with the thermodynamic entropy is.

Finally, another interesting question that has received attention is the connection between microscopic dynamics and thermodynamics in the semiclassical limit. In general, in classical systems there is no simple analog to the instantaneous energy levels, the key concept in analysis of quantum systems. Such an analog, however, exists in the case of periodic motion. Then in the semiclassical limit the stationary levels are found from the Bohr quantization [or more accurately from the WKB approximation (Landau and Lifshitz, 1981)], which states that the reduced action in the stationary orbit should be quantized. In classical mechanics it is known that the reduced action is an adiabatic invariant, i.e., it is conserved for the adiabatic evolution (Landau and Lifshitz,

1982). From the previous discussion applied to quantum systems we can deduce that near singularities, such as second order phase transitions, conservation of adiabatic invariants should be violated and this is indeed the case (Landau and Lifshitz, 1982). Altland *et al.* (2009), and Itin and Törmä (2009a, 2009b) analyzed slow dynamics for a particular many-body generalization of the Landau-Zener model (closely related to the Dicke model) in the semiclassical limit. It was found that the nontrivial power law scaling of the number of excitations in this system [similar to Eq. (1)] follows from the changes of adiabatic invariants near the singularity, which in turn corresponds to a quantum critical point in the thermodynamic limit. It is interesting that quantum fluctuations in this problem entered only through the initial distribution of the adiabatic invariants but not through the equations of motion. The corresponding truncated Wigner approximation turned out to be accurate in all regimes of the dynamics (Altland *et al.*, 2009; Kiegel, 2009). It would be important to understand precise connections between transitions among microscopic energy levels in the quantum case and changes of suitable generalizations of adiabatic invariants in the classical limit.

C. Slow dynamics in gapped and gapless systems.

Up to now we have discussed the universal dynamics resulting from the variation of a control parameter λ through a quantum critical point. However, the dynamics of interacting quantum systems has interesting regimes even when the system is fully gapped or gapless for the entire duration of the protocol. The classification of these regimes is important in order to understand dissipation and to develop optimal protocols minimizing nonadiabatic effects. Many of these questions are currently a subject of intense theoretical research in different contexts, from quantum computation to transport.

The general formulas (6) and (7), which describe the density of excitations $P_{\rm ex}(v)$ generated by a variation of the control parameter, tell us that if the system remains fully gapped throughout the evolution, then $P_{\rm ex}$ and $n_{\rm ex}$ will have a quadratic scaling with v whenever the susceptibility χ_{2r+2} evaluated at the initial and final couplings is finite (Rigolin *et al.*, 2008; De Grandi and Polkovnikov, 2010). A similar argument shows that the heat Q is also quadratic in v if χ_{2r+1} is finite. This quadratic scaling is characteristic of any quantum system. We point out that in the standard Landau-Zener problem in the slow limit $\Delta^2 \gg v$, if we start in the ground state at $t \to -\infty$ and let the system evolve up to $t \to +\infty$, the transitions to excited states are exponentially suppressed as a result of destructive interference between multiple transitions (Vitanov and Garraway, 1996; Vitanov, 1999). At the same time in the intermediate stages of the evolution the transition probability reaches much higher values which scale only quadratically with the rate v. For example, if one considers a process which starts at $t_0 \to -\infty$, the transition probability to the instantaneous excited state at the moment t in the slow limit $v/\Delta^2 \ll 1$ can be given by (Vitanov, 1999)

$$p_{\rm ex} \approx \frac{v^2\Delta^2}{16[g(t)^2+\Delta^2]^3}. \tag{11}$$

If $t > 0$, there is an additional exponential term which leads to Eq. (3) at $t \to +\infty$. If the process starts at $t_0 = 0$ exactly in the symmetric point, where $g(0) = 0$, the quadratic asymptotics (11) is also recovered (Cucchietti *et al.*, 2007). This scaling occurs as a result of the discontinuity of the first derivative of $g(t)$ at the moment where the process starts or ends or following a discontinuity in any other point of the evolution [see, e.g., Damski and Zurek (2006) and Divakaran *et al.* (2010) for particular cases]. Likewise if there is a discontinuity in the second order derivative of $g(t)$, asymptotically the transition probability in the LZ problem scales quadratically with acceleration. More generally for the protocol $g(t) = g_0 + v(t-t_0)^r/r!\theta(t-t_0)$, where $\theta(t)$ is the step function, one can show that (De Grandi and Polkovnikov, 2010)

$$p_{\rm ex}(t \to \infty) \approx \frac{v^2\Delta^2}{16(g_0^2+\Delta^2)^{2r+1}}. \tag{12}$$

As discussed in the previous section this formula applies even to sudden transitions ($r = 0$), where it reduces to the result of the ordinary perturbation theory. The same expression applies to the reverse process. If both the initial and final couplings are finite then the resulting transition probability is asymptotically determined by the sum of probabilities associated with discontinuities of derivatives of $g(t)$ at the initial and final times of the evolution plus additional interference terms which highly oscillate in the slow limit. We point out that in the LZ problem (and in general in gapped systems) one can suppress power law asymptotics of the transition probability by starting and ending the protocol infinitely smoothly, e.g., $g(t) \sim g_0 + g_1 \exp[-\tau/(t-t_0)]\theta(t-t_0)$. In this case only the nonanalytic term in the transition probability survives and we are back to Eq. (3), where v is the time derivative of $g(t)$ near the symmetric point where $g(t) = 0$.

In gapless systems the situation becomes qualitatively different. In this case the adiabatic susceptibilities can diverge leading to nonanalytic dependence of the corresponding quantities on v, as in the case of the crossing of a quantum critical point. For example, it is straightforward to see that for marginal perturbations in a generic gapless phase the scaling dimension of the adiabatic susceptibility $\dim[\chi_{2r+2}] = d - 2zr$. It becomes negative in low dimensions $d < 2zr$ leading to a nonanalytic scaling of the density of excitations with v. Thus depending on dimensionality in gapless systems, one expects at least two different regimes of the response of the system to a slow external perturbation: analytic and nonanalytic. These regimes were first suggested by Polkovnikov and Gritsev (2008) together with a third regime where adiabaticity is violated in the thermodynamic limit and Q or $n_{\rm ex}$ becomes proportional to a power of the system size or some other large length scale associated with some irrelevant operator. In this regime, which can be realized in low-dimensional bosonic systems, the scaling Eq. (1) is violated. At the moment it is unclear how generic it is and what sets the scaling of various quantities.

A close inspection of the adiabatic susceptibility shows that in general the analytic (quadratic) part of the heat and energy of excitations on v comes from the high-energy (or ultraviolet) part of the spectrum, while the nonanalytic part comes from low energies. This was indeed shown to be the case in several situations, from the sine-Gordon model

(De Grandi *et al.*, 2010a) to the Falicov-Kimball model (Eckstein and Kollar, 2010) and the turning on interactions in a Fermi liquid (Moeckel and Kehrein, 2010). As pointed out above the ultraviolet transitions can be suppressed by avoiding discontinuities in $\lambda(t)$ and its derivatives. However, this is not necessarily the case for the low-energy nonanalytic contribution. To see this we need to reexpress the excess energy (or density of excitations) in terms of the total time of the process τ. Doing this it was found that in an insulating, gapped phase, the details of the protocol are important and smoother protocols lead to a suppression of nonadiabatic effects, while in a gapless phase making $\lambda(t)$ smoother does not affect the heating (Eckstein and Kollar, 2010). This result can again be understood by analyzing the scaling dimension of the susceptibility χ_{2r+1}. According to our discussion for generic gapless systems its scaling dimension is negative when $d+z<2zr$. Then $Q\sim v^{(d+z)/zr}$ [this result immediately follows from Eq. (10) by taking the limit $\nu\to\infty$]. On the other hand, v is related to the total quench time as $v\sim 1/\tau^r$. Thus we see that in this case $Q\sim 1/\tau^{(d+z)/z}$, i.e., indeed independent on r. On the other hand, for positive scaling dimension of χ_{2r+1} which is the case for $d+z>2zr$ and which is always true in gapped systems we have $Q\sim v^2\sim 1/(\tau)^{2r}$. Since in the adiabatic limit τ is large, we see that indeed the heat can be suppressed by increasing r and making $\lambda(t)$ smoother. An interesting open question is finding an optimal protocol for minimizing the nonadiabatic effects within given time τ. It is plausible that the optimal power is determined by a vanishing scaling dimension of the corresponding adiabatic susceptibility χ_{2r+1}. The questions of finding protocols minimizing nonadiabatic effects for gapped systems (with the possibility of crossing isolated quantum critical points) were also addressed by approximately minimizing the transition probability and identifying the Riemannian metric tensor underlying the adiabatic evolution (Rezakhani *et al.*, 2009; Rezakhani *et al.*, 2010). Studying the optimization of the protocol taking a system through a QCP it was found the optimal exponent of $\lambda(t)\sim|t|^r\mathrm{sign}(t)$ near the QCP scales logarithmically with the quench time, $r\propto\ln(\tau)$ (Barankov and Polkovnikov, 2008). This result was also extended to systems with external confining potential (Collura and Karevski, 2010).

D. Effects of finite temperature

In the discussion above we always implicitly assumed that the system is initially prepared in the ground state. An interesting and genuine question is how finite temperature effects modify the picture. In isolated systems temperature enters through initial conditions: The system is prepared in the initial finite temperature equilibrium state and is then dynamically driven out of equilibrium. How is the response of the system affected by the initial thermal fluctuations? One naturally expects that while the transitions to high-energy states (quadratic in v) will not be affected by small temperatures in the system, the transitions to the low-energy states, which determine the nonanalytic contribution to heat and density of excitations, will be sensitive to temperature. De Grandi *et al.* (2010a, 2010b) [see also Gritsev and Polkovnikov (2010)], studying a particular sine-Gordon model in the two limits where it could be mapped to free bosons and free fermions, showed that the statistics of quasiparticles enters the scaling of both Q and n_{ex} making dynamics more adiabatic (compared to the zero temperature case) for fermions due to Pauli blocking and less adiabatic for bosons due to Bose enhancement. These results were not yet extended to generic interacting systems.

Another aspect of thermalization, the influence of the coupling to an environment setting the temperature on the slow dynamics near quantum critical points, has been studied by Fubini *et al.* (2007), Mostame *et al.* (2007), Patanè *et al.* (2008), Patanè, Amico *et al.* (2009), and Patanè, Silva *et al.* (2009). This setup allows one to analyze the effects of thermal smearing and of dephasing and/or dissipation on the dynamics of a quantum critical system. Using a combination of kinetic equations and scaling arguments it was found that in this situation the excess energy has two universal contributions, one still given by Eq. (10), while the second involving a universal power of temperature replacing the universal power of v (Patanè *et al.*, 2008).

E. Open problems

While the physics described above is definitely an important example of the emergence of universality in the dynamics of interacting quantum systems, it is evidently a piece, albeit important, of the puzzle that has to be composed in order to understand to which extent the standard concepts of statistical physics can be applied to nonequilibrium problems. Understanding the meaning of relevance or irrelevance of a perturbation in generic nonequilibrium processes, extending the notion of universality to nonequilibrium systems, as well as the concept of the renormalization group, is a task that certainly requires the solution of many specific problems, and a close comparison between experiments and theory.

So far most of the theoretical research focused on analyzing slow dynamics for global quenches, where the external perturbation couples to the whole system. How these results can be extended to local or spatially nonuniform perturbations is an open question. At one extreme limit, one can imagine performing a quench only locally. Then the rest of the system could be seen as a thermal bath. Analysis of the dynamics of a special case transverse-field Ising model where the tuning parameter linearly depends on both time and space has shown that excitations are generally suppressed by nonuniformity of the tuning parameter (Dziarmaga and Rams, 2010). This suggests that quantitative and qualitative differences may emerge when some of the symmetries of the system, e.g., translational, are broken in the quench process.

Another important issue concerns the connections between adiabaticity in thermodynamics and microscopic dynamics. One of the consequences of the thermodynamic adiabatic theorem is that no heat can be generated in an isolated system during an infinitesimally slow process. More generally, according to the second law of thermodynamics in Thompson's (Kelvin's) form for any cyclic process, the system can increase only its energy, i.e., the heat should be always non-negative as long as one starts in equilibrium. This statement, which is obvious if the system is initially in the ground state, has been proven microscopically for a class of passive initial states

(Allahverdyan and Nieuwenhuizen, 2002; Thirring, 2002; Allahverdyan and Nieuwenhuizen, 2005; Boksenbojm *et al.*, 2009), whose initial density matrix is stationary (diagonal) and a monotonically decreasing function of energy: $(\rho_n - \rho_m)(\epsilon_m - \epsilon_n) \geq 0$. This statement also directly follows from analyzing transitions between microscopic energy levels (Polkovnikov, 2008b). Likewise many statements of thermodynamics related to behavior of entropy including the second law and fundamental thermodynamic relations are recovered using the concept of diagonal entropy (Polkovnikov, 2010a). At the same time there are many open questions remaining: What are the time scales involved in the definition of adiabaticity? How can one microscopically define adiabatic time scales in interacting systems and why are these time scales much shorter than the inverse distance between many-body levels [see, e.g., the discussion by Balian (1991)]? And finally, what is the role of integrability in nonequilibrium thermodynamics? These questions are closely connected to the microscopic origin of conventional dissipation, which in turn is also likely related to the combination of nonadiabatic creation of the elementary excitations and their following relaxation or dephasing. From the discussion above, we can anticipate anomalous dissipation near critical points and in gapless low-dimensional systems.

III. EFFECTS OF INTEGRABILITY AND ITS BREAKING: ERGODICITY AND THERMALIZATION

We now turn to one of the most natural questions to be addressed when studying the dynamics of a closed many-body quantum system: Are interactions within the system sufficient to make the system behave ergodically? If we focus on local degrees of freedom, e.g., a few spins in a spin chain, can the rest of the system always be thought of as an effective thermal bath? And if this is not possible, are there some observable effects on the system dynamics? While these questions are definitely connected to quantum ergodicity (Goldstein *et al.*, 2010), a topic with a long history dating back to the early days of quantum mechanics (von Neumann, 1929; Pauli and Fierz, 1937; Mazur, 1969; Suzuki, 1971; Peres, 1984; Deutsch, 1991; Srednicki, 1994), the past few years have brought a great deal of progress on the context of closed many-body systems. The main motivation came from recent experiments on low-dimensional cold atomic gases described in Sec. IV in this Colloquium (Greiner *et al.*, 2002b; Kinoshita *et al.*, 2006). The experimental availability of essentially closed (on the time scales of experiments), strongly correlated systems together with the awareness of the conceptual importance of these issues in a number of areas (e.g., transport problems, many-body localization, integrable and nonintegrable dynamics) have stimulated much interest on *quantum thermalization*. Below we give a synthetic view on a number of recent important developments on this subject, starting with the discussions of the general concepts of ergodicity and thermalization and then moving to the discussion of many-body systems and integrability.

A. Quantum and classical ergodicity

While the idea of ergodicity is well defined in classical mechanics, the concept of *quantum ergodicity* is somewhat less precise and intuitive. Classically, an interacting system of N particles in d dimensions is described by a point X in a $(2dN)$-dimensional phase space. The intuitive content of the word "ergodic," i.e., the equivalence of phase space and time averages, can then be formalized by requiring that if we select an initial condition X_0 having initial energy $H(X_0) = E$, where H is the Hamiltonian of the system, then

$$\overline{\delta[X - X(t)]} \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \delta(X - X(t)) = \rho_{\mathrm{mc}}(E), \qquad (13)$$

where $\rho_{\mathrm{mc}}(E)$ is the microcanonical density of the system on the hypersurface of the phase space of constant energy E, and $X(t)$ is the phase space trajectory with initial condition X_0. Of course, if this condition is satisfied by all trajectories, then it is also true for every observable. We immediately see that in order to have ergodicity, the dynamics cannot be arbitrary: The trajectories $X(t)$ have to uniformly cover the energy hypersurface for almost every initial condition X_0.

The most obvious quantum generalization of this notion of ergodicity is arduous (von Neumann, 1929). We first define a quantum microcanonical density matrix: Given a Hamiltonian with eigenstates $|\Psi_\alpha\rangle$ of energies E_α, a viable definition of the microcanonical ensemble is obtained by coarse graining the spectrum on energy shells of width δE, sufficiently large to contain many states but small on macroscopic scales. Denoting by $\mathcal{H}(E)$ the set of eigenstates of H having energies between E and $E + \delta E$, we define $\hat{\rho}_{\mathrm{mc}}(E) = \sum_{\alpha\in\mathcal{H}(E)} 1/\mathcal{N} |\Psi_\alpha\rangle\langle\Psi_\alpha|$, where $\mathcal{N}$ is the total number of states in the microcanonical shell. We now ask the most obvious question: Given a generic initial condition made out of states in a microcanonical shell, $|\Psi_0\rangle = \sum_{\alpha\in\mathcal{H}(E)} c_\alpha |\Psi_\alpha\rangle$, is the long time average of the density matrix of the system given by the microcanonical density matrix? The answer to this question for a quantum system is, unlike in the classical case, almost always no, as J. von Neumann realized already in 1929 (von Neumann, 1929). More precisely, if we assume the eigenstates of the system not to be degenerate, the time average is

$$\overline{|\Psi(t)\rangle\langle\Psi(t)|} = \sum_\alpha |c_\alpha|^2 |\Psi_\alpha\rangle\langle\Psi_\alpha| = \hat{\rho}_{\mathrm{diag}}, \qquad (14)$$

where $|\Psi(t)\rangle$ is the time evolved of $|\Psi_0\rangle$. This object is known in the modern literature as the diagonal ensemble (Rigol *et al.*, 2007, 2008; Rigol, 2009). Note that now the most obvious definition of ergodicity, i.e., the requirement $\rho_{\mathrm{mc}} = \rho_{\mathrm{diag}}$, implies that $|c_\alpha|^2 = 1/\mathcal{N}$ for every α, a condition that can be satisfied only for a special class of states. Quantum ergodicity in the strict sense above is therefore almost never realizable (von Neumann, 1929; Goldstein *et al.*, 2010).

Our common sense and expectations, which frequently fail in the quantum realm, make us nevertheless believe that, in contrast with the arguments above, macroscopic many-body systems should behave ergodically almost always, unless some special conditions are met (e.g., integrability). The key to understanding ergodicity is therefore to look at quantum systems in a different way, shifting the focus on *observables* rather than on the states themselves (von Neumann, 1929; Mazur, 1969; Peres, 1984). Given a set of *macroscopic*

observables $\{M_\beta\}$ a natural expectation from an ergodic system would be for every $|\Psi_0\rangle$ on a microcanonical shell $\mathcal{H}(E)$

$$\langle\Psi(t)|M_\beta(t)|\Psi(t)\rangle \rightarrow_{t\rightarrow+\infty} \mathrm{Tr}[M_\beta\hat{\rho}_{\mathrm{mc}}] \equiv \langle M_\beta\rangle_{\mathrm{mc}}, \quad (15)$$

i.e., that looking at macroscopic observables long after the time evolution started makes the system appear ergodic for every initial condition we may choose in $\mathcal{H}(E)$. One needs to take a certain care in defining the infinite time limit here, since literally speaking it does not exist in finite systems because of quantum revivals. A proper way to understand this limit is to require that Eq. (15) holds in the long time limit at almost all times. Mathematically this means that the mean square difference between the left-hand side and the right-hand side of Eq. (15) averaged over long times is vanishingly small for large systems (Reimann, 2008). To avoid dealing with these issues ergodicity can be defined using the time average, i.e., requiring that

$$\overline{\langle\Psi(t)|M_\beta(t)|\Psi(t)\rangle} = \mathrm{Tr}[M_\beta\hat{\rho}_{\mathrm{diag}}] = \langle M_\beta\rangle_{\mathrm{mc}}. \quad (16)$$

Note that if the expectation value of M_β relaxes to a well-defined state in the sense described above, this state will coincide with the time averaged state and the two definitions of ergodicity Eqs. (15) and (16) will be equivalent. If the conditions above are satisfied then in loose terms $\hat{\rho}_{\mathrm{mc}}$ can be considered as equivalent to $\hat{\rho}_{\mathrm{diag}}$. von Neumann proved that if the system satisfies some natural requirements (e.g., absence of resonances), and the set $\{M_\beta\}$ is constructed in such a way as to define *macrostates* of the system, which obviously requires the observables to be coarse grained on the various microcanonical shells $\mathcal{H}(E)$ and mutually commuting, then a form of ergodicity is observed (sometimes referred to as a *normal typicality*). In particular, for every $|\Psi_0\rangle$ and almost every set $\{M_\beta\}$ the diagonal and microcanonical ensembles are equivalent (von Neumann, 1929; Goldstein *et al.*, 2010). More recently it was proven that the whole density matrix of a small subsystem of a larger system which is placed initially in a typical eigenstate is described by the canonical ensemble (Popescu *et al.*, 2006). Gogolin *et al.* (2011) further extended these results to the problem of measurement and decoherence. Particular care is nevertheless needed in relating these statements to the dynamics and thermalization of actual many-body systems, since physical initial conditions in quenched system almost never correspond to eigenstates of a new Hamiltonian.

B. Nonergodic behavior in integrable systems: The generalized Gibbs ensemble

While the statements above are very general, their application to specific systems is not at all straightforward. Looking at a concrete many-body system, it is of primary interest not just to find out whether in principle a set of macroscopic observables that behave ergodically exists, but whether specific and natural observables, such as the magnetization for spin chains, density for cold atomic gases, or various correlation functions behave ergodically or not. In this respect, experiments tell us that ergodicity is not at all guaranteed (Kinoshita *et al.*, 2006) if the closed system is integrable or nearly integrable. While this fact was expected (Mazur, 1969; Girardeau, 1969; Barouch *et al.*, 1970; Girardeau, 1970; Suzuki, 1971), recent research on the dynamics of integrable systems has focused on finding ways to predict the asymptotical states taking into account integrability, i.e., the presence of many constants of motion.

We now discuss how this can be done qualitatively using the simplest example of an integrable system, a periodic harmonic chain of finite length described by the Hamiltonian

$$H = \sum_{j=1}^{M-1}\left[\frac{p_j^2}{2m} + \frac{m\nu^2}{2}(x_j - x_{j+1})^2\right], \quad (17)$$

where x_j are deviations of particles from the equilibrium positions and p_j are their momenta; we use the identification $x_M \equiv x_0$. Imagine that initially we deform the system in a particular way and ask how this deformation evolves with time. We note that since this is a harmonic system described by linear equations of motion, the following analysis also applies to quantum systems. From elementary physics we know that the initial deformation splits into normal modes characterized by the quasimomenta $q_n = 2\pi n/M$, where n is integer $n \in [0, M-1]$, and the dispersion $\omega_q = 2\nu \sin q/2$. This system obviously does not thermalize even at long times because there is no energy exchange between modes. This does not imply though that it cannot reach a well-defined asymptotic state in the long time limit (Barthel and Schollwöck, 2008; Cramer and Eisert, 2010). To illustrate this point consider, for example, the displacement of the jth atom at time t after some initial displacement:

$$x_j(t) = \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} x_{q_n}(t)e^{iq_n j}. \quad (18)$$

$x_q(t) = A_q \cos[\omega_q t]$, where A_q is a complex amplitude determined by initial conditions (for simplicity we assumed an initial stationary state). We now analyze qualitatively the dynamics of this system. At short times, provided that the initial modulation is smooth and only modes with small momenta ($q \ll \pi$) are excited, we can linearize the spectrum $\omega_q \approx \nu q$. Clearly in this case we recover periodic motion of the wave packet with a period equal to the ratio of the system size and sound velocity: $T = M/\nu$. This persistent motion is characteristic of the absence of any relaxation. However, as time gets longer deviations of the dispersion from linear become more important. In particular, when $t^*(\omega_{\bar{n}+1} + \omega_{\bar{n}-1} - 2\omega_{\bar{n}}) \sim 1$, where $\omega_{\bar{n}}$ is the central frequency of the wave packet, correlations between phases among the different modes are lost and they can be treated as essentially random numbers. For our model $t^* \sim M^2/\omega_{\bar{n}}$. At long times $t \gg t^*$ the different momentum modes become uncorrelated and the system reaches the asymptotic stationary state in a sense we defined earlier (it can be found close to that state at almost all times). For this asymptotic state the only relevant information about the initial conditions is encoded in the M mode amplitudes $|A_q|$ or equivalently in their squares $|A_q|^2$ proportional to the occupancies of the modes of energy E_q, which are the integrals of motion. Note that there are special modes corresponding to momenta q and $-q$ which are exactly degenerate. The correlations between A_q and $A_{-q} = A_q^\star$ thus never disappear and in general one needs to fix M additional constraints representing the relative phases of the complex

amplitudes. For example, if the initial configuration is symmetric $x_j = x_{-j}$ then A_q is real, meaning that the phases of all modes are identical. Then it is easy to check that with this constraint $\langle x_j^2(t)\rangle$ acquires spatial dependence on j even in the long time limit. This dependence cannot be recovered by fixing only mode occupancies. Only when these phases are unimportant, e.g., they average to zero or if there are no degeneracies between the normal modes, the asymptotic state is fully fixed by the integrals of motion. Thus in contrast with ergodic systems where only the energy needs to be fixed, the long time behavior of our integrable model can be reproduced by fixing the M integrals of motion and possibly $\sim M$ other constraints if there are degeneracies. While the number of commuting (local) integrals of motion is large, equal to the system size M, it is vastly smaller than the total number of states which scales with M exponentially.

We now see how these considerations are transposed in many-body systems, focusing on another simple integrable model, the quantum Ising chain (Sachdev, 1999) described by the Hamiltonian $H_0 = -\sum_i \sigma_i^x \sigma_{i+1}^x + g\sigma_i^z$. Here $\sigma_i^{x,z}$ are the spin operators at site i and g is the strength of the transverse field. This model gives one of the simplest examples of quantum phase transition, with a quantum critical point at $g_c = 1$ separating two mutually dual gapped phases, a quantum paramagnet ($g > g_c$) and a ferromagnet ($g < g_c$).

In the quantum Ising chain the *local* transverse magnetization $M^x = \sum_i \sigma_i^x$ is a nonergodic operator (Mazur, 1969; Girardeau, 1969, 1970; Barouch *et al.*, 1970). To see this, it is useful to employ a Jordan-Wigner transformation that reduces the problem to a free fermion model (Sachdev, 1999). In terms of the fermionic operators c_k relative to modes of momentum $k = 2\pi n/L$ the Hamiltonian takes the form

$$H = 2\sum_{k>0}[g - \cos(k)](c_k^\dagger c_k - c_{-k}c_{-k}^\dagger) + i\sin(k)(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k). \tag{19}$$

Under this mapping the transverse magnetization becomes $M^x = -2\sum_{k>0}(c_k^\dagger c_k - c_{-k}c_{-k}^\dagger)$. The eigenmodes γ_k of energy $E_k = 2\sqrt{[g-\cos(k)]^2 + \sin(k)^2}$ diagonalizing the Hamiltonian are related to the fermionic operators c_k by a Bogoliubov rotation, $c_k = \cos(\theta_k)\gamma_k - i\sin(\theta_k)\gamma_{-k}^\dagger$, with $\tan(2\theta_k) = \sin(k)/[g - \cos(k)]$. In the Heisenberg representation the operators γ_k acquire simple time dependence: $\gamma_k(t) = \gamma_k(0)\exp[-iE_k t]$. As in the previous problem of a harmonic chain if the energies E_k are incommensurate, at sufficiently long times different momentum modes become statistically independent from each other. This statement does not apply to modes with opposite momenta k and $-k$ which have identical energies. However, if these correlations are not important, then in the long time limit (see below) each mode can be characterized by the conserved quantity $n_k = \langle\gamma_k^\dagger\gamma_k\rangle$. We now consider unitary dynamics of the transverse magnetization starting with a generic initial condition $|\Psi_0\rangle$. The time evolution of the operator $M^x(t)$ expressed in terms of the eigenmodes of the Hamiltonian is

$$M^x(t) = -2\sum_{k>0}\cos(2\theta_k)(\gamma_k^\dagger\gamma_k - \gamma_{-k}\gamma_{-k}^\dagger) + i\sin(2\theta_k)(\gamma_{-k}\gamma_k e^{-2iE_k t} - \gamma_k^\dagger\gamma_{-k}^\dagger e^{2iE_k t}). \tag{20}$$

In the long time limit only the diagonal terms in the sum survive, while the off-diagonal ones, describing creation or destruction of two fermions, average to zero. Therefore, for any initial condition $|\Psi_0\rangle$ the asymptotic value attained by the transverse magnetization is

$$\overline{\langle M^x(t)\rangle} = -2\sum_{k>0}\cos(2\theta_k)(\langle\gamma_k^\dagger\gamma_k\rangle - \langle\gamma_{-k}\gamma_{-k}^\dagger\rangle). \tag{21}$$

This asymptotic value is therefore perfectly described by the set of the occupation numbers n_k.

The result above leads one to conjecture that the asymptotic state is described by a Gibbs-like statistical ensemble of the type (Rigol *et al.*, 2007)

$$\rho_G = \frac{e^{-\sum_k \lambda_k\gamma_k^\dagger\gamma_k}}{Z}, \tag{22}$$

where the Lagrange multipliers λ_k are fixed by requiring that $n_k \equiv \langle\Psi_0|\gamma_k^\dagger\gamma_k|\Psi_0\rangle = \mathrm{Tr}[\rho_G\gamma_k^\dagger\gamma_k] = \langle\gamma_k^\dagger\gamma_k\rangle_G$. The ensemble defined in Eq. (22) can be seen as a particular case of the ensemble

$$\hat{\rho}_G = \frac{e^{-\sum_\alpha \lambda_\alpha I_\alpha}}{Z}, \tag{23}$$

known as the generalized Gibbs ensemble (GGE) or maximum entropy ensemble introduced by Jaynes (1957) to describe the equilibrium state of a system possessing N constant of motions I_k. A recent conjecture (Rigol *et al.*, 2007) proposed to use the GGE to describe the asymptotic state of a generic quantum integrable model. However, this proposal had to face two obvious subtleties. First it needs to be specified how to choose the I_k in Eq. (23). Indeed, if all constants of motion would be admissible, including nonlocal ones, then one would obviously and tautologically describe the asymptotic state (for both integrable and nonintegrable systems), as one can easily see by choosing as I_k the projectors onto the eigenstates of the Hamiltonian. The way out comes, however, by observing that in standard thermodynamics the Gibbs ensemble emerges for small subsystems from the assumption of statistical independence between sufficiently large subsystems. In this derivation the additivity of a conserved quantity, energy, plays a crucial role. This is the reason why the probability of a given configuration is exponential in energy and not, e.g., in energy squared (Kardar, 2007). Similar arguments apply to any additive integrals of motion so that statistical independence and invariance of the ensemble to the choice of a subsystem of an integrable system puts strong constraints on the choice of the integrals of motion in GGE when the latter is applied to subsystems of an integrable system. In this respect the average occupation numbers of different momentum modes used in Eq. (22) become approximately additive for small subsystems. This approximate additivity of integrals of motion was recently discovered by Cassidy *et al.* (2011) for another integrable system of one-dimensional hard-core bosons. In particular, it was noted that the integrals of motion I_α and the Lagrange multipliers λ_α in that case can be written as smooth functions of α/N implying that in large systems the argument of the exponent entering Eq. (23) can be written in the extensive (additive) form: $\sum_\alpha \lambda_\alpha I_\alpha \approx L\int_0^1 d\xi,\ I(\xi)\lambda(\xi)$, where

$\xi = \alpha/L$. This suggests that GGE can be defined through a smooth function $\lambda(\xi)$, which replaces the temperature in the ergodic systems.

There is a second subtlety in applying the GGE to quantum systems. Here the most natural definition of integrability is based on the requirement that the system has well-defined quasiparticles that maintain their identity upon scattering [see Caux and Mossel (2011) for a more detailed discussion], i.e., scattering is purely elastic and there is no production of particles or dissipation associated with it (Sutherland, 2004; Mussardo, 2009). This notion can be made precise in continuum integrable models, such as the Luttinger liquid or the sinh-Gordon model, which can emerge as low-energy descriptions of other integrable models, such as the critical XXZ chain and the Lieb-Liniger gas. In these systems it is natural to associate the $\hat{I}_\alpha$ to the occupation numbers of the quasiparticle states. More specifically, considering a generic one-dimensional relativistically invariant integrable system with say a single species of quasiparticles of mass m, energy $E = m\cosh(\theta)$, and momentum $p = m\sinh(\theta)$ (θ is the rapidity), the quasiparticles can be described by annihilation operators $\hat{A}(\theta)$ satisfying the algebra $\hat{A}(\theta_i)\hat{A}(\theta_j) = S(\theta_i - \theta_j)\hat{A}(\theta_j)\hat{A}(\theta_i)$, where S is the S matrix of the two particle scattering. Similar relations are valid for the products of creation and creation-annihilation operators (Mussardo, 2009). Since the Hamiltonian is by definition diagonal in $\hat{A}(\theta)$, $H = \int d\theta E(\theta)A^\dagger(\theta)A(\theta)$, and every eigenstate can be written as $|\theta_1, \ldots, \theta_n\rangle = A^\dagger(\theta_1)\cdots A^\dagger(\theta_n)|0\rangle$, with $\theta_1 > \cdots > \theta_n$, in this case it is rather natural to postulate the form

$$\hat{\rho}_G = \frac{e^{-\int d\theta \lambda(\theta)A^\dagger(\theta)A(\theta)}}{Z}, \tag{24}$$

for the generalized Gibbs ensemble (Fioretto and Mussardo, 2010). This ensemble is a direct generalization of the GGE for the quantum Ising model, where $S = -1$. For this general class of integrable systems and a specific class of *translationally invariant* initial states it was indeed shown that the long time limit of the average of local operators is well described by this ensemble (Fioretto and Mussardo, 2010). Such initial states can be written as

$$|\Psi_0\rangle = \mathcal{N} e^{-\int d\theta K(\theta)A^\dagger(\theta)A^\dagger(-\theta)}, \tag{25}$$

which in turn are similar to the so-called *integrable boundary states* in statistical field theory (Ghoshal and Zamolodchikov, 1994). Such states naturally emerge in experimentally relevant systems, for example, when studying dephasing in split quasi-1D condensates (Gritsev *et al.*, 2007) or in the quantum Ising model, when studying a quantum quench from a transverse field γ_i to a transverse field γ_f (Silva, 2008).

An interesting idea related to the GGE was suggested by Gurarie (1995) to explain the steady state of a driven nearly integrable system. It was shown that the steady state distribution of the wave amplitudes corresponding to different momenta [see Zakharov *et al.* (1992) for details] can be obtained by taking the probability density $\rho \propto \exp[-F]$, where F is a (complex) combination of the approximate integrals of motion found perturbatively. In terms of this ensemble one recovers the correct power law distribution of the amplitudes of waves with the momentum and other observables.

Another view toward elucidating the validity of the generalized Gibbs ensemble has been pursued for special quenches in a 1D Bose-Hubbard model (Cramer *et al.*, 2008) and in integrable systems with free quasiparticles (Barthel and Schollwöck, 2008). It was shown that, upon tracing all degrees of freedom of the system outside a small region of space and under specific conditions, the local density matrix tends asymptotically to $\hat{\rho}_G$. More recently a series of recent theoretical (Flesch *et al.*, 2008) and experimental (Trotzky *et al.*, 2011) works on the dynamics of Bose-Hubbard models has proven the relaxation of local observables in this system to a maximum entropy ensemble consistent with the constraints of the dynamics. A hint toward the generalization of $\hat{\rho}_G$ for Bethe ansatz integrable systems was proposed by Barthel and Schollwöck (2008). The GGE was also tested in a number of models, from Luttinger liquids (Cazalilla, 2006; Iucci and Cazalilla, 2009) and free bosonic theories (Calabrese and Cardy, 2007) to integrable hard-core boson models (Rigol *et al.*, 2007) and Hubbard-like models (Eckstein and Kollar, 2008; Kollar and Eckstein, 2008). In all cases, it was shown to correctly predict the asymptotic momentum distribution functions for a variety of systems and quantum quenches.

At this point is should be stressed that as discussed before the GGE does not always give a complete description of the asymptotic state of the system. In the simple example of the harmonic chain we saw that for generic initial conditions it is necessary to specify $2N$ real constants or N complex amplitudes in order to correctly describe the asymptotic state even if we focus exclusively on local observables. For a quantum Ising chain, moreover, ρ_G can be interpreted as a grand-canonical distribution with an energy dependent chemical potential $\mu_k = E_k - \lambda_k$. It is evident now that if we consider the correlations of $\delta n_k = \gamma_k^\dagger\gamma_k - \langle\gamma_k^\dagger\gamma_k\rangle$, the occupation numbers of different eigenmodes, the GGE necessarily predicts $\langle\delta n_k \delta n_{k'}\rangle = 0$. Likewise the GGE predicts the correlators of the type $\langle\gamma_k^\dagger(t)\gamma_{-k}(t)\rangle$ are always equal to zero. For a generic initial state $|\Psi_0\rangle$ both statements are not necessarily true: by breaking translational invariance in the initial state one could have $\langle\Psi_0|\delta n_k \delta n_{k'}|\Psi_0\rangle \neq 0$ and $\langle\Psi_0|\gamma_k^\dagger\gamma_{-k}|\Psi_0\rangle \neq 0$. Note that the mere survival of off-diagonal correlations of this type when the evolution starts with a nontranslationally invariant state signals in a sense the integrability of the model, i.e., the existence of well-defined quasiparticles γ_k. Indeed, following the argument of Gangardt and Pustilnik (2008), if the Hamiltonian of the system is translationally invariant but integrability is broken, the off-diagonal correlators are expected to decay to zero for any initial condition, thereby restoring the translational invariance in the asymptotic state. Finally, note that off-diagonal correlations might influence the asymptotics of physically relevant observables: A simple example is the asymptotic value of $\langle[M^x(t)]^2\rangle$, which for a generic nontranslationally invariant condition $|\Psi_0\rangle$ cannot be predicted using the GGE.

An important open question is to understand under which general circumstances the GGE can be applied. For free fermionic and bosonic systems the GGE was argued to hold

for local observables (Barthel and Schollwöck, 2008; Cramer *et al.*, 2008). For more general integrable systems this is not evident at all. For example, in the case of the quantum Ising chain the two significant observables, the transverse magnetization σ_i^x and the order parameter σ_j^z, are local in the spin representation. However, this locality does not translate directly to their representation in terms of the quasiparticles of the model: While σ_i^x retains a local character in terms of γ_i, σ_i^z does not. Will the asymptotic dynamics of any local operator be represented by the GGE, or just that of local operators in the quasiparticle fields? Do the symmetries of the initial state play any role in this? Answering these questions appears to be crucial to understand the role of integrability in the dynamics of many-body systems.

Another important question is whether all natural observables of an integrable system behave necessarily nonthermally. The answer to it appears to be no, as pointed out recently (Rossini *et al.*, 2009). The key to understanding this issue seems to be again *locality* with respect to the quasiparticles diagonalizing the model. Thus in the quantum Ising model it was shown that while the transverse magnetization is nonergodic, the correlators of the order parameter σ_z following a quench of the transverse field relax as in a thermal state with an effective temperature T_{eff} set by the initial energy of the system $E = \langle\Psi_0(g_i)|H(g_f)|\Psi_0(g_i)\rangle$. At low T_{eff} this relaxation appears to be universal, i.e., determined only by the low-energy scattering properties of quasiparticles (Rossini *et al.*, 2009). Analogous studies for an XXZ chain hint toward a different behavior of local and nonlocal operators with respect to quasiparticles (Canovi *et al.*, 2011). The situation is much less clear for quenches with high effective temperatures, where the universal character of the low-energy theory is lost (Barmettler *et al.*, 2009).

C. Breaking integrability: Eigenstate thermalization

When integrability is explicitly broken with a strong enough perturbation one naturally expects ergodic behavior to emerge for all observables (Kollath *et al.*, 2007; Manmana *et al.*, 2007; Rigol *et al.*, 2008; Rigol, 2009; Roux, 2009, 2010). The quest for the necessary conditions for thermalization to occur (i.e., how strongly should integrability be broken, which spectral properties should the system display) is an important problem in many different fields, from mathematical and statistical physics to quantum chaos (Peres, 1984; Deutsch, 1991; Srednicki, 1994; Srednicki, 1999; Rigol *et al.*, 2008). In classical systems the intense research on this subject was stimulated by the study of dynamics of a nonlinear chain of coupled oscillators by FPU (Fermi *et al.*, 1955), where instead of thermalization regular quasiperiodic oscillations were observed. Later it was realized that the FPU problem was nearly integrable and that there was a finite threshold for the chaotic behavior (Campbell *et al.*, 2005). In quantum systems the situation was far less clear: While different views on this issue emerged from time to time, the key toward a clear understanding of quantum thermalization appears to be linked to the emergence of quantum chaotic behavior (Peres, 1984). In particular, it was proposed that the emergence of thermal behavior is linked to the pseudorandom form of natural observables once represented in the eigenbasis of the Hamiltonian (Peres, 1984). This observation was made more precise by conjecturing that thermalization in quantum chaotic systems occurs eigenstate by eigenstate, i.e., the expectation value of a natural observable $\langle\Psi_\alpha|A|\Psi_\alpha\rangle$ on an eigenstate $|\Psi_\alpha\rangle$ is a smooth function of its energy E_α being essentially constant on each microcanonical energy shell (Deutsch, 1991; Srednicki, 1994, 1999). If this happens, then ergodicity and thermalization in the asymptotic state follow for every initial condition sufficiently narrow in energy (e.g., localized in a microcanonical shell), as one can easily understand using the diagonal ensemble. This hypothesis is known as *eigenstate thermalization* (ETH).

In order to understand how eigenstate thermalization can emerge, consider a quantum gas of N particles of mass m with hard-core interactions (Srednicki, 1994). Srednicki pointed out that in the time evolution of this system starting with an initial condition $|\Psi_0\rangle$ sufficiently narrow in energy, the momentum distribution will always relax to the Maxwell-Boltzmann distribution $f_{\text{MB}}(p)$ as long as the eigenstates of the system $|\Psi_\alpha\rangle$ can be considered as pseudorandom superpositions of plane waves, i.e., they have a diffusive nature in phase space. This requirement should be satisfied as a result of the chaoticity of the system, the so-called Berry's conjecture. Calling $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ the coordinates of the particles and $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ their momenta, Berry's conjecture states that the eigenstates have the form

$$\Psi_\alpha(\mathbf{X}) = \mathcal{N}\int d\mathbf{P} A_\alpha(\mathbf{P})\delta(\mathbf{P}^2 - 2mE_\alpha)e^{i\mathbf{P}\cdot\mathbf{X}}, \tag{26}$$

with $A_\alpha(\mathbf{P})$ being pseudorandom variables with Gaussian statistics, $\langle A_\alpha(\mathbf{P})A_\beta(\mathbf{P}')\rangle = \delta_\alpha^\beta\delta^{(3N)}(\mathbf{P}+\mathbf{P}')/\delta(\mathbf{P}^2 - \mathbf{P}'^2)$. Note that we are disregarding the symmetrization of the wave function, see Srednicki (1994) for a discussion of this aspect. If the properties above are assumed it is easy to prove that on average in the thermodynamic limit the momentum distribution function is

$$\langle f(\mathbf{p})\rangle = \int d\mathbf{p}_2 d\mathbf{p}_3 \ldots \langle|\Psi_\alpha(\mathbf{p}, \mathbf{p}_2, \ldots)|^2\rangle = \frac{e^{-(\mathbf{p}^2/2mkT)}}{(2\pi mkT)^{3/2}} = f_{\text{MB}}(p), \tag{27}$$

where the temperature is set by the equipartition law as $E_\alpha = \frac{3}{2}NkT$. Note that this is expected to happen for every eigenstate of energy close to E_α, as required by the ETH. Hence, thermal behavior will follow for every initial condition sufficiently narrow in energy.

For generic many-body systems, such as Hubbard-like models and spin chains, the close relation between breaking of integrability and quantum chaotic behavior is a known fact (Poilblanc *et al.*, 1993). In particular, finite size many-body integrable systems are characterized by the Poisson spectral statistics while the gradual breaking of integrability by a perturbation leads to a crossover to the Wigner-Dyson statistics. The latter is typically associated, in mesoscopic systems or billiards, with diffusive behavior and can be taken as a signature of quantum chaos (Imry, 1997). In many-body disordered systems the emergence of the Wigner-Dyson statistics was argued to be an indicator of the transition between metallic (ergodic) and insulating (nonergodic)

phases (Mukerjee *et al.*, 2006; Oganesyan and Huse, 2007). Inspired by these close analogies, recent studies gave a boost to our understanding of the crossover from nonergodic to thermal behavior as integrability is gradually broken and of the origin of ergodicity and thermalization in systems sufficiently far from integrability (Kollath *et al.*, 2007; Manmana *et al.*, 2007; Rigol *et al.*, 2008; Rigol, 2009; Biroli *et al.*, 2010). In particular, a careful study of the asymptotics of density-density correlators and momentum distribution function for hard-core bosons in 1D showed that the transition from nonthermal to thermal behavior in finite size systems takes the form of a crossover controlled by the strength of the integrability breaking perturbation and the system size (Rigol, 2009). Moreover, there is a universality in state-to-state fluctuations of simple observables in this crossover regime (Neuenhahn and Marquardt, 2010), which goes hand in hand with an analogous transition from Poisson to Wigner-Dyson level statistics (Rigol and Santos, 2010; Santos and Rigol, 2010a). When integrability is broken by sufficiently strong perturbation ergodic behavior emerges (Rigol, 2009; Neuenhahn and Marquardt, 2010; Rigol and Santos, 2010), which in turn appears to be related to the validity of the ETH (Rigol *et al.*, 2008). In this context, the anomalous, nonergodic behavior of integrable models has been reinterpreted as originating from wide fluctuations of the expectation value of natural observables around the microcanonical average (Biroli *et al.*, 2010).

All these statements apply to the asymptotic (or time averaged) state. So far the relaxation in time, in particular, in the thermodynamic limit, has received much less attention. In a series of studies of relaxation in fermionic Hubbard models subject to quenches in the interactions strength it has been argued that for sufficiently rapid quenches relaxation toward thermal equilibrium occurs through a prethermalized phase (Moeckel and Kehrein, 2008, 2010). Similar two step dynamics occurs in quenches of coupled superfluids where initial fast ''light cone'' dynamics leads to a prethermalized steady state, which then slowly decays to the thermal equilibrium through the vortex-antivortex unbinding (Mathey and Polkovnikov, 2010). Burkov *et al.* (2007)) predicted and later observed experimentally an unusual subexponential in time decay of correlation functions (Hofferberth *et al.*, 2007) for relaxational dynamics of decoupled 1D bosonic systems.

D. Outlook and open problems: Quantum KAM threshold as a many-body delocalization transition?

The arguments above clearly pointed to the connection between thermalization in strongly correlated systems and thermalization in chaotic billiards. This analogy, however, rather than being the end of a quest, opens an entire new kind of questions, which are a current focus of both theoretical and experimental research. In particular, we do know that in a number of models of strongly correlated particles eigenstate thermalization is at the root of thermal behavior (Rigol *et al.*, 2008). What is the cause of eigenstate thermalization in a generic many-body system, i.e., the analog of the diffusive eigenstates in phase space of Berry's conjecture? And more importantly, while in a finite size system the transition from nonergodic to ergodic behavior takes the form of a crossover, but what happens in the thermodynamic limit? Is the transition from ergodic to nonergodic behavior still a crossover or is it sharp (a quantum KAM threshold)?

Research on these questions has just started. An interesting idea that has recently emerged is that the study of the transition from integrability to nonintegrability in quantum many-body systems is deeply connected to another important problem at the frontier of condensed matter physics: the concept of many-body localization (Altshuler *et al.*, 1997; Basko *et al.*, 2006), which extends the original work of Anderson (1958) on single-particle localization. We note that related ideas were put forward in studying energy transfer in interacting harmonic systems in the context of large organic molecules (Logan and Wolynes, 1990; Leitner and Wolynes, 1996). More specifically, it was noted that a transition from localized to delocalized states in either real space (Pal and Huse, 2010) or more generally in quasiparticle space (Canovi *et al.*, 2011) is closely connected to a corresponding transition from thermal to nonthermal behavior in the asymptotics of significant observables. For weakly perturbed integrable models, the main characteristic of the observables to display such transition appears again to be their locality with respect to the quasiparticles (Canovi *et al.*, 2011). This connection with many-body localization becomes more clear on the basis of a recently proposed way to quantify the transition from nonergodic to ergodic behavior in many-body systems (Olshanii and Yurovsky 2009) considered an integrable Hamiltonian H_0 with a weak nonintegrable perturbation λV. Formulating essentially a generalization of Berry's conjecture and making some additional assumptions, they showed that the deviations from thermal behavior in the expectation value of observables can be quantified according to

$$\overline{\langle\Psi(t)|A|\Psi(t)\rangle} - \langle A\rangle_{\mathrm{mc}} \approx \eta[\overline{\langle\Psi_0(t)|A|\Psi_0(t)\rangle_0} - \langle A\rangle_{\mathrm{mc}}],$$

where $|\Psi_0(t)\rangle = \exp[-i\mathcal{H}_0 t]|\Psi_0\rangle$ is the time evolved state with respect to the integrable Hamiltonian, while $|\Psi(t)\rangle = \exp[-i(\mathcal{H}_0 + \lambda\mathcal{V})t]|\Psi_0\rangle$ evolves with the nonintegrable one. The key ingredient in this formula is the parameter η, defined as the average over the microcanonical shell of the inverse participation ratio $\eta_\alpha = \sum_n |\langle\varphi_n|\Psi_\alpha\rangle|^4$, where $|\varphi_n\rangle$ are the eigenstates of the integrable Hamiltonian $\mathcal{H}_0$ and $|\Psi_\alpha\rangle$ are the eigenstates of $\mathcal{H}_0 + \lambda\mathcal{V}$. Note that when the system is close to integrability $\eta \simeq 1$, but as the strength of $\mathcal{V}$ increases, η is roughly proportional to the inverse of the number of states $\mathcal{N}$ hybridized by the perturbation.

Using this formula it is now possible to understand how many-body localization enters the scenario (Canovi *et al.*, 2011; Pal and Huse, 2010): An abrupt transition at a certain λ_c from localized to delocalized states in quasiparticle space corresponds to a sharp decrease of the inverse participation ratio η from a value $O(1)$ to a value negligibly small and tending to zero in the thermodynamic limit, essentially $O(1/\mathcal{N}(\lambda))$, where $\mathcal{N}(\lambda)$ is the total number of states in an energy window whose width is on the order of the matrix elements of the perturbation. This would lead to an abrupt transition from nonthermal to thermal behavior at λ_c, a transition qualitatively corresponding to the physics of a quantum KAM threshold. Note that on the delocalized side of this transition the eigenstates are expected to be of a

diffusive nature (in quasiparticle space), i.e., a natural generalization of the form postulated by Berry's conjecture. Such a transition has been studied extensively in a confined electronic system, following a seminal paper by Althsuler, Gefen, Kamenev, and Levitov (Altshuler *et al.*, 1997) and on interacting electron systems with localized single-particle states (Basko *et al.*, 2006). While the dependence $\eta(\lambda)$ was analyzed numerically in certain small systems (Canovi *et al.*, 2011; Neuenhahn and Marquardt, 2010; Santos and Rigol, 2010a, 2010b), eventual emergence of a sharp KAM-like threshold in the thermodynamic limit remains an open question.

The ETH also suggested a new way of looking at quantum relaxational dynamics as dephasing in the many-body basis. In particular, the information about the asymptotic state is fully contained in the diagonal elements of the density matrix, which do not change in time if the Hamiltonian is constant. As such the process of time evolution in this picture is equivalent to averaging of oscillating off-diagonal elements of the density matrix to zero. In a way this picture is different from conventional thinking based on kinetic theory of thermalization through collisions between quasiparticles and the time evolution of their distribution function. This apparent difference is hidden in the complicated structure of the many-body eigenstates. Our intuition is based on thinking about dynamics in the approximate basis, e.g., a basis of independent quasiparticles. The precise relation between the many-body and kinetic approaches to thermalization is still an open question. Another potentially intriguing possibility is understanding thermalization as a renormalization group process, where time evolution results in averaging over high-energy degrees of freedom. If one deals with an approximate noninteracting basis then because of interactions the process of eliminating high-energy states affects dynamics of low-energy modes and hence in renormalization of the low-energy dynamics. Mathey and Polkovnikov (2010)) showed that such a renormalization process indeed can explain real time dynamics in a two-dimensional sine-Gordon model and the emerging nonequilibrium Kosterlitz-Thouless transition. Moeckel and Kehrein (2009) put forward similar ideas to analyze dynamics of interacting fermions using the flow equation method. At the moment it is unclear whether by using such a real time renormalization group one can analyze relaxational long time dynamics in generic interacting systems.

IV. EXPERIMENTAL PROGRESS IN QUANTUM DYNAMICS IN COLD ATOMS AND OTHER SYSTEMS

As mentioned, the study of nonequilibrium dynamics of quantum many-body systems has been increasingly motivated by a series of advances in the field of ultracold atomic and molecular gases. Because of the confluence of various features, these mesoscopic quantum systems are in many ways near-ideal systems for the study of nonequilibrium quantum phenomena.

First, quantum gases exhibit a remarkably high degree of isolation from environmental sources of decoherence and dissipation. Thus, to an excellent approximation, during the duration of experiments they are regarded as closed quantum systems. Further, the dilute nature of these gases and exceptionally low temperatures result in long time scales of dynamical effects (typically on the order of milliseconds or longer) allowing for time-resolved studies of nonequilibrium processes resulting from phase-coherent many-body dynamics. Such studies are hardly possible in conventional condensed matter systems.

Second, an array of techniques was developed to dynamically tune various parameters of the Hamiltonian governing these quantum gases. This has made it possible to realize various prototypical nonequilibrium processes such as quantum quenches discussed above. Quenches across phase transitions have been realized to investigate the onset and formation of long range order and the mechanism underlying the spontaneous formation of topological defects. The latter is closely related to the KZ mechanism described earlier in the text. A quantitative experimental study of the defect, entropy, and energy production resulting from such quantum quenches should allow for an accurate comparison with the theoretical predictions.

Last, the ability to engineer and experimentally realize model Hamiltonians of archetypal correlated systems coupled with a detailed knowledge of the microscopic interactions make ultracold atomic gases a tantalizing system for applications ranging from the quantum simulation of strongly correlated systems to the adiabatic quantum computation. In addition to the form of the model Hamiltonian, experimental control can also be achieved over the effective dimensionality of the ultracold gas making it possible to investigate the nontrivial interplay between fluctuations, interactions, and dimensionality.

From a technological perspective, there is an increasing thrust toward engineering ultracold atomic many-body systems for applications in quantum metrology (Vengalattore *et al.*, 2007; Estéve *et al.*, 2008; Meiser *et al.*, 2008; Appel *et al.*, 2009; Leroux *et al.*, 2010; Riedel *et al.*, 2010). A deeper understanding of the dynamics of interacting many-body systems and the mechanisms of decoherence and dissipation in these systems is of crucial importance in this context.

Motivated by these factors, a number of experiments have been performed in recent years using ultracold quantum gases to investigate topics including quantum coherent dynamics in optical lattices, quenches across quantum phase transitions, and thermalization in low-dimensional systems. For the purposes of this Colloquium, we distinguish between classes of nonequilibrium experiments in terms of both the general protocol and the questions being addressed by these experiments: (i) *nonequilibrium states* of many-body atomic systems wherein the high degree of isolation of the atomic system from the environment allows for the creation of metastable or highly excited many-body states with long lifetimes, (ii) *quantum quench* experiments in which one or more parameters of the Hamiltonian are changed rapidly to create an out-of-equilibrium state of the many-body system, and (iii) *dynamical tuning* of the Hamiltonian in order to study quantum coherent dynamics of an interacting many-body system.

These experimental advances have stimulated an active theoretical research in the area of nonequilibrium quantum

dynamics in interacting many-body systems. Among the issues most debated in recent literature is the relation between thermalization in isolated quantum systems and quantum integrability. In this regard, a recent pioneering study on thermalization in 1D Bose gases was performed by Kinoshita *et al.* (2006). In this experiment, a blue detuned 2D optical lattice was used to create arrays of tightly confined tubes of ultracold ^{87}Rb atoms. The depth of the lattice potential far exceeded the energy of the ultracold gas ensuring negligible tunneling among the tubes. The array of tubes was then placed in a superposition of states of momentum $\pm 2p_0$ by the application of a transient optical phase grating. The imparted kinetic energy was small compared to the energy required to excite the atoms to the higher transverse states and the gases remained one dimensional. This out-of-equilibrium system was then allowed to evolve for variable durations before the momentum distribution was probed by absorption imaging of the gas (see Fig. 2).

It was found that, while the initial momentum distributions exhibit some dephasing on account of trap anharmonicities, the dephased distribution remains non-Gaussian even after thousands of collisions. This is in distinct contrast to the Gaussian distributions observed when the 2D optical lattice is adiabatically imposed on an equilibrium 3D Bose gas. This remarkable observation that the nonequilibrium Bose gases do not equilibrate on the time scales of the experiment appears consistent with the fact that this system is a close experimental realization of a Lieb-Liniger gas with pointlike collisional interactions, an integrable quantum system in which only elastic pairwise interactions can occur. Apparently the experimental technicalities such as anharmonicities or the axial potential are insufficient to sufficiently lift integrability in this system.

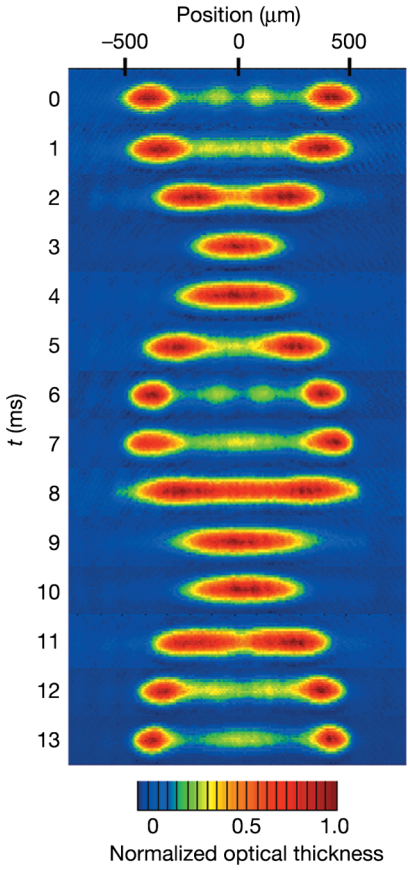

FIG. 2 (color online). Time-of-flight absorption images of an ensemble of 1D Bose gases. Ultracold atoms are confined in arrays of 1D optical traps. Optical pulses are used to place the atoms in a superposition of $\pm 2\hbar k$ momentum states. The gas is then allowed to evolve for variable durations before being released from the trap and photographed to reveal the momentum distribution. The false color in each image is rescaled to show detail. The non-Gaussian nature of the momentum distribution clearly indicates an absence of thermalization. Adapted from Kinoshita *et al.*, 2006.

In addition to unambiguously showing the absence of thermalization within experimental time scales in this model realization of the Lieb-Liniger gas, this study also points toward addressing more general questions on integrability and ergodicity. Starting from an integrable system, modifications such as the addition of finite range interactions, tunneling between the 1D tubes, and the imposition of axial potentials can tunably lift integrability and analyze emergence of irreversability and thermalization. This experiment largely motivated much of the theoretical work discussed in the previous section.

Another issue that has attracted much attention is the search for universal effects either in the nonequilibrium dynamics following a quantum quench or in the adiabatic dynamics near a quantum critical point which was described earlier. In particular, the issue of nonadiabatic dynamics near quantum phase transitions has been the focus of recent experimental studies on condensate formation in a dilute, weakly interacting Bose gas that is rapidly cooled past the BEC phase transition (Weiler *et al.*, 2008). This process was found to be accompanied by the spontaneous formation of topological defects, i.e., vortices, in the nascent superfluid. This can be phenomenologically understood as being due to the formation of isolated superfluid regions of a characteristic size ξ, each with a random relative phase. These isolated regions then gradually merge to give rise to global phase coherence. In this process, regions which enclose phase loops of 2π are constrained by the nature of the superfluid, i.e., the continuity of the wave function, to have a vanishing superfluid density at the core. Thus, the KZ mechanism predicts a density of vortices that scales as $1/\xi^2$.

In this experiment, a magnetically trapped thermal gas of ^{87}Rb atoms was cooled by radio frequency (rf) evaporation to temperatures below the BEC transition temperature. The quench rate, i.e., the rate of cooling, was controlled by varying the rate at which the rf frequency was ramped down. Following a brief duration of equilibration, vortices are detected by absorption imaging of the gas after ballistic expansion. Allowing for some uncertainty in the ability to discern a vortex due to line-of-sight integration in these images, it was found that about one-quarter of the images showed at least one vortex core.

The rate of cooling during the quench was limited by the collision rate between atoms in the trapped gas during evaporative cooling. This resulted in a limited dynamic range for the quench rate. Also, the rapid decrease of the thermal fraction following the formation of the condensate led to a low damping rate for the vortices. A faster quench rate, realized through a trap with tighter confinement or increased

density or via sympathetic cooling with another species, could result in the observation of an increased number of vortices during the quench. In turn, this would potentially allow for quantitative tests of the predicted scaling of the vortex number with the quench rate and the extraction of dynamic critical exponents.

While the formation of a superfluid by quenching the temperature is seeded by thermal fluctuations, ultracold atomic gases also potentially allow for the realization of phase transitions initiated purely by quantum fluctuations (Greiner *et al.*, 2002a; Sadler *et al.*, 2006).

A particularly intriguing study of a quench past such quantum phase transitions was carried out in a degenerate $F = 1$ spinor Bose gas of ^{87}Rb (Sadler *et al.*, 2006). These gases, with a spin degree of freedom arising from a nonzero hyperfine spin F, are quantum fluids that may simultaneously exhibit the phenomena of magnetism and superfluidity, both of which result from symmetry breaking and long range order. Owing to rotational symmetry, the contact interactions between two atoms can be characterized by the total spin of the colliding pair. In the case of a $F = 1$ spinor gas, these interactions give rise to a mean field energy given by $n(c_0 + c_2\langle \mathbf{F}\rangle^2)$, where the coupling strengths $c_{0,2}$ are related to the s-wave scattering lengths in the total spin $f = 0, 2$ channels (Ho, 1998; Ohmi and Machida, 1998). In addition to the mean field interactions, a finite external magnetic field B imposes a quadratic Zeeman energy (QZE) that scales as $q\langle F_z^2\rangle$ with $q = (\mu_B B)^2/4\Delta_{\mathrm{hf}}$, where μ_B is the Bohr magneton and Δ_{hf} is the energy splitting between the ground state hyperfine manifolds.

For a $F = 1$ condensate of ^{87}Rb, the competing influences of the spin-dependent interaction and the QZE give rise to a continuous quantum phase transition between a ''polar'' and a ferromagnetic phase. Rapidly tuning the external magnetic field from large values ($q \gg |c_2 n|$) to small values ($q \ll |c_2 n|$) quenches the spinor gas from the polar phase to the ferromagnetic phase. The ensuing growth of ferromagnetic domains was directly detected by *in situ* imaging (see Fig. 1). It was found that the resulting texture of ferromagnetic domains was spatially inhomogeneous and characterized by a typical length scale that was related to the spin healing length $\xi = \hbar/\sqrt{2m|c_2 n|}$. Concurrent with the appearance of these domains, the spin textures revealed the spontaneous formation of polar-core spin vortices. These topological defects are characterized by a nonzero spin current but no mass current. The origin of these spin vortices is also rooted in the KZ mechanism. It was shown that, for slow quenches, the number of such vortices is expected to scale as $\tau^{-1/6}$, where τ is the time over which the spinor gas is swept into the ferromagnetic state (Saito *et al.*, 2007).

The weak spin-dependent interactions inherent to this spinor gas also allow for nondestructive detection of the vortices and studies of their dynamics. In addition, the weak coupling between the spin and mass degrees of freedom make it straightforward to realize extremely low spin temperatures to examine the role of quantum fluctuations in seeding this phase transition (Klempt *et al.*, 2010). These features make spinor quantum fluids a rich system to investigate the quench dynamics and KZ mechanism past quantum phase transitions between different magnetically ordered phases. In addition, corrections to the KZ scaling imposed by long range interactions (Vengalattore *et al.*, 2008), conservation laws, and finite temperature effects can also be studied.

Yet another range of experimental studies is made possible by the tunability of atomic interactions using a Feshbach resonance. This technique allows the rapid dynamic control of the s-wave scattering length by means of a time-varying external magnetic field. This ability was utilized in a recent study of a strongly interacting two-component Fermi mixture (Jo *et al.*, 2009). Starting from an initially weak, repulsive interaction between the two fermionic species, the interactions were rapidly increased by tuning the magnetic field to the vicinity of the Feshbach resonance. The subsequent decrease in the atomic loss rate, the increase in the size of the trapped gas, and the increase in kinetic energy as measured in time-of-flight images were interpreted as indications of the Stoner transition to a ferromagnetic state. However, in a later theoretical work this interpretation was questioned and an alternative explanation based on rapid molecule formation was suggested (Babadi *et al.*, 2009). Thus, a direct *in situ* measurement of local magnetization is necessary to understand whether or not ferromagnetism plays a role in this experiment.

In addition to the thermalization dynamics across phase transitions, the long coherence times inherent to ultracold gases also make it possible to study the quantum coherent dynamics of many-particle systems. A particularly dramatic instance of such coherent many-body dynamics was illustrated in the collapse and revival of the matter wave field of a Bose condensate (Greiner *et al.*, 2002b). Here the interaction-induced dynamical evolution of a matter wave field was clearly revealed in the multiple matter wave interference patterns obtained after releasing the gas from the lattice. This work has also been extended to the time-resolved observation of superexchange processes in optical ''superlattice'' potentials (Trotzky *et al.*, 2008). Similar demonstrations of collisional coherence have also been shown in spinor Bose gases (Chang *et al.*, 2005; Kronjäger *et al.*, 2005). Because of the internal degrees of freedom in a spinor gas, the dynamics in this fluid is due to coherent spin-mixing collisions. In a trapped gas that is well described by the single mode approximation, these coherent collisions can lead to the periodic and reversible formation of condensates in initially unpopulated spin states.

Further, in certain situations, coherent interactions can also lead to quantum correlations (Sorensen *et al.*, 2001). Schemes that might realize such entangled many-particle states have received attention due to potential applications in quantum information processing and metrology. The dynamical evolution of such entangled states in the presence of quantum or thermal fluctuations is obviously of great interest. A recent experiment investigated this evolution in low-dimensional two-component Bose gases with adjustable interactions (Widera *et al.*, 2008), finding that quantum fluctuations play a crucial role in the phase diffusion dynamics of low-dimensional systems. More recently, the dynamical control of a Bose-Einstein condensate confined in a strongly driven optical lattice was demonstrated (Lignier *et al.*, 2007). By periodically modulating the lattice potential, the tunneling

parameter J was shown to be suppressed in a phase-coherent manner opening the possibility of driving quantum phase transitions using this technique.

The isolation of ultracold atomic gases from external sources of dissipation also makes it possible to study relaxation dynamics driven purely by intrinsic mechanisms. Such mechanisms should set the time scales for adiabatic quantum computing or the simulation of strongly correlated lattice models. A recent experiment along these lines investigated the evolution of excited states of the repulsive Fermi-Hubbard system (Strohmaier *et al.*, 2010). Here doubly occupied lattice sites (doublons) were created by modulating the lattice, and the subsequent decay of the system to thermal equilibrium was monitored over time. It was shown that the lifetime of these doublons scales exponentially with the ratio of the interaction energy to kinetic energy, in fair agreement with theoretical predictions. It was argued that the dominant mechanism driving this relaxation was a high-order scattering process involving several fermions (Sensarma *et al.*, 2010).

While this Colloquium places an emphasis on experiments involving ultracold atomic gases, there is a range of other mesoscopic quantum systems which also lend themselves to studies on quantum nonequilibrium dynamics. For completeness, we briefly review a few of these systems here. Defect formation following a quench was first studied in the context of vortices in liquid crystals (Chuang *et al.*, 1991). This has since been followed by similar studies in various mesoscopic systems including isolated superconducting loops where the defects assume the form of spontaneous fluxoids (Monaco *et al.*, 2009), superconducting thin films (Maniv *et al.*, 2003), and multi-Josephson junction loops (Monaco *et al.*, 2002; Monaco *et al.*, 2006). A cumulative view of these studies indicates that the influence of finite size effects, thermal fluctuations, and dimensionality on the production of topological defects by the KZ mechanism is as yet unclear and a topic that warrants further study.

Another potential system for the study of nonequilibrium dynamics of many-particle states arises from rapid advances in the field of photonics. There have been several proposals (Greentree *et al.*, 2006; Chang *et al.*, 2008) for the dynamical creation of strongly correlated photonic states using photon-photon interactions mediated by a nonlinear optical medium. The realization of states such as a Tonks gas of photons has been proposed using hollow-core optical fibers, tapered optical fibers, photons in coupled cavities, and surface plasmons on conducting nanowires. Such strongly correlated photon states should have applications in metrology, subshot noise interferometry, and the quantum emulation of exotic spin models.

V. OUTLOOK

One of the ultimate goals of the new field of quantum dynamics is to develop a systematic understanding of nonequilibrium phenomena in strongly interacting quantum many-body systems. A few of the most significant open questions are readily identified: How can we classify nonequilibrium behavior in closed many-body systems? What is the general relation between integrability and dynamics? What is the dynamical effect of a gradual breaking of integrability? What are the effects of dissipation on these nonequilibrium processes? Can we understand time evolution of interacting systems through the renormalization group? Answering these and other questions allied with systematic, quantitative studies of possible nonequilibrium quantum phase transitions and the extraction of dynamical critical exponents are just a few of the many tantalizing programs to be pursued. The rapidly developing sophistication and precision of ultracold atomic experiments and other experimental systems should allow for close and direct comparison between theoretical predictions and *ad hoc* experiments.

The realization of robust techniques for the experimental study of such systems and the development of theoretical tools to describe nonequilibrium many-body processes should bode for tantalizing opportunities in this nascent field, potentially leading to a deeper understanding of the principles governing nonequilibrium many-body phenomena and establishing robust connections between microscopic dynamics and statistical physics.

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