$f(R)$ theories of gravity

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Modified gravity theories have received increased attention lately due to combined motivation coming from high-energy physics, cosmology, and astrophysics. Among numerous alternatives to Einstein's theory of gravity, theories that include higher-order curvature invariants, and specifically the particular class of $f(R)$ theories, have a long history. In the last five years there has been a new stimulus for their study, leading to a number of interesting results. Here $f(R)$ theories of gravity are reviewed in an attempt to comprehensively present their most important aspects and cover the largest possible portion of the relevant literature. All known formalisms are presented—metric, Palatini, and metric affine—and the following topics are discussed: motivation; actions, field equations, and theoretical aspects; equivalence with other theories; cosmological aspects and constraints; viability criteria; and astrophysical applications.

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CONTENTS

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I. INTRODUCTION

A. Historical

As we approach the closing of a century after the introduction of the theory of general relativity (GR) in

1915, questions related to its limitations are becoming more and more pertinent. However, before coming to the contemporary reasons for challenging a theory as successful as Einstein's, it is worth mentioning that it took only four years from its introduction for people to start questioning its unique status among gravitation theories. Indeed, it was just 1919 when Weyl and 1923 when Eddington (the very man who three years earlier had provided the first experimental verification of GR by measuring light bending during a solar eclipse) started considering modifications of the theory by including higher-order invariants in its action (Weyl, 1919; Eddington, 1923).

These early attempts were triggered mainly by scientific curiosity and a will to question and therefore understand the then newly proposed theory. It is quite straightforward to realize that it is not very appealing to complicate the action and, consequently, the field equations with no apparent theoretical or experimental motivation. However, the motivation was soon to come.

Beginning in the 1960s indications appeared that complicating the gravitational action might indeed have its merits. GR is not renormalizable and, therefore, cannot be conventionally quantized. In 1962, Utiyama and DeWitt showed that renormalization at one loop demands that the Einstein Hilbert action be supplemented by higher-order curvature terms (Utiyama and DeWitt, 1962). Later on, Stelle showed that higher-order actions are indeed renormalizable (but not unitary) (Stelle, 1977). More recent results show that, when quantum corrections or string theory are taken into account, the effective low-energy gravitational action admits higher-order curvature invariants (Birrell and Davies, 1982; Buchbinder *et al.*, 1992; Vilkovisky, 1992).

Such considerations stimulated the interest of the scientific community in higher-order theories of gravity, i.e., modifications of the Einstein-Hilbert action in order to include higher-order curvature invariants with respect to the Ricci scalar [see Schmidt (2007) for a historical review and a list of references to early work]. However, the relevance of such terms in the action was considered to be restricted to very strong gravity regimes and they were expected to be strongly suppressed by small couplings, as one would expect when simple effective field theory considerations are taken into account. Therefore, corrections to GR were considered to be important only at scales close to the Planck scale and, consequently, in the early universe or near black hole singularities—and indeed there are relevant studies, such as the well-known curvature-driven inflation scenario (Starobinsky, 1980) and attempts to avoid cosmological and black hole singularities (Shahid-Saless, 1990; Brandenberger, 1992, 1993, 1995; Mukhanov and Brandenberger, 1992; Brandenberger *et al.*, 1993; Trodden *et al.*, 1993). However, it was not expected that such corrections could affect the gravitational phenomenology at low energies and consequently large scales such as, for instance, in the late universe.

B. Contemporary motivation

More recently, new evidence coming from astrophysics and cosmology has revealed a quite unexpected picture of the universe. Our latest data sets coming from different sources, such as the cosmic microwave background radiation and supernovae surveys, seem to indicate that the energy budget of the universe is the following: 4% ordinary baryonic matter, 20% *dark matter*, and 76% *dark energy* (Riess *et al.*, 2004; Eisenstein *et al.*, 2005; Astier *et al.*, 2006; Spergel *et al.*, 2007). The term "dark matter" refers to an unknown form of matter, which has the clustering properties of ordinary matter but has not yet been detected in the laboratory. The term "dark energy" is reserved for an unknown form of energy which not only has not been detected directly but also does not cluster as ordinary matter does. More rigorously, one could use the various energy conditions (Wald, 1984) to distinguish dark matter and dark energy: Ordinary matter and dark matter satisfy the strong-energy condition, whereas dark energy does not. Additionally, dark energy seems to very closely resemble a cosmological constant. Due to its dominance over matter (ordinary and dark) at present times, the expansion of the universe seems to be an accelerated one, contrary to past expectations.[1]

Note that this late-time speedup comes to be added to an early-time accelerated epoch as predicted by the inflationary paradigm (Guth, 1981; Linde, 1990; Kolb and Turner, 1992). The inflationary epoch is needed to address the *horizon, flatness*, and *monopole problems* (Misner, 1968; Weinberg, 1972; Linde, 1990; Kolb and Turner, 1992) as well as to provide the mechanism that generates primordial inhomogeneities acting as seeds for the formation of large-scale structures (Mukhanov, 2003). Recall also that, in between these two periods of acceleration, there should be a period of decelerated expansion, so that the more conventional cosmological eras of radiation domination and matter domination can take place. Indeed, there are stringent observational bounds on the abundances of light elements, such as deuterium, helium, and lithium, which require that big bang nucleosynthesis, the production of nuclei other than hydrogen, takes place during radiation domination (Burles *et al.*, 2001; Carroll and Kaplinghat, 2002). On the other hand, a matter-dominated era is required for structure formation to occur.

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[1]Recall that from GR in the absence of the cosmological constant and under the standard cosmological assumptions (spatial homogeneity, isotropy, etc.) one obtains the second Friedmann equation,

$$\ddot{a}/a = -(4\pi G/3)(\rho + 3P), \tag{1}$$

where a is the scale factor, G is the gravitational constant, and ρ and P are the energy density and the pressure of the cosmological fluid, respectively. Therefore, if the strong-energy condition $\rho + 3P \geqslant 0$ is satisfied, there can be no acceleration (gravity is attractive).

Puzzling observations do not stop here. Dark matter makes its appearance not only in cosmological data but also in astrophysical observations. The "missing mass" question had already been posed in 1933 for galaxy clusters (Zwicky, 1933) and in 1959 for individual galaxies (Kahn and Woltjer, 1959), and a satisfactory final answer has been pending ever since (Rubin and Ford, 1970; Bosma, 1978; Rubin *et al.*, 1980; Persic *et al.*, 1996; Moore, 2001; Ellis, 2002).

One therefore has to admit that our current picture of the evolution and the matter-energy content of the universe is at least surprising and definitely calls for an explanation. The simplest model that adequately fits the data creating this picture is the so-called concordance or Λ cold dark matter (ΛCDM) model, supplemented by some inflationary scenario, usually based on some scalar field called the inflaton. Besides not explaining the origin of the inflaton or the nature of dark matter itself, the ΛCDM model is burdened with the well-known cosmological constant problems (Weinberg, 1989; Carroll, 2001a): the magnitude problem, according to which the observed value of the cosmological constant is extravagantly small to be attributed to the vacuum energy of matter fields, and the coincidence problem, which can be summed up in the following question: since there is just an extremely short period of time in the evolution of the universe in which the energy density of the cosmological constant is comparable with that of matter, why is this happening today when we are present to observe it?

These problems make the ΛCDM model more of an empirical fit to the data whose theoretical motivation can be regarded as quite poor. Consequently, there have been several attempts either to directly motivate the presence of a cosmological constant or to propose dynamical alternatives to dark energy. Unfortunately, none of these attempts are problem-free. For instance, the so-called anthropic reasoning for the magnitude of Λ (Carter, 1974; Barrow and Tipler, 1986), even when placed onto firmer ground through the idea of the "anthropic or string landscape" (Susskind, 2003), still makes many physicists feel uncomfortable due to its probabilistic nature. On the other hand, simple scenarios for dynamical dark energy, such as quintessence (Peebles and Ratra, 1988; Ratra and Peebles, 1988; Wetterich, 1988; Ostriker and Steinhardt, 1995; Caldwell *et al.*, 1998; Carroll, 1998; Bahcall *et al.*, 1999; Wang *et al.*, 2000), do not seem to be as well motivated theoretically as one would desire.[2]

Another perspective for resolving the issues described above, which might appear as more radical to some, is the following: Gravity is by far the dominant interaction at cosmological scales and therefore it is the force governing the evolution of the universe. Could it be that our description of the gravitational interaction at the relevant scales is not sufficiently adequate and stands at the root of all or some of these problems? Should we consider modifying our theory of gravitation and, if so, would this help in avoiding dark components and answering the cosmological and astrophysical riddles?

It is rather pointless to argue whether such a perspective would be better or worse than any of the other solutions already proposed. It is definitely a different way to address the same problems and, as long as these problems do not find a plausible, well-accepted, and simple solution, it is worth pursuing all alternatives. Additionally, questioning of the gravitational theory itself definitely has its merits: it helps us to obtain a deeper understanding of the relevant issues and of the gravitational interaction, it has a high chance to lead to new physics, and it has worked in the past. Recall that the precession of Mercury's orbit was at first attributed to some unobserved ("dark") planet orbiting inside Mercury's orbit, but was actually explained only after the passage from Newtonian gravity to GR.

C. $f(R)$ theories as toy theories

Even if one decides that modifying gravity is the way to go, this is not an easy task. To begin with, there are numerous ways to deviate from GR. Setting aside the early attempts to generalize Einstein's theory, most of which have been shown to be nonviable (Will, 1981), and the best-known alternative to GR, scalar-tensor theory (Brans and Dicke, 1961; Dicke, 1962; Bergmann, 1968; Nordtvedt, 1970; Wagoner, 1970; Faraoni, 2004a), there are still numerous proposals for modified gravity in contemporary literature. Typical examples are Dvali-Gabadadze-Porrati gravity (Dvali *et al.*, 2000), braneworld gravity (Maartens, 2004), tensor-vector-scalar theory (Bekenstein, 2004), and Einstein-Aether theory (Jacobson and Mattingly, 2001). The subject of this review is a different class of theories, $f(R)$ theories of gravity. These theories come about by a straightforward generalization of the Lagrangian in the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \tag{2}$$

where $\kappa \equiv 8\pi G$, G is the gravitational constant, g is the determinant of the metric, and R is the Ricci scalar ($c=\hbar=1$), to become a general function of R, i.e.,

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R). \tag{3}$$

Before we go further into the discussion of the details and the history of such actions—this will happen in the forthcoming section—some remarks are in order. We have already mentioned the motivation coming from high-energy physics for adding higher-order invariants to the gravitational action, as well as a general motivation coming from cosmology and astrophysics for seeking generalizations of GR. There are, however, still two

[2]We are referring here not only to the fact that the mass of the scalar turns out to be many orders of magnitude smaller than any of the masses of the scalar fields usually encountered in particle physics but also to the inability to motivate the absence of any coupling of the scalar field to matter (there is no mechanism or symmetry preventing this) (Carroll, 2001b).

questions that might be troubling the reader. The first one is the following: Why specifically $f(R)$ actions and not more general ones, which include other higher-order invariants, such as $R_{\mu\nu}R^{\mu\nu}$?

The answer to this question is twofold. First, there is simplicity: $f(R)$ actions are sufficiently general to encapsulate some of the basic characteristics of higher-order gravity, but at the same time they are simple enough to be easy to handle. For instance, viewing f as a series expansion, i.e.,

$$f(R) = \cdots + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + \cdots, \qquad (4)$$

where the α_i and β_i coefficients have the appropriate dimensions, we see that the action includes a number of phenomenologically interesting terms. In brief, $f(R)$ theories make excellent candidates for toy theories—tools from which one gains some insight into such gravity modifications. Second, there are serious reasons to believe that $f(R)$ theories are unique among higher-order gravity theories in the sense that they seem to be the only ones that can avoid the long-known and fatal Ostrogradski instability (Woodard, 2007).

The second question calling for an answer is related to a possible loophole that one may have already spotted in the motivation presented: How can high-energy modifications of the gravitational action have anything to do with late-time cosmological phenomenology? Would not effective field theory considerations require that the coefficients in Eq. (4) be such as to make any corrections to the standard Einstein-Hilbert term important only near the Planck scale?

Conservative thinking would give a positive answer. However, one also has to stress two other serious factors: first, there is a large ambiguity about how gravity really works at small scales or high energies. Indeed, there are certain results already in the literature claiming that terms responsible for late-time gravitational phenomenology might be predicted by some more fundamental theory, such as string theory [see, for instance, Nojiri and Odintsov (2003b)]. On the other hand, one should not forget that the observationally measured value of the cosmological constant corresponds to some energy scale. Neither effective field theory nor any other high-energy theory consideration has thus far been able to predict or explain it. Yet it stands as an experimental fact and putting the number in the right context can be crucial in explaining its value. Therefore, in any phenomenological approach, it seems inevitable that some parameter will appear to be unnaturally small at first (the mass of a scalar, a coefficient of some expansion, etc., according to the approach). The real question is whether this initial "unnaturalness" can still be explained.

In other words, the motivation for infrared modifications of gravity in general and $f(R)$ gravity in particular is, to some extent, hand waving. However, the importance of the issues leading to this motivation and our inability to find other, successful, more straightforward, and maybe better-motivated ways to address them, combined with the significant room for speculation which our quantum gravity candidates leave, have triggered an increase of interest in modified gravity that is probably reasonable.

To conclude, when all of the above concerns are taken into account, $f(R)$ gravity should be neither overestimated nor underestimated. It is an interesting and relatively simple alternative to GR from the study of which some useful conclusions have been derived already. However, it is still a toy theory, as already mentioned; an easy-to-handle deviation from Einstein's theory to be used mostly in order to understand the principles and limitations of modified gravity. Similar considerations apply to modification of gravity in general: We are probably far from concluding whether it is the answer to our problems at the moment. However, in some sense, such an approach is bound to be fruitful since, even if it only leads to the conclusion that GR is the only correct theory of gravitation, it will still have helped us both to understand GR better and to secure our faith in it.

II. ACTIONS AND FIELD EQUATIONS

As can be found in many textbooks—see, for example, Misner *et al.* (1973) and Wald (1984)—there are actually two variational principles that one can apply to the Einstein-Hilbert action in order to derive Einstein's equations: the standard metric variation and a less standard variation dubbed the Palatini variation [even though it was Einstein and not Palatini who introduced it (Ferraris *et al.*, 1982)]. In the latter the metric and the connection are assumed to be independent variables and one varies the action with respect to both of them (we will see how this variation leads to Einstein's equations shortly), under the important assumption that the matter action does not depend on the connection. The choice of the variational principle is usually referred to as a formalism, so one can use the terms metric (or second-order) formalism and Palatini (or first-order) formalism. However, even though both variational principles lead to the same field equation for an action whose Lagrangian is linear in R, this is no longer true for a more general action. Therefore, it is intuitive that there will be two versions of $f(R)$ gravity, according to which variational principle or formalism is used. Indeed this is the case: $f(R)$ gravity in the metric formalism is called *metric $f(R)$ gravity* and $f(R)$ gravity in the Palatini formalism is called *Palatini $f(R)$ gravity* (Buchdahl, 1970).

Finally, there is actually even a third version of $f(R)$ gravity: *metric-affine $f(R)$ gravity* (Sotiriou and Liberati, 2007a, 2007b). This comes about if one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection. Clearly, metric-affine $f(R)$ gravity is the most general of these theories and reduces to metric or Palatini $f(R)$ gravity if further assumptions are made. In this section we present the actions and field equations of all three versions of $f(R)$

gravity and point out their differences. We also clarify the physical meaning behind the assumptions that discriminate them.

For an introduction to metric $f(R)$ gravity see Nojiri and Odintsov (2007a), for a shorter review of metric and Palatini $f(R)$ gravities see Capozziello and Francaviglia (2008), and for an extensive analysis of all versions of $f(R)$ gravity and other alternative theories of gravity see Sotiriou (2007b).

A. Metric formalism

Beginning from the action (3) and adding a matter term S_M, the total action for $f(R)$ gravity takes the form

$$S_{\rm met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\psi), \tag{5}$$

where ψ collectively denotes the matter fields. Variation with respect to the metric gives, after some manipulations and modulo surface terms,

$$f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(R) = \kappa T_{\mu\nu}, \tag{6}$$

where, as usual,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{7}$$

a prime denotes differentiation with respect to the argument, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric, and $\Box \equiv \nabla^\mu\nabla_\mu$. Metric $f(R)$ gravity was first rigorously studied by Buchdahl (1970).[3]

It has to be stressed that there is a mathematical jump in deriving Eq. (6) from the action (5) having to do with the surface terms that appear in the variation: As in the case of the Einstein-Hilbert action, the surface terms do not vanish just by fixing the metric on the boundary. For the Einstein-Hilbert action, however, these terms gather into a total variation of a quantity. Therefore, it is possible to add a total divergence to the action in order to "heal" it and arrive at a well-defined variational principle [this is the well-known Gibbons-Hawking-York surface term (York and James, 1972; Gibbons and Hawking, 1977)]. Unfortunately, the surface terms in the variation of the action (3) do not consist of a total variation of some quantity and it is not possible to heal the action by just subtracting some surface term before performing the variation.

The way out comes from the fact that the action includes higher-order derivatives of the metric and, therefore, it should be possible to fix more degrees of freedom on the boundary than those of the metric itself. There is no unique prescription for such a fixing in the literature so far. Note also that the choice of fixing is not void of physical meaning since it will be relevant for the Hamiltonian formulation of the theory. However, the field equations (6) will be unaffected by the fixing chosen and, from a purely classical perspective, such as the one followed here, the field equations are all that one needs [see Sotiriou (2007b) for a more detailed discussion on these issues].

Setting aside the complications of the variation we can now focus on the field equations (6). These are obviously fourth-order partial differential equations in the metric since R already includes second derivatives of the latter. For an action that is linear in R, the fourth-order terms—the last two on the left-hand side—vanish and the theory reduces to GR.

Notice also that the trace of Eq. (6),

$$f'(R)R - 2f(R) + 3\Box f' = \kappa T, \tag{8}$$

where $T = g^{\mu\nu}T_{\mu\nu}$, relates R with T differentially and not algebraically as in GR, where $R = -\kappa T$. This is already an indication that the field equations of $f(R)$ theories will admit a larger variety of solutions than Einstein's theory. As an example, we mention here that the Jebsen-Birkhoff theorem, stating that the Schwarzschild solution is the unique spherically symmetric vacuum solution, no longer holds in metric $f(R)$ gravity. Without going into detail, we stress that $T=0$ no longer implies that $R=0$ or is even constant.

Equation (8) will turn out to be useful in studying various aspects of $f(R)$ gravity, notably its stability and weak-field limit. For the moment, we use it to make some remarks about maximally symmetric solutions. Recall that maximally symmetric solutions lead to a constant Ricci scalar. For R=const and $T_{\mu\nu}=0$, Eq. (8) reduces to

$$f'(R)R - 2f(R) = 0, \tag{9}$$

which, for a given f, is an algebraic equation in R. If $R=0$ is a root of this equation and one takes this root, then Eq. (6) reduces to $R_{\mu\nu}=0$ and the maximally symmetric solution is Minkowski space-time. On the other hand, if the root of Eq. (9) is $R=C$, where C is a constant, then Eq. (6) reduces to $R_{\mu\nu}=g_{\mu\nu}C/4$ and the maximally symmetric solution is de Sitter or anti–de Sitter space depending on the sign of C, just as in GR with a cosmological constant.

Another issue that should be stressed is that of energy conservation. In metric $f(R)$ gravity the matter is minimally coupled to the metric. One can therefore use the usual arguments based on the invariance of the action under diffeomorphisms of the space-time manifold [coordinate transformations $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ followed by a pullback, with the field ξ^μ vanishing on the boundary of the space-time region considered, leave the physics unchanged; see Wald (1984)] to show that $T_{\mu\nu}$ is divergence-free. The same can be done at the level of the field equations: a "brute force" calculation reveals that the left-hand side of Eq. (6) is divergence-free

[3]Specific attention to higher-dimensional $f(R)$ gravity was given by Gunther *et al.* (2002, 2003, 2005) and Saidov and Zhuk (2006, 2007).

(generalized Bianchi identity) implying that $\nabla_\mu T^{\mu\nu}=0$ (Koivisto, 2006a).[4]

Finally, we note that it is possible to write the field equations in the form of Einstein equations with an effective stress-energy tensor composed of curvature terms moved to the right-hand side. This approach is questionable in principle (the theory is not Einstein's theory and it is artificial to force upon it an interpretation in terms of Einstein equations) but in practice it has been proved to be useful in scalar-tensor gravity. Specifically, Eq. (6) can be written as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \frac{\kappa T_{\mu\nu}}{f'(R)} + g_{\mu\nu}\frac{[f(R) - Rf'(R)]}{2f'(R)} + \frac{[\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)]}{f'(R)} \tag{10}$$

or

$$G_{\mu\nu} = \frac{\kappa}{f'(R)}(T_{\mu\nu} + T^{(\mathrm{eff})}_{\mu\nu}), \tag{11}$$

where the quantity $G_{\mathrm{eff}} \equiv G/f'(R)$ can be regarded as the effective gravitational coupling strength, in analogy to what is done in scalar-tensor gravity—positivity of G_{eff} (equivalent to the requirement that the graviton is not a ghost) imposes that $f'(R)>0$. Moreover,

$$T^{(\mathrm{eff})}_{\mu\nu} \equiv \frac{1}{\kappa}\left[\frac{f(R) - Rf'(R)}{2}g_{\mu\nu} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right] \tag{12}$$

is an effective stress-energy tensor which does not have the canonical form quadratic in the first derivatives of the field $f'(R)$ but contains terms linear in the second derivatives. The effective energy density derived from it is not positive definite and none of the energy conditions holds. Again, this situation is analogous to that occurring in scalar-tensor gravity. The effective stress-energy tensor (12) can be put in the form of a perfect fluid energy-momentum tensor, which will turn out to be useful in Sec. IV.

B. Palatini formalism

We have mentioned that the Einstein equations can be derived using, instead of the standard metric variation of the Einstein-Hilbert action, the Palatini formalism, i.e., an independent variation with respect to the metric and an independent connection (Palatini variation). The action is formally the same but now the Riemann tensor and the Ricci tensor are constructed with the independent connection. Note that the metric is not needed to obtain the latter from the former. For clarity of notation, we denote the Ricci tensor constructed with this independent connection as $\mathcal{R}_{\mu\nu}$ and the corresponding Ricci scalar[5] is $\mathcal{R}=g^{\mu\nu}\mathcal{R}_{\mu\nu}$. The action now takes the form

$$S_{\mathrm{Pal}} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu},\psi). \tag{13}$$

GR will come about, as we will see shortly, when $f(\mathcal{R}) = \mathcal{R}$. Note that the matter action S_M is assumed to depend only on the metric and the matter fields and not on the independent connection. This assumption is crucial for the derivation of Einstein's equations from the linear version of the action (13) and is the main feature of the Palatini formalism.

It has been mentioned that this assumption has consequences for the physical meaning of the independent connection (Sotiriou, 2006b, 2006d; Sotiriou and Liberati, 2007b). We now elaborate on this. Recall that an affine connection usually defines parallel transport and the covariant derivative. On the other hand, the matter action S_M is supposed to be a generally covariant scalar which includes derivatives of the matter fields. Therefore, these derivatives ought to be covariant derivatives for a general matter field. Exceptions exist, such as a scalar field, for which a covariant and a partial derivative coincide, and the electromagnetic field, for which one can write a covariant action without the use of the covariant derivative [it is the exterior derivative that is actually needed; see the next section and Sotiriou and Liberati (2007b)]. However, S_M should include all possible fields. Therefore, the assumption that S_M is independent of the connection can imply one of two things (Sotiriou, 2006d): either we are restricting ourselves to specific fields or we are implicitly assuming that it is the Levi-Civita connection of the metric that actually defines parallel transport. Since the first option is implausibly limiting for a gravitational theory, we are left with the conclusion that the independent connection $\Gamma^\lambda{}_{\mu\nu}$ does not define parallel transport or the covariant derivative and the geometry is actually pseudo-Riemannian. The covariant derivative is actually defined by the Levi-Civita connection of the metric $\{^\lambda{}_{\mu\nu}\}$.

This also implies that Palatini $f(R)$ gravity is a *metric theory* in the sense that it satisfies the metric postulates (Will, 1981). We now clarify this: matter is minimally coupled to the metric and not coupled to any other fields. Once again, as in GR or metric $f(R)$ gravity, one could use diffeomorphism invariance to show that the stress-energy tensor is conserved by the covariant derivative defined with the Levi-Civita connection of the metric, i.e., $\nabla_\mu T^{\mu\nu}=0$ (but $\bar{\nabla}_\mu T^{\mu\nu}\neq 0$). This can also be shown using the field equations, which we present shortly, in order to calculate the divergence of $T_{\mu\nu}$ with respect to the Levi-Civita connection of the metric and show that it vanishes (Barraco *et al.*, 1999, Koivisto

[4]Energy-momentum complexes in the spherically symmetric case have been computed by Multamaki *et al.* (2008).

[5]The term "$f(R)$ gravity" is used generically for a theory in which the action is some function of some Ricci scalar, not necessarily R.

2006a).[6] Clearly then Palatini $f(R)$ gravity is a metric theory according to the definition of Will (1981) [not to be confused with the term "metric" in metric $f(R)$ gravity, which simply refers to the fact that one varies only the action with respect to the metric]. Conventionally thinking, as a consequence of the covariant conservation of the matter energy-momentum tensor, test particles should follow geodesics of the metric in Palatini $f(R)$ gravity. This can be seen by considering a dust fluid with $T_{\mu\nu}=\rho u_\mu u_\nu$ and projecting the conservation equation $\nabla^\beta T_{\mu\beta}=0$ onto the fluid four-velocity u^β. Similarly, theories that satisfy the metric postulates are supposed to satisfy the Einstein equivalence principle as well (Will, 1981). Unfortunately, things are more complicated here and therefore we set this issue aside for the moment. We will return to it and attempt to fully clarify it in Secs. VI.B and VI.C.2. For now, we proceed with our discussion of the field equations.

Varying the action (13) independently with respect to the metric and the connection, respectively, and using

$$\delta\mathcal{R}_{\mu\nu} = \bar{\nabla}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \bar{\nabla}_\nu \delta\Gamma^\lambda{}_{\mu\lambda} \tag{14}$$

we obtain

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{15}$$

$$-\bar{\nabla}_\lambda[\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}] + \bar{\nabla}_\sigma[\sqrt{-g}f'(\mathcal{R})g^{\sigma(\mu}]\delta^{\nu)}_\lambda = 0, \tag{16}$$

where $T_{\mu\nu}$ is defined in the usual way as in Eq. (7), $\bar{\nabla}_\mu$ denotes the covariant derivative defined with the independent connection $\Gamma^\lambda{}_{\mu\nu}$, and $(\mu\nu)$ and $[\mu\nu]$ denote symmetrization or antisymmetrization over the indices μ and ν, respectively. Taking the trace of Eq. (16), it can be easily shown that

$$\bar{\nabla}_\sigma[\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}] = 0, \tag{17}$$

which implies that we can bring the field equations into the more economical form

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa G T_{\mu\nu}, \tag{18}$$

$$\bar{\nabla}_\lambda[\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}] = 0. \tag{19}$$

It is now easy to see how the Palatini formalism leads to GR when $f(\mathcal{R})=\mathcal{R}$; in this case $f'(\mathcal{R})=1$ and Eq. (19) becomes the definition of the Levi-Civita connection for the initially independent connection $\Gamma^\lambda{}_{\mu\nu}$. Then, $\mathcal{R}_{\mu\nu}=R_{\mu\nu}$, $\mathcal{R}=R$, and Eq. (18) yields Einstein's equations. This reproduces the result that can be found in textbooks (Misner *et al.*, 1973; Wald, 1984). Note that, in the Palatini formalism for GR, the fact that the connection turns out to be the Levi-Civita one is a dynamical feature instead of an *a priori* assumption.

It is now evident that generalizing the action to be a general function of $\mathcal{R}$ in the Palatini formalism is just as natural as it is to generalize the Einstein-Hilbert action in the metric formalism.[7] Remarkably, even though the two formalisms give the same results for linear actions, they lead to different results for more general actions (Buchdahl, 1970; Shahid-Saless, 1987; Burton and Mann, 1998a, 1998b; Querella, 1999; Exirifard and Sheikh-Jabbari, 2008).

Finally, we present some useful manipulations of the field equations. Taking the trace of Eq. (18) yields

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T. \tag{20}$$

As in the metric case, this equation will prove very useful later on. For a given f, it is an algebraic equation in $\mathcal{R}$. For all cases in which $T=0$, including vacuum and electrovacuum, $\mathcal{R}$ will therefore be a constant and a root of the equation

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0. \tag{21}$$

We will not consider cases for which this equation has no roots since it can be shown that the field equations are then inconsistent (Ferraris *et al.*, 1992). Therefore, choices of f that lead to this behavior should simply be avoided. Equation (21) can also be identically satisfied if $f(\mathcal{R}) \propto \mathcal{R}^2$. This very particular choice for f leads to a conformally invariant theory (Ferraris *et al.*, 1992). As is apparent from Eq. (20), if $f(\mathcal{R}) \propto \mathcal{R}^2$ then only conformally invariant matter, for which $T=0$ identically, can be coupled to gravity. Matter is not generically conformally invariant though, and so this particular choice of f is not suitable for a low-energy theory of gravity. We will therefore not consider it further [see Sotiriou (2006b) for a discussion].

Next, we consider Eq. (19). We define a metric conformal to $g_{\mu\nu}$ as

$$h_{\mu\nu} \equiv f'(\mathcal{R})g_{\mu\nu}. \tag{22}$$

It can easily be shown that[8]

$$\sqrt{-h}h^{\mu\nu} = \sqrt{-g}f'(\mathcal{R})g^{\mu\nu}. \tag{23}$$

Then, Eq. (19) becomes the definition of the Levi-Civita connection of $h_{\mu\nu}$ and can be solved algebraically to give

$$\Gamma^\lambda{}_{\mu\nu} = h^{\lambda\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \tag{24}$$

or, equivalently, in terms of $g_{\mu\nu}$,

[6]Energy supertensors and pseudotensors in Palatini $f(R)$ gravity were studied by Ferraris *et al.* (1992); Borowiec *et al.* (1994, 1998), and Barraco *et al.* (1999) and alternative energy definitions were given by Deser and Tekin (2002, 2003a, 2003b, 2007).

[7]See, however, Sotiriou (2007b) for further analysis of the $f(R)$ action and how it can be derived from first principles in the two formalisms.

[8]This calculation holds in four dimensions. When the number of dimensions D is different from 4 then, instead of using Eq. (22), the conformal metric $h_{\mu\nu}$ should be introduced as $h_{\mu\nu} \equiv [f'(\mathcal{R})]^{2/(D-2)} g_{\mu\nu}$ in order for Eq. (23) to still hold.

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{f'(\mathcal{R})} g^{\lambda\sigma}\{\partial_\mu[f'(\mathcal{R})g_{\nu\sigma}] + \partial_\nu[f'(\mathcal{R})g_{\mu\sigma}] - \partial_\sigma[f'(\mathcal{R})g_{\mu\nu}]\}. \quad (25)$$

Given that Eq. (20) relates $\mathcal{R}$ algebraically with T, and since we have an explicit expression for $\Gamma^{\lambda}{}_{\mu\nu}$ in terms of $\mathcal{R}$ and $g^{\mu\nu}$, we can in principle eliminate the independent connection from the field equations and express them in terms of only the metric and the matter fields. Actually, the fact that we can algebraically express $\Gamma^{\lambda}{}_{\mu\nu}$ in terms of the latter two already indicates that these connections act as some sort of auxiliary field. We explore this further in Sec. III. For the moment, we take into account how the Ricci tensor transforms under conformal transformations and write

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2}\frac{1}{[f'(\mathcal{R})]^2}[\nabla_\mu f'(\mathcal{R})][\nabla_\nu f'(\mathcal{R})] - \frac{1}{f'(\mathcal{R})}\left(\nabla_\mu\nabla_\nu - \frac{1}{2}g_{\mu\nu}\Box\right)f'(\mathcal{R}). \quad (26)$$

Contraction with $g^{\mu\nu}$ yields

$$\mathcal{R} = R + \frac{3}{2[f'(\mathcal{R})]^2}[\nabla_\mu f'(\mathcal{R})][\nabla^\mu f'(\mathcal{R})] + \frac{3}{f'(\mathcal{R})}\Box f'(\mathcal{R}). \quad (27)$$

Note the difference between $\mathcal{R}$ and the Ricci scalar of $h_{\mu\nu}$ due to the fact that $g_{\mu\nu}$ is used here for the contraction of $\mathcal{R}_{\mu\nu}$.

Replacing Eqs. (26) and (27) in Eq. (18) and after some easy manipulations, one obtains

$$G_{\mu\nu} = \frac{\kappa}{f'}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\mathcal{R} - \frac{f}{f'}\right) + \frac{1}{f'}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f' - \frac{3}{2}\frac{1}{f'^2}\left[(\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2}g_{\mu\nu}(\nabla f')^2\right]. \quad (28)$$

Notice that, assuming that we know the root of Eq. (20), $\mathcal{R}=\mathcal{R}(T)$, we have completely eliminated the independent connection from this equation. Therefore, we have successfully reduced the number of field equations to 1 and at the same time both sides of Eq. (28) depend only on the metric and the matter fields. In a sense, the theory has been brought into the form of GR with a modified source.

We can now straightforwardly deduce the following:

- When $f(\mathcal{R})=\mathcal{R}$, the theory reduces to GR, as discussed previously.
- For matter fields with $T=0$, due to Eq. (21), $\mathcal{R}$ and consequently $f(\mathcal{R})$ and $f'(\mathcal{R})$ are constants and the theory reduces to GR with a cosmological constant and a modified coupling constant G/f'. If we denote the value of $\mathcal{R}$ when $T=0$ as $\mathcal{R}_0$, then the value of the cosmological constant is

$$\frac{1}{2}\left(\mathcal{R}_0 - \frac{f(\mathcal{R}_0)}{f'(\mathcal{R}_0)}\right) = \frac{\mathcal{R}_0}{4}, \quad (29)$$

  where we have used Eq. (21). Besides vacuum, $T=0$ also for electromagnetic fields, radiation, and any other conformally invariant type of matter.
- In the general case $T\neq 0$, the modified source on the right-hand side of Eq. (28) includes derivatives of the stress-energy tensor, unlike in GR. These are implicit in the last two terms of Eq. (28) since f' is in practice a function of T given that[9] $f'=f'(\mathcal{R})$ and $\mathcal{R}=\mathcal{R}(T)$.

The serious implications of this last observation will become clear in Sec. VI.C.1.

C. Metric-affine formalism

As pointed out, the Palatini formalism of $f(R)$ gravity relies on the crucial assumption that the matter action does not depend on the independent connection. We also argued that this assumption relegates this connection to the role of some sort of auxiliary field, and the connection carrying the usual geometrical meaning—parallel transport and definition of the covariant derivative—remains the Levi-Civita connection of the metric. All of these statements will be supported further in the forthcoming sections, but for the moment consider what would be the outcome if we decided to be faithful to the geometrical interpretation of the independent connection $\Gamma^{\lambda}{}_{\mu\nu}$: this would imply that we would define the covariant derivatives of the matter fields with this connection and, therefore, we would have $S_M = S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi)$. The action of this theory, dubbed metric-affine $f(R)$ gravity (Sotiriou and Liberati, 2007b), would then be [note the difference with respect to the action (13)]

$$S_{\rm MA} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi). \quad (30)$$

1. Preliminaries

Before going further and deriving field equations from this action we need to clarify certain issues. First, since now the matter action depends on the connection, we should define a quantity representing the variation of S_M with respect to the connection, which mimics the definition of the stress-energy tensor. We call this quantity the *hypermomentum* and it is defined as (Hehl and Kerling, 1978)

[9]Note that, apart from special cases such as a perfect fluid, $T_{\mu\nu}$ and consequently T already include first derivatives of the matter fields, given that the matter action has such a dependence. This implies that the right-hand side of Eq. (28) will include at least second derivatives of the matter fields and possibly up to third derivatives.

$$\Delta_\lambda{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta\Gamma^\lambda{}_{\mu\nu}}. \quad (31)$$

Additionally, since the connection is now promoted to the role of a completely independent field, it is interesting to consider not placing any restrictions on it. Therefore, besides dropping the assumption that the connection is related to the metric, we also drop the assumption that the connection is symmetric. It is useful to define the following quantities: the nonmetricity tensor

$$Q_{\mu\nu\lambda} \equiv -\bar{\nabla}_\mu g_{\nu\lambda}, \quad (32)$$

which measures the failure of the connection to covariantly conserve the metric, the trace of the nonmetricity tensor with respect to its last two (symmetric) indices, which is called the Weyl vector,

$$Q_\mu \equiv \tfrac{1}{4}Q_{\mu\nu}{}^{\nu}, \quad (33)$$

and the Cartan torsion tensor

$$S_{\mu\nu}{}^{\lambda} \equiv \Gamma^\lambda{}_{[\mu\nu]}, \quad (34)$$

which is the antisymmetric part of the connection.

By allowing a nonvanishing Cartan torsion tensor we are allowing the theory to include torsion naturally. Even though this brings complications, it has been considered by some to be an advantage for a gravity theory since some matter fields, such as Dirac fields, can be coupled to gravity in a way that might be considered more natural (Hehl *et al.*, 1995): one might expect that at some intermediate- or high-energy regime the spin of particles might interact with the geometry (in the same sense that macroscopic angular momentum interacts with geometry), and torsion can arise. Theories with torsion have a long history, probably starting with the Einstein-Cartan(-Sciama-Kibble) theory (Cartan, 1922, 1923, 1924; Kibble, 1961; Sciama, 1964; Hehl *et al.*, 1976). In this theory, as well as in other theories with an independent connection, some part of the connection is still related to the metric (e.g., the nonmetricity is set to zero). In our case, the connection is left completely unconstrained and is to be determined by the field equations. Metric-affine gravity with the linear version of the action (30) was initially proposed by Hehl and Kerling (1978) and the generalization to $f(\mathcal{R})$ actions was considered by Sotiriou and Liberati (2007a, 2007b).

Unfortunately, if the connection is left completely unconstrained a complication follows. Consider the projective transformation

$$\Gamma^\lambda{}_{\mu\nu} \to \Gamma^\lambda{}_{\mu\nu} + \delta^\lambda{}_\mu \xi_\nu, \quad (35)$$

where ξ_ν is an arbitrary covariant vector field. One can easily show that the Ricci tensor will correspondingly transform as

$$\mathcal{R}_{\mu\nu} \to \mathcal{R}_{\mu\nu} - 2\partial_{[\mu}\xi_{\nu]}. \quad (36)$$

However, given that the metric is symmetric, this implies that the curvature scalar does not change,

$$\mathcal{R} \to \mathcal{R}, \quad (37)$$

i.e., $\mathcal{R}$ is invariant under projective transformations. Hence the Einstein-Hilbert action or any other action built from a function of $\mathcal{R}$, such as the one used here, is projective invariant in metric-affine gravity. However, the matter action is not generically projective invariant and this would be the cause of an inconsistency in the field equations.

One could try to avoid this problem by generalizing the gravitational action in order to break projective invariance. This can be done in several ways, such as allowing for the metric to be nonsymmetric as well, or adding higher-order curvature invariants or terms including the Cartan torsion tensor [see Sotiriou (2007b) and Sotiriou and Liberati (2007b) for a more detailed discussion]. However, if one wants to stay within the framework of $f(R)$ gravity, which is our subject here, then there is only one way to cure this problem: to somehow constrain the connection. In fact, it is evident from Eq. (35) that, if the connection were symmetric, projective invariance would be broken. However, one does not have to take such a drastic measure.

To understand this issue further, we should reexamine the meaning of projective invariance. This is very similar to gauge invariance in electromagnetism. It tells us that the corresponding field, in this case the connections $\Gamma^\lambda{}_{\mu\nu}$, can be determined from the field equations up to a projective transformation [Eq. (35)]. This invariance can therefore be broken by fixing some degrees of freedom of the field, similarly to gauge fixing. The number of degrees of freedom that we need to fix is obviously the number of the components of the four-vector used for the transformation, i.e., simply 4. In practice, this means that we should start by assuming that the connection is not the most general that one can construct but satisfies some constraints.

Since the degrees of freedom that we need to fix are 4 and seem to be related to the nonsymmetric part of the connection, the most obvious prescription is to demand that $S_\mu = S_{\sigma\mu}{}^{\sigma}$ be equal to zero, which was first suggested by Sandberg (1975) for a linear action and shown to work also for an $f(\mathcal{R})$ action by Sotiriou and Liberati (2007b).[10] Note that this does not mean that $\Gamma_{\mu\sigma}{}^{\sigma}$ should vanish, but merely that $\Gamma_{\mu\sigma}{}^{\sigma} = \Gamma_{\sigma\mu}{}^{\sigma}$. This constraint can easily be imposed by adding a Lagrange multiplier B^μ. The additional term in the action will be

$$S_{\rm LM} = \int d^4x\sqrt{-g}B^\mu S_\mu. \quad (38)$$

The action (30) with the addition of the term in Eq. (38) is, therefore, the action of the most general metric-affine $f(R)$ theory of gravity.

[10]The proposal of Hehl and Kerling (1978) to fix part of the nonmetricity, namely, the Weyl vector Q_μ, in order to break projective invariance works only when $f(\mathcal{R}) = \mathcal{R}$ (Sotiriou, 2007b; Sotiriou and Liberati, 2007b).

2. Field equations

We are now ready to vary the action and obtain field equations. Due to space limitations, we will not present the various steps of the variation here. Instead we give

$$\delta\mathcal{R}_{\mu\nu} = \bar{\nabla}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \bar{\nabla}_\nu \delta\Gamma^\lambda{}_{\mu\lambda} + 2\Gamma^\sigma{}_{[\nu\lambda]}\delta\Gamma^\lambda{}_{\mu\sigma}, \tag{39}$$

which is useful to those wanting to repeat the variation as an exercise, and we also stress our definitions for the covariant derivative,

$$\bar{\nabla}_\mu A^\nu{}_\sigma = \partial_\mu A^\nu{}_\sigma + \Gamma^\nu{}_{\mu\alpha}A^\alpha{}_\sigma - \Gamma^\alpha{}_{\mu\sigma}A^\nu{}_\alpha, \tag{40}$$

and for the Ricci tensor of an independent connection,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}. \tag{41}$$

The outcome of independent variation with respect to the metric, the connection, and the Lagrange multiplier is, respectively,

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{42}$$

$$\begin{aligned}&\frac{1}{\sqrt{-g}}\{-\bar{\nabla}_\lambda[\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}] + \bar{\nabla}_\sigma[\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}]\delta^\nu{}_\lambda\}\\ &\quad + 2f'(\mathcal{R})(g^{\mu\nu}S_{\lambda\sigma}{}^\sigma - g^{\mu\rho}S_{\rho\sigma}{}^\sigma\delta^\nu{}_\lambda + g^{\mu\sigma}S_{\sigma\lambda}{}^\nu)\\ &\quad = \kappa(\Delta_\lambda{}^{\mu\nu} - B^{[\mu}\delta^{\nu]}{}_\lambda),\end{aligned} \tag{43}$$

$$S_{\mu\sigma}{}^\sigma = 0. \tag{44}$$

Taking the trace of Eq. (43) over the indices μ and λ and using Eq. (44), one obtains

$$B^\mu = \tfrac{2}{3}\Delta_\sigma{}^{\sigma\mu}. \tag{45}$$

Therefore, the final form of the field equations is

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{46}$$

$$\begin{aligned}&\frac{1}{\sqrt{-g}}\{-\bar{\nabla}_\lambda[\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}] + \bar{\nabla}_\sigma[\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}]\delta^\nu{}_\lambda\}\\ &\quad + 2f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^\nu = \kappa\left(\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\Delta_\sigma{}^{\sigma[\nu}\delta^{\mu]}{}_\lambda\right),\end{aligned} \tag{47}$$

$$S_{\mu\sigma}{}^\sigma = 0. \tag{48}$$

Next, we examine the role of $\Delta_\lambda{}^{\mu\nu}$. By splitting Eq. (47) into a symmetric and an antisymmetric part and performing contractions and manipulations, it can be shown that (Sotiriou and Liberati, 2007b)

$$\Delta_\lambda{}^{[\mu\nu]} = 0 \Rightarrow S_{\mu\nu}{}^\lambda = 0. \tag{49}$$

This straightforwardly implies two things: (a) Any torsion is introduced by matter fields for which $\Delta_\lambda{}^{[\mu\nu]}$ is nonvanishing and (b) torsion is not propagating, since it is given algebraically in terms of the matter fields through $\Delta_\lambda{}^{[\mu\nu]}$. It can therefore be detected only in the presence of such matter fields. In the absence of the latter, space-time will have no torsion.

In a similar fashion, one can use the symmetrized version of Eq. (47) to show that the symmetric part of the hypermomentum $\Delta_\lambda{}^{(\mu\nu)}$ is algebraically related to the nonmetricity $Q_{\mu\nu\lambda}$. Therefore, matter fields with nonvanishing $\Delta_\lambda{}^{(\mu\nu)}$ will introduce nonmetricity. However, in this case things are slightly more complicated because part of the nonmetricity is also due to the functional form of the Lagrangian itself [see Sotiriou and Liberati (2007b)].

We will not perform a detailed study of different matter fields and their role in metric-affine gravity. We refer the reader to the analysis of Sotiriou and Liberati (2007b) for details and we restrict ourselves to the following remarks. Obviously, there are certain types of matter fields for which $\Delta_\lambda{}^{\mu\nu}=0$. Characteristic examples are the following:

- A scalar field since in this case the covariant derivative can be replaced with a partial derivative. Therefore, the connection does not enter the matter action.
- The electromagnetic field (and gauge fields in general) since the electromagnetic field tensor $F_{\mu\nu}$ is defined in a covariant manner using the exterior derivative. This definition remains unaffected when torsion is included [this can be related to gauge invariance; see Sotiriou and Liberati (2007b) for a discussion].

On the contrary, particles with spin, such as Dirac fields, generically have a nonvanishing hypermomentum and will, therefore, introduce torsion. A more complicated case is that of a perfect fluid with vanishing vorticity. If we set torsion aside or if we consider a fluid describing particles that would initially not introduce any torsion then, as for a usual perfect fluid in GR, the matter action can be written in terms of three scalars: the energy density, the pressure, and the velocity potential (Schakel, 1996; Stone, 2000). Therefore, such a fluid will lead to a vanishing $\Delta_\lambda{}^{\mu\nu}$. However, complications arise when torsion is taken into account: Even though it can be argued that the spins of the individual particles composing the fluids will be randomly oriented and therefore the expectation value for the spin should add up to zero, fluctuations around this value will affect space-time (Hehl *et al.*, 1976; Sotiriou and Liberati, 2007b). Of course, such effects will be largely suppressed, especially in situations in which the energy density is small, such as late-time cosmology.

It should be evident by now that, due to Eq. (49), the field equations of metric $f(R)$ gravity reduce to Eqs. (15) and (16) and, ultimately, to the field equations of Palatini $f(R)$ gravity [Eqs. (18) and (19)] for all cases in which $\Delta_\lambda{}^{\mu\nu}=0$. Consequently, *in vacuo*, where also $T_{\mu\nu}=0$, they will reduce to the Einstein equations with an effective cosmological constant given by Eq. (29), as discussed in Sec. II.B for Palatini $f(R)$ gravity.

In conclusion, metric-affine $f(R)$ gravity appears to be the most general case of $f(R)$ gravity. It includes enriched phenomenology, such as matter-induced nonmetricity and torsion. It is worth stressing that torsion comes quite naturally, since it is actually introduced by particles with spin (excluding gauge fields). Remarkably, the theory reduces to GR *in vacuo* or for conformally invariant types of matter, such as the electromagnetic field, and departs from GR in the same way that Palatini $f(R)$ gravity does for most matter fields that are usually studied as sources of gravity. However, at the same time, it exhibits new phenomenology in less-studied cases, such as in the presence of Dirac fields, which include torsion and nonmetricity. Finally, we repeat once more that Palatini $f(R)$ gravity, despite appearances, is really a metric theory according to the definition of Will (1981) (and the geometry is *a priori* pseudo-Riemannian).[11] On the contrary, metric-affine $f(R)$ gravity is not a metric theory (hence the name). Consequently, it should also be clear that $T^{\mu\nu}$ is not divergence-free with respect to the covariant derivative defined with the Levi-Civita connection (nor with $\bar{\nabla}_\mu$ actually). However, the physical meaning of this last statement is questionable and deserves further analysis since in metric-affine gravity $T_{\mu\nu}$ does not really carry the usual meaning of a stress-energy tensor (for instance, it does not reduce to the special relativistic tensor at an appropriate limit and at the same time there is also another quantity, the hypermomentum, which describes matter characteristics).

III. EQUIVALENCE WITH BRANS-DICKE THEORY AND CLASSIFICATION OF THEORIES

In the same way that one can make variable redefinitions in classical mechanics in order to bring an equation describing a system to a more attractive, or easy to handle, form (and in a similar way to changing coordinate systems), one can also perform field redefinitions in a field theory in order to rewrite the action or the field equations.

There is no unique prescription for redefining the fields of a theory. One can introduce auxiliary fields, perform renormalizations or conformal transformations, or even simply redefine fields to one's convenience.

It is important to mention that, at least within a classical perspective such as the one followed here, two theories are considered to be dynamically equivalent if, under a suitable redefinition of the gravitational and matter fields, one can make their field equations coincide. The same statement can be made at the level of the action. Dynamically equivalent theories give exactly the same results when describing a dynamical system that falls within the purview of these theories. There are clear advantages in exploring the dynamical equivalence between theories: we can use results already derived for one theory in the study of another, equivalent, theory.

The term "dynamical equivalence" can be considered misleading in classical gravity. Within a classical perspective, a theory is fully described by a set of field equations. When we are referring to gravitation theories, these equations describe the dynamics of gravitating systems. Therefore, two dynamically equivalent theories are actually just different representations of the same theory (which also makes it clear that all allowed representations can be used on an equal footing).

The issue of distinguishing between truly different theories and different representations of the same theory (or dynamically equivalent theories) is an intricate one. It has serious implications and has been the cause of many misconceptions in the past, especially when conformal transformations are used in order to redefine the fields (e.g., the Jordan and Einstein frames in scalar-tensor theory). It goes beyond the scope of this review to present a detailed analysis of this issue. We refer the interested reader to the literature and specifically to Sotiriou *et al.* (2008) and references therein for a detailed discussion. Here we simply mention that, given that they are handled carefully, field redefinitions and different representations of the same theory are perfectly legitimate and constitute useful tools for understanding gravitational theories.

In what follows, we review the equivalence between metric and Palatini $f(R)$ gravity with specific theories within the Brans-Dicke class with a potential. It is shown that these versions of $f(R)$ gravity are nothing but different representations of Brans-Dicke theory with Brans-Dicke parameter $\omega_0=0$ and $-3/2$, respectively. We comment on this equivalence and on whether preference to a specific representation should be an issue. Finally, we use this equivalence to perform a classification of $f(R)$ gravity.

A. Metric formalism

It was noticed quite early that metric quadratic gravity can be cast into the form of a Brans-Dicke theory, and it did not take long for these results to be extended to more general actions that are functions of the Ricci scalar of the metric (Teyssandier and Tourrenc, 1983; Barrow, 1988; Barrow and Cotsakis, 1988; Wands, 1994) [see also Cecotti (1987), Wands (1994), and Flanagan (2003) for the extension to theories of the type $f(R,\Box^k R)$ with $k\geqslant 1$ of interest in supergravity]. This equivalence has been reexamined recently due to the increased interest in metric $f(R)$ gravity (Chiba, 2003; Flanagan, 2004a; Sotiriou, 2006b). We now present this equivalence in some detail.

We work at the level of the action but the same approach can be used to work directly at the level of the field equations. We begin with metric $f(R)$ gravity. For convenience we rewrite here the action (5),

$$S_{\rm met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\psi). \tag{50}$$

One can introduce a new field χ and write the dynamically equivalent action,

[11] As mentioned in Sec. II.B, although the metric postulates are manifestly satisfied, there are ambiguities regarding the physical interpretation of this property and its relation with the Einstein equivalence principle (see Sec. VI.C.1).

$$S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[f(\chi)+f'(\chi)(R-\chi)] + S_M(g_{\mu\nu},\psi). \quad (51)$$

Variation with respect to χ leads to

$$f''(\chi)(R-\chi)=0. \quad (52)$$

Therefore, $\chi=R$ if $f''(\chi)\neq 0$, which reproduces the action (5).[12] Redefining the field χ by $\phi=f'(\chi)$ and setting

$$V(\phi)=\chi(\phi)\phi - f(\chi(\phi)), \quad (53)$$

the action takes the form

$$S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[\phi R - V(\phi)] + S_M(g_{\mu\nu},\psi). \quad (54)$$

This is the Jordan-frame representation of the action of a Brans-Dicke theory with Brans-Dicke parameter $\omega_0 = 0$. An $\omega_0=0$ Brans-Dicke theory [sometimes called "massive dilaton gravity" (Wands, 1994)] was originally proposed by O'Hanlon (1972a) in order to generate a Yukawa term in the Newtonian limit and has occasionally been considered in the literature (Deser, 1970; Anderson, 1971; O'Hanlon, 1972a; O'Hanlon and Tupper, 1972; Fujii, 1982; Barber, 2003; Davidson, 2005; Dabrowski *et al.*, 2007). It should be stressed that the scalar degree of freedom $\phi=f'(\chi)$ is quite different from a matter field; for example, like all nonminimally coupled scalars, it can violate all of the energy conditions (Faraoni, 2004a).

The field equations corresponding to the action (54) are

$$G_{\mu\nu} = \frac{\kappa}{\phi}T_{\mu\nu} - \frac{1}{2\phi}g_{\mu\nu}V(\phi) + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi), \quad (55)$$

$$R = V'(\phi). \quad (56)$$

These field equations could have been derived directly from Eq. (6) using the same field redefinitions that were mentioned above for the action. By taking the trace of Eq. (55) in order to replace R in Eq. (56), one gets

$$3\Box\phi + 2V(\phi) - \phi\frac{dV}{d\phi} = \kappa T. \quad (57)$$

This last equation determines the dynamics of ϕ for given matter sources.

The condition $f''\neq 0$ for the scalar-tensor theory to be equivalent to the original $f(R)$ gravity theory can be seen as the condition that the change of variable $\phi=f'(R)$ needed to express the theory as a Brans-Dicke one [Eq. (54)] be invertible, i.e., $d\phi/dR=f''\neq 0$. This is a sufficient but not necessary condition for invertibility: it is only necessary that $f'(R)$ be continuous and one to one (Olmo, 2007). By looking at Eq. (52), it is seen that $f''\neq 0$ implies $\phi=f'(R)$ and the equivalence of the actions (3) and (51). When f'' is not defined, or it vanishes, the equality $\phi=f'(R)$ and the equivalence between the two theories cannot be guaranteed (although this it is not *a priori* excluded by $f''=0$).

Finally, we mention that, as usual in Brans-Dicke theory and more general scalar-tensor theories, one can perform a conformal transformation and rewrite the action (54) in what is called the Einstein frame (as opposed to the Jordan frame). Specifically, with the conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R)g_{\mu\nu} \equiv \phi g_{\mu\nu} \quad (58)$$

and the scalar field redefinition $\phi=f'(R)\to\tilde{\phi}$ with

$$d\tilde{\phi} = \sqrt{\frac{2\omega_0+3}{2\kappa}}\frac{d\phi}{\phi}, \quad (59)$$

a scalar-tensor theory is mapped into the Einstein frame in which the "new" scalar field $\tilde{\phi}$ couples minimally to the Ricci curvature and has canonical kinetic energy, as described by the gravitational action,

$$S^{(g)} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - U(\tilde{\phi})\right]. \quad (60)$$

For the $\omega_0=0$ equivalent of metric $f(R)$ gravity we have

$$\phi \equiv f'(R) = e^{\sqrt{2\kappa/3}\tilde{\phi}}, \quad (61)$$

$$U(\tilde{\phi}) = \frac{Rf'(R)-f(R)}{2\kappa(f'(R))^2}, \quad (62)$$

where $R=R(\tilde{\phi})$, and the complete action is

$$S'_{met} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - U(\tilde{\phi})\right] + S_M(e^{-\sqrt{2\kappa/3}\tilde{\phi}}\tilde{g}_{\mu\nu},\psi). \quad (63)$$

A direct transformation to the Einstein frame, without the intermediate passage from the Jordan frame, has been discovered by Whitt (1984) and Barrow and Cotsakis (1988).

We stress once more that the actions (5), (54), and (63) are nothing but different representations of the same theory.[13] Additionally, there is nothing exceptional about the Jordan or the Einstein frame of the Brans-Dicke representation, and one can actually find infinitely many conformal frames (Flanagan, 2004a; Sotiriou *et al.*, 2008).

B. Palatini formalism

Palatini $f(R)$ gravity can also be cast in the form of a Brans-Dicke theory with a potential (Flanagan, 2004b;

[12] The action is sometimes called "R regular" by mathematical physicists if $f''(R)\neq 0$ [e.g., Magnano and Sokolowski (1994)].

[13] This has been an issue of debate and confusion [see, for example, Faraoni and Nadeau (2007)].

Olmo, 2005b; Sotiriou, 2006b). As a matter of fact, beginning from the Palatini $f(R)$ action, repeated here for convenience,

$$S_{\rm Pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu},\psi), \tag{64}$$

and following exactly the same steps as before, i.e., introducing a scalar field χ, which we later redefine in terms of ϕ, we obtain

$$S_{\rm Pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[\phi\mathcal{R} - V(\phi)] + S_M(g_{\mu\nu},\psi). \tag{65}$$

Even though the gravitational part of this action is formally the same as that of the action (54) this action is not a Brans-Dicke one with $\omega_0=0$: $\mathcal{R}$ is not the Ricci scalar of the metric $g_{\mu\nu}$. However, we have already seen that the field equation (18) can be solved algebraically for the independent connection yielding Eq. (25). This implies that we can replace the connection in the action without affecting the dynamics of the theory (the independent connection is essentially an auxiliary field). Alternatively, we can directly use Eq. (27), which relates R and $\mathcal{R}$. Therefore, the action (65) can be rewritten, modulo surface terms, as

$$S_{\rm Pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\phi R + \frac{3}{2\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + S_M(g_{\mu\nu},\psi). \tag{66}$$

This is the action of a Brans-Dicke theory with Brans-Dicke parameter $\omega_0=-3/2$. The corresponding field equations obtained from the action (66) through variation with respect to the metric and the scalar are

$$G_{\mu\nu} = \frac{\kappa}{\phi}T_{\mu\nu} - \frac{3}{2\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\right) + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V}{2\phi}g_{\mu\nu}, \tag{67}$$

$$\Box\phi = \frac{\phi}{3}(R - V') + \frac{1}{2\phi}\nabla^\mu\phi\nabla_\mu\phi. \tag{68}$$

Once again, we can use the trace of Eq. (67) in order to eliminate R in Eq. (68) and relate ϕ directly to the matter sources. The outcome is

$$2V - \phi V' = \kappa T. \tag{69}$$

Finally, one can also perform the conformal transformation (58) in order to rewrite the action (66) in the Einstein frame. The result is

$$S'_{\rm Pal} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - U(\phi)\right] + S_M(\phi^{-1}\tilde{g}_{\mu\nu},\psi), \tag{70}$$

where $U(\phi)=V(\phi)/(2\kappa\phi^2)$. Note that here we have not used any redefinition for the scalar.

To conclude, we have established that Palatini $f(R)$ gravity can be cast into the form of an $\omega_0=-3/2$ Brans-Dicke theory with a potential.

C. Classification

The scope of this section is to present a classification of the different versions of $f(R)$ gravity. However, before we do so, some remarks are in order.

First, we use the Brans-Dicke representation of both metric and Palatini $f(R)$ gravities to comment on the dynamics of these theories. This representation makes it transparent that metric $f(R)$ gravity has just one extra scalar degree of freedom with respect to GR. The absence of a kinetic term for the scalar in the action (54) or in Eq. (56) should not mislead us to think that this degree of freedom does not carry dynamics. As can be seen by Eq. (57), ϕ is dynamically related to the matter fields and, therefore, it is a dynamical degree of freedom. Of course, one should also not fail to mention that Eq. (56) does constrain the dynamics of ϕ. In this sense metric $f(R)$ gravity and $\omega_0=0$ Brans-Dicke theory differ from the general Brans-Dicke theories and constitute a special case. On the other hand, in the $\omega_0=-3/2$ case, which corresponds to Palatini $f(R)$ gravity, the scalar ϕ appears to have dynamics in the action (66) or in Eq. (68). However, once again this is misleading since, as is clear from Eq. (69), ϕ is in this case algebraically related to the matter and, therefore, carries no dynamics of its own [indeed the field equations (67) and (69) could be combined to give Eq. (28), eliminating ϕ completely]. As a remark, we state that the equivalence between Palatini $f(R)$ gravity and $\omega_0=-3/2$ Brans-Dicke theory and the clarifications just made highlight two issues already mentioned: the facts that Palatini $f(R)$ gravity is a metric theory according to the definition of Will (1981) and that the independent connection is actually some sort of auxiliary field.

The fact that the dynamics of ϕ are not transparent at the level of the action in both cases should not come as a surprise: ϕ is coupled to the derivatives of the metric (through the coupling with R) and therefore partial integrations to "free" $\delta\phi$ or $\delta g^{\mu\nu}$ during the variation are bound to generate dynamical terms even if they are not initially present in the action. The $\omega_0=-3/2$ case is even more intricate because the dynamical terms generated through this procedure exactly cancel the existing one in the action.

We already saw an example of how different representations of the theory can highlight some of its characteristics and be useful for our understanding of it. The equivalence between $f(R)$ gravity and Brans-Dicke theory will turn out to be useful in the forthcoming sections.

Until now we have not discussed any possible equivalence between Brans-Dicke theory and metric-affine $f(R)$ gravity. However, it is quite straightforward to see that there cannot be any. Metric-affine $f(R)$ gravity is not a metric theory and, consequently, it cannot be cast into the form of one, for instance, Brans-Dicke theory. For the sake of clarity, we state that one could still start from the action (30) and follow the steps of the previous section to bring its gravitational part into the form of the

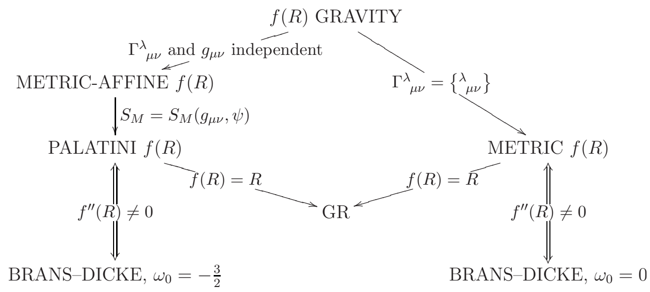

FIG. 1. Classification of $f(R)$ theories of gravity and equivalent Brans-Dicke theories. The flowchart shows the list of assumptions that are needed to arrive at the various versions of $f(R)$ gravity and GR beginning from the the general $f(R)$ action. It also includes the equivalent Brans-Dicke classes. From Sotiriou, 2006b.

action (66). However, the matter action would have an explicit dependence on the connection. Additionally, one would not be able to use Eq. (27) to eliminate $\mathcal{R}$ in favor of R, since this holds only in Palatini $f(R)$ gravity.

In conclusion, metric-affine $f(R)$ gravity is the most general case of $f(R)$ gravity. Imposition of further assumptions can lead to both metric and Palatini $f(R)$ gravity, which can be cast into the form of $\omega_0=0$ and $\omega_0=-3/2$ Brans-Dicke theories with a potential. In both cases, restriction of the functional form of the action leads to GR. These results are summarized in Fig. 1.

D. Why $f(R)$ gravity then?

Since $f(R)$ gravity in both the metric and Palatini formalisms can acquire a Brans-Dicke theory representation, one might be led to ask two questions: first, why should we consider the $f(R)$ representation and not just work with the Brans-Dicke one, and, second, why, since we know a lot about Brans-Dicke theory, should we regard $f(R)$ gravity as unexplored or interesting?

The answer to the first question is quite straightforward. There is actually no reason to prefer either of the two representations—at least as far as classical gravity is concerned. There can be applications where the $f(R)$ representation can be more convenient and applications where the Brans-Dicke representation is more convenient. One should probably mention that habit affects our taste and therefore an $f(R)$ representation seems more appealing to relativists due to its more apparent geometrical nature, whereas the Brans-Dicke representation seems more appealing to particle physicists. This issue can have theoretical implications. To give an example: If $f(R)$ gravity is considered as a step toward a more complicated theory, which generalization would be more straightforward will depend on the chosen representation [see also Sotiriou *et al.* (2008) for a discussion].

Whether $f(R)$ theories of gravity are unexplored and interesting or just an already studied subcase of Brans-Dicke theory is a more practical question that certainly deserves a direct answer. It is indeed true that scalar-tensor theories and, more precisely, Brans-Dicke theory are well-studied theories that have been extensively used in many applications, including cosmology. However, the specific choices $\omega_0=0$, $-3/2$ for the Brans-Dicke parameter are quite exceptional, as mentioned in the previous section. It is also worthwhile pointing out the following: (a) As far as the $\omega_0=0$ case is concerned, one can probably speculate that it is the apparent absence of the kinetic term for the scalar in the action that did not seem appealing and prevented the study of this theory. (b) The $\omega_0=-3/2$ case leads to a conformally invariant theory in the absence of the potential [see Sotiriou (2006b), and references therein], which constituted the initial form of Brans-Dicke theory, and hence it was considered nonviable (a coupling with nonconformally invariant matter is not feasible). However, in the presence of a potential, the theory no longer has this feature. Additionally, most calculations that are done for a general value of ω_0 in the literature actually exclude $\omega_0=-3/2$, mainly because, merely for simplicity, they are done in such a way that the combination $2\omega_0+3$ appears in a denominator (see also Sec. V.A).

In any case, the conclusion is that the theories in the Brans-Dicke class that correspond to metric and Palatini $f(R)$ gravity had not yet been explored before the recent reintroduction of $f(R)$ gravity and, as will also become clear later, several of their special characteristics when compared with more standard Brans-Dicke theories were revealed through studies of $f(R)$ gravity.

IV. COSMOLOGICAL EVOLUTION AND CONSTRAINTS

We now turn our attention to cosmology, which motivated the recent surge of interest in $f(R)$ gravity in order to explain the current cosmic acceleration without the need for dark energy. Before reviewing how $f(R)$ gravity might provide a solution to the more recent cosmological riddles, we stress that the following criteria must be satisfied in order for an $f(R)$ model to be theoretically consistent and compatible with cosmological observations and experiments. The model must have the correct cosmological dynamics, exhibit the correct behavior of gravitational perturbations, and generate cosmological perturbations compatible with the cosmological constraints from the cosmic microwave background, large-scale structure, big bang nucleosynthesis, and gravity waves. These are independent requirements to be studied separately, and they must all be satisfied.

A. Background evolution

In cosmology, the identification of our universe with a Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime is largely based on the high degree of isotropy measured in the cosmic microwave background; this identification relies on a formal result known as the Ehlers-Geren-Sachs (EGS) theorem (Ehlers *et al.*, 1968) which is a kinematical characterization of FLRW spaces stating that, if a congruence of timelike freely falling observers sees an isotropic radiation field, then (assuming that isotropy holds about every spatial point) the space-time is spatially homogeneous and isotropic and therefore a FLRW one. This applies to a universe filled with any perfect fluid that is geodesic and barotropic (Ellis, Matravers, and Treciokas, 1983; Ellis, Treciokas, and Matravers, 1983; Clarkson and Barrett, 1999). Moreover, an "almost-EGS theorem" holds: space-times that are close to satisfying the EGS conditions are close to FLRW universes in an appropriate sense (Stoeger *et al.*, 1995). One would expect that the EGS theorem would be extended to $f(R)$ gravity; indeed, its validity for the (metric) theory

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}] + S_M \qquad (71)$$

was proved by Maartens and Taylor (1994) and Taylor and Maartens (1995) and the generalization to arbitrary metric $f(R)$ gravity was given by Rippl *et al.* (1996). The validity of the EGS theorem can also be seen through the equivalence between $f(R)$ and Brans-Dicke theory: the theorem was extended to scalar-tensor theories by Clarkson *et al.* (2001, 2003). Since metric and Palatini $f(R)$ gravities are equivalent to $\omega=0$ and $-3/2$ Brans-Dicke theories, respectively, it seems that the results of Clarkson *et al.* (2001, 2003) can be considered as straightforward generalizations of the EGS theorem in both versions of $f(R)$ gravity as well. However, in the case of Palatini $f(R)$ gravity there is still some doubt regarding this issue due to complications in averaging (Flanagan, 2004b).

1. Metric $f(R)$ gravity

From the discussion above, it is valid to use the FLRW line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \qquad (72)$$

as a local description of space-time at cosmological scales, where (t,r,θ,ϕ) are comoving coordinates. We remind the reader that $k=-1,0,1$ according to whether the universe is hyperspherical, spatially flat, or hyperbolic and that $a(t)$ is called the scale factor. Part of the standard approach, which we follow here as well, is to use a perfect fluid description for matter with stress-energy tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}, \qquad (73)$$

where u^μ denotes the four-velocity of an observer comoving with the fluid and ρ and P are the energy density and pressure of the fluid, respectively.

Note that the value of k is an external parameter. As in other works in the literature, in what follows we choose $k=0$, i.e., we focus on a spatially flat universe. This choice in made in order to simplify the equations and should be viewed skeptically. It is sometimes claimed that such a choice is favored by the data. However, this is not entirely correct. Even though the data [e.g., Spergel *et al.* (2007)] indicate that the current value of Ω_k is very close to zero, it should be stressed that this does not really reveal the value of k itself. Since

$$\Omega_k = -\frac{k}{a^2H^2}, \qquad (74)$$

the current value of Ω_k is sensitive to the current value of $a(t)$, i.e., to the amount of expansion the universe has undergone after the big bang. A significant amount of expansion can easily drive Ω_k very close to zero. The success of the inflationary paradigm is exactly that it explains the flatness problem—how the universe became so flat—in a dynamical way, allowing us to avoid fine tuning the parameter k (the value $k=0$ is statistically exceptional).

The above having been said, the choice of $k=0$ for simplicity is not a dramatic departure from generality when it comes to late-time cosmology. If it is viewed as an approximation and not as a choice of an initial condition, then one can say that, since Ω_k as inferred from observations is very close to zero at current times, the terms related to k will be subdominant in the Friedmann or generalized Friedmann equations and therefore one could choose to discard them by setting $k=0$, without great loss of accuracy. In any case, results derived under the assumption that $k=0$ should be considered preliminary until the influence of the spatial curvature is precisely determined, since there are indications that even a very small value of Ω_k may have an effect on them [see, for instance, Clarkson *et al.* (2007)].

Returning to our discussion, inserting the flat FLRW metric in the field equations (6) and assuming that the stress-energy tensor is that of Eq. (73) yield

$$H^2 = \frac{\kappa}{3f'}\left[\rho + \frac{Rf'-f}{2} - 3H\dot{R}f''\right], \qquad (75)$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left[P + (\dot{R})^2f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right]. \qquad (76)$$

With some hindsight, we assume that $f'>0$ in order to have a positive effective gravitational coupling and $f''>0$ to avoid the Dolgov-Kawasaki instability (Dolgov and Kawasaki, 2003a; Faraoni, 2006a) discussed in Sec. V.B.

A significant part of the motivation for $f(R)$ gravity is that it can lead to accelerated expansion without the need for dark energy (or an inflaton field). An easy way to see this is to define an effective energy density and pressure of the geometry as

$$\rho_{\rm eff} = \frac{Rf'-f}{2f'} - \frac{3H\dot{R}f''}{f'}, \qquad (77)$$

$$P_{\rm eff}=\frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f-Rf')}{f'}, \tag{78}$$

where $\rho_{\rm eff}$ has to be non-negative in a spatially flat FLRW space-time, as follows from the inspection of Eq. (75) in the limit $\rho \to 0$. Then, *in vacuo*, Eqs. (75) and (76) can take the form of the standard Friedmann equation,

$$H^2=\frac{\kappa}{3}\rho_{\rm eff}, \tag{79}$$

$$\frac{\ddot{a}}{a}=-\frac{\kappa}{6}[\rho_{\rm eff}+3P_{\rm eff}]. \tag{80}$$

Hence, *in vacuo* the curvature correction can be viewed as an effective fluid.[14]

The effective equation of state parameter $w_{\rm eff}$ of modified gravity can be expressed as

$$w_{\rm eff}\equiv\frac{P_{\rm eff}}{\rho_{\rm eff}}=\frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f-Rf')}{(Rf'-f)/2-3H\dot{R}f''}. \tag{81}$$

Since the denominator on the right-hand side of Eq. (81) is strictly positive, the sign of $w_{\rm eff}$ is determined by its numerator. In general, for a metric $f(R)$ model to mimic the de Sitter equation of state $w_{\rm eff}=-1$, it must be

$$\frac{f'''}{f''}=\frac{\dot{R}H-\ddot{R}}{(\dot{R})^2}. \tag{82}$$

We also give two simple examples that can be found in the literature for demonstrative purposes and without considering their viability. First, one can consider the function f to be of the form $f(R)\propto R^n$. It is quite straightforward to calculate $w_{\rm eff}$ as a function of n if the scale factor is assumed to be a generic power law $a(t)=a_0(t/t_0)^\alpha$ [a general $a(t)$ would lead to a time-dependent $w_{\rm eff}$] (Capozziello *et al.*, 2003). The result is

$$w_{\rm eff}=-\frac{6n^2-7n-1}{6n^2-9n+3} \tag{83}$$

for $n\neq 1$, and α is given is terms of n as

$$\alpha=\frac{-2n^2+3n-1}{n-2}. \tag{84}$$

A suitable choice of n can lead to a desired value for $w_{\rm eff}$. For instance, $n=2$ yields $w_{\rm eff}=-1$ and $\alpha=\infty$, as expected, considering that quadratic corrections to the Einstein-Hilbert Lagrangian were used in the well-known Starobinsky inflation (Starobinsky, 1980).

The second example we refer to is a model of the form $f(R)=R-\mu^{2(n+1)}/R^n$, where μ is a suitably chosen parameter (Carroll *et al.*, 2004). In this case, and once again if the scale factor is assumed to be a generic power law, $w_{\rm eff}$ can again be written as a function of n (Carroll *et al.*, 2004),

$$w_{\rm eff}=-1+\frac{2(n+2)}{3(2n+1)(n+1)}. \tag{85}$$

The most typical model within this class is that with $n=1$ (Carroll *et al.*, 2004), in which case $w_{\rm eff}=-2/3$. Note that, in this class of models, a positive n implies the presence of a term inversely proportional to R in the action, contrary to the situation for the R^n models.

In terms of the quantity $\phi(R)\equiv f'(R)$ one can rewrite Eq. (81) as

$$w_{\rm eff}=-1+2\frac{(\ddot{\phi}-H\dot{\phi})}{R\phi-f-6H\dot{\phi}}=-1+\frac{\kappa(\ddot{\phi}-H\dot{\phi})}{3\phi H^2} \tag{86}$$

and

$$\rho_{\rm eff}+P_{\rm eff}=\frac{\ddot{\phi}-H\dot{\phi}}{\phi}=\frac{\dot{\phi}}{\phi}\frac{d}{dt}\left[\ln\left(\frac{\dot{\phi}}{a}\right)\right]. \tag{87}$$

An exact de Sitter solution corresponds to $\dot{\phi}=f''(R)\dot{R}=0$ or to $\dot{\phi}=Ca(t)=Ca_0e^{H_0t}$, where $C\neq 0$ is an integration constant. However, the second solution for $\phi(t)$ is not acceptable because it leads to $f''(R)\dot{R}=Ca_0e^{H_0t}$, which is not correct because the left-hand side is time independent (for a de Sitter solution), while the right-hand side depends on time.

One could impose energy conditions for the effective stress-energy tensor (12) of $f(R)$ gravity. However, this is not very meaningful from the physical point of view since it is well known that effective stress-energy terms originating from the geometry by rewriting the field equations of alternative gravities as effective Einstein equations do, in general, violate all the energy conditions [e.g., Faraoni (2004a)].[15] Also, the concept of gravitational energy density is, anyway, ill defined in GR and in all metric theories of gravity as a consequence of the equivalence principle. Moreover, the violation of the energy conditions makes it possible to have $\dot{H}>0$ and bouncing universes (Carloni *et al.*, 2006; Novello and Bergliaffa, 2008).

The field equations are clearly of fourth order in $a(t)$. When matter is absent [a situation of interest in early-time inflation or in a very late universe completely dominated by $f(R)$ corrections], $a(t)$ appears only in the combination $H\equiv\dot{a}/a$. Since the Hubble parameter H is a cosmological observable, it is convenient to adopt it as

[14] Note the following subtlety though: if we had included matter it would enter the Friedmann equations with a modified coupling κ/f'. In general this effective fluid representation is used only for demonstrative purposes and should not be overestimated or misinterpreted.

[15] Santos *et al.* (2007) derived the null-energy condition and the strong-energy condition for metric $f(R)$ gravity using the Raychaudhuri equation and imposing that gravity be attractive, whereas for the weak-energy condition and the dominant-energy condition an effective stress-energy tensor that includes the matter was used. Perez Bergliaffa (2006) followed a different approach in which the standard energy conditions on matter were used in an attempt to constrain $f(R)$ gravity.

the (only) dynamical variable; then the field equations (75) and (76) are of third order in H. This elimination of a is not possible when $k \neq 0$ or when a fluid with density $\rho = \rho(a)$ is included in the picture.

Regarding the dynamical field content of the theory, the fact that quadratic corrections to the Einstein-Hilbert action introduce a massive scalar field was noted by Utiyama and DeWitt (1962), Stelle (1977, 1978), Strominger (1984), Buchbinder *et al.* (1992), and Vilkovisky (1992); this applies to any $f(R)$ gravity theory in the metric formalism; [see, e.g., Ferraris *et al.* (1988), Hindawi *et al.* (1996), and Olmo (2007)]. The metric tensor contains, in principle, various degrees of freedom: spin-2 modes and vector and scalar modes, which can all be massless or massive. In GR we find only the massless graviton but, when the action is allowed to include terms that depend on R, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, other modes show up. In $f(R)$ gravity, a massive scalar mode appears, which is evident in the equivalence with scalar-tensor theory (see Sec. III). As discussed in Sec. III.C, the scalar field $\phi = R$ is dynamical in the metric formalism and nondynamical in the Palatini formalism.

2. Palatini $f(R)$ gravity

As mentioned there has been some concern as to whether the homogeneity approximation can justify the use of the FLRW metric as a cosmological solution in Palatini $f(R)$ gravity (Flanagan, 2004b) [see also Li *et al.* (2009)]. Therefore, even though it is standard practice in the literature to assume a FLRW background and a perfect fluid description for matter when studying cosmology in Palatini $f(R)$ gravity [e.g., Vollick (2003), Meng and Wang (2004a, 2004c, 2005), Allemandi *et al.* (2004), Allemandi, Borowiec, *et al.* (2005), Amarzguioui *et al.* (2006), and Sotiriou (2006a, 2006e)], and we are going to review this approach here, the reader should approach it with some reasonable skepticism until this issue is clarified further.

Under the assumptions that the space-time is indeed described at cosmological scales by the FLRW metric [Eq. (72)], that the stress-energy tensor of matter is that of Eq. (73), and that $k=0$, easy manipulations reveal that the field Eqs. (18) and (19) yield the following modified Friedmann equation [see, for instance, Meng and Wang (2004a) and Sotiriou (2006e)]:

$$\left(H + \frac{1}{2}\frac{\dot{f}'}{f'}\right)^2 = \frac{1}{6}\frac{\kappa(\rho+3P)}{f'} + \frac{1}{6}\frac{f}{f'}, \tag{88}$$

where the overdot denotes differentiation with respect to coordinate time. Note that when f is linear, $f'=1$ and, therefore, $\dot{f}'=0$. Taking into account Eq. (20), one can easily show that in this case Eq. (88) reduces to the standard Friedmann equation.

We avoid representing the extra terms in Eq. (88) with respect to the standard Friedmann equation as an effective stress-energy density and pressure since, as it is not very difficult to see, the former equation does not carry more dynamics than the latter. Indeed, assume as usual that the cosmological fluid is composed of pressureless dust ($P_m=0$) and radiation ($P_r=\rho_r/3$), and $\rho=\rho_m+\rho_p$ and $P=P_m+P_r$, where ρ_m, ρ_p and P_m, P_r denote the energy density and the pressure of dust and radiation, respectively. Due to Eq. (20) and the fact that for radiation $T=0$, it is quite straightforward to derive an algebraic relation between $\mathcal{R}$ and the energy density of the dust. Combining this with energy conservation, one obtains (Sotiriou, 2006e)

$$\dot{\mathcal{R}} = -\frac{3H(\mathcal{R}f' - 2f)}{\mathcal{R}f'' - f'}. \tag{89}$$

This equation can be used to replace $\dot{\mathcal{R}}$ in Eq. (88), yielding

$$H^2 = \frac{1}{6f'}\frac{2\kappa\rho + \mathcal{R}f' - f}{\left[1 - \frac{3}{2}f''(\mathcal{R}f' - 2f)/f'(\mathcal{R}f'' - f')\right]^2}. \tag{90}$$

Considering now that, due to Eq. (20), $\mathcal{R}$ is just an algebraic function of ρ_m, it is easy to realize that Eq. (90) is actually just the usual Friedmann equation with a modified source. The functional form of f will determine how the dynamics will be affected by this modification.

It seems, therefore, quite intuitive that by tampering with the function f one can affect the cosmological dynamics in a prescribed way. Indeed, it has been shown that for $f(\mathcal{R}) = \mathcal{R} - \alpha^2/3\mathcal{R}$ one approaches a de Sitter expansion as the density goes to zero (Vollick, 2003). In order to match observations of the expansion history, one needs to choose $\alpha \sim 10^{-67}$ $\mathrm{eV}^2 \sim 10^{-53}$ m^{-2}. Additionally, in regimes for which $\kappa\rho \gg \alpha$, Eq. (90) reduces with high precision to the standard Friedmann equation. The above can easily be verified by replacing this particular choice of f in Eq. (90).

One could, of course, consider more general functions of $\mathcal{R}$. It would be of particular interest to have positive powers of $\mathcal{R}$ higher than the first power added in the action (since one could think of the Lagrangian as a series expansion). Indeed this has been considered (Meng and Wang, 2004a, 2004c, 2005; Sotiriou, 2006a, 2006e). However, it can be shown that such terms do not really lead to interesting phenomenology as in metric $f(R)$ gravity: for instance, they cannot drive inflation as here there are no extra dynamics and inflation cannot end gracefully (Meng and Wang, 2004c; Sotiriou, 2006a), unlike in the scenario proposed by Starobinsky (1980) in the metric formalism. As a matter of fact, it is more likely that positive powers of $\mathcal{R}$ will lead to no interesting cosmological phenomenology unless their coefficients are large enough to make the models nonviable (Sotiriou, 2006a).

B. Cosmological eras

As stated in the Introduction, the recent flurry of theoretical activity on $f(R)$ models derives from the need to explain the present acceleration of the universe discovered with supernovae of type Ia (Filippenko and Riess, 1998; Perlmutter *et al.*, 1998; Riess *et al.*, 1998, 1999, 2004; Schmidt *et al.*, 1998; Knop *et al.*, 2003; Tonry *et al.*, 2003; Borris *et al.*, 2004; Astier *et al.*, 2006). We have seen in the previous section how $f(R)$ gravity can achieve cosmic acceleration and an effective equation of state parameter $w_{\rm eff} \sim -1$; on the other hand, it was already known from R^2-inflationary scenarios of the early universe that this is possible, so we are actually witnessing a resurrection of this theoretical possibility in models of the late universe—this parallels the use of scalar fields to drive early inflation or late-time acceleration in quintessence models. There are also attempts to unify early inflation and late time acceleration in modified gravity (Nojiri and Odintsov, 2007d, 2007e, 2008a, 2008b, 2008c; Bamba and Odintsov, 2008). However, any model attempting to explain the cosmic speedup at late times should not spoil the successes of the standard cosmological model, which requires a definite sequence of eras to follow each other, including (1) early inflation, (2) a radiation era during which big bang nucleosynthesis occurs, (3) a matter era, (4) the present accelerated epoch, and (5) a future era.

Big bang nucleosynthesis is well constrained—see Kneller and Steigman (2004), Clifton and Barrow (2005a), Brookfield *et al.* (2006), Lambiase and Scarpetta (2006), Nakamura *et al.* (2006), and Evans *et al.* (2008) for such constraints on $f(R)$ models. The matter era must last long enough to allow the primordial density perturbations generated during inflation to grow and become the structures observed in the universe today. The future era is usually found to be a de Sitter attractor solution or to be truncated at a finite time by a big rip singularity.

Furthermore, there must be smooth transitions between consecutive eras, which may not happen in all $f(R)$ models. In particular, the exit from the radiation era has been studied and claimed to originate problems for many forms of $f(R)$ in the metric formalism, including $f=R-\mu^{2(n+1)}/R^n$, $n>0$ (Brookfield *et al.*, 2006; Capozziello, Nojiri, Odintsov, and Troisi, 2006; Nojiri and Odintsov, 2006; Amendola *et al.* 2007; Amendola, Polarski, and Tsujikawa, 2007a) [but not in the Palatini formalism (Fay *et al.*, 2007; Carvalho *et al.*, 2008)]. However, the usual model $f(R)=R-\mu^4/R$ with "bad" behavior was studied using singular perturbation methods (Evans *et al.*, 2008) and a sufficiently long matter era was definitely found.

Moreover, one can always find choices of the function $f(R)$ with the correct cosmological dynamics in the following way: one can prescribe the desired form of the scale factor $a(t)$ and integrate a differential equation for $f(R)$ that produces the desired scale factor (Capozziello, Cardone, and Troisi, 2005; Capozziello, Nojiri, Odintsov, and Troisi, 2006; de la Cruz-Dombriz and Dobado, 2006; Multamaki and Vilja, 2006a; Nojiri and Odintsov, 2006, 2007b, 2007c; Faulkner *et al.*, 2007; Fay, Nesseris, and Perivolaropoulos, 2007; Fay, Tavakol, and Tsujikawa, 2007; Hu and Sawicki, 2007a, 2007b; Song *et al.*, 2007). In general, this "designer $f(R)$ gravity" produces forms of the function $f(R)$ that are rather contrived. Moreover, the prescribed evolution of the scale factor $a(t)$ does not determine uniquely the form of $f(R)$ but, at best, only a class of $f(R)$ models (Multamaki and Vilja, 2006a; Sokolowski, 2007b, 2007a; Starobinsky, 2007). Therefore, the observational data providing information on the history of $a(t)$ are not sufficient to reconstruct $f(R)$: one needs additional information, which may come from cosmological density perturbations. There remains a caveat on being careful to terminate the radiation era and allowing a matter era that is sufficiently long for scalar perturbations to grow.

While sometimes it is possible to find exact solutions to the cosmological equations, the general behavior of the solutions can only be assessed with a phase space analysis, which constitutes a powerful tool in cosmology (Wainwright and Ellis, 1997; Coley, 2003). In a spatially flat FLRW universe the dynamical variable is the Hubble parameter H, and a convenient choice of phase space variables in this case is (H,R). Then, for any form of the function $f(R)$, the phase space is a two-dimensional curved manifold embedded in the three-dimensional space $(H,R,\dot{R})$ with de Sitter spaces as fixed points (de Souza and Faraoni, 2007); the structure of the phase space is simplified with respect to that of general scalar-tensor cosmology (Faraoni, 2005b).

Studies of the phase space of $f(R)$ cosmology (not limited to the spatially flat FLRW case) were common in the pre-1998 literature on R^2 inflation (Starobinsky, 1980; Muller *et al.*, 1990; Amendola *et al.*, 1992; Capozziello *et al.*, 1993). The presence or absence of chaos in metric $f(R)$ gravity was studied by Barrow and Cotsakis (1989, 1991). Such studies with dynamical system methods have become widespread with the recent surge of interest in $f(R)$ gravity to explain the present cosmic acceleration. Of course, detailed phase space analyses are possible only for specific choices of the function $f(R)$ (Easson, 2004; Nojiri and Odintsov, 2004b; Carloni *et al.*, 2005, 2009; Carroll *et al.*, 2005; Clifton and Barrow, 2005b; Sami *et al.*, 2005; Clifton, 2006a, 2007; Leach *et al.*, 2006, 2007; Amendola, Polarski, and Tsujikawa, 2007a, 2007b; Amendola *et al.*, 2007; Carloni and Dunsby, 2007; Fay, Nesseris, and Perivolaropoulos, 2007; Fay, Tavakol, and Tsujikawa, 2007; Goheer *et al.*, 2007, 2008; Li and Barrow, 2007; Abdelwahab *et al.*, 2008; Amendola and Tsujikawa, 2008; Carloni, Capozziello, *et al.*, 2008).

C. Dynamics of cosmological perturbations

It is not sufficient to obtain the correct dynamics of the background cosmological model for the theory to be viable: in fact, the FLRW metric can be obtained as a solution of the field equations of most gravitation theories, and it is practically impossible to discriminate between $f(R)$ gravity and dark energy theories [or between different $f(R)$ models] using only the unperturbed FLRW cosmological model, i.e., using only probes that are sensitive to the expansion history of the universe. By contrast, the growth of cosmological perturbations is sensitive to the theory of gravity adopted and constitutes a possible avenue to discriminate between dark energy and modified gravity. A change in the theory of gravity affects the dynamics of cosmological perturbations and, among other things, the imprints that these leave in the cosmic microwave background (which currently provide the most sensitive cosmological probe) and in galaxy surveys (White and Kochanek, 2001; Sealfon *et al.*, 2005; Shirata *et al.*, 2005, 2007; Knox *et al.*, 2006; Koivisto, 2006b; Koyama and Maartens, 2006; Li and Chu, 2006; Skordis *et al.*, 2006; Stabenau and Jain, 2006; Li and Barrow, 2007; Song *et al.*, 2007; Tsujikawa, 2007; Zhang *et al.*, 2007). This is the origin of various efforts to constrain $f(R)$ gravity with cosmic microwave background data (Appleby and Battye, 2007; Hu and Sawicki, 2007b; Li and Barrow, 2007; Li, Barrow, and Mota, 2007; Starobinsky, 2007; Amendola and Tsujikawa, 2008; Carloni, Dunsby, and Troisi, 2009; Pogosian and Silvestri, 2008; Tsujikawa, 2008; Tsujikawa *et al.*, 2008; Wei and Zhang, 2008).

Most of these works are restricted to specific choices of the function $f(R)$, but a few general results have also been obtained. The growth and evolution of local scalar perturbations, which depend on the theory of gravity employed, were studied in metric $f(R)$ gravity theories which reproduce GR at high curvatures in various papers (Carroll *et al.*, 2006; Song *et al.*, 2007; de la Cruz-Dombriz *et al.*, 2008) by assuming a scale factor evolution typical of a ΛCDM model. Vector and tensor modes are unaffected by $f(R)$ corrections. It is found that $f''(R)>0$ is required for the stability of scalar perturbations (Song *et al.*, 2007), which matches the analysis of Sec. V.B.2 in a locally de Sitter background. The corrections to the Einstein-Hilbert action produce qualitative differences with respect to Einstein gravity: they lower the large-angle anisotropy of the cosmic microwave background and may help explain the observed low quadrupole and they produce different correlations between the cosmic microwave background and galaxy surveys (Song *et al.*, 2007). Further studies challenge the viability of $f(R)$ gravity in comparison with the ΛCDM model: Bean *et al.* (2007) found that large-scale density fluctuations are suppressed in comparison to small scales by an amount incompatible with the observational data. This makes it impossible to fit simultaneously large-scale data from the cosmic microwave background and small-scale data from galaxy surveys. Also, a quasistatic approximation used in a previous analysis (Zhang, 2007) is found to be invalid.

de la Cruz-Dombriz *et al.* (2008) studied the growth of matter density perturbations in the longitudinal gauge using a fourth-order equation for the density contrast $\delta\rho/\rho$, which reduces to a second-order one for subhorizon modes. The quasistatic approximation, which does not hold for general forms of the function $f(R)$, is, however, found to be valid for those forms of this function that describe successfully the present cosmic acceleration and pass the Solar System tests in the weak-field limit. It is interesting that the relation between the gravitational potentials in the metric, which are responsible for gravitational lensing, and the matter overdensities depends on the theory of gravity; a study of this relation in $f(R)$ gravity (as well as in other gravitational theories) is given by Zhang *et al.* (2007).

Cosmological density perturbations in the Palatini formalism have been studied by Amarzguioui *et al.* (2006), Carroll *et al.* (2006), Koivisto (2006b, 2007), Koivisto and Kurki-Suonio (2006), Lee (2007, 2008), Li, Chan, and Chu (2007), and Uddin *et al.* (2007). Two different formalisms developed by Hwang and Noh (2002), Lue *et al.* (2004), and Koivisto and Kurki-Suonio (2006), were compared for the model $f(R)=R-\mu^{2(n+1)}/R^n$, and it was found that the two models agree for scenarios that are "close" (in parameter space) to the standard concordance model but give different results for models that differ significantly from the ΛCDM model. Although this is not something to worry about in practice (all models aiming at explaining the observational data are close to the standard concordance model), it signals the need to test the validity of perturbation analyses for theories that do differ significantly from GR in some aspects.

V. OTHER STANDARD VIABILITY CRITERIA

In addition to having the correct cosmological dynamics and the correct evolution of cosmological perturbations, the following criteria must be satisfied in order for an $f(R)$ model to be theoretically consistent and compatible with experiment. The model must have the correct weak-field limit at both the Newtonian and post-Newtonian levels, i.e., one that is compatible with the available Solar System experiments; be stable at the classical and semiclassical levels (the checks performed include the study of a matter instability, of gravitational instabilities for de Sitter space, and of a semiclassical instability with respect to black hole nucleation); not contain ghost fields; and admit a well-posed Cauchy problem. These independent requirements are discussed separately in the following.

A. Weak-field limit

It is obvious that a viable theory of gravity must have the correct Newtonian and post-Newtonian limits. Indeed, since the modified gravity theories of current interest are explicitly designed to fit the cosmological observations, Solar System tests are more stringent than the cosmological ones and constitute a real test bed for these theories.

1. The scalar degree of freedom

It is clear from the equivalence between $f(R)$ and Brans-Dicke gravities discussed in Sec. III that the former contains a massive scalar field ϕ [see Eqs. (54) and (66)]. While in the metric formalism this scalar is dynamical and represents a genuine degree of freedom, it is nondynamical in the Palatini case. We therefore consider the role of the scalar field in the metric formalism as it will turn out to be crucial for the weak-field limit. Using the notations of Sec. III.A, the action is given by Eq. (54) and the corresponding field equations by Eq. (55).

Equation (52) for χ has no dynamical content because it only enforces the equality $\chi=R$. However, $\chi=R$ is indeed a dynamical field that satisfies the wave equation,

$$3f''(\chi)\Box\chi + 3f'''(\chi)\nabla_\alpha\chi\nabla^\alpha\chi + \chi f'(\chi) - 2f(\chi) = \kappa T. \quad (91)$$

When $f''\neq 0$ a new effective potential $W(\chi)\neq V(\chi)$ can be introduced, such that

$$\frac{dW}{d\chi} = \frac{\kappa T - \chi f'(\chi) + 2f(\chi)}{3f''(\chi)}. \quad (92)$$

The action can be seen as a Brans-Dicke action with $\omega_0=0$ if the field $\phi\equiv f'(\chi)=f'(R)$ is used instead of χ as the independent Brans-Dicke field,

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[\phi R - V(\phi)] + S^{(m)}, \quad (93)$$

where $V(\phi)$ is given by Eq. (53).

Now one may think of studying the dynamics and stability of the model by looking at the shape and extrema of the effective potential $V(\chi)$ but this would be misleading because the dynamics of χ are not regulated by $V(\chi)$ [indeed, the wave equation (91) does not contain V] but are subject to the strong constraint $\chi=R$, and R [or $f'(R)$] is ruled by the trace equation (8).

The following example shows how the use of the potential $V(\chi)$ can be misleading. As is well known, the effective mass of a scalar field (corresponding to the second derivative of the potential evaluated at the minimum) controls the range of the force mediated by this field. Thus, when studying the weak-field limit of the theory it is important to know the range of the dynamical scalar field $\chi=R$ present in the metric formalism in addition to the metric field $g_{\mu\nu}$, as this field can potentially violate the post-Newtonian constraints obtained from Solar System experiments if the scalar field gives observable effects at the relevant scales. One way to avoid Solar System constraints, however, is to have χ have a sufficiently short range (see Sec. V.A.2 for more details). Consider the example $f(R)=R+aR^2$, with a a positive constant. By naively taking the potential, one obtains

$$V(\chi) = a\chi^2 \equiv \frac{m_1^2}{2}\chi^2 \quad (94)$$

with effective mass squared, $m_1^2=2a$. Then, the small values of a generated by quantum corrections to GR imply a small mass m_1, and a long-range field χ might be detectable at Solar System scales (Jin *et al.*, 2006; Chiba *et al.*, 2007; Olmo, 2007). However, this conclusion is incorrect because m_1 is not the physical mass of χ. The true effective mass is obtained from the trace equation (8) ruling the evolution of R which, for $f(R)=R+aR^2$, reduces to

$$\Box R - \frac{R}{6a} = \frac{\kappa T}{6a}, \quad (95)$$

and the identification of the mass squared of $\chi=R$ as

$$m^2 = \frac{1}{6a} \quad (96)$$

is straightforward.[16] A small enough value of a now leads to a large value of m and a short range for[17] χ. The situation is, however, more complicated; the chameleon effect due to the dependence of the effective mass on the curvature may change the range of the scalar (Faulkner *et al.*, 2007; Starobinsky, 2007).

For a general $f(R)$ model, the effective mass squared of $\chi=R$ is obtained in the weak-field limit by considering a small, spherically symmetric, perturbation of de Sitter space with constant curvature R_0. One finds

$$m^2 = \frac{1}{3}\left(\frac{f_0'}{f_0''} - R_0\right). \quad (97)$$

This equation coincides with Eq. (6) of Muller *et al.* (1990), with Eq. (26) of Olmo (2007), and with Eq. (17) of Navarro and Van Acoleyen (2007). It also appears in a calculation of the propagator for $f(R)$ gravity in a locally flat background [Eq. (8) of Nunez and Solganik (2004)]. The same expression is recovered in a gauge-invariant stability analysis of de Sitter space (Faraoni and Nadeau, 2005) reported in Sec. V.B.2 below.

Another possibility is to consider the field $\phi\equiv f'(R)$ instead of $\chi=R$ and to define the effective mass of ϕ using the *Einstein-frame* scalar-tensor analog of $f(R)$ gravity instead of its Jordan-frame cousin already discussed (Chiba, 2003). By the conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R)g_{\mu\nu} \equiv \phi g_{\mu\nu} \quad (98)$$

and the scalar field redefinition $\phi=f'(R)\to\tilde{\phi}$ with

$$d\tilde{\phi} = \sqrt{\frac{2\omega_0+3}{2\kappa}}\frac{d\phi}{\phi}, \quad (99)$$

a scalar-tensor theory is mapped to the Einstein frame in which the new scalar field $\tilde{\phi}$ couples minimally to the Ricci curvature and has canonical kinetic energy, as described by the action

$$S^{(g)} = \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - U(\tilde{\phi})\right] + S_M(e^{-\sqrt{2\kappa/3}\tilde{\phi}}\tilde{g}_{\mu\nu},\psi) \quad (100)$$

[16]It was already noted by Stelle (1978) that an R^2 correction to the Einstein-Hilbert Lagrangian generates a Yukawa correction to the Newtonian potential—this has to be kept small at macroscopic scales by giving it a short range.

[17]The deflection of light by the Sun in GR plus quadratic corrections was studied by calculating the Feynman amplitudes for photon scattering, and it was found that, to linearized order, this deflection is the same as in GR (Accioly *et al.*, 1999).

(note once more the nonminimal coupling of the matter in the Einstein frame). For the $\omega_0=0$ equivalent of metric $f(R)$ gravity we have

$$\phi \equiv f'(R) = e^{\sqrt{2\kappa/3}\tilde{\phi}}, \tag{101}$$

$$U(\tilde{\phi}) = \frac{Rf'(R) - f(R)}{2\kappa(f'(R))^2}, \tag{102}$$

where $R=R(\tilde{\phi})$. By use of $d\tilde{\phi}/d\phi=\sqrt{3/2\kappa}f''/f'$, the effective mass of $\tilde{\phi}$ is defined by

$$\tilde{m}_{\text{eff}}^2 \equiv \frac{d^2U}{d\tilde{\phi}^2} = \frac{1}{3}\left[\frac{1}{f''} + \frac{\tilde{\phi}}{f'} - \frac{4f}{(f')^2}\right] \tag{103}$$

[this equation appears in the footnote of Chiba (2003)]. By assuming a de Sitter background with constant curvature $R_0=12H_0^2=f_0/6f_0'$, this turns into

$$\tilde{m}_{\text{eff}}^2 = \frac{1}{3f_0'}\left(\frac{f_0'}{f_0''} - R_0\right) = \frac{m_{\text{eff}}^2}{f_0'}. \tag{104}$$

In the Einstein frame, it is not the mass $\tilde{m}$ of a particle or a field that is measurable but rather the ratio $\tilde{m}/\tilde{m}_u$ between $\tilde{m}$ and the Einstein-frame unit of mass $\tilde{m}_u$, which is varying, scaling as $\tilde{m}_u=[f'(R)]^{-1/2}m_u=\phi^{-1/2}m_u$, where m_u is the constant unit of mass in the Jordan frame (Dicke, 1962; Faraoni *et al.*, 1999; Faraoni and Nardone, 2007). Therefore,

$$\frac{\tilde{m}_{\text{eff}}^2}{\tilde{m}_u^2} = \frac{m_{\text{eff}}^2}{m_u^2}. \tag{105}$$

In practice, $\phi\equiv f'(R)$ is dimensionless and its value must be of order unity in order to obtain the gravitational coupling strength measured in the Solar System; as a result, the Einstein-frame metric $\tilde{g}_{\mu\nu}$ and the Jordan-frame metric $g_{\mu\nu}$ are almost equal, and the same applies to $\tilde{m}_u, m_u$ and to $\tilde{m}_{\text{eff}}, m_{\text{eff}}$, respectively. Then, the only relevant difference between the Einstein and Jordan frames is the scalar field redefinition $\phi \to \tilde{\phi}$.

2. Weak-field limit in the metric formalism

Having discussed the field content of the theory, we are now ready to discuss the weak-field limit. Having the correct weak-field limit at the Newtonian and post-Newtonian levels is essential for theoretical viability.

From the beginning, works on the weak-field (Newtonian and post-Newtonian) limit of $f(R)$ gravity led to opposite results appearing in the literature (Accioly *et al.*, 1999; Rajaraman, 2003; Dick, 2004; Easson, 2004; Soussa and Woodard, 2004; Capozziello and Troisi, 2005; Clifton and Barrow, 2005a, 2006; Navarro and Van Acoleyen, 2005, 2006; Olmo, 2005a, 2005b; Barrow and Clifton, 2006; Capozziello *et al.*, 2006, 2007b, 2008; Cembranos, 2006; Multamaki and Vilja, 2006b, 2008; Shao *et al.*, 2006; Baghram *et al.*, 2007; Hu and Sawicki, 2007b; Ruggiero and Iorio, 2007; Zhang, 2007; Capozziello and Tsujikawa, 2008; Iorio, 2009). Moreover, a certain lack of rigor in checking the convergence of series used in the expansion around a de Sitter background often left doubts even on results that, *a posteriori*, turned out to be correct (Sotiriou, 2006c).

Using the equivalence between $f(R)$ and scalar-tensor gravity, Chiba originally suggested that all $f(R)$ theories are ruled out (Chiba, 2003). This claim was based on the fact that metric $f(R)$ gravity is equivalent to an $\omega_0=0$ Brans-Dicke theory, while the observational constraint is $|\omega_0|>40\,000$ (Bertotti *et al.*, 2003). This is not quite the case and the weak-field limit is more subtle than it appears, as the discussion of the previous section might have already revealed: The value of the parametrized post-Newtonian (PPN) formalism parameter γ, on which the observational bounds are directly applicable, is practically independent of the mass of the scalar only when the latter is small (Wagoner, 1970). In this case, the constraints on γ can indeed be turned into constraints on ω_0. However, if the mass of this scalar is large, it dominates over ω_0 in the expression of γ and drives its value to unity. The physical explanation of this fact, as mentioned, is that the scalar becomes short ranged and therefore has no effect at Solar System scales. Additionally, there is even the possibility that the effective mass of the scalar field itself is actually scale dependent. In this case, the scalar may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there, while being effectively light at cosmological scales. This is the *chameleon mechanism*, well known in quintessence models (Khoury and Weltman, 2004b, 2004b).

Given the above, it is worth examining these issues in more detail. Even though early doubts about the validity of the dynamical equivalence with scalar-tensor theory in the Newtonian limit (Faraoni, 2006b; Kainulainen, Piilonen, *et al.*, 2007) have now been dissipated (Faraoni, 2007b), a direct approach that does not resort to the scalar-tensor equivalence is preferable as the equivalence could in principle hide things (Olmo, 2005b). This was given in the metric formalism, first in the special case (Erickcek *et al.*, 2006) $f(R)=R-\mu^4/R$ [which is already ruled out by the Ricci scalar instability (Dolgov and Kawasaki, 2003a; Faraoni, 2006a)] and in the case $f(R)=R^n$ using light deflection and other Solar System experiments[18] (Clifton and Barrow, 2005a, 2006; Barrow and Clifton, 2006; Zakharov *et al.*, 2006). Only later was the case of a general function $f(R)$ discussed (Jin *et al.*, 2006; Chiba *et al.*, 2007; Olmo, 2007). Chiba's result based on the scalar-tensor equivalence eventually turned out to be valid subject to certain assumptions which are not always satisfied (Jin *et al.*, 2006; Chiba *et al.*, 2007; Olmo, 2007)—see below. This method, however, does not apply to the Palatini version of $f(R)$ gravity.

In what follows we adhere to, but streamline, the discussion of Chiba *et al.* (2007) with minor modifications in

[18]The perihelion precession in modified gravity is studied by Baghram *et al.* (2007), Iorio and Ruggiero (2007, 2008), Schmidt (2008), and Iorio (2009).

order to compute the PPN parameter γ for metric $f(R)$ gravity [see also Olmo (2007)]. We consider a spherically symmetric, static, noncompact body embedded in a background de Sitter universe; the latter can exist in an adiabatic approximation in which the evolution of the universe is very slow in comparison with local dynamics. The condition for the existence of a de Sitter space with $R_{\mu\nu}=R_0 g_{\mu\nu}/4$ and constant curvature $R_0=12H_0^2$ is

$$f_0' R_0 - 2f_0 = 0, \quad H_0 = \sqrt{\frac{f_0}{6f_0'}}. \tag{106}$$

The line element is

$$ds^2 = -[1+2\Psi(r) - H_0^2 r^2]dt^2 + [1+2\Phi(r) + H_0^2 r^2]dr^2 + r^2 d\Omega^2 \tag{107}$$

in Schwarzschild coordinates, where the post-Newtonian potentials $\Psi(r)$ and $\Phi(r)$ are treated as small perturbations.[19] The goal is to compute the PPN parameter $\gamma=-\Psi/\Phi$ by solving the equations satisfied by these potentials. A linearized analysis is performed assuming

$$|\Psi(r)|,|\Phi(r)| \ll 1, \quad r \ll H_0^{-1}, \tag{108}$$

and

$$R(r) = R_0 + R_1(r), \tag{109}$$

where the deviation $R_1(r)$ of the Ricci curvature from the constant R_0 is also a small perturbation.[20]

Three assumptions are made:

Assumption 1: $f(R)$ is analytical at R_0.

Assumption 2: $mr \ll 1$, where m is the effective mass of the scalar degree of freedom of the theory. In other words, this scalar field (the Ricci curvature, which is an extra dynamical quantity in the metric formalism) must have a range longer than the size of the Solar System—if it is much shorter than, say, 0.2 mm (Hoyle *et al.*, 2001), the presence of this scalar is effectively hidden from Solar System and terrestrial experiments. In this case, this field could not have cosmological effects at late times but could be important only in the very early universe at high curvatures, e.g., in Starobinsky-like inflation (Starobinsky, 1980).

Assumption 3: the pressure $P \simeq 0$ for the energy-momentum of the local starlike object. The trace of the corresponding energy-momentum tensor reduces to $T_1 \simeq -\rho$.

[19]Isotropic coordinates are usually employed in the study of the weak-field limit of spherically symmetric metrics; however, the difference is irrelevant to first order in Ψ and Φ (Olmo, 2007).

[20]The solution derived for the spherically symmetric metric is valid only when $mr \ll 1$, where m is the effective mass of the scalar. If this assumption is not made, then [for example, according to Jin *et al.* (2006)], it would seem that quantum corrections in $f(R)=R+aR^2$ with $a \simeq 10^{-24}$ GeV^{-2} are ruled out by Solar System constraints, which is not the case because these corrections are equivalent to a *massive* scalar field with short range that is not constrained by the available data.

By expanding $f(R)$ and $f'(R)$ around R_0, the trace equation (153) reduces to

$$3f_0'' \Box R_1 + (f_0'' R_0 - f_0')R_1 = \kappa T_1, \tag{110}$$

where $T=T_1$ since T is zero in the background. For a static, spherically symmetric body, $R_1=R_1(r)$ and $\Box R_1 = \nabla^2 R_1 = r^{-2}(d/dr)(r^2 dR_1/dr)$. The reduced trace equation (110) then becomes

$$\nabla^2 R_1 - m^2 R_1 = -\frac{\kappa\rho}{3f_0''}, \tag{111}$$

where

$$m^2 = \frac{f_0' - f_0'' R_0}{3f_0''}. \tag{112}$$

By using $R_0=12H_0^2=2f_0/f_0'$, this reduces to

$$m^2 = \frac{(f_0')^2 - 2f_0 f_0''}{3f_0' f_0''}. \tag{113}$$

This equation is found in various other treatments of perturbations of de Sitter space (Nunez and Solganik, 2004; Faraoni and Nadeau, 2005; Navarro and Van Acoleyen, 2007; Olmo, 2007).

Assumption 2 that the scalar R_1 is light, which enables the $f(R)$ theory to produce significant cosmological effects at late times, also allows one to neglect[21] the term $m^2 R_1$ in Eq. (111). The Green's function of the equation $\nabla^2 R_1 = -\kappa\rho/3f_0''$ is then $G(r)=-1/4\pi r$ and the solution is $R_1 \simeq \int d^3\vec{x}'[-\kappa\rho(r')/3f_0'']G(r-r')$, which yields

$$R_1 \simeq \frac{\kappa M}{12\pi f_0'' r} \quad (mr \ll 1). \tag{114}$$

Now, the condition $m^2 r^2 \ll 1$ yields

$$\frac{1}{3}\left|\frac{f_0'}{f_0''} - R_0\right| r^2 \ll 1 \tag{115}$$

and, using $H_0 r \ll 1$,

$$\left|\frac{f_0'}{f_0''}\right| r^2 \ll 1. \tag{116}$$

We now use the full field equations (6); by expanding $f(R)$ and $f'(R)$ and using $f_0=6H_0^2 f_0'$ we get

$$\delta^\alpha_\beta f_0'' \Box R_1 + f_0'(R^\alpha_\beta - 3H_0^2 \delta^\alpha_\beta) - \frac{f_0'}{2} R_1 \delta^\alpha_\beta - f_0'' \nabla^\alpha \nabla_\beta R_1 + f_0'' R_1 R^\alpha_\beta = \kappa T^\alpha_\beta. \tag{117}$$

Using again the assumption $H_0 r \ll 1$, the d'Alembertian $\Box$ becomes ∇^2_η and, for $(\mu,\nu)=(0,0)$,

[21]Although Chiba *et al.* (2007) provided Green's functions in both cases $m^2>0$ and $m^2<0$, the latter corresponds to a spacetime instability and is unphysical. This is irrelevant in the end because only the case $m^2 \to 0$ is necessary and used in the calculation (Faraoni and Lanahan-Tremblay, 2008).

$$f_0'(R_0^0-3H_0^2)-\frac{f_0'}{2}R_1+f_0''R_1R_0^0+f_0''\nabla^2R_1=-\kappa\rho. \quad (118)$$

By computing $R_0^0=3H_0^2-\nabla^2\Psi(r)$ and dropping terms $f_0''H_0^2R_1\ll f_0'\nabla^2\Psi$, etc., we obtain

$$f_0'\nabla^2\Psi(r)+\frac{f_0'}{2}R_1-f_0''\nabla^2R_1=\kappa\rho. \quad (119)$$

Recalling that $\nabla^2R_1\simeq-\kappa\rho/3f_0''$ for $mr\ll1$, one obtains

$$f_0'\nabla^2\Psi(r)=\frac{2\kappa\rho}{3}-\frac{f_0'}{2}R_1. \quad (120)$$

Equation (120) can be integrated from $r=0$ to $r>r_0$ (where r_0 is the radius of the starlike object) to obtain, using Gauss' law,

$$\frac{d\Psi}{dr}=\frac{\kappa}{6\pi f_0'}\frac{\kappa M}{48\pi f_0''r^2}-\frac{C_1}{r^2}, \quad (121)$$

where $M(r)=4\pi\int_0^{r_0}dr'(r')^2\rho(r')$. The integration constant C_1 must be set to zero to guarantee regularity of the Newtonian potential at $r=0$. The potential $\Psi(r)$ then becomes

$$\Psi(r)=-\frac{\kappa M}{6\pi f_0'r}-\frac{\kappa M}{48\pi f_0''}r. \quad (122)$$

The second term on the right-hand side is negligible; in fact,

$$\left|\frac{\kappa Mr/48\pi f_0''}{-\kappa M/6\pi f_0'r}\right|=\left|\frac{f_0'}{8f_0''}\right|r^2\ll1 \quad (123)$$

and

$$\Psi(r)\simeq-\frac{\kappa M}{6\pi f_0'r}. \quad (124)$$

We now find the second potential $\Phi(r)$ appearing in the line element (107). Using the field equations (6) with $(a,b)=(1,1)$,

$$f_0'(R_1^1-3H_0^2)-\frac{f_0'}{2}R_1-f_0''\nabla^1\nabla_1R_1+f_0''R_1R_1^1+f_0''\Box R_1=\kappa T_1^1, \quad (125)$$

with $T_1^1\simeq0$ outside the star, and

$$R_1^1\simeq3H_0^2-\frac{d^2\Psi}{dr^2}+\frac{2}{r}\frac{d\Phi}{dr}, \quad (126)$$

$$g^{11}\nabla_1\nabla_1R_1\simeq\frac{d^2R_1}{dr^2}, \quad (127)$$

and neglecting higher-order terms, one obtains [Eq. (22) of Chiba *et al.* (2007)]

$$f_0'\left(-\frac{d^2\Psi}{dr^2}+\frac{2}{r}\frac{d\Phi}{dr}\right)-\frac{f_0'R_1}{2}+\frac{2f_0''}{r}\frac{dR_1}{dr}\simeq0. \quad (128)$$

Now, using Eq. (114) for R_1, one concludes that the third term in Eq. (128) is negligible in comparison with the fourth term. In fact,

$$\left|\frac{f_0'R_1/2}{2f_0''/rdR_1/dr}\right|\simeq\left|\frac{f_0'}{f_0''}\right|r^2\ll1. \quad (129)$$

Then, again using Eq. (114) for dR_1/dr and Eq. (124) for $\Psi(r)$, one obtains

$$\frac{d\Phi}{dr}=-\frac{\kappa M}{12\pi f_0'r}, \quad (130)$$

which is immediately integrated to

$$\Phi(r)=\frac{\kappa M}{12\pi f_0'r}. \quad (131)$$

The post-Newtonian metric (107) therefore gives the PPN parameter γ as

$$\gamma=-\frac{\Phi(r)}{\Psi(r)}=\frac{1}{2}. \quad (132)$$

This is a gross violation of the experimental bound $|\gamma-1|<2.3\times10^{-5}$ (Bertotti *et al.*, 2003) and agrees with the calculation of the PPN parameter $\gamma=(\omega_0+1)/(\omega_0+2)$ found using the equivalence of metric $f(R)$ gravity with an $\omega_0=0$ Brans-Dicke theory (Chiba, 2003).

The results of Chiba *et al.* (2007) have been reproduced by Olmo (2007), who worked in isotropic coordinates with a slightly different approach. Kainulainen, Piilonen, *et al.* (2007) obtained spherically symmetric interior solutions matched to the exterior solutions of metric $f(R)$ gravity and confirmed the result $\gamma=1/2$.

a. Limits of validity of the previous analysis

One can contemplate various circumstances in which the assumptions above are not satisfied and the previous analysis breaks down. It is important to ascertain whether these are physically relevant situations. There are three main cases to consider.

i. The case of nonanalytic $f(R)$. While Chiba *et al.* (2007) considered functions $f(R)$ that are analytic at the background value R_0 of the Ricci curvature, the situation in which this function is not analytical was contemplated by Jin *et al.* (2006). Assuming that $f(R)$ has an isolated singularity at $R=R_s$, it can be expressed as the sum of a Laurent series,

$$f(R)=\sum_{n=0}^{+\infty}a_n(R-R_s)^n. \quad (133)$$

Jin *et al.* (2006) noted that it must be $R\neq R_s$ in the dynamics of the universe because a constant curvature space with $R=R_s$ *cannot* be a solution of the field equations. Therefore, one can approximate the solution adiabatically with a de Sitter space with constant curvature $R_0\neq R_s$. The function $f(R)$ is analytical here and the previous discussion applies. This is not possible if $f(R)$ has an essential singularity, for example, if $f(R)=R-\mu^2\sin[\mu^2/(R-\lambda)]$ (Jin *et al.*, 2006). There is, of course, no reason other than Occam's razor to exclude this possibility.

ii. Short-range scalar field. If the assumption $mr \ll 1$ is not satisfied, the scalar is massive. If its range is sufficiently short, it is effectively hidden from experiments probing deviations from Newton's law and from other Newtonian and post-Newtonian experiments in the solar neighborhood. This is the case of quadratic quantum corrections to Einstein's gravity, e.g., $f(R)=R+\alpha R^2$. If the effective mass is $m \geq 10^{-3}$ eV (corresponding to a fifth-force range less than ~ 0.2 mm, the shortest scale currently accessible to weak-field experiments), this correction is undetectable and yet it can still have large effects in the early inflationary universe (Starobinsky, 1980). However, it cannot work as a model for late-time acceleration.

iii. Chameleon behavior. The *chameleon effect* (Khoury and Weltman, 2004a, 2004b), originally discovered in scalar field models of dark energy, occurs when the effective mass m of the scalar degree of freedom is a function of the curvature (or, better, of the energy density of the local environment), so that m can be large at Solar System and terrestrial curvatures and densities and small at cosmological curvatures and densities—effectively, it is short ranged in the Solar System and it becomes long ranged at cosmological densities, thus causing the acceleration of the universe. The chameleon effect can be applied to metric $f(R)$ gravity (Cembranos, 2006; Faulkner *et al.*, 2007; Navarro and Van Acoleyen, 2007; Starobinsky, 2007), with the result that theories of the kind (Carroll *et al.*, 2004; Amendola, Polarski, and Tsujikawa, 2007a, 2007b; Amendola *et al.*, 2007),

$$f(R) = R - (1-n)\mu^2\left(\frac{R}{\mu^2}\right)^n \tag{134}$$

are compatible with the observations in the region of the parameter space $0 < n \leq 0.25$ with μ sufficiently small (Faulkner *et al.*, 2007). Precisely, using the Cassini bound on the PPN parameter γ (Bertotti *et al.*, 2003), the constraint

$$\frac{\mu}{H_0} \leq \sqrt{3}\left[\frac{2}{n(1-n)}\right]^{1/[2(1-n)]} 10^{(-6-5n)/[2(1-n)]} \tag{135}$$

is obtained (Faulkner *et al.*, 2007). Fifth-force experiments give the bounds

$$\frac{\mu}{H_0} \leq \sqrt{1-n}\left[\frac{2}{n(1-n)}\right]^{1/2(1-n)} 10^{(-2-12n)/(1-n)}. \tag{136}$$

Preferred values seem to be $m \simeq 10^{-50}$ eV$\sim 10^{-17} H_0$ (Faulkner *et al.*, 2007). Note that $n > 0$, which guarantees $f'' > 0$, is required for Ricci scalar stability ($n=0$ reduces the model to GR with a cosmological constant, but avoidance of the latter was exactly the reason why dark energy and modified gravity were introduced in the first place).

These models work to explain the current cosmic acceleration because, for small curvatures R, the correction in R^n with $n<1$ is larger than the Einstein-Hilbert term R and comes to dominate the dynamics. On the negative side, these theories are observationally indistinguishable from a cosmological constant and they have been dubbed "vanilla $f(R)$ gravity" (Amendola, Polarski, and Tsujikawa, 2007a, 2007b; Amendola *et al.*, 2007; Faulkner *et al.*, 2007; Amendola and Tsujikawa, 2008). However, they still have the advantage of avoiding a fine-tuning problem in Λ at the price of a much smaller fine tuning of the parameter μ. As for all modified gravity and dark energy models, they do not address the cosmological constant problem.

The weak-field limit of metric $f(R)$ theories which admit a *global* Minkowski solution around which to linearize was studied by Clifton (2008). These theories [including, e.g., analytic functions $f(R)=\Sigma_{n=1}^{+\infty} a_n R^n$] are not motivated by late-time cosmology and the Minkowski global solution, although present, may not be stable (Clifton and Barrow, 2005a), which in practice detracts from the usefulness of this analysis. Several new post-Newtonian potentials are found to appear in addition to the two usual ones (Clifton, 2008).

3. Weak-field limit in the Palatini formalism

Early works on the weak-field limit of Palatini $f(R)$ gravity often led to contradictory results and to several technical problems as well (Barraco and Hamity, 2000; Dominguez and Barraco, 2004; Meng and Wang, 2004b; Allemandi, Francaviglia, *et al.* 2005; Olmo, 2005a, 2005b, 2007; Sotiriou, 2006c; Allemandi and Ruggiero, 2007; Bustelo and Barraco, 2007; Kainulainen, Reijonen, and Sunhede, 2007; Ruggiero and Iorio, 2007; Ruggiero, 2009) which seem to have been clarified by now.

First, there seems to have been some confusion in the literature about the fact that Palatini $f(R)$ gravity reduces to GR with a cosmological constant in vacuum and the consequences that this can have on the weak-field limit and Solar System tests. It is, of course, true (see Sec. II.B) that *in vacuo* Palatini $f(R)$ gravity will have the same solutions of GR plus a cosmological constant and, therefore, the Schwarzschild–(anti–)de Sitter solution will be the unique vacuum spherically symmetric solution by (see also Sec. VI.C.1 for a discussion of the Jebsen-Birkhoff theorem). This was interpreted by Allemandi and Ruggiero (2007) and Ruggiero and Iorio (2007) as an indication that the only parameter that can be constrained is the effective cosmological constant and, therefore, models that are cosmologically interesting (for which this parameter is very small) trivially satisfy Solar System tests. However, even if one sets aside the fact that a weak-gravity regime is possible inside matter as well, such claims cannot be correct: they would completely defeat the purpose of performing a parametrized post-Newtonian expansion for any theory for which one can establish uniqueness of a spherically symmetric solution, as in this case we would be able to judge Solar System viability just by considering this vacuum solution (which would be much simpler).

Indeed, the existence of a spherically symmetric vacuum solution, irrespective of its uniqueness, does not suffice to guarantee a good Newtonian limit. For instance, the Schwarzschild–de Sitter solution has two free parameters; one of them can be associated with the effective cosmological constant in a straightforward manner (using the asymptotics). However, it is not clear how the second parameter, which in GR is identified with the mass of the object in the Newtonian regime, is related to the internal structure of the object in Palatini $f(R)$ gravity. The *assumption* that it represents the mass defined in the usual way is not, of course, sufficient. One would have to actually match the exterior solution to a solution describing the interior of the Sun within the realm of the theory in order to express the undetermined parameter in the exterior solution in terms of known physical quantities, such as Newton's constant and the Newtonian mass. The essence of the derivation of the Newtonian limit of the theory consists also of deriving such an explicit relation for this quantity and showing that it agrees with the Newtonian expression. The parametrized post-Newtonian expansion is nothing but an alternative way to do that without having to solve the full field equations. Therefore, it is clear that more information than the form of the vacuum solution is needed in order to check whether the theory can satisfy the Solar System constraints.

However, some early attempts toward Newtonian and post-Newtonian expansions were also flawed. Barraco and Hamity (2000) and Meng and Wang (2004b) performed, for instance, a series expansion around a de Sitter background in order to derive the Newtonian limit. Writing

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1, \tag{137}$$

where $\mathcal{R}_0$ is the Ricci curvature of the background and $\mathcal{R}_1$ is a correction, one is tempted to expand in powers of $\mathcal{R}_1/\mathcal{R}_0$, regarding the latter as a small quantity. Since one needs the quantities $f(\mathcal{R}_0+\mathcal{R}_1)$ and $f'(\mathcal{R}_0+\mathcal{R}_1)$, the usual approach is to Taylor-expand around $\mathcal{R}=\mathcal{R}_0$ and keep only the leading order terms in $\mathcal{R}_1$. However, it has been shown by Sotiriou (2006c) that this cannot be done for most cosmologically interesting models because $\mathcal{R}_1/\mathcal{R}_0$ is not small.

Take as an example the model (Vollick, 2003)

$$f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon_2^2}{\mathcal{R}} \tag{138}$$

and $\epsilon_2 \sim 10^{-67}\text{eV}^2 \sim 10^{-53}\ \text{m}^{-2}$. Expanding as

$$f(\mathcal{R}) = f(\mathcal{R}_0) + f'(\mathcal{R}_0)\mathcal{R}_1 + \tfrac{1}{2}f''(\mathcal{R}_0)\mathcal{R}_1^2 + \cdots \tag{139}$$

and using Eq. (138) one obtains

$$f(\mathcal{R}) = f(\mathcal{R}_0) + \left(1 + \frac{\epsilon_2^2}{\mathcal{R}_0^2}\right)\mathcal{R}_1 - \frac{1}{2}\frac{2\epsilon_2^2}{\mathcal{R}_0^3}\mathcal{R}_1^2 + \cdots, \tag{140}$$

where now $\mathcal{R}_0 = \epsilon_2$. It is then easy to see that the second term on the right-hand side is of the order of $\mathcal{R}_1$, whereas the third term is of the order of $\mathcal{R}_1^2/\epsilon_2$. Therefore, in order to truncate before the third term, one needs $\mathcal{R}_1 \gg \mathcal{R}_1^2/\epsilon_2$ or

$$\epsilon_2 \gg \mathcal{R}_1. \tag{141}$$

This is not a stringent constraint: $\mathcal{R}_0 \sim \epsilon_2$ and so this is the usual condition for linearization.

We return now to the trace (20). For the model under consideration,

$$\mathcal{R} = \tfrac{1}{2}(-\kappa T \pm \sqrt{\kappa^2 T^2 + 12\epsilon_2^2}). \tag{142}$$

According to Eq. (142), the value of $\mathcal{R}$, and consequently $\mathcal{R}_1$, is algebraically related to T and, whether or not the condition (141) is satisfied critically depends on the value of the energy density. To demonstrate this, pick the mean density of the Solar System, $\rho \sim 10^{-11}\ \text{g/cm}^3$, which satisfies the weak-field limit criteria. For this value, $|\epsilon_2/\kappa T| \sim 10^{-21}$, where $T \sim -\rho$. The "physical" branch of the solution (142) is the one with positive sign because, given that $T<0$, it ensures that matter leads to a standard positive curvature in strong gravity. Then,

$$\mathcal{R} \sim -\kappa T - \frac{3\epsilon_2^2}{\kappa T} \tag{143}$$

and $\mathcal{R}_1 \sim -\kappa T \sim \kappa\rho$. Thus, $\epsilon_2/\mathcal{R}_1 \sim 10^{-21}$ and it is evident that the required condition does not hold for some typical densities related to the Newtonian limit.

The situation does not improve even with the "unphysical" branch of Eq. (142) with a negative sign. In fact, in this case $\mathcal{R}_1 \sim \epsilon_2(3\epsilon_2/\kappa T + \sqrt{3})$ and the correction to the background curvature is of the order ϵ_2 and not much smaller than that, as would be required in order to truncate the expansion (140). Barraco and Hamity (2000) overlooked this fact and only linear terms in $\mathcal{R}_1$ were kept in the expansion of $f(\mathcal{R})$ and $f'(\mathcal{R})$ around $\mathcal{R}_0$. Meng and Wang (2004b) did not take this properly into account at the outset, even though this fact is noticed in the final stages of the analysis and is actually used, keeping again only linear terms [see, e.g., Eq. (11) of Meng and Wang (2004b)].

However, the algebraic dependence of $\mathcal{R}$ on the density does not only signal a problem for the approaches just mentioned. It actually implies that the outcome of the post-Newtonian expansion itself depends on the density, as shown by Olmo (2005a, 2005b) and Sotiriou (2006c). Consider, for instance, along the lines of Sotiriou (2006a, 2006c), the conformal metric

$$h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu} \tag{144}$$

which was introduced in Sec. II.B [cf. Eq. (22)]. In terms of this metric, the field equations can be written in the form

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}h_{\mu\nu} + (f'-1)\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2f'}h_{\mu\nu}\right) = \kappa T_{\mu\nu}, \tag{145}$$

and $\mathcal{R}_{\mu\nu}$ is the Ricci tensor of the metric $h_{\mu\nu}$. It is evident that, if $f'=1$, then $h_{\mu\nu}$ and $g_{\mu\nu}$ coincide and Eq. (145) yields Einstein's equation. However, since $\mathcal{R}$ and consequently $f'(\mathcal{R})$ are functions of the energy density, due to Eq. (20), the deviation of f' from unity will always depend on the energy density and the functional form of

f. Therefore, one can definitely find some function f which, for some range of energy densities, will give $f'=1$ to high precision. However, for the same function f, there will be large deviations from $f'=1$ at a different density range. This dependence of the weak-field limit on the energy density is a novel characteristic of Palatini $f(R)$ gravity.

This dependence can be made explicit if the problem is approached via the equivalent Brans-Dicke theory (Olmo, 2005a, 2005b). Note that the usual bounds coming from Solar System experiments do not apply in the $\omega_0=-3/2$ case, which is equivalent to Palatini $f(R)$ gravity. This is because the standard treatment of the post-Newtonian expansion of Brans-Dicke theory, which one uses to arrive at such bounds, is critically based on the assumption that $\omega_0 \neq -3/2$ and the term $2\omega_0+3$ frequently appears as a denominator. It is not necessary to make this assumption, of course, in order to derive a post-Newtonian expansion but it is a convenient choice, which allows for this otherwise general treatment. Therefore, a different approach, such as the one followed in Olmo (2005b), was indeed required for the $\omega_0=-3/2$ case. Following the standard assumptions of a post-Newtonian expansion around a background specified by a cosmological solution (Will, 1981), the following relations were derived for the post-Newtonian limit:

$$-\frac{1}{2}\nabla^2[h_{00}^1-\Omega(T)]=\frac{\kappa\rho-V(\phi)}{2\phi}, \tag{146}$$

$$-\frac{1}{2}\nabla^2[h_{ij}^1+\delta_{ij}\Omega(T)]=\left[\frac{\kappa\rho+V(\phi)}{2\phi}\right], \tag{147}$$

where V is the potential of the scalar field ϕ and $\Omega(T)\equiv\ln[\phi/\phi_0]$ The subscript 0 in ϕ_0, and in any other quantity in the rest of this subsection, denotes that it is evaluated at $T=0$.

The solutions of Eqs. (146) and (147) are

$$h_{00}^{(1)}(t,\vec{x})=\frac{2G_{\rm eff}M_\odot}{r}+\frac{V_0}{6\phi_0}r^2+\Omega(T), \tag{148}$$

$$h_{ij}^{(1)}(t,\vec{x})=\left[\frac{2\gamma G_{\rm eff}M_\odot}{r}-\frac{V_0}{6\phi_0}r^2-\Omega(T)\right]\delta_{ij}, \tag{149}$$

where $M_\odot\equiv\phi_0\int d^3\vec{x}\rho(t,\vec{x})/\phi$. The effective Newton constant $G_{\rm eff}$ and the post-Newtonian parameter γ are defined as

$$G_{\rm eff}\equiv\frac{G}{\phi_0}\left(1+\frac{M_V}{M_\odot}\right), \tag{150}$$

$$\gamma\equiv\frac{M_\odot-M_V}{M_\odot+M_V}, \tag{151}$$

where $M_V\equiv\kappa^{-1}\phi_0\int d^3\vec{x}[V_0/\phi_0-V(\phi)/\phi]$.

As stated differently by Olmo (2005b), if the Newtonian mass is defined as $M_N\equiv\int d^3\vec{x}\rho(t,\vec{x})$, the requirement that a theory has a good Newtonian limit is that $G_{\rm eff}M_\odot$ equals GM_N, where N denotes Newtonian, and $\gamma\simeq 1$ to very high precision. Additionally, the second term on the right-hand side of both Eqs. (148) and (149) should be negligible since it plays the role of a cosmological constant term. $\Omega(T)$ should also be small and have a negligible dependence on T.

Even though it is not impossible, as mentioned, to prescribe f such that all of the above are satisfied for some range of densities within matter (Sotiriou 2006c), this does not seem possible over the wide range of densities relevant for the Solar System tests. As a matter of fact, Ω is nothing but an algebraic function of T and therefore of the density (since ϕ is an algebraic function of $\mathcal{R}$). The presence of the $\Omega(T)$ term in Eqs. (148) and (149) signals an algebraic dependence of the post-Newtonian metric on the density. This direct dependence of the metric on the matter field is not only surprising but also seriously problematic. Besides the fact that it is evident that the theory cannot have the proper Newtonian limit for all densities (the range of densities for which it will fail depends on the functional form of f), consider the following: What happens to the post-Newtonian metric if a very weak point source (approximated by a delta function) is taken into account as a perturbation? And will the post-Newtonian metric be continuous when going from the interior of a source to the exterior, as it should be?

We refrain from further analysis of these issues here since evidence coming from considerations different from the post-Newtonian limit, which we review shortly, will be of significant help. We return to this discussion in Sec. VI.C.2.

B. Stability issues

In principle, several kinds of instabilities need to be considered to make sure that $f(R)$ gravity is a viable alternative to GR (Chiba, 2005; Wang, 2005; Calcagni *et al.*, 2006; De Felice *et al.*, 2006; Sokolowski, 2007a, 2007b).

The Dolgov-Kawasaki (Dolgov and Kawasaki, 2003a) instability in the matter sector, specific to metric $f(R)$ gravity, imposes restrictions on the functional form of f and is discussed below. More generally, it is believed that a stable ground state, the existence of which is necessary in a gravitational theory, should be highly symmetric, such as the de Sitter, Minkowski, or perhaps Einstein static space. Instabilities of de Sitter space in the gravity sector have been found by Faraoni (2004b, 2004c, 2005a), Faraoni and Nadeau (2005), Barrow and Hervik (2006a), and Dolgov and Pelliccia (2006) [see also Barrow and Ottewill (1983) and Muller *et al.* (1990) for pre-1998 discussions], while stability in first-loop quantization of $f(R)$ gravity and with respect to black hole nucleation was studied by Cognola *et al.* (2005, 2008), Paul and Paul (2005, 2006), and Cognola and Zerbini (2006). The linear stability of de Sitter space with respect to homogeneous perturbations in generalized theories of the form $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})$ was studied by Cognola and Zerbini (2008). The stability of the Einstein static space in metric $f(R)$ gravity with respect to homogeneous perturbations was studied by Boehmer, Hollenstein, and Lobo (2007) while stability of this space with respect to inhomogeneous isotropic perturbations was established, with a gauge-invariant formalism and under certain conditions, by Goswami *et al.* (2008).

1. Ricci stability in the metric formalism

In the metric formalism, Dolgov and Kawasaki discovered an instability in the prototype model $f(R)=R-\mu^4/R$ (now called the Dolgov-Kawasaki, Ricci scalar, or matter instability), which manifests itself on an extremely short time scale and is sufficient to rule out this model (Dolgov and Kawasaki, 2003a). Their result was confirmed by Nojiri and Odintsov (2003a, 2004b), in which it was also shown that adding to this specific $f(R)$ an R^2 term removes this instability. The instability was rediscovered by Baghram *et al.* (2007) for a specific form of the function $f(R)$. The analysis of this instability is generalized to arbitrary $f(R)$ theories in the metric formalism in the following way (Faraoni, 2006a).

We parametrize the deviations from Einstein gravity as

$$f(R) = R + \epsilon\varphi(R), \tag{152}$$

where ϵ is a small parameter with the dimensions of a mass squared and φ is arranged to be dimensionless (in the example $f=R-\mu^4/R$, one has $\epsilon=\mu^2$, $\varphi=-\mu^2/R$, and $\mu\simeq H_0\approx 10^{-33}$ eV).

Using the trace equation (8),

$$3\Box f'(R) + f'(R)R - 2f(R) = \kappa T, \tag{153}$$

and evaluating $\Box f'$,

$$\Box R + \frac{\varphi'''}{\varphi''}\nabla^\alpha R\nabla_\alpha R + \frac{(\epsilon\varphi'-1)}{3\epsilon\varphi''}R = \frac{\kappa T}{3\epsilon\varphi''} + \frac{2\varphi}{3\varphi''}. \tag{154}$$

We assume that $\varphi''\neq 0$: if $\varphi''=0$ on an interval then the theory reduces to GR. Isolated zeros of φ'', at which the theory is "instantaneously GR," are in principle possible but will not be considered here.

Consider a small region of space-time in the weak-field regime and approximate *locally* the metric and the curvature by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad R = -\kappa T + R_1, \tag{155}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $|R_1/\kappa T|\ll 1$. This inequality excludes the case of conformally invariant matter with $T=0$, a situation considered later. Equation (155) yields, to first order in R_1,

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa\varphi'''}{\varphi''}\dot{T}\dot{R}_1 + \frac{2\kappa\varphi'''}{\varphi''}\vec{\nabla}T\cdot\vec{\nabla}R_1 + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon}-\varphi'\right)R_1 = \kappa\ddot{T} - \kappa\nabla^2 T - \frac{(\kappa T\varphi'+2\varphi)}{3\varphi''}, \tag{156}$$

where $\vec{\nabla}$ and ∇^2 are the gradient and Laplacian in Euclidean three-dimensional space, respectively, and an overdot denotes differentiation with respect to time. The function φ and its derivatives are now evaluated at $R=-\kappa T$. The coefficient of R_1 in the fifth term on the left-hand side is the square of an effective mass and is dominated by the term $(3\epsilon\varphi'')^{-1}$ due to the extremely small value of ϵ needed for these theories to reproduce the correct cosmological dynamics. Then, the scalar mode R_1 of the $f(R)$ theory is stable if $\varphi''=f''>0$ and unstable if this effective mass is negative, *i.e.*, if $\varphi''=f''<0$. The time scale for this instability to manifest is estimated to be of the order of the inverse effective mass $\sim 10^{-26}$ s in the example $\epsilon\varphi(R)=-\mu^4/R$ (Dolgov and Kawasaki, 2003a). The small value of φ'' gives a large effective mass and is responsible for the small time scale over which the instability develops.

Consider now matter with vanishing trace T of the stress-energy tensor. In this case Eq. (156) becomes

$$\ddot{R}_1 + \frac{\varphi'''}{\varphi''}\dot{R}_1{}^2 - \nabla^2 R_1 - \frac{\varphi'''}{\varphi''}(\vec{\nabla}R_1)^2 + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon}-\varphi'\right)R_1 = \frac{2\varphi}{3\varphi''}. \tag{157}$$

Again, the effective mass term is $\sim(3\epsilon\varphi'')^{-1}$, which has the sign of f'', and the previous stability criterion is recovered. The stability condition $f''(R)\geqslant 0$ is useful to veto $f(R)$ gravity models.[22]

When $f''<0$, the instability of these theories can be interpreted, following Eq. (156), as an instability in the gravity sector. Equivalently, since it appears inside matter when R starts deviating from T [see Eq. (155)], it can be seen as a matter instability [this is the interpretation taken in Dolgov and Kawasaki (2003a)]. Whether the instability arises in the gravity or matter sector seems to be a matter of interpretation.

The instability of stars made of any type of matter in theories with $f''<0$ and sufficiently small is confirmed with a different approach (a generalized variational principle) by Seifert (2007), in which the time scale for instability found by Dolgov and Kawasaki in the $1/R$ model is also recovered. The stability condition $f''\geqslant 0$ is recovered in studies of cosmological perturbations (Sawicki and Hu, 2007).

The stability condition $f''(R)\geqslant 0$, expressing the fact that the scalar degree of freedom is not a ghost, can be given a simple physical interpretation (Faraoni, 2007b). Assume that the effective gravitational coupling $G_{\text{eff}}(R)\equiv G/f'(R)$ is positive; then, if G_{eff} increases with the curvature, i.e.,

[22] Nojiri (2004) and Multamaki and Vilja (2006a) hinted towards the stability criterion, but did not fully derive it because a decomposition in orders of ϵ^{-1} was not performed.

$$\frac{dG_{\rm eff}}{dR} = \frac{-f''(R)G}{[f'(R)]^2} > 0, \tag{158}$$

at large curvature the effect of gravity becomes stronger, and since R itself generates larger and larger curvature via Eq. (153), the effect of which becomes stronger and stronger because of an increased $G_{\rm eff}(R)$, a positive feedback mechanism acts to destabilize the theory. There is no stable ground state if a small curvature grows and grows without limit and the system runs away. If instead the effective gravitational coupling *decreases* when R increases, which is achieved when $f''(R)>0$, a negative feedback mechanism operates which compensates for the increase in R and there is no running away of the solutions. These considerations have to be inverted if $f'<0$, which can happen only if the effective energy density $\rho_{\rm eff}$ also becomes negative. This is not a physically meaningful situation because the effective gravitational coupling becomes negative and the tensor and scalar fields of metric $f(R)$ gravity become ghosts (Nunez and Solganik, 2004).

GR, with $f''(R)=0$ and $G_{\rm eff}$=const, is the borderline case between the two behaviors corresponding to stability ($f''>0$) and instability ($f''<0$), respectively.

Remarkably, besides the Dolgov-Kawasaki instability which manifests itself in the linearized version of Eq. (153), there are also recent claims that R can be driven to infinity due to strong nonlinear effects related to the same equation (Appleby and Battye, 2008; Frolov, 2008; Tsujikawa, 2008). More specifically, Tsujikawa (2008) found an oscillating mode as a solution to the perturbed version of Eq. (153). This mode appears to dominate over the matter-induced mode as one goes back into the past and, therefore, it can violate the stability conditions. Equation (153) was studied by Frolov (2008) with the use of a convenient variable redefinition but without resorting to any perturbative approach. It was found that there exists a singularity at a finite field value and energy level. The strongly nonlinear character of the equation allows R to easily reach the singularity in the presence of matter. As noticed by Appleby and Battye (2008), since when it comes to cosmology the singularity lies in the past, it can in principle be avoided by choosing appropriate initial conditions and evolving forward in time. This, of course, might result in a hidden fine-tuning issue.

All three studies mentioned consider models in which $f(R)$ includes, besides the linear term, only terms that become important at low curvatures. It is the form of the effective potential governing the motion of R, which depends on the functional form of $f(R)$, that determines how easy it is to drive R to infinity (Frolov, 2008). Therefore, it seems interesting to study how the presence of terms that become important at large curvatures, such as positive powers of R, could affect these results. Finally, it would be interesting to see in detail how these findings manifest themselves in the case of compact objects and whether there is any relation between this issue and the Dolgov-Kawasaki instability.

2. Gauge-invariant stability of de Sitter space in the metric formalism

One can consider the generalized gravity action

$$S = \int d^4x\sqrt{-g}\left[\frac{f(\phi,R)}{2} - \frac{\omega(\phi)}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)\right], \tag{159}$$

incorporating both scalar-tensor gravity [if $f(\phi,R)=\psi(\phi)R$] and modified gravity (if the scalar field ϕ is absent and $f_{RR}\neq 0$). In a spatially flat FLRW universe the vacuum field equations assume the form

$$H^2 = \frac{1}{3f'}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{Rf'}{2} - \frac{f}{2} + V - 3H\dot{f}'\right), \tag{160}$$

$$\dot{H} = -\frac{1}{2f'}(\omega\dot{\phi}^2 + \ddot{F} - H\dot{f}'), \tag{161}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega}\left(\frac{d\omega}{d\phi}\dot{\phi}^2 - \frac{\partial f}{\partial\phi} + 2\frac{dV}{d\phi}\right) = 0, \tag{162}$$

where $f'\equiv\partial f/\partial\phi$, $F\equiv\partial f/\partial R$, and an overdot denotes differentiation with respect to t. We choose (H,ϕ) as dynamical variables; then the stationary points of the dynamical system [Eqs. (160)–(162)] are de Sitter spaces with constant scalar field (H_0,ϕ_0). The conditions for these de Sitter solutions to exist are

$$6H_0^2 f_0' - f_0 + 2V_0 = 0, \tag{163}$$

$$\left.\frac{df}{d\phi}\right|_0 - 2\left.\frac{dV}{d\phi}\right|_0 = 0, \tag{164}$$

where $f_0'\equiv f'(\phi_0,R_0)$, $f_0\equiv f(\phi_0,R_0)$, $V_0\equiv V(\phi_0)$, and $R_0=12H_0^2$. The phase space is a curved two-dimensional surface embedded in a three-dimensional space (de Souza and Faraoni, 2007).

Inhomogeneous perturbations of de Sitter space have been studied using the covariant and gauge-invariant formalism of Bardeen (1980), Ellis and Bruni (1989), and Ellis *et al.* (1989, 1990) in a version provided by Hwang (1990a, 1990b, 1997, 1998) and Hwang and Noh (1996) for generalized gravity. The metric perturbations are defined by

$$g_{00} = -a^2(1+2AY), \quad g_{0i} = -a^2BY_i, \tag{165}$$

$$g_{ij} = a^2[h_{ij}(1+2H_LY) + 2H_TY_{ij}]. \tag{166}$$

Here Y are scalar spherical harmonics, h_{ij} is the three-dimensional metric of the FLRW background, $\hat{\nabla}_i$ is the covariant derivative of h_{ij}, and k is the eigenvalue of $\hat{\nabla}_i\hat{\nabla}^iY=-k^2Y$. Y_i and Y_{ij} are vector and tensor harmonics satisfying

$$Y_i = -\frac{1}{k}\hat{\nabla}_iY, \quad Y_{ij} = \frac{1}{k^2}\hat{\nabla}_i\hat{\nabla}_jY + \frac{1}{3}Yh_{ij}, \tag{167}$$

respectively. The Bardeen gauge-invariant potentials are

$$\Phi_H = H_L + \frac{H_T}{3} + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right), \tag{168}$$

$$\Phi_A = A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) + \frac{a}{k}\left[\dot{B} - \frac{1}{k}(a\dot{H}_T)^{\cdot}\right], \tag{169}$$

the Ellis-Bruni variable is

$$\Delta\phi = \delta\phi + \frac{a}{k}\dot{\phi}\left(B - \frac{a}{k}\dot{H}_T\right), \tag{170}$$

and analogous gauge-invariant variables Δf, $\Delta f'$, and ΔR satisfy first-order equations given by Hwang (1990a, 1990b, 1997, 1998) and Hwang and Noh (1996), which simplify significantly in the de Sitter background (H_0, ϕ_0) (Faraoni, 2004b, 2005a; Faraoni and Nadeau, 2005).

To first order and in the absence of ordinary matter, vector perturbations do not appear (Hwang, 1990a, 1990b, 1997, 1998; Hwang and Noh, 1996), and de Sitter space is always stable with respect to first-order tensor perturbations. Focusing on scalar perturbations, modified gravity corresponds to $\phi \equiv 1$ and $f = f(R)$ with $f''(R) \neq 0$ in Eq. (159). The gauge-invariant perturbations Φ_H (from which one easily obtains Φ_A and ΔR) satisfy

$$\ddot{\Phi}_H + 3H_0\dot{\Phi}_H + \left(\frac{k^2}{a^2} - 4H_0^2 + \frac{f_0'}{3f_0''}\right)\Phi_H = 0 \tag{171}$$

(Faraoni, 2004b, 2005a; Faraoni and Nadeau, 2005), where the term k^2/a^2 can be dropped at late times and for long-wavelength modes. Linear stability ensues if the coefficient of Φ_H is non-negative, i.e. [using Eq. (163)], if[23]

$$\frac{(f_0')^2 - 2f_0 f_0''}{f_0' f_0''} \geqslant 0. \tag{173}$$

The only term containing the comoving wave vector k in Eq. (171) becomes negligible at late times and/or for zero-momentum modes and thus the spatial dependence effectively disappears. In fact, Eq. (173) coincides with the stability condition that can be obtained by a straightforward homogeneous perturbation analysis of Eqs. (160) and (161). As a result, in the stability analysis of de Sitter space in modified gravity, inhomogeneous perturbations can be ignored and the study can be restricted to the simpler homogeneous perturbations, which are free of the notorious gauge-dependence problems. This result, which could not be reached *a priori* but relies on the inhomogeneous perturbation analysis, holds only for de Sitter spaces and not for different attractor (e.g., power-law) solutions that may be present in the phase space. The stability condition (173) is equivalent to the condition that the scalar field potential in the Einstein frame of the equivalent Brans-Dicke theory has a minimum at the configuration identified by the de Sitter space of curvature R_0 (Sokolowski, 2007b).

[23]The generalization of the condition (173) to D space-time dimensions, derived by Rador (2007) for homogeneous perturbations, is

$$\frac{(D-2)(f_0')^2 - Df_0 f_0''}{f_0' f_0''} \geqslant 0. \tag{172}$$

As an example, consider the prototype model

$$f(R) = R - \frac{\mu^4}{R}. \tag{174}$$

The background de Sitter space has $R_0 = 12H_0^2 = \sqrt{3}\mu^2$ and the stability condition (173) is never satisfied: this de Sitter solution is always unstable. An improvement is obtained by adding a quadratic correction to this model

$$f(R) = R - \frac{\mu^4}{R} + aR^2. \tag{175}$$

Then, the condition for the existence of a de Sitter solution is again $R_0 = \sqrt{3}\mu^2$, while the stability condition (173) is satisfied if $a > 1/3\sqrt{3}\mu^2$, in agreement with Nojiri and Odintsov (2003a, 2004b) who used independent methods.

Different definitions of stability lead to different, albeit close, stability criteria for de Sitter space [see Cognola *et al.* (2005, 2008) for the semiclassical stability of modified gravity, Bertolami (1987) for scalar-tensor gravity, and Seifert (2007) for a variational approach applicable to various alternative gravities].

3. Ricci stability in the Palatini formalism

For Palatini $f(R)$ gravity the field equations (18) and (19) are of second order and the trace equation (20) is

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T, \tag{176}$$

where $\mathcal{R}$ is the Ricci scalar of the nonmetric connection $\Gamma^{\mu}_{\nu\sigma}$ (and not that of the metric connection $\{^{\ \mu}_{\nu\sigma}\}$ of $g_{\mu\nu}$). In contrast to the metric case, Eq. (176) is not an evolution equation for $\mathcal{R}$; it is not even a differential equation but rather an *algebraic* equation in $\mathcal{R}$ once the function $f(\mathcal{R})$ is specified. This is also the case in GR, in which the Einstein field equations are of second order and taking their trace yields $R = -\kappa T$. Accordingly, the scalar field ϕ of the equivalent $\omega_0 = -3/2$ Brans-Dicke theory is not dynamical. Therefore, the Dolgov-Kawasaki instability cannot occur in Palatini $f(R)$ gravity (Sotiriou, 2007a).

4. Ghost fields

Ghosts (massive states of negative norm that cause apparent lack of unitarity) appear easily in higher-order gravities. A viable theory should be ghost-free: the presence of ghosts in $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})$ gravity has been studied by Utiyama and DeWitt (1962), Stelle (1977, 1978), Strominger (1984), Buchbinder *et al.* (1992), Vilkovisky (1992), Codello and Percacci (2006), De Felice (2007), and De Felice and Hindmarsh (2007). Due to the Gauss-Bonnet identity, if the initial action is linear in $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, one can reduce the theory under consideration to[24] $f(R, R_{\mu\nu}R^{\mu\nu})$ which, in general, contains a massive spin-2 ghost field in addition to the usual massless graviton and the massive scalar. $f(R)$ theories have no ghosts (Utiyama and DeWitt, 1962; Stelle, 1978, 1977; Strominger, 1984; Ferraris *et al.*, 1988; Buchbinder *et al.*, 1992; Vilkovisky, 1992), and the stability condition $f''(R) \geq 0$ of Dolgov and Kawasaki (2003a) and Faraoni (2006a) essentially amounts to a guarantee that the scalaron is not a ghost. Theories of the kind $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ in general are plagued by ghosts [this is the case, for example, of conformal gravity, as noticed long before the 1998 discovery of the cosmic acceleration (Riegert, 1984)], but models with only $f(R, R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ terms in the action have been claimed to be ghost-free (Comelli, 2005; Navarro and Van Acoleyen, 2006).

C. The Cauchy problem

A physical theory must have predictive power and, to this extent, a well-posed initial value problem is a required feature. GR satisfies this requirement for most reasonable forms of matter (Wald, 1984). The well-posedness of the Cauchy problem for $f(R)$ gravity is an open issue. Using harmonic coordinates, Noakes showed that theories with action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2) + S_M \tag{177}$$

in the metric formalism have a well-posed initial value problem *in vacuo* (Noakes, 1983). Using the dynamical equivalence with the scalar-tensor theory (54) when $f''(R) \neq 0$, the well-posedness of the Cauchy problem can be reduced to the analogous problem for Brans-Dicke gravity with $\omega_0 = 0$ (metric formalism) or $\omega_0 = -3/2$ (Palatini formalism). The fact that the initial value problem is well posed was demonstrated for particular scalar-tensor theories by Cocke and Cohen (1968) and Noakes (1983) and a general analysis has recently been presented (Salgado, 2006; Salgado *et al.*, 2008). This work, however, does not cover the $\omega_0 = 0, -3/2$ cases.

A system of 3+1 equations of motion is said to be well formulated if it can be rewritten as a system of equations that are of only first order in both time and space derivatives. When this set can be put in the full first-order form

$$\partial_t\vec{u} + M^i\nabla_i\vec{u} = \vec{S}(\vec{u}), \tag{178}$$

where $\vec{u}$ collectively denotes the fundamental variables h_{ij}, K_{ij}, etc., introduced below, M^i is called the *characteristic matrix* of the system, and $\vec{S}(\vec{u})$ describes source terms and contains only the fundamental variables but not their derivatives. The initial value formulation is well posed if the system of partial differential equations is *symmetric hyperbolic* (i.e., the matrices M^i are symmetric) and *strongly hyperbolic* if s_iM^i has a real set of eigenvalues and a complete set of eigenvectors for any 1-form s_i and obeys some boundedness conditions [see Solin (2006)].

The Cauchy problem for metric $f(R)$ gravity is well formulated and is well posed *in vacuo* and with matter, as shown below. For Palatini $f(R)$ gravity, instead, the Cauchy problem is unlikely to be well formulated or well posed unless the trace of the matter energy-momentum tensor is constant, due to the presence of higher derivatives of the matter fields in the field equations and to the impossibility of eliminating them (see below).

A systematic covariant approach to scalar-tensor theories of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{\psi(\phi)R}{2\kappa} - \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - W(\phi)\right] + S_M \tag{179}$$

is due to Salgado (2006), who showed that the Cauchy problem of these theories is well posed in the absence of matter and well formulated otherwise. With the exception of $\omega_0 = -3/2$, as we will see later, most of Salgado's results can be extended to the more general action

$$S = \int d^4x\sqrt{-g}\left[\frac{\psi(\phi)R}{2\kappa} - \frac{\omega(\phi)}{2}\partial^\alpha\phi\partial_\alpha\phi - W(\phi)\right] + S_M, \tag{180}$$

which contains the additional coupling function $\omega(\phi)$ (which is different from the Brans-Dicke parameter ω_0) (Lanahan-Tremblay and Faraoni, 2007).

The field equations, after setting $\kappa = 1$, are

$$\begin{aligned} G_{\mu\nu} = {} & \frac{1}{\psi}[\psi''(\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi) + \psi'(\nabla_\mu\nabla_\nu\phi \\ & - g_{\mu\nu}\Box\phi)] + \frac{1}{\psi}\left[\omega\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right)\right. \\ & \left. - W(\phi)g_{\mu\nu} + T^{(m)}_{\mu\nu}\right], \end{aligned} \tag{181}$$

$$\omega\Box\phi + \frac{\psi'}{2}R - W'(\phi) + \frac{\omega'}{2}\nabla^\alpha\phi\nabla_\alpha\phi = 0, \tag{182}$$

where a prime denotes differentiation with respect to ϕ. Equation (181) can be cast in the form of the effective Einstein equation $G_{\mu\nu} = T^{(\mathrm{eff})}_{\mu\nu}$, with the effective stress-energy tensor (Salgado, 2006)

$$T^{(\mathrm{eff})}_{\mu\nu} = \frac{1}{\psi(\phi)}(T^{(\psi)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}) \tag{183}$$

and

$$\begin{aligned} T^{(\psi)}_{\mu\nu} = {} & \psi''(\phi)(\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\nabla^\beta\phi\nabla_\beta\phi) \\ & + \psi'(\phi)(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi), \end{aligned} \tag{184}$$

[24] Furthermore, $R_{\mu\nu}R^{\mu\nu}$ can be expressed in terms of R^2 in a FLRW background (Wands, 1994).

$$T^{(\phi)}_{\mu\nu} = \omega(\phi)\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\beta\phi\nabla_\beta\phi\right) - W(\phi)g_{\mu\nu}. \tag{185}$$

The trace of the effective Einstein equations yields

$$\Box\phi = \left\{\psi\left[\omega + \frac{3(\psi')^2}{2\psi}\right]\right\}^{-1}\left\{\frac{\psi' T^{(m)}}{2} - 2\psi' W(\phi) + \psi W'(\phi) + \left[-\frac{\omega'\psi}{2} - \frac{\psi'}{2}(\omega + 3\psi'')\right]\nabla^c\phi\nabla_c\phi\right\}. \tag{186}$$

The 3+1 Arnowitt-Deser-Misner formulation of the theory proceeds by introducing lapse, shift, extrinsic curvature, and gradients of ϕ (Wald, 1984; Reula, 1998; Salgado, 2006). Assume that a time function t exists such that the space-time $(M, g_{\mu\nu})$ admits a foliation with hypersurfaces Σ_t of constant t with unit timelike normal n^a. The three-metric and projection operator on Σ_t are $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and $h^\alpha{}_\beta$, respectively. Moreover,

$$n^\mu n_\mu = -1, \quad h_{\alpha\beta}n^\beta = h_{\alpha\beta},$$
$$n^\alpha = 0, \quad h_\alpha{}^\beta h_{\beta\gamma} = h_{\alpha\gamma}. \tag{187}$$

The metric is then

$$ds^2 = -(N^2 - N^iN_i)dt^2 - 2N_i dt dx^i + h_{ij}dx^i dx^j \tag{188}$$

$(i,j=1,2,3)$, where $N>0$, $n_\alpha = -N\nabla_\alpha t$, $N^\alpha = -h^\alpha{}_\beta t^\beta$ is the shift vector, while t^α obeys $t^\alpha\nabla_\alpha t = 1$ and $t^\alpha = -N^\alpha + Nn^\alpha$ so that $N = -n_\alpha t^\alpha$ and $N^\alpha n_\alpha = 0$. The extrinsic curvature of Σ_t is

$$K_{\alpha\beta} = -h_\alpha{}^\gamma h_\beta{}^\delta \nabla_\gamma n_\delta \tag{189}$$

and the three-dimensional covariant derivative of $h_{\alpha\beta}$ on Σ_t is defined by

$$D_i^{(3)} T^{\alpha_1 \cdots}{}_{\beta_1 \cdots} = h^{\alpha_1}{}_{\gamma_1} \cdots h^{\delta_1}{}_{\beta_1} \cdots h^\mu{}_i \nabla_\mu^{(3)} T^{\gamma_1 \cdots}{}_{\delta_1 \cdots} \tag{190}$$

for any 3-tensor ${}^{(3)}T^{\mu_1 \cdots}{}_{\nu_1 \ldots}$, with $D_i h_{\mu\nu} = 0$. The spatial gradient of the scalar ϕ is $Q_\mu \equiv D_\mu\phi$ (where D_μ denotes the covariant derivative of $h_{\mu\nu}$), while its momentum is $\Pi = \mathcal{L}_n\phi = n^\nu\nabla_\nu\phi$ and

$$K_{ij} = -\nabla_i n_j = -\frac{1}{2N}\left(\frac{\partial h_{ij}}{\partial t} + D_iN_j + D_jN_i\right), \tag{191}$$

$$\Pi = \frac{1}{N}(\partial_t\phi + N^\gamma Q_\gamma), \tag{192}$$

$$\partial_t Q_i + N^l\partial_l Q_i + Q_l\partial_i N^l = D_i(N\Pi). \tag{193}$$

The effective stress-energy tensor $T^{(\mathrm{eff})}_{\alpha\beta}$ is decomposed as

$$T^{(\mathrm{eff})}_{\alpha\beta} = \frac{1}{\psi}(S_{\alpha\beta} + J_\alpha n_\beta + J_\beta n_\alpha + En_\alpha n_\beta), \tag{194}$$

where

$$S_{\alpha\beta} \equiv h_\alpha{}^\gamma h_\beta{}^\delta T^{(\mathrm{eff})}_{\gamma\delta} = \frac{1}{\psi}(S^{(\psi)}_{\alpha\beta} + S^{(\phi)}_{\alpha\beta} + S^{(m)}_{\alpha\beta}), \tag{195}$$

$$J_\alpha \equiv -h_\alpha{}^\gamma T^{(\mathrm{eff})}_{\gamma\delta}n^\delta = \frac{1}{\psi}(J^{(\psi)}_\alpha + J^{(\phi)}_\alpha + J^{(m)}_\alpha), \tag{196}$$

$$E \equiv n^\alpha n^\beta T^{(\mathrm{eff})}_{\alpha\beta} = \frac{1}{\psi}(E^{(\psi)} + E^{(\phi)} + E^{(m)}). \tag{197}$$

Its trace is $T^{(\mathrm{eff})} = S - E$, where $S \equiv S^\mu{}_\mu$. The Gauss-Codazzi equations then yield the Hamiltonian constraint (Wald, 1984; Salgado, 2006)

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 2E, \tag{198}$$

the vector constraint

$$D_lK^l{}_i - D_iK = J_i, \tag{199}$$

and the dynamical equations

$$\partial_t K^i{}_j + N^l\partial_l K^i{}_j + K^i{}_l\partial_j N^l - K^l{}_j\partial_l N^i + D^iD_jN - {}^{(3)}R^i{}_jN - NKK^i{}_j = \frac{N}{2}[(S-E)\delta^i_j - 2S^i_j], \tag{200}$$

where $K \equiv K^i{}_i$. The trace of Eq. (200) yields

$$\partial_t K + N^l\partial_l K + {}^{(3)}\Delta N - NK_{ij}K^{ij} = \frac{N}{2}(S+E), \tag{201}$$

where ${}^{(3)}\Delta \equiv D^iD_i$. Our purpose is to eventually eliminate all second derivatives. $\Box\phi$, which is present in Eqs. (195)–(197), can actually be eliminated using Eq. (186), provided that $\omega \neq -3(\psi')^2/(2\psi)$.

To be more precise, a direct calculation yields the ψ and ϕ quantities of Eqs. (195)–(197),

$$E^{(\psi)} = \psi'(D^\mu Q_\mu + K\Pi) + \psi'' Q^2, \tag{202}$$

$$J^{(\psi)}_\alpha = -\psi'(K^\gamma_\alpha Q_\gamma + D_\alpha\Pi) - \psi''\Pi Q_\alpha, \tag{203}$$

$$S^{(\psi)}_{\alpha\beta} = \psi'(D_\alpha Q_\beta + \Pi K_{\alpha\beta} - h_{\alpha\beta}\Box\phi) - \psi''[h_{\alpha\beta}(Q^2 - \Pi^2) - Q_\alpha Q_\beta], \tag{204}$$

where $Q^2 \equiv Q^\nu Q_\nu$, while

$$S^{(\psi)} = \psi'(D_\nu Q^\nu + K\Pi - 3\Box\phi) + \psi''(3\Pi^2 - 2Q^2), \tag{205}$$

and

$$E^{(\phi)} = \frac{\omega}{2}(\Pi^2 + Q^2) + W(\phi), \tag{206}$$

$$J^{(\phi)}_\mu = -\omega\Pi Q_\mu, \tag{207}$$

$$S^{(\phi)}_{\alpha\beta} = \omega Q_\alpha Q_\beta - h_{\alpha\beta}\left[\frac{\omega}{2}(Q^2 - \Pi^2) + W(\phi)\right], \tag{208}$$

$$S^{(\phi)} = \frac{\omega}{2}(3\Pi^2 - Q^2) - 3W(\phi). \tag{209}$$

The Hamiltonian and the vector constraints become

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} - \frac{2}{\psi}\left[\psi'(D_\mu Q^\mu + K\Pi) + \frac{\omega}{2}\Pi^2 + \frac{Q^2}{2}(\omega + 2\psi'') \right] = \frac{2}{\psi}[E^{(m)} + W(\phi)], \tag{210} $$

$$ D_l K^l{}_i - D_i K + \frac{1}{\psi}[\psi'(K_i{}^c Q_c + D_i\Pi) + (\omega + \psi'')\Pi Q_i] = \frac{J_i^{(m)}}{\psi}, \tag{211} $$

respectively, and the dynamical equation (200) is

$$ \partial_t K^i{}_j + N^l\partial_l K^i{}_j + K^i{}_l\partial_j N^l - K_j{}^l\partial_l N^i + D^i D_j N - {}^{(3)}R^i{}_j N - NKK^i{}_j + \frac{N}{2\psi}[\psi''(Q^2 - \Pi^2) + 2W(\phi) + \psi'\Box\phi]\delta^i_j + \frac{N\psi'}{\psi}(D^i_j + \Pi K^i{}_j)\frac{N}{\psi}(\omega + \psi'')Q^i Q_j = \frac{N}{2\psi}[(S^{(m)} - E^{(m)})\delta^i_j - 2S^{(m)i}{}_j], \tag{212} $$

with trace

$$ \partial_t K + N^l\partial_l K + {}^{(3)}\Delta N - NK_{ij}K^{ij} - \frac{N\psi'}{\psi}(D^\mu Q_\mu + \Pi K) + \frac{N}{2\psi}[\psi'' Q^2 - (2\omega + 3\psi'')\Pi^2] = \frac{N}{2\psi}(-2W(\phi) - 3\psi'\Box\phi + S^{(m)} + E^{(m)}), \tag{213} $$

where (Salgado, 2006)

$$ \mathcal{L}_n\Pi - \Pi K - Q^\mu D_\mu(\ln N) - D_\mu Q^\mu = -\Box\phi = -\frac{1}{\psi[\omega + 3(\psi')^2/2\psi]}\left\{ \frac{\psi' T^{(m)}}{2} - 2\psi' W(\phi) \times + \psi W'(\phi) + \left[\frac{-\omega'\psi}{2} - (\omega + 3\psi'')\frac{\psi'}{2} \right]\nabla^\mu\phi\nabla_\mu\phi \right\}. \tag{214} $$

In vacuo, the initial data $(h_{ij}, K_{ij}, \phi, Q_i, \Pi)$ on an initial hypersurface Σ_0 obey Eqs. (210) and (211), $Q_i = D_i\phi$, $D_i Q_j = D_j Q_i$. In the presence of matter, the variables $E^{(m)}$, $J_a^{(m)}$, $S_{ab}^{(m)}$ must also be assigned on the initial hypersurface Σ_0. Fixing a gauge corresponds to specifying the lapse and the shift vector. The system (210)–(213) contains only first-order derivatives in both space and time once the d'Alembertian $\Box\phi$ is written in terms of ϕ, $\nabla^\mu\phi\nabla_\mu\phi$, ψ, and its derivatives by means of Eq. (186) or (214). As mentioned, this can be done whenever $\omega \neq -3(\psi')^2/(2\psi)$. As pointed out by Salgado (2006) for the specific case $\omega = 1$, and which can now be generalized for any $\omega \neq -3(\psi')^2/(2\psi)$, the reduction to a first-order system shows that the Cauchy problem is well posed *in vacuo* and well formulated in the presence of reasonable matter.

Now consider the results specific to $f(R)$ gravity. Recall that Brans-Dicke theory, which is of interest for us due to its equivalence with $f(R)$ gravity, corresponds to $\omega(\phi) = \omega_0/\phi$, with $\psi(\phi) = \phi$ and $W \to 2V$. This yields the constraints

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} - \frac{2}{\phi}\left[D^\mu Q_\mu + K\Pi + \frac{\omega_0}{2\phi}(\Pi^2 + Q^2) \right] = \frac{2}{\phi}[E^{(m)} + V(\phi)], \tag{215} $$

$$ D_l K^l{}_i - D_i K + \frac{1}{\phi}\left(K_i{}^l Q_l + D_i\Pi + \frac{\omega_0}{\phi}\Pi Q_i \right) = \frac{J_i^{(m)}}{\phi}, \tag{216} $$

the dynamical equations

$$ \partial_t K^i{}_j + N^l\partial_l K^i{}_j + K^i{}_l\partial_j N^l - K_j{}^l\partial_l N^i + D^i D_j N - {}^{(3)}R^i{}_j N - NKK^i{}_j + \frac{N}{2\phi}\delta^i_j[2V(\phi) + \Box\phi] + \frac{N}{\phi}(D^i Q_j + \Pi K^i{}_j) + \frac{N\omega_0}{\phi^2}Q^i Q_j = \frac{N}{2\phi}[(S^{(m)} - E^{(m)})\delta^i_j - 2S^{(m)i}{}_j], \tag{217} $$

$$ \partial_t K + N^l\partial_l K + {}^{(3)}\Delta N - NK_{ij}K^{ij} - \frac{N}{\phi}(D^\nu Q_\nu + \Pi K) - \frac{\omega_0 N}{\phi^2}\Pi^2 = \frac{N}{2\phi}[-2V(\phi) - 3\Box\phi + S^{(m)} + E^{(m)}], \tag{218} $$

with

$$ \left(\omega_0 + \frac{3}{2}\right)\Box\phi = \frac{T^{(m)}}{2} - 2V(\phi) + \phi V'(\phi) + \frac{\omega_0}{\phi}(\Pi^2 - Q^2). \tag{219} $$

The condition $\omega \neq -3(\psi')^2/(2\psi)$, which needs to be satisfied in order for one to be able to use Eq. (186) in order to eliminate $\Box\phi$, can be written in the Brans-Dicke theory notation as $\omega_0 \neq -3/2$. One could of course have guessed that by looking at Eq. (219). Therefore, metric $f(R)$ gravity, which is equivalent to $\omega_0 = 0$ Brans-Dicke gravity, has a well-formulated Cauchy problem in general and is well posed *in vacuo*. Further work by Salgado *et al.* (2008) established the well-posedness of the Cauchy problem for scalar-tensor gravity with $\omega = 1$ in the presence of matter; this can be translated into the well posedness of metric $f(R)$ gravity with matter along the lines established above.

How about Palatini $f(R)$ gravity, which, corresponding to $\omega_0 = -3/2$, is exactly the case that the constraint $\omega_0 \neq -3/2$ excludes? Actually, for this value of the Brans-Dicke parameter, Eq. (69), and consequently Eq. (219), include no derivatives of ϕ. Therefore, one can actually solve algebraically for ϕ. [The same could be done using Eq. (186) in the more general case where ω is a function of ϕ when $\omega = -3(\psi')^2/(2\psi)$.] We will not consider cases for which Eq. (69) has no roots or when it is identically satisfied *in vacuo*. These cases lead to theories for which, in the Palatini $f(R)$ formulation, Eq. (21) has no roots or when it is identically satisfied *in vacuo* respectively. As mentioned in Sec. II.B, the first case leads to inconsistent field equations and the second to a conformally invariant theory (Ferraris *et al.*, 1992); see also Sotiriou (2006b) for a discussion.

Now, *in vacuo* one can easily show that the solutions of Eq. (69) or (219) will be of the form ϕ=const. Therefore, all derivatives of ϕ vanish and one concludes that $\omega_0=-3/2$ Brans-Dicke theory or Palatini $f(R)$ gravity has a well-formulated and well-posed Cauchy problem.[25] This could have been expected, as noticed by Olmo and Singh (2009), considering that Palatini $f(R)$ gravity reduces to GR with a cosmological constant *in vacuo*.

In the presence of matter, things are more complicated. The solutions of Eq. (69) or (219) will give ϕ as a function of T, the trace of the stress-energy tensor. This can still be used to replace ϕ in all equations but it will lead to terms such as $\Box T$. Therefore, for the Cauchy problem to be well formulated in the presence of matter, one not only has to assume that the matter is "reasonable," in the sense that the matter fields satisfy a quasilinear, diagonal, second-order hyperbolic system of equations [see Wald (1984)], but also to require that the matter field equations are such that they allow us to express all derivatives of T present in Eqs. (215)–(218) for $\omega_0=-3/2$ in terms of only first derivatives of the matter fields. It seems highly implausible that this requirement can be fulfilled for generic matter fields. This seems to imply that $\omega_0=-3/2$ Brans-Dicke theory and Palatini $f(R)$ gravity are unlikely to have a well-formulated Cauchy problem in the presence of matter fields. However, more precise conclusions can be drawn only if specific matter fields are considered on a case by case basis. The complications arising from the appearance of derivatives of T and consequently higher derivatives of the matter fields in the equations, and which seem to be critical for whether the Cauchy problem can be well formulated in the presence of matter, will be better understood in Sec. VI.C.2.

VI. CONFRONTATION WITH PARTICLE PHYSICS AND ASTROPHYSICS

A. Metric $f(R)$ gravity as dark matter

Although most recent motivation for $f(R)$ gravity originates from the need to find alternatives to the mysterious dark energy at cosmological scales, several authors adopt the same perspective at galactic and cluster scales, using metric $f(R)$ gravity as a substitute for dark matter (Capozziello *et al.*, 2004; Capozziello, Cardone, Carloni, *et al.*, 2005; Capozziello, Cardone, and Troisi, 2006, 2007; Capozziello, Nojiri, and Odintsov, 2006; Capozziello, Troisi, and Cardone, 2007; Iorio and Ruggiero, 2007, 2008; Martins and Salucci, 2007; Saffari and Sobouti, 2007; Jhingan *et al.*, 2008; Nojiri and Odintsov, 2008a; Zhao and Li, 2008). Given the equivalence between $f(R)$ and scalar-tensor gravity, these efforts resemble previous attempts to model dark matter using scalar fields (Matos and Guzman, 2000; Matos and Urena-Lopez, 2001, 2007; Matos *et al.*, 2001; Alcubierre *et al.*, 2002a, 2002b; Rodriguez-Meza and Cervantes-Cota, 2004; Bernal and Matos, 2005; Rodriguez-Meza *et al.*, 2005, 2007; Bernal and Guzman, 2006a, 2006b, 2006c; Cervantes-Cota, Rodriguez-Meza, Gabbasov, and Klapp, 2007; Cervantes-Cota, Rodriguez-Meza, and Nunez, 2007).

Most works concentrate on models of the form $f(R)=R^n$. A theory of this form with $n=1-\alpha/2$ was studied by Mendoza and Rosas-Guevara (2007) and Saffari and Sobouti (2007) using spherically symmetric solutions to approximate galaxies. The fit to galaxy samples yields

$$\alpha=(3.07\pm 0.18)\times 10^{-7}\left(\frac{M}{10^{10}M_{\odot}}\right)^{0.494}, \tag{220}$$

where M is the mass appearing in the spherically symmetric metric (the mass of the galaxy). Notice that having α depending on the mass of each individual galaxy straightforwardly implies that one cannot fit the data for all galactic masses with the same choice of $f(R)$. This makes the whole approach highly implausible.

Capozziello *et al.* (2004), Capozziello, Cardone, *et al.* (2005), Capozziello, Cardone, and Troisi (2006, 2007), and Capozziello, Troisi, and Cardone (2007) computed weak-field limit corrections to the Newtonian galactic potential and the resulting rotation curves; when matched to galaxy samples, a best fit yields $n\simeq 1.7$. Martins and Salucci (2007) performed a χ^2 fit using two broader samples, finding $n\simeq 2.2$ [see also Boehmer, Harko, and Lobo (2008a) for a variation of this approach focusing on the constant velocity tails of the rotation curves]. All these values of the parameter n are in violent contrast with the bounds obtained by Clifton and Barrow (2005a, 2006), Barrow and Clifton (2006), and Zakharov *et al.* (2006) and have been shown to violate also the current constraints on the precession of perihelia of several Solar System planets (Iorio and Ruggiero, 2007, 2008). In addition, the consideration of vacuum metrics used in these works in order to model the gravitational field of galaxies is highly questionable.

The potential obtained in the weak-field limit of $f(R)$ gravity can affect other aspects of galactic dynamics as well: the scattering probability of an intruder star and the relaxation time of a stellar system were studied by Hadjimichef and Kokubun (1997), originally motivated by quadratic corrections to the Einstein-Hilbert action.

[25] This was missed by Lanahan-Tremblay and Faraoni (2007), who claimed that the Cauchy problem is not well posed because the constraint $\omega_0\neq-3/2$ does not allow for the use of Eq. (219) in order to eliminate $\Box\phi$. Note also that in the absence of a potential [there is no corresponding Palatini $f(R)$ gravity] $\omega_0=-3/2$ Brans-Dicke theory does not have a well-posed Cauchy problem, as noticed by Noakes (1983), because this theory is conformally invariant and ϕ is indeterminate.

B. Palatini $f(R)$ gravity and the conflict with the standard model

One important and unexpected shortcoming of Palatini $f(R)$ gravity is that it appears to be in conflict with the standard model of particle physics in the sense that it introduces nonperturbative corrections to the matter action (or the field equations) and strong couplings between gravity and matter in the local frame and at low energies. The reason that we call this shortcoming unexpected is that, judging by the form of the action (13), Palatini $f(R)$ gravity is, as mentioned, a metric theory of gravity in the sense that matter is only coupled minimally to the metric. Therefore, the stress-energy tensor is divergence-free with respect to the metric covariant derivative, the metric postulates (Will, 1981) are satisfied, the theory apparently satisfies the Einstein equivalence principle, and the matter action should trivially reduce locally to that of special relativity.

We now see how this conflict comes about. This issue was first pointed out by Flanagan (2004b) using Dirac particles for the matter action as an example and later on studied again by Iglesias *et al.* (2007) who assumed that the matter action is that of the Higgs field [see also Olmo (2008)]. Both calculations use the equivalent Brans-Dicke theory and are performed in the Einstein frame. Although the use of the Einstein frame has been criticized (Vollick, 2004),[26] it is equivalent to the Jordan frame and both are perfectly suitable for performing calculations (Flanagan, 2004a) [see also the discussion in Sec. III and Faraoni and Nadeau (2007) and Sotiriou *et al.* (2008)].

Nevertheless, since test particles are supposed to follow geodesics of the Jordan-frame metric, it is this metric which becomes approximately flat in the laboratory reference frame. Therefore, when the calculations are performed in the Einstein frame they are less transparent since the actual effects could be confused with frame effects, and vice versa. Consequently, for simplicity and clarity, we present the calculation in the Jordan frame, as it appears in Barausse *et al.* (2008b). We begin from the action (66), which is the Jordan-frame equivalent of Palatini $f(R)$ gravity, and we take matter to be represented by a scalar field H (e.g., the Higgs boson), the action of which reads

$$S_M = \frac{1}{2\hbar}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu H\partial_\nu H - \frac{m_H^2}{\hbar^2}H^2\right) \tag{221}$$

(in units in which $G=c=1$). As an example, we choose $f(R)=\mathcal{R}-\mu^4/\mathcal{R}$ (Vollick, 2003; Carroll *et al.*, 2004). For this choice of f, the potential is $V(\phi)=2\mu^2(\phi-1)^{1/2}$. To go to the local frame, we expand the action to second order around vacuum. The vacuum of the action (66) with Eq. (221) as a matter action is $H=0$, $\phi=4/3$ [using Eq. (69)], and $g_{\mu\nu}\simeq\eta_{\mu\nu}$ ($\mu^2\sim\Lambda$ acts as an effective cosmological constant, so its contribution in the local frame can be safely neglected).

However, when one tries to use a perturbative expansion for ϕ, things stop being straightforward: ϕ is algebraically related to the matter fields as is obvious from Eq. (69). Therefore, one gets $\delta\phi\sim T/\mu^2\sim m_{\rm H}^2\delta H^2/\hbar^3\mu^2$ at energies lower than the Higgs mass ($m_H \sim 100-1000$ GeV). Replacing this expression in the action (66) perturbed to second order, one immediately obtains that the effective action for the Higgs scalar is

$$S_M^{\rm eff} \simeq \int d^4x\sqrt{-g}\frac{1}{2\hbar}\left(g^{\mu\nu}\partial_\mu\delta H\partial_\nu\delta H - \frac{m_H^2}{\hbar^2}\delta H^2\right) \times\left[1+\frac{m_H^2\delta H^2}{\mu^2\hbar^3}+\frac{m_H^2(\partial\delta H)^2}{\mu^4\hbar^3}\right] \tag{222}$$

at energies $E\ll m_H$. Taking into account the fact that $\mu^2\sim\Lambda\sim H_0^2$, where $H_0^{-1}=4000$ Mpc is the Hubble radius and $\delta H\sim m_H$ because $E\ll m_H$, it is not difficult to estimate the order of magnitude of the corrections: at an energy $E=10^{-3}$ eV (corresponding to the length scale $L=\hbar/E=2\times10^{-4}$ m), the first correction is of the order

$$m_H^2\delta H^2/\mu^2\hbar^3 \sim (H_0^{-1}/\lambda_H)^2(m_H/M_P)^2 \gg 1,$$

where $\lambda_H=\hbar/m_H\sim2\times10^{-19}-2\times10^{-18}$ m is the Compton wavelength of the Higgs and $M_P=(\hbar c^5/G)^{1/2}=1.2\times10^{19}$ GeV is the Planck mass. The second correction is of the order

$$m_H^2(\partial\delta H)^2/\mu^4\hbar^3 \sim (H_0^{-1}/\lambda_{\rm XH})^2(m_H/M_P)^2(H_0^{-1}/L)^2 \gg 1.$$

Clearly, it is unacceptable to have such nonperturbative corrections to the local frame matter action.

An alternative way to see the same problem would be to replace $\delta\phi\sim m_H^2\delta H^2/\hbar^3\mu^2$ directly in Eq. (66). Then the coupling of matter to gravity is described by the interaction Lagrangian

$$\mathcal{L}_{\rm int} \sim \frac{m_H^2\delta H^2}{\hbar^3}\left(\delta g+\frac{\partial^2\delta g}{\mu^2}\right) \sim \frac{m_H^2\delta H^2}{\hbar^3}\delta g\left[1+\left(\frac{H_0^{-1}}{L}\right)^2\right]. \tag{223}$$

This clearly exhibits the fact that gravity becomes nonperturbative at microscopic scales.

It is obvious that the algebraic dependence of ϕ on the matter fields stands at the root of this problem. We have still not given any explanation for the "paradox" of seeing such a behavior in a theory which apparently sat-

[26] Note that in the case of Flanagan (2004b) in which fermions are used as the matter fields, one could decide to couple the independent connection to them by allowing it to enter the matter action and define the covariant derivative (which would be equivalent to assuming that the spin connection is an independent variable in a tetrad formalism) as noted by Vollick (2005). Although the results of Flanagan (2004b) would cease to hold in this case, this cannot be considered a problem: clearly in this case we would be talking about a different theory, namely, metric-affine $f(R)$ gravity (Sotiriou and Liberati, 2007b).

isfies the metric postulates in both the $f(R)$ and the Brans-Dicke representations. However, this will become clear in Sec. VI.C.2.

C. Exact solutions and relevant constraints

1. Vacuum and nonvacuum exact solutions

We now turn our attention to exact solutions starting from metric $f(R)$ gravity. We mentioned in Sec. II.A that, as can be seen easily from the form of the field equations (6), the maximally symmetric vacuum solution will be either Minkowski space-time, if $R=0$ is a root of Eq. (9), or de Sitter and anti–de Sitter space-time, depending on the sign of the root of the same equation. Things are slightly more complicated for vacuum solutions with less symmetry: Using Eq. (6) it is easy to verify that any vacuum solution ($R_{\mu\nu}=\Lambda g_{\mu\nu}$, $T_{\mu\nu}=0$) of Einstein's theory with a (possibly vanishing) cosmological constant, including black hole solutions, is a solution of metric $f(R)$ gravity [except for pathological cases for which Eq. (9) has no roots]. However, the converse is not true.

For example, when spherical symmetry is imposed, the Schwarzschild metric is a solution of metric $f(R)$ gravity if $R=0$ in vacuum. If R is constant *in vacuo*, then Schwarzschild–(anti–)de Sitter space-time is a solution. As mentioned though, the Jebsen-Birkhoff theorem (Weinberg, 1972; Wald, 1984) does not hold in metric $f(R)$ gravity [unless, of course one wishes to impose further conditions, such as that R is constant (Capozziello *et al.*, 2008)]. Therefore, other solutions can exist as well. An interesting finding is that the cosmic no-hair theorem valid in GR and in pure $f(R)$ gravity is not valid, in general, in theories of the form

$$S=\frac{1}{2\kappa}\int d^4x\sqrt{-g}(R+\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}-2\Lambda), \qquad (224)$$

for which exact anisotropic solutions that continue to inflate anisotropically have been found (Barrow and Hervik, 2006a, 2006b) [see also Maeda (1988) and Kluske and Schmidt (1996)]. However, isotropization during inflation occurs in mixed $f(\phi,R)$ models (Maeda *et al.*, 1989).

In addition to the exact cosmological solutions explored for the purpose of explaining the current cosmic acceleration [see, e.g., Abdalla *et al.* (2005); Clifton and Barrow (2005b, 2006); Modak *et al.* (2005); Barrow and Clifton (2006); Clifton (2006a, 2007); Capozziello *et al.* (2007a); Capozziello and De Felice (2008); and Vakili (2008) for an approach based on Noether symmetries; see Carloni *et al.* (2006) for bouncing solutions and the conditions that they satisfy], exact spherically symmetric solutions of metric $f(R)$ gravity have been explored in the literature, with most recent studies being motivated by the need to understand the weak-field limit of cosmologically motivated theories.

Regarding nonvacuum solutions, the most common matter source is a perfect fluid. Fluid dynamics in metric $f(R)$ gravity was studied by Maartens and Taylor (1994), Taylor and Maartens (1995), Rippl *et al.* (1996), and Mohseni Sadjadi (2008). Spherically symmetric solutions were found by Whitt (1985), Mignemi and Wiltshire (1992), Bronnikov and Chernakova (2005a, 2005b, 2005c), Clifton (2006a, 2006b), Multamaki and Vilja (2006b, 2007, 2008), Bustelo and Barraco (2007), and Capozziello *et al.* (2008). We regret not being able to present these solutions extensively here due to space limitations and refer the interested reader to the literature for more details.

Stability issues for spherically symmetric solutions were discussed by Seifert (2007). In the theory

$$S=\int d^4x\frac{\sqrt{-g}}{\kappa}[R-\alpha R^2-\beta R_{\mu\nu}R^{\mu\nu}]+\epsilon\chi, \qquad (225)$$

where α, β, and ϵ are constants and χ is the Gauss-Bonnet invariant, the Schwarzschild metric is a solution, and the stability of Schwarzschild black holes was studied by Whitt (1985). Surprisingly, it was found that the massive ghost graviton present in this theory stabilizes small-mass black holes against quantum instabilities [see also Myers and Simon (1988, 1989)]. In the case $\beta=\epsilon=0$, which reduces the theory (225) to a quadratic $f(R)$ gravity, the stability criterion found by Whitt (1985) reduces to $\alpha<0$, which corresponds again to $f''(R)>0$. For $\alpha=0$ we recover GR, in which black holes are stable classically [but not quantum mechanically due to Hawking radiation and their negative specific heat, a feature that persists in $f(R)$ gravity], so the classical stability condition for Schwarzschild black holes can be enunciated as $f''(R)\geqslant 0$.

We now turn our attention to Palatini $f(R)$ gravity. In this case things are simpler *in vacuo*: as we saw in Sec. II.B, the theory reduces in this case (or more precisely even for matter fields with $T=$const, where T is the trace of the stress-energy tensor) to GR with a cosmological constant, which might as well be zero for some models (Ferraris *et al.*, 1992, 1994; Borowiec *et al.*, 1998; Barraco *et al.*, 1999). Therefore, it is quite straightforward that Palatini $f(R)$ gravity will have the same vacuum solutions as GR with a cosmological constant. Also, the Jebsen-Birkhoff theorem (Weinberg, 1972; Wald, 1984) is valid in the Palatini formalism (Kainulainen, Reijonen, and Sunhede, 2007; Barausse *et al.*, 2008a, 2008b, 2008c).

Cosmological solutions in quadratic gravity were obtained by Shahid-Saless (1990, 1991). Spherically symmetric interior solutions in the Palatini formalism can be found using the generalization of the Tolman-Oppenheimer-Volkoff equation valid for these theories, as found by Barraco and Hamity (2000), Bustelo and Barraco (2007), and Kainulainen, Reijonen, and Sunhede (2007). Indeed, such solutions have been found and matched with the unique exterior (anti–)de Sitter solution (Barraco and Hamity, 1998, 2000; Bustelo and Barraco, 2007; Kainulainen, Piilonen, *et al.*, 2007; Kainulainen, Reijonen, and Sunhede, 2007). Nevertheless, a matching between exterior and interior that can lead to a sensible solution throughout space-time is not always feasible, and this seems to have serious consequences for the viability of $f(R)$ gravity (Barausse *et al.*, 2008a, 2008b, 2008c). This is discussed extensively in the next section.

We close this section with some remarks on black hole solutions. As is clear from the above discussion, all black hole solutions of GR (with a cosmological constant) will also be solutions of $f(R)$ in both the metric and the Palatini formalism [see also Barausse and Sotiriou (2008) and Psaltis *et al.* (2008)]. However, in the Palatini formalism they will constitute the complete set of black hole solutions of the theory, whereas in the metric formalism other black hole solutions can exist in principle, as the Jebsen-Birkhoff theorem does not hold. For a discussion on black hole entropy in $f(R)$ gravity see Jacobson *et al.* (1994, 1995) and Vollick (2007).[27]

2. Surface singularities and the incompleteness of Palatini $f(R)$ gravity

In Secs. V.A.3, V.C, and VI.B, we already spotted three serious shortcomings of Palatini $f(R)$ gravity, namely, the algebraic dependence of the post-Newtonian metric on the density, the complications with the initial value problem in the presence of matter, and a conflict with particle physics. In this section we study static spherically symmetric interior solutions and their matching to the unique exterior with the same symmetries, the Schwarzschild–de Sitter solution, along the lines of Barausse *et al.* (2008a, 2008b, 2008c). As we will see, the three problems mentioned earlier are actually very much related and stem from a specific characteristic of Palatini $f(R)$ gravity, which the discussion of this section will help us pin down.

A common way of arriving at a full description of a space-time that includes matter is to solve the field equations separately inside and outside the sources and then match the interior and exterior solutions using appropriate junction conditions [called Israel junction conditions in GR (Israel, 1966)]. This is what we are going to attempt here. We already know the exterior solution so, for the moment, we focus on the interior. Since we assume that the metric is static and spherically symmetric, we can write it in the form

$$ds^2 \equiv -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2 d\Omega^2. \tag{226}$$

We can then replace this metric in the field equations of Palatini $f(R)$ gravity, preferably in Eq. (28), which is the simplest of all the possible reformulations. Assuming also a perfect fluid description for matter with $T_{\mu\nu}=(\rho+P)u_\mu u_\nu + P g_{\mu\nu}$, where ρ is the energy density, P is the pressure, and u^μ is the fluid four velocity, and representing d/dr with a prime,[28] one arrives at

$$A' = \frac{-1}{1+\gamma}\left(\frac{1-e^B}{r} - \frac{e^B}{F}8\pi G r P + \frac{\alpha}{r}\right), \tag{227}$$

$$B' = \frac{1}{1+\gamma}\left(\frac{1-e^B}{r} + \frac{e^B}{F}8\pi G r \rho + \frac{\alpha+\beta}{r}\right), \tag{228}$$

$$\alpha \equiv r^2\left[\frac{3}{4}\left(\frac{F'}{F}\right)^2 + \frac{2F'}{rF} + \frac{e^B}{2}\left(\mathcal{R} - \frac{f}{F}\right)\right], \tag{229}$$

$$\beta \equiv r^2\left[\frac{F''}{F} - \frac{3}{2}\left(\frac{F'}{F}\right)^2\right], \quad \gamma \equiv \frac{rF'}{2F}, \tag{230}$$

where $F \equiv \partial f/\partial R$. To determine an interior solution we need a generalization of the Tolman-Oppenheimer-Volkoff (TOV) hydrostatic equilibrium equation. This has been derived for Palatini $f(R)$ gravity by Barraco and Hamity (2000), Bustelo and Barraco (2007), and Kainulainen, Reijonen, and Sunhede (2007). Defining $m_{\text{tot}}(r) \equiv r(1-e^{-B})/2$ and using Euler's equation

$$P' = -\frac{A'}{2}(P+\rho), \tag{231}$$

one can use Eqs. (227) and (228) to arrive at the generalized TOV equations,

$$P' = -\frac{1}{1+\gamma}\frac{(\rho+P)}{r(r-2m_{\text{tot}})}\left(m_{\text{tot}} + \frac{4\pi r^3 P}{F} - \frac{\alpha}{2}(r-2m_{\text{tot}})\right), \tag{232}$$

$$m'_{\text{tot}} = \frac{1}{1+\gamma}\left(\frac{4\pi r^2 \rho}{F} + \frac{\alpha+\beta}{2} - \frac{m_{\text{tot}}}{r}(\alpha+\beta-\gamma)\right). \tag{233}$$

We now have three differential equations, namely, Eqs. (231)–(233), and four unknown functions, namely, A, m_{tot} (or B), P, and ρ. The missing piece is the information about the microphysics of the matter configuration under investigation. In the case of a perfect fluid, this is effectively given by an equation of state (EOS). A one-parameter EOS relates the pressure directly to the energy density, i.e., $P=P(\rho)$. A simple form of such an EOS is a polytropic equation of state $P=k\rho_0^\Gamma$, where ρ_0 is the rest-mass density and k and Γ are constants. This is the case that we consider here. Note that the rest-mass density can be expressed in terms of the energy density ρ and the internal energy U as $\rho_0=\rho-U$. Assuming an adiabatic transformation and using the first law of ther-

[27] See also Eling *et al.* (2006) for a derivation of the field equations of metric $f(R)$ gravity based on thermodynamical arguments applied to local Rindler horizons.

[28] In this section we modify our standard notation for economy and a prime denotes differentiation with respect to the radial coordinate instead of differentiation with respect to the argument of the function.

modynamics, one can express the internal energy in terms of the pressure, i.e., $U=P/(\Gamma-1)$. Therefore, the polytropic EOS can be rewritten as

$$\rho = \left(\frac{P}{k}\right)^{1/\Gamma} + \frac{P}{\Gamma-1}, \tag{234}$$

giving a direct link between P and ρ.

Without specifying the interior solution, we can already examine the appropriate matching conditions needed. One needs continuity of the metric and of A' on the surface of the matter configuration (A is given by a second-order differential equation). Since we know that the exterior solution is unique and is the Schwarzschild–de Sitter solution with a cosmological constant equal to $\mathcal{R}_0/4$, where $\mathcal{R}_0$ is the vacuum value of $\mathcal{R}$ (see Sec. II.B), we can directly write for the exterior

$$\exp[-B(r)] = \ell \exp[A(r)] = 1 - 2m/r - \mathcal{R}_0 r^2/12, \tag{235}$$

where ℓ and m are integration constants to be fixed by requiring continuity of the metric coefficients across the surface, which is implicitly defined by $r=r_{\rm out}$, where $P=\rho=0$. Using the definition of $m_{\rm tot}(r)$ this gives, in the exterior,

$$m_{\rm tot}(r) = m + \frac{r^3}{24}\mathcal{R}_0. \tag{236}$$

On the other hand, based on the exterior solution, one gets on the surface

$$A'(r_{\rm out}) = \frac{2(r_{\rm out}^3\mathcal{R}_0 - 12m)}{r_{\rm out}(\mathcal{R}_0 r_{\rm out}^3 - 12 r_{\rm out} + 24m)}. \tag{237}$$

Assuming that, approaching the surface from the interior, A and $m_{\rm tot}$ indeed take the correct values required for the matching, it can be shown that continuity of A' across the surface requires $F'(r_{\rm out})=0$ for $r\to r_{\rm out}^-$ (Barausse *et al.*, 2008a). Additionally, if this is the case then (Barausse *et al.*, 2008a)

$$m'_{\rm tot}(r_{\rm out}) = \frac{2F_0\mathcal{R}_0 r_{\rm out}^2 + (r_{\rm out}^3\mathcal{R}_0 - 8m_{\rm tot})\mathcal{C}'}{16F_0}, \tag{238}$$

where

$$\mathcal{C} = \frac{dF}{dP}(P+\rho) = \frac{dF}{d\rho}\frac{d\rho}{dP}(P+\rho). \tag{239}$$

We now examine the behavior of $m'_{\rm tot}$ at the surface for different values of the polytropic index Γ. For $1<\Gamma<3/2$, $\mathcal{C}'=d\mathcal{C}/dPP' \propto d\mathcal{C}/dP(P+\rho)\to 0$ at the surface so that Eq. (238) is finite and it gives continuity of $m'_{\rm tot}$ across the surface [cf. Eq. (236)]. However, for $3/2<\Gamma<2$, $\mathcal{C}'\to\infty$ as the surface is approached, provided that $dF/d\mathcal{R}(\mathcal{R}_0)\neq 0$ and $d\mathcal{R}/dT(T_0)\neq 0$ [note that these conditions are satisfied by generic forms of $f(\mathcal{R})$, i.e., whenever an $\mathcal{R}^2$ term or a term inversely proportional to $\mathcal{R}$ is present]. Therefore, even though $m_{\rm tot}$ remains finite (as can be shown using the fact that $P'=0$ at the surface), the divergence of $m'_{\rm tot}$ drives to infinity the Riemann tensor of the metric $R_{\mu\nu\sigma\lambda}$ and curvature invariants, such as R or $R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda}$, as can easily be checked.[29] Clearly, such a singular behavior is bound to give rise to unphysical phenomena, such as infinite tidal forces at the surface (cf. the geodesic deviation equation), which would destroy anything present there. We are, therefore, forced to conclude that no physically relevant solution exists for any polytropic EOS with $3/2<\Gamma<2$.

The following points about the result just presented should be stressed:

- The sufficient condition for the singularity to occur is that a polytropic EOS with $3/2<\Gamma<2$ should adequately describe just the outer layer of the matter configuration (and not necessarily the whole configuration).
- In practice, there is no dependence of the result on the functional form of $f(\mathcal{R})$ [a few unrealistic exceptions can be found in Barausse *et al.* (2008a)] so what is revealed is a generic aspect of Palatini $f(R)$ gravity as a class of theories.
- The singularities discussed are not coordinate, but true singularities, as can be easily verified by checking that curvature invariants diverge.
- The only assumptions made regard the EOS and the symmetries. Thus, the result applies to all regimes ranging from Newtonian to strong gravity.

We now interpret these results. Obviously, one could object to the use of the polytropic EOS. Even though it is extensively used for simple stellar models, it is not a very realistic description for stellar configurations. However, one does not necessarily have to refer to stars in order to check whether the issue discussed here leaves an observable signature. Consider two very well-known matter configurations which are exactly described by a polytropic EOS: a monoatomic isentropic gas and a degenerate nonrelativistic electron gas. For both of those cases $\Gamma=5/3$, which is well within the range for which the singularities appear. Additionally, both of these configuration can be very well described even with Newtonian gravity. Yet Palatini $f(R)$ gravity fails to provide a reasonable description. Therefore, one could think of such matter configurations as gedanken experiments which reveal that Palatini $f(R)$ gravity is at best incomplete (Barausse *et al.*, 2008a, 2008b, 2008c).

On the other hand, the use of the polytropic EOS requires a perfect fluid approximation for the description of matter. One may therefore wish to question whether the length scale on which the tidal forces become important is larger than the length scale for which the fluid approximation breaks down (Kainulainen, Piilonen, *et al.*, 2007). However, quantitative results for tidal forces have been given by Barausse *et al.* (2008b), and it has been shown that the length scales at which the tidal forces become relevant are indeed larger than

[29]This fact seems to have been missed by Barraco and Hamity (2000).

would be required for the fluid approximation to break down. The observable consequences on stellar configurations have also been discussed there. To this, one could also add that a theory that requires a full description of the microscopic structure of the system in order to provide a macroscopic description of the dynamics is not very appealing anyway.

In any case, it should be stated that the problem discussed is not specific to the polytropic EOS. The use of the latter only simplifies the calculation and allows an analytic approach. The root of the problem actually lies with the differential structure of Palatini $f(R)$ gravity.

Consider the field equations in the form (28): it is not difficult to see that these are second-order partial differential equations in the metric. However, since f is a function of $\mathcal{R}$ and $\mathcal{R}$ is an algebraic function of T due to Eq. (20), the right-hand side of Eq. (28) includes second derivatives of T. Now T, being the trace of the stress-energy tensor, will include up to first-order derivatives of the matter fields (assuming that the matter action has to lead to second-order field equations when varied with respect to the matter fields). Consequently, Eq. (28) can be up to third order in the matter fields.

In GR and most of its alternatives, the field equations are only of first order in the matter fields. This guarantees that gravity is a cumulative effect: the metric is generated by an integral over the matter sources and, therefore, any discontinuities (or even singularities) in the latter and their derivatives, which are allowed, will not become discontinuities or singularities of the metric, which are not allowed [see Barausse *et al.* (2008b) for a detailed discussion]. This characteristic is not present in Palatini $f(R)$ gravity and creates an algebraic dependence of the metric on the matter fields.

The polytropic description not only does not cause this problem but, as a matter of fact, it makes it less acute, simply because in the fluid approximation the stress-energy tensor does not include derivatives of the matter fields and effectively "smooths out" the matter distribution. Actually, the fact that the metric is extremely sensitive to rapid changes of the matter field has been exhibited also in the interior of stars described by realistic tabulated EOSs in Barausse *et al.* (2008a).

One should not be puzzled by the fact that this awkward differential structure of Palatini $f(R)$ gravity is not manifest in the $f(\mathcal{R})$ formulation of the theory [and the field equations (18) and (19)]. We have mentioned that the independent connection is actually an auxiliary field and the presence of auxiliary fields can always be misleading when it comes to the dynamics. In fact, it just takes a closer look to realize that the Palatini $f(\mathcal{R})$ action does not contain any derivatives of the metric and is of only first order in the derivatives of the connection. Now, given that the connection turns out to be an auxiliary field and can be algebraically related to derivatives of the matter and of the metric, it no longer comes as a surprise that the outcome is a theory with higher differential order in the matter than the metric.

By now, the fact that the post-Newtonian metric turns out to be algebraically dependent on the density, as discussed in Sec. V.A.3, should no longer sound surprising: it is merely a manifestation of the problem discussed here in the weak-field regime. The fact that it is unlikely that the Cauchy problem will be well formulated in the presence of matter also originates from the same feature of Palatini $f(R)$ gravity, as mentioned. Similarly, the fact that a theory, which manifestly satisfies the metric postulates and therefore is expected to satisfy the equivalence principle, actually exhibits unexpected phenomenology in local nongravitational experiments and conflicts with the standard model, as shown in Sec. VI.B also ceases to be a puzzle: the algebraic dependence of the connection on the derivatives of matter fields (as the former is an auxiliary field) makes the matter enter the gravitational action through the "back door." This introduces strong couplings between matter and gravity and self-interactions of the matter fields, which manifest themselves in the local frame. Alternatively, if one completely eliminates the connection (or the scalar field in the equivalent Brans-Dicke representation) at the level of the action or attempts to write down an action which leads to the field equations (28) directly through metric variation, then this action will have to include higher-order derivatives of the matter field and self-interactions in the matter sector. In this sense, the $f(\mathcal{R})$ representation is simply misleading [see also Sotiriou *et al.* (2008) for a general discussion of representation issues in gravitational theories].

D. Gravitational waves in $f(R)$ gravity

By now it is clear that the metric tensor of $f(R)$ gravity contains, in addition to the usual massless spin-2 graviton, a massive scalar that shows up in gravitational waves in the metric version of these theories (in the Palatini version, this scalar is not dynamical and does not propagate). A scalar gravitational wave mode is familiar from scalar-tensor gravity (Will, 1981), to which $f(R)$ gravity is equivalent. Because this scalar is massive, it propagates at a speed lower than the speed of light and of massless tensor modes and is, in principle, detectable in the arrival times of signals from an exploding supernova when gravitational wave detectors are sufficiently sensitive [this possibility has been pointed out as a discriminator between tensor-vector-scalar theories and GR (Kahya and Woodard, 2007)]. This massive scalar mode is longitudinal and is of dipole nature to lowest order (Will, 1981; Corda, 2007). The study of its generation, propagation, and detection falls within the purview of scalar-tensor gravity (Will, 1981). The propagation of gravitational waves in the specific model $f(R)=R^n$ was studied by Mendoza and Rosas-Guevara (2007) where the massive scalar mode is missed, however.

The generation of gravitational waves specifically in $f(R)$ gravity has not received much attention in the literature. Even though the fact that the black hole solutions of GR will also be solutions of metric $f(R)$ gravity (without the converse being true) implies that determining the geometry around a black hole is unlikely to provide evidence for such modifications of gravity (Psaltis *et* al., 2008), solutions describing perturbed black holes do behave differently and could, therefore, leave a detectable imprint on gravitational wave radiation (Barausse and Sotiriou, 2008). Note the analogy to the fact that cosmological FLRW solutions are shared by most gravitational theories, but cosmological perturbations reveal more about the underlying theory of gravity than the exact solutions themselves. Additionally, gravitational radiation from binary systems would probably be more revealing than that coming from perturbed black holes when it comes to modified gravity.

The detection of gravitational waves generated in the theories $f(R)=1/R$ [already ruled out by Solar System data (Clifton and Barrow, 2005a, 2006; Barrow and Clifton, 2006)] and $f(R)=R+aR^2$ were studied by Corda and De Laurentis (2007) and Corda (2007), and Corda (2008), respectively.

The study of cosmological gravitational waves in $f(R)$ gravity is perhaps more promising than that of astrophysically generated waves. The stochastic gravitational wave background produced in the early universe was analyzed by Capozziello, Corda, and De Laurentis (2007) and Capozziello *et al.* (2008). The authors of this last reference consider the model $f(R)=R^{1+\delta}$ and derive an evolution equation for the metric perturbations $h_{ij}=h(t)e^{ik_l k^l}e_{ij}$ in a background FLRW universe with scale factor $a(t)=a_0(t/t_0)^n$,

$$\ddot{h}+\frac{(3n-2\delta)}{t}\dot{h}+k^2a_0\left(\frac{t}{t_0}\right)^2 nh=0. \tag{240}$$

This can be solved in terms of Bessel functions; plots of these wave amplitudes are reported in Capozziello *et al.* (2008) for various values of the parameter δ, but the limit $0\leqslant\delta<7.2\times10^{-19}$ obtained by Clifton and Barrow (2005a, 2006) and Barrow and Clifton (2006) leaves little hope of detecting $f(R)$ effects in the gravitational wave background.

Ananda *et al.* (2008) gave a covariant and gauge-invariant description of gravitational waves in a perturbed FLRW universe filled with a barotropic perfect fluid in the toy model $f(R)=R^n$. The perturbation equations are solved (again, in terms of Bessel functions of the first and second kind) in the approximation of scales much larger or much smaller than the Hubble radius H^{-1}, showing a high sensitivity of the tensor-mode evolution to the value of the parameter n. In particular, a tensor mode is found that grows during the radiation-dominated era, with potential implications for detectability in advanced space interferometers. This study, and others of this kind expected to appear in future literature, are in the spirit of discrimination between dark energy and dark gravity or even between different $f(R)$ theories (if this class is taken seriously) when gravitational wave observations become available: as remarked, this is not possible by consideration of only unperturbed FLRW solutions.

VII. SUMMARY AND CONCLUSIONS

A. Summary

While we have presented $f(R)$ gravity as a class of toy theories, various elevate modified gravity, in one or the other of its incarnations corresponding to specific choices of the function $f(R)$, to the role of a fully realistic model to be compared in detail with cosmological observations and to be distinguished from other models. A large fraction of the work in the literature is actually devoted to specific models corresponding to definite choices of the function $f(R)$ and to specific parametrizations.

Besides the power law and power series of R models, which we have mentioned extensively, some other typical examples are functions that contain terms such as $\ln(\lambda R)$ (Nojiri and Odintsov, 2004b; Perez Bergliaffa, 2006) or $e^{\lambda R}$ (Carloni *et al.*, 2006; Bean *et al.*, 2007; Song *et al.*, 2007; Abdelwahab *et al.*, 2008) or are more involved functions of R, such as $f(R)=R-a(R-\Lambda_1)^{-m}+b(R-\Lambda_2)^n$, with $n,m,a,b>0$ (Nojiri and Odintsov, 2003a). Some models have actually been tailored to pass all or most of the known constraints, such as the one proposed by Starobinsky (2007), where $f(R)=R+\lambda R_0[(1+R^2/R_0^2)^{-n}-1]$ with $n,\lambda>0$ and R_0 being of the order of H_0^2. Here we have tried to avoid considering specific models and have attempted to collect general model-independent results, with the viewpoint that these theories are to be seen more as toy theories than definitive and realistic models.

We are now ready to summarize the main results on $f(R)$ gravity. On the theoretical side, we have explored all three versions of $f(R)$ gravity: metric, Palatini, and metric-affine. Several issues concerning dynamics, degrees of freedom, matter couplings, etc, have been extensively discussed. The dynamical equivalence between both metric and Palatini $f(R)$ gravity and Brans-Dicke theory has been, and continues to be, a useful tool to study these theories, given some knowledge of the aspects of interest in scalar-tensor gravity. At the same time, the study of $f(R)$ gravity itself has provided new insight into the two previously unexplored cases of Brans-Dicke theory with $\omega_0=0$ and $-3/2$. We have also considered most of the applications of $f(R)$ gravity to both cosmology and astrophysics. Finally, we have explored a large number of possible ways to constrain $f(R)$ theories and check their viability. In fact, many avatars of $f(R)$ have been shown to be subject to potentially fatal troubles, such as a grossly incorrect post-Newtonian limit, short time scale instabilities, the absence of a matter era, conflict with particle physics or astrophysics, etc.

To avoid repetition, we will not attempt to summarize here all of the theoretical issues, the applications, or the constraints discussed. This, besides being redundant, would not be very helpful to the reader, as, in most cases, the insight gained cannot be summarized in a sentence or two. Specifically, some of the constraints that have been derived in the literature are not model or parametrization independent (and the usefulness of some parametrizations is questionable). This does not allow for them to be expressed in a straightforward manner through simple mathematical equations applicable directly to a general function $f(R)$. Particular examples of such constraints are those coming from cosmology (background evolution, perturbations, etc.).

However, we have encountered cases in which clear-cut viability criteria are indeed easy to derive. We would, therefore, like to make a specific mention of those. A brief list of results that are quick and easy to use is the following

- In metric $f(R)$ gravity, the Dolgov-Kawasaki instability is avoided if and only if $f''(R) \geqslant 0$. The stability condition of de Sitter space is expressed by Eq. (173).
- Metric $f(R)$ gravity might pass the weak-field limit test and at the same time constitute an alternative to dark energy only if the chameleon mechanism is effective—this restricts the possible forms of the function $f(R)$ in a way that cannot be specified by a simple formula.
- Palatini $f(R)$ gravity suffered multiple deaths due to the differential structure of its field equations. These conclusions are essentially model independent. (However, this theory could potentially be fixed by adding extra terms quadratic in the Ricci and/or Riemann tensors, which would raise the order of the equations.)
- Metric-affine gravity as an extension of the Palatini formalism is not sufficiently developed yet. At the moment of writing, it is not clear whether or not it suffers from the same problems that afflict the Palatini formalism.

Of course, as mentioned, the situation is often more involved and cannot be summarized with a quick recipe.

B. Extensions of and new perspectives on $f(R)$ gravity

We have treated $f(R)$ gravity here as a toy theory and, as stated in the Introduction, one of its merits is its relative simplicity. However, we have seen a number of viability issues related to such theories. One obvious way to address this issue is to generalize the action even further in order to avoid these problems, at the cost of increased complexity. Several extensions of $f(R)$ gravity exist. Analyzing them in detail goes beyond the scope of this review, but we make a brief mention of the most straightforward of them.

We have discussed the possibility of having higher-order curvature invariants, such as $R_{\mu\nu}R^{\mu\nu}$, in the action. In fact, from a dimensional analysis perspective, the terms R^2 and $R_{\mu\nu}R^{\mu\nu}$ should appear at the same order. However, theories of this sort seem to be burdened with what is called the Ostrogradski instability (Woodard, 2007). Ostrogradski's theorem states that there is a linear instability in the Hamiltonians associated with Lagrangians, which depend on higher than first-order derivatives in such a way that the dependence cannot be eliminated by partial integration (Ostrogradski, 1850). $f(R)$ gravity seems to be the only case that manages to avoid this theorem (Woodard, 2007) and it obviously does not seem very appealing to extend it in a way that will spoil this.[30]

The alert reader has probably noticed that the above holds true only for metric $f(R)$ gravity. In Palatini $f(R)$ gravity [and metric-affine $f(R)$ gravity], as mentioned earlier, one could add more dynamics to the action without having to worry about making it second order in the fields. Recall that, in practice, the independent connection is an auxiliary field. For instance, the term $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ still contains only first derivatives of the connection. In fact, since we have traced the root of some of the most crucial viability issues of Palatini $f(R)$ gravity to the lack of dynamics in the gravity sector, such generalizations could actually help by promoting the connection from the role of an auxiliary field to that of a truly dynamical field (Barausse *et al.*, 2008b). Such generalizations have been considered by Li, Barrow, and Mota (2007).

Another extension of metric $f(R)$ gravity that appeared recently is that in which the action also includes an explicit coupling between R and the matter fields. Bertolami *et al.* (2007), Bertolami and Paramos (2008), and Boehmer *et al.* (2008b) considered the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{f_1(R)}{2} + [1 + \lambda f_2(R)]L_m\right\}, \qquad (241)$$

where L_m is the matter Lagrangian and $f_{1,2}$ are (*a priori* arbitrary) functions of the Ricci curvature R. Since the matter is not minimally coupled to R, such theories will not lead to energy conservation and will generically exhibit a violation of the equivalence principle (which could potentially be controlled by the parameter λ).

The motivation for considering such an action spelled out by Bertolami *et al.* (2007) was that the nonconservation of energy could lead to extra forces, which in turn might give rise to phenomenology similar to modified Newtonian dynamics (MOND) gravity (Milgrom, 1983) on galactic scales. Other variants of this action have also been considered elsewhere: in Nojiri and Odintsov (2004a), as an alternative to dark energy by setting $f_1(R)=R$ and keeping only the nonminimal coupling of matter to the Ricci curvature; in Dolgov and Kawasaki (2003b) and Mukohyama and Randall (2004), where the idea of making the kinetic term of a (minimally coupled) scalar field dependent on the curvature, while keeping $f_1(R)=R$, was exploited in attempts to cure the cosmological constant problem. Bertolami and Paramos (2008) studied the consequences of such a theory for stellar equilibrium and generalized constraints in order to

[30] However, one could consider adding a function of the Gauss-Bonnet invariant $\mathcal{G}=R^2-4R_{\mu\nu}R^{\mu\nu}-R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ (Nojiri and Odintsov, 2005; Cognola *et al.*, 2006).

avoid the instability discussed in Sec. V.B.1 were derived by Faraoni (2007a). The viability of theories with such couplings between R and matter is still under investigation. However, the case in which both f_1 and f_2 are linear has been shown to be non-viable (Sotiriou, 2008) and, for the more general case of the action (241), serious doubts have been expressed (Sotiriou and Faraoni, 2008) on whether extra forces are indeed present in galactic environments and, therefore, whether this theory can really account for the MOND-like phenomenology that initially motivated its use in Bertolami *et al.* (2007) as a substitute for dark matter.

One could also consider extensions of $f(R)$ gravity in which extra fields appearing in the action couple to different curvature invariants. A simple example with a scalar field is the action (159), which is sometimes called *extended quintessence* (Perrotta *et al.*, 2000) similarly to the extended inflation realized with Brans-Dicke theory. However, such generalizations lie beyond the scope of this review.

Finally, it is worth mentioning a different perspective on $f(R)$ gravity. It is common in the literature reviewed here to treat $f(R)$ gravity as an *exact* theory: the generalized action is used to derive field equations, the solutions of which describe the exact dynamics of the gravitational field (in spite of the fact that the action might be only an approximation and the theory merely a toy theory). A different approach (Bel and Sirousse Zia, 1985; Simon, 1990) that was recently revived by DeDeo and Psaltis (2008) is that of treating metric $f(R)$ as an effective field theory: That is, assuming that the extra terms are an artifact of some expansion of which we are considering only the leading order terms. Now, when we consider a correction to the usual Einstein-Hilbert term, this correction has to be suppressed by some coefficient. This approach assumes that this coefficient controls the order of the expansion and, therefore, the field equations and their solutions are only to be trusted to the order with which that coefficient appears in the action (higher orders are to be discarded). Such an approach is based on two assumptions: first, some power (or function) of the coefficient of the correction considered should be present in all terms of the expansion; second, the extra degrees of freedom [which manifest themselves as higher-order derivatives in metric $f(R)$ gravity] are actually an artifact of the expansion (and there would be a cancellation if all orders were considered). This way, one can do away with these extra degrees of freedom just by proper power counting. Since many of the viability issues troubling higher-order actions are related to the presence of such degrees of freedom (e.g., classical instabilities), removing these degrees of freedom could certainly alleviate many problems (DeDeo and Psaltis, 2008). However, the assumptions on which this approach is based should not be underestimated either. For instance, early results that showed renormalization of higher-order actions were based on an exact treatment, i.e., it is fourth-order gravity that is renormalizable (Stelle, 1977). Even though, from one hand, the effective field theory approach seems reasonable (these actions are regarded as low-energy limits of a more fundamental theory anyway), there is no guarantee that extra degrees of freedom should indeed not be present in a nonperturbative regime.

C. Concluding remarks

Our goal was to present a comprehensive but still thorough review of $f(R)$ gravity in order to provide a starting point for the reader less experienced in this field and a reference guide for the expert. However, even though we have attempted to cover all angles, no review can replace an actual study of the literature itself. It seems inevitable that certain aspects of $f(R)$ might have been omitted or analyzed less than rigorously, and therefore the reader is urged to resort to the original sources.

Although many shortcomings of $f(R)$ gravity have been presented, which may reduce the initial enthusiasm with which one might have approached this field, the fact that such theories are mostly considered as toy theories should not be missed. The fast progress in this field, especially in the last five years, is probably obvious by now. And very useful lessons, which have helped significantly in the understanding of (classical) gravity, have been learned in the study of $f(R)$ gravity. In this sense, the statement made in the Introduction that $f(R)$ gravity is a very useful toy theory seems to be fully justified. Remarkably, there are still unexplored aspects of $f(R)$ theories or their extensions, such as those mentioned in the previous section, which can turn out to be fruitful.

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