Colloquium: Area laws for the entanglement entropy

J. Eisert

Institute of Physics and Astronomy, University of Potsdam, 14469 Potsdam, Germany;
Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, United Kingdom;
and Institute for Mathematical Sciences, Imperial College London, Exhibition Road, London SW7 2PG, United Kingdom

M. Cramer and M. B. Plenio

Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, United Kingdom
and Institut für Theoretische Physik, Albert-Einstein-Allee 11, Universitat Ulm, D-89069 Ulm, Germany

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Physical interactions in quantum many-body systems are typically local: Individual constituents interact mainly with their few nearest neighbors. This locality of interactions is inherited by a decay of correlation functions, but also reflected by scaling laws of a quite profound quantity: the entanglement entropy of ground states. This entropy of the reduced state of a subregion often merely grows like the boundary area of the subregion, and not like its volume, in sharp contrast with an expected extensive behavior. Such "area laws" for the entanglement entropy and related quantities have received considerable attention in recent years. They emerge in several seemingly unrelated fields, in the context of black hole physics, quantum information science, and quantum many-body physics where they have important implications on the numerical simulation of lattice models. In this Colloquium the current status of area laws in these fields is reviewed. Center stage is taken by rigorous results on lattice models in one and higher spatial dimensions. The differences and similarities between bosonic and fermionic models are stressed, area laws are related to the velocity of information propagation in quantum lattice models, and disordered systems, nonequilibrium situations, and topological entanglement entropies are discussed. These questions are considered in classical and quantum systems, in their ground and thermal states, for a variety of correlation measures. A significant proportion is devoted to the clear and quantitative connection between the entanglement content of states and the possibility of their efficient numerical simulation. Matrix-product states, higher-dimensional analogs, and variational sets from entanglement renormalization are also discussed and the paper is concluded by highlighting the implications of area laws on quantifying the effective degrees of freedom that need to be considered in simulations of quantum states.

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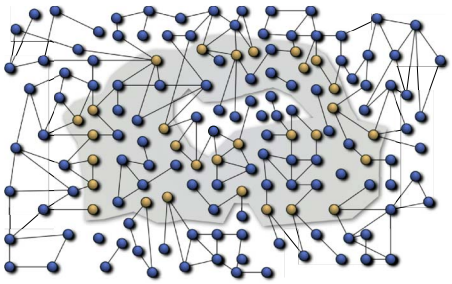

FIG. 1. (Color online) A lattice L with a distinguished set $I \subset L$ (shaded area). Vertices depict the boundary ∂I of I with surface area $s(I)=|\partial I|$.

I. INTRODUCTION

In classical physics concepts of entropy quantify the extent to which we are uncertain about the exact state of a physical system at hand or, in other words, the amount of information that is lacking to identify the microstate of a system from all possibilities compatible with the macrostate of the system. If we are not quite sure what microstate of a system to expect, notions of entropy will reflect this lack of knowledge. Randomness, after all, is always and necessarily related to ignorance about the state.

In quantum mechanics positive entropies may arise even without an objective lack of information. To see this, consider a quantum lattice systems (see, e.g., Fig. 1) as an example for a quantum many-body system where each of the vertices i of the lattice L is associated with an individual quantum system. This quantum many-body system is thought to be in its nondegenerate pure ground state $\rho=|\psi\rangle\langle\psi|$ at zero temperature which has vanishing *von Neumann entropy*

$$S(\rho) = -\mathrm{tr}[\rho \log_2 \rho].$$

We now distinguish a region of this quantum lattice system, denoting its sites with the set I and all other sites with $O=L\backslash I$. If we consider the reduced state $\rho_I = \mathrm{tr}_O[\rho]$ of the sites of the region I, the state will not be pure in general and will have a nonvanishing von Neumann entropy $S(\rho_I)$.[1]

In contrast to thermal states this entropy does not originate from a lack of knowledge about the microstate of the system. Even at zero temperature we encounter a nonzero entropy. This entropy arises because of a fundamental property of quantum mechanics: entanglement. This quite intriguing trait of quantum mechanics gives rise to correlations even in situations where the randomness cannot be traced back to a mere lack of knowledge. The mentioned quantity, the entropy of a subregion, is called *entanglement entropy* or *geometric entropy* and in quantum information *entropy of entanglement*, which represents an operationally defined entanglement measure for pure states [for recent reviews see Plenio and Virmani (2007) and Horodecki *et al.* (2009)].

In the context of quantum field theory, questions of scaling of entanglement entropies in the size of I have some tradition. Seminal work on the geometric entropy of the free Klein-Gordon field (Bombelli *et al.*, 1986; Srednicki, 1993) and subsequent work on conformal field theories (Callan and Wilczek, 1994; Fiola *et al.*, 1994; Holzhey *et al.*, 1994; Hawking *et al.*, 2001; Calabrese and Cardy, 2004) was driven in part by the intriguing suggested connection to the Bekenstein-Hawking black hole entropy (Bekenstein, 1973, 2004; Hawking, 1974).

In recent years, studies of properties of the entanglement entropy in this sense have enjoyed a revival initiated by Audenaert *et al.* (2002); Osborne and Nielsen (2002), Osterloh *et al.* (2002), and Vidal *et al.* (2003). Importantly, this renewed activity is benefitting from the new perspectives and ideas of quantum information theory, and from the realization of their significance for the understanding of numerical methods and especially their efficiency for describing quantum many-body physics. Quantum information theory also provides novel conceptual and mathematical techniques for determining properties of the geometric entropy analytically.

At the heart of these studies are questions like: What role do genuine quantum correlations—entanglement—play in quantum many-body systems? Typically, in such investigations, one abstracts to a large extent from the microscopic specifics of the system: Quite in the spirit of studies of *critical phenomena*, one thinks less of detailed properties, but is rather interested in the *scaling* of the entanglement entropy when the distinguished region grows in size. In fact, for quantum chains this scaling of entanglement as genuine quantum correlations—*a priori* very different from the scaling of two-point correlation functions—reflects to a large extent the critical behavior of the quantum many-body system, and shares some relationship to conformal charges.

At first sight one might be tempted to think that the entropy of a distinguished region I will always possess an extensive character. Such a behavior is referred to as a *volume scaling* and is observed for thermal states. Intriguingly, for typical ground states, however, this is not at all what one encounters: Instead, one typically finds an *area law*, or an area law with a small (often logarithmic) correction: This means that if one distinguishes a region, the scaling of the entropy is merely linear in the boundary area of the region. The entanglement entropy is then said to fulfill an area law. It is the purpose of this Colloquium to review studies on area laws and the scaling of the entanglement entropy in a nontechnical manner.

The main four motivations to approach this question are as follows.

- The holographic principle and black hole entropy:

[1]Of interest are also other entropies, such as the *Renyi entropies*, $S_\alpha(\rho)=(1-\alpha)^{-1}\log_2 \mathrm{tr}[\rho^\alpha]$ with $\alpha \geqslant 0$. For $\alpha \searrow 1$ the usual von Neumann entropy is recovered. In particular, in the context of simulatability, Renyi entropies for arbitrary α play an important role.

The historical motivation to study the entanglement or geometric entropy stems from considerations of black hole physics: It has been suggested by Bombelli *et al.* (1986) and Srednicki (1993) that the area law of the geometric entropy for a discrete version of a massless free scalar field—then numerically found for an imaginary sphere in a radial symmetry—could be related to the physics of black holes (Hawking *et al.*, 2001) in particular the Bekenstein-Hawking entropy of a black hole which is proportional to its boundary surface. It has been muted that the *holographic principle* (Bousso, 2002)—the conjecture that the information contained in a volume of space can be represented by a theory which lives in the boundary of that region—could be related to the area law behavior of the entanglement entropy in microscopic theories.

- Distribution of quantum correlations in quantum many-body systems: Area laws also say something on how quantum correlations are distributed in ground states of local quantum many-body systems. Interactions in quantum many-body systems are typically local, which means that systems interact only over a short distance with a finite number of neighbors. The emergence of an area law then provides support for the intuition that short ranged interactions require that quantum correlations between a distinguished region and its exterior are established via its boundary surface. That a strict area law emerges is by no means obvious from the decay of two-point correlators, as we will see. Quantum phase transitions are governed by quantum fluctuations at zero temperature, so it is more than plausible to observe signatures of *criticality* on the level of entanglement and *quantum correlations*. This situation is now particularly clear in one-dimensional (1D) systems (Audenaert *et al.*, 2002; Fannes *et al.*, 2003; Vidal *et al.*, 2003, 2007; Calabrese and Cardy, 2004, 2006a; Jin and Korepin, 2004; Latorre *et al.*, 2004, 2005; Dür *et al.*, 2005; Eisert and Cramer, 2005; Farkas and Zimboras, 2005; Its *et al.*, 2005; Keating and Mezzadri, 2005; Barthel *et al.*, 2006; Eisert and Osborne, 2006; Cardy *et al.*, 2007; Casini and Huerta, 2007; Franchini *et al.*, 2007; Hastings, 2007a; Amico *et al.*, 2008) but progress has also been made in higher-dimensional systems (Hein *et al.*, 2004; Plenio *et al.*, 2005; Bravyi *et al.*, 2006; Cramer and Eisert, 2006; Cramer *et al.*, 2006; Fradkin and Moore, 2006; Kitaev and Preskill, 2006; Riera and Latorre, 2006; Verstraete *et al.*, 2006; Wolf, 2006; Farkas and Zimboras, 2007), with rigorous area laws specifically for quasifree bosonic (Plenio *et al.*, 2005; Cramer and Eisert, 2006; Cramer *et al.*, 2006) and fermionic (Gioev and Klich, 2006; Wolf, 2006; Cramer *et al.*, 2007; Farkas and Zimboras, 2007) systems, as well as in disordered systems (Refael and Moore, 2007).
- Complexity of quantum many-body systems and their simulation: One of the key motivations for studying area laws stems from a quite practical context: the numerical simulation of quantum many-body systems. In fact, if there is little entanglement in a ground state of many-body systems, one might suspect on intuitive grounds that one can describe this ground state with relatively few parameters. More specifically, for one-dimensional systems one would expect numerical algorithms like the powerful density-matrix renormalization group method (White, 1992; Schollwöck, 2005) (DMRG) to perform well if the ground state contains a small amount of entanglement. This suspicion can in fact be made rigorous (Peschel, 2004; Verstraete and Cirac, 2006; Hastings, 2007a; Schuch *et al.*, 2008a) as it turns out that the scaling of entanglement specifies how well a given state can be approximated by a matrix-product state (Fannes *et al.*, 1992; Schollwöck, 2005) as generated in DMRG. It is hence not the decay behavior of correlation functions as such that matters here, but in fact the scaling of entanglement.
- Topological entanglement entropy: The topological entanglement entropy is an indicator of *topological order* (Wen, 1989; Witten, 1998; Nussinov and Ortiz, 2009), a new kind of order in quantum many-body systems that cannot be described by local order parameters (Kitaev and Preskill, 2006; Levin and Wen, 2006; Haque *et al.*, 2007; Papanikolaou *et al.*, 2007; Nussinov and Ortiz, 2009). Lattice models having a nonvanishing topological entanglement entropy may be seen as lattice instances of topological quantum field theories. Here a global feature is detected by means of the scaling of geometric entropies.

In this Colloquium we do not have space to give an account of all known derivations of area laws for the entanglement entropy. However, we will try not to merely remain at a superficial level and only state results, but will explain a number of key techniques and arguments. When we label main statements as "theorems" this is done to highlight their special role, to make it easier to follow the line of reasoning. For details of arguments and proofs, often technically involved, we refer the interested reader to the original work. The reason for the technicality of proofs originates from the type of question that is posed: to distinguish a region of a lattice breaks the translational symmetry of the problem—even in a translationally invariant setting. While numerical studies are sometimes easier to come by, analytical argument can be technically involved, even for quasifree models. In this paper, we discuss the study of entanglement entropy primarily (i) from the viewpoint of quantum information theory, (ii) with an emphasis on rigorous and analytical results, and (iii) the implications on the efficiency of numerical simulation.

II. LOCAL HAMILTONIANS AND AREA LAWS

Throughout this paper, we consider quantum many-body systems on a lattice. Such quantum lattice systems are ubiquitous in the condensed matter context (Vojta, 2003), where they play a key role in obtaining an under-

standing of material properties from a microscopic basis. Lattices systems are also of considerable importance in the study of quantum field theories where a lattice provides a natural ultraviolet cutoff and facilitates numerical simulations of quantum fields (Montvay and Münster, 1994). One could think of systems of strongly correlated electron systems or lattice vibrations of a crystal lattice. With the advent of research on *cold atoms* in optical lattices, quantum lattice systems can also be prepared in laboratory conditions with an unprecedented degree of control (Bloch *et al.*, 2008).

We will consider—at least in parts of this paper—general lattices. Each vertex of the *lattice* is associated with a quantum system, such as a spin, a bosonic, or a fermionic system. It is convenient to think of this lattice as a simple graph $G=(L,E)$ with vertices L, and the edge set E labeling neighborhood relations. G could be the graph representing a one-dimensional chain with periodic boundary conditions, and in fact a good proportion of this paper will deal with such quantum chains. For later purposes, it will be convenient to think in terms of such a slightly more general picture, however. The Hilbert space of the total many-body system is then the tensor product $\mathcal{H}=\otimes_{j\in L}\mathcal{H}_j$, where $\mathcal{H}_j$ is the Hilbert space associated with the physical system on lattice site j. On such a lattice, one has $\mathrm{dist}(j,k)$ for $j,k\in L$ as the natural graph theoretical distance, which is the length of the shortest path connecting j and k. For a cubic lattice of dimension $\mathcal{D}$ with periodic boundary conditions, in turn, $\mathrm{dist}(j,k)=\Sigma_{d=1}^{\mathcal{D}}|j_d-k_d|$, where the components of $j,k\in L$ are taken modulo the base length of the cubic lattice.

We are concerned largely with local Hamiltonians on lattices. This means that the physical system associated with a specific lattice site will interact only with its neighbors and not with all sites of the lattice. The total Hamiltonian can hence be written as

$$H=\sum_{X\subset L} H_X,$$

where H_X has a compact support X, independent of the system size, that is the number of lattice sites denoted by $|L|$.

The boundary surface area $s(I)$ of a distinguished region I of the lattice L can be defined in a natural fashion on such a graph as the cardinality of the set of boundary points

$$\partial I=\{i\in I:\ \text{there is a } j\in L\setminus I \text{ with } \mathrm{dist}(i,j)=1\}, \quad (1)$$

so $s(I)=|\partial I|$; see Fig. 1. Throughout the paper, unless defined specifically otherwise, we say that the entanglement entropy satisfies an area law if

$$S(\rho_I)=O(s(I)).$$

This means that the entropy of the reduced state ρ_I scales at most as the boundary area of the region I.

Before we dive into the details of known results on area laws in quantum many-body systems, we appreciate how unusual it is for a quantum state to satisfy an area law. In fact, a quantum state picked at random will exhibit a different scaling behavior. If one has a lattice system with d-dimensional constituents and divides it into a subsystem $I\subset L$ and the complement $O=L\setminus I$, then one may consider the expected entanglement entropy of I for the natural choice, the unitarily invariant Haar measure. One finds (Page, 1993; Foong and Kanno, 1994; Sen, 1996)

$$\mathbb{E}[S(\rho_I)]>|I|\log_2(d)-\frac{d^{|I|-|O|}}{2\log_2(2)}.$$

That is, asymptotically, the typical entropy of a subsystem is almost maximal, and hence linear in the number of constituents $|I|$. Hence a "typical" quantum state will asymptotically satisfy a volume law, and not an area law. As we will see that area laws are common for ground states of quantum many-body systems, we find that in this sense ground states are very nongeneric. This fact is heavily exploited in numerical approaches to study ground states of strongly correlated many-body systems: One does not have to vary over all quantum states in variational approaches, but merely over a much smaller set of states that are good candidates of approximating ground states of local Hamiltonians well; that is, states that satisfy an area law.

III. ONE-DIMENSIONAL SYSTEMS

Most known results on area laws refer to one-dimensional chains such as *harmonic* or *spin chains*. This emphasis is no surprise: After all, a number of physical ideas—like the Jordan-Wigner transformation—as well as mathematical methods—such as the theory of Toeplitz determinants and Fisher-Hartwig techniques—are specifically tailored to one-dimensional translationally invariant systems.

If we distinguish a contiguous set of quantum systems of a chain, a block $I=\{1,\ldots,n\}$, the boundary of the block consists of only one (two) site(s) for open (periodic) boundary conditions. An area law then clearly means that the entropy is upper bounded by a constant independent of the block size n and the lattice size $|L|$, i.e.,

$$S(\rho_I)=O(1). \quad (2)$$

We will see that in quantum chains a clear picture emerges concerning the scaling of the entanglement entropy. Whether an area law holds or not will largely depend on whether the system is at a quantum critical point or not. We summarize what is known in one-dimensional systems at the end of the detailed discussion of quantum chains, starting with bosonic harmonic chains.

A. Bosonic harmonic chain

Bosonic harmonic quantum systems, as well as fermionic models and their quantum spin chain counterparts like the XY model, play a seminal role in the study of

quantum many-body systems. Harmonic lattice systems model discrete versions of Klein-Gordon fields, vibrational modes of crystal lattices, or of trapped ions and serve generally as lowest-order approximations to anharmonic systems. The fact that they are integrable renders even sophisticated questions like the scaling of the geometric entropy in instances amenable to fully analytical study, even in higher spatial dimensions. In fact, in the latter case these so-called quasifree models are the only settings that allow for rigorous analytical results so far. Hence, they do form the central object of consideration to explore what should be expected concerning general scaling laws.

The Hamiltonian for a harmonic lattice L is given by

$$H = \frac{1}{2}\sum_{i,j\in L}(p_i P_{i,j} p_j + x_i X_{i,j} x_j), \tag{3}$$

where $X,P\in\mathbb{R}^{|L|\times|L|}$ are real, symmetric, and positive matrices determining the coupling structure of the systems. The canonical operators x_i,p_i satisfy the canonical commutation relations $[x_j,p_k]=i\delta_{j,k}$. In terms of the bosonic annihilation operators $b_j=(x_j+ip_j)/\sqrt{2}$ the Hamiltonian (3) reads

$$H = \frac{1}{2}\sum_{i,j}(b_i^\dagger A_{i,j} b_j + b_i A_{i,j} b_j^\dagger + b_i B_{i,j} b_j + b_i^\dagger B_{i,j} b_j^\dagger), \tag{4}$$

where $A=(X+P)/2$, $B=(X-P)/2$. Ground and thermal states of the above Hamiltonian are fully characterized by the second moments of the canonical operators, while first moments vanish (Eisert and Plenio, 2003) (entanglement properties of the state are invariant under changes of first moments anyway). The second moments define the *covariance matrix*

$$\Gamma_{i,j} = \langle\{r_i,r_j\}\rangle = \langle r_i r_j\rangle + \langle r_j r_i\rangle, \tag{5}$$

where $r=(x_1,\ldots,x_{|L|},p_1,\ldots,p_{|L|})$ is the vector of canonical operators. The covariance matrix of the ground state of Eq. (3) is given by $\Gamma=\Gamma_x\oplus\Gamma_p$, where $\Gamma_p=X^{1/2}(X^{1/2}PX^{1/2})^{-1/2}X^{1/2}$ and $\Gamma_x=\Gamma_p^{-1}$; see Cramer and Eisert (2006) and Schuch *et al.* (2006). On the level of covariance matrices unitary operations express themselves as symplectic transformations S that preserve the commutation relations $\sigma_{k,l}=i[r_k,r_l]$, i.e., $S\sigma S^T=\sigma$. Importantly, Williamson's theorem states that for any strictly positive matrix $A\in\mathbb{R}^{2N\times 2N}$ there exist a symplectic transformation S such that $SAS^T=D$, where D is a diagonal matrix with the same spectrum as the positive square roots of $(i\sigma A)^2$. The eigenvalues d_i of D are called the *symplectic eigenvalues* of A.

Now, what is the entanglement content of the ground state? To answer this we need to define entanglement measures and compute them in terms of the properties of the covariance matrix. The first of these is of course the entropy of entanglement. Williamson's theorem shows that any function of a state that is unitarily invariant is fully determined by the symplectic eigenvalues. Notably, the entropy of a Gaussian state ρ with symplectic eigenvalues $d_1,\ldots,d_N$ of the covariance matrix of ρ is given by

$$S(\rho) = \sum_{j=1}^{N}\left(\frac{d_j+1}{2}\log_2\frac{d_j+1}{2} - \frac{d_j-1}{2}\log_2\frac{d_j-1}{2}\right).$$

A key ingredient in the analytical work is another full entanglement measure that was defined in quantum information theory, the *logarithmic negativity* (Życzkowski *et al.*, 1998; Eisert and Plenio, 1999; Eisert, 2001; Vidal and Werner, 2002; Plenio, 2005; Plenio and Virmani, 2007). It is defined as

$$E_N(\rho,I) = \log_2\|\rho^{\Gamma_I}\|_1,$$

where $\|A\|_1=\mathrm{tr}[(A^\dagger A)^{1/2}]$ is the trace norm and ρ^{Γ_I} is the partial transpose of ρ with respect to the interior I. The partial transpose with respect to the second subsystem is defined as $(|i\rangle\langle k|\otimes|j\rangle\langle l|)^{\Gamma_2}=|i\rangle\langle k|\otimes|l\rangle\langle j|$. On the level of covariance matrices the partial transpose is partial time reversal, i.e., $p_i\mapsto -p_i$ if $i\in I$ while x_i remains invariant. Then for ρ with covariance matrix $\Gamma=\Gamma_x\oplus\Gamma_p$ we find that ρ^{Γ_I} has covariance matrix $\Gamma'=\Gamma_x\oplus(F\Gamma_p F)$, where the diagonal matrix F has entries $F_{i,j}=\pm\delta_{i,j}$, depending on whether a coordinate is in I or O: Then one finds for a state with covariance matrix $\Gamma=\Gamma_x\oplus\Gamma_p$ the logarithmic negativity (Audenaert *et al.*, 2002; Cramer, 2006)

$$E_N(\rho,I) = \frac{1}{2}\sum_{k=1}^{|L|}\log_2\max\{1,\lambda_k(\Gamma_p^{-1}F\Gamma_x^{-1}F)\},$$

where λ_k denote the eigenvalue. The logarithmic negativity has two key features. Mathematically, the importance of $E_N(\rho,I)$ is due to

$$E_N(\rho,I) \geqslant S(\rho_I), \tag{6}$$

which holds for all pure states ρ. This *upper bound* for the entanglement entropy is simpler to compute as one does not have to look at spectra of reductions ρ_I but of the full system. This renders a study of area laws possible even in higher-dimensional systems. Second, in contrast to the entropy of entanglement, the negativity is also an entanglement measure for mixed states, such as thermal states and provides an upper bound on other important measures of mixed state entanglement (Bennett *et al.*, 1996; Vedral and Plenio, 1998; Christandl and Winter, 2004; Plenio and Virmani, 2007).

All of the above holds for general lattices L but for the moment we focus on the one-dimensional setting; that is, $L=\{1,\ldots,N\}$ where N is even to allow us to consider the symmetrically bisected chain $I=\{1,\ldots,N/2\}$ with periodic boundary conditions and $P=1$. We concentrate on the ground state and discuss thermal states later.[2] It is worth noting that in higher spatial dimension

[2]We do not discuss the entanglement properties in excited states here as this area has not been explored in detail so far (Stelmachovic and Buzek, 2004; Das and Shankaranarayanan, 2006).

the natural analog of this setting, the half-space, is of some importance as it allows for a reduction in the problem in question to the 1D case discussed here (Cramer *et al.*, 2007). Furthermore, the scaling behavior of the entanglement of the half-chain has direct consequences on the availability of efficient representations of the state by means of matrix-product states as discussed later. For a general nearest-neighbor coupling this means that X is the circulant matrix,

$$X = \mathrm{circ}(a,b,0,\ldots,0,b), \tag{7}$$

as a consequence of translational invariance. b specifies the coupling strength, a defines the on-site term, $\lambda_{\min}(X)=a-2|b|$, i.e., positivity demands $a>2|b|$, and the energy gap above the ground state is given by $\Delta E=\lambda_{\min}^{1/2}(XP)=(a-2|b|)^{1/2}$. For the logarithmic negativity of the symmetrically bisected half-chain we find (Audenaert *et al.*, 2002) the following.

Theorem 1 (Exact negativity of the half-chain). Consider a Hamiltonian of a harmonic chain on $L=\{1,\ldots,N\}$ with periodic boundary conditions, $P=1$, and nearest-neighbor interactions as in Eq. (7). Then the entanglement entropy of the symmetrically bisected chain and the logarithmic negativity satisfy

$$S(\rho_I) \leqslant E_N(\rho,I) = \frac{1}{4}\log_2\left(\frac{a+2|b|}{a-2|b|}\right) = \frac{1}{2}\log_2\left(\frac{\|X\|^{1/2}}{\Delta E}\right), \tag{8}$$

where $\|\cdot\|$ is the operator norm and $\Delta E=\lambda_{\min}^{1/2}(X)$.

The quantity $\|X\|$ will later be related to the speed of sound in the system. This expression for the block entanglement quantified with respect to the negativity is exact and no approximation. This was the first rigorous area law for a lattice system, complementing earlier seminal work for fields (Callan and Wilczek, 1994). Remarkably, this expression is entirely independent of N, the system size. The most important observation here is that an area law holds, which can be expressed in terms of the spectral gap in the system: Whenever the system is noncritical in the sense that the energy gap ΔE satisfies $\Delta E \geqslant c>0$ with a system size independent constant c, a one-dimensional area law will hold. The above link of entanglement entropy and spectral gap in the system can be established in much more generality and we delay this discussion to later.

The argument leading to Theorem 1 is involved, and for details see Audenaert *et al.* (2002). The interesting aspect of this proof is that the spectrum of the half-chain cannot be obtained analytically, thus not allowing for a direct computation of the entanglement content. Instead, it is the particular combination of spectral values of the partial transpose entering $E_N(\rho,I)$ itself that can be explicitly computed. The proof makes heavy use of the symmetry of the problem, namely, the invariance under a flip of the two half-chains.

This result suggested that the locality of the interaction in the gapped model is inherited by the locality of entanglement, a picture that was also later confirmed in more generality. Note that the above bound is a particularly tight one, and that it may well suggest what prefactor in terms of the energy gap and speed of sound one might expect in general area laws, as discussed later.

We now consider an important model for which the energy gap vanishes in the thermodynamical limit $N\to\infty$: Taking $a=m^2+2N^2$, $b=-N^2$, identifying lattice sites by $i=xN$, and the canonical operators by $x_i=N^{-1/2}\phi(x)$, $p_i=N^{-1/2}\pi(x)$, one obtains the Klein-Gordon field Hamiltonian

$$H = \frac{1}{2}\int_0^1 dx\left[\pi^2(x)+\left(\frac{\partial}{\partial x}\phi(x)\right)^2+m^2\phi^2(x)\right], \tag{9}$$

in the field limit $N\to\infty$. [For a detailed discussion of the continuum limit for the Klein-Gordon field, see also Botero and Reznik (2004)]. From Eq. (8) for the entanglement, we immediately obtain

$$E_N(\rho,I) = \frac{1}{4}\log_2\left(1+\frac{4N^2}{m^2}\right) \xrightarrow[N\to\infty]{} \frac{1}{2}\log_2\left(\frac{2N}{m}\right). \tag{10}$$

This is a striking difference to the area laws that we have observed earlier; now the entanglement does not saturate but diverges with the length of the half-chain.[3] The behavior observed here will be mirrored by a similar logarithmic divergence in critical quantum spin chains and fermionic systems. This will be discussed in the following section.

B. Fermionic chain and the *XY* model

Following the initial work on bosonic models of Audenaert *et al.* (2002), similar questions were explored in fermionic systems and the associated spin models. The numerical studies in Vidal *et al.* (2003) and Latorre *et al.* (2004) presented a significant first step in this direction. Their key observation, later confirmed rigorously (Jin and Korepin, 2004; Its *et al.* 2005; Keating and Mezzadri, 2005) using techniques sketched in this section, is that the scaling of the entanglement entropy as a function of the block size appears to be related to the system being quantum critical or not. Again, for a gapped system, away from a quantum critical point, the entanglement entropy would saturate; i.e., an area law holds. In turn in all cases when the system was critical, the numerical study indicated that the entanglement entropy grows beyond all bounds. More specifically, it grows logarithmically with the block size. This behavior is also consistent with the behavior of geometric entropies in conformal field theory (Callan and Wilczek, 1994; Holzhey *et al.*, 1994), which applies to the critical points of the models discussed by Korepin (2003), Vidal *et al.* (2003), Latorre *et al.* (2004), and Franchini *et al.* (2008). The intriguing aspect here is that being critical or not is not only reflected by the scaling of expectation values of two-point correlators, but in fact by the ground-state entanglement, so genuine quantum correlations.

[3]Compare also the divergence of the entanglement entropy in collectively interacting chains (Unanyan *et al.*, 2007).

This section defines the setting, introduces the basic concepts required, and outlines the rigorous results in more detail. Fermionic quasifree models, that is, Hamiltonians that are quadratic in fermionic operators f_i and $f_i^\dagger$,

$$H = \frac{1}{2}\sum_{i,j\in L}(f_i^\dagger A_{i,j} f_j - f_i A_{i,j} f_j^\dagger + f_i B_{i,j} f_j - f_i^\dagger B_{i,j} f_j^\dagger) \tag{11}$$

may be treated by similar analytical techniques and follow similar intuition to the bosonic case. In Eq. (11), to ensure Hermiticity of the Hamiltonian, $A^T=A$ and $B^T=-B$ must hold for the matrices A and B defining the coupling. The role of the canonical coordinates is taken by the Majorana operators $x_j=(f_j^\dagger+f_j)/\sqrt{2}$ and $p_j=i(f_j^\dagger-f_j)/\sqrt{2}$, while the role of symplectic transformations is taken by orthogonal transformations. The energy gap above the ground state is given by the smallest nonzero singular value of $A+B$.

Note that, in contrast to the bosonic case, the ground state is $2^{|L|-\mathrm{rank}(A+B)}=:q$-fold degenerate. We define the ground-state expectation $\langle\cdot\rangle=\mathrm{tr}[\cdot P_0]/q$, where P_0 projects onto the ground-state sector. Then, as in the bosonic case, the ground state is fully characterized by two-point correlations embodied in the covariance matrix with entries

$$-i\Gamma_{i,j} = \langle [r_i, r_j]\rangle = \langle r_i r_j\rangle - \langle r_j r_i\rangle,$$

where now $r=(x_1,\ldots,x_{|L|},p_1,\ldots,p_{|L|})$ collects Majorana operators. One then finds

$$\Gamma = \begin{pmatrix} 0 & -V \\ V^T & 0 \end{pmatrix}, \quad V = |A+B|^+(A+B), \tag{12}$$

where $+$ indicates the Moore-Penrose generalized inverse of a matrix (Horn and Johnson, 1985), i.e., for a unique ground state one simply has $V=|A+B|^{-1}(A+B)$. The entropy of a contiguous block I of fermions in the ground state can be expressed in terms of the singular values σ_k of the principle submatrix V_I of V. One finds $S(\rho_I)=\Sigma_k f(\sigma_k)$, where

$$f(x) = -\frac{1-x}{2}\log_2\left(\frac{1-x}{2}\right) - \frac{1+x}{2}\log_2\left(\frac{1+x}{2}\right). \tag{13}$$

All the above holds for general lattices but for the moment we turn to a discussion of $L=\{1,\ldots,N\}$.

We have started the discussion on the level of fermionic operators to highlight the similarity to the bosonic case. It is important to note, however, that these fermionic models share a close relationship to natural spin models in the 1D setting. This is revealed by the Jordan-Wigner transformation which relates fermionic operators with spin operators according to

$$\sigma_i^z = 1 - 2f_i^\dagger f_i, \quad \frac{\sigma_i^x + i\sigma_i^y}{2} = \prod_{k=1}^{i-1}(1-2f_k^\dagger f_k) f_i, \tag{14}$$

where $\sigma_i^x,\sigma_i^y,\sigma_i^z$ denote the Pauli operators associated with site $i\in L$. The fermionic model (11) is hence equivalent to a spin model with short- or long-range interactions.

The most important model of this kind is the XY model with a transverse magnetic field, with nearest-neighbor interaction, $A_{i,i}=\lambda$, $A_{i,j}=-1/2$ if $\mathrm{dist}(i,j)=1$, and $B_{i,j}=-B_{j,i}=\gamma/2$ for $\mathrm{dist}(i,j)=1$. This gives rise to

$$H = -\frac{1}{2}\sum_{\langle i,j\rangle}\left(\frac{1+\gamma}{4}\sigma_i^x\sigma_j^x + \frac{1-\gamma}{4}\sigma_i^y\sigma_j^y\right) - \frac{\lambda}{2}\sum_{i\in L}\sigma_i^z, \tag{15}$$

where $\langle i,j\rangle$ denotes summation over nearest neighbors, γ is the anisotropy parameter, and λ is an external magnetic field.[4] Once again, translational invariance of the model means that the spectrum can be readily computed by means of a discrete Fourier transform. One obtains

$$E_k = \{[\lambda - \cos(2\pi k/N)]^2 + \gamma^2\sin^2(2\pi k/N)\}^{1/2},$$

for $k=1,\ldots,N$. This is a well-known integrable model (Lieb *et al.*, 1961; Barouch and McCoy, 1971).

In the plane defined by (γ,λ) several critical lines can be identified: Along the lines $|\lambda|=1$ and on the line segment $\gamma=0$, $|\lambda|\leqslant 1$, the system is critical, $\lim_{N\to\infty}\Delta E(N)=0$. For all other points in the (γ,λ) plane there exists a $c>0$ independent of N such that $\Delta E\geqslant c$. The class of models with $\gamma=1$ is called Ising model. The most important case subsequently is the isotropic case of the XY model, then often referred to as the XX model or the isotropic XY model. This is the case when $\gamma=0$. The XX model is critical whenever $|\lambda|\leqslant 1$. The XX model is equivalent to the Bose-Hubbard model in the limit of hard-core bosons, so the Bose-Hubbard model with the additional constraint that each site can be occupied by at most a single boson.

Assume that we have a nondegenerate ground state, such that the entropy of entanglement $S(\rho_I)$ really quantifies the entanglement content. For the translation-invariant system at hand, the entries $V_{i,j}=V_{i-j}$ of V are given by

[4]Note that the boundary conditions give rise to a (sometimes overlooked) subtlety here. For open boundary conditions in the fermionic model, the Jordan-Wigner transformation relates the above fermionic model to the spin model in Eq. (15) with open boundary conditions. For periodic boundary conditions, the term $f_N^\dagger f_1$ is replaced by the operator $[\Pi_{j=1}^N(2f_j^\dagger f_j-1)]f_N^\dagger f_1$. Hence, the periodic fermionic model does not truly correspond to the periodic XY model (De Pasquale and Facchi, 2009). Importantly, the degeneracy of ground states is affected by this. For a degenerate ground state, the entanglement of formation (Bennett *et al.*, 1996), the relative entropy of entanglement (Vedral and Plenio, 1998), the distillable entanglement (Bennett *et al.*, 1996), or the logarithmic negativity of the ground-state sector are the appropriate entanglement quantifiers (Plenio and Virmani, 2007; Horodecki *et al.*, 2009), and no longer the entropy of entanglement. Only in the case that for large subsystems n one can almost certainly locally distinguish the finitely many different ground states, the entropy of entanglement for each of the degenerate ground states still gives the correct value for the entanglement of a subsystem.

$$V_l = \frac{1}{|L|}\sum_{k=1}^{|L|} g_k e^{2\pi i l k/|L|}, \quad g_k = \lambda_k\left(\frac{A+B}{|A+B|}\right). \tag{16}$$

The entanglement properties of the model are encoded in the numbers g_k. For $N=|L|\to\infty$, we can write

$$V_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, g(\phi) e^{il\phi},$$

for $|l|\leqslant N/2$, where $g:[0,2\pi)\to\mathbb{C}$ is called the symbol of V. Note that the *Fermi surface* is defined by the discontinuities of the symbol. In order to evaluate the entropy of a reduction $S(\rho_l)$, we merely have to know the singular values of $n\times n$ submatrices $V_l=V_n$ of V; see Eq. (13). For isotropic models, i.e., for $B=0$, V then being symmetric, the singular values are the absolute values of the eigenvalues. In other words, in order to understand the correlation and entanglement structure of sub-blocks of such systems, one has to understand properties of matrices the entries of which are of the form $T_{i,j}=T_{i-j}$. Such matrices are called Toeplitz matrices. An $n\times n$ Toeplitz matrix is entirely defined by the $2n-1$ numbers T_l, $l=1-n,\ldots,n-1$.

The spectral values $\lambda_1(T_n),\ldots,\lambda_n(T_n)$ of T_n are the zeros of the characteristic polynomial

$$\det(T_n-\lambda\mathbb{1}) = \prod_{k=1}^{n}[\lambda_k(T_n)-\lambda],$$

so in order to grasp the asymptotic behavior of the spectrum of T_n, it is sufficient to know the asymptotic behavior of this determinant expression.[5] The mathematical theory of determinants of such Toeplitz matrices is much developed. The Fisher-Hartwig theorem provides exactly the tools to study the asymptotic behavior of Toeplitz determinants in terms of the symbol. Crudely speaking, what matters are the zeros of g and the jumps: Once g is written in what can be called a normal form, one can "read off" the asymptotic behavior of the sequence of Toeplitz determinants defined by this symbol. Note that the matrices $V_n-z\mathbb{1}$ take here the role of T_n. The exact formulation of the Fisher-Hartwig theorem is presented in the Appendix.

This machinery was used by Jin and Korepin (2004), Eisert and Cramer (2005), Its *et al.* (2005), and Keating and Mezzadri (2005) to evaluate the asymptotic behavior of the block entropy for the critical XX model and other isotropic models. In the first paper introducing this idea (Jin and Korepin, 2004), in fact, there is a single jump from 1 to −1 in the symbol defining the Toeplitz matrices (and no zeros), which gives rise to the prefactor of 1/3 of $\log_2(n)$ in the formula for the entanglement entropy in the XX model. This prefactor—which emerges here rather as a consequence of mathematical properties of the symbol—is related to the conformal charge of the underlying conformal field theory. In more general isotropic models, as pointed out by Keating and Mezzadri (2005), the number of jumps determines the prefactor in the entanglement scaling. Hence in such quasifree isotropic fermionic models the connection between criticality and a logarithmic divergence is transparent and clear: If there is no Fermi surface at all, and hence no jump in the symbol, the system will be gapped and hence noncritical. Then, the entropy will saturate to a constant.

In contrast, in case there is a Fermi surface this will lead to jumps in the symbol, and the system is critical. In any such case one will find a logarithmically divergent entanglement entropy. The prefactor is determined by the number of jumps. So more physically speaking, what matters is the number of boundary points of the Fermi surface in the interval $[0,2\pi)$. So—if one can say so in a simple one-dimensional system—the "topology of the Fermi surface determines the prefactor." This aspect will be discussed in more detail later. Jin and Korepin (2004), Its *et al.* (2005), and Keating and Mezzadri (2005) found the following.

Theorem 2 (Critical quasifree fermionic chains). Consider a family of quasifree isotropic fermionic Hamiltonians with periodic boundary conditions as in Eq. (11) with $B=0$. Then, the entanglement entropy of a block of $I=\{1,\ldots,n\}$ continuous spins scales as

$$S(\rho_I) = \xi\log_2(n) + O(1).$$

$\xi>0$ is a constant that can be related to the number of jumps in the symbol (defined above). This applies, e.g., to the scaling of the entanglement entropy in the XX spin model, for which

$$S(\rho_I) = \tfrac{1}{3}\log_2(n) + O(1).$$

The constant ξ is not to be mistaken for the conformal charge which will be discussed later. These arguments correspond to the isotropic model with $B=0$, where the Fisher-Hartwig machinery can be conveniently applied. In contrast, the anisotropic case, albeit innocent looking, is overburdened with technicalities. Then, in order to compute the singular values of submatrices of V as in Eq. (12), it is no longer sufficient to consider Toeplitz matrices, but block Toeplitz matrices where the entries are conceived as 2×2 matrices. This setting has been studied by Its *et al.* (2005) in the case of a noncritical anisotropic system, finding again a saturation of the entanglement entropy and by Its *et al.* (2007), where the prefactor of the area law for the entanglement entropy in the gapped XX model was computed rigorously. Franchini *et al.* (2008) also discussed other Renyi entropies in this model.

[5] Once this quantity is known, one can evaluate the entropy by means of

$$S(\rho_I) = \sum_{k=1}^{n} f(|\lambda_k|) = \lim_{\varepsilon\to 0}\oint \frac{dz}{2\pi i} f_\varepsilon(z)\frac{\partial}{\partial z}\ln\det(V_n - z\mathbb{1}). \tag{17}$$

Here the integration path in the complex plane has been chosen to contain all eigenvalues $\lambda_k(V_n)$. The function f_ε is a continuation of f: We require that $\lim_{\varepsilon\to 0} f_\varepsilon(z)=f(|z|)$, including the parameter ε such that f_ε is analytic within the contour of integration.

Using an idea that originates from the concept of single-copy entanglement all these technicalities may be avoided and we can prove that the entanglement entropy diverges at least logarithmically in the case of a critical (anisotropic) Ising model. The Ω notation just means that there is asymptotically a lower bound with this behavior.[6]

Theorem 3 (Divergence for the critical Ising model). The entanglement entropy in the critical Ising model scales as

$$S(\rho_l) = \Omega(\log_2 n). \tag{18}$$

The starting point leading to this result from Eisert and Cramer (2005) is a lower bound in the operator norm of ρ_l leading to

$$\|\rho_l\|_\infty = -\log_2 \det[(\mathbb{1} + V_n)/2] \geqslant -\tfrac{1}{2}\log_2|\det(V_n)|. \tag{19}$$

This makes a big difference: we now no longer need the singular values of V_n (which would lead to an enormously complicated block Toeplitz expression, for a case for which the Fisher-Hartwig conjecture has not yet been proven). Instead—as the absolute value of the determinant is just as well the product of the absolute values of the eigenvalues as of the singular values—we can use the ordinary Fisher-Hartwig machinery to get an asymptotic handle on eigenvalues. For the critical Ising model, we can find an explicit factorization of the Fisher-Hartwig symbol, in terms of a function reflecting a single discontinuity and an analytical function. Using again a proven instance of the Fisher-Hartwig conjecture (Libby, 1990)—albeit a different one than used in the case of an isotropic model—one finds the bound as in Eq. (18); for details see Eisert and Cramer (2005). The entanglement in two blocks of the critical Ising model has been studied by Facchi *et al.* (2008).

Another useful starting point to obtain bounds to entanglement entropies in fermionic systems is to make use of quadratic bounds to the entropy function: Such quadratic bounds immediately translate to a bound to the function f in the expression of the entropy of a fermionic state in terms of the covariance matrix as in Eq. (13) as

$$(1-x^2)^{1/2} \geqslant f(x) \geqslant 1 - x^2. \tag{20}$$

This immediately translates to a bound of the form $\mathrm{tr}[(\mathbb{1}-V_I V_I^T)^{1/2}] \geqslant S(\rho_l) \geqslant \mathrm{tr}[\mathbb{1}-V_I V_I^T]$, where V_I is the submatrix of V associated with the interior I (Fannes *et al.*, 2003). These bounds have also been exploited in higher-dimensional analysis (Wolf, 2006).

A method to obtain area laws, in particular, for symmetrically bisected quantum chains, is the so-called method of *corner transfer matrices*. This method was first used by Calabrese and Cardy (2004) for the computation of the entanglement entropy, using ideas going back to Peschel *et al.* (1999). The infinite sum of Calabrese and Cardy (2004) could be performed in Peschel (2004), also giving rise to a formula for the entanglement entropy in the XX model. This idea has also applied to further models by Weston (2006).

To conclude the discussion of critical quasifree fermionic models, we note that the correspondence of being critical (gapped) and having a logarithmically divergent (saturating) entanglement entropy holds true for local systems only. If one allows for long-ranged interactions, then one can indeed find gapped noncritical models that exhibit a logarithmically divergent entanglement entropy (Eisert and Osborne, 2006).

Theorem 4 (Gapped model with long-range interactions). There exist models with long-range interactions, the coupling strength being bounded by $r/\mathrm{dist}(j,k)$ for some constant $r>0$, such that for some constant $\xi>0$,

$$S(\rho_l) = \xi \log_2(n) + O(1).$$

Hence, being gapped—albeit having power-law correlations—does not necessarily imply an area law. If one allows for long-range interactions (and a *fractal structure* of the Fermi surface), one can show that one can even approach arbitrarily well a volume law for the entanglement entropy (Fannes *et al.*, 2003; Farkas and Zimboras, 2005). Interestingly, states that are defined by *quantum expanders* can have exponentially decaying correlations and still have large entanglement, as proven by Ben-Aroya and Ta-Shma (2007) and Hastings (2007b). These models again give rise to long-range Hamiltonians, but they still clearly demonstrate a strong distinction between correlations and entanglement.

C. General gapped local spin models

We now turn to the discussion of general 1D gapped spin models with local interactions, where each site supports a d-dimensional quantum system. As stated rigorously in the theorem below for such models an area law always holds (Hastings, 2007a). The proof is deeply rooted in the existence of Lieb-Robinson bounds which have also been essential in the proof of the exponential decay of correlation functions in gapped local models (Hastings and Koma, 2006; Nachtergaele and Sims, 2006).[7]

As we allow for arbitrary d, it is sufficient to consider Hamiltonians on the chain $L=\{1,\ldots,N\}$ that have interactions only to nearest neighbors. Then

[6] $f(n)=\Omega(g(n))$ if $\exists C>0, n_0: \forall \quad n>n_0: |Cg(n)| \leqslant |f(n)|$.

[7] This result is compatible with an earlier result of an area law in 1D gapped quantum field theories, based on the c theorem presented by Calabrese and Cardy (2004). This work also connected the role of the boundary points between regions I and O with the cluster decomposition in quantum field theory. In the gapless system with open boundaries, the entropy is then half of the one in the situation of having periodic boundary conditions.

$$H = \sum_{j \in L} H_{j,j+1}, \tag{21}$$

where $H_{j,j+1}$ is supported on sites j and $j+1$. We also impose a constraint of finite-interaction strength in that the operator norm $\|H_{i,i+1}\| \leq J$ for some $J>0$. Then Hastings (2007a) found the following.

Theorem 5 (Area law for gapped spin chains). Consider a local Hamiltonian H as in Eq. (21) with finite interaction strength. Suppose H has a unique ground state with a spectral gap ΔE to the first excited state. As before consider the block $I=\{1,\ldots,n\}$. Then,

$$S(\rho_I) \leq S_{\max} = c_0 \xi \log(6\xi) \log(d) 2^{6\xi \log(d)}, \tag{22}$$

for some numerical constant $c_0>0$ of order unity, and where $\xi=\max(2v/\Delta E, \xi_C)$, v is the velocity of sound, and ξ_C, which is of the order of unity.

The proof of this statement is quite intricate (Hastings, 2007a) and well beyond the scope in this paper. At its heart is the way locality enters by virtue of the Lieb-Robinson theorem. It is a statement on the existence of a *speed of sound* in local Hamiltonian systems with finite-dimensional constituents: Imagine we single out two disjoint sets X,Y from a lattice, and consider observables A and B that have support only on X and Y, respectively. Then $[A,B]=0$. If we evolve A with time under a local Hamiltonian H, it is no longer exactly true that $A(t)$ and B commute: $A(t)$ will be significantly supported on more and more sites, "melting away," and developing a long tail in support. For short times or large distances between sets X and Y, the commutator of $A(t)$ and B will be very small. How small exactly is governed by the Lieb-Robinson theorem (Lieb and Robinson, 1972; Hastings and Koma, 2006; Nachtergaele and Sims, 2006; Hastings, 2007a).

Theorem 6 (Lieb-Robinson Theorem). Let H be as in Eq. (21) a local Hamiltonian with a finite interaction strength. Then there exists a velocity of sound $v>0$ and $\mu,c>0$ such that for any two operators A and B with support on disjoint sets X and Y we have that

$$\|[A(t),B]\| \leq c\|A\|\,\|B\| \exp\{-\mu[\mathrm{dist}(X,Y) - v|t|]\}, \tag{23}$$

where the distance between sets is taken to be $\mathrm{dist}(X,Y)=\min_{i\in X, j\in Y}(|i-j|)$, and where

$$A(t) = e^{iHt} A e^{-iHt}.$$

The velocity v is of order J.

This statement, natural as it may seem when viewed with a reasonable physical intuition, is a rigorous, and profound statement on how locality manifests itself in quantum lattice systems. From this bound, the decay of correlation functions in gapped models can be proven (Hastings and Koma, 2006; Nachtergaele and Sims, 2006), the quantization of the Hall conductance for interacting electrons (Hastings and Michalakis, 2009) an area law as above (Hastings, 2007a), as well as statements concerning propagation of quantum information and correlations through local dynamics (Bravyi *et al.*, 2006).[8] Lieb-Robinson bounds also feature in the proof of a higher-dimensional Lieb-Schultz-Mattis theorem (Hastings, 2004; Nachtergaele and Sims, 2007).

We will later, in Sec. IV.G, encounter another consequence of the Lieb-Robinson theorem, namely, that quenched nonequilibrium systems generically satisfy area laws when starting from a product state and undergoing time evolution under a local Hamiltonian. This perspective receives much attention in the context of nonequilibrium dynamics of quantum many-body systems. Here the Lieb-Robinson theorem is also the basis for the functioning of numerical *light cone methods* to study time evolution of quantum many-body systems (Osborne, 2006, 2007a, 2007b; Hastings, 2008), in which effectively only the essential part inside the causal cone is simulated.

D. Results from conformal field theory

In critical models the correlation length diverges and the models become scale invariant and allow for a description in terms of conformal field theories. According to the universality hypothesis, the microscopic details become irrelevant for a number of key properties. These universal quantities then depend only on basic properties such as the symmetry of the system, or the spatial dimension. Models from the same universality class are characterized by the same fixed-point Hamiltonian under renormalization transformations, which is invariant under general rotations. Conformal field theory then describes such continuum models, which have the symmetry of the *conformal group* (including translations, rotations, and scalings). The universality class is characterized by the central charge c, a quantity that roughly quantifies the "degrees of freedom of the theory." For free bosons $c=1$, whereas the Ising universality class has $c=1/2$.

Once a model is known to be described by a conformal field theory, powerful methods are available to compute universal properties, and entanglement entropies (or even the full reduced spectra) of subsystems.[9] This approach applies for $(1+1)$-dimensional systems; that is, with one spatial dimension. In the seminal work of Holzhey *et al.* (1994) the entanglement entropy in $1+1$ dimensions has been calculated [see also Callan and Wilczek (1994) and Fiola *et al.* (1994)]. Calabrese and Cardy (2004) and Cardy *et al.* (2007) put this into a more general context, and also allow for noncontiguous regions I.

[8] The assumption that we have a spin system, meaning finite-dimensional local constituents, is crucial here.

[9] Conformal field theory provides—in this context specifically in $1+1$ dimensions—a powerful repertory of methods to compute quantities that are otherwise inaccessible especially for nonintegrable models. From a mathematical physics perspective, it is the lack of a rigorous proof of the relationship between the lattice model and the conformal field theory that makes such a treatment nonrigorous.

The local spectra of the reductions are discussed in Orus (2005), Orus *et al.* (2006), and Calabrese and Lefevre (2008). Block-block entanglement is also discussed in Franca and Capelle (2008) and Marcovitch *et al.* (2009). For a short nontechnical review, see Calabrese and Cardy (2006a).

The starting point of the computations is the observation that powers of the reduced density matrix ρ_I^n for any positive integer n can be computed. The series $\mathrm{tr}[\rho_I^n] = \Sigma_j \lambda_j(\rho_I)^n$ is absolute convergent and analytic for all $\mathrm{Re}(n) > 1$. The derivative exists, and hence one can make use of

$$S(\rho_I) = \lim_{n \searrow 1} -\frac{1}{\ln 2} \frac{\partial}{\partial n} \mathrm{tr}[\rho_I^n]$$

to compute the entanglement entropy. This procedure is typically referred to as "replica trick." This leads in 1+1 dimensions to the expression (Holzhey *et al.*, 1994)

$$S(\rho_I) = \frac{c}{3} \log_2(l/a) + O(1), \tag{24}$$

where c is as above the central charge, l is the length of a single interval forming region I, and a is an *ultraviolet cutoff*, corresponding to a lattice spacing, to avoid an ultraviolet divergence; cfp. Eq. (10). The above constant C is nothing but $c/3$. This divergence is also removed using the mutual information (Casini and Huerta, 2007), see Sec. V.B. The offset constant in Eq. (24) is nonuniversal. So the logarithmic divergence of the entanglement entropy in the length of the interval is recovered here. From the expression given in Calabrese and Cardy (2004) for ρ_I^n, one also finds for the Renyi entropies for $\alpha > 1$,

$$S_\alpha(\rho_I) = \frac{c}{6}(1 + 1/\alpha)\log_2(l/a) + O(1).$$

If one is close to the critical point, where the correlation length $\xi > 0$ is large but finite, one can often still effectively describe the system by a conformal field theory. One then obtains for the entanglement entropy (Calabrese and Cardy, 2004) [compare also Casini (2005)]

$$S(\rho_I) \to \frac{c}{3} \log_2(\xi/a).$$

E. Disordered spin chains

Natural systems will generally exhibit a certain amount of *quenched disorder*, which means that the model parameters are drawn randomly and the resulting correlation functions or entanglement entropies $\mathbb{E}[S(\rho_I)]$ have to be considered as being averaged over the *a priori* distributions, with average $\mathbb{E}$. The critical behavior of quantum spin chains with "quenched" disorder is remarkably different from its counterpart in the corresponding pure case, in several respects. Hence, it is only natural to ask whether the scaling of the entanglement entropy is influenced by having some disorder in the model. This question has first been posed by Refael and Moore (2004) for the spin-1/2 random antiferromagnetic Heisenberg model,

$$H = \sum_{j \in L} J_j(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z),$$

with $\{J_j\}$ drawn from a suitable continuous distribution. The low-energy properties of this model, along with the random XX model, are described by what is called a random-singlet phase (Laflorencie, 2005). Using a real-space renormalization group approach (Refael and Moore, 2004), the intuition can be developed that in this phase, singlets form in a random fashion, distributed over all length scales. The entanglement entropy of a subblock is hence obtained by effectively counting the singlets that cross the boundary of the subblock. This intuition has been further developed by Refael and Moore (2007). Within the framework of a real-space renormalization group approach—it is shown that the averaged entanglement entropy for a large class of disordered models scales like

$$\mathbb{E}[S(\rho_I)] = \frac{\gamma}{3} \log_2(n) + O(1).$$

In this class one observes universal behavior in the scaling of the averaged entanglement entropy. The intuition elaborated to above is further corroborated by work on the random antiferromagnetic XXZ chain (Hoyos *et al.*, 2007), described by a Hamiltonian

$$H = \sum_{j \in L} J_j(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta_j \sigma_j^z \sigma_{j+1}^z),$$

where $\{J_j\}$ are positive uncorrelated random variables drawn from some probability distribution and the uncorrelated anisotropy parameters $\{\Delta_j\}$ are also taken from a probability distribution. In this work, the observation is further explored that the scaling of the averaged entanglement entropy can be universal, even if correlation amplitudes are not, in that they would manifest themselves only in nonleading order terms in the entanglement entropy. This intuition has also been further corroborated by Lin *et al.* (2007) and Yu *et al.* (2008), on the entanglement entropy in a two-dimensional (2D) situation.

From a fully rigorous perspective, the entanglement entropy in the random Ising model—for which Refael and Moore (2004) found a scaling with the effective central charge of $\log(\sqrt{2})$—has been revisited with methods and ideas of *percolation theory* (Grimmett *et al.*, 2007). This approach is more limited than the Fisher-Hartwig techniques in terms of the class of models that can be considered—the Ising model only—but is more powerful in that also disordered systems with no translational invariance can be considered.

Theorem 7 (Nontranslationally invariant Ising model). Consider the Ising model

$$H = -\frac{1}{2}\sum_{j,k\in L,\mathrm{dist}(j,k)=1} \lambda_{j,k} X_j X_k - \sum_{j\in L} \delta_j Z_j,$$

where $\lambda_{j,k} \geqslant 0$ and $\delta_j \geqslant 0$ are the spin-coupling and external field intensities, respectively, which may depend on the lattice site in a nontranslationally invariant system. The total number of sites is, $N=2m+n+1$, with $\{1,\ldots,n\}$ the distinguished region. Then there exist γ,α,C with properties as in the subsequent footnote.[10] If $\gamma > 4\ln 2$, then there exist constants $c_1, c_2 > 0$ depending only on γ such that

$$S(\rho_l) \leqslant c_1 \log_2(n) + c_2 \tag{26}$$

for $m \geqslant 0$, so the entanglement entropy is at most logarithmically divergent.

The general picture that emerges is that the entanglement entropy scales as in the nonrandom case, but with a different prefactor in the logarithmic divergence. This seems natural, as the disorder tends to "localize" excitations, and, hence, with faster decaying correlations one would expect less entanglement to be present in the system. Yet, there are exceptions: cases in which one does find a logarithmic divergence, but with a larger prefactor compared to the nonrandom case. This includes the random quantum Potts model with spin dimension d: Here for the very large dimension of $d > 41$ one finds a larger factor (Santachiara, 2006). The exploration and complete classification of the role of disorder to entanglement properties of ground states—including noncritical and higher-dimensional models—remains an interesting challenge.

F. Matrix-product states

Matrix-product states (MPS) play a central role in the context of area laws for the entanglement entropy. They form the class of states that is at the root of the workhorse of simulating strongly correlated quantum chains—DMRG. This link will be elaborated upon in Sec. VI. Here we focus on the entanglement and correlation properties of MPS. In the original sense, MPS are states defined on quantum chains consisting of N sites, each constituent being a d-level system. There are several ways of defining and introducing MPS, the relationship of which may not be entirely obvious. This is also the reason that it was left unnoticed for some time that MPS—as being generated in DMRG—and finitely correlated states (Fannes *et al.*, 1992)—as being considered in the mathematical physics literature—are up to translational invariance essentially the same objects.

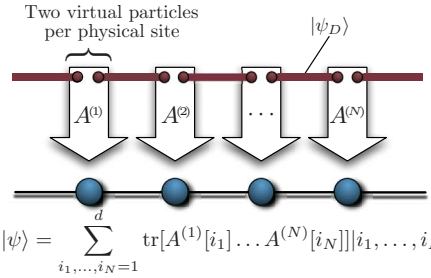

FIG. 2. (Color online) The valence-bond picture underlying matrix-product states.

One way of looking at MPS is via a valence-bond picture: For each of the constituents one introduces a virtual substructure consisting of two particles. Per site with Hilbert space $\mathbb{C}^d$, one associates a Hilbert space $\mathbb{C}^D \otimes \mathbb{C}^D$ for some D. This D is sometimes referred to as the dimension of the correlation space, or D called the auxiliary or virtual dimension.

These D-dimensional virtual systems are thought to be prepared to a maximally entangled state with each one particle of each of the neighbors, arranged on a ring (see Fig. 2). In other words, one starts from a pure state defined by the state vector $|\psi_D\rangle^{\otimes n}$, where we have defined the maximally entangled state vector as

$$|\psi_D\rangle = \frac{1}{\sqrt{D}}\sum_{k=1}^{D} |k,k\rangle. \tag{27}$$

Then, one applies a local linear map to each of the pairs of systems associated with every physical constituent in the center of the chain,

$$A^{(k)} = \sum_{j=1}^{d}\sum_{a,b=1}^{D} A^{(k)}_{a,b}[j]|j\rangle\langle a,b|, \tag{28}$$

where $k=1,\ldots,N$. This procedure will prepare a certain class of states: indeed the MPS. We may conceive for each site $k \in L$ the collection of complex numbers $A^{(k)}_{a,b}[j]$ as the elements of d matrices $A^{(k)}[1],\ldots,A^{(k)}[d]$. For a quantum spin chain with $d=2$, we hence simply have two matrices $A^{(k)}[1]$, $A^{(k)}[2]$ per site. This procedure of locally projecting to the physical dimension d gives rise to state vectors of the form

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} \mathrm{tr}[A^{(1)}[i_1]\cdots A^{(N)}[i_N]]|i_1,\ldots,i_N\rangle. \tag{29}$$

This is the most frequently used form of representing matrix-product states. For open boundary conditions in a chain $L=\{1,\ldots,N\}$, $A^{(1)}[i_1]$ and $A^{(N)}[i_N]$ are row and column vectors, respectively. MPS are described by only a number of parameters polynomial in the system size, $O(dND^2)$, in contrast to the scaling of the dimension of

[10] Let $\lambda,\delta \in (0,\infty)$ and write $\theta=\lambda/\delta$. There exist constants $\alpha, C \in (0,\infty)$—depending on θ only—and a constant $\gamma=\gamma(\theta)$ satisfying $0<\gamma<\infty$ if $\theta<1$, such that, for all $n \geqslant 1$,

$$\|\rho_N^n - \rho_M^n\| \leqslant \min\{2, Cn^\alpha e^{-\gamma N}\}, \quad 2 \leqslant N \leqslant M. \tag{25}$$

Here ρ_N^n denotes the reduced state of n sites in a system of total size N. One may find such a γ satisfying $\gamma\to\infty$ as $\theta \searrow 0$.

the full Hilbert space $(\mathbb{C}^d)^{\otimes N}$, which is exponential in N.[11]

The particular construction of MPS immediately shows that MPS satisfy an area law. In fact, it follows trivially from their definition (see also Fig. 2) that

$$S(\rho_I) \leqslant 2\log_2(D),$$

so the entanglement entropy is always bounded from above by a constant in N. Hence MPS have an in-built area law property. As shown earlier the ground states of a variety of Hamiltonians exhibit exactly such an area scaling when the system is noncritical but a logarithmic divergence when the system is critical. This already suggests that MPS may be a good description for ground states of noncritical systems but that this description may become less efficient in critical systems. Indeed, it will be discussed and highlighted later that systems satisfying an area law can be economically represented as MPS so that MPS with a small auxiliary dimension D can indeed typically approximate ground states of local Hamiltonians.

G. Single-copy entanglement

The entanglement entropy—occupying center stage in this paper—quantifies entanglement in a precise sense: for pure states it is the *distillable entanglement* (Plenio and Virmani, 2007; Horodecki *et al.*, 2009), so the rate with which one can locally extract maximally entangled pairs from a supply of identically prepared systems. Specifically, local refers here to a subsystem I of the system, but to a collective operation on many identically prepared states. In a quantum many-body system, needless to say, this means that one performs operations that are local to all constituents in I collectively in all specimens at hand.

When having the entanglement content in mind, one can equally reasonably ask how much entanglement is contained in a single spin chain. The concept of single-copy entanglement grasps this notion of distilling entanglement from a single specimen of a quantum spin chain with certainty.

If D is the largest integer such that one can deterministically transform a state into the maximally entangled state $|\psi_D\rangle\langle\psi_D|$ [see Eq. (27)] by local operations and classical communication (LOCC), i.e.,

$$\rho \mapsto |\psi_D\rangle\langle\psi_D|, \tag{30}$$

one assigns the value $E_1=\log_2(D)$ to the state as its single-copy entanglement. For pure states, such transformations on the level of specimens are perfectly well understood (Jonathan and Plenio, 1999; Nielsen, 1999; Vidal, 1999) and are linked to the well-established theory of majorization in linear algebra (Horn and Johnson, 1985). For our present purposes, for a pure state $\rho=|\psi\rangle\langle\psi|$, we find that Eq. (30) holds if and only if $\|\rho_I\| \leqslant 1/D$. Hence,

$$E_1(\rho_I) = \log_2(\lfloor \|\rho_I\|^{-1} \rfloor).$$

This, in turn, means that single-copy entanglement can be derived from the α-Renyi entropy of the reduction in the limit of large α. A surprising insight is that in critical systems we do not only find a local spectrum leading to the logarithmic divergence of the entanglement entropy but that there is more structure to the spectrum, governing all of its Renyi entropies. For example, for quasifree models, we find that once the entanglement entropy diverges, so does the single-copy entanglement, with a prefactor that is asymptotically exactly half the value of the entanglement entropy (Eisert and Cramer, 2005; Orus *et al.*, 2006).

Theorem 8 (Single-copy entanglement). Consider a family of quasifree fermionic Hamiltonians as in Theorem 2. Then, whenever the entanglement entropy scales as

$$S(\rho_I) = \xi\log_2(n) + O(1),$$

for some constant $\xi>0$, then the scaling of the single-copy entanglement is found to be

$$E_1(\rho_I) = \frac{\xi}{2}\log_2(n) + O(1).$$

This means that exactly half the entanglement can be distilled from a single critical chain rather than what is available as a rate in the asymptotic setting (Peschel and Zhao, 2005; Orus *et al.*, 2006). This finding has also been corroborated by the behavior of all critical models for which the local spectra can be described by their conformal field theory in quite some generality (Orus *et al.*, 2006). Zhou *et al.* (2006) studied Renyi entropies in boundary critical phenomena, and hence also arrived at a relationship between the entanglement entropy and the single-copy entanglement. Riera and Latorre (2006) considered the entropy loss along the renormalization group trajectory driven by the mass term in free massive theories, and also discussed the single-copy entanglement in such situations. Keyl *et al.* (2006) studied the

[11] If one allows D to (exponentially) grow with the system size, one can show that actually every state vector from $(\mathbb{C}^d)^{\otimes N}$ can be represented as a MPS of the form as in Eq. (29). It is important to note that MPS cannot only be described with linearly many parameters in the system size: One can also efficiently compute local properties from them, which is a property not merely following from the small number of parameters to define them. For expectation values of observables having a nontrivial support on sites $k,\ldots,k+l\in L$, we find $\langle S_k\cdots S_{k+l}\rangle = \mathrm{tr}[E_1^{(1)}\cdots E_1^{(k-1)}E_{S_k}^{(k)}\cdots E_{S_{k+l}}^{(k+l)}E_1^{(k+l+1)}\cdots E_1^{(N)}]$, where the transfer operators are defined as

$$E_S^{(l)} = \sum_{j,k=1}^{d} \langle j|S|k\rangle [A^{(l)}[k]\otimes (A^{(l)}[j])_*],$$

the star denoting complex conjugation. The decay of correlation functions can also be studied: If all matrices are the same per site, $A^{(k)}[j]=:A[j]$ for all $j\in L$, and similarly define E_1, then one finds $\langle S_kS_{k+l}\rangle - \langle S_k\rangle\langle S_{k+l}\rangle = O(|\lambda_2(E_1)|^{l-1})$, where $\lambda_2(E_1)$ denotes the second to largest eigenvalue of the transfer operator of the identity E_1.

situation of single-copy entanglement in the situation of bipartite systems between blocks when there is a gap of a finite number of sites between the two blocks. Interestingly, there are critical models in which the single-copy entanglement still diverges in this sense.[12]

H. Summary of one-dimensional systems

In a nutshell, the situation in one-dimensional translation invariant models is quite clear: If a system is local and gapped, an area law always holds rigorously. In many specific models, prefactors can be computed. In contrast, if the interactions may be long ranged, area laws may be violated. For critical lattice models for which one can directly evaluate the entanglement entropy, a logarithmic divergence is encountered. This picture is supported by the findings of conformal field theory. The situation will be less transparent and more intricate in higher-dimensional models. In any case, in the light of the previous findings one may be tempted to formulate the following conjecture on the numerical bound on the right-hand side of the previously discussed area law.

Conjecture 1 (Area bound in one dimension). There exists a function $f:\mathbb{R}^+\to\mathbb{R}^+$—equipped with further suitable properties—such that in any gapped one-dimensional model, we have

$$S(\rho_l)\leqslant f(v/\Delta E),$$

where ΔE is the spectral gap and v is the speed of sound as used in the Lieb-Robinson bound.

Indeed, while most explicit studies do indicate a behavior linear in $\log(1/\Delta E)$ of the entanglement entropy—the above-mentioned quasifree models—one can construct models (Gottesman and Hastings, 2009; Irani, 2009) for which one finds a dependence which is polynomial in $1/\Delta E$.

IV. AREA LAWS IN HIGHER DIMENSIONS

For a chain, to satisfy an area law for the entanglement entropy means simply that it saturates with increasing block sizes. Needless to say, the notion of having entropic quantities scaling like the boundary area of a subregion becomes specifically relevant in the case of higher dimensions: Then the boundary of the region I is a nontrivial object. Now we are in a position to approach the question: Given a ground state of a quantum many-body system, does the entanglement entropy of a subregion I fulfill an area law? This question has been initiated by Bombelli *et al.* (1986) and Srednicki (1993), where also a numerical answer has been found.

The answer to this question for ground states is very much developed in the case of quasifree bosonic or fermionic models. Even in such systems, the rigorous answer to this question will turn out to be technically quite involved. The reason for these technicalities is essentially rooted in the fact that one distinguishes a subregion, thus breaking translational symmetry of translationally invariant systems, and analytical methods are hard to come by. The first rigorous higher-dimensional area law has been proven by Plenio *et al.* (2005), with refinements for arbitrary harmonic interactions by Cramer and Eisert (2006) and Cramer *et al.* (2006) so that for such bosonic free models the problem can be considered solved in all generality forming a "laboratory" of what one should expect in general systems.

For critical fermionic models (Barthel *et al.*, 2006; Gioev and Klich, 2006; Li *et al.*, 2006; Wolf, 2006; Cramer *et al.*, 2007; Farkas and Zimboras, 2007) one can find small violations of area laws: the area law is then only satisfied up to a logarithmic correction. In this section, we discuss quasifree models in great detail. Beyond such quasifree models, no rigorous results are known for states at zero temperature, with the exception of classes of states that satisfy an area law by their very construction, and a subsection will specifically be devoted to those.

The models discussed here, however, do provide a clear intuition: Whenever one has a gapped and local model, and hence a length scale provided by the correlation length, one should reasonably expect an area law to hold. In cases where the number of eigenstates with vanishing energy density is not exponential in the volume—a technical condition in its own right—one can even prove an area law with at most a logarithmic correction from a sufficient decay of correlations (Masanes, 2009). The converse is not true, as we see later, and one can have area laws even for critical systems in which the correlation length does not provide a length scale. For systems at nonzero temperatures, by contrast, the entropy of entanglement forms neither a meaningful measure of entanglement nor for quantum correlations. For appropriately defined measures of correlations, however, one can restore an area law which holds in generality for a large class of systems.

A. Quasifree bosonic and fermionic models: Sufficient conditions for an area law

We follow the general description of Cramer *et al.* (2006), where we think of the model being defined on a general lattice L specified by a general simple graph. We consider quadratic bosonic Hamiltonians as in Eq. (4) and quadratic fermionic Hamiltonians as in Eq. (11). The key step is to relate correlation functions to entropic quantities. As before in the case of a harmonic chain, it is very involved to think of the entropy of entanglement itself. What comes to our help, however, is again that we can use the logarithmic negativity as an

[12]Note that the single-copy entanglement still grasps bipartite entanglement in a quantum many-body system. Identifying scaling laws for genuine multiparticle entanglement is an interesting enterprise in its own right. Notably, Botero and Reznik (2007) and Orus *et al.* (2008) considered the geometric entanglement (the logarithm of the largest Hilbert-Schmidt scalar product with a pure product state) and relate it to the conformal charge of the underlying model.

upper bound to the entanglement entropy [see Eq. (6)]. The logarithmic negativity is easier to treat analytically, as we can at all times refer to the full system, and not to subsystems I. In fact, we find that the logarithmic negativity can be bounded from above by the L_1 norm of a submatrix of the covariance matrix (Plenio *et al.*, 2005). For fermions in turn, the entropy may be bounded directly using the bound in Eq. (20).

Theorem 9 (Entropic bounds from matrix norms). The entanglement entropy of ground states of quadratic bosonic Hamiltonians as in Eq. (4) satisfies

$$S(\rho_I) \leqslant E_N(\rho,I) \leqslant 8\|\Gamma_x\| \sum_{i\in I, j\in O} |\langle p_i p_j\rangle|.$$

The entanglement entropy of unique ground states of quadratic fermionic Hamiltonians as in Eq. (11) satisfies

$$S(\rho_I) \leqslant 2 \sum_{i\in I, j\in O} |\langle f_i^\dagger f_j\rangle + \langle f_i f_j\rangle|.$$

This is a key tool towards proving the main theorem: We can reduce the evaluation of an entropic quantity to a counting argument over terms that can be evaluated from two-point correlators. Note that the use of the logarithmic negativity results in an important simplification of the problem. This shows that ideas from quantum information theory indeed help in finding proofs of statements of the scaling of the entanglement entropy.

We are now in the position to state the bound of the scaling of the ground-state entanglement in the boundary area $s(I)$, Eq. (1), of the distinguished region I (Cramer, 2006; Cramer and Eisert, 2006; Cramer *et al.*, 2006). It is remarkable that merely the decay of two-point correlations matter here, and that even some critical models will give rise to an area law, as long as the algebraic decay of correlations is sufficiently strong.

Theorem 10 (Quadratic Hamiltonians on general lattices). Let $\eta=\mathcal{D}+1+2\varepsilon$, $\varepsilon>0$, and assume that the ground state is unique and fulfills for $i,j\in L$, $i\neq j$, and some constant K_0,

$$\frac{K_0}{\mathrm{dist}^\eta(i,j)} \geqslant \begin{cases} |\langle p_i p_j\rangle| & \text{for bosons} \\ |\langle f_i^\dagger f_j\rangle + \langle f_i f_j\rangle| & \text{for fermions.} \end{cases}$$

Then

$$S(\rho_I) \leqslant K_0 c_{\mathcal{D}} \zeta(1+\varepsilon) s(I) \times \begin{cases} \|\Gamma_x\| & \text{for bosons} \\ 1 & \text{for fermions,} \end{cases}$$

where ζ is the Riemann zeta function and the constant $c_{\mathcal{D}}$ depends only on the dimension of the lattice.

A general version of what one should expect to be true provides the connection to the spectral gap: For gapped models the correlation functions decay exponentially with the graph theoretical distance. One cannot apply the Lieb-Robinson theorem to prove this, unfortunately, as the involved operators are unbounded. Hence, a technique that is applicable to describe clustering of correlations in such models had to be developed. The ideas of the proof go back to Benzi and Golub (1999), generalized to arbitrary lattices by Cramer (2006) and Cramer *et al.* (2006). Key ideas of the proof are polynomial approximations in the sense of Bernstein's theorem. For a thorough discussion of clustering of correlations in translation-invariant harmonic systems, see Schuch *et al.* (2006). For general lattices and gapped quadratic bosonic and fermionic Hamiltonians, one finds that two-point correlation functions decay exponentially. Together with the above theorem this leads to an area law whenever the model is gapped.

Corollary 1 (Area law for gapped quasifree models). The entanglement entropy of ground states of local gapped models of the type of Eq. (4) for bosons and of Eq. (11) for fermions for arbitrary lattices $G=(L,E)$ and arbitrary regions I satisfies for a suitable constant $\xi>0$,

$$S(\rho_I) \leqslant \xi s(I).$$

B. Logarithmic correction to an area law: Critical fermions

What can we say about situations in which the previous sufficient conditions are not satisfied? Specifically, how is the scaling of the entanglement entropy modified in the case of critical fermionic models? This is the question that we will focus on in this section. Following the bosonic result in Plenio *et al.* (2005) and Cramer *et al.* (2006), the entanglement entropy in fermionic models was first studied by Wolf (2006) for cubic lattices. Here the quadratic bound in Eq. (20) plays an important role, to relate bounds to the entropy to feasible expressions of the covariance matrix of the ground state. Here not quite an area law, but only one up to a logarithmic correction is found. The results can be summarized as follows.

Theorem 11 (Violation of area laws for critical fermions). For a cubic sublattice $I=\{1,\ldots,n\}^{\times\mathcal{D}}$ and an isotropic quasifree model as in Eq. (11) with a Fermi sea of nonzero measure and a finite nonzero surface there exist constants $c_0,c_1>0$ such that the ground state fulfils

$$c_0 n^{\mathcal{D}-1}\log_2(n) \leqslant S(\rho_I) \leqslant c_1 n^{\mathcal{D}-1}\log_2^2(n).$$

The stated lower bound makes use of the assumption that the Fermi surface is finite (and of a technical assumption that the sets representing the states cannot have nontrivial irrelevant directions); assumptions both of which can be removed (Farkas and Zimboras, 2007).

This fermionic quasifree case already exhibits a quite complex phase diagram (Barthel *et al.*, 2006; Li *et al.*, 2006). At the same time, Gioev and Klich (2006) formulated a similar result, based on a conjecture on the validity of Fisher-Hartig-type scaling results for higher-dimensional equivalents of Toeplitz matrices as further numerically corroborated by Barthel *et al.* (2006). A logarithmic divergence is not directly inconsistent with the picture suggested in a conformal field theory setting, as relativistic conformal field theories do not have a Fermi surface (Ryu and Takayanagi, 2006a). It is still intriguing that critical fermions do not satisfy an area law, but have logarithmic corrections. In this sense, critical fermionic models could be said to be "more strongly entangled" than critical bosonic models.

C. Difference between critical fermions and bosons: Half-spaces

The scaling of block entropies for bosons and fermions in higher spatial dimensions hence exhibit remarkable differences. Consider the case of a cubic lattice of $n^{\mathcal{D}}$ sites with periodic boundary conditions and $I=\{1,\ldots,m\}\times\{1,\ldots,n\}^{\times\mathcal{D}-1}$ (without limitation of generality we distinguish the first spatial dimension). Then one may transform the Hamiltonian to a system of mutually uncoupled one-dimensional chains using a unitary discrete Fourier transform. After this decoupling procedure the entanglement between I and O is given by a sum of the entanglement between the sites $I'=\{1,\ldots,m\}$ and $O'=\{m+1,\ldots,n\}$ of the $n^{\mathcal{D}-1}$ individual chains

$$\frac{S(\rho_I)}{s(I)}=\frac{1}{n^{\mathcal{D}-1}}\sum_{i=1}^{n^{\mathcal{D}-1}}S(\rho_{I'}^i).$$

We start with a discussion of fermions and focus on the isotropic setting [$B=0$ in Eq. (11)]. After taking the limit $n\to\infty$, the asymptotic behavior in m of the entanglement $S(\rho_{I'}^i)$ can then be read off Theorem 2 to yield the following statement [for technical details see Cramer *et al.* (2007)].

Theorem 12 (Prefactor for fermionic half-spaces). Asymptotically, the entanglement entropy of fermionic isotropic models of half-spaces satisfies

$$\lim_{n\to\infty}\frac{S(\rho_I)}{s(I)}=\frac{\log_2(m)}{6}\sum_{s=1}^{\infty}sv_s+O(1).$$

Here

$$v_s=\frac{|\{\phi\in[0,2\pi)^{\mathcal{D}-1}|\sigma(\phi)=s\}|}{(2\pi)^{D-1}}$$

is the integral over individual chains ϕ with s discontinuities $\sigma(\phi)$ in their symbol.

Hence, one encounters a logarithmic divergence in m of the entanglement entropy and the prefactor depends on the topology of the Fermi surface: The symbols exhibit discontinuities on the Fermi surface. If the Fermi surface is of measure zero (i.e., the set of solutions to $\lambda_\phi=0$, $\phi\in[0,2\pi)^{\mathcal{D}}$, is countable, as, e.g., in the critical bosonic case discussed below), we have $v_s=0$ and the system obeys the area law.

For bosons, we discuss the important case of $m=n/2$ for the Klein-Gordon Hamiltonian as in Eq. (9). After the transformation to uncoupled chains, one finds Hamiltonians for the individual chains that correspond to a nearest-neighbor coupling matrix X of the form as in Theorem 1, which yields

$$\frac{E_N(\rho,I)}{s(I)}=\int\frac{d\varphi}{4(2\pi)^{D-1}}\log_2\left(\frac{\mathcal{D}-\sum_{d=1}^{\mathcal{D}-1}\cos(\varphi_d)+1}{\mathcal{D}-\sum_{d=1}^{\mathcal{D}-1}\cos(\varphi_d)-1}\right)$$

in the limit $n\to\infty$. This expression is independent of the mass m and finite: For $\mathcal{D}=2$, it evaluates to $\log_2(3+2\sqrt{2})/4$ and similarly for $\mathcal{D}>2$. Hence, despite being critical, the system obeys an area law, in contrast to the fermionic case (for $m=n/2$ the entanglement for a critical fermionic system would diverge in n).

Hence, in quasifree critical models it matters whether a system is bosonic or fermionic when it comes to the question whether or not an area law holds. The above results confirm the numerical analysis of Srednicki (1993) for critical bosonic theories, and of Li *et al.* (2006) for two-dimensional fermionic systems. Motivated by these findings, Ding *et al.* (2008) numerically studied the nonleading order terms of an area law in nodal fermionic systems: It was found that in noncritical regimes, the leading subarea term is a negative constant, whereas in critical models one encounters a logarithmic additive term. A lesson from these higher-dimensional considerations is that the simple relationship between criticality and a violation of an area law is hence no longer valid for local lattice models in $\mathcal{D}>1$.

D. Entanglement in bosonic thermal states

In this section, we discuss area laws for notions of entanglement in Gibbs states,

$$\rho_\beta=\frac{\exp(-\beta H)}{\mathrm{tr}[\exp(-\beta H)]}$$

for some inverse temperature $\beta>0$. The second moments matrix, the covariance matrix, is then found to be $\Gamma=\Gamma_x\oplus\Gamma_p$ (Cramer and Eisert, 2006):

$$\Gamma_x=X^{-1/2}(X^{1/2}PX^{1/2})^{1/2}(\mathbb{1}+G)X^{-1/2},$$

$$\Gamma_p=X^{1/2}(X^{1/2}PX^{1/2})^{-1/2}(\mathbb{1}+G)X^{1/2},$$

$$G=2\{\exp[\beta(X^{1/2}PX^{1/2})^{1/2}]-\mathbb{1}\}^{-1}.$$

Using the methods of Benzi and Golub (1999) and Cramer and Eisert (2006) one again finds the suitable decay of correlations, which can be translated into an area law for the entanglement content. Here the result—taken from Cramer and Eisert (2006) and Cramer *et al.* (2006)—is stated in terms of the logarithmic negativity.

Theorem 13 (Entanglement in thermal bosonic states). The logarithmic negativity of thermal states of quadratic finite-ranged bosonic Hamiltonians as in Eq. (4) for $[X,P]=0$ satisfies $E_N(\rho,I)\leqslant\xi s(I)$ for a suitable constant $\xi>0$.

Since the logarithmic negativity is an upper bound to the entanglement of formation and hence the *distillable entanglement* (Plenio and Virmani, 2007; Horodecki *et al.*, 2009), this implies an area law for these quantities as

well. It is important to stress that the entropy of a subregion as such no longer reasonably quantifies entanglement between that subregion and the rest of the lattice: even classically correlated separable states will in general have a positive entropy of the reduced state. The latter quantity is then indeed extensive and fulfills a volume law, unlike the entanglement content. Area laws in thermal states have been further studied by Anders and Winter (2008), where an emphasis has been put on identifying regions where the states become separable. Cavalcanti *et al.* (2008) and Ferraro *et al.* (2008) investigated thermal bound entanglement—entanglement that is not distillable—in bosonic quadratic and spin systems.

E. Results from conformal field theory

In systems with more than one spatial dimension, the situation is more intricate, and there is no general expression known for entanglement entropies in $(d+1)$-dimensional conformal field theories. For interesting steps into a description of systems with two spatial dimensions in the framework of conformal field theory, see Fradkin and Moore (2006), and Ryu and Takayanagi (2006a). For a class of critical models in two spatial dimensions (including the quantum dimer model), it was found that $S(\rho_I)=2f_s(L/a)+cg\log(L/a)+O(1)$, where L is the length of the boundary area, f_s is an area law prefactor that is interpreted as a boundary free energy, and g is a coefficient that depends on the geometric properties of the partition into I and O. That is, in addition to a nonuniversal area law, one finds a universal logarithmically divergent correction. For a further discussion of steps towards a full theory of entanglement entropies in $(d+1)$-dimensional conformal field theories, see Fradkin and Moore (2006) and Ryu and Takayanagi (2006a).

F. States satisfying area laws by construction: Projected entangled pair states, graph states, and entanglement renormalization

In this section, we discuss classes of states that have the area law already built into their very construction. In this sense, they grasp the entanglement structure of local higher-dimensional models. These are projected entangled pair states, so matrix-product states in higher dimensions, and states from entanglement renormalization. They are designed to be variational states well approximating true ground states of local many-body systems: As was already true for matrix product states, they form a complete set of variational states. Yet, typically, for a much smaller, polynomial or constant, number of variational parameters they often deliver a very good approximation. In projected entangled pair states, locality is respected in just the same way as for MPS. Entanglement renormalization, in turn, is based on a scale-invariant tree structure, intercepted by disentangling steps, which in higher dimensions nevertheless leads to an area law for the entanglement entropy.

Projected entangled pair states (PEPS) can be thought of as being prepared as MPS in higher-dimensional cubic lattices $L=\{1,\ldots,N\}^{\times\mathcal{D}}$, or in fact to any lattice defined by any undirected simple graph $G=(L,E)$. In this valence bond construction, one again associates a physical space with Hilbert space $\mathbb{C}^d$ with each of the vertices L of G. Then, one places a maximally entangled pair of dimension $D\times D$ [see Eq. (27)] for some positive integer D between any two vertices that are connected by an edge $e\in E$. For a cubic lattice, one hence starts from a cubic grid of maximally entangled state vectors. Then, one applies a linear map $P^{(k)}:\mathbb{C}^D\otimes\cdots\otimes\mathbb{C}^D\to\mathbb{C}^d$ to each physical site, as

$$P^{(k)}=\sum_{j=1}^{d}\sum_{i_1,\ldots,i_{|S_1(k)|}}^{D}A^{(k)}_{j,i_1,\ldots,i_{e_k}}|j\rangle\langle i_1,\ldots,i_{|S_1(k)|}|.$$

Here $|S_1(k)|$ is the vertex degree of the vertex $k\in L$. The resulting state vector as such becomes

$$|\psi\rangle=\sum_{i_1,i_2,\ldots,i_{|L|}=1}^{d}\mathcal{C}[\{A^{(k)}_{i_l}\}_l]|i_1,i_2,\ldots,i_{|L|}\rangle,$$

where $\mathcal{C}$ denotes a contraction of all higher-order tensors with respect to the edges E of the graph. This amounts to a summation over all indices associated with connected vertices. The objects $A^{(k)}$ are hence tensors of an order that corresponds to the vertex degree of the lattice (a second-order tensor—a matrix—for a one-dimensional chain, a third-order tensor in hexagonal lattices, a fourth-order tensor in cubic lattices with $\mathcal{D}=2$, and so on). This construction is the natural equivalent of the valence bond construction for matrix-product states as explained in Eq. (28). This ansatz as such is the one of tensor-product states that is due to Martin-Delgado *et al.* (2001) which in turn is generalizing earlier work on AKLT-type valence bond states in two dimensions by Niggeman *et al.* (1997) and Hieida *et al.* (1999). The generated class of states is referred to as projected entangled pair states (Verstraete and Cirac, 2004a) reflecting the preparation procedure. PEPS states are sometimes also in higher dimensions simply referred to as matrix-product states (Hastings, 2007b). This ansatz has proven to provide a powerful and rich class of states. Importantly, Verstraete and Cirac (2004a) provided a first simulation method based on PEPS.[13]

This class of states is complete, in that any state of a given finite lattice can be arbitrarily well approximated by such a state if D is sufficiently large. Clearly, to compute local observables in such an ansatz, one has to contract this instance of a tensor network which in 2D is

[13]Note that 1D MPS based on a suitable order of the constituents do not form a good approximation for 2D models. This is essentially rooted in the observation that one should expect an area law for the entanglement entropy in gapped 2D models. For a discussion of subsystems spectra in 2D integrable models see Chung and Peschel (2000) and, for a discussion on DMRG, see Verstraete and Cirac (2004b).

actually computationally hard.[14] It is, however, possible to provide approximation techniques, related to the DMRG approach, that allow for the contraction of the tensor network and then for the computation of the expectation values of local observables (Isacsson and Syljuasen 2006; Verstraete *et al.*, 2006; Murg *et al.*, 2007; Verstraete *et al.*, 2008).

A particularly simple yet important subset of the projected entangled pair states is constituted by the so-called graph states (Schlingemann and Werner, 2002; Hein *et al.*, 2004, 2006). They are instances of stabilizer states (Gottesman, 1997; Audenaert and Plenio, 2005) which can be thought of as being prepared in the following fashion: On any graph $G=(L,E)$, one associates each vertex with a $\mathbb{C}^2$ spin. This spin is prepared in $|+\rangle = (|0\rangle+|1\rangle)/2^{1/2}$. Then, one applies a phase gate

$$U = |0,0\rangle\langle 0,0| + |0,1\rangle\langle 0,1| + |1,0\rangle\langle 1,0| - |1,1\rangle\langle 1,1|$$

to each pair of vertices that are connected by an edge. This phase gate corresponds to an Ising interaction. Clearly, this construction makes sense for any simple graph, and this is a subset of the above projected entangled pair states. Graph states readily satisfy an area law by construction (Hein *et al.*, 2004; Hamma *et al.*, 2005) as one merely needs to count the edges over the boundary of a distinguished region to obtains the entanglement entropy, then obviously linear in the boundary area.[15]

Graph states may be generalized to weighted-graph states (Anders *et al.*, 2006; Hein *et al.*, 2006; Plenio, 2007), where the edges may carry a different weight, and in turn generalize to the ansatz of a renormalization algorithm with graph enhancement (Hübener *et al.*, 2009), being a strict superset of matrix-product states and weighted-graph states, one that can nevertheless be efficiently contracted. As the graph defining the (weighted-) graph state does not need to have the same structure as the graph of the physical system whose quantum state we would like to describe, (weighted-) graph states may describe volume scaling on the level of the physical system. This makes them particularly suitable for simulation of time evolution, where no area law can be expected to hold.

Yet a different class of many-body states with applications in the simulation of quantum spin systems is given by the states generated by entanglement renormalization (MERA) (Vidal, 2007). This is a class of states the construction of which is inspired by a renormalization scheme. Consider a tree tensor network with the physical sites at the end. This can be efficiently contracted. Yet, when decimating, say, two spins of one layer to a single "superspin" in the next layer in a single step of a renormalization procedure, one loses information about the state. The idea of a MERA ansatz is to allow for disentangling unitaries, effectively removing entanglement from a state, before doing a renormalization step.

More specifically, consider a cubic lattice $L=\{1,\ldots,N\}^{\times\mathcal{D}}$ in some dimension $\mathcal{D}$, embodying $N^{\mathcal{D}}$ sites. Each site $j\in L$ is associated with a physical system with Hilbert space $\mathbb{C}^d$. The MERA is essentially a unitary tensor network of depth $O(\log(N))$, preparing $|\psi\rangle$ from $|0\rangle^{\otimes N}$. It consists of layers of isometries—performing the renormalization step—and disentanglers, which minimize the entanglement in each step before the next renormalization step. This renormalization step may be labeled with a fictitious time parameter. Each of the unitary disentanglers $U\in U(d^m)$ in the disentangling layer has a finite support on m sites. In the simplest possible realization of a MERA this would be $m=2$. The unitaries can be taken to be different in each layer, and also different from each other within the layer. Unlike PEPS, they do not give rise to strictly translationally invariant states, even if all unitaries are taken to be identical in each layer.

Such a procedure can be defined for cubic lattices of any dimension $\mathcal{D}$. In $\mathcal{D}=1$, one does not in fact observe an area law, but typically a logarithmic divergence of the entanglement entropy, quite like in critical spin systems. Indeed, the MERA ansatz as a scale-invariant ansatz is expected to be suitable to approximate critical systems well, and numerical simulations based on the MERA ansatz corroborate this intuition (Evenbly and Vidal, 2007; Dawson *et al.*, 2008; Rizzi *et al.*, 2008; Evenbly and Vidal, 2009). A precise connection between homogeneous instances of a MERA ansatz and conformal field theory is established in Giovanette *et al.* (2009) and Pfeifer *et al.* (2009). In more than one dimension, $\mathcal{D}>1$, MERA again satisfy an area law, as a moment of thought reveals: One encounters linearly many unitaries over a boundary that have entangling power, rendering the computation of an upper bound to the entanglement

[14]In fact, it is known that the exact contraction of such a tensor network is contained in the complexity class $\#P$ complete (Schuch *et al.*, 2007). Clearly, this means that no algorithm is known with polynomial running time.

[15]Conversely the difficulty of actually contracting tensor networks, even if they correspond to states that approximate ground states satisfying area laws well, is that such states can have computational power for quantum computing. Indeed, certain graph states or cluster states—as they are called for a cubic lattice—are universal resources for quantum computing: Quantum computing can be done by applying local measurements onto single sites of such a cluster state, without the need of additional unitary control. This computational model—known as one-way computing (Raussendorf and Briegel, 2001)—can also be understood as a teleportation scheme in virtual qubits (Verstraete and Cirac, 2004). The tensor networks that occur when performing Pauli measurements can still be efficiently contracted, but not under arbitrary measurements, leading to universal computation. The program of using general projected entangled pair states in quantum computing based on measurements has been pursued (Gross and Eisert, 2007; Gross *et al.*, 2007), giving rise to new measurement-based quantum computational models. This also highlights how the disadvantage of having no classical efficient description can be made an advantage: One can at each instance of the computation not efficiently compute the outcome, but on a physical system realizing this model one could efficiently simulate any quantum computer.

entropy a combinatorial problem. Despite this observation, first numerical work on fermionic instances of MERA ansatz appears to deliver promising results (Corboz *et al.*, 2009; Pineda *et al.*, 2009).

Theorem 14 (Area laws for PEPS, graph states, and MERA). For any finite dimension D of the virtual systems, the entanglement entropy of a projected entangled pair state satisfies $S(\rho_I) \leqslant s(I)D$, where as before $s(I)$ denotes the surface area of I on a graph. Hence, also graph states with a fixed vertex degree satisfy area laws. A family of states from entanglement renormalization will also satisfy an area law for cubic lattices with $\mathcal{D} \geqslant 2$, and a logarithmic divergence in $\mathcal{D}=1$.

Interestingly, based on a PEPS description, one can construct critical models that still satisfy an area law in $\mathcal{D}=2$ (Verstraete *et al.*, 2006), resembling the situation for critical quasifree bosonic systems. The validity of an area law follows trivially from construction, so the technical part in the argument amounts to showing that a model is critical. Verstraete *et al.* (2006) showed this by employing a quantum-classical correspondence: Take a classical two-body spin Hamiltonian of the form $H(\sigma_1,\ldots,\sigma_N)=\Sigma_{\mathrm{dist}(i,j)=1}h(\sigma_i,\sigma_j)$, $\sigma_i=1,\ldots,d$. This Hamiltonian will have at some inverse temperature $\beta>0$ a partition function $Z=\Sigma_\sigma e^{-\beta H(\sigma)}$. From this classical partition function, a quantum state can be constructed using the Boltzmann weights as superposition coefficients,

$$|\psi_{H,\beta}\rangle = \frac{1}{Z^{1/2}} \sum_{\sigma_1,\ldots,\sigma_N} e^{-\beta H(\sigma_1,\ldots,\sigma_N)/2} |\sigma_1,\ldots,\sigma_N\rangle.$$

This state vector has the properties that for diagonal observables it gives rise to the same expectation values and correlation functions as the corresponding classical model does, it has a simple representation as a PEPS for $D=d$, and it is—as any PEPS—the ground state of a local Hamiltonian. The classical model can then be chosen such that the appropriate decay of correlation functions follows. This construction delivers critical spin models that nevertheless satisfy an area law.

G. Quenches and nonequilibrium dynamics

A physical setting that receives much recent attention is the nonequilibrium dynamics of quantum many-body systems. A specifically interesting setting is the one of a sudden quench (Eisert *et al.*, 2004; Plenio *et al.*, 2004; Calabrese and Cardy, 2005, 2006b, 2007a, 2007b; Zurek *et al.*, 2005; De Chiara *et al.*, 2006; Eisert and Osborne, 2006; Rodriguez *et al.*, 2006; Eisler and Peschel, 2007; Kollath *et al.*, 2007; Barthel and Schollwöck, 2008; Cramer, Dawson, *et al.*, 2008; Cramer, Flesch, *et al.*, 2008; Eisler *et al.*, 2008; Fagotti and Calabrese, 2008; Hastings, 2008; Schuch *et al.*, 2008): Here the initial condition is the nondegenerate ground state of some local Hamiltonian H, with state vector $|\psi\rangle$. Then, one suddenly (locally) alters the system parameters to a new Hamiltonian V. Since $|\psi\rangle$ will typically no longer be an eigenvalue of H, one arrives at a nonequilibrium situation: The state vector's time evolution is simply given by $|\psi(t)\rangle = e^{-itV}|\psi\rangle$. Studies of instances of such complex nonequilibrium many-body dynamics and questions of the dynamics of quantum phase transitions are enjoying a renaissance recently, not the least due to the advent of the high degree of control over quantum lattice systems with cold atoms in optical lattices.[16]

For finite times, infinite quenched systems satisfy an area law in the entanglement entropy (Calabrese and Cardy, 2005; Bravyi *et al.*, 2006; Eisert and Osborne, 2006) (strictly speaking, whenever one considers time evolution under local finite-dimensional Hamiltonians starting from product states). For finite systems this holds true for times that are sufficiently small compared to the system size over the speed of sound. The intuition is that when suddenly switching to a new Hamiltonian, local excitations will be created. These excitations will propagate through the lattice, but—except from an exponentially suppressed tail—at most with the Lieb-Robinson velocity of Theorem 6 (Bravyi *et al.*, 2006; Eisert and Osborne, 2006; Hastings and Koma, 2006; Hastings, 2008). This is yet again a consequence for the approximate locality in quantum lattice systems, similar to the situation in relativity and implies that correlations can only slowly build up, resulting in an area theorem. In turn, such a quench does in general give rise to a linear increase in the entanglement entropy, a statement that is provably correct, and has been encountered in numerous numerical studies on quenched nonequilibrium systems (Calabrese and Cardy, 2005; Bravyi *et al.*, 2006; Eisert and Osborne, 2006; Barthel and Schollwöck, 2008; Cramer *et al.*, 2008; Schuch *et al.*, 2008). In fact, finite subsystems can locally relax in time, to appear as if they were in a thermal state (Cramer *et al.*, 2008). These results may be summarized in the following statement.

Theorem 15 (Area laws in nonequilibrium systems). Let $|\psi\rangle$ be a product initial state vector, and H a local Hamiltonian. Then, for any time $t>0$ there exist constants $c_0, c_1 > 0$ such that for any subset I the entanglement entropy of the time evolved reduction ρ_I of $\rho(t) = e^{-itH}|\psi\rangle\langle\psi|e^{itH}$ satisfies

$$S(\rho_I(t)) \leqslant c_0 s(I) + c_1. \tag{31}$$

Specifically, this is true for any local Hamiltonian on a cubic lattice in dimension $\mathcal{D}$. This means that for any constant time, the entanglement entropy satisfies what is called an *area law*. In turn, there are product initial state vectors $|\psi\rangle$ of one-dimensional spin chains, local Hamiltonians H, and constants $c_2, c_3, c_4, L_0, s_0, t_0 > 0$ such that

$$S(\rho_I(t)) \geqslant c_2 t + c_3,$$

for $L \geqslant L_0$ and $s \geqslant s_0$ and $t_0 \leqslant t \leqslant c_4 s$, for $I=\{1,\ldots,s\}$.

That is, for any fixed time t, one encounters an area law for the entanglement entropy, but the prefactor can

[16]The situation of locally perturbing the state and hence generating a nonequilibrium situation has been considered (Calabrese and Cardy, 2007a; Eisler and Peschel, 2007; Eisler *et al.*, 2008), where an area law is always expected to hold.

grow linearly in time. In fact, by a suitable choice of blocks, one can show that a lower bound grows linearly in time. This fact is responsible for the hardness of simulating time evolution of quantum many-body systems using instances of the DMRG approach: to represent such states faithfully, exponential resources are then required. Similar bounds give rise to statements on the minimal time needed in order to prepare states with topological order using local Hamiltonians (Bravyi *et al.*, 2006).

There is an interesting localization effect of entanglement under quenched disorder, linking to the previous discussion on ground-state entanglement in disordered systems. Whereas one obtains from Lieb-Robinson bound the estimate in time

$$S(\rho_I(t)) \leqslant c_0|t| + c_1$$

for suitable constants $c_0, c_1 > 0$, in the disordered one-dimensional XY spin chain this bound is replaced by the tighter bound

$$S(\rho_I(t)) \leqslant c_0 \log_2(N|t|) + c_1,$$

again for appropriate constants (Burrell and Osborne, 2007). This means that due to quenched disorder the growth of entanglement is merely logarithmic in time, not linear. There is an intuitive explanation for this: The linear sound cone provided by the Lieb-Robinson bounds is replaced by a logarithmically growing or even a constant one, leading to a suppressed entanglement propagation. A similar behavior is observed under time-dependent fluctuating disorder (Burrell *et al.*, 2009).

H. Topological entanglement entropy

The topological entanglement entropy is a quantity that is constructed in a fashion that enables it to characterize quantum many-body states that exhibit topological order, a concept introduced by Wen (1989, 1990) [see also Witten (1989) and Wen (1995)]. On both sides of a critical point in a system undergoing a quantum phase transition, the quantum many-body system may have a different kind of quantum order; but this order is not necessarily one that is characterized by a local order parameter: In systems of, say, two spatial dimensions, topological order may occur. Topological order manifests itself in a degeneracy of the ground-state manifold that depends on the topology of the entire system and the quasiparticle excitations then show an exotic type of anyonic quasiparticle statistics. These are features that make topologically ordered systems interesting for quantum computation, when exactly this degeneracy can be exploited in order to achieve a quantum memory robust against local fluctuations. They even allow in theory for robust instances of quantum computation, then referred to as topological quantum computation (Freedman *et al.*, 2003; Kitaev, 2003).

The topological entanglement entropy is now designed as an instrument to detect such topological order. Introduced by Kitaev and Preskill (2006) and Levin and Wen (2006), it received much attention recently (Furukawa and Misguich, 2007; Haque *et al.*, 2007; Papanikolaou *et al.*, 2007; Aguado and Vidal, 2008; Hemma *et al.*, 2008; Kargarian, 2008; Li and Haldane, 2008). The details of the relationship between positive topological entanglement entropy and topological quantum order are discussed in Nussinov and Ortiz (2009).

Kitaev and Preskill (2006) considered a disk in the plane I with boundary length L. This disk is thought to be much larger than the correlation length, and it is hence assumed that an area law in the above sense holds. The entanglement entropy of ρ_I will then have the form

$$S(\rho_I) = \alpha L - \gamma + O(1), \tag{32}$$

where the last term vanishes in the limit $L \to \infty$. The prefactor α is nonuniversal and ultraviolet divergent. However, $\gamma > 0$ is an additive constant which is universal and characterizes a global feature of the entanglement in the ground state. This quantity is referred to as topological entanglement entropy by Kitaev and Preskill (2006). To avoid ambiguities when distinguishing the constant term from the linear one in Eq. (32), Kitaev and Preskill (2006) made use of the following construction: The plane is divided into four regions, each of them large compared to the correlation length. A, B, and C are arranged as neighboring each other in three identical subparts of a disk. D is the exterior of the disk. The respective reductions to the parts are denoted as ρ_A and ρ_{AB} to regions A and jointly A and B, respectively. The topological entropy S_{Topo} is then defined as

$$\begin{aligned} S_{\text{Topo}} = {} & S(\rho_A) + S(\rho_B) + S(\rho_C) - S(\rho_{AB}) \\ & - S(\rho_{BC}) - S(\rho_{AC}) + S(\rho_{ABC}). \end{aligned} \tag{33}$$

This is a linear combination of entropies of reductions, constructed specifically in a way such that the dependencies on the length of the respective boundaries of regions cancel. It is not directly meant as an information theoretical quantity, although the differences of entropies resemble a mutual information expression. Also, slightly different definitions with similar properties are conceivable, and, indeed, the independent proposal of Levin and Wen (2006) made use of an alternative combination of entropies. The important aspect here is the above-mentioned cancellation of the boundary term. Taking the behavior as in Eq. (32) for granted, one indeed finds

$$S_{\text{Topo}} = -\gamma.$$

From the way S_{Topo} is constructed it is a topological invariant, and depends only on a universal quantity (unaltered under smooth deformations, as long as one stays away from critical points), and on how the regions are located with respect to each other, but not on their specific geometry (again assuming that the correlation length is much smaller than the regions and does not matter). Interestingly, topological order is hence a global property that is detected by the entanglement entropy. This construction can also readily be used in numerical studies. The explicit computation of how the entanglement entropy detects the presence of topological order

in an actually time-dependent model undergoing a quantum phase transition from a spin-polarized to a topologically ordered phase has been systematically explored by Hamma *et al.* (2008), further strengthening the findings of Kitaev and Preskill (2006).

Since its proposal, this and related quantities have been considered in a number of contexts. A natural candidate to explore this concept is the toric code state of Kitaev (2003): Consider for this a square lattice $I=\{1,\ldots,n\}^{\times 2}$ with periodic boundary conditions, and place the physical two-dimensional quantum spins on the vertices of this lattice.[17] This lattice is tiled into two sublattices of different color, red and white. Every white p and red plaquette s is then associated with one of the commuting operators

$$A_s = \prod_{j\in\partial s} \sigma_j^z, \quad B_p = \prod_{j\in\partial p} \sigma_j^x, \tag{34}$$

respectively, with nontrivial support on four spins each, where as before $\sigma_i^x,\sigma_i^y,\sigma_i^z$ denote the Pauli operators supported on i. The Hamiltonian of the system—a local Hamiltonian—is then taken to be

$$H = -\sum_s A_s - \sum_p B_p.$$

This is a gapped and frustration-free Hamiltonian. It is also straightforward to verify that for any closed path g the operator $\Pi_{j\in g}\sigma_j^z$ commutes with all operators in Eq. (34). The ground-state manifold depends on the topology of the lattice and is in the chosen case fourfold degenerate. The topological entanglement entropy, evaluated for this toric code state, gives $\gamma=\log(2)$. The ground states can readily be cast into a PEPS language, as done in Verstraete *et al.* (2006). An analysis of how topological order can be grasped in a language of entanglement renormalization or MERA has been performed in Aguado and Vidal (2008): Indeed, the topological degrees of freedom can then be distilled to the top of the tensor network.

An equally important explicit and closely related model is the loop model on a honeycomb lattice (Kitaev, 2003). Ground states of more general string-net lattice models can also often be expressed in terms of remarkably simple tensor networks (Buerschaper *et al.*, 2008; Gu *et al.*, 2008, 2009). Entanglement entropies of topological color codes (Bombin and Martin-Delgado, 2008) have been studied (Kargarian, 2008). Equivalents of the topological entanglement entropy for finite temperature—where the very robustness can be probed—have been considered and introduced (Castelnovo and Chamon, 2007; Iblisdir *et al.*, 2009): Notably, for Gibbs states it still makes sense to consider quantities of the type as in Eq. (33), only with the respective entropies replaced by mutual information grasping correlations instead of entanglement, as discussed in Sec. V. It is found that the interplay between thermal effects, topological order, and the size of the lattice indeed give rise to well-defined scaling relations.

The study of entanglement entropies in fractional quantum Hall states in a spherical geometry has been initiated by Haque *et al.* (2007) and Zozulya *et al.* (2007) considered Abelian-Laughlin states as well as Moore-Read states, where also rigorous upper bounds for particle entanglement entropies have been derived. Particle partitioning entanglement in itinerant many-particle systems has been studied (Zozulya *et al.*, 2008). The MPS representation of the Laughlin wave function has been derived (Iblisdir *et al.*, 2007). The tolopogical entanglement of integer quantum Hall states has been computed (Rodriguez and Sierra, 2009). Topological entanglement Renyi entropies have been considered in Flammia *et al.* (2009). Similar quantities in Chern-Simons theories—the best understood topological field theories—have been identified (Dong *et al.*, 2008). The suggestion that the full spectrum of H in $\rho_I=e^{-H}$ should be considered to detect topological order has been proposed (Li and Haldane, 2008). As being certified by this list of recent developments, studies of entanglement entropies as indicators of topological order are still under rapid development.

I. Relationship to black hole entropy

As mentioned one of the particularly intriguing motivations for the study of area laws of the entanglement entropy is the relationship to the area dependence of the black hole entropy. The Bekenstein-Hawking area law (Bardeen *et al.*, 1973; Bekenstein, 1973; Hawking, 1974) suggests that a black hole carries an entropy that is proportional to its horizon area $\mathcal{A}$,

$$S_{\mathrm{BH}} = \frac{kc^3\mathcal{A}}{4G\hbar}.$$

Hence, according to this relationship, the (thermodynamical) entropy of a black hole is just a quarter of its area measured in Planck units (Bousso, 2002), i.e., when $k=c=G=\hbar=1$. For the sum of this black hole entropy and the matter entropy S_{Matter} a second law of thermodynamics is proposed to hold. Such a generalized second law of thermodynamics led to the suggestion that one would have a "spherical entropy bound" for matter: In asymptotically flat space-time, any weakly gravitating matter system would satisfy $S_{\mathrm{Matter}}\leqslant 2\pi kEr/\hbar c$, interestingly not containing G. E denotes the total mass energy of the system, whereas r denotes the smallest radius of a sphere that contains the matter system at hand. The range in which one can expect the validity of such a law has been discussed by Bousso (2002).

The linear relationship between the boundary area and the (thermodynamical) entropy—formally, the two equations look identical—suggests that one may expect a close relationship between these area laws: On the one hand, for the (von Neumann) entanglement entropy of a subregion of a free quantum field in flat space-time, on

[17] Equivalently, one can place the physical spins on the edges and formulate the operators $\{A_s\}$ and $\{B_p\}$ as nontrivially supported on the respective four spins associated with vertices and plaquettes.

the other hand, for the black hole entropy. This intriguing connection was first suggested and explored by Bombelli *et al.* (1986) and Srednicki (1993) and extended by Callan and Wilczek (1994), Fiola *et al.* (1994), and Holzhey *et al.* (1994). Indeed, there are physical arguments that make the reduction in the situation of having a scalar field in a static spherically symmetric space-time to a scalar field in flat space-time plausible (Das *et al.*, 2008). The exact status of the relationship between these quantities (or to what extent they are related by originating from a common cause—the general locality of interactions) is still subject to debate.[18]

This relationship has been employed to take steps in computing the entanglement entropy in higher-dimensional conformal field theories: The correspondence relating a $(d+2)$-dimensional anti–de Sitter (AdS) space to a $(d+1)$-dimensional conformal field theory (CFT) (Witten, 1998; Aharony *et al.*, 2000) has been used to study the Bekenstein formula in the AdS context (Ryu and Takayanagi, 2006a, 2006b); see also Casini and Huerta (2007). In this way, the above formula is used as a tool to compute the geometric entropy in a plausible fashion in situations where the exact computation is not known to be possible using the tools of conformal field theory.

The holographic principle—dating back to 't Hooft (1985) and Susskind (1995)—goes even further, and suggests that generally all information that is contained in a volume of space can be represented by information that resides on the boundary of that region. For an extensive review, see Bousso (2002).

V. AREA LAWS FOR CLASSICAL SYSTEMS AND FOR TOTAL CORRELATIONS

A. Classical harmonic systems

Throughout we have been concerned with quantum systems on a lattice. What if we have classical systems on a lattice, could one still expect an area law to hold? Obviously, the concept of entanglement is no longer meaningful. Also, the Shannon entropy of, say, the marginal distribution of a distinguished region I would not quantify correlations in a reasonable fashion. What is worse, in the case of harmonic classical systems on a lattice, when thinking in terms of phase-space cells, this quantity is burdened with the usual Gibbs paradox. However, it does make perfect sense to talk about classical correlations in classical systems, the appropriate quantity grasping such correlations being the mutual information:

Given a probability distribution p on the lattice L, one can quantify the correlations between the marginals with respect to a distinguished region I and its complement O by means of the mutual information. This tells us how much information can be obtained on O from measurements in I, and equally on I by measurements in O. This quantity enjoys a number of very natural properties. The mutual information is always positive—there can be no negative correlations—and will vanish exactly if the probability distribution factorizes, in which case one cannot learn anything about O from I. Given the marginals p_I and p_O of the probability distribution p on I and O, respectively, the mutual information is defined as

$$I(I:O) = S(p_I) + S(p_O) - S(p), \tag{35}$$

where $S(p) = -\Sigma_j p_j \log_2(p_j)$ is the standard information theoretical Shannon entropy. It is noteworthy that the mutual information does not suffer from the Gibbs paradox as shown below. How does the mutual information scale with the size of a region in the case of a harmonic coupled classical system? The subsequent statement clarifies this situation. Consider a classical harmonic lattice system, with Hamiltonian

$$H = \frac{1}{2}\left(\sum_{j\in L} p_j^2 + \sum_{j,k\in L} x_j V_{j,k} x_k\right), \tag{36}$$

where now $x=(x_1,\ldots,x_N)$ and $p=(p_1,\ldots,p_N)$ are the vectors of classical position and momentum variables of classical oscillators arranged on a cubic lattice $L=\{1,\ldots,N\}^{\times D}$. The phase-space coordinates are then $\xi=(x,p)$. The matrix $V\in\mathbb{R}^{|L|\times|L|}$ with a finite-ranged interaction defines the interaction.

The state of the system is defined by the phase-space density, so a classical distribution $\rho:\mathbb{R}^{N^D}\to\mathbb{R}^+$. For any nonzero inverse temperature $\beta>0$, this phase-space distribution is nothing but

$$\rho_\beta(\xi) = \frac{1}{Z} e^{-\beta H(\xi)}, \quad Z = \int d\xi e^{-\beta H(\xi)}.$$

To define the mutual information, following the standard procedure, we split the phase space into cubic cells each with a volume h^{2N^D}, with $h>0$ some constant. From the phase-space density, we can then identify a discrete probability distribution, from an average of the phase-space density over these cells, $p_j=\int_{\text{Cell}} d\xi\rho(\xi)$ for $j\in L$. The discrete classical entropy is then defined as the Shannon entropy of this probability distribution as

$$S_C(h) = -\sum_{j\in L} p_j \log_2(p_j).$$

We now return to the situation of having a lattice system with an interior I and an exterior O. The respective discrete classical entropies are defined as $S_I(h)$ and $S_O(h)$. Obviously, the values of these entropies will depend on the choice of h, and in the limit $h\to 0$ they will diverge, logarithmically in h. This is a familiar observation in classical statistical physics, the divergence being resolved in the third law of thermodynamics. Here we are, however, interested in classical correlations, as being quantified in terms of the mutual information which in the

[18]For a review on this connection see Das *et al.* (2008), for a calculation of the one-loop correction to the Bekenstein-Hawking entropy in the presence of matter fields and its relationship to the geometric entropy see Susskind and Uglum (1994), and for an entanglement-based view of the Bekenstein-Hawking entropy law see Brustein *et al.* (2006) and Ryu and Takayanagi (2006a).

limit of $h \to 0$ is well defined. Hence we can define the classical mutual information of a harmonic lattice system as $I(I:O) = \lim_{h\to 0}[S_I(h) + S_O(h) - S_C(h)]$. We are now in the position to state the area theorem for classical harmonic systems (Cramer *et al.*, 2006).

Theorem 16 (Correlations in classical harmonic systems). Consider a harmonic lattice system with Hamiltonian as in Eq. (36) on a general lattice $G=(L,E)$. Then the classical mutual information $I(I:O)$ of the Gibbs state at some inverse temperature $\beta > 0$ satisfies an area law,

$$I(I:O) = O(s(I)).$$

The interesting aspect of this proof (Cramer *et al.*, 2006) is that it relates this question of the classical mutual information to a quantity that arises in the quantum case where the coupling matrix V_x is replaced by V_x^2, and is hence a simple corollary of earlier results on quantum systems, now with a coupling that is replaced by the squared coupling matrix. Hence, a "quantum proof" can be applied to establish a statement on classical lattice systems. The lesson to learn is that whenever one has local interactions—even in classical systems—one should not be too surprised if this manifests itself in an area law in the correlations.

B. Classical correlations quantum spin models

The situation is even simpler for finite-dimensional constituents. Indeed, in contrast to the overburdening technicalities that render the question of area laws in higher-dimensional quantum systems at zero temperature so difficult, the situation here can be clarified with hardly any mathematics at all: An elegant, but simple argument shows that total correlations in quantum (and classical) systems at nonzero temperatures always satisfy an area law. This is a statement on correlations—not entanglement, in contrast to the discussion of Sec. IV.D—in thermal states $\rho_\beta = e^{-\beta H}/Z$ for some $\beta > 0$ for classical or quantum systems (Wolf *et al.*, 2008). The relevant quantity grasping correlations is again the mutual information

$$I(I:O) = S(\rho_I) + S(\rho_O) - S(\rho), \tag{37}$$

where S stands either for the von Neumann quantum entropy or for the Shannon entropy of the probability distribution. The classical variant was first discussed in Cramer *et al.* (2006), and the quantum version in Casini and Huerta (2004, 2007). Casini and Huerta (2004) introduced this quantity to avoid divergencies of the entanglement entropy in quantum field theory: In a similar fashion as above, regulators will in fact cancel each other, and the familiar ultraviolet divergence in the quantum field limit disappears.

Interestingly, a general statement on the scaling of correlations at nonzero temperature in terms of Eq. (37) can be derived which holds for any spin model with local dimension d [see Bratteli and Robinson (1979) and also Wolf *et al.* (2008)].

Theorem 17 (Classical correlations at nonzero temperature). Consider a classical or a quantum system with finite local dimension d defined on a translation-invariant lattice $G=(L,E)$. Consider the Gibbs state at some inverse temperature $\beta > 0$ of a local Hamiltonian H with two-site interactions. In the classical case, where each of the lattice sites corresponds to a spin with configuration space $\mathbb{Z}_d$,

$$I(I:O) \leqslant |s(I)| \log_2(d). \tag{38}$$

For a quantum system with local Hilbert spaces $\mathbb{C}^d$, the mutual information satisfies the area law

$$I(I:O) \leqslant \beta \|h\| \, |s(I)|, \tag{39}$$

where $\|h\|$ is the largest eigenvalue of all Hamiltonians across the boundary of I and O.

This statement is valid in remarkable generality, given the simplicity of the argument. We focus on quantum systems in the following. One can write the Hamiltonian H having two-site interactions as $H = H_I + H_\partial + H_O$, where H_I and H_O collect all interaction terms within the regions, whereas H_∂ stands for terms connecting the two regions. The Gibbs state ρ_β for some inverse temperature $\beta > 0$ minimizes the free energy $F(\rho) = \mathrm{tr}[H\rho] - S(\rho)/\beta$. Therefore,

$$F(\rho_\beta) \leqslant F(\rho_I \otimes \rho_O),$$

from which $I(I:O) \leqslant \beta \, \mathrm{tr}[H_\partial(\rho_I \otimes \rho_O - \rho_\beta)]$ is obtained. As the right-hand side depends only on terms coupling the inside to the outside, i.e., surface terms, Eq. (39) follows straightforwardly. A naive limit $\beta \to \infty$ will not yield an area law for zero temperature, as the right-hand side of Eq. (39) then clearly diverges, but for any finite temperature one obtains a bound.

VI. CONNECTION TO SIMULATABILITY

There is an intimate connection between area laws for the entanglement entropy and questions of the simulatability of quantum many-body systems. The fact that there is "little entanglement" in a system that satisfies an area law is at the core of the functioning of powerful numerical techniques such as the density-matrix renormalization group (DMRG) methods. To describe the large research field of numerical simulation using DMRG-type methods is beyond the scope of the present review. Instead, we concentrate on the direct relationship between the "effective degrees of freedom" that must be considered when classically describing quantum systems.

A. Numerical simulations with the density-matrix renormalization group method

This connection is particularly clear in one-dimensional systems, that is, for quantum spin chains.

Indeed, one can say that the fact that ground states of gapped systems satisfy an area law—and to a lesser extent that critical systems merely have a logarithmic divergence of the entanglement entropy—is responsible for the success of the density-matrix renormalization approach. Matrix-product states also satisfy a one-dimensional area law. As MPS are underlying the DMRG approach this suggests that the entanglement content of a state and the best possible performance of a DMRG approach can be intimately linked.

Historically, DMRG was born out of an idea of renormalization, where one iteratively identifies the relevant degrees of freedom, grasping the essential physics of the problem, when going from one step of the procedure to the next one. This general idea goes back to the real-space renormalization group approach, presented by Wilson (1975) in the mid-1970s. This approach was particularly successful in the numerical assessment of the Kondo problem, whereas for other problems results were not quite what was hoped for. The birth of the DMRG approach as such was related to a clear analysis of the strengths and weaknesses of the real-space renormalization group approach to study the low-energy properties of quantum many-body systems (White and Noack, 1992). White (1992) was the one in which the DMRG method was introduced. Since then, this method has seen a standard method in the numerical study of strongly correlated quantum many-body systems. For a recent review, see Schollwöck (2005).

Initially, the formulation of DMRG was based on the above renormalization idea. However, in the following years it became clear that DMRG generates matrix-product states, an insight reported in Östlund and Rommer (1995) for the thermodynamical limit of DMRG, and in Dukelsky *et al.* (1998) for finite-size DMRG methods with the latter placing a particular emphasis on exploiting a rotational symmetry in variational approaches. Peschel *et al.* (1999) gave an early overview over variational ansatz with matrix-product states and the relationship with the DMRG idea. Östlund and Rommer (1995) hinted at the possibility for treating period boundary conditions in the MPS picture but chose translation invariant matrices. Verstraete *et al.* (2004) relaxed this constraint to demonstrate that a suitable formulation significantly outperforms standard DMRG for periodic boundary conditions in terms of memory requirements.

Hence, DMRG—in its several variants—can be seen as a variational method, where the optimization problem

$$\begin{aligned} &\text{minimize} \quad \langle\psi|H|\psi\rangle, \\ &\text{subject to} \quad |\psi\rangle \in (\mathbb{C}^d)^{\otimes N}, \end{aligned} \tag{40}$$

impractical already because of its exponentially large feasible set, is replaced by a variant of an optimization problem over a polynomially large set

$$\begin{aligned} &\text{minimize} \quad \langle\psi|H|\psi\rangle, \\ &\text{subject to} \\ &\quad |\psi\rangle \in (\mathbb{C}^d)^{\otimes N} \text{ is an MPS vector of dimension } D. \end{aligned} \tag{41}$$

In this variant, or more accurately in each of these variants, one does not attempt in one go to identify the global optimum, but rather effectively iteratively solves for the local matrices involved. Such an iteration will then certainly converge (albeit, strictly speaking, not necessarily to the global minimum).[19]

B. Approximation of states with matrix-product states

Any such method can then only be as good as the best possible MPS can approximate the true ground state at hand. This, in fact, is related to the entanglement content, in that it matters whether or not the true ground state satisfies an area law or not. In light of previous discussions, this connection is not that surprising anymore: after all MPS satisfy an area law for the entanglement entropy. Hence, one aims at approximating ground states with states that have in this sense little entanglement, and those states can be well approximated by MPS that satisfy an area law in the first place.

This connection has been hinted at already in the first work on DMRG (White, 1992), where the spectrum of the half-chain has been considered and put into relationship with the "truncation error" in DMRG. This is a key figure of merit of the quality of an approximation in a step, so unity minus the weight of those terms being kept in a step of the iteration.

This connection between the decay of spectral values of half-chains, the more rapid the decay the better can DMRG perform, has been made more precise and fleshed out in Peschel *et al.* (1999). Latorre *et al.* (2004) emphasized the relationship to criticality in this context. Riera and Latorre (2006) presented a short review on this question. In more recent quantitative approaches, the optimal approximation that can possibly be obtained by a MPS of a given D is considered. Denote with $\mathcal{H}_N = (\mathbb{C}^d)^{\otimes N}$ the Hilbert space of a quantum chain of length N. MPS are considered as defined in Eq. (29) for open boundary conditions. Given a family $\{|\psi_N\rangle\}_N$ of state vectors, it is said that it can be approximated efficiently by MPS if for every $\delta > 0$ there exists a sequence $|\psi_{N,D(N)}\rangle$ of MPS with $D(N) = O((\text{poly}_\delta(N))$ such that

$$\| |\psi_N\rangle\langle\psi_N| - |\psi_{N,D(N)}\rangle\langle\psi_{N,D(N)}| \|_1 \leqslant \delta,$$

where $\|\cdot\|_1$ denotes the usual trace norm. In contrast, it is said that this sequence cannot be approximated efficiently by MPS if there exists some $\delta > 0$ such that no sequence of MPS with $D(N)$ growing polynomially can approximate $|\psi\rangle\langle\psi|$ up to a small error δ in trace norm (Schuch *et al.*, 2008b).

[19] For mixed state simulations, see Verstraete *et al.* (2004) and Datta and Vidal (2007).

Theorem 18 (Approximatability with MPS). Consider sequences of state vectors $\{|\psi_N\rangle\}_N \in \mathcal{H}_N$ of a quantum chain of length N, and denote as before the reduced state of a block $I=\{1,\ldots,n\}$ of length n with ρ_I. If the sequence of ρ_I satisfies an area law for a Renyi entropy S_α for $\alpha<1$,

$$S_\alpha(\rho_I) = O(1),$$

then the sequence $\{|\psi_N\rangle\}_N$ is efficiently approximable by MPS. In contrast, if the von Neumann entropy $S_1(\rho_I) = \Omega(n)$, so grows at least linearly with the block size, then it cannot be approximated efficiently by MPS. This means that states satisfying a volume law cannot be approximated. The same holds true if any Renyi entropy S_α for some $\alpha>1$ grows at least as $S_\alpha(\rho_I)=\Omega(n^\kappa)$ for some $\kappa<1$. Otherwise, the connection is undetermined, in that examples for both approximable and inapproximable states can be found.

This statement clarifies the connection between the entanglement content and the possibility of describing states with matrix-product states. The validity of an area law implies that there is sufficiently little entanglement in the state such that an economical description in terms of matrix-product states is possible. The success of DMRG is related to the fact that gapped systems satisfy an area law. Even if the system is critical, the logarithmic divergence still allows for a relatively economical description in terms of matrix-product states. The fact that Renyi entropies for α smaller than or larger than unity feature here may be seen rather as a technical detail. The general message is clear: The arealike entanglement scaling, with or without small corrections, allows for an efficient approximation in D for matrix-product states.

To reiterate the point made in Sec. IV.G: Quenched, nonequilibrium systems can indeed fall exactly into the category of having an effectively linearly growing block entropy, so are characterized by a volume law for the entanglement entropy. More precisely, we face the interesting situation that while for each time we have an area law in n, the constant in the upper bound grows in time such that for a suitable choice for the sub-block, one arrives effectively at a volume law, as made precise in Theorem 15. This has severe practical implications: For small times, t-DMRG (Daley *et al.*, 2004; Vidal, 2004; White and Feiguin, 2004; Schollwöck, 2005; Kollath *et al.*, 2007; Cramer *et al.*, 2008) the time-dependent version of DMRG, can accurately keep track of the dynamics of the system. This is a variant in which one essentially makes a Lie-Trotter approximation of the time evolution operator, and then approximates in each time step the resulting state vector by an MPS, following Vidal (2004). The functioning of this algorithm can essentially be traced back to the observation that an arbitrarily good approximation to the propagator can be established with polynomial computational resources in the system size (Osborne, 2006). In time, however, one will eventually encounter typically an exponential increase in the number of degrees of freedom to be kept in order to faithfully describe the state. This eventually limits the time up to which one can numerically simulate time evolution using a variant of DMRG. The increase in the entanglement content also eventually limits classical simulations of quantum adiabatic algorithms based on MPS, which nevertheless perform often impressively well [for a numerical analysis, see Banuls *et al.* (2006)]. It is interesting to note, however, that this complexity does not necessarily translate in the difficulty of following the time evolution of specific observables when evolving them in the Heisenberg picture using t-DMRG. Then, in some cases the Heisenberg time evolution can be carried out exactly for finite bond dimension and arbitrary long times (Prosen and Pizorn, 2007; Znidaric *et al.*, 2008; Clark *et al.*, 2009; Hartmann *et al.*, 2009).

There are numerical simulation methods that allow for the simulation of certain quantum states that do not satisfy an area law. MERA already allows for a logarithmic divergence of the entanglement entropy in one-dimensional systems. Weighted graph-state-based approaches (Anders *et al.*, 2006) and its 1D variant, the renormalization algorithm with graph enhancement (Hübener *et al.*, 2009), can cope with instances of volume laws for the entanglement entropy, the latter in one dimension, and the former in arbitrary spatial dimensions. Early work on the simulation of a particular kind of discrete time evolution, namely, the application of random unitary circuits, suggests that this may be a promising approach for the "efficient simulation of quantum many-body systems beyond area laws."

We end with a note from the computer science rather than from the physics perspective: The fact that a true ground state is, strictly speaking, well approximated by an MPS does not necessarily mean that DMRG will also efficiently find this best approximation. In practice, DMRG works well, and it typically produces good and reasonable results. It is remarkable how well this approximation is found in the iterative scheme as being pursued by any DMRG algorithm: After all, the full problem (41) is a nonconvex polynomial global optimization problem of very high order ($\langle\psi|H|\psi\rangle$ is of degree N^2 in D). Still, by local variations and sweeping one achieves good results. The ultimate reason for this impressive performance is yet to be ultimately understood.

Having said that, the worst case complexity of the problem of finding the best approximation can be computationally difficult in the sense of computer science. In fact, the class of problem of keeping some matrices fixed and varying over a finite subset has in worst case instances that are NP hard (Eisert, 2006). In nontranslation invariant settings, one even finds that if one could efficiently identify the best possible MPS approximation, one could efficiently solve NP-hard problems (Schuch *et al.*, 2008). Even more strongly put, the problem of approximating the ground-state energy of a system composed of a chain of quantum systems is QMA complete (Aharonov *et al.*, 2009).

This should be seen as a warning sign: The functioning of variational algorithms such as DMRG is essentially based on heuristics, and in worst case one can encounter hard problems. The energy landscape is then so rugged

that one gets stuck in local optima. Still, while it is important to acknowledge that DMRG is, strictly speaking, not certifiable, it is still true that it works very well in practice and is one of the pillars of the numerical assessment of strongly correlated systems in one dimension.

C. Implications on higher-dimensional simulations

For higher-dimensional systems, tensor-product states or PEPS, as well as those of MERA, satisfy area laws, as discussed in Sec. IV.F. This fact suggests that when minimizing $\langle\psi|H|\psi\rangle$ for an $N\times N$ lattice subject to $|\psi\rangle \in (\mathbb{C}^d)^{\otimes N^2}$ being a PEPS or MERA described by polynomially many real parameters, one encounters a good approximation whenever the system at hand already satisfies an area law. In light of the fact that even critical two-dimensional systems can satisfy an area law, this would mean that they can be well described by PEPS or MERA with relatively few parameters. Numerical work in case of PEPS indicates that this is indeed the case (Verstraete and Cirac, 2004; Isacsson and Syljuasen, 2006; Murg *et al.*, 2007; Verstraete *et al.*, 2008).

A rigorous result similar to Theorem 18, yet, is still lacking for PEPS or MERA. The intuition developed so far, however, is in one way or the other quite certainly right: Whenever an area law is satisfied, PEPS with small bond dimension should give rise to a reasonably good approximation. Here subtle aspects are rather connected to the fact that the exact contraction of the tensor networks of PEPS, and hence the computation of expectation values, is inefficient, and that approximate contractions have to be employed. Suitable subsets, such as the class of string states, can always be efficiently contracted, giving rise to variational sets in higher-dimensional systems (Schuch *et al.*, 2008). The Osborne (2007a) method also gives rise to certifiable approximations of 2D ground states for a class of models, exploiting quasiadiabatic evolutions.

As before, one has to distinguish the variational set from the computational method of varying over this set. Usually, one has to find practical and heuristically suitable methods of solving a global optimization problem over many variables. Several strategies may be followed when varying over suitable sets to simulate higher-dimensional strongly correlated systems: One may use local variations such as in DMRG, imaginary time evolution, or flow methods (Dawson *et al.*, 2008), making use of gradient flow and optimal control ideas to vary over the manifold of unitary gates that describe the variational set of states at hand. For MERA, the same intuition should hold true. Here the approaches implemented so far are focused on one-dimensional systems (Evenbly and Vidal, 2007; Dawson *et al.*, 2008; Rizzi *et al.*, 2008), but the ideas are also applicable in higher dimensions. It would be an interesting approach to systematically explore the performance of the simulation of higher-dimensional strongly correlated systems using a MERA approach.

VII. PERSPECTIVES

In this Colloquium, we presented the state of affairs in the study of area laws for entanglement entropies. As pointed out, this research field is presently enjoying much attention for a number of reasons and motivations. Yet, needless to say, there are numerous open questions that are to be studied, of which we mention a few to highlight further perspectives:

- Can one prove that gapped higher-dimensional general local lattice models always satisfy an area law?
- In higher-dimensional systems, critical systems can both satisfy and violate an area law. What are further conditions to ensure that critical systems satisfy an area law? What is the exact role of the Fermi surface in the study of area laws in fermionic critical models?
- Can one compute scaling laws for the mutual information for quasifree systems?
- For what 1D models beyond quasifree and conformal settings can one find rigorous expressions for the entanglement entropy?
- Under what precise conditions do quenched disordered local models lead to having "less entanglement"?
- What are the further perspectives of using conformal methods for systems with more than one spatial dimension?
- Can the link between the Bekenstein formula in the AdS context and the scaling of geometric entropies in conformal field theories be sharpened?
- To what extent is having a positive topological entropy and encountering topological order one to one?
- How can the relationship between satisfying an area law and the efficient approximation of ground states with PEPS be rigorously established?
- What efficiently describable states satisfy an area law, such that one can still efficiently compute local properties?
- Are there further instances for 1D systems satisfying an area law that allow for certifiable approximations in terms of matrix-product states?

These questions only touch upon the various perspectives that open up in this context. The quantitative study of a research area that could be called "Hamiltonian complexity"[20] is just beginning to emerge. The puzzle of how complex quantum many-body systems are, and how many effective degrees of freedom are exploited by nature, is still one of the intriguing topics in the study of interacting quantum systems.

[20]This term has been coined by B. M. Terhal.

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APPENDIX: FISHER-HARTWIG THEOREM

In this appendix we present an important technical result concerning the asymptotic behavior of Toeplitz matrices (Böttcher and Silbermann, 2006).

Lemma 1 (Fisher-Hartwig). Consider a sequence of $n\times n$ Toeplitz matrices $\{T_n\}_n$ with entries $(T_n)_{i,j}=(T_n)_{i-j}$,

$$(T_n)_l=\frac{1}{2\pi}\int_0^{2\pi} d\phi\, g(\phi)e^{-il\phi},$$

generated by $g:[0,2\pi)\to\mathbb{C}$. Let g be of the form

$$g(\phi)=b(\phi)\prod_{r=1}^{R} t_{\beta_r}(\phi-\phi_r)u_{\alpha_r}(\phi-\phi_r),$$

with $t_\beta(\phi)=e^{-i\beta(\pi-\phi)}$, $u_\alpha=[2-2\cos(\phi)]^\alpha$, $\mathrm{Re}(\alpha)>-1/2$, and $b:[0,2\pi)\to\mathbb{C}$ a smooth nonvanishing function with winding number zero. Then (Basor, 1978; Böttcher and Silbermann, 1985; Libby, 1990), for $|\mathrm{Re}(\alpha_r)|<1/2$ and $|\mathrm{Re}(\beta_r)|<1/2$ or $R=1$, $\alpha=0$ $|\mathrm{Re}(\beta)|<5/2$, the asymptotic behavior of the determinant of T_n is given by

$$\lim_{n\to\infty}\frac{\det(T_n)}{EG^n n^{\Sigma_{r=1}^R(\alpha_r^2-\beta_r^2)}}=1,$$

where $E=O(1)$ in n and

$$G=\exp\left(\frac{1}{2\pi}\int_0^{2\pi} d\phi\,\log_2[b(\phi)]\right).$$

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