Conifolds and geometric transitions

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(Published 21 October 2008)

Conifold geometries have recieved much attention in string theory and string-inspired cosmology recently, in particular the Klebanov-Strassler background that is known as the "warped throat." This paper provides a pedagogical explanation for the singularity resolution in this geometry and emphasizes its connection to geometric transitions. The first part focuses on the gauge theory dual to the Klebanov-Strassler background, including the T-dual intersecting branes description. Then, a connection to the Gopakumar-Vafa conjecture for open-closed string duality is presented and a series of papers verifying this model on the supergravity level is summarized. An appendix provides extensive background material about conifold geometries. Special attention is given to their complex structures and the supersymmetry conditions on the background flux in constructions with fractional D3-branes on the singular (Klebanov-Tseytlin) and resolved (Pando Zayas–Tseytlin) conifolds are reevaluated. In agreement with earlier results, it is shown that only the singular solution allows a supersymmetric flux. However, the importance of using the correct complex structure to reach this conclusion is emphasized.

DOI: 10.1103/RevModPhys.80.1419 PACS number(s): 11.25.Tq

CONTENTS

I. INTRODUCTION

The geometric transition between conifold geometries is an example of a string theory duality between compactifications on different geometrical backgrounds. Initial arguments came from two different angles: a generalization of AdS/CFT (the correspondence between string theory on anti–de Sitter space and a conformal field theory on the boundary of that space) via the Klebanov and Strassler (2000) model and independently as a duality between open and closed topological strings by Gopakumar and Vafa (1999). A common embedding has since been found in IIB, IIA, and M-theory (Atiyah *et al.*, 2001; Cachazo *et al.*, 2001; Dasgupta *et al.*, 2001; Vafa, 2001; Dasgupta, Oh, *et al.*, 2002) and the geometric transition has been confirmed on the supergravity level (Becker *et al.*, 2004, 2006; Alexander *et al.*, 2005; Dasgupta *et al.*, 2006; Knauf, 2007). This paper gives a comprehensive overview of the different ideas underlying geometric transitions and reviews the lengthy supergravity calculations of the latter references.

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The model of Klebanov and Strassler is based on a series of papers (Gubser and Klebanov, 1998; Klebanov and Witten, 1998; Klebanov and Nekrasov, 2000; Klebanov and Tseytlin, 2000) generalizing the AdS/CFT correspondence (Gubser *et al.*, 1998; Maldacena, 1998; Witten, 1998). Instead of the $\mathcal{N}=4$ superconformal field theory one obtains from considering $AdS_5\times S^5$, theories with less supersymmetry can be found by taking $AdS_5 \times M^5$, where M^5 is some five-dimensional manifold. One can break conformal invariance by introducing fractional D3-branes instead of (only) D3-branes. These are objects that wrap compact cycles in the internal manifold and therefore appear effectively three-dimensional. Once conformal invariance is broken, the gauge theory exhibits a running coupling. The coupling constant is related to the NS-NS (Neveu-Schwarz) B field in the string theory dual. One approaches the far IR limit of the gauge theory as the radial coordinate in the supergravity dual approaches zero. The manifold M_5 considered in this model is the base of a conifold, so there is a singularity at $r=0$. This does not mean that the far IR limit of the gauge theory is not well defined. On the contrary, knowledge of the strong-coupling behavior of the dual super-Yang-Mills (SYM) theory led Klebanov and Strassler to the following conclusion: since SYM exhibits gaugino condensation and chiral symmetry breaking [which breaks the U(1) symmetry down to $\mathbb{Z}_2$] in the far IR, the dual string theory background has to be modified for $r\to 0$ in order to reflect this symmetry property. The singularity is smoothed out, giving a manifold which looks like the conifold at large radial distances, but approaches a finite three-sphere at the tip of the cone. This manifold is called the "deformed conifold" and has precisely the required symmetry property, i.e., it is only invariant under $\mathbb{Z}_2$, where the (singular) conifold was invariant under a full U(1). We summarize this in Fig. 1.

Gopakumar and Vafa (1999) also considered conifold geometries, but they were interested in topological string amplitudes. They showed that the open A model on the deformed conifold (with a blown-up S^3) agrees with the closed A model on the resolved conifold (with a blown up S^2) on the level of topological string partition functions. One has to identify the correct parameters from each theory: roughly speaking, the size of the three-cycle in the deformed geometry (its complex structure modulus) is identified with the size of the two-cycle in the resolved geometry (its Kähler modulus). Via mirror symmetry, the same can be said for the B model, but here the roles of deformed and resolved conifold are exchanged. The connection to the KS model becomes apparent if one embeds this B model into IIB superstring theory, as done by Vafa (2001). Before the geometric transition, D5-branes wrap the nonvanishing two-cycle in the resolved conifold and appear as fractional D3-branes that carry an $\mathcal{N}=1$ SU(N) SYM.

This seems to be precisely the picture one finds at the "bottom of the duality cascade" in the KS model: for the model with N D3 *and M* fractional D3-branes the gauge group is SU(N)$\times$SU($N+M$) and the RG flow is such

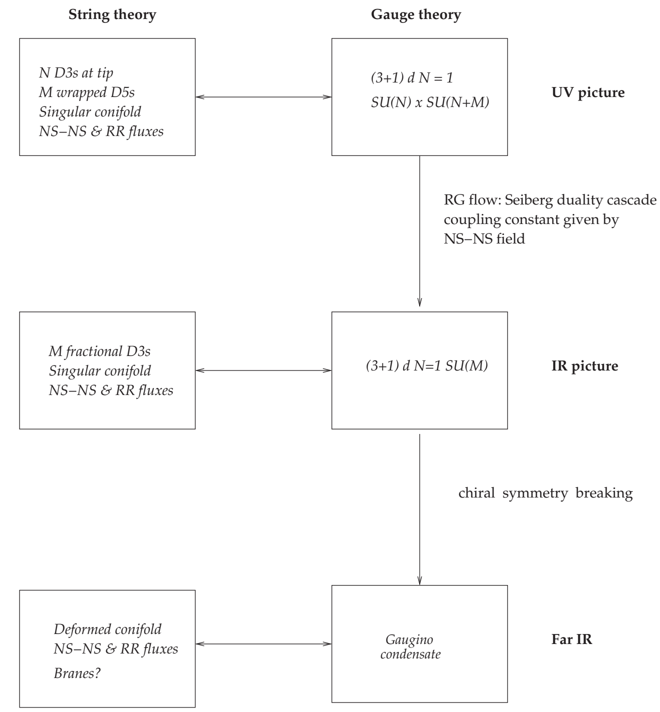


FIG. 1. The Klebanov-Strassler model.

that one of the theories flows towards strong and the other towards weak coupling. This translates via Seiberg duality to an SU($N-M$)$\times$SU(N) theory, where the two gauge group factors now exhibit the opposite running coupling behavior. Ergo, one can follow a "cascade" of such Seiberg dualities, where in each step the gauge group factor drops by M. If N is a multiple of M (we discuss the more general case in Sec. II.A.2) all the regular D3-branes will "cascade away" and the gauge group becomes SU(M), like in the Vafa setup. The only difference is that Vafa considers a resolved conifold, whereas KS started with the singular version. Nevertheless, the picture they both find in the IR is very similar: Vafa also argues that in the large M limit the string theory background is given by a deformed conifold. He gives a more concise description of this picture: since the topological string argument was based on an open-closed duality, there is no equivalent for the D-branes in the dual closed theory. The geometric transition conjectured by Vafa is therefore a duality between a background with D-branes (on which the gauge degrees of freedom propagate) and a background with only fluxes (where a geometrical parameter enters into the flux-generated superpotential to give the correct confining IR behavior). We therefore conclude that the D-branes in the KS model should also disappear once the singular conifold is traded against its deformation; see Fig. 2 for comparison.

All the transitions discussed so far take either the singular or resolved conifold to the deformed conifold by blowing up a nontrivial two-cycle. Thus, what we have really been discussing is the "conifold transition." As all conifold geometries are cones over $S^2\times S^3$, this can be depicted as in Fig. 3.

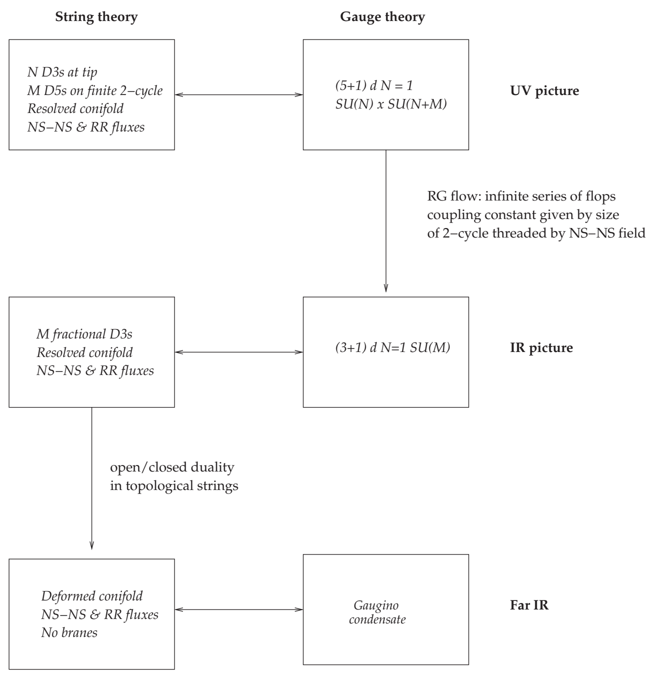


FIG. 2. Vafa's model.

Generalizations could be imagined for other manifolds that allow for nontrivial two- and three-cycles. In fact, when trying to confirm Vafa's picture on the supergravity level, we immediately encountered generalizations of the conifold (Becker *et al.*, 2004). These come about because under T-duality the original geometry attains a twisting of its fibration structure by the B field. The resulting manifold is therefore non-Kähler and we review its construction in Sec. III.C. The B field is a key ingredient in the Vafa and KS models, as its radial dependence is what gives rise to the running coupling (Klebanov and Nekrasov, 2000). It cannot be avoided when introducing fractional D3-branes. Therefore either the IIA or IIB embedding of the geometric transition will have such a B field and its mirror (or T-dual) will be non-Kähler.

The outline of this paper is as follows. In Sec. II we review the Klebanov-Strassler and Gopakumar-Vafa models. Section II.A explains the gauge theory duals of regular and fractional $D3$-branes, as well as the duality cascade and the singularity resolution via chiral symmetry breaking. The discussion of Gopakumar and Vafa's model, Sec. II.B, starts with a short review of topological string theory and states their conjecture. References are provided for detailed calculations, as they alone would fill an entire paper. We do, however, discuss the embedding of the open-closed duality into superstring theory and review the derivation of the flux-generated superpotential in Sec. II.C. The IIB and IIA pictures can be connected to an M-theory background in which the geometric transition manifests itself as a flop; this is called Vafa's duality chain in Sec. II.C.2. Both IIB models, the KS and the Vafa model, have an intuitive description in T-dual IIA theory, where the conifold background turns into a pair of NS5-branes. This picture is useful for observing the gauge theory construction and the unification of the geometric transition with the cascading solution in M-theory (Dasgupta *et al.*, 2001; Dasgupta, Oh, and Tatar 2002; Dasgupta, Oh, *et al.*, 2002). We review the arguments in Sec. II.D before we summarize the supergravity analysis of the duality chain in Sec. III. We walk the reader through the main steps, as there appear some nontrivial issues along the way, but we try to be as nontechnical as possible.

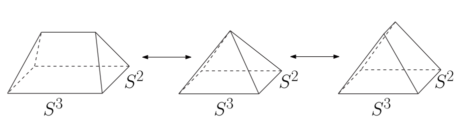


FIG. 3. The conifold transition. All three geometries share the base $S^2 \times S^3$, but the S^3 of the deformed conifold (left) remains finite. In the conifold transition it can be shrunk to zero size to give the singular conifold (center), from which blowing up the S^2 gives the resolved conifold (right).

An extensive appendix summarizes known facts about conifold geometries. In Appendix A.4 we give the choice of complex structures that make all three conifold metrics Ricci flat and Kähler and we use this in Appendix A.5 to evaluate the supersymmetry requirements for known supergravity solutions for fractional D3-branes on conifold geometries. We agree with earlier results of Cvetic *et al.* (2003) that the KS model on the singular conifold preserves supersymmetry, whereas the Pando Zayas and Tseytlin (2000) solution for D5-branes wrapped on the resolution of the conifold does not.

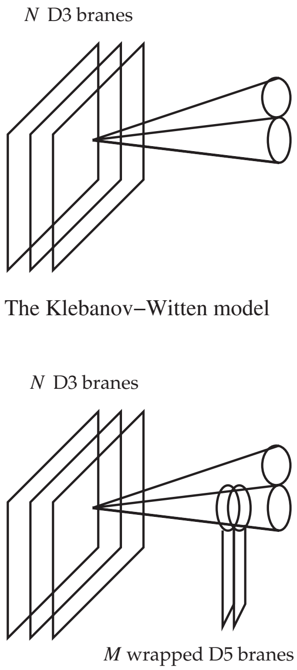
II. EVIDENCE FOR GEOMETRIC TRANSITIONS

A. Gauge theory argument from Klebanov-Strassler

In the Klebanov-Strassler model (Klebanov and Strassler, 2000) a configuration of N D3-branes and M fractional D3-branes on a singular conifold geometry is considered. The D3-branes sit at the singular point of the conifold, while the fractional branes arise from wrapping M D5-branes on the vanishing two-cycle of the conifold. The gauge theory on the branes is nonconformal, and in the IR is given by an SU(M) theory which exhibits chiral symmetry breaking and gaugino condensation, suggesting that the correct dual of the gauge theory in the IR limit is a deformed conifold. In this section we review the argument for this duality from the Klebanov-Strassler model. We begin by constructing the gauge theory of the Klebanov-Witten model in which no fractional branes are present and the gauge theory is conformal, and then proceed to the nonconformal case corresponding to the presence of wrapped D5-branes.

1. The Klebanov-Witten model

First consider the Klebanov-Witten model (Klebanov and Witten, 1998) in which a stack of D3-branes is placed at the tip of a conifold (see Fig. 4). As in the original scenario of the AdS/CFT conjecture, we expect a duality between the gauge theory on the branes and



(a) The Klebanov–Witten model

(b) The Klebanov–Strassler model

FIG. 4. Comparing the Klebanov-Witten and Klebanov-Strassler setups.

the gravity theory, found by taking the near-horizon limit. The near-horizon geometry in this case is $AdS_5 \times T^{1,1}$ where $T^{1,1}$ is the base of the conifold, a Sasaki-Einstein manifold. The reader not familiar with conifold geometries should consult the Appendix for more details. It is the coset space $[\mathrm{SU}(2)\times \mathrm{SU}(2)]/\mathrm{U}(1)$ and has topology $S^2 \times S^3$ (as shown in Fig. 3). The metric of $T^{1,1}$ was found by Candelas and de la Ossa (1990) and is given by

$$d\Sigma^2_{T^{1,1}} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{2.1}$$

so that the metric on the (singular) conifold is

$$ds^2 = dr^2 + r^2 d\Sigma^2_{T^{1,1}}. \tag{2.2}$$

The conifold is a noncompact Calabi-Yau manifold. Note that although the conifold is singular, the supergravity solution for the configuration of N D3-branes at the tip is given by

$$ds^2 = H^{-1/2}(r) ds^2_{0123} + H^{1/2}(r)(dr^2 + r^2 d\Sigma^2_{T^{1,1}}) \tag{2.3}$$

and is nonsingular everywhere since $H(r) = 1 + L^4/r^4$, with $L^4 = 4\pi g_S N(\alpha')^2$.

We would like to study the gauge theory on the D3-branes. To do this we will use the symmetries of the conifold to find a convenient set of coordinates that can be promoted to fields. We will thus construct a gauge theory with exactly the conifold symmetries and find the correct gauge theory on the branes by adding, first, one D-brane and then generalizing to the case where there is a stack of N D3-branes on the conifold tip. We begin by rewriting the defining equation for the conifold (A2),

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \quad z_i \in \mathbb{C}^4 \tag{2.4}$$

as

$$\det Z_{ij} = 0, \tag{2.5}$$

where $Z_{ij} = (1/\sqrt{2})\Sigma_n \sigma^n_{ij} z_n$, with σ^n the Pauli matrices for $n=1,2,3$ and $\sigma^4 = \imath\mathbf{1}$. The defining equation for the conifold is now $\det Z = 0$, where we have a choice of coordinates[1]

$$Z = \begin{pmatrix} z_3 + \imath z_4 & z_1 - \imath z_2 \\ z_1 + \imath z_2 & -z_3 + \imath z_4 \end{pmatrix} = \begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{pmatrix}. \tag{2.6}$$

As explained in Appendix A.1, the conifold is invariant under $\mathrm{SO}(4) \approx \mathrm{SU}(2) \times \mathrm{SU}(2)$, where SO(4) acts on the z_i in Eq. (2.4) and the SU(2)s act on the i and j indices in Eq. (2.5), respectively. In addition, there is a U(1) symmetry under which all the z_i in Eq. (2.4) are rotated by the same phase. It is easy to see that the metric (2.1) possesses the same symmetries. Each SU(2) acts on one $\{\theta_i, \phi_i, \psi_i\}$ [where ψ in Eq. (2.1) is given by $\psi_1 - \psi_2$] while the U(1) symmetry corresponds to invariance under shifts in ψ. To find the gauge theory we will consider D3-branes on this space and study the low-energy field theory of the modes localized on the brane. The moduli space of vacua of this theory should be exactly N copies of the manifold—in this case the conifold—modulo the action of the permutation group (since the D3-branes are identical). The global symmetry group of the gauge theory on the D3-branes will therefore be $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

The coordinates A_i and B_j, $i,j = 1,2$ give a useful parametrization of the conifold. The A_i are rotated into each other under one SU(2) and the B_j transform under the other. As they stand, these coordinates represent eight real degrees of freedom. However, in describing the conifold using A_1, A_2, B_1, and B_2 we have invariance under

$$A_i \to \lambda A_i, \quad \lambda \in \mathbb{C},$$

$$B_j \to \frac{1}{\lambda} B_j. \tag{2.7}$$

We fix the magnitude and phase of λ separately. To fix the magnitude, we impose

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0. \tag{2.8}$$

This removes one degree of freedom.[2] To fix the phase, we have to divide by U(1), which means we make the identifications

[1]The main references for this section are Klebanov and Witten (1998); Klebanov and Strassler (2000); and Strassler (2005). A recent useful reference is Klebanov and Murugan (2007).

[2]It is then by further dividing by the scale invariance that one obtains the base, see Eq. (A4).

$$A_i \sim e^{\imath\alpha} A_i,$$

$$B_j \sim e^{-\imath\alpha} B_j, \tag{2.9}$$

leaving six degrees of freedom. In other words, no more conditions are required in order for these coordinates to give a complete description of the conifold. Thus, we have arrived at a description of the conifold given by the four coordinates A_i, B_j and the magnitude constraint (2.8), subject to the identifications (2.9). This description is equivalent to the equation of the conifold given by Eq. (2.5).

Next, we turn to the gauge theory. We can formulate the field theory in terms of A_i and B_i because they have the symmetries we want; we promote them to chiral superfields. Consider then an $\mathcal{N}=1$ supersymmetric theory with U(1) gauge group and chiral superfields $\tilde{A}_i, \tilde{B}_j$ with $i,j=1,2$, such that the $\tilde{A}_i$ have charge 1 and the $\tilde{B}_j$ have charge −1. The superpotential is given by

$$W = \lambda \det(\tilde{A}_i \tilde{B}_j) = \lambda(\tilde{A}_1\tilde{B}_1\tilde{A}_2\tilde{B}_2 - \tilde{A}_1\tilde{B}_2\tilde{A}_2\tilde{B}_1), \tag{2.10}$$

which preserves $SU(2)\times SU(2)\times U(1)_R$. The D-term condition for supersymmetry is given by $D=0$, where

$$D = -\tfrac{1}{2}\sum_i [q_{\tilde{A}_i}\tilde{A}_i^*\tilde{A}_i + q_{\tilde{B}_i}\tilde{B}_i^*\tilde{B}_i] + \xi, \tag{2.11}$$

with ξ the coefficient of the Fayet-Iliopoulos term and q_i the U(1) charge of the relevant field. When ξ is zero[3] this is exactly our conifold constraint (2.8). The moduli space of vacua is found by dividing by the gauge group U(1) which is equivalent to imposing Eq. (2.9). Now we claim that, for a single D3-brane on a conifold, the gauge theory whose moduli space exactly matches the conifold is in fact given by a gauge group $U(1)\times U(1)$, so the fields now have charges (1,−1) and (−1,1), respectively, under the two U(1) groups. The two D-term conditions [one for each U(1)] both yield Eq. (2.8). The superpotential (2.10) is unchanged and zero, so there are no F-term conditions

$$F_i = -\frac{\partial W}{\partial \tilde{A}_i} = 0,$$

and this theory's moduli space is the conifold.

When we place instead of a single brane a stack of N D3-branes on the conifold, the gauge group becomes[4] $SU(N)\times SU(N)$ so the superpotential no longer vanishes. The superpotential is given by

$$W = \lambda \operatorname{tr} \det(\tilde{A}_i \tilde{B}_j), \tag{2.12}$$

where the trace is now necessary because the superfields carry a gauge index for each $U(N)$ and should therefore be treated as matrices. The $\tilde{A}$ and $\tilde{B}$ fields are now bifundamental fields transforming in the $(N,\bar{N})$ and $(\bar{N},N)$ representations of the gauge groups, respectively. Together with the D-term conditions and the gauge invariance they give a description of the moduli space which matches the description of the conifold in terms of A and B arrived at above. The F-term equations

$$\tilde{B}_1\tilde{A}_i\tilde{B}_2 - \tilde{B}_2\tilde{A}_i\tilde{B}_1 = 0,$$

$$\tilde{A}_1\tilde{B}_j\tilde{A}_2 - \tilde{A}_2\tilde{B}_j\tilde{A}_1 = 0, \tag{2.13}$$

together with the D-term conditions (2.11), can be shown to have a solution if and only if $\tilde{A}$ and $\tilde{B}$ can be simultaneously diagonalized (Strassler, 2005). In this case the superpotential vanishes, giving exactly Eq. (2.5). The theory flows to a nontrivial infrared fixed point (Klebanov and Witten, 1998). We have seen how the symmetry group $SU(2)\times SU(2)\times U(1)$ acts on A_i, B_j. The U(1) global symmetry of the conifold manifests as what is called a $U(1)_R$ symmetry in the gauge theory, under which

$$(A_i, B_j) \to e^{R\alpha}(A_i, B_j). \tag{2.14}$$

The R-charges of the fields A_i and B_j are found by imposing conformal invariance[5] and are equal to 1/2. This leads to an R-charge of 2 for the superpotential, as required. The extra U(1) symmetry expressed in Eq. (2.9) is referred to in the gauge theory as baryonic symmetry.

2. The duality cascade

Having discussed the Klebanov-Witten model in detail, we are now in a position to study the Klebanov-Strassler model (Klebanov and Strassler, 2000). As shown in Fig. 4, the difference between the two setups is that in the configuration considered by Klebanov and Strassler there are M D5-branes wrapping the vanishing two-cycle of the conifold. These are effectively D3-branes with fractional charge, called fractional branes. Their effect is to render the dual gauge theory nonconformal, with many interesting consequences.

This setup was studied by Gubser and Klebanov (1998) and Klebanov and Nekrasov (2000), where the running of the gauge coupling was found. An exact solution, including backreactions, was given by Klebanov and Tseytlin (2000), but the details of the duality cascade, chiral symmetry breaking, and confinement were elucidated in Klebanov and Strassler (2000), which we follow closely here. A comprehensive review is given by Strassler (2005).

Consider first the gravity theory. The fractional branes are magnetic sources for the three-form field strength (F_3 is the Hodge dual of $F_7 = dC_6$)

[3] In fact $\xi \neq 0$ corresponds to a resolved conifold, as discussed by Klebanov and Murugan (2007).

[4] Actually it is $U(N)\times U(N)$ but the U(1) factors decouple in the IR.

[5] Setting the β functions equal to zero yields anomalous dimensions −1/2 for the fields, and one can make use of the relation $\dim \mathcal{O} = 1 + \frac{1}{2}\gamma_{\mathcal{O}} = 3R_{\mathcal{O}}/2$ for an operator $\mathcal{O}$ to solve for the R-charges.

$$F_3 = M\omega_3, \tag{2.15}$$

where

$$\omega_3 = \tfrac{1}{2}d\psi \wedge (\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2) + \tfrac{1}{2}d\phi_1 \wedge d\phi_2 \wedge (\cos\theta_1 \sin\theta_2 d\theta_2 + \sin\theta_1 \cos\theta_2 d\theta_1) \tag{2.16}$$

is the three-form dual to the nontrivial three-cycle of the conifold. In addition there are N units of five-form flux due to the D3-branes, but the total five-form RR (Ramond-Ramond) field strength also has a contribution from the NS-NS B-field, which is necessarily present in the supergravity solution:

$$\tilde{F}_5 = dC_4 + B_2 \wedge F_3, \tag{2.17}$$

$$B_2 = 3g_s M\omega_2 \ln\left(\frac{r}{r_0}\right). \tag{2.18}$$

Here $\omega_2 = \frac{1}{2}(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2)$ is the two-form on the two-cycle wrapped by the D5s and r_0 is some UV scale.

From the gauge theory point of view, the presence of the M fractional branes changes the gauge group[6] from $SU(N)\times SU(N)$ to $SU(N+M)\times SU(N)$. The field content is still given by four chiral superfields A_i and B_j, while the superpotential is given by Eq. (2.12). However, the relative gauge coupling depends on B_2 and therefore on r (Klebanov and Nekrasov, 2000; Klebanov and Tseytlin, 2000):

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim \frac{1}{g_s}\left[\int_{S^2} B_2 - \frac{1}{2}\right]. \tag{2.19}$$

According to the usual AdS/CFT dictionary, r maps to the renormalization-group (RG) scale in the dual gauge theory. Thus, this theory is no longer conformal.

The gauge couplings $1/g_1^2$ and $1/g_2^2$ flow in opposite directions. Facing a divergence in one we can continue by performing a Seiberg duality transformation (Seiberg, 1995), under which we obtain an $SU(N)\times SU(N-M)$ theory which resembles closely the theory we started with. We can see this by noting the running of the five-form flux under which the D3-branes are charged [Eq. (2.17)]. Since C_4 is oriented along the world-volume of the D3-branes in the 0123 directions, we can write the self-dual five-form

$$\tilde{F}_5 = \mathcal{F}_5 + \star\mathcal{F}_5$$

in terms of

$$\mathcal{F}_5 = N(r)\mathrm{vol}(T^{1,1}).$$

Here we used $\omega_2 \wedge \omega_3 \sim \mathrm{vol}(T^{1,1})$ and defined

$$N(r) = N + \frac{3}{2\pi}g_s M^2 \ln\left(\frac{r}{r_0}\right).$$

So the number of colors in the theory has become a scale-dependent quantity. $N(r)$ will decrease in units of M, i.e., the running gives rise to a flow under which $SU(N+M)\times SU(N) \to SU(N)\times SU(N-M)$. This process will continue until the gauge group is $SU(2M)\times SU(M)$ or $SU(M)$, corresponding to a situation in which only the M fractional D3-branes remain—the five-form flux has decreased to zero indicating that the N D3-branes have "cascaded away." The process is called a duality cascade, since the $SU(N)$ and $SU(N-M)$ theories are related by a Seiberg duality (Seiberg, 1995).

3. Chiral symmetry breaking and deformation of the conifold

When $N-kM$ approaches zero a more careful analysis is needed; the cascade must stop because negative N is physically nonsensical. At sufficiently small r the solution becomes singular. By studying the far IR of the gauge theory, Klebanov and Strassler argued that this singularity is removed by the IR dynamics, via a gluino condensate which breaks the anomaly-free $\mathbb{Z}_{2M}$ R-symmetry in the theory to $\mathbb{Z}_2$. The expectation value acquired by the gaugino condensate maps to the deformation parameter of the conifold: μ in Eq. (A15). Thus, the singularity is removed in the gravity picture by blowing up the S^3 of $T^{1,1}$.

Although the metric of the KS setup has a continuous $U(1)_R$ symmetry, the full supergravity solution is only invariant under a Z_{2M} subgroup of this, under which

$$(A_i, B_j) \to (A_i, B_j)\exp\left(\frac{2\pi \imath n}{4M}\right) \tag{2.20}$$

and the superpotential rotates by $e^{2\pi\imath n/M}$.

The theory has a moduli space with M independent branches in the IR. To see this, we probe the space with a single additional D3-brane. Since in the far IR the five-form field strength has cascaded to zero and only the M fractional D3-branes remain, the gauge group is $SU(M+1)\times SU(1)$ or $SU(M+1)$. The fields are A_i and B_j, $i,j=1,2$, in the $M+1$ and $\overline{M+1}$ representations, and the superpotential is of the form of Eq. (2.12). We can write this in terms of the gauge invariant $N_{ij}=A_iB_j$, which one can think of as a meson.

At low energy the theory is described by the Affleck-Dine-Seiberg superpotential (Affleck *et al.*, 1984)

$$W_L = \lambda N_{ij}N_{kl}\varepsilon^{ik}\varepsilon^{jl} + (M-1)\left[\frac{2\Lambda^{3M+1}}{N_{ij}N_{kl}\varepsilon^{ik}\varepsilon^{jl}}\right]^{1/(M-1)}$$

to which the only solutions for a supersymmetric vacuum are

$$(N_{ij}N_{kl}\varepsilon^{ik}\varepsilon^{jl})^M = \frac{2\Lambda^{3M+1}}{\lambda^{M-1}}. \tag{2.21}$$

In other words, the theory in the far IR has a moduli space with M independent branches. The $\mathbb{Z}_{2M}$ R-symmetry permutes these branches, rotating

[6]See Gubser and Klebanov (1998); Klebanov and Nekrasov (2000); and Klebanov and Tseytlin (2000). An easy way to understand this generalization is given in the brane construction discussion in Sec. II.D.

$N_{ij}N_{kl}\varepsilon^{ik}\varepsilon^{jl}$ by $e^{2\pi i M}$, so the R-symmetry is spontaneously broken from $\mathbb{Z}_{2M}$ to $\mathbb{Z}_2$. Thus, the gaugino condensate is responsible for the chiral symmetry breaking. To see how this symmetry breaking smooths out the singularity in the IR, see Sec. II.A.1 in which we promoted coordinates to chiral superfields. We now go the other way, using N_{ij} to find a coordinate description of the geometry in terms of the gauge theory fields. In the classical theory, $\det N_{ij}=0$, which should be compared to Eq. (2.5). In this case the probe brane is moving on the singular conifold. However, as we have seen in the Klebanov-Strassler model the field theory in the IR gives

$$\det N_{ij} = \left(\frac{\Lambda^{3M+1}}{(2\lambda)^{M-1}} \right)^{1/M}. \qquad (2.22)$$

Thus the probe brane moves not on the singular but on the deformed conifold, where the deformation is given by the gaugino condensate responsible for chiral symmetry breaking. It is the deformed conifold which gives the correct moduli for the field theory and which is the correct background geometry for the supergravity solution that is dual to the gauge theory in the far IR.

It was pointed out by Gubser *et al.*(2004) that the IR of the KS background should be thought of as the next-to-last step of the cascade. Studying the field theory dual to this $\mathrm{SU}(2M)\times\mathrm{SU}(M)$, one finds that baryonic operators, instead of mesonic ones as for mobile D3-branes, acquire a vacuum expectation value. This background is therefore said to lie on the baryonic branch.

B. Open-closed duality

A different perspective on geometric transitions comes from topological string theory. Gopakumar and Vafa (1999) conjectured a duality between an open and a closed topological string theory that live on different backgrounds. The connection to the Klebanov-Strassler model became apparent after Vafa (2001) embedded this duality into superstring theory. We only review the arguments and refer the reader to the original work (Bershadsky *et al.*, 1994; Gopakumar and Vafa, 1999; Vafa, 2001) or other reviews (Neitzke and Vafa, 2004; Auckly and Koshkin, 2007) for details. Here we define topological string amplitudes and explain the difference between open and closed topological string theories. The reader familiar with topological string theory may want to skip ahead to the Gopakumar-Vafa conjecture explained in Sec. II.B.2.

1. Topological sigma models and string theory

Throughout this section we restrict ourselves to the case $H_3=0$. We follow the review by Neitzke and Vafa (2004); see, e.g., Eguchi and Yang (1990); Witten (1988a, 1988b); and Marino (2005) for details.

String theory is intrinsically linked to sigma models. We can view string theory as the description of a two-dimensional world-sheet Σ propagating through a ten-dimensional target space M. The sigma model that describes this theory deals with maps $\phi:\Sigma\to M$. These maps can be promoted to chiral superfields Φ that have ϕ in their lowest component and obey the two-dimensional sigma model action

$$S = -\tfrac{1}{4}\int d\tau d\sigma d^2\theta [g_{ij}(\Phi) + B_{ij}(\Phi)] D^\alpha \Phi^i D_\alpha \Phi^j, \qquad (2.23)$$

with indices $i,j=1,\ldots,d$ parametrizing the target space and

$$D_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} + i\rho^\mu \theta_\alpha \partial_\mu, \qquad (2.24)$$

where ρ^μ is a two-dimensional γ matrix and θ_α a two-component Grassmann-valued spinor. Chiral superfields are defined by $\bar{D}^\alpha\Phi^i=0$. Written in terms of $\mathcal{N}=1$ superfields, this action has explicit $\mathcal{N}=1$ supersymmetry. In the case $H=dB=0$ it has further nonmanifest supersymmetry if and only if the target space is Kähler (Zumino, 1979).

Considering a sigma model that contains not only chiral but also twisted chiral superfields, one can find additional supersymmetry with $H\neq 0$ even if the target is not Kähler. This was proposed by Gates *et al.* (1984) more than 20 years ago and has recently found an embedding in generalized complex geometry (Hitchin, 2003). It turns out that the target manifolds in this model define a (twisted) generalized Kähler structure (Gualtieri, 2003). Stimulated by this new mathematical language, there has been tremendous progress in defining generalized (topological) sigma models (Kapustin and Li, 2004; Lindstrom *et al.*, 2005, 2007; Bredthauer *et al.*, 2006; Pestun, 2007).

Topological string theory integrates not only over all maps ϕ but also over all metrics on Σ; this is often called a sigma model coupled to two-dimensional gravity. Classically, the sigma model action depends only on the conformal class of the metric, so the integral over metrics reduces to an integral over conformal (or complex) structures on Σ.

The sigma model (2.23) with Kähler target can be made topological by a procedure called "twisting" (Witten, 1988b), which basically shifts the spin of all operators by $1/2$ their R-charge. There are two conserved supercurrents for the two world-sheet supersymmetries that are nilpotent

$$(G^\pm)^2 = 0, \qquad (2.25)$$

so one might be tempted to use these as BRST operators and build cohomologies of states, but they have spin $3/2$. The twist shifts their spin by half their R-charge to obtain spin 1 operators:

$$S_{\mathrm{new}} = S_{\mathrm{old}} + \tfrac{1}{2}q, \qquad (2.26)$$

where q is the U(1) R-charge of the operator in question. Classically, the theory has a vector $\mathrm{U}(1)_V$ symmetry and an axial $\mathrm{U}(1)_A$ symmetry. Twisting by $\mathrm{U}(1)_V$ gives the so-called A model; twisting by $\mathrm{U}(1)_A$ gives the B model. The $\mathrm{U}(1)_A$ might suffer from an anomaly unless

$c_1(M)=0$, which leads to the requirement that the target must be a Calabi-Yau manifold for the B model. One could now define $Q=G^+$ or $Q=G^-$ and use this nilpotent operator as a BRST operator, i.e., restrict one's attention to observables which are annihilated by Q.

Before doing so we note a special feature of $\mathcal{N}=(2,2)$ supersymmetry. Since left and right movers basically decouple, we can split any of the operators $G^\pm$ into two commuting copies, one for left and one for right movers. In terms of complex coordinates we denote the left movers as holomorphic $G^\pm$ and the right movers as antiholomorphic $\overline{G^\pm}$. This makes the (2,2) supersymmetry more apparent. Now twisting can be defined for left and right movers independently and we obtain in principle four models, depending on which we choose as BRST operators:

$$\text{A model:}\quad (G^+,\bar{G}^+),\quad \text{B model:}\quad (G^+,\bar{G}^-),$$

$$\bar{\text{A}}\text{ model:}\quad (G^-,\bar{G}^-),\quad \bar{\text{B}}\text{ model:}\quad (G^-,\bar{G}^+).$$

Of these four models, only two are actually independent, since the correlators for A (B) and for $\bar{\text{A}}$ ($\bar{\text{B}}$) are related by complex conjugation. So we ignore $\bar{\text{A}}$ and $\bar{\text{B}}$ in the following.

Starting with this setup, one can now discuss observables in topological theories. It turns out that $Q+\bar{Q}$ in the A model reduces to the differential operator $d=\partial+\bar{\partial}$ on M, i.e., the states of the theory lie in the de Rham cohomology. A "physical state" constraint requires states to be in $H^{(1,1)}(M)$ only, which corresponds to deformations of the Kähler structure on M. One can also show that correlators are independent of the complex structure modulus of M, since the corresponding operators are Q exact (they decouple from the computation of string amplitudes).

In the B model the relevant cohomology is that of $\bar{\partial}$ with values in $\Lambda^*(TM)$, i.e., the observables are (0,1)-forms with values in the tangent bundle TM. These correspond to complex structure deformations. One can also show that in this case correlation functions are independent of Kähler moduli. So each of the two topological models depends only on half the moduli,

$$\text{A model on } M\text{:}\quad \text{depends on Kähler moduli of } M;$$

$$\text{B model on } M\text{:}\quad \text{depends on complex structure moduli of } M.$$

In this sense both models describe topological theories because they only depend on the topology of the target, not its metric. It can also be shown that the relevant path integral $\int e^{-S}$ simplifies tremendously compared to ordinary field theories. The path integral localizes on Q-invariant configurations. These are simply constant maps $\phi:\Sigma\to M$ with $d\phi=0$ for the B model and holomorphic maps $\bar{\partial}\phi=0$ for the A model. In this sense the B model is simpler than the A model, because the string world-sheet "reduces to a point" on M; its correlation functions are those of a field theory on M. They compute quantities determined by the periods of the holomorphic three-form $\Omega^{(3,0)}$, which are sensitive to complex structure deformations.

The holomorphic maps in the A model are called "world-sheet instantons." Each world-sheet instanton is weighted by

$$\exp\left(\int_C (J+iB)\right),$$

where $t=J+iB\in H^2(M,\mathbb{C})$ is the complexified Kähler parameter and C is the image of the string world-sheet in M. Summing over all instantons makes this theory more complicated than the B model, but only in the sense that it is not local on M and does not straightforwardly reduce to a field theory on M. In summary, the A-model moduli are complexified volumes of two-cycles, while the B-model moduli are the periods of Ω.

We now discuss the relation of these topologically twisted sigma models to string theory. As mentioned previously, string theory sums not only over all possible maps $\phi:\Sigma\to M$, as discused for the sigma models above, but also over all possible metrics on Σ. The latter actually reduces to a sum over the moduli space of genus g Riemann surfaces. The topological string free energy is then defined as a sum over all genera

$$\mathcal{F}=\sum_{g=0}^{\infty} g_s^{2-2g}F_g, \tag{2.27}$$

with the string coupling g_s and F_g the amplitude for a fixed genus g. The string partition function is given by $\mathcal{Z}=\exp\mathcal{F}$.

The relevant quantities for the topological string theory are therefore the genus g partition functions. Already at genus zero one finds much interesting information about M. In the A model the genus zero free energy is

$$F_0=\int_M J\wedge J\wedge J+\text{instanton corrections}. \tag{2.28}$$

The first term corresponds to the classical contribution of the world-sheet theory: it gives the leading order contribution in which the string world-sheet reduces to a point. The instanton term contains a sum over all homology classes $H_2(M,\mathbb{Z})$ of the image of the world-sheet, each weighted by the complexified area, and a sum over "multiwrappings" in which the map $\Sigma\to M$ is not one to one.

To define the genus zero free energy in the B model requires more effort. We already noted that the relevant moduli are periods of $\Omega\in H^3(M,\mathbb{C})$. This cohomology can be decomposed as

$$H^3=H^{3,0}\oplus H^{2,1}\oplus H^{1,2}\oplus H^{0,3}. \tag{2.29}$$

For a Calabi-Yau threefold the Hodge numbers are given by $h^{3,0}=h^{0,3}=1$, because there is one unique holomorphic three-form, and $h^{2,1}=h^{1,2}$. Therefore $H^3(M,\mathbb{C})$ has real dimension $2h^{1,2}+2$. It is customary to choose a symplectic basis of three-cycles A^i and B_j with intersection numbers

$$A^i \cap A^j = 0, \quad B_i \cap B_j = 0, \quad A^i \cap B_j = \delta^i_j, \tag{2.30}$$

with $i,j=1,\ldots,h^{1,2}+1$. One can then define homogeneous coordinates on the moduli space of complex structure deformations by

$$X^i := \int_{A^i} \Omega. \tag{2.31}$$

This gives $h^{1,2}+1$ complex coordinates, although the moduli space only has dimension $h^{1,2}$. This overcounting is due to the fact that Ω is only unique up to overall rescaling, so the same is true for the coordinates defined this way. Therefore they carry the name "homogeneous coordinates." There are also $h^{1,2}+1$ periods over B cycles

$$\hat{F}_i := \int_{B^i} \Omega. \tag{2.32}$$

Due to the relation between A and B cycles, there must be a relation between the periods. In other words, we can express $\hat{F}_i$ as a function of X^j:

$$\hat{F}_i = \hat{F}_i(X^j). \tag{2.33}$$

One can prove that these satisfy an integrability condition,

$$\frac{\partial}{\partial X^i}\hat{F}_j = \frac{\partial}{\partial X^j}\hat{F}_i, \tag{2.34}$$

which allows us to define a new function F via

$$\hat{F}_i = \frac{\partial}{\partial X^i} F. \tag{2.35}$$

This function is actually the genus zero free energy of the B model. It is simply given by

$$F = \tfrac{1}{2} X_i \hat{F}^i. \tag{2.36}$$

In general, the sum over all world-sheet configurations is too hard to carry out explicitly. There are nevertheless some tools that enable one to calculate topological string amplitudes. For example, mirror symmetry between the A and B models can be used to compute amplitudes in the model of choice (usually the B model since it does not obtain instanton corrections) and then extrapolating the result to the mirror theory. We are more interested in a duality between open and closed strings, which enables one in principle to calculate the free energy at all genera for a particular class of noncompact geometries, e.g., conifolds. To describe open topological strings we must explain what we mean by topological branes that appear as boundaries of Σ.

A D-brane corresponds to a boundary condition for Σ that is BRST invariant. In the A model this implies that the boundary should be mapped to a Lagrangian submanifold[7] L of M. If we allow open strings to end on L, we say that the D-branes are wrapped on L. Having a stack of N D-branes on L corresponds to including a weighting factor N for each boundary.

D-branes carry gauge theories in physical string theory (we use "physical" for the target space perspective to distinguish it from toplogical string theory). The same is true for topological branes. In the A model it turns out that one can actually compute the exact string field theory, which is again a topological theory: U(N) Chern-Simons theory (Witten, 1986, 1995). Its action in terms of the U(N) gauge connection A is given by

$$S = \int_L \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A). \tag{2.37}$$

This action might still obtain instanton corrections, but Witten (1995) showed that in the special case where $L = S^3$ there are none. This is fascinating, because the S^3 in the deformed conifold (which is also T^*S^3) is such a Lagrangian submanifold.

In physical superstring theory, D-branes are sources for RR fluxes. So under what quantity are topological branes charged? The only fluxes available are the Kähler two-form J and the holomorphic three-form Ω. Wrapping a topological brane on a Lagrangian subspace L of M (in the A model) creates a flux through a two-cycle C which "links" L. This link means that $C=\partial S$ for some three-cycle S that intersects L once, so C is homologically trivial in M, although it becomes nontrivial if considered as a cycle in $M\backslash L$. This implies that $\int_C J=0$ since J is closed and C trivial.

Wrapping N branes on L has the effect of creating a Kähler flux through C

$$\int_C J = N g_s, \tag{2.38}$$

because the branes act as a δ function source for the two-form, i.e., J is no longer closed on L, but $dJ = N g_s \delta(L)$. Similarly, a B-model brane on a holomorphic two-cycle Y induces a flux of Ω through the three-cycle linking Y. In principle we could also wrap branes on zero-, four-, or six-cycles in the B model, but there is no field candidate those branes could be charged under. This suggests a privileged role for two-cycles.

2. The Gopakumar-Vafa conjecture

After all these preliminaries we are now ready to explain the geometric transition on conifolds. This is a duality between open and closed topological strings (it has been shown that they give rise to the same string parti-

[7]This means L has half the dimension of M and the Kähler form restricted to L vanishes or in other words it is an isotropic submanifold of maximal dimension (half of that of the ambient Calabi-Yau manifold). However, a more generic condition that allows for a nonflat gauge field on the brane was derived by Kapustin and Li (2003).

tion function) which has profound physical consequences. The dual gauge theory from the open string sector is $\mathcal{N}=1$ SYM in $d=4$. The IR dynamics of this gauge theory can be obtained either from the open or from the closed string sector. In this sense, both string theory backgrounds are dual; they compute the same superpotential because they have the same topological string partition function. The key to this duality in the gauge theory is to identify parameters from the open string theory with parameters from the closed string theory. In the IR it will be the gluino condensate which is identified with either the Kähler or complex structure modulus of the closed or open string theory background.

The geometric transition in question (Gopakumar and Vafa, 1999) considers the A model on the deformed conifold T^*S^3. As noted by Witten (1995), the exact partition function of this theory is simply given by U(N) Chern-Simons theory. The *closed* A model on this geometry is trivial, because it has no Kähler moduli. But the T^*S^3 contains a Lagrangian three-cycle $L=S^3$ on which we can wrap branes in the open A model. This creates a flux Ng_s of J through the two-cycle C which links L, in this case $C=S^2$. It is thus natural to conjecture that this background is dual to a background with only flux through this two-cycle. The resolved conifold is the logical candidate for this dual background as it looks asymptotically like the deformed conifold, but has a finite S^2 at the tip of the cone. This led Gopakumar and Vafa to the following conjecture:

The open A model on the deformed conifold T^*S^3 with N branes wrapping the S^3 is dual to the closed A model on the resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\rightarrow\mathbb{CP}^1$, where the size of the $\mathbb{CP}^1$ is determined by $t=Ng_s$.

There are no branes anymore in the dual geometry; there is simply no three-cycle which they could wrap. The passage from one geometry to the other is called a "geometric transition" or "conifold transition" in this case. The agreement of the partition function on both sides was shown by Gopakumar and Vafa (1999) for arbitrary 't Hooft coupling $\lambda=Ng_s$ and to all orders in $1/N$. In this sense, this duality is an example of a large N duality as suggested by 't Hooft: for large N holes in the Riemann surface of Feynman diagrams are "filled in" or "condensed," where one takes $N\rightarrow\infty$ with g_s fixed. Gopakumar and Vafa (1999) matched the free energy F_g at every genus g via the identification of the 't Hooft coupling

$$i\lambda = Ng_s \text{ (open)} \leftrightarrow i\lambda = t \text{ (closed)}, \tag{2.39}$$

where t is the complexified Kähler parameter of the S^2 in the resolved conifold and identification of the 't Hooft coupling for open strings is dictated by the Chern-Simons theory on S^3.

Beyond that, it was also shown that coupling to gravity (to the metric[8]) and the Wilson loops take the same form for the open and closed theories. The two topological string theories described here correspond to the different limits $\lambda\rightarrow 0$ and $\lambda\rightarrow\infty$, but they are conjectured to describe the same string theory (with the same small g_s) only on different geometries.

C. The Vafa model

1. Embedding the Gopakumar-Vafa model in superstrings and superpotential

This scenario has an embedding in physical type IIA string theory. Starting with N D6-branes on the S^3 of the deformed conifold we find a dual background with flux through the S^2 of the resolved conifold. The Calabi-Yau manifold breaks 3/4 of the supersymmetry (which leaves eight supercharges), therefore the theory on the world-volume of the branes has $\mathcal{N}=1$ supersymmetry (the branes break another half). There is a U(N) gauge theory on the branes [in the low-energy limit of the string theory the U(1) factor decouples and we have effectively SU(N)]. As described in the last section, these wrapped branes create flux and therefore a superpotential.[9] This superpotential is computed from topological strings, but we need a gauge theory parameter in which to express it. The relevant superfield for $\mathcal{N}=1$ SU(N) is S, the chiral superfield with a gaugino bilinear in its bottom component. We want to express the free energy F_g in terms of S. Since there will be contributions from world-sheets with boundaries, we can arrange this into a sum over holes h:

$$F_g(S) = \sum_{h=0}^{\infty} F_{g,h} S^h. \tag{2.40}$$

It turns out that the genus zero term computes the pure gauge theory, i.e., pure SYM. Higher genera are related to gravitational corrections.

As discussed above, the open topological string theory is given by Chern-Simons on T^*S^3, which has no Kähler modulus. The superpotential created by the open topological amplitude of genus zero was found by Vafa (2001) to be

$$W^{\text{open}} = \int d^2\theta \frac{\partial F_0^{\text{open}(S)}}{\partial S} + \alpha S + \beta, \tag{2.41}$$

with $\alpha,\beta=$const and αS the explicit annulus contribution ($h=2$).

[8]It might seem contradictory that there can be a coupling to the metric when discussing topological models. The classical Chern-Simons action is independent of the background, but at the quantum level such a coupling can arise. On the closed side there are possible IR divergences, anomalies for noncompact manifolds, that depend on the boundary metric of these manifolds.

[9]We set the string coupling $g_s=1$ throughout most of this section.

Although the topological model is not sensitive to any flux through a four- or six-cycle, in the superstring theory the corresponding RR forms F_4 and F_6 can be turned on. On the closed string side this corresponds to a superpotential

$$W^{\mathrm{closed}} = \int F_2 \wedge (J + \imath B) \wedge (J + \imath B) + i \int F_4 \wedge (J + \imath B) + \int F_6, \tag{2.42}$$

where $J+\imath B$ is the complexified Kähler class, whose period over the basis two-cycle gives the complex Kähler modulus t (of the resolved conifold). According to Vafa (2001) the topological string amplitude is not modified by these fluxes. The genus zero topological string amplitude F_0 determines the size of the four-cycle to be $\partial F_0 / \partial t$ (Vafa, 2001). If we have M, L, and P units of two-, four- and six-form flux, respectively, the superpotential yields after integration

$$W^{\mathrm{closed}} = M \frac{\partial F_0}{\partial t} + itL + P. \tag{2.43}$$

Note that requiring $W=0$ and $\partial_t W=0$ fixes P and L in terms of M and t. M is of course fixed by the number of branes in the open string theory.

This looks similar to the superpotential for the open theory (2.41). As already discussed the topological string amplitudes agree,

$$F^{\mathrm{open}} = F^{\mathrm{closed}}, \tag{2.44}$$

if one identifies the relevant parameters as in Eq. (2.39). To map the superpotentials onto each other we have to identify S with t and α, β with the flux quantum numbers iL, P. It is clear from the gauge theory side that α (or L) is related to a shift in the bare coupling of the gauge theory. In particular, to agree with the bare coupling to all orders we require $iL = V/g_s$, where V is the volume of the S^3 that the branes are wrapped on. This gives an interesting relation between the size V of the blown up S^3 (open) and the size t of the blown up S^2 (closed):

$$(e^t - 1)^N = \mathrm{const} \times e^{-V}. \tag{2.45}$$

This indicates that for small N ($N/V \ll 1$) the description with D-branes wrapped on S^3 is good (since $t \to 0$), whereas for large N the description with blown up S^2 is good (since $V \to -\infty$ does not make sense). It should be clear from our discussion that after the S^3 has shrunk to zero size there cannot be any D6-branes in the background, but RR fluxes are turned on.

We conclude this section with an explicit derivation of the Veneziano-Yankielowicz superpotential in type IIB (Cachazo *et al.*, 2001), from the perspective of the closed string theory. This theory is mirror dual to the IIA scenario discussed above,[10] and so the open string theory lives on a resolved conifold background and the closed theory with fluxes lives on the deformed conifold described by

$$f = \sum_{i=1}^{4} z_i^2 - \mu^2 = 0. \tag{2.46}$$

On the deformed conifold there is only one compact A cycle (the S^3) with N units of RR flux and one noncompact B cycle with α units of NS-NS flux. (This is the same setup as in the KS model where B_2 also threads through the noncompact cycle.) We therefore define the superfield S as the period of the A cycle and its dual period $\hat{F}$:

$$S = \int_A \Omega, \quad \hat{F} = \int_B \Omega. \tag{2.47}$$

The holomorphic three-form is given by

$$\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4}{df} = \frac{dz_1 \wedge dz_2 \wedge dz_3}{2z_4} = \frac{dz_1 \wedge dz_2 \wedge dz_3}{2\sqrt{\mu^2 - z_1^2 - z_2^2 - z_3^2}}. \tag{2.48}$$

Viewing the three-cycles A and B as two-spheres (spanned by a real subspace of z_2, z_3) fibered over z_1, one can integrate Ω over S^2, resulting in a one-form

$$\int_{S^2} \Omega = \frac{1}{2\pi i} \sqrt{z_1^2 - \mu^2} dz_1. \tag{2.49}$$

The compact A cycle, projected to the z_1 plane, becomes an interval $(-\mu, \mu)$, so that the A period results in

$$S = \frac{1}{2\pi \imath} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} dz_1 \sqrt{z_1^2 - \mu^2} = \frac{\mu}{4}. \tag{2.50}$$

The noncompact B cycle is projected to (μ, ∞). We therefore introduce a cutoff Λ_0 such that

$$\hat{F} = \frac{1}{2\pi i} \int_{\mu}^{\Lambda_0^{3/2}} dz_1 \sqrt{z_1^2 - \mu^2} = \frac{1}{2\pi i} \left(\frac{\Lambda_0^3}{2} - S + S \ln \frac{S}{\Lambda_0^3} \right) + \mathcal{O}\left(\frac{1}{\Lambda_0} \right). \tag{2.51}$$

With N units of flux through the A cycle and α units of flux through the B cycle the flux-generated superpotential (Vafa, 2001)

$$W_{\mathrm{eff}} = -2\pi i \sum_i (N_i \hat{F}_i + \alpha_i S_i) \tag{2.52}$$

becomes

$$W_{\mathrm{eff}} = N(S \ln \Lambda_0^3 + S - \ln S) - 2\pi i \alpha S. \tag{2.53}$$

The first and last terms can be combined by replacing Λ_0 with the *physical* scale of the theory. This is because α is related to the bare coupling of the four-dimensional $\mathcal{N}=1$ SU(N) gauge theory (as discussed in Sec. II.B.2) via $2\pi i \alpha = 8\pi^2 / g_0^2$, but the coupling of $\mathcal{N}=1$ SYM exhibits a logarithmic running

[10]The same result could be obtained in IIA, but the IIB treatment is not complicated by instantons.

$$\frac{8\pi^2}{g^2(\Lambda_0)} = 3N \ln \Lambda_0 + \text{const.} \tag{2.54}$$

The term $3N \ln \Lambda_0 - 2\imath\pi\alpha$ is precisely what shows up as the coefficient of S in Eq. (2.53). α can therefore be absorbed into Λ_0 by introducing the physical scale Λ and the superpotential becomes

$$W_{\text{eff}} = NS\left(1 - \ln \frac{S}{\Lambda^3}\right), \tag{2.55}$$

which is precisely the Veneziano-Yankielowicz superpotential (Veneziano and Yankielowicz, 1982) suggested for four-dimensional $\mathcal{N}=1$ SU(N) super-Yang-Mills theory, where S plays the role of the glueball field. The vacuum of the theory exhibits all the known phenomena of gaugino condensation, chiral symmetry breaking, and domain walls. This is a remarkable result and the first example where string theory produces the correct superpotential of a gauge theory.

The discussion of the IR limit deserves a word of caution. We argued earlier that both the KS and the Vafa model reach an SU(M) gauge theory. This suggests that the open string background in Vafa's scenario is actually the IR limit of the KS cascade, which then in the far IR is dual to the closed string background. This is not quite accurate, as the UV limit of Vafa's scenario does not coincide with the UV limit of KS. Instead, it approaches a (5+1)-dimensional theory. The UV limit of Vafa's scenario should rather correspond to a Maldacena-Nunez (2001) (MN) type of solution. However, the MN UV limit does not quite fit into interpolating scenarios known as the KS baryonic branch (Papadopoulos and Tseytlin, 2001; Gubser *et al.*, 2004), which is a one-parameter family of backgrounds obtained by deformations of the original KS solution (Butti *et al.*, 2005; Dymarsky *et al.*, 2006; Benna *et al.*, 2007). These SU(3) structure backgrounds are all expected to have a dual field theory description with vaccum expectation values of baryonic operators turned on. The MN background, however, does not share this property (Gubser *et al.*, 2004). In terms of the interpolating solution of Butti *et al.* (2005), it can only be reached in an infinite limit of the parameter space and can therefore not be interpreted as on the same "branch" as the KS solution.[11] Although we have illustrated various similarities between the KS cascade and Vafa's geometric transition, one should keep this subtle distinction in the UV behavior in mind.

2. Vafa's duality chain

We summarize the superstring picture of the conifold transition: In type IIA we start with N D6-branes on the S^3 of the deformed geometry and find as its dual N units of two-form flux through the S^2 of the resolved conifold. In the mirror type IIB, N D5-branes wrapping the S^2 of the resolved conifold are dual (in the large N limit) to a

[11]We thank I. Klebanov for discussion on this point.

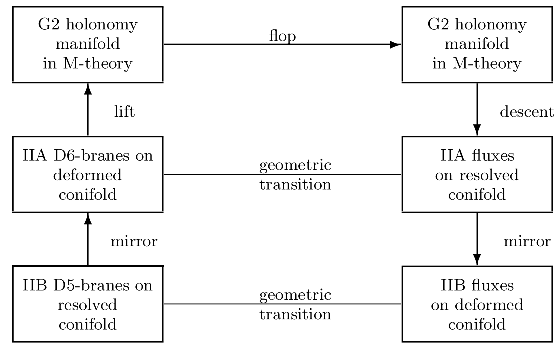


FIG. 5. Vafa's duality chain. Following the arrows through a series of mirror symmetries and a flop transition, we can verify the geometric transition as conjectured for IIA and IIB.

background without D-branes but with the three-form flux turned on. The geometry after transition is given by the deformed conifold with blown up S^3. In both cases we have to identify the complex structure modulus of the deformed conifold with the Kähler modulus of the resolved conifold or, roughly speaking, the size of the S^3 with the size of the S^2.

The type IIA scenario can be lifted to M-theory where the deformed and resolved conifolds are related by a flop transition. In seven dimensions both manifolds stem from a manifold with G_2 holonomy and symmetry $SU(2)\times SU(2)\times U(1)$. Topologically, the manifold in question is equivalent to a cone over $S^3\times\tilde{S}^3$ that has a U(1) fiber on which one can reduce to $d=6$. One can either reduce on a fiber that belongs to an S^3 of vanishing size (this yields a six-dimensional manifold with blown up $\tilde{S}^3$, the deformed conifold) or on a fiber that belongs to an $\tilde{S}^3$ of finite size (this gives a finite size $\tilde{S}^2$ in six dimensions, the resolved geometry).[12] In other words, both scenarios are related by an exchange of the finite-size $\tilde{S}^3$ with the vanishing S^3, which is called a "flop transition."

A cone over $S^3\times\tilde{S}^3$ is given by $\mathbb{R}^+\times S^3\times\tilde{S}^3$ which is equivalent to $\mathbb{R}^4\times\tilde{S}^3$. The topology of this manifold can be viewed as (Atiyah *et al.*, 2001)

$$(u_1^2+u_2^2+u_3^2+u_4^2)-(v_1^2+v_2^2+v_3^2+v_4^2)=V, \tag{2.56}$$

with $u_i, v_i \in \mathbb{R}$. For $V>0$ the blown up $\tilde{S}^3$ is described by the u_i, and the v_i correspond to $\mathbb{R}^4$. For $V<0$ their roles are exchanged. The flop transition can then be viewed as a sign flip in V or as an exchange of the two S^3. In the presence of G flux in M-theory the volume V can be complexified to $V+iG$, so that even for the transition point $V=0$ the singularity is avoided.

Using these arguments, one can follow a "duality chain," as depicted in Fig. 5, which leads from D-branes

[12]Furthermore, modding out by a $\mathbf{Z}_N$ in both cases gives a singularity corresponding to N D6-branes or a nonsingular solution with N units of flux, respectively (Atiyah *et al.*, 2001).

in IIB through mirror symmetry to IIA, then via an M-theory flop to the closed string side in IIA and back to IIB via another mirror symmetry. Precisely this chain was followed by Becker *et al.* (2004, 2006); Alexander *et al.* (2005); Knauf (2007) and will be reviewed in Sec. III.

D. Brane constructions

We saw in Sec. II.A.1 that finding gauge theories on singular geometries like the conifold can be difficult. A more intuitive way to arrive at these gauge theories is to T-dualize the type IIB picture of branes on a conifold along a direction perpendicular to the branes. As we shall see, this gives type IIA configurations of branes suspended between orthogonal NS5-branes, from which the gauge theory on the branes can be easily deduced as in the Hanany-Witten setup (Hanany and Witten, 1997). In addition, lifting these IIA configurations to M-theory allows one to reproduce the dualities conjectured via gauge or topological string theory arguments in the KS and Vafa setups, respectively. This was done by Dasgupta *et al.* (2001) for the geometric transition [see also Hori *et al.* (1998); Dasgupta, Oh, and Tatar (2002)], and by Dasgupta, Oh, *et al.* (2002) for the Klebanov-Strassler model [following Dasgupta and Mukhi (1999a, 1999b)]. In both cases use was made of Witten's MQCD (an M-brane description of QCD) methods (Witten, 1997a, 1997b) based on lifting to M-theory configurations of D4-branes suspended between NS5-branes.

In this section we discuss first the brane configuration dual to string theory on conifold geometries and then in turn the relevant brane configurations dual to both the Klebanov-Strassler and the geometric transition (as embedded in type IIB) setups. In the brane configuration picture, and even more so in the pictures lifted to M-theory which allow us to track the duality in each case, we see similarities between the two arguments. The brane configuration picture is useful for understanding several aspects of the theories in a different way, and for highlighting deep similarities between the two scenarios. However, it has some limitations which should be kept in mind and which we mention as they arise.

1. The T-dual of a conifold

As shown by Bershadsky *et al.* (1996) and Hanany and Urangam (1998), a conifold can be described by two degenerating tori varying over a $\mathbb{P}^1$ base. When two T-dualities are performed, one over a cycle of each torus, the conifold gives rise to a pair of orthogonal NS5-branes. A configuration of orthogonal NS5-branes is also found upon performing a single T-duality along the U(1) fibration of the conifold, as shown explicitly by Dasgupta and Mukhi (1999a). To see this, consider the conifold equation in the form given in Eq. (A24):

$$xy - uv = 0. \tag{2.57}$$

We can define coordinates in such a way that $x=x^4+\imath x^5$ and $y=x^8+\imath x^9$. Here we map θ_1,ϕ_1 to x^4,x^5 and θ_2,ϕ_2 to x^8,x^9. Following the literature, we make the identifications $x^6=\psi$ and $x^7=r$ for the rest of this section. We perform the T-duality along x^6. Our reason for adopting this change of coordinates is that in the dual brane picture x^4,x^5,x^8, and x^9 are no longer compact directions.

Upon T-dualizing, the NS5-branes will arise on the degeneration loci of the fibration (Bershadsky *et al.*, 1996; Uranga, 1999). The fiber degenerates at the conifold singularity, where $x=y=u=v=0$. A convenient choice of coordinates allows us to take the fiber as degenerating to two intersecting complex lines given by

$$xy = 0. \tag{2.58}$$

In the T-dual picture this curve then defines the intersection of the two NS5-branes: one extends along x ($y=u=v=0$) and the other along y ($x=u=v=0$). For the deformed conifold,

$$xy - uv = \mu^2, \tag{2.59}$$

xy can no longer be zero at the tip but is given by

$$xy = \mu^2. \tag{2.60}$$

This gives the intersection curve of the NS5-branes in the corresponding T-dual picture. In both cases x^6 and x^7 parametrize possible separations between the NS5-branes. If it is present an NS-NS B field on a vanishing two-cycle in the conifold picture maps to separation in x^6 of the NS5-branes (Karch *et al.*, 1998), while a finite S^2 implies a separation in x^7 given by the resolution parameter.

Dasgupta and Mukhi (1999a, 1999b) and Uranga (1999) exploited this T-duality for the construction of a type IIA brane configuration dual to the Klebanov-Witten and Klebanov-Strassler setups, further studied by Dasgupta, Oh, and Tatar *et al.* (2002) and Dasgupta, Oh, *et al.* (2002) where Witten's MQCD methods were used to track the conjectured geometric transition via the M-theory description. This was also applied to Vafa's geometric transition [see also Dasgupta *et al.* (2001)]. A discussion of the general approach was given by Karch *et al.* (1998) where the references for several key results are provided.

2. The Klebanov-Strassler setup via brane configurations

We begin with the Klebanov-Witten setup and then proceed to the Klebanov-Strassler scenario and discuss what becomes of the M fractional D3-branes under the T-duality. In Sec. II.A.1 we argued that while an $\mathcal{N}=1$ supersymmetric theory with U(1) gauge group reproduced the parametrization of the conifold given by the fields A_i,B_j and Eqs. (2.8) and (2.9), the gauge theory whose moduli space corresponded to a single D3-brane on the conifold had gauge group $\mathrm{U}(1)\times\mathrm{U}(1)$ [which generalized to $\mathrm{U}(N)\times\mathrm{U}(N)$ for N D3-branes]. The appearance of two gauge group factors can be understood by T-dualizing the conifold setup in a direction perpendicular to the branes. Under a T-duality along ψ or x^6 a conifold will give rise to a pair of NS5-branes (Dasgupta

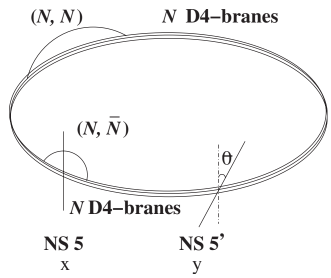


FIG. 6. The IIA brane configuration dual to the Klebanov-Witten setup.

and Mukhi, 1999a).[13] The presence of D3-branes as in the Klebanov-Witten setup yields a configuration of D4-branes stretching between these NS5-branes, along the T-dual direction x^6, as shown in Fig. 6.

We label one NS-brane NS5 and the other NS5′. In Fig. 6, the 0123 and radial directions are suppressed. The branes are orthogonal: NS5 extends in x and NS5′ extends in y. The NS5-branes are much heavier than the D4-branes because they are infinite in all of their world-volume directions and so they can be treated classically (Hanany and Witten, 1997; Witten, 1997b). We study the gauge theory on the D4-branes where the NS5-branes are considered as fixed and the positions of the D4-branes parametrize the moduli space of the theory. The D4-branes have only four infinite directions, so the field theory living on them is effectively 3+1 dimensional. It is the same as that living on the D3-branes at the tip of the conifold in the IIB picture.

As should be clear from Fig. 6, the gauge theory is easily deduced. The configuration preserves four supercharges, and therefore describes an $\mathcal{N}=1$ supersymmetric gauge theory on the D4-branes. Gauge fields transforming as (N,N) or $(\bar{N},\bar{N})$ in the adjoint representation correspond to open strings ending on the D4-branes between the NS5-branes, while matter fields transforming in the bifundamental representations $(N,\bar{N})$ or $(\bar{N},N)$ correspond to strings stretching between D4-branes on opposite sides of an NS5-brane. The separation of the NS5-branes in the x^6 direction is given by the NS-NS two-form, and is not specified by the geometry.[14] Thus generically the N D4-branes are broken into two segments by the NS5-branes, and the gauge group is U(N) $\times$U(N), as claimed in Sec. II.A.1.

For $\mathcal{N}=1$ supersymmetry (SUSY) the NS5-branes must be perpendicular. If NS5′ is rotated in the (x,y) plane so that they are parallel, the gauge theory on the D4-branes has $\mathcal{N}=2$ supersymmetry. In this case, the D4-branes can move in x (the scalars corresponding to their fluctuations in these directions fit into the adjoint representation of $\mathcal{N}=2$). This is no longer true once there is a relative rotation of the NS5-branes, since any separation of the D4-branes in x would lead to twisting of the D4-branes that would break supersymmetry completely. When the NS5-branes are perpendicular, there are no moduli for motion of the D4-branes in the directions of the NS5-branes. Furthermore, the angle between the NS5-branes serves as a SUSY-breaking parameter. In fact, this angle (θ) is related to the mass of the adjoint chiral multiplets by $\mu=\tan\theta$ (Barbon, 1997; Witten, 1997b; Hori *et al.*, 1998). Thus, the adjoints can be integrated out of the superpotential when the NS5-branes are perpendicular, and agreement with the Klebanov-Witten setup is also found at the superpotential level (Dasgupta and Mukhi, 1999a). On the other hand, the D4-branes on either side of each NS5-brane cannot be split from each other, so the bifundamental fields remain massless. This T-duality therefore allows us to rederive the gauge theory dual of the Klebanov-Witten setup.

To study the brane configuration dual to the Klebanov-Strassler model we need to know how a fractional D3-brane will transform under a T-duality along a direction perpendicular to it. In a five-dimensional space-time, D3-branes and fractional D3-branes are domain walls (Gubser and Klebanov, 1998; Gubser, 1999). There is a jump in the five-form flux when one crosses them, and this is related in AdS compactifications to the number of branes on which the gauge theory lives. Thus, crossing a domain wall corresponds to increasing or decreasing the rank of the gauge group in the dual gauge theory. It is easy to see in the brane configuration setup dual to the Klebanov-Witten scenario that addition of a single D3-brane will increase the rank of the gauge group from SU(N)$\times$SU(N) to SU(N+1)$\times$SU(N+1). However, we will soon see that the M wrapped branes of the KS model map to D4-branes extended only along part of x^6 between NS5 and NS5′ (Karch *et al.*, 1998; Dasgupta and Mukhi 1999b). Thus, fractional D3-branes function as exotic domain walls, the crossing of which will take the gauge group of the dual field theory from SU(N)$\times$SU(N) to SU(N+1)$\times$SU(N). With the presence of M wrapped D5-branes, the brane configuration shown in Fig. 7 results.

The action of the T-duality on the M wrapped D5-branes of the KS model can be deduced by noting that the S^2 they wrap is the difference in the sense of homology between the two S^2s which form the base of the U(1) fibration giving $T^{1,1}$, i.e., $S^2=S_1^2-S_2^2$, where the first S^2 is parametrized by θ_1 and ϕ_1 and the second is parametrized by θ_2,ϕ_2. The cohomology H^2 is given by two-forms which are closed but not exact. Candidate representatives of H^2 are $\sin\theta_1 d\theta_1\wedge d\phi_1 \pm \sin\theta_2 d\theta_2\wedge d\phi_2$, both

[13]Explicit calculation at the supergravity level by Dasgupta and Mukhi (1999a) takes the T-dual of the NS5-brane configuration described and does not exactly reproduce a conifold but something similar, where x^4,x^5 and x^8,x^9 remain two-planes instead of the required fibered two-spheres. This means that the SU(2)$\times$SU(2) global symmetry is not directly visible in the brane construction. Other arguments, coming from the gauge theory and the fact that this symmetry is regained in the M-theory lift nevertheless support the geometrical interpretation given here (Dasgupta and Mukhi, 1999a).

[14]As expected, the separation of the NS5-branes is related to the coupling(s) of the dual field theory.

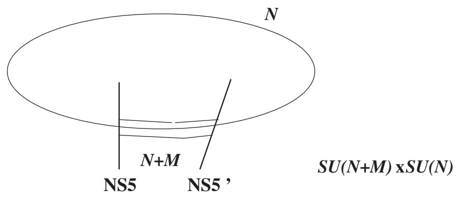


FIG. 7. The IIA brane configuration dual to the Klebanov-Strassler setup.

of which live only on the two S^2 factors in $T^{1,1}$ and are independent of the U(1) fiber. One might think that they are exact, since they can be written as $d(\cos\theta_1 d\phi_1 \pm \cos\theta_2 d\phi_2)$, but these expressions are not globally defined because ϕ_i is not well defined at the poles. Dasgupta and Mukhi (1999b) argue that the term with the plus sign is actually exact, since it can be written as $d(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \sim de^{\psi}$, where e^{ψ} is one of the five vielbeins and is globally defined, because ψ exhibits a shift symmetry. Therefore it is the minus term which is a representative of the second cohomology:

$$\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2.$$

This means that the dual object to the domain wall in the type IIA picture is something that carries a charge away from the (x^4, x^5) plane and deposits it on the (x^8, x^9) plane.[15] A D4-brane with one end on NS5 and the other on NS5′ performs this function. Thus an exotic domain wall maps under this T-duality into a D4-brane stretched only part of the way along x^6.

Next, we use the brane picture to study the gauge theory on the D4-branes. The coupling constant of the gauge theory is determined by the distance between the branes in x^6 (Karch *et al.*, 1998; Dasgupta and Mukhi, 1999a, 1999b) which is given by the B field and therefore matches arguments from Secs. II.A.2 and II.C.1. For our product gauge group setup,

$$\frac{1}{g_1^2} = \frac{l_6 - a}{g_s}, \tag{2.61}$$

$$\frac{1}{g_2^2} = \frac{a}{g_s}, \tag{2.62}$$

where g_1 and g_2 are the couplings of the two theories, g_s is the string coupling, and l_6 is the circumference of the x^6 circle so that a is the separation between the NS5-branes. We see that it is exactly the presence of the M fractional branes that breaks conformal invariance: In the IIA description, a nonzero β function arises in the gauge theory when the NS5-branes are bent. This occurs because the end points of the D4-branes on the NS5-branes introduce a dependence of a on θ_i, ϕ_i. Thus, the positions of the NS5-branes should really be measured far from the D4-branes. As shown by Witten (1997b), a will only have a well-defined limiting value when the NS5-brane has an equal number of D4-branes on either side. For the KS brane configuration, this is not the case—in other words, the branes are bent in r (the only available direction, suppressed in the figures). This introduces a dependence of a on r, from which one can rederive the running of the gauge theory coupling constant(s).

In addition, the duality cascade is also observed in the brane configuration picture. For an early reference see Elitzur *et al.* (1997) and also Uranga (1999). The bending of the NS5-branes means that, at some point far from the suspended D4-branes, a will vanish, implying a divergence of one coupling constant. To avoid it one must move the NS5-branes, pulling one through the other around the circle. As NS5′ approaches NS5, the N D4-branes occupy the entire x^6, and the $N+M$ branes shrink to zero size. When NS5′ is pulled through NS5, the N branes double up, while the branes that were originally between NS5 and NS5′ regrow with the opposite orientation as antibranes. These can partially annihilate the $2N$ branes, leaving $N-M$ branes on the expanding segment, as shown in Fig. 8. The system now has a dual description in terms of a gauge theory with gauge group $SU(N-M)\times SU(N)$, so that moving the branes through each other corresponds to performing a Seiberg duality.

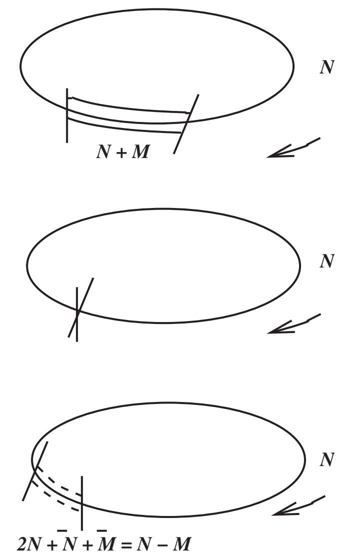


FIG. 8. A Seiberg duality transformation in the dual brane picture. Note that here $N=1$ and $M=2$ so in the final picture two of the branes are actually antibranes (denoted by dashed lines).

The branes continue to be bent, so one is led to repeat the motion around the circle. As in the duality cascade described in Sec. II.A.3, the process continues until only the NS5-branes with M D4-branes stretching between them are left and the gauge group is just $SU(M)$.

To see how this configuration maps to the deformed conifold, we have to lift it to M-theory. This allows us to study the nonperturbative dynamics of the theory. The first such analysis was performed by Witten, for the case of $\mathcal{N}=2$ theories (Witten, 1997b), but it was generalized

[15]Recall that θ_1, ϕ_1 map to x^4, x^5, and θ_2, ϕ_2 map to x^8, x^9. The U(1) fiber is ψ which maps to x^6 and is the direction in which we perform the T-duality.

to $\mathcal{N}=1$ by Brandhuber *et al.* (1997); Witten (1997a); and Hori *et al.* (1998) and it is these results which apply most directly to our case (in particular elliptic $\mathcal{N}=1$ models). Both D4- and NS5-branes map to M5-branes in M-theory, with the D4-branes corresponding to M5-branes wrapped on the eleventh dimension x^{10}. We define

$$t = \exp\left(-\frac{x^6}{R} + \imath x^{10}\right), \tag{2.63}$$

where R is the radius of the eleventh dimension.[16] In the case that no fractional branes are present, the D4- and NS5-branes lift to a configuration with three separate components. The fully wrapped N D4-branes become M5-branes wrapping x^6 and x^{10} or t. These N toroidal branes are described by the equations

$$x^N = 0,$$
$$y^N = 0, \tag{2.64}$$

which have to be supplemented with the equations describing the lifted NS5-branes: $y=0$, $t=1$ (NS5) and $x=0$, $t=e^{-a/R}$ (NS5′). For the brane dual picture at the bottom of the cascade when the D3-branes have cascaded away and the gauge group is just SU(M), the M suspended (fractional) branes join with the NS5-branes to become a single object in the M-theory description (Dasgupta and Mukhi, 1999b). This introduces dynamical effects into the model. This setup for pure $\mathcal{N}=1$ SYM was studied by Witten (1997a), but in the limit where $x^6 \to \infty$. The NS5-branes by themselves would lift to $xy=0$, which describes the conifold. The Klebanov-Strassler configuration of NS5-branes and fractional branes lifts to an M5-brane described by Dasgupta, Oh, *et al.* (2002) as

$$xy = \zeta,$$
$$t = x^M, \tag{2.65}$$

where $\zeta \in \mathbb{C}$ is some complex parameter. If we now try to continue the cascade by shrinking the distance x^6 between the NS5-branes, we find $t=e^{\imath x^{10}}$, i.e., t now parametrizes a circle. The embedding of the M5-brane into (x,y,t) must be holomorphic in order for SUSY to be preserved but as there is no nonconstant holomorphic map into S^1, t must be constant. Then Eq. (2.65) becomes

$$xy = \zeta,$$
$$t = t_0. \tag{2.66}$$

This object is exactly the lifted deformed conifold dual described by Eq. (2.60). The appropriate RR flux arises from the usual one-form one obtains by dimensional reduction of the M-theory lift to IIA on a twisted circle. The flux is then T-dualized to F_3 in IIB.

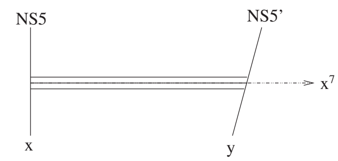


FIG. 9. The IIA brane configuration dual to the Vafa setup.

3. The Vafa setup via brane configurations

The duality put forward by Vafa was studied from the brane configuration point of view by Dasgupta *et al.* (2001), Dasgupta, Oh, and Tatar (2002), Dasgupta, Oh, *et al.* (2002). We begin with the IIB embedding setup, in which N D5-branes wrap the finite S^2 of a resolved conifold. Under a T-duality in x^6, this maps to a IIA configuration of N D4-branes stretching between two orthogonal NS5-branes, similar to the brane configuration dual of the KS setup.[17] There are two differences between the KS and Vafa brane configuration duals. The NS5-branes in the Vafa setup are also separated in x^7, with the separation in x^7 given by the size of S^2 at the tip of the conifold, the resolution parameter. More precisely, the two NS5-branes will be separated along $z=x^6+\imath x^7$ since B_2 controls the separation in x^6 (Dasgupta *et al.*, 2001). In the IR limit, the separation in x^6 will be negligible. In addition, the direction along which the D4-branes stretch between NS5 and NS5′ is noncompact, as shown in Fig. 9.

Vafa's geometric transition duality as expressed in the type IIB embedding can be seen directly in the IIA brane description (Dasgupta *et al.*, 2001). The easiest way to track it is "backwards," beginning with the final IIB picture of a deformed conifold with fluxes and no branes and asking what its T-dual (along x^6) is. Then we will shrink the S^3 to zero size and blow up S^2, both in the IIA brane configuration, and finally T-dualize back to IIB to find the resolved conifold with wrapped D5-branes. Note that the orthogonal NS5-branes described as T-dual to type IIB string theory on a singular conifold in Sec. II.D.1 are coincident in x^6 and x^7. The T-dual of a deformed conifold consists of two NS5-branes intersecting on a curve $xy=\mu^2$. However, when there is an RR flux F_3 through the S^3 in the IIB picture, the T-dual will be modified. Since F_3 has one component in the direction of the T-duality (ψ or x^6), the IIA picture will have a two-form flux F_2 in the x,y directions.[18] The geo-

[16]For our case we should keep in mind that t is not periodic in x^6 so we should only use it for a finite range of values.

[17]This should not be confused with the mirror IIA embedding of Vafa's, in which D6-branes wrap the S^3 of a deformed conifold. Any reference to T-duals or T-dualities in this section is to those along ψ or x^6 which take us between the IIB embedding setup described above and the IIA brane configuration dual shown in Fig. 9.

[18]This can be seen from $F_3 = N\omega_3$, where ω_3 is given by Eq. (2.16).

metric transition must involve shrinking the S^3 to zero size. Then x and y describe vanishing spheres. This implies an infinite flux per unit area, which singularity is resolved by the creation of a D4-brane. Here $F_2=dA$ couples to the world volume of the NS5-brane through

$$\int A \wedge \star d\phi = \int F_2 \wedge C_4, \tag{2.67}$$

where ϕ is one of the periodic scalars on the world-volume of the NS5. The correct SUSY-preserving source for C_4 in this case is a D4-brane, which intersects NS5 and NS5′ in the requisite four dimensions. Since we complete the geometric transition by blowing up the two-sphere, it must stretch between the NS5-branes. Here we see the fundamental connection between the KS and geometric transition pictures most clearly. The two directions available for the D4-brane to stretch along are x^6 and x^7. Growing it in the x^6 direction only results in the pretransition Klebanov-Strassler IIA brane setup, while growing it in the $z=x^6+\imath x^7$ direction gives exactly the brane configuration dual of the resolved conifold with D5s wrapping the S^2. We are able to begin with the deformed conifold picture in type IIB, T-dualize it to type IIA and find after the relevant transitions the T-dual either of the KS picture or of the Vafa picture, depending on whether the direction in which the dual NS5-branes are separated is compact or not.

The geometric transition can also be followed in M-theory (Dasgupta *et al.*, 2001), where the argument now runs in the opposite direction to that of the previous paragraph. The pretransition configuration of D4-branes stretching between orthogonal NS5-branes lifts to a single M5-brane with a complicated world-volume structure given by $R^{1,3}\times\Sigma$, where Σ is a complex curve defined by

$$xy = \zeta,$$

$$t = x^N. \tag{2.68}$$

This time we shrink S^2 to effect the transition, and find again that t must parametrize a circle and is therefore constant. This implies $x^N y^N=\zeta^N$, i.e., $\Sigma\to\Sigma_k$, where $t=t_0$ and

$$xy = \zeta \exp\left(\frac{2\pi\imath k}{N}\right).$$

We obtain N degenerate curves which are no longer the M-theory lift of D4-branes, but correspond to a closed string background, the deformed conifold. The main difference between the pretransition setups in M-theory is that the KS M5-brane is wrapped on a torus parametrized by $x^6+\imath x^{10}$ while the M5-brane of the Vafa setup is wrapped on a cylinder $x^7+\imath x^{10}$, at least in the IR.

III. SUPERGRAVITY TREATMENT AND NON-KÄHLER DUALITY CHAIN

In the last section we reviewed three arguments for geometric transition: one based on gauge theory, one on topological strings, and the third on brane constructions and MQCD methods. The "duality chain" derived by Atiyah *et al.* (2001); Vafa (2001), see Fig. 5, suggests a seemingly straightforward way to verify geometric transitions on the supergravity level. Exploiting the well-known fact that mirror symmetry on Calabi-Yau manifolds (CYs) can be realized (in a certain limit) by three T-dualities, see Strominger-Yau-Zaslow (SYZ) (Strominger *et al.*, 1996), and that a T-duality for a given metric and set of background fields is performed by applying Buscher's rules (Buscher, 1987, 1988), one should be able to explicitly formulate the supergravity solution corresponding to all links in the above chain. Proving more subtle than naively expected, this analysis was nevertheless carried out (Becker *et al.*, 2004, 2006; Alexander *et al.*, 2005; Dasgupta *et al.*, 2006; Knauf, 2007) and we review it here.

The issues that make the supergravity treatment nontrivial are the following.

- Resolved and deformed conifolds are only approximately mirror to each other. Whereas the resolved conifold does indeed admit a T^3 fiber for T-dualizing, the deformed conifold possesses less symmetry. It is therefore only possible to recover a deformed conifold mirror in the large complex structure limit. This was discussed by, e.g., Knauf (2007) in Chap. 2.
- Taking the back-reaction of RR and NS-NS fluxes (or D-branes) into account changes the IIB supergravity solution. Instead of D5-branes wrapping a resolved conifold, the manifold is only *conformally* a Calabi-Yau manifold. The manifold acquires a warp factor $h(r)$, depending on the radial direction. Strictly speaking this spoils the SYZ argument, but this problem can be overcome by working in the "local limit," i.e., restricting the metric to a patch over which $h(r)$ does not vary too much. This approach is inherent to all references (Becker *et al.*, 2004, 2006; Alexander *et al.*, 2005; Dasgupta *et al.*, 2006; Knauf, 2007) and we will comment on its shortcomings.
- We have seen that a supergravity solution like the Klebanov-Strassler model (with fractional branes) necessarily contains nontrivial NS-NS flux. Assuming the SYZ argument still holds and performing T-dualities along the isometry directions of the resolved conifold, the metric and the B field get mixed. This phenomenon can be used to argue that the mirror of a CY in the presence of NS-NS flux is no longer CY and it was postulated that these manifolds are half-flat (Gurrieri *et al.*, 2003).[19] It has now been established (Grana *et al.*, 2004a, 2004b, 2005) that the most general $\mathcal{N}=1$ SUSY-preserving backgrounds are generalized CYs as described by Gualtieri (2003);

[19]In contrast to a CY, which is characterized by a closed fundamental two-form J and a closed holomorphic three-form $\Omega=\Omega_+ + i\Omega_-$, half-flat manifolds only obey $d(J\wedge J)=0$ and $d\Omega_-=0$, and are therefore a special class of non-Kähler manifolds ($dJ\neq 0$).

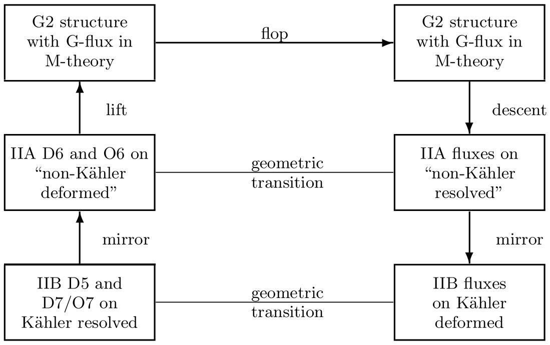


FIG. 10. The modified duality. The backgrounds in IIA have to be replaced by non-Kähler versions of the deformed and resolved conifolds and the $\mathcal{M}$-theory lift does not possess G_2 holonomy anymore.

Hitchin (2003); they contain half-flat manifolds as a subclass. Even if we do not start with a simple torus, we encounter the same phenomenon of twisted fibers and the IIA solutions in Fig. 5 have to be replaced by non-Kähler backgrounds.

- All that said, there remains another problem: no SUSY-preserving background for D5-branes on the resolved conifold is known (without other ingredients). The one derived by Pando Zayas and Tseytlin (2000) was shown by Cvetic *et al.* (2003) to break SUSY completely; we review the argument in Appendix A.5. This problem was avoided by Becker *et al.* (2006); Knauf (2007) by constructing a IIB solution from F-theory, which in addition to the D5-branes contains D7 and O7 planes. Note that this subtlety is not visible in the local limit and does not alter the calculations by Becker *et al.* (2004); Alexander *et al.* (2005) much. Nevertheless, the whole analysis with all fluxes that are consistent with this orientifold construction was repeated by Knauf (2007).

To treat these subtleties simultaneously, we first review mirror symmetry *à la* SYZ and explain the local limit. After that we describe the orientifold setup used by Becker *et al.* (2006); Knauf (2007) and then discuss the whole duality chain, as depicted in Fig. 10.

A. Mirror symmetry between the resolved and deformed conifolds

The resolved and deformed conifolds describe asymptotically a cone over $S^2\times S^3$, but the singularity at $r=0$ is smoothed out to an S^2 or S^3, respectively. The Ricci-flat Kähler metric of the resolved conifold was derived by Candelas and de la Ossa (1990); Pando Zayas and Tseytlin (2000):

$$ds^2_{\rm res}=\tilde\gamma' d\tilde r^2+\frac{\tilde\gamma'}{4}\tilde r^2(d\tilde\psi+\cos\tilde\theta_1 d\tilde\phi_1+\cos\tilde\theta_2 d\tilde\phi_2)^2 + \frac{\tilde\gamma}{4}(d\tilde\theta_1^2+\sin^2\tilde\theta_1 d\tilde\phi_1^2)+\frac{\tilde\gamma+4a^2}{4}(d\tilde\theta_2^2 + \sin^2\tilde\theta_2 d\tilde\phi_2^2), \tag{3.1}$$

where $(\tilde\phi_i,\tilde\theta_i)$ are the usual Euler angles on S^2, $\tilde\psi=0\cdots 4\pi$ is a U(1) fiber over these two spheres, and $\tilde\gamma$ is a function[20] of $\tilde r$ that goes to zero as $\tilde r\to 0$. The constant a is called the resolution parameter because it produces a finite size prefactor for the $(\tilde\phi_2,\tilde\theta_2)$ sphere at $\tilde r=0$. This metric clearly has three isometries related to shift symmetries in the coordinates $\tilde\psi$, $\tilde\phi_1$, and $\tilde\phi_2$. These are the appropriate Killing directions as the metric was constructed to be invariant under $SU(2)\times SU(2)\times U(1)$ (Candelas and de la Ossa, 1990); see Appendix A.1 and A.3 for a brief review.

The deformed conifold metric, on the other hand, is given by (Minasian and Tsimpis, 2000; Ohta and Yokono, 2000; Papadopoulos and Tseytlin, 2001)

$$\begin{aligned}ds^2_{\rm def}=\hat\Gamma\bigg[&\frac{4d\tilde r^2}{\tilde r^2(1-\mu^4/\tilde r^4)}+(d\tilde\psi+\cos\tilde\theta_1 d\tilde\phi_1 + \cos\tilde\theta_2 d\tilde\phi_2)^2\bigg]+\frac{\hat\gamma}{4}[(\sin\tilde\theta_1^2 d\tilde\phi_1^2+d\tilde\theta_1^2)\\ &+(\sin\tilde\theta_2^2 d\tilde\phi_2^2+d\tilde\theta_2^2)]+\frac{\hat\gamma\mu^2}{2\tilde r^2}[\cos\tilde\psi(d\tilde\theta_1 d\tilde\theta_2 - \sin\tilde\theta_1\sin\tilde\theta_2 d\tilde\phi_1 d\tilde\phi_2)+\sin\tilde\psi(\sin\tilde\theta_1 d\tilde\phi_1 d\tilde\theta_2\\ &+\sin\tilde\theta_2 d\tilde\phi_2 d\tilde\theta_1)],\end{aligned} \tag{3.2}$$

with the deformation parameter μ and a similar function $\hat\gamma(\tilde r)$. $\hat\Gamma$ can be read off from Eq. (A19). This metric exhibits the same structure of a $\tilde\psi$ fibration over two spheres, but there are additional cross terms in the last two lines. We see that the U(1) symmetry associated with shifts $\tilde\psi\to\tilde\psi+k$ is absent. This is not a peculiarity of our coordinate choice but an inherent property of the deformed conifold. As discussed in Sec. II.A.2, the deformed conifold breaks the U(1) symmetry of the singular conifold (which the resolved conifold also exhibits).

Clearly, the manifolds cannot be mirrors of each other in the SYZ sense: one possesses a T^3 fibration and the other one does not. From the point of view of cohomology, mirror symmetry implies an exchange of odd and even cohomologies; more precisely, $h^{1,2}\leftrightarrow h^{1,1}$. This is expressed in the exchange of the blown-up two-cycle of the resolved conifold with the blown-up three-cycle of the deformed conifold. However, Aganagic *et al.* (2000); Hori *et al.* (2000), made an attempt to find the mirror of the resolved conifold, and the resulting manifold was found to differ from the deformed conifold in that some coordinates are elements of $\mathbb{C}^*$ instead of $\mathbb{C}$. The mirror manifold can be described by $x_1+x_2+x_1x_2e^t+1-uv=0$, where $x_i\in\mathbb{C}^*$ and $u,v\in\mathbb{C}$ and t is the size of the blown-up S^2 in the original resolved geometry. The mirror symmetry between the two manifolds only becomes

[20]The function $\tilde\gamma$ is related to the Kähler potential $\tilde{\mathcal{F}}$ as $\tilde\gamma=\tilde r^2\,\partial\tilde{\mathcal{F}}/\partial r^2$, and similarly for $\hat\gamma$ below, see Eq. (A11).

exact in the limit where the size of the S^2 and S^3 shrinks to zero.

The way to realize mirror symmetry via T-duality even in the absence of isometry directions is to go to the large complex structure limit (Strominger *et al.*, 1996) that takes us away from the singular fibers. We can still apply SYZ if the base is large compared to the T^3 fiber. If we identify $(\tilde{r},\tilde{\theta}_1,\tilde{\theta}_2)$ as the base coordinates and $(\tilde{\psi},\tilde{\phi}_1,\tilde{\phi}_2)$ as the coordinates of the T^3 fiber in the resolved metric (3.1), we can T-dualize along the latter.

It was furthermore shown by Becker *et al.* (2004) that the large complex structure limit has to be imposed "by hand," i.e., the coordinates $(\tilde{\theta}_1,\tilde{\theta}_2)$ receive a large boost. This would be a nontrivial manipulation of the metric that is not guaranteed to preserve the Calabi-Yau property, which is why the following analysis will be presented in local coordinates in which this boost amounts to a coordinate redefinition. See Chap. 2 of Knauf (2007) for details; we review the results.

We restrict our analysis to a small neighborhood of $(r_0,\langle z\rangle,\langle\phi_1\rangle,\langle\phi_2\rangle,\langle\theta_1\rangle,\langle\theta_2\rangle)$ by introducing

$$\tilde{r}=r_0+\frac{r}{\sqrt{\tilde{\gamma}_0'}},\quad \tilde{\psi}=\langle z\rangle+\frac{2z}{\sqrt{\tilde{\gamma}_0'}r_0},$$

$$\tilde{\phi}_1=\langle\phi_1\rangle+\frac{2x}{\sqrt{\tilde{\gamma}_0}\sin\langle\theta_1\rangle},\quad \tilde{\theta}_1=\langle\theta_1\rangle+\frac{2\theta_1}{\sqrt{\tilde{\gamma}_0}},$$

$$\tilde{\phi}_2=\langle\phi_2\rangle+\frac{2y}{\sqrt{(\tilde{\gamma}_0+4a^2)}\sin\langle\theta_2\rangle},$$

$$\tilde{\theta}_2=\langle\theta_2\rangle+\frac{2\theta_2}{\sqrt{(\tilde{\gamma}_0+4a^2)}},\tag{3.3}$$

where $\tilde{\gamma}_0$ is constant, namely, $\tilde{\gamma}(\tilde{r})$ evaluated at $\tilde{r}=r_0$. The coordinates $(r,z,x,y,\theta_1,\theta_2)$ describe small deviations from these expectation values and we call them "local coordinates" henceforth. In these local coordinates the metric on the resolved conifold takes a simple form (in lowest order in local coordinates):

$$ds^2=dr^2+(dz+Adx+Bdy)^2 + (dx^2+d\theta_1^2)+(dy^2+d\theta_2^2),\tag{3.4}$$

where we have defined

$$A=\sqrt{\frac{\tilde{\gamma}_0'}{\tilde{\gamma}_0}}r_0\cot\langle\theta_1\rangle,$$

$$B=\sqrt{\frac{\tilde{\gamma}_0'}{(\tilde{\gamma}_0+4a^2)}}r_0\cot\langle\theta_2\rangle.\tag{3.5}$$

Note that at lowest order in local coordinates these are simply constants. Dasgupta *et al.* (2006) showed that at linear order in r the θ_i dependences can be resummed to give precisely cot θ_i instead of $\cot\langle\theta_i\rangle$, but for illustrative purposes we stick to the simplest case of lowest order in local coordinates in this paper. For later convenience we define

$$\alpha=(1+A^2+B^2)^{-1}.\tag{3.6}$$

The metric (3.4) is easily T-dualized along x, y, and z (which correspond to the former isometry directions $\tilde{\psi},\tilde{\phi}_1,\tilde{\phi}_2$) with the help of Buscher's rules (Buscher, 1987, 1988). To illustrate the large complex structure limit, consider again Eq. (3.4), which can be written as

$$ds^2=dr^2+(dz+Adx+Bdy)^2+|d\chi_1|^2+|d\chi_2|^2,\tag{3.7}$$

with the two tori

$$d\chi_1=dx+\tau_1 d\theta_1,\quad d\chi_2=dy+\tau_2 d\theta_2.\tag{3.8}$$

In Eq. (3.4) the complex structures are simply $\tau_1=\tau_2=i$. Note that these tori are local versions of the spheres in Eq. (3.1), since locally a sphere resembles a degenerate torus.[21] The large complex structure limit is then given by letting

$$\tau_1\to i-f_1,\quad \tau_2\to i-f_2,\tag{3.9}$$

with real and large $f_{1,2}$. With the choice $f_1^2=f_2^2=\alpha/\varepsilon$ one recovers the mirror metric in type IIA [taking $\varepsilon\to 0$ and rotating the (y,θ_2) torus—see Sec. II.B of Knauf (2007)]

$$\begin{aligned}d\tilde{s}^2={}&dr^2+\alpha^{-1}(dz-\alpha Adx-\alpha Bdy)^2+\alpha(1+B^2)(dx^2\\&+d\theta_1^2)+\alpha(1+A^2)(dy^2+d\theta_2^2)+2\alpha AB[\cos\langle z\rangle\\&\times(d\theta_1 d\theta_2-dxdy)+\sin\langle z\rangle(dxd\theta_2+dyd\theta_1)],\end{aligned}\tag{3.10}$$

which matches the local limit of a deformed conifold. To see this, simply rewrite Eq. (3.2) in local coordinates similar to those of Eq. (3.3) (but A and B will differ). There is one subtle difference, though: the two spheres (tori) parametrized by (x,θ_1) and (y,θ_2) are not of the same size as in the CY metric (3.2). Knauf (2007) discussed how the mirror symmetry becomes exact in the limit where the resolution and deformation parameters approach zero, as expected by Hori *et al.* (2000); but since this means having vanishingly small two-or three-cycles ("close to the transition point"), this is a regime where the base cannot be large compared to the T^3 fiber, i.e., we cannot expect SYZ to work. This is why the large complex structure boost by hand became necessary.

B. IIB orientifold and resolved conifold

The $SL(2,\mathbb{Z})$ symmetry of IIB string theory has been proposed to have a geometrical interpretation in terms of F-theory (Vafa, 1996). Consider an elliptically fibered Calabi-Yau fourfold K which is a toroidal fiber bundle over a base B. Even though K is a smooth manifold, there will be points in the base where the fiber becomes singular and its complex structure parameter τ can have nontrivial monodromy around these points. An F-theory compactification on K refers to a compactification of type IIB on B, where the IIB axion-dilaton $\lambda=\chi+ie^{-\phi}$ is

[21]The appearance of tori instead of spheres is also consistent with the dual brane pictures as described in Sec. II.D.

identified with the geometrical parameter τ (Vafa, 1996). This leads to orientifolds in IIB (Sen, 1996, 1997); see Dabholkar (1997) for a detailed review. In our case, the base B is an orientifold of the resolved conifold[22]

$$\frac{B}{\Omega(-1)^{F_L} I_2}, \tag{3.11}$$

where F_L indicates the left-moving fermion number, Ω stands for the world-sheet parity operator, and I_2 is a target space parity that inverts both coordinates of the toroidal fiber.

In general, τ varies over the base resulting in a nonconstant field λ. However, there are possible scenarios that allow for a constant solution of λ (Dasgupta and Mukhi, 1996; Sen, 1996, 1997). These solutions are characterized by 24 singularities in the function describing the elliptic fibration. In the special case where these singularities appear at four different locations with a multiplicity of six, λ is constant. The singularities are interpreted as 24 seven-branes in F-theory and give rise to four orientifold seven-planes with four D7-branes on top of each (to cancel their charges), located at the four orientifold fixed points.

If we now wrap D5-branes on the S^2 of the resolved geometry, we obtain an intersecting D5/D7-brane scenario on a IIB orientifold. This preserves supersymmetry because it can form a bound state (Gava *et al.* 1997). The bound-state metric was derived by Dasgupta *et al.* (2007) and agrees with the local limit used here. We will not make explicit use of the bound-state description in order to keep this section illustrative. The metric of the base B has to resemble the resolved conifold locally, but globally it will also contain singularities that correspond to the 7-branes. Adding D5-branes creates warp factors. To incorporate these effects we make the following generic ansatz for the base:

$$\begin{aligned} ds^2 = {} & h_0(\tilde{r})d\tilde{r}^2 + h_1(\tilde{r})(d\tilde{\psi} + \cos\tilde{\theta}_1 d\tilde{\phi}_1 + \cos\tilde{\theta}_2 d\tilde{\phi}_2)^2 \\ & + h_2(\tilde{r})d\tilde{\theta}_1^2 + h_3(\tilde{r})\sin^2\tilde{\theta}_1 d\tilde{\phi}_1^2 \\ & + h_4(\tilde{r})d\tilde{\theta}_2^2 + h_5(\tilde{r})\sin^2\tilde{\theta}_2 d\tilde{\phi}_2^2, \end{aligned} \tag{3.12}$$

which allows, in particular, for the two spheres to be asymmetric and squashed. This ansatz is motivated by the idea that in the absence of D-branes and fluxes we should recover the Kähler metric. Also, for $\tilde{r}\to\infty$ the warp factors should approach 1, so we suppress any $\theta_{1,2}$ dependence in the functions h_i although it would not influence the following local analysis. We again define local coordinates and absorb the prefactors $h_i(\tilde{r})$, which gives the same simple form of the local metric

$$\begin{aligned} ds^2 = {} & dr^2 + (dz + A dx + B dy)^2 \\ & + (dx^2 + d\theta_1^2) + (dy^2 + d\theta_2^2), \end{aligned} \tag{3.13}$$

where we kept the names A and B for the constants, but they are now more generically defined. Apart from this redefinition of A and B, the mirror symmetry analysis will be completely unchanged from Sec. III.A. The mirror is then given by Eq. (3.10), which is the local limit of a deformed conifold. We show shortly the consistency of this construction with an orientifold in IIA.

We need our IIB background to be invariant under the orientifold action, which is given by $\Omega(-1)^{F_L}I_{ij}$. Since the IIB background is invariant under $\Omega(-1)^{F_L}$, we require the metric to be invariant under space-time parity I_{ij} of the two coordinates x_i and x_j over which the fiber degenerates. Furthermore, we require the IIA orientifold metric that results after three T-dualities to resemble the deformed conifold. This severely restricts the possibilities for x_i and x_j.

The choice we adopt is that the F-theory torus is fibered nontrivially over the two-torus (x,θ_1). This is actually the only choice that preserves all the symmetries we require (Becker *et al.*, 2006). The D5-branes are wrapped on the two-torus (or sphere) given by (y,θ_2) [recall from Eq. (3.1) that this is the sphere that remains finite as $\tilde{r}\to 0$]. This means that under three T-dualities

$$\text{IIB on } \frac{B}{\Omega(-1)^{F_L} I_{x\theta_1}} \xrightarrow{T_{xyz}} \text{IIA on } \frac{B'}{\Omega(-1)^{F_L} I_{yz\theta_1}}.$$

We find the following system of D-branes and orientifold planes in type IIB:

D5:	0	1	2	3	–	–	–	–	y	θ_2
D7/O7:	0	1	2	3	r	z	–	–	y	θ_2

which turns after three T-dualities along x, y, and z into IIA with

D6:	0	1	2	3	–	z	x	–	–	θ_2
D6/O6:	0	1	2	3	r	–	x	–	–	θ_2

One can see that the metric (3.13) is indeed invariant under $I_{x\theta_1}$ (remember that A contains $\cot\langle\theta_1\rangle$, so it is odd under this parity) and the mirror will be symmetric under $I_{yz\theta_1}$ after we impose some restrictions on the B-field components.

Note that the D7-branes extend along the noncompact direction r. A similar brane configuration on the *singular* conifold was considered by Ouyang (2004), but it was not constructed from F-theory.[23] It was shown there how strings stretching between D7- and D5-branes (or D6 and D6) give rise to a global symmetry. It is not a gauge symmetry because of the large volume factor associated with the D7-branes extending along the noncompact direction r. We call the D7- or D6-branes that originate from F-theory "flavor branes" to distinguish them from the D5- or D6-branes that carry the gauge theory.

[22] B will not be a Calabi-Yau threefold anymore, since K is a Calabi-Yau manifold, but it is still Kähler (Becker *et al.*, 2006).

[23] This analysis has recently been extended to the resolved conifold (Dasgupta *et al.*, 2008)

C. Mirror symmetry with NS-NS flux and "non-Kähler deformed conifold"

Having argued that Eq. (3.4) or equivalently Eq. (3.13) is the correct local metric for D7/O7- and D5-branes on the resolved conifold, we now turn to the first link in the duality chain. As mentioned, the mirror symmetry analysis from Sec. III.A is changed once we take fluxes into account. The full RR spectrum that is consistent with the IIB orientifold was studied by Knauf (2007). We only focus on the NS-NS sector here, as it alone is relevant for the geometry. It should also have become clear that we can only present a local analysis here, for two reasons: (1) The mirror symmetry argument between resolved and deformed conifold fails globally; and (2) we do not know the full F-theory solution, in other words, the functions $h_i(\tilde{r})$ in Eq. (3.12) remain unknown.

For the NS-NS flux we make the most generic ansatz that is consistent with our orientifold, with one exception: we only allow for electric NS-NS flux [magnetic NS-NS flux leads in general to nongeometrical solutions (Hellerman *et al.*, 2004; Flournoy *et al.*, 2005; Hull 2005; Shelton *et al.*, 2005; Dabholkar and Hull 2006)], i.e., B-field components that have only one leg along T-duality directions:[24]

$$B_2^{\mathrm{IIB}} = b_{z\theta_1} dz \wedge d\theta_1 + b_{x\theta_2} dx \wedge d\theta_2 + b_{y\theta_1} dy \wedge d\theta_1. \tag{3.14}$$

In general, the coefficients $b_{z\theta_1}$, $b_{x\theta_2}$, and $b_{y\theta_1}$ can depend on all base coordinates (r,θ_1,θ_2) to preserve the background's isometries.

This B field has nontrivial consequences when we perform T-dualities along x, y, and z. We will not merely find a local version of the deformed conifold, but a manifold with *twisted fibers* that is clearly the local limit of a non-Kähler version of the deformed conifold.

The reason why mirror symmetry with NS-NS field gives rise to a non-Kähler manifold is easy to illustrate in the SYZ picture. T-duality mixes the B field and metric. In the presence of NS-NS flux, Buscher's rules (Buscher, 1987, 1988) read

$$\tilde{G}_{yy} = \frac{1}{G_{yy}}, \quad \tilde{G}_{\mu y} = \frac{B_{\mu y}}{G_{yy}},$$

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}, B_{\nu y}}{G_{yy}},$$

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}},$$

$$\tilde{B}_{\mu y} = \frac{G_{\mu y}}{G_{yy}}, \tag{3.15}$$

so cross terms in the metric are traded against the corresponding components in B_2 and vice versa. Therefore the T^3 fibers acquire a twisting by B_2-dependent one-forms under T-duality that we denote by

$$dx \to d\hat{x} = dx - b_{x\theta_2} d\theta_2,$$

$$dy \to d\hat{y} = dy - b_{y\theta_1} d\theta_1,$$

$$dz \to d\hat{z} = dz - b_{z\theta_1} d\theta_1. \tag{3.16}$$

This does not mean that $d\hat{x}$, etc. are exact; in general the B field is nonconstant. Apart from this modification, we perform the same steps as in Sec. III.A: we boost the complex structure as in Eq. (3.9) and take the limit $\varepsilon \to 0$.

Then we find the mirror in IIA to be

$$\begin{aligned} d\tilde{s}^2 = {} & dr^2 + \alpha^{-1}[d\hat{z} - \alpha A d\hat{x} - \alpha B d\hat{y}]^2 + \alpha(1+B^2)[d\theta_1^2 \\ & + d\hat{x}^2] + \alpha(1+A^2)[d\theta_2^2 + d\hat{y}^2] + 2\alpha AB \cos\langle z\rangle \\ & \times [d\theta_1 d\theta_2 - d\hat{x} d\hat{y}] + 2\alpha AB \sin\langle z\rangle [d\hat{x} d\theta_2 \\ & + d\hat{y} d\theta_1], \end{aligned} \tag{3.17}$$

with α defined in Eq. (3.6). We therefore conjecture the local resolved conifold to be mirror dual to a local non-Kähler deformed conifold with twisted fibers that make this metric inherently non-Kähler.

In the absence of a B field this is a Kähler background, since in this local version all coefficients in the metric are constants. With a B-field-dependent fibration the fundamental two-form will in general no longer be closed because it will depend on derivatives of b_{ij}. A more thorough analysis of this geometry was attempted by Knauf (2007), but it remains somewhat incomplete because we lack the knowledge of a global background. Strictly speaking, we only know the metric in a small patch and have no global information about the manifold. We can, however, make some predictions on what we expect for the global solution, since supersymmetry imposes restrictions on allowed non-Kähler manifolds.

We were able to show that this metric admits a symplectic structure, but we were not able to find a half-flat structure [with quite a generic ansatz for the almost complex structure, see Chap. 4 in Knauf (2007)]. This is not in contradiction with the results from Gurrieri *et al.* (2003), where the mirror of a torus was found to be half-flat. Our IIB starting background is not a Calabi-Yau manifold; it is at best a conformal Calabi-Yau manifold. Conformal Calabi-Yau manifolds are complex manifolds,[25] and there is ample evidence in the literature—see, e.g., Chuang *et al.* (2007); Jeschek (2004)—that the mirror of a complex manifold is symplectic (which can be half-flat at the same time, but does not have to be). Furthermore, our background includes RR flux and therefore has to fulfill different SUSY re-

[24]Without loss of generality we do not include components involving dr since components of the three-form field strength like $dr \wedge dx \wedge d\theta_i$ can be obtained from $\partial_r b_{xi}(r) dr \wedge dx \wedge d\theta_i$.

[25]This is obvious from their SU (3) torsion classes. See, e.g., Chiossi and Salamon (2002); Lopes Cardoso *et al.* (2003).

quirements than a purely geometrical background, i.e., it does not lift to a manifold with G_2 holonomy, as half-flat manifolds do, but only to one with G_2 structure.

D. Completing the duality chain

With the metric (3.17) we are now ready to follow the duality chain through the M-theory flop and back to type IIB.

1. M-theory lift

Note that there is no longer any NS-NS flux in our IIA background since it was completely "used up" under T-duality and became part of the metric, but there is RR flux, dual to the RR three- and five-forms of the original IIB background. This flux means that we have to lift the ten-dimensional solution on a twisted fiber instead of a circle and that there will be G flux in M-theory. Becker *et al.* (2004) and Knauf (2007) showed that the RR flux does not change during the flop, so we only consider the RR one-form potential that enters into the 11-dimensional metric. This potential can be written as

$$C_1 = \Delta_1 d\hat{x} - \Delta_2 d\hat{y}, \tag{3.18}$$

where Δ_1 and Δ_2 depend on the assumptions made about the IIB RR forms in the beginning. [Recall the definition of the twisted fibers $d\hat{x}$, $d\hat{y}$, $d\hat{z}$ from Eq. (3.16).] As usual in the presence of a gauge field C_1 and dilaton ϕ [which remains unchanged under three T-dualities (Becker *et al.* 2004; Knauf, 2007)], type IIA on a manifold X is lifted to M-theory on a twisted circle via

$$ds^2_{\mathcal{M}} = e^{-2\phi/3} ds^2_X + e^{4\phi/3}(dx_{11} + C_1)^2, \tag{3.19}$$

with x_{11} parametrizing the extra dimension with radius R: $x_{11} \in (0, 2\pi R)$. In the limit $R \to 0$ we recover ten-dimensional IIA theory. The gauge fields in our case enter into the metric so that it becomes

$$\begin{aligned} ds^2_{\mathcal{M}} = {} & e^{-2\phi/3} dr^2 + e^{-2\phi/3}\alpha^{-1}(dz - \alpha A d\hat{x} - \alpha B d\hat{y})^2 \\ & + e^{4\phi/3}(dx_{11} + \Delta_1 d\hat{x} - \Delta_2 d\hat{y})^2 + e^{-2\phi/3}[\alpha(1+B^2) \\ & \times (d\theta_1^2 + d\hat{x}^2) + \alpha(1+A^2)(d\theta_2^2 + d\hat{y}^2)] \\ & + e^{-2\phi/3} 2\alpha A B[\cos\langle z\rangle(d\theta_1 d\theta_2 - d\hat{x} d\hat{y}) + \sin\langle z\rangle \\ & \times (d\hat{x} d\theta_2 + d\hat{y} d\theta_1)]. \end{aligned} \tag{3.20}$$

The two fibration terms in the first line are of special interest. They are similar in structure, even more so if one introduces new coordinates ψ_1 and ψ_2 via

$$dz = d\psi_1 - d\psi_2 \quad \text{and} \quad dx_{11} = d\psi_1 + d\psi_2. \tag{3.21}$$

This choice is particularly convenient for performing the flop.

2. Flop

The metric of all three conifold geometries can be written in terms of two sets of SU(2) left-invariant one-forms; see Appendix A.4. In terms of Euler angles on two S^3s these left-invariant one-forms are given as[26]

$$\begin{aligned} \sigma_1 &= \cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1, \\ \sigma_2 &= -\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1, \\ \sigma_3 &= d\psi_1 + \cos\theta_1 d\phi_1, \\ \Sigma_1 &= \cos\psi_2 d\theta_2 - \sin\psi_2 \sin\theta_2 d\phi_2, \\ \Sigma_2 &= -\sin\psi_2 d\theta_2 - \cos\psi_2 \sin\theta_2 d\phi_2, \\ \Sigma_3 &= d\psi_2 - \cos\theta_2 d\phi_2. \end{aligned} \tag{3.22}$$

The Calabi-Yau metrics for resolved and deformed conifolds are written in these vielbeins as (with the definition $\psi = \psi_1 - \psi_2$)

$$\begin{aligned} ds^2_{\text{def}} = {} & A^2 \sum_{i=1}^{2} (\sigma_i - \Sigma_i)^2 + B^2 \sum_{i=1}^{2} (\sigma_i + \Sigma_i)^2 \\ & + C^2(\sigma_3 - \Sigma_3)^2 + D^2 dr^2, \\ ds^2_{\text{res}} = {} & \tilde{A}^2 \sum_{i=1}^{2} (\sigma_i)^2 + \tilde{B}^2 \sum_{i=1}^{2} (\Sigma_i)^2 + \tilde{C}^2(\sigma_3 - \Sigma_3)^2 \\ & + \tilde{D}^2 dr^2, \end{aligned} \tag{3.23}$$

with the coefficients A, B, etc. determined by Kähler and Ricci flatness conditions, see Eqs. (A30) and (A19). This clearly shows that the deformed conifold is completely symmetric under a $\mathbb{Z}_2$ that acts as $\sigma_i \leftrightarrow \Sigma_i$, whereas the resolved conifold does not have this symmetry since $\tilde{A} \neq \tilde{B}$.

Both geometries can be lifted to a G_2 holonomy manifold, a cone over $S^3 \times \tilde{S}^3$, where S^3 describes a sphere with vanishing size at the tip of the cone, whereas $\tilde{S}^3$ remains finite. As described in Sec. II.C.2, the flop corresponds to an exchange $S^3 \leftrightarrow \tilde{S}^3$. In terms of vielbeins, the flop simply amounts to an exchange $\sigma_i \leftrightarrow \Sigma_i$, since each S^3 is described by a set of SU(2) left-invariant one-forms; but note that this also implies that the U(1) fiber along which one reduces to $d=6$ is contained in either σ_3 or Σ_3, i.e., it is given by either ψ_1 or ψ_2, but not by $x_{11} = \psi_1 + \psi_2$ as defined in Eq. (3.21).

This discussion was for Calabi-Yau metrics. The "non-Kähler deformed conifold" found in Sec. III.C does not have two S^2s of the same size. We therefore need to use

[26] We follow the conventions of Cvetic *et al.* (2002a, 2002b), but our notation differs by $\phi_2 \to -\phi_2$.

a more general ansatz. Cvetic *et al.* (2002a, 2002b) showed that there exists a one-parameter family of G_2-holonomy metrics (that includes the lift of the resolved and deformed conifolds[27]) of the form

$$ds^2 = dr^2 + a^2[(\Sigma_1 + \xi\sigma_1)^2 + (\Sigma_2 + \xi\sigma_2)^2] + b^2(\sigma_1^2 + \sigma_2^2) + c^2(\Sigma_3 - \sigma_3) + f^2(\Sigma_3 + g_3\sigma_3)^2, \quad (3.24)$$

where $\psi_1 - \psi_2$ was identified as the eleventh direction by Cvetic *et al.* (2002b), i.e., the limit $c \to 0$ corresponds to a reduction to ten dimensions. This metric has less symmetry than the metric in Atiyah *et al.* (2001), for which the flop was discussed. Note that the parameter ξ describes an asymmetry between the two S^2 in a deformed metric. It seems therefore appropriate to adopt this ansatz for our purposes.

Of course our metric (3.20) does not describe $S^3 \times S^3$ principal orbits. Recall that our coordinates x, y, z are nontrivially fibered due to the B-field components which entered into the metric. We can nevertheless adopt the ansatz (3.24) with a different definition of vielbeins:

$$\sigma_1 = \cos\psi_1 d\theta_1 + \sin\psi_1 d\hat{x},$$

$$\sigma_2 = -\sin\psi_1 d\theta_1 + \cos\psi_1 d\hat{x},$$

$$\sigma_3 = d\psi_1 - \alpha A d\hat{x},$$

$$\Sigma_1 = \cos\psi_2 d\theta_2 - \sin\psi_2 d\hat{y},$$

$$\Sigma_2 = -\sin\psi_2 d\theta_2 - \cos\psi_2 d\hat{y},$$

$$\Sigma_3 = d\psi_2 + \alpha B d\hat{y}. \quad (3.25)$$

The flop has to be different from the case considered by Atiyah *et al.* (2001), since we do not want to exchange the roles of ψ_1 and ψ_2, but exchange x_{11} and z as these are the naturally fibered coordinates in Eq. (3.20). Furthermore, we have the asymmetry factor ξ, so that our metric does not exhibit the $\mathbb{Z}_2$ symmetry $\sigma_i \leftrightarrow \Sigma_i$ as the lift of the Calabi-Yau deformed conifold does. As explained in Sec. III.C of Knauf (2007), the flop in our conventions corresponds to

$$\sigma_3 - \Sigma_3 \leftrightarrow \sigma_3 + \Sigma_3,$$

$$\sigma_i \to \Sigma_i,$$

$$\Sigma_i \to \xi(\sigma_i - \Sigma_i), \quad (3.26)$$

with $i = 1,2$. This results in the following metric after the flop:[28]

$$ds^2 = e^{-2\phi/3}dr^2 + e^{-2\phi/3}\frac{\alpha A^2 B^2}{1 + A^2}(d\theta_1^2 + d\hat{x}^2) + e^{-2\phi/3}\frac{1}{1 + A^2}(d\theta_2^2 + d\hat{y}^2) + e^{-2\phi/3}\alpha^{-1}(dx_{11} - \alpha A d\hat{x}\alpha B d\hat{y})^2 + e^{4\phi/3}(dz - \alpha A d\hat{x} - \alpha B d\hat{y})^2, \quad (3.27)$$

which can now be reduced along the same x_{11} to the IIA background after transition.

3. M-theory reduction

Dimensional reduction on the same x_{11} does not give the same metric as before the flop. Instead, we find

$$ds^2_{\mathrm{IIA}} = dr^2 + e^{2\phi}[(dz - b_{z\theta_1}d\theta_1) - \alpha A(dx - b_{x\theta_2}d\theta_2) - \alpha B(dy - b_{y\theta_1}d\theta_1)]^2 + C[d\theta_1^2 + (dx - b_{x\theta_2}d\theta_2)^2] + D[d\theta_2^2 + (dy - b_{y\theta_1}d\theta_1)^2], \quad (3.28)$$

where the fibration structure is given explicitly as a reminder that the original IIB B field is contained in this metric. We introduce another set of symbols for the metric components giving the spheres:

$$C = \frac{\alpha A^2 B^2}{1 + A^2}, \quad D = \frac{1}{1 + A^2},$$

$$\text{and } \alpha_0^{-1} = CD + \alpha^2 e^{2\phi}(CB^2 + DA^2) \quad (3.29)$$

in analogy with the definition of A, B, and α in Eqs. (3.5) and (3.6).

This manifold is non-Kähler in precisely the same spirit as the "non-Kähler deformed conifold" before the flop (3.17). Comparing it to Eq. (3.13) shows that it also possesses the characteristic metric of a resolved conifold (locally). We therefore call this manifold a "non-Kähler resolved conifold" and claim it to be transition dual to the metric (3.17). The latter is a manifold with D6-branes wrapping a three-cycle, whereas the former describes a blown up two-cycle with fluxes on it.

4. The final mirror

We can now "close the duality chain" by performing another mirror which takes us back to IIB. We expect to recover a Kähler background similar to the Klebanov-Strassler model (Klebanov and Strassler, 2000), since we started with a Kähler manifold in IIB. In principle the analysis follows the same steps as when T-dualizing the resolved conifold from IIB to IIA without NS-NS flux in Sec. III.A, only now our starting metric is the non-Kähler version of the resolved conifold (3.28).

T-dualizing this background along x, y, and z is tedious but nevertheless straightforward. See Sec. III.D of Knauf (2007) for the details. We observe that the same mechanism that converted the B field into metric cross terms now serves to restore $b_{x\theta_2}$, $b_{y\theta_1}$, and $b_{z\theta_1}$ as the B

[27]In particular, Cvetic *et al.* (2002b) solved the differential equations for the r-dependent coefficients a, b, c, f, g_3, and ξ and showed that the resulting Kähler form looks like that for the resolved conifold. It was not considered how a flop between resolved and deformed conifolds can be performed.

[28]Here we used an explicit gauge choice for the RR one-form: $C_1 = -\alpha A d\hat{x} + \alpha B d\hat{y}$.

field and the metric is completely free of any B-field-dependent fibration. The final IIB metric after transition is

$$d\tilde{s}^2_{IIB} = dr^2 + \frac{e^{-2\phi}}{\alpha_0 CD}[dz + \alpha_0 \alpha A D e^{2\phi} dx + \alpha_0 \alpha B C e^{2\phi} dy]^2 + \alpha_0 (D + \alpha^2 B^2 e^{2\phi}) \times (dx^2 + \zeta d\theta_1^2) + \alpha_0 (C + \alpha^2 A^2 e^{2\phi})(dy^2 + d\theta_2^2) + 2\alpha_0 \alpha^2 A B e^{2\phi}[\cos\langle z\rangle (d\theta_1 d\theta_2 - dx dy) + \sin\langle z\rangle (dx d\theta_2 + dy d\theta_1)], \quad (3.30)$$

where we have introduced the "squashing factor"

$$\zeta = \frac{C - \alpha^2 A^2 \tilde{\beta}_1^2}{\alpha_0 (D + \alpha^2 B^2 e^{2\phi})}. \quad (3.31)$$

We therefore find that the final IIB metric after the flop (3.30) is not quite a deformed conifold due to the asymmetry in the (x, θ_1) sphere or torus. In the local version presented above it is, of course, Kähler (all coefficients are constant), but we cannot make any statement about the global behavior. Remember that we do not have the global metric for our starting background with D7/O7- and D5-branes.

The cross terms in the metric (3.28) are now converted into B-field components using the same mechanism of Buscher's rules (3.15). One recovers the B field (3.14) we started with in IIB before the transition. The same holds true for the RR flux. The flux does not change under geometric transition (Becker *et al.*, 2004; Knauf, 2007), confirming the picture advocated by Vafa (2001).

In conclusion, we have shown that we can construct a new pair of IIA string theory backgrounds that are non-Kähler and deviate from the deformed and resolved conifolds in a precise manner: the T^3 fibers are twisted by the B field. They are related by a geometric transition because their respective lifts to M-theory are related by a flop. The IIB backgrounds (3.13) and (3.30), on the other hand, are Kähler and are also transition dual, based on mirror symmetry.

IV. DISCUSSION

We have presented a supergravity analysis confirming Vafa's duality chain, see Fig. 5, with the inclusion of non-Kähler manifolds in type IIA. These manifolds are non-Kähler due to a twisting of their fibers by the B field that is introduced via T-duality. Thus, they should fall into the classification of T-folds (Hull, 2005), where the transition functions of a manifold are allowed to take values in the T-duality group $O(d,d;Z)$ or into generalized complex geometries (Gualtieri, 2003; Hitchin, 2003). They are only trivial examples though, as we have only considered T-duality with a B field of (1,1)-type (also called "electric"). Thus, our backgrounds are still geometric, i.e., true manifolds rather than T-folds, and their generalized complex structure is not of a mixed type, but purely symplectic [as the IIB background we started with was complex and it is by now well established that mirror symmetry with electric NS-NS flux connects complex and symplectic manifolds (Jeschek, 2004; Chuang *et al.*, 2007)]. A symplectic structure was found in terms of SU(3) torsion classes; see Becker *et al.* (2006); Knauf (2007) for details.

We still lack a global description for these manifolds, as mirror symmetry between the resolved and deformed conifolds forced us to adopt a local limit. In addition, there is no known global description for our IIB starting background: D5-branes wrapped on the resolution of the conifold. The Pando Zayas–Tseytlin solution (Pando Zayas and Tseytlin, 2000) suggested for this case explicitly breaks supersymmetry, as explained in Appendix A.5. We circumvented this problem by viewing the IIB background as an orientifold stemming from F-theory. This background contains additional D7-branes and O7-planes, but allows for a supersymmetric background with D5-branes. We left the ansatz for the fluxes generic, as long as they are invariant under the orientifold operation. It would of course be more satisfying to find a global background with all these properties and explicitly confirm its supersymmetry.

Once we introduce additional D7-branes, one may ask two questions: Can the D5- and D7-branes form a supersymmetric bound state? And do these branes introduce additional symmetry into the dual gauge theory? Both questions have been answered affirmatively; see Dasgupta *et al.* (2007) and Becker *et al.* (2006), respectively. Dasgupta *et al.* (2007) found the metric of a D5/D7 bound state on a resolved conifold geometry exactly, in the sense that all back-reactions neglected in earlier papers were taken into account. Difficulties inherent in solving the equations of motion were circumvented by a U-duality chain which took as its starting point a D1/F1 bound state or (m,n) string. The result matches earlier conjectures in certain limits, completing the supergravity description of the Vafa duality chain starting point as found from the F-theory setup. The 7-branes from F-theory lead to a global symmetry group depending on the special degeneration of the F-theory torus over the base. Becker *et al.* (2006) argued the symmetry group to be $SU(2)^{16}$. This was due to the fact that in IIB every orientifold fixed point contributes four D7-branes giving rise to an SO(8) that is broken by Wilson lines to $SO(4) \times SO(4) \simeq SU(2)^4$. This is consistent with the IIA orientifold that contains eight fixed points, each accompanied by two D6-branes. The symmetry group generated by eight stacks of D6-branes is therefore $SO(4)^8 \simeq SU(2)^{16}$. Dasgupta and Mukhi (1996) showed that one can even construct the exceptional gauge groups E_6, E_7, and E_8, which are of particular interest for grand unified theories (GUTs). The D7-branes do not give rise to a gauge symmetry because they extend along the noncompact radial direction and therefore suffer from a large-volume suppression. A similar setup was suggested by Ouyang (2004), but there the flavor branes did not have

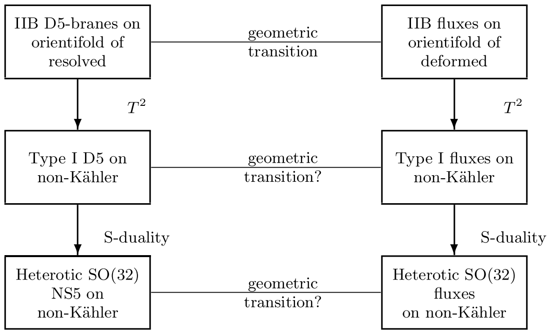


FIG. 11. The heterotic duality chain. Following the arrows one can construct non-Kähler backgrounds in type I and heterotic theory that are dual to the type IIB backgrounds before and after transition. This implies that the new backgrounds are also transition duals in some sense.

an *F*-theory origin and were treated in a probe approximation.

The superpotential in our flux backgrounds also remains to be calculated. One remarkable result from Vafa (2001) was to show that the flux-generated superpotential does indeed agree (at lowest order) with the Veneziano-Yankielowicz superpotential for super-Yang-Mills (SYM) theory. This superpotential receives corrections from field theory (Farrar *et al.*, 1998; Cerdeno *et al.*, 2003) as well as from string theory considerations (Dijkgraaf and Vafa, 2002). One issue we would like to address in the future is whether a generalized superpotential (taking the non-Kähler structure of the target manifold into account) might be better suited to reproduce these corrections. Furthermore, one should study the additional global symmetry. The field theory analog to the Veneziano-Yankielowicz superpotential for an SU(N) theory with matter is given by the Affleck-Dine-Seiberg superpotential (Affleck *et al.*, 1984). It would be interesting to see if we could reproduce this superpotential [as in the case of a Calabi-Yau orbifold $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$ (Imeroni and Lerda, 2003)] or if we would find an extension to it when including the flux due to D7-branes. We need the precise supergravity solution to see which fluxes are actually turned on. In our setup, the charge of the D7-branes is immediately canceled by the orientifold planes. We would have to move the orientifold planes away from the flavor branes to observe their effect. This would lead to nonperturbative corrections.

Another generalization of our duality chain was suggested by Alexander *et al.* (2005). One can exploit the idea of a IIB orientifold to go to the orientifold limit, i.e., type I. Another S-duality takes us to heterotic SO(32) and we find two non-Kähler backgrounds that must in a certain sense be dual to each other, since they are individually U-dual to the IIB backgrounds for which we confirmed the geometric transition picture, see Fig. 11. The orientifold operator we have to choose here is different from the one used in Sec. III.B and the heterotic backgrounds will therefore not resemble non-Kähler versions[29] of conifolds anymore. See Alexander *et al.* (2005) or Knauf (2007) for details.

The interpretation of this duality is still unresolved. Since heterotic string theory does not contain any open strings, the interpretation as an open-closed duality fails. We think this is a case where the geometric transition changes the vector bundle in a way that it requires the introduction of NS5-branes as localized sources for anomaly cancellation (before transition). It would also be interesting to study the underlying topological (0,2) theory. Results from Kapustin (2005); Sharpe (2005); Witten (2005); and Katz and Sharpe (2006) should prove useful here.

We would also like to gain a better understanding of how our IIA non-Kähler backgrounds fit into generalized complex geometry. We have not explicitly shown that our manifolds are (twisted) generalized Calabi-Yau manifolds [i.e., possess a (twisted) closed pure spinor (Grana *et al.*, 2004a, 2004b, 2005) or have SU(3) $\times$SU(3) structure; see Grana *et al.* (2007), and references therein], which is the most general condition for all non-Kähler backgrounds with fluxes to preserve supersymmetry [there is a second pure spinor, which is not closed, but its derivative is proportional to the RR field strengths; NS-NS flux and dilaton enter into the (twisted) d operator]. With much progress having been made in the field of generalized topological sigma models (Kapustin and Li, 2004; Lindstrom *et al.*, 2005) and topological string theory (Kapustin, 2004; Kapustin and Li, 2005; Pestun, 2007) one could hope to repeat the analysis of Bershadsky *et al.* (1994) on these kinds of backgrounds and show agreement of the open and closed *generalized* topological partition functions. This is complicated by the fact that we also have RR flux in our model, whose role in topological string theory is still not well understood.

To make contact with phenomenology, one would need to compactify the six-dimensional manifolds mentioned here. Since our analysis was performed in a local limit anyway, it would still hold if the conifold bulk was cut off and replaced with a compact Calabi-Yau manifold. This is similar in spirit to the cosmological models working with the "warped Klebanov-Strassler throat." Indeed, once we compactify we would also be forced to introduce extra objects for charge cancellation. If these were anti-D-branes, we would find ourselves in the realm of nonsupersymmetric, potentially viable cosmological models. Another phenomenologically appealing direction is the study of more realistic gauge groups, like the standard model or simply QCD, in terms of geometric transitions. For the future one might hope that open-closed duality can teach us something about the strong coupling behavior of confining gauge theories.

[29]These manifolds are non-Kähler because we perform two T-dualities before S-dualizing to heterotic. The aim is to "use up" all the NS-NS field of the IIB theory, so that only RR flux is left, which becomes NS-NS flux in the heterotic theory and to convert the D5/D7 system into D5/D9-branes.

ACKNOWLEDGMENTS

We thank our collaborators on the various projects that entered into this review, Stephon Alexander, Katrin Becker, Melanie Becker, Keshav Dasgupta, Marc Grisaru, Sheldon Katz, and Radu Tatar, and we are very grateful to Keshav Dasgupta, Andrew Frey, and Igor Klebanov for many helpful discussions. We thank Arvind Murugan for illuminating correspondence. R.G. was supported in part by a Chalk-Rowles fellowship. The work of A.K. was supported by an NSERC grant. A.K. thanks the Galileo Galilei Institute for Theoretical Physics in Florence, where this work was completed, and the organizers of the workshop "String- and M-theory approaches to particle physics and cosmology" for creating an enjoyable and stimulating atmosphere, as well as the INFN for partial financial support.

APPENDIX: CONIFOLDS

The (singular) conifold is a cone over a five-dimensional base and is a Calabi-Yau threefold. There are two "relatives" of the conifold, in which the singularity has been smoothed out in two different ways: one is called a "resolved conifold," with a blown up S^2 at the tip of the cone; the other is the "deformed conifold," in which the singularity is blown up into an S^3. All three manifolds look asymptotically the same, like a cone over $S^2 \times S^3$. Their metrics then take the form

$$ds^2 = dr^2 + r^2 ds^2_{\text{base}}. \tag{A1}$$

Candelas and de la Ossa (1990) showed that all three possess a Kähler metric and are Ricci flat and that one can pass continuously from one geometry to another. This is despite the fact that they are topologically different, which is seen, e.g., from the Euler numbers: $\chi(S^3) = 0$, $\chi(\text{point}) = 1$, and $\chi(S^2) = 2$. This transition is called a "conifold transition" and can be visualized as in Fig. 3. The deformed conifold on the left approaches the singular conifold as the S^3 shrinks to zero size and the resolved conifold is obtained by blowing up the orthogonal S^2.

We now review the symmetry properties and Ricci-flat Kähler metrics on all three manifolds, as well as discuss their complex structures. Useful references are Candelas and de la Ossa (1990); Minasian and Tsimpis (2000); Pando Zayas and Tseytlin (2000); and Papadopoulos and Tseytlin (2001).

1. The singular conifold

Just as a two-dimensional cone is embedded in real three-dimensional space as $x^2 + y^2 - z^2 = 0$, a real six-dimensional conifold can be expressed in terms of three complex coordinates, and is therefore embedded in $\mathbb{C}^4$ as

$$\sum_{i=1}^{4} z_i^2 = 0. \tag{A2}$$

This describes a surface which is smooth apart from a singularity at $z_i = 0$. The space has an $SO(4) \approx SU(2) \times SU(2)$ symmetry by which the z_i are rotated into each other, and a $U(1)$ which rotates all z_i by the same phase. There is also a scaling symmetry given by the transformation $z_i \to t z_i, t \in \mathbb{C}^\star$. By choosing $z_i = x_i + \imath y_i$, we can rewrite Eq. (A2) as

$$\sum_{i=1}^{4} x_i y_i = 0, \quad \sum_{i=1}^{4} (x_i^2 - y_i^2) = 0. \tag{A3}$$

The x_i describe a three-sphere for any y_i, with vanishing radius at $y_i = 0$, and the coordinates y_i are orthogonally fibered to them. Therefore, the space is given by $T^* S^3$.

To find the base of the conifold we take its intersection with a three-sphere of radius r:

$$\sum_{i=1}^{4} |z_i|^2 = \sum_{i=1}^{4} (x_i^2 + y_i^2) = r^2, \tag{A4}$$

which removes the scaling symmetry $z_i \to t z_i$. The resulting five-dimensional space is a Sasaki-Einstein manifold[30] called $T^{1,1}$. Together with Eq. (A3) we see that Eq. (A4) gives a three-sphere of radius $r/\sqrt{2}$ parametrized by x_i, whereas the y_i describe a two-sphere fibered over the S^3. Since all such fibrations are trivial, the topology of the base $T^{1,1}$ is $S^2 \times S^3$ (Candelas and de la Ossa, 1990).

$T^{1,1}$ also has a coset space description as $SU(2) \times SU(2)/U(1)$. To see this, define

$$W = \frac{1}{\sqrt{2}} \sum_n \sigma^n z_n, \tag{A5}$$

with σ^n the Pauli matrices for $n = 1,2,3$ and $\sigma^4 = \imath \mathbf{1}$ so that

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} z_3 + \imath z_4 & z_1 - \imath z_2 \\ z_1 + \imath z_2 & -z_3 + \imath z_4 \end{pmatrix}.$$

Then the defining equation for the conifold (A2) and the base (A4) can be written as

$$\det W = 0, \tag{A6}$$

$$\operatorname{tr} W^\dagger W = r^2. \tag{A7}$$

By rescaling $Z = W/r$ these become

$$\det Z = 0,$$

$$\operatorname{tr} Z^\dagger Z = 1.$$

Given a particular solution Z_0, say $Z_0 = \frac{1}{2}(\sigma_1 + \imath \sigma_2)$, the general solution can be written as

[30] The base of a Kähler cone is a Sasakian manifold and the base of a Ricci-flat cone is an Einstein manifold, so the base of a Calabi-Yau cone is a Sasaki-Einstein manifold.

$$Z = LZ_0R^\dagger, \tag{A8}$$

where

$$L = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad R = \begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix}. \tag{A9}$$

$L, R \in \mathrm{SU}(2)$ so $|a|^2+|b|^2=|k|^2+|l|^2=1$. Thus, we have shown that $\mathrm{SU}(2)\times\mathrm{SU}(2)$ acts transitively on the base. When $(L,R)=(\Theta,\Theta^\dagger)$ with

$$\Theta = \begin{pmatrix} e^{\imath\theta} & 0 \\ 0 & e^{-\imath\theta} \end{pmatrix},$$

Z_0 is left fixed. This means that we can identify (L,R) and $(L\Theta, R\Theta^\dagger)$, i.e., the base is the coset space $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)=S^3\times S^3/\mathrm{U}(1)$ with topology $S^2\times S^3$ and symmetry group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$.

We now turn to the discussion of the Kähler metric on the singular conifold. The metric on a complex manifold is Kähler if and only if it can be written as

$$g_{\mu\bar{\nu}} = \partial_\mu\partial_\nu\mathcal{F},$$

where $\mathcal{F}$ is the Kähler potential. If this potential is to be invariant under the action of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ it can only be a function of r^2, so

$$g_{\mu\bar{\nu}} = (\partial_\mu\partial_{\bar{\nu}}r^2)\mathcal{F}' + (\partial_\mu r^2)(\partial_{\bar{\nu}}r^2)\mathcal{F}'',$$

where the prime indicates a derivative with respect to r^2. In terms of W,

$$ds^2 = \mathcal{F}' \,\mathrm{Tr}(dW^\dagger dW) + \mathcal{F}''|\mathrm{Tr}\, W^\dagger dW|^2. \tag{A10}$$

To find the condition where our metric will be in addition Ricci flat, we need the Ricci tensor, which takes the form $R_{\mu\bar{\nu}}=\partial_\mu\partial_{\bar{\nu}}\ln(\det g_{\mu\bar{\nu}})$ on a Kähler manifold. Define a function

$$\gamma(r) = r^2\mathcal{F}', \tag{A11}$$

then requiring Ricci flatness leads to

$$\gamma(r) = r^{4/3}. \tag{A12}$$

After a rescaling $r\to\tilde{r}=\sqrt{3/2}r^{2/3}$ one recovers a metric of the form (A1) from Eq. (A10). The metric of the base has a useful description in terms of Euler angles. Choosing the following parametrization of L and R in Eq. (A9):

$$a = \cos\frac{\theta_1}{2}e^{i/2(\psi_1+\phi_1)}, \quad k = \cos\frac{\theta_2}{2}e^{i/2(\psi_2+\phi_2)},$$

$$b = \sin\frac{\theta_1}{2}e^{i/2(\psi_1-\phi_1)}, \quad l = \sin\frac{\theta_2}{2}e^{i/2(\psi_2-\phi_2)}, \tag{A13}$$

where ψ_i, ϕ_i, θ_i are the Euler angles of each SU(2), one obtains from Eq. (A10)

$$ds^2_{T^{1,1}} = \tfrac{1}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \tfrac{1}{6}\sum_{i=1}^{2}(d\theta_1^2 + \sin^2\theta_i d\phi_i^2). \tag{A14}$$

This form of the metric shows explicitly the two spheres (θ_i,ϕ_i) and the U(1) fiber over them, parametrized by $\psi=\psi_1+\psi_2$. We also observe that the U(1) symmetry [discussed after Eq. (A2)] manifests itself as a shift symmetry in ψ.

2. The deformed conifold

One way to repair the singularity of a conifold is by deformation in which the defining equation (A2) near $r=0$ is replaced by

$$\sum_{i=1}^{4} z_i^2 = \mu^2. \tag{A15}$$

By taking again the intersection with the three-sphere to find the base, one finds $2x_i^2=\mu^2+r^2$, i.e., a finite S^3 remains at $r=0$. This is called a *deformed conifold*. Note that the U(1) symmetry of the singular conifold (corresponding to a rotation $z_i\to e^{i\alpha}z_i$ with constant phase α for all i) is broken to a $\mathbb{Z}_2$ that sends $z_i\to -z_i$.

In terms of the matrix W as defined in Eq. (A5) the deformed conifold is given by

$$\det W = -\mu^2/2$$

and as in Eq. (A7) we define a radial coordinate via $r^2=\mathrm{Tr}(W^\dagger W)$. Splitting the z_i into real and imaginary parts we obtain

$$r^2 = \sum_{i=1}^{4}(x_i^2+y_i^2), \quad \mu^2 = \sum_{i=1}^{4}(x_i^2-y_i^2), \tag{A16}$$

which implies that r ranges from μ to ∞, but it is also clear that the deformed conifold is still the cotangent bundle over a three-sphere T^*S^3, only that the S^3 has a minimal size. The S^3 never shrinks to zero. A particular solution is found to be

$$W_\mu = \begin{pmatrix} \frac{\mu}{\sqrt{2}} & \sqrt{r^2-\mu^2} \\ 0 & -\frac{\mu}{\sqrt{2}} \end{pmatrix} \tag{A17}$$

and the general solution is obtained by setting $W = LW_\mu R^\dagger$. For $r\neq\mu$ the stability group is again U(1). So for each $r\neq\mu$ the surfaces $r=$const are again $S^2\times S^3$. Note, however, that for $r=\mu$ the matrix W_μ is proportional to σ_3 and is invariant under an entire SU(2). Thus, the "origin" of coordinates $r=\mu$ is in fact an $\mathrm{SU}(2)=S^3$.

Again we define a Kähler potential $\hat{\mathcal{F}}$ and $\hat{\gamma}=r^2\hat{\mathcal{F}}$. Then the metric is given by Eq. (A10) and the condition for Ricci flatness becomes (Candelas and de la Ossa 1990)

$$r^2(r^4-\mu^4)(\hat{\gamma}^3)' + 3\mu^4\hat{\gamma}^3 = 2r^8. \quad (A18)$$

This can be integrated and one finds that for $r \to \infty$ the function $\hat{\gamma}$ approaches $r^{4/3}$, which agrees with the singular conifold solution. So asymptotically (for large r) the two spaces look the same. In terms of Euler angles (A13) the metric is explicitly given as (Minasian and Tsimpis, 2000; Papadopoulos and Tseytlin, 2001) [see also Ohta and Yokono (2000)]

$$\begin{aligned} ds^2_{\rm def} = \Bigg[& (r^2\hat{\gamma}' - \hat{\gamma})\left(1 - \frac{\mu^4}{r^4}\right) + \hat{\gamma}\Bigg]\Bigg(\frac{dr^2}{r^2(1-\mu^4/r^4)} \\ & + \frac{1}{4}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\Bigg) \\ & + \frac{\hat{\gamma}}{4}[(\sin\theta_1^2 d\phi_1^2 + d\theta_1^2) + (\sin\theta_2^2 d\phi_2^2 + d\theta_2^2)] \\ & + \frac{\hat{\gamma}\mu^2}{2r^2}[\cos\psi(d\theta_1 d\theta_2 - \sin\theta_1 \sin\theta_2 d\phi_1 d\phi_2) \\ & + \sin\psi(\sin\theta_1 d\phi_1 d\theta_2 + \sin\theta_2 d\phi_2 d\theta_1)], \quad (A19) \end{aligned}$$

where we would need to rescale r to ensure that $\hat{\gamma}$ has dimension r^2. Note that even the metric now shows the absence of the U(1) symmetry formerly associated with shifts in ψ. As discussed at the beginning of this section, this is not an accident of the parametrization we chose, but inherent to the defining equation (A15) of the deformed conifold.

3. The resolved conifold

Another way to repair the conifold singularity is to resolve it by blowing up a two-sphere. Upon defining new variables

$$x = z_1 + \imath z_2, \quad (A20)$$

$$y = z_2 + \imath z_1, \quad (A21)$$

$$u = z_3 - \imath z_4, \quad (A22)$$

$$v = z_4 - \imath z_3, \quad (A23)$$

the conifold equation (A2) becomes

$$xy - uv = 0. \quad (A24)$$

This is equivalent to requiring nontrivial solutions to

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \quad (A25)$$

in which ξ_1, ξ_2 are not both zero. So, for $(u,v,x,y) \neq 0$ (away from the tip), they describe again a conifold; but at $(u,v,x,y)=0$ this is solved by any pair (ξ_1,ξ_2). Due to the overall scaling freedom $(\xi_1,\xi_2) \sim (\lambda\xi_1, \lambda\xi_2)$ we can mod out by this equivalence class and (ξ_1,ξ_2) actually describe a $\mathbb{CP}^1 \sim S^2$ at the tip of the cone. Therefore the resolved conifold is depicted as $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. We work in a patch where $\xi_2/\xi_1 = \lambda$ is a good inhomogeneous coordinate on $\mathbb{CP}^1$. Hence

$$W = \begin{pmatrix} -u\lambda & u \\ -y\lambda & y \end{pmatrix}. \quad (A26)$$

The radial coordinate is defined as in Eq. (A4) and becomes

$$r^2 = \mathrm{Tr}\, W^\dagger W = \sigma\Lambda, \quad (A27)$$

with $\sigma = |u|^2 + |y|^2$ and $\Lambda = 1 + |\lambda|^2$. The Kähler potential $\mathcal{K}$ in this case is not only a function of r^2, but

$$\mathcal{K} = \tilde{\mathcal{F}} + 4a^2 \ln\Lambda, \quad (A28)$$

with $\tilde{\mathcal{F}}$ a function of r^2 and a a constant, the resolution parameter. This gives the metric on the resolved conifold

$$ds^2 = \tilde{\mathcal{F}}'\,\mathrm{Tr}(dW^\dagger dW) + \tilde{\mathcal{F}}''|\mathrm{Tr}\, W^\dagger dW|^2 + 4a^2\frac{|d\lambda|^2}{\Lambda^2}.$$

This reduces to the singular conifold metric when $a \to 0$. We again define $\tilde{\gamma} = r^2\tilde{\mathcal{F}}$. Then Ricci flatness requires

$$\tilde{\gamma}'\tilde{\gamma}(\tilde{\gamma} + 4a^2) = 2r^2/3, \quad (A29)$$

which can be solved for $\tilde{\gamma}(r)$. In terms of the Euler angles (A13) with $\psi = \psi_1 + \psi_2$, this metric was derived by Pando Zayas and Tseytlin (2000) to be[31]

$$\begin{aligned} ds^2_{\rm res} = & \tilde{\gamma}' dr^2 + \frac{\tilde{\gamma}'}{4} r^2 (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ & + \frac{\tilde{\gamma}}{4}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{\tilde{\gamma} + 4a^2}{4}(d\theta_2^2 \\ & + \sin^2\theta_2 d\phi_2^2), \quad (A30) \end{aligned}$$

with $\tilde{\gamma} = \tilde{\gamma}(r)$ going to zero like r^2 and $\tilde{\gamma} = \partial\tilde{\gamma}/\partial r^2$. a is called the resolution parameter because it determines the size of the blown up S^2 at $r=0$. This illustrates that the (θ_2, ϕ_2) sphere is the only part of the metric that remains finite as we approach the tip at $r=0$.

It is convenient to define a new radial coordinate via $\rho^2 = 3/2\gamma$. Using Eq. (A29), the Ricci-flat metric with appropriate dimensions can be written as

$$\begin{aligned} ds^2_{\rm res} = & \frac{\kappa(\rho)}{9}\rho^2(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{\rho^3}{6}(d\theta_1^2 \\ & + \sin^2\theta_1 d\phi_1^2) + \frac{\rho^2 + 6a^2}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ & + \kappa(\rho)^{-1} d\rho^2, \quad (A31) \end{aligned}$$

with $\kappa(\rho) = (\rho^2 + 9a^2)/(\rho^2 + 6a^2)$. It is interesting that there is another Kähler metric on the resolved conifold which is related to this one by a flop, basically corresponding to the exchange of the two S^2.

[31]Again we need to rescale the radial coordinate such that $\tilde{\gamma}$ has dimension r^2.

4. Complex structures of conifolds

In this section we explore a set of vielbeins that does not only give rise to the Ricci-flat Kähler metric on all three conifold geometries, but also makes the closed Kähler and holomorphic three-form apparent. We follow the convention of Cvetic *et al.* (2002b), which is similar to Papadopoulos and Tseytlin (2001), but note that the simpler set of vielbeins advocated by Klebanov and Strassler (2000) and Minasian and Tsimpis (2000) does not produce a closed holomorphic three-form for the deformed conifold. For a good introduction on basic concepts of complex differential geometry see, for example, Nakahara (2003).

The vielbeins are deduced from the symmetry group SU(2)×SU(2). In terms of Euler angles on the corresponding two S^3s, we choose left-invariant one-forms on the conifold base:

$$\sigma_1 = \cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1,$$

$$\sigma_2 = -\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1,$$

$$\sigma_3 = d\psi_1 + \cos\theta_1 d\phi_1,$$

$$\Sigma_1 = \cos\psi_2 d\theta_2 + \sin\psi_2 \sin\theta_2 d\phi_2,$$

$$\Sigma_2 = -\sin\psi_2 d\theta_2 + \cos\psi_2 \sin\theta_2 d\phi_2,$$

$$\Sigma_3 = d\psi_2 + \cos\theta_2 d\phi_2. \qquad \text{(A32)}$$

They satisfy a Maurer-Cartan equation $d\sigma_i = -i/2\varepsilon_i^{jk}\sigma_j\wedge\sigma_k$ and similarly for the Σ_i. Papadopoulos and Tseytlin (2001) used only five angles and $\psi_1=\psi_2=\psi/2$. This is sufficient for the six-dimensional conifolds, but Cvetic *et al.* (2002b) lifted these geometries to a unified solution in M-theory. It was shown that all three conifold geometries give rise to one G_2 holonomy metric. The eleventh direction is identified with $\psi_1-\psi_2$ and therefore the coordinate choice $\psi_1+\psi_2=\psi$ and $\psi_1-\psi_2=0$ can indeed be viewed as a dimensional reduction from seven to six dimensions.

a. Singular conifold

The one-forms (A32) give rise to vielbeins on the six-dimensional conifold:

$$e_1 = \frac{r}{\sqrt{6}}\sigma_1, \quad e_2 = \frac{r}{\sqrt{6}}\sigma_2,$$

$$e_3 = \frac{r}{\sqrt{6}}\Sigma_1, \quad e_4 = \frac{r}{\sqrt{6}}\Sigma_2,$$

$$e_5 = \frac{r}{3}(\sigma_3+\Sigma_3), \quad e_6 = dr, \qquad \text{(A33)}$$

and the metric is diagonal in these vielbeins:

$$ds^2 = \sum_{i=1}^{6} e_i^2 = dr^2 + \frac{r^2}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \frac{r^2}{6}\sum_{i=1}^{2}(d\theta_1^2 + \sin^2\theta_i d\phi_i^2). \qquad \text{(A34)}$$

Here we identify $\psi=\psi_1+\psi_2$. Note that the odd combination of $\psi_1-\psi_2$ does not appear and we recover the by-now-familiar structure of the base—an S^2 fibered over an S^3—although we started out with coordinates for two S^3s.

An (almost) complex structure on this real six-dimensional manifold is defined by choosing complex vielbeins

$$E_1 = e_1 + \iota e_2, \quad E_2 = e_3 + \iota e_4, \quad E_3 = e_5 + \iota e_6. \qquad \text{(A35)}$$

In terms of these complex vielbeins, the fundamental two-form J and holomorphic three-form Ω are defined as

$$J^{(1,1)} = \frac{\iota}{2}(E_1\wedge\bar{E}_1 + E_2\wedge\bar{E}_2 + E_3\wedge\bar{E}_3), \qquad \text{(A36)}$$

$$\Omega^{(3,0)} = E_1\wedge E_2\wedge E_3. \qquad \text{(A37)}$$

For the singular conifold their coordinate expressions are still fairly simple (again, only the even combination of ψ_1 and ψ_2 appears):

$$J = -\frac{r}{3}dr\wedge(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) - \frac{r^2}{6}(\sin\theta_1 d\phi_1\wedge d\theta_1 + \sin\theta_2 d\phi_2\wedge d\theta_2),$$

$$\begin{aligned}\Omega = {}& \frac{r^2}{6}(\cos\psi - \iota\sin\psi)dr\wedge[\sin\theta_1 d\theta_2\wedge d\phi_1 - \sin\theta_2 d\theta_1\wedge d\phi_2 + \iota(d\theta_1\wedge d\theta_2 - \sin\theta_1\sin\theta_2 d\phi_1\wedge d\phi_2)]\\ & + \frac{r^3}{18}(\cos\psi - \iota\sin\psi)[d\theta_1\wedge d\theta_2\wedge(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) - \sin\theta_1\sin\theta_2 d\phi_1\wedge d\phi_2\wedge d\psi\\ & - \iota(\sin\theta_1 d\theta_2\wedge d\phi_1 - \sin\theta_2 d\theta_1\wedge d\phi_2)\wedge d\psi - \iota(\sin\theta_1\cos\theta_2 d\theta_2 + \cos\theta_1\sin\theta_2 d\theta_1)\wedge d\phi_1\wedge d\phi_2],\end{aligned}$$

and one can easily show that

$$dJ = 0 \quad \text{and} \quad d\Omega = 0. \tag{A38}$$

Together these relations imply that the almost complex structure is actually integrable, so the closure of the fundamental two-form means that this manifold is Kähler. For a Kähler manifold the closure of Ω means furthermore that it is a Calabi-Yau manifold [see, e.g., Chiossi and Salamon (2002)].

The complex structure induced by these vielbeins is of course identical with the one from the holomorphic coordinates z_i used in Eq. (A2) to define the singular conifold. One finds that, up to a numerical factor, the holomorphic three-form can be expressed as

$$\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_4}, \tag{A39}$$

which agrees with the above coordinate expression if the holomorphic coordinates are parametrized as

$$x = r^{3/2} e^{\iota/2(\psi - \phi_1 - \phi_2)} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2},$$

$$y = r^{3/2} e^{\iota/2(\psi + \phi_1 + \phi_2)} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2},$$

$$u = r^{3/2} e^{\iota/2(\psi + \phi_1 - \phi_2)} \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2},$$

$$v = r^{3/2} e^{\iota/2(\psi - \phi_1 + \phi_2)} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2}. \tag{A40}$$

We also make use of the coordinate redefinition (A20) to relate these coordinates to z_i:

$$z_1 = \frac{1}{2}(x - \iota y),$$

$$z_2 = \frac{1}{2\iota}(x + \iota y),$$

$$z_3 = \frac{1}{2}(u + v),$$

$$z_4 = \frac{-1}{2\iota}(u - v).$$

For practical computations these coordinates are not very useful, as they are the homogeneous ones. The real coordinates make the structure of the six-dimensional manifold much more transparent and the vielbeins serve as a convenient basis for all sorts of differential forms, like fluxes.

b. Resolved conifold

The same complex structure (A35) can be used for the resolved conifold. We only have to scale the vielbeins according to the metric:

$$e_1 = \frac{\rho}{\sqrt{6}}\sigma_1, \quad e_2 = \frac{\rho}{\sqrt{6}}\sigma_2,$$

$$e_3 = \frac{\sqrt{\rho^2 + 6a^2}}{\sqrt{6}}\Sigma_1, \quad e_4 = \frac{\sqrt{\rho^2 + 6a^2}}{\sqrt{6}}\Sigma_2,$$

$$e_5 = \frac{\rho}{3}\sqrt{\frac{\rho^2 + 9a^2}{\rho^2 + 6a^2}}(\sigma_3 + \Sigma_3), \quad e_6 = \sqrt{\frac{\rho^2 + 6a^2}{\rho^2 + 9a^2}}d\rho, \tag{A41}$$

then the metric remains diagonal and we recover Eq. (A31) with $\psi = \psi_1 + \psi_2$. The fundamental two-form (A36) is found to be

$$J = -\frac{\rho}{3}d\rho \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) - \frac{\rho^2}{6}\sin\theta_1 d\phi_1 \wedge d\theta_1 - \frac{\rho^2 + 6a^2}{6}\sin\theta_2 d\phi_2 \wedge d\theta_2 \tag{A42}$$

and is closed, as is the holomorphic three-form one obtains from Eq. (A37):

$$\begin{aligned}\Omega = {} & \frac{\rho(\rho^2 + 6a^2)}{6\sqrt{\rho^2 + 9a^2}}(\cos\psi - i\sin\psi)d\rho \wedge [\sin\theta_1 d\theta_2 \wedge d\phi_1 - \sin\theta_2 d\theta_1 \wedge d\phi_2 + i(d\theta_1 \wedge d\theta_2 - \sin\theta_1 \sin\theta_2 d\phi_1 \wedge d\phi_2)] \\ & + \frac{\rho^2}{18}\sqrt{\rho^2 + 9a^2}(\cos\psi - i\sin\psi)[d\theta_1 \wedge d\theta_2 \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) - \sin\theta_1 \sin\theta_2 d\phi_1 \wedge d\phi_2 \wedge d\psi \\ & - i(\sin\theta_1 d\theta_2 \wedge d\phi_1 - \sin\theta_2 d\theta_1 \wedge d\phi_2) \wedge d\psi - i(\sin\theta_1 \cos\theta_2 d\theta_2 + \cos\theta_1 \sin\theta_2 d\theta_1) \wedge d\phi_1 \wedge d\phi_2].\end{aligned} \tag{A43}$$

So this complex structure also fulfills the Calabi-Yau conditions. The corresponding homogeneous holomorphic coordinates in this case read

$$x = (9a^2\rho^4 + \rho^6)^{1/4} e^{i/2(\psi-\phi_1-\phi_2)} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2},$$

$$y = (9a^2\rho^4 + \rho^6)^{1/4} e^{i/2(\psi+\phi_1+\phi_2)} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2},$$

$$u = (9a^2\rho^4 + \rho^6)^{1/4} e^{i/2(\psi+\phi_1-\phi_2)} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2},$$

$$v = (9a^2\rho^4 + \rho^6)^{1/4} e^{i/2(\psi-\phi_1+\phi_2)} \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}.$$

They lead to the same holomorphic three-form with the definition (A39).

c. Deformed conifold

For the deformed conifold the story is more complicated. The metric is not diagonal in the vielbeins (A32) and we have to define linear combinations of them such that

$$e_1 = \frac{\sqrt{\hat\gamma}}{2}(\alpha\sigma_1 - \beta\Sigma_1), \quad e_2 = \frac{\sqrt{\hat\gamma}}{2}(\alpha\sigma_2 + \beta\Sigma_2),$$

$$e_3 = \frac{\sqrt{\hat\gamma}}{2}(-\beta\sigma_1 + \alpha\Sigma_1), \quad e_4 = \frac{\sqrt{\hat\gamma}}{2}(\beta\sigma_2 + \alpha\Sigma_2),$$

$$e_5 = \frac{1}{2}\sqrt{(r^2\hat\gamma' - \hat\gamma)(1-\mu^4/r^4) + \hat\gamma}(\sigma_3 + \Sigma_3),$$

$$e_6 = \frac{\sqrt{(r^2\hat\gamma' - \hat\gamma)(1-\mu^4/r^4) + \hat\gamma}}{r\sqrt{1-\mu^4/r^4}}dr, \tag{A44}$$

where $\alpha^2+\beta^2=1$ has to hold for the metric to turn out correctly. With these linear combinations one recovers Eq. (A19) from

$$ds^2 = \sum_{i=1}^{6} e_i^2.$$

For the metric to also be Ricci flat and Kähler, the coefficients α and β are determined to be

$$\alpha = \tfrac{1}{2}\sqrt{1+\mu^2/r^2} + \tfrac{1}{2}\sqrt{1-\mu^2/r^2}, \quad \beta = \frac{\mu^2}{2r^2\alpha}. \tag{A45}$$

The complex structure is defined as in Eq. (A35) and again gives rise to a Calabi-Yau manifold.

With the choice (A45) the Kähler form amounts to

$$J = -\frac{r^6\hat\gamma' + \mu^4\hat\gamma - r^2\mu^4\hat\gamma'}{2r^5\sqrt{1-\mu^4/r^4}}dr\wedge(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) + \frac{\hat\gamma}{4}\sqrt{1-\frac{\mu^4}{r^4}}(\sin\theta_1 d\theta_1\wedge d\phi_1 + \sin\theta_2 d\theta_2\wedge d\phi_2), \tag{A46}$$

which is easily shown to be closed (recall that the prime indicates derivative with respect to r^2). The expression for the holomorphic three-form is

$$\begin{aligned}\Omega ={}& 2\mathcal{T}\frac{\mathcal{S}\cos\psi - i\sin\psi}{r\mathcal{S}}dr\wedge(\sin\theta_1 d\theta_2\wedge d\phi_1 - \sin\theta_2 d\theta_1\wedge d\phi_2) + \frac{2\iota\mathcal{T}}{r\mathcal{S}}(\cos\psi - i\mathcal{S}\sin\psi)dr\wedge(d\theta_1\wedge d\theta_2\\ &- \sin\theta_1\sin\theta_2 d\phi_1\wedge d\phi_2) - \frac{2\mu^2\mathcal{T}}{r^3\mathcal{S}}dr\wedge(\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2) + \frac{\iota\mu^2\mathcal{T}}{r^2}[\sin\theta_1 d\theta_1\wedge d\phi_1\wedge(d\psi\\ &+ \cos\theta_2 d\phi_2) - \sin\theta_2 d\theta_2\wedge d\phi_2\wedge(d\psi + \cos\theta_1 d\phi_1)] + \mathcal{T}(\iota\cos\psi + \mathcal{S}\sin\psi)[\sin\theta_2 d\theta_1\wedge d\phi_2\wedge(d\psi + \cos\theta_1 d\phi_1)\\ &- \sin\theta_1 d\theta_2\wedge d\phi_1\wedge(d\psi + \cos\theta_2 d\phi_2)] + \mathcal{T}(\mathcal{S}\cos\psi - \iota\sin\psi)[d\theta_1\wedge d\theta_2\wedge(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\\ &- \sin\theta_1\sin\theta_2 d\phi_1\wedge d\phi_2\wedge d\psi],\end{aligned} \tag{A47}$$

where $\mathcal{S}=\sqrt{1-\mu^4/r^4}$ and $\mathcal{T}=\hat\gamma\sqrt{\hat\gamma+(r^2\hat\gamma'-\hat\gamma)(1-\mu^4/r^4)}/8$. To show that it is indeed closed one needs to make use of Eq. (A18).

For the deformed conifold we can use the same holomorphic coordinates as for the singular conifold (A40), but the three-form (A39) now reads

$$\Omega = \frac{dz_1\wedge dz_2\wedge dz_3}{\sqrt{\mu^2 - z_1^2 - z_2^2 - z_3^2}}. \tag{A48}$$

As a side remark, we note that the much simpler vielbeins from Klebanov and Strassler (2000),

$$g_1 = -\sin\theta_1 d\phi_1,$$

$$g_2 = d\theta_1,$$

$$g_3 = -\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2,$$

$$g_4 = \cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2,$$

$$g_5 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2,$$

$$g_6 = dr, \tag{A49}$$

will never give a closed holomorphic three-form on the deformed conifold, even with a generic ansatz for a linear combination of these vielbeins.[32] In other words, they are not compatible with the holomorphic coordinates (A40). They do work for the singular and resolved conifold, because they happen to give the same two- and three-forms. So there is more than one choice of vielbeins that allows for a Calabi-Yau metric. However, if we wish to pass from one geometry to the other, we prefer to employ a complex structure that allows for all three of them to be Calabi-Yau manifolds. The set of vielbeins (A49) was also used by Minasian and Tsimpis (2000). Caution should therefore be used with their solutions, in particular from the viewpoint of supersymmetry.

5. Fluxes on conifolds

Having studied the complex structure of conifold geometries, we can now turn to the question of what types of fluxes are allowed on them. In type IIB compactifications the background three-form flux $G_3 = F_3 + \tau H_3$ ($\tau = C_0 + ie^{-\phi}$ is the axion-dilaton) has to obey a self-duality condition (Giddings *et al.*, 2002)

$$*_6 G_3 = iG_3, \tag{A50}$$

where $*_6$ indicates the Hodge dual in six dimensions. Supersymmetry requires G_3 to be of type (2,1) and primitive (Gubser 2000; Grana and Polchinski, 2001), i.e., that it satisfy $G_3 \wedge J = 0$. Cvetic *et al.* (2003) showed that the solution of Klebanov and Strassler (2000) for D5-branes on the singular conifold fulfills these requirements, whereas the Pando Zayas–Tseytlin (2000) (PT) solution for D5-branes on the resolved conifold does not. The latter has a (1,2) part in addition to the allowed (2,1).

Although we agree with the result obtained by Cvetic *et al.* (2003), we question the complex structure they use. Following Pando Zayas and Tseytlin they take the simplest set of vielbeins that would give the right resolved metric (A30),[33] i.e.,

$$\epsilon_1 = \frac{\rho}{\sqrt{6}} d\theta_1, \quad \epsilon_2 = \frac{\rho}{\sqrt{6}} \sin\theta_1 d\phi_1,$$

$$\epsilon_3 = \frac{\sqrt{\rho^2 + 6a^2}}{\sqrt{6}} d\theta_2, \quad \epsilon_4 = \frac{\sqrt{\rho^2 + 6a^2}}{\sqrt{6}} \sin\theta_2 d\phi_2,$$

$$\epsilon_5 = -\sqrt{\frac{\rho^2 + 6a^2}{\rho^2 + 9a^2}} d\rho,$$

$$\epsilon_6 = \frac{\rho}{3}\sqrt{\frac{\rho^2 + 9a^2}{\rho^2 + 6a^2}} (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2), \tag{A51}$$

and then show that the fluxes from Pando Zayas and Tseytlin (2000) have not only a (2,1) but also a (1,2) part with respect to the complex structure:

$$E_1 = \epsilon_1 + \iota\epsilon_2, \quad E_2 = \epsilon_3 + \iota\epsilon_4, \quad E_3 = \epsilon_5 + \iota\epsilon_6.$$

Note, however, that this choice is not the right one to observe the Calabi-Yau property. The choice leads to a closed fundamental two-form, but the holomorphic three-form,

$$\begin{aligned}\Omega = &-\frac{\rho(\rho^2+6a^2)}{6\sqrt{\rho^2+9a^2}} d\rho \wedge [\iota(\sin\theta_2 d\theta_1 \wedge d\phi_2 \\ &- \sin\theta_1 d\theta_2 \wedge d\phi_1) + (d\theta_1 \wedge d\theta_2 \\ &- \sin\theta_1 \sin\theta_2 d\phi_1 \wedge d\phi_2)] \\ &+ \frac{\rho^2}{18}\sqrt{\rho^2+9a^2}[\iota d\theta_1 \wedge d\theta_2 \wedge (d\psi + \cos\theta_1 d\phi_1 \\ &+ \cos\theta_2 d\phi_2) - \iota \sin\theta_1 \sin\theta_2 d\phi_1 \wedge d\phi_2 \wedge d\psi \\ &+ (\sin\theta_1 d\theta_2 \wedge d\phi_1 - \sin\theta_2 d\theta_1 \wedge d\phi_2) \wedge d\psi \\ &+ (\sin\theta_1 \cos\theta_2 d\theta_2 \\ &+ \cos\theta_1 \sin\theta_2 d\theta_1) \wedge d\phi_1 \wedge d\phi_2],\end{aligned}$$

lacks the $\cos\psi - \iota\sin\psi$ terms compared to Eq. (A43). It is therefore not closed but instead

$$\begin{aligned}d\Omega = &\frac{\rho(\rho^2+6a^2)}{6\sqrt{\rho^2+9a^2}} d\rho \wedge d\psi \wedge [\iota(d\theta_1 \wedge d\theta_2 \\ &- \sin\theta_1 \sin\theta_2 d\phi_1 \wedge d\phi_2) - (\sin\theta_2 d\theta_1 \wedge d\phi_2 \\ &- \sin\theta_1 d\theta_2 \wedge d\phi_1)] + \frac{\rho^2}{18}\sqrt{\rho^2+9a^2}[\\ &- d\theta_1 \wedge d\theta_2 \wedge d\psi \wedge (\cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \\ &+ \iota(\sin\theta_1 \cos\theta_2 d\theta_2 \\ &+ \cos\theta_1 \sin\theta_2 d\theta_1) \wedge d\phi_1 \wedge d\phi_2 \wedge d\psi],\end{aligned}$$

which will never vanish. This statement remains true for the singular conifold, so this complex structure should not be used for analyzing the KS flux either. We therefore believe that care should be taken when using the analysis of Cvetic *et al.* (2003).

We can use our knowledge from Appendix A.4 to find out to which cohomology group the flux from Pando Zayas and Tseytlin (2000) belongs. The imaginary self-dual three-form flux $G_3 = F_3 + iH_3$ found in Pando Zayas and Tseytlin (2000) is given by[34]

[32] This statement was confirmed with MATHEMATICA for arbitrary r-dependent coefficients.

[33] Note that there is a typo in Eq. (6.5) of Cvetic *et al.* (2003).

[34] This is a solution with constant dilaton, which can therefore be set to zero, and vanishing axion. Furthermore, there is a typo in Eq. (4.3) in Pando Zayas and Tseytlin (2000), concerning the sign of F_3.

$$H_3 = d\rho \wedge [F_1'(\rho)\sin\theta_1 d\theta_1 \wedge d\phi_1 + F_2'(\rho)\sin\theta_2 d\theta_2 \wedge d\phi_2] \tag{A52}$$

and

$$F_3 = P(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \wedge (\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2), \tag{A53}$$

where P is a constant (to ensure $dF_3=0$) and $F_1'(\rho)$ and $F_2'(\rho)$ were determined from the equations of motion to be

$$F_1'(\rho) = 3P\frac{\rho}{\rho^2+9a^2}, \quad F_2'(\rho) = -3P\frac{(\rho^2+6a^2)^2}{\rho^3(\rho^2+9a^2)}.$$

We now use the vielbeins from Appendix A.4.b and invert Eq. (A41) to solve for the coordinate differentials. We then find the flux in terms of vielbeins

$$G_3 = \frac{18P\sqrt{\rho^2+6a^2}}{\rho^3\sqrt{\rho^2+9a^2}}(e_1\wedge e_2\wedge e_5 - \iota e_3\wedge e_4\wedge e_6) - \frac{18P(e_3\wedge e_4\wedge e_5 - \iota e_1\wedge e_2\wedge e_6)}{\rho\sqrt{\rho^2+6a^2}\sqrt{\rho^2+9a^2}}. \tag{A54}$$

The vielbein notation is convenient to see that this flux is indeed imaginary self-dual[35] (remarkable since Pando Zayas and Tseytlin also used the wrong set of vielbeins). The Hodge dual is found by

$$*_6(e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}) = \epsilon_{i_1 i_2\cdots i_k}{}^{i_{k+1}\cdots i_6} e_{i_{k+1}}\wedge\cdots\wedge e_{i_6}$$

and does not involve any factors of $\sqrt{g}$. We use the convention that $\varepsilon_{123456}=\varepsilon_{123}{}^{456}=1$. With the usual complex structure (A35) the Pando Zayas–Tseytlin flux becomes

$$G_3 = -\frac{9\iota P}{\rho^3\sqrt{\rho^2+9a^2}\sqrt{\rho^2+6a^2}}\times[(\rho^2+3a^2)\times(E_1\wedge E_3\wedge\bar{E}_1 - E_2\wedge E_3\wedge\bar{E}_2) - 3a^2(E_1\wedge\bar{E}_1\wedge\bar{E}_3 + E_2\wedge\bar{E}_2\wedge\bar{E}_3)]. \tag{A55}$$

We make several observations: This flux is neither primitive[36] nor is it of type (2,1). It has a (1,2) and a (2,1) part. With just a (1,2) part present we could have made this flux supersymmetric by a different choice of complex structure, but as it stands, this flux indeed breaks supersymmetry, as claimed by Cvetic *et al.* (2003). Apart from that, in the limit $a\to 0$ the (1,2) part vanishes, the flux becomes primitive and we recover the singular conifold. This seems to indicate that the resolution parameter forbids a supersymmetric supergravity solution for wrapped D5-branes on the resolution in the presence of flux.

We close this section by repeating the flux analysis for the KS model, which in the region away from the tip agrees with the singular conifold solution first advocated by Klebanov and Tseytlin (2000) (KT). Again, we agree with the result of Cvetic *et al.* (2003), but we maintain that a complex structure that allows for a closed holomorphic three-form on the singular conifold should have been used. We use the set (A33) with the same complex structure as in Eq. (A35). The three-form flux $G_3=F_3+\iota H_3$ with

$$F_3 = \frac{M}{2}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \wedge (\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2),$$

$$H_3 = \frac{3}{2Mr}dr\wedge(\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2)$$

becomes

$$G_3 = -\frac{9\iota M}{2r^3}(E_1\wedge E_3\wedge\bar{E}_1 - E_2\wedge E_3\wedge\bar{E}_2), \tag{A56}$$

where M indicates the number of fractional D3-branes in the KT model; see Sec. II.A.2. It is also easy to check that this flux is indeed primitive ($J\wedge G_3=0$). Also, the resulting five-form flux $F_5=dC_4+B_2\wedge F_3$ can be made self-dual by choosing

$$dC_4 = d[h^{-1}(r)]\wedge dx_0\wedge\cdots\wedge dx_3 = *_{10}(B_2\wedge F_3), \tag{A57}$$

where the ten-dimensional Hodge dual is to be taken with respect to to the warped metric

$$ds^2 = h^{-1/2}(r)\eta^{\mu\nu}dx_\mu dx_\nu + h^{1/2}(r)(dr^2 + r^2 ds^2_{T^{1,1}}).$$

Thus we have confirmed that the KT model preserves supersymmetry in the correct complex structure. We also see that in the limit where the two-cycle in the resolved conifold shrinks to zero, the flux in the PT solution agrees with the singular conifold solution of KT.

[35]The self-duality should be checked with respect to the *warped* resolved conifold, but since we consider a three-form flux on a six-dimensional manifold, the appropriate factors of the warp factor drop out when taking the Hodge dual.

[36]Since $J=\frac{\iota}{2}\Sigma_i(E_i\wedge\bar{E}_i)$ it follows immediately that $J\wedge G_3$ has a nonvanishing $E_1\wedge E_2\wedge\bar{E}_1\wedge\bar{E}_2\wedge\bar{E}_3$ part that is proportional to a^2.

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