Nuclear Physics

A. Stationary States of Nuclei[1]

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[1] The present issue contains Part A of this report. Part B, treating the dynamics of nuclei, particularly transmutations, will appear in a later issue of these *Reviews*.

I. Fundamental Properties of Nuclei

§1. Charge, Weight (A4)[2]

OF all properties of the nucleus the charge is by far the most important for *atomic* physics. It determines the number of electrons of an atom in its neutral state, the energy levels of the atom, its chemical actions; in short, all the properties of the atom except for very small corrections such as hyperfine structure, isotope shift of spectral lines, etc.

It is well known that the charge of any nucleus is an integral multiple of the charge e of the proton, let us say Ze. Every integer Z corresponds to a certain chemical element. All elements corresponding to values of Z from 1 to 92 have been actually found in nature, except the elements 85 and 87.[3] Recently, the elements 93 and probably 94 have been produced by disintegration experiments (F6, F9).

With our present knowledge about atomic physics, the nuclear charge of an element can be easily inferred from its chemical or spectroscopic properties (periodic system). The most direct measurement of the nuclear charge of an atom consists, however, in the determination of its x-ray spectrum, in other words, of the binding energy of its inner electrons. Another method is based on the large-angle scattering of α-particles or protons by the nuclei of the element. It is less accurate but of great historical importance because it was this experiment which led Rutherford to the concept of the nuclear atom (R15, C4).

The second nuclear quantity, the atomic weight, was known long before the existence of nuclei was discovered. Moreover, it was suggested as early as 1813 by Prout that all atomic weights are integral multiples of the weight of the hydrogen atom. Later, it was seen that this rule held more accurately when one-sixteenth of the atomic weight of oxygen was taken as the unit. However, some bad exceptions from the integral—weight—rule remained, e.g., chlorine.

We know now that all the elements not conforming to this rule, and, indeed, many of those apparently conforming, consist of several isotopes. The nuclei of two isotopic atoms possess the same charge, but different weight. Since the charge alone determines the chemical and spectroscopic properties of the atom, two isotopes have the same chemical behavior and (practically) the same spectrum. The atomic weight of every isotope is very nearly an integer, while the mixed element which is found in nature will, of course, in general[4] have a non-integral atomic weight.

The analysis of the isotopic constitution of an element as well as the determination of the atomic weights of the single isotopes requires the use of a mass spectrograph (A4). Mainly through the work of Aston, we know at present about 280 different isotopes which occur in nature, corresponding to about 3 isotopes per element. The highest number of isotopes for any single element is found for Sn (10 isotopes).

The atomic weights of the known isotopes run from 1 (light hydrogen) up to 238 (uranium). From 1 to 212, there exists at least one isotope for every integral atomic weight,[5] with the only exception of the atomic weights 5 and 8. In many cases, the same atomic weight is found for two isotopes of two different elements; e.g., one of the isotopes of argon as well as one of the calcium isotopes has the atomic weight 40. Such nuclei which have equal atomic weight but different nuclear charge are called isobars. There are 44 pairs of such isobars known, excluding about 10 pairs for which one of the two isobars is doubtful. In at least two cases, the existence of three isobars seems to be definitely established (atomic weight 96 (Zr, Mo, Ru), and atomic weight 124: tin, tellurium and xenon). The actual number of existing pairs of isobars is certainly larger than the number found thus far, because the isobars in the region of the rare earths are practically unknown. This is because it is very difficult to separate the various rare

[2] A letter and a number, e.g., A4, are used for references to original papers. A list of references is found at the end of the paper.

[3] The discovery of illinium (61) seems to be still disputed.

[4] In some cases all the isotopes except one are very rare, e.g., the isotopes 17 and 18 of oxygen compared to O^{16}. Then the atomic weight of the natural mixed element is approximately equal to that of the abundant isotope and therefore is nearly an integer.

[5] Cf. J. Mattauch (M11).

earths and thus to tell whether a given mass found in the mass spectrograph is an isotope of the element actually investigated or of another rare earth which occurs as an impurity.

The notation accepted for denoting a given isotope, consists in putting the atomic weight of the isotope as a superscript on the chemical symbol of the element the isotope belongs to, e.g., H^2, A^{40}, Te^{124}. Some writers put in addition the nuclear charge in the lower left-hand corner, thus: ${}_{18}A^{40}$, or ${}^{40}_{18}A$. We shall not do so because the charge is uniquely determined by the chemical symbol. It need hardly be noted that the upper index is not the *exact* atomic weight of the isotope which deviates slightly from an integer but the "mass number," i.e., the integer nearest to the atomic weight.

The atomic weights of the natural elements are remarkably independent of the source where the element is found. This means that the various

TABLE I. *Known stable isotopes.*

A = Atomic weight, Z = Nuclear charge, Ch = Chemical symbol.

A	Z	Ch	A	Z	Ch	A	Z	Ch	A	Z	Ch	A	Z	Ch	A	Z	Ch
1	1	H	49	22	Ti		38	Sr	119	50	Sn	156	64	Gd	200	80	Hg
2	1	H	50	22	Ti	88	38	Sr	120	50	Sn	157	64	Gd	201	80	Hg
3	1	H		24	Mo	89	39	Y	121	51	Sb	158	64	Gd	202	80	Hg
4	2	He	51	23	V	90	40	Zr	122	50	Sn	159	65	Tb	203	81	Tl
5	—	—	52	24	Cr	91	40	Zr		52	Te	160	64	Gd	204	80	Hg
6	3	Li	53	24	Cr	92	40	Zr	123	51	Sb	161	66	Dy		82	Pb
7	3	Li	54	24	Cr		42	Mo		52	Te	162	66	Dy	205	81	Tl
8	—	—		26	Fe	93	41	Cb	124	50	Sn	163	66	Dy	206	82	Pb
9	4	Be	55	25	Mn	94	40	Zr		52	Te	164	66	Dy			(U Pb)
10	5	B	56	26	Fe		42	Mo		54	Xe	165	67	Ho	207	82	Pb
11	5	B	57	26	Fe	95	42	Mo	125	52	Te	166	68	Er			(AcPb)
12	6	C	58	28	Ni	96	40	Zr	126	52	Te	167	68	Er	208	82	Pb
13	6	C	59	27	Co		42	Mo		54	Xe	168	68	Er			(ThPb)
14	7	N	60	28	Ni		44	Ru	127	53	I	169	69	Tu	209	83	Bi
15	7	N	61	28	Ni	97	42	Mo	128	52	Te	170	68	Er	210	84	Po
16	8	O	62	28	Ni	98	42	Mo		54	Xe	171	70	Yb			(Ra F)
17	8	O	63	29	Cu		44	Ru	129	54	Xe	172	70	Yb	211	—	
18	8	O	64	30	Zn	99	44	Ru	130	52	Te	173	70	Yb	212	84	Th C′
19	9	F	65	29	Cu		(43	Ms?)		54	Xe	174	70	Yb	213	83	Ac C
20	10	Ne	66	30	Zn	100	42	Mo	131	54	Xe	175	71	Cp	214	84	Ra C′
21	10	Ne	67	30	Zn		44	Ru	132	54	Xe	176	70	Yb	215	—	
22	10	Ne	68	30	Zn	101	44	Ru	133	55	Cs		72	Hf	216	84	Th A
23	11	Na	69	31	Ga	102	44	Ru	134	54	Xe	177	72	Hf	217	84	Ac A
24	12	Mg	70	30	Zn		46	Pd		56	Ba	178	72	Hf	218	84	Ra A
25	12	Mg		32	Ge	103	45	Rh	135	56	Ba	179	72	Hf	219	—	
26	12	Mg	71	31	Ga	104	44	Ru	136	54	Xe	180	72	Hf	220	86	ThEm
27	13	Al	72	32	Ge		46	Pd		56	Ba	181	73	Ta	221	86	AcEm
28	14	Si	73	32	Ge	105	46	Pd	137	56	Ba	182	74	W	222	86	RaEm
29	14	Si	74	32	Ge	106	46	Pd	138	56	Ba	183	74	W	223	—	
30	14	Si		34	Se		48	Cd	139	57	La	184	74	W	224	88	Th X
31	15	P	75	33	As	107	47	Ag	140	58	Ce	185	75	Re	225	88	Ac X
32	16	S	76	32	Ge	108	46	Pd	141	59	Pr	186	74	W	226	88	Ra
33	16	S		34	Se		48	Cd	142	58	Ce		76	Os	227	—	
34	16	S	77	34	Se	109	47	Ag		60	Nd	187	75	Re	228	90	RdTh
35	17	Cl	78	34	Se	110	48	Cd	143	60	Nd	188	76	Os	229	90	RdAc
36	18	A		36	Kr	111	48	Cd	144	60	Nd	189	76	Os	230	90	Io
37	17	Cl	79	35	Br	112	48	Cd		62	Sm	190	76	Os	231	—	
38	18	A	80	34	Se		50	Sn	145	60	Nd	191	77	Ir	232	90	Th
39	19	K		36	Kr	113	48	Cd	146	60	Nd	192	76	Os	233	91	Pa
40	18	A	81	35	Br		49	In		(62	Sm)		78	Pt	234	92	U II
	20	Ca	82	34	Se	114	48	Cd	147	62	Sm	193	77	Ir	235	92	U?
41	19	K		36	Kr		50	Sn	148	62	Sm	194	78	Pt	236	—	
42	20	Ca	83	36	Kr	115	49	In	149	62	Sm	195	78	Pt	237	—	
43	20	Ca	84	36	Kr		50	Sn	150	62	Sm	196	78	Pt	238	92	U I
44	20	Ca		38	Sr	116	48	Cd	151	63	Eu		80	Hg			
45	21	Sc	85	37	Rb		50	Sn	152	62	Sm	197	79	Au			
46	22	Ti	86	36	Kr	117	50	Sn	153	63	Eu	198	78	Pt			
47	22	Ti		38	Sr	118	48	Cd	154	62	Sm		80	Hg			
48	22	Ti	87	37	Rb*		50	Sn	155	64	Gd	199	80	Hg			

Remarks: β-radioactive isotopes have not been included in the table, except Rb^{87}, which is possibly β-active, and Ac C which has a very weak β-activity besides a strong α-activity. Radioactive α-emitters have generally been included.

isotopes occur always in about the same ratio in the natural element. Exceptions from this rule are hydrogen, lead and, to a very small extent, boron. The content of heavy hydrogen H^2 (deuterium) in H from different sources, varies from about 1 part in 3500 to 1 in 5000. This great variation is due to the very large relative difference in atomic weights. For boron, this relative difference is, of course, much smaller than for hydrogen but larger than for most other elements. The ratio $B^{10} : B^{11}$ has been found (B29) to vary from about 1 : 4 to 1 : $3\frac{3}{4}$. On the other hand, lead in uranium ores is produced by the radioactive decay of uranium and is thus, in an ideal case, the pure isotope Pb^{206}.

Since every isotope contains its own characteristic nucleus, the separation of isotopes is very important for nuclear physics. The most complete separation is achieved in the mass spectrograph; the main disadvantage being that extremely high currents or long times are required to separate an appreciable amount of an element into its isotopes. Light and heavy hydrogen can be separated comparatively easily by repeated electrolysis of water. Repeated diffusion has been successful for neon. Repeated chemical actions, distillations, etc., are all possible methods but it is hard to obtain more than a partial separation of the isotopes with their help, except in the case of hydrogen.

It remains to interpret the rules of (nearly) integral atomic weights and (exactly) integral nuclear charge where the latter is always smaller than the former. The interpretation which suggests itself is that any nucleus consists of particles of unit atomic weight some of which are positively charged while others are neutral. In fact, we know two nuclei of atomic weight unity, *viz.*, the proton and the neutron (C5), the first bearing unit positive charge while the second is neutral. We are thus led to the hypothesis that every nucleus consists of protons and neutrons;[6] (H7, also H2, I1). The total number of elementary particles, protons and neutrons together, is then equal to the atomic weight of the nucleus or more exactly to its mass number A, i.e., the integer nearest to the atomic weight. The number of protons must be given by the nuclear charge Z, wherefrom the number of neutrons follows as being

$$N = A - Z. \tag{1}$$

Since for many of the lighter nuclei the number of neutrons is approximately equal to the number of protons, it is sometimes useful to introduce the "isotopic number" I (cf. H2), i.e., the excess of the number of neutrons over the number of protons, *viz.*,

$$I = N - Z = A - 2Z. \tag{2}$$

§2. Energy (A4, A5, B5, B13)

The mass spectrograph has shown that the atomic weight of every nucleus is approximately an integer, thus giving support to the hypothesis that any nucleus is constituted of neutrons and protons. At the same time, however, the mass spectrograph revealed that the atomic weights of separated isotopes are not exactly integers, e.g., $H^1 = 1.00807$, $Li^7 = 7.0164$, $Kr^{80} = 79.926$ and $Tl^{205} = 205.037$. It is seen that the lightest atoms have atomic weights M slightly higher, those of medium atomic weight (from about 20 up to 200) such somewhat lower and the very heaviest atoms again weights slightly higher than the next integer A.

The difference between exact atomic weight M and "mass number" A, the so-called "mass defect"

$$-\Delta = A - M \tag{3}$$

is far outside the experimental error, being for the lightest atoms about 100 times, for the heavier ones about 10 times the probable error. On the other hand, the mass defect is much too small and depends much too regularly on the mass number A to allow the abandoning of the rule of integral atomic weights and thus of our hypothesis about the constitution of the nuclei. It must therefore be concluded that protons and neutrons bound together in a nucleus have a weight different, and more precisely smaller, than the same number of free protons and neutrons. This can be interpreted by Einstein's law of equivalence of mass and energy, as show-

[6] This hypothesis has been used for the first time as the basis of a thorough nuclear theory by Heisenberg (H7); however, it had been suggested earlier as a convenient manner of describing the existing isotopes by Harkins (H2) and Rutherford.

ing that the binding of neutrons and protons in a nucleus decreases the total energy. Thus the mass defect gives us direct information about the binding energy of the particles in a nucleus.

This information is extremely useful.[7] It serves to determine the total binding energy of the elementary particles of a nucleus by comparing its weight to that of an equal number of neutrons and protons. It also serves to determine the binding energy of the last neutron, proton or α-particle in a nucleus by comparing the weight of the nucleus to that of another nucleus containing one neutron, proton or α-particle, respectively, less than the given nucleus. In this way, it can be decided whether a given nucleus is stable or not. Furthermore, it can be deduced whether a given nuclear reaction will be endothermic or exothermic, thus providing a great help to the experimental investigator of nuclear reactions, etc.

As an example, let us take the nucleus Li^6. Its atomic weight (B13) is 6.01614. The nucleus consists of three neutrons, of atomic weight 1.00845 each, and three protons (atomic weight of the hydrogen atom 1.00807). These six particles together in the free state would have a weight of 6.04956 which is 0.03342 units more than the weight of the Li^6 atom. The binding energy of Li^6 is thus 0.03342 "mass units," one mass unit being the energy corresponding, according to Einstein's law of equivalence, to one-sixteenth of the mass of the oxygen atom. To convert this energy into more familiar units, we note that the energy corresponding to the mass m, is mc^2 according to Einstein's law, c being the velocity of light.

$$\text{Thus}\quad 1 \text{ mass unit} = 8.99\cdot 10^{20}\cdot(1/16)\cdot M_O \text{ ergs} = 1.49\cdot 10^{-3} \text{ erg},$$

where $M_O = 2.64\cdot 10^{-22}$ gram is the mass of the oxygen atom O^{16}.

More useful than ergs are electron volts as units of nuclear energies, because the kinetic energies of projectiles used for nuclear disintegrations are measured directly in these units. Let $-10e$ denote the charge of the electron in *international* coulombs, so that $-e/q$ is the electronic charge in absolute electromagnetic units, and q the conversion factor of international into absolute coulombs.[8] Let furthermore pq be the conversion factor of international into absolute volts; then the energy in international electron volts corresponding to one mass unit is

$$M = \frac{c^2 M_O}{16eq}\,\frac{10^{-8}}{pq} = \frac{10^{-9}c^2}{Fpq^2},$$

where $F = 10e/(M_O/16)$ is the "Faraday." With Birge's values for the constants, we find

$$M = \frac{10^{-9}\cdot(2.99796\pm 0.00004)^2\cdot 10^{20}}{(96494\pm 1)(1.00051\pm 0.00002)\times(0.99995\pm 0.00005)^2}\text{ev},$$

$$= 931.05\pm 0.15 \text{ MV} \qquad (4)$$

where MV denotes million electron volts.[9] (The greatest uncertainty arises from the conversion factor q.)

With this conversion factor, we find that the binding energy of the Li^6 nucleus, compared to the free elementary particles, is

$$0.03342\cdot 931.0 = 31.11 \text{ MV}.$$

To assure the stability of Li^6, it must also be lighter than, e.g., H^3+He^3 or H^2+He^3, or generally, lighter than the sum of any two nuclei which between them contain as many neutrons and protons as Li^6. We find for the atomic weights:

[7] It is, in fact, the only point in which our information about nuclear properties is superior to the information about atomic properties: The total binding energy of all the electrons in an atom, which knowledge is quite worth while, can only be inferred by measuring the successive ionization potentials of the atom, a procedure extremely hard to carry out with heavy atoms; there is no way of determining this total binding energy directly. The reason why the nuclear binding energies show up in the atomic weights, while the binding energies of the atomic electrons practically do not, is of course the very large magnitude of nuclear binding energies, *viz.*, several million electron volts per nuclear particle compared to about 300,000 electron volts binding energy of all the electrons in the uranium atom together.

[8] Cf. Birge, Rev. Mod. Phys. **1**, 1 (1929).

[9] It seems to us that a newly introduced abbreviation should be as short as possible without giving rise to confusion with other abbreviations. It seems unnecessary to show explicitly in the abbreviation that *electron* volts are meant especially since this follows clearly from the text in any given case. On the other hand, it has always been customary to denote volts by a capital V, also mega- (or million) by a capital M, in contrast to "milli"- (which is usually denoted by m).

$He^4=4.00336$	$He^4=3.01699$
$H^2=2.01423$	$H^3=3.01610$
6.01759	6.03309
$Li^6=6.01614$	$Li^6=6.01614$
bind. en. = 0.00145 mass unit	0.01685 m.u.
= 1.35 MV	= 15.69 MV

Thus we see that the Li^6 nucleus is very stable against spontaneous disintegration into a He^3 and a H^3 nucleus, but much less stable against disintegration into an α-particle and a deuteron.

It has probably been noted that we have given the *atomic* weights of all the particles concerned rather than the nuclear weights. The atomic weight of an atom of nuclear charge Z contains, besides the weight of the nucleus, that of Z orbital electrons (atomic weight of one electron 0.000548). In ordinary nuclear reactions, the nuclear charge must always balance up, so that the atoms produced in the reaction contain just as many orbital electrons as the atoms originally present. Thus no error is introduced in the energy balance of a nuclear reaction if atomic masses are used instead of nuclear masses, since the number of electrons contained in the atomic masses does not change in the reaction. The same is true in the stability considerations carried out above. A given nucleus must always be compared to a number of other nuclei whose total charge is equal to its own charge.

Even for the energy evolved in radioactive β-decay, the use of atomic rather than nuclear masses is legitimate: If a nucleus of charge Z transforms into one of charge $Z+1$ plus a negative electron, the energy set free is obviously equal[10] to the mass of the nucleus Z, minus the sum of the masses of nucleus $Z+1$ and one electron. Now the atomic weight of atom Z contains the weight of Z electrons, while the weight of atom $Z+1$ contains the weight of one more electron; thus the difference of the two atomic weights is equal to the difference of the nuclear weights minus the mass of one electron, which is exactly the energy[10] set free in the β-decay. The mass of the neutrino (cf. §39) has been assumed to be zero.

Only in the case of positron radioactivity the difference of the weights of original and final atom does not give immediately the energy set free. Let us assume that an atom of nuclear charge Z and *atomic* weight M_Z emits a positron and thus transforms into an atom $Z-1$ with atomic weight M_{Z-1}. Calling the electron mass m, the masses of the two *nuclei* are M_Z-Zm and $M_{Z-1}-(Z-1)m$. The energy set free[10] is equal to the difference of these nuclear masses, minus the mass of the positron emitted, *viz.*,

$$E=(M_Z-Zm)-(M_{Z-1}-(Z-1)m)-m$$
$$=M_Z-M_{Z-1}-2m. \tag{5}$$

The energy evolved is thus equivalent to the difference of the atomic weights of the two atoms, minus *twice* the mass of an electron, or 0.00110 mass unit. The factor 2 comes in, because firstly a positron is created, and secondly one more electron is contained in the atomic weight M_Z than in M_{Z-1}.

It is sometimes useful to define the mass defect per elementary particle contained in the nucleus, the so-called packing fraction

$$P=\Delta/A=(M-A)/A. \tag{6}$$

Fig. 1 gives the packing fraction as a function of the mass number. The packing fractions of proton and neutron are +0.00807 and +0.00845, respectively. The packing fraction then decreases with increasing mass number, indicating stronger binding of the nuclear particles. P reaches, by definition, zero for O^{16}; then it becomes negative and almost constant, equal to −0.001 over a large region. This indicates that the mass defect and therefore the binding energy of all the nuclei from O up to Hg is very nearly proportional to the number of particles in the nucleus, which is a very important theorem (cf. §7). The total binding energy of any given nucleus in this region is

$$Z\Delta_H+(A-Z)\Delta_n-\Delta_A\approx A\left(\frac{P_H+P_n}{2}-P_A\right)$$
$$=0.009A \text{ mass units}=8\tfrac{1}{2}A \text{ MV}, \tag{7}$$

where Δ_H, Δ_n, Δ_A are the "mass excesses" of proton, neutron and atom A, as defined in (6); $P_H=\Delta_H$, $P_n=\Delta_n$ and P_A are the respective packing fractions; and it has been taken into account

[10] The factor c^2 is omitted.

that the number of protons in atom A is approximately equal to the number of neutrons, thus: $Z \approx A - Z \approx \frac{1}{2}A$. The value of the binding energy (7) is small compared to the energy corresponding to the nuclear mass (cf. (4)). Therefrom we may conclude that the velocities of nuclear particles are small compared to the velocity of light, so that nonrelativistic quantum mechanics can be applied to the motion of neutrons and protons in the nucleus.

§3. Size (G2)

The radii of nuclei range from about 2 or $3 \cdot 10^{-13}$ cm for the α-particle, up to about $9 \cdot 10^{-13}$ cm for the uranium nucleus. It seems that the volume of a nucleus is approximately proportional to its mass number, so that the volume per elementary particle is about the same in every nucleus. Only the very lightest nuclei seem to be exceptions from this rule.

The size of nuclei has been determined by the interaction of nuclei with small nuclear entities, such as proton, neutron, deuteron, α-particle. If a nucleus and a positively charged particle are a considerable distance apart, there is an electrostatic repulsion between them, calculable from the well-known Coulomb law. However, when the two particles get closer together, it is found that deviations from Coulomb's law set in, and finally an attractive force is found to exist between nucleus and external particle. The existence of this attractive force can be inferred from the fact that nuclear particles can become attached to an existing nucleus, forming a new stable nucleus. The point where the repulsion changes into an attraction gives a possible definition of the nuclear radius. It seems that the specifically nuclear forces between nucleus and external particle fall off very rapidly when the distance between the two becomes larger than the nuclear radius, or, more exactly, the sum of the radii of nucleus and particle, so that the Coulomb law holds if the distance is only just larger than this sum of the radii. Thus the boundary of a nucleus is quite well defined; better, at any rate, than the boundary of an atom.

The interaction between a neutron and nucleus is, of course, zero at large distances, the neutron being not subject to electric forces. In this case, there is only the "specifically nuclear" interaction which sets in when the distance of the two particles becomes of the order of the nuclear radius or smaller.

The most exact determination of nuclear radii is afforded by the lifetime of radioactive nuclei, emitting α-rays (cf. Chapter IX). The information about the sizes of light nuclei has been obtained from the scattering of α-particles and protons in the nuclear fields and from the probability of nuclear disintegrations as a function of the energy of the bombarding particles (Chapter X). Similar results can be obtained from the scattering of fast neutrons by nuclei (Chapter XII).

The size of the nuclei is of the order of magnitude which one should expect from the binding energy of the nuclear particles, assuming that the nucleus consists of protons and neutrons. The wave-length of a proton or neutron (mass M) is

$$\lambda = 2\pi\hbar/Mv = 2\pi\hbar(2ME_{\text{kin}})^{-\frac{1}{2}}, \qquad (8)$$

where $2\pi\hbar$ is Planck's constant, v the velocity and E_{kin} the kinetic energy of the particle. It is safe to assume that E_{kin} is of the same order of magnitude as the binding energy ϵ of the particle which is, according to the end of §2, equal to about $8\frac{1}{2}$ MV for atoms of medium atomic weight. We obtain therefore[11]

$$\lambda\!\!\!^{-} = \lambda/2\pi = \hbar(2M\epsilon)^{-\frac{1}{2}}$$
$$= 1.04 \cdot 10^{-27}(2 \cdot 1.66 \cdot 10^{-24} \cdot 8.5 \cdot 10^{6} \cdot 1.59 \cdot 10^{-12})^{-\frac{1}{2}}$$
$$= 1.55 \cdot 10^{-13} \text{ cm}.$$

This is of the same order but somewhat smaller than nuclear radii. $\lambda\!\!\!^{-}$ should, indeed, be somewhat smaller than the nuclear radius because heavier nuclei contain many protons and neutrons, most of which must be in excited quantum states. Their wave functions must then have several nodes inside the nucleus. Thus the observed size of nuclei again lends support to the hypothesis that protons and neutrons are the elementary particles constituting a nucleus.

If we want to apply the considerations corresponding to Eq. (8) to electrons, we have to use the relativistic relation between wave-length

[11] For obtaining estimates of the order of magnitude, $\lambda\!\!\!^{-}$ is a much more suitable quantity than the wave-length λ.

and kinetic energy; *viz.*,

$$\lambda\!\!\!^{-}=\hbar/p=\hbar c[(2mc^2+E_{\rm kin})E_{\rm kin}]^{-\frac{1}{2}} \qquad (9)$$

(m=electron mass, p=momentum). The radius of middle-sized nuclei, *viz.*, $5\cdot10^{-13}$, is certainly larger than $\lambda\!\!\!^{-}$ according to the foregoing. Therefore, if we put $\lambda\!\!\!^{-}=5\cdot10^{-13}$, we certainly obtain too small a value for $E_{\rm kin}$. We find then

$$E_{\rm kin}>2\pi\hbar c/\lambda=1.05\cdot10^{-27}\cdot3\cdot10^{10}/5\cdot10^{-13}\\ =6.3\cdot10^{-5}\text{ erg}=40\text{ MV}.$$

This value is much larger than nuclear binding energies, and the actual value of $E_{\rm kin}$ should be even bigger. This does not seem plausible and is thus a strong point against the assumption of the existence of electrons in the nucleus. A more rigorous disproof of this assumption will be given in §38.

§4. Statistics (E1)

Any kind of particles in nature obeys either the Fermi-Dirac or the Bose-Einstein statistics. In the first case, the Pauli principle holds for the particular sort of particles under consideration, i.e., there can never be two particles of this kind in the same quantum state. Notable examples are electrons (positive and negative), protons, and, as we shall see, neutrons and neutrinos (§39). It seems, in fact, to turn out that every "elementary" particle obeys Fermi statistics.

Particles obeying Bose statistics are allowed to be in the same quantum state; indeed, they have even what may be called a preference for being in the same state. Photons are the most well-known example; deuterons, α-particles and a great many other nuclei also belong to this category.

The most rigorous and most fruitful definition of the statistics is based upon properties of the wave function of a system of particles of a given kind. Let us suppose we have n identical particles 1, 2, $\cdots$, i, $\cdots n$; then the wave function describing their motion will be a certain function $\psi(x_1\cdots x_i\cdots x_n)$ depending on the coordinates[12] of the particles. If the coordinates of two particles are interchanged, e.g., the coordinate x_j of particle j inserted in place of x_i, and x_i in place of x_j, another function of the coordinates is obtained. This new function is (a) identical with the original function, if the particles obey Bose statistics, (b) identical except for a change of sign, if the particles obey Fermi statistics. Functions of kind (a) are called symmetrical in the particles, such of kind (b) antisymmetrical. In formulae, we have

$$\begin{aligned}&\psi(x_1\cdots x_j\cdots x_i\cdots x_n)\\&=+\psi(x_1\cdots x_i\cdots x_j\cdots x_n)\ \text{(Bose statistics)},\\&\psi(x_1\cdots x_j\cdots x_i\cdots x_n)\\&=-\psi(x_1\cdots x_i\cdots x_j\cdots x_n)\ \text{(Fermi statistics)}.\end{aligned} \qquad (10)$$

The experimental determination of the statistics of nuclei is based on alternating intensities in rotational band spectra, most conveniently of diatomic molecules. The theory of this determination will be described in §47. It has been found that the proton obeys Fermi statistics, the deuteron, the α-particle, the nuclei of N^{14}, O^{16}, etc., Bose statistics, the Li^7 nucleus again Fermi statistics. From the observations the general rule *can be inferred that all nuclei with even atomic weight follow Bose statistics, those of odd atomic weight Fermi statistics.* This is in accord with the assumption that all nuclei are composed of protons and neutrons, and that the *neutron has Fermi statistics.*

To prove this statement, we have to show that a system composed of elementary particles each obeying Fermi statistics, obeys Bose or Fermi statistics according to whether the number of elementary particles in the system is even or odd. We consider two systems (nuclei), α and β, each containing m elementary particles of one sort (protons) and n particles of another sort (neutrons). We assume that the first m protons and the first n neutrons constitute the nucleus α which is situated near the point r_α, the second set of m protons and n neutrons are bound up in the second nucleus β near r_β. This state of affairs will be described by a certain wave function depending on the coordinates of all particles. We now exchange one proton of nucleus α and one of nucleus β; when doing this, the wave function of the whole system is multiplied by -1 because the protons obey Fermi statistics. We then exchange another pair of particles and continue this process until all

[12] x_i is meant to symbolize all three coordinates of the particle i, and, if the particles have spin, also the spin coordinate.

the m protons and n neutrons originally constituting nucleus α have been brought to the point r_β and *vice versa*. Every exchange multiplies the wave function by -1, thus after the exchange of all $m+n$ particles of nucleus α against the $m+n$ particles constituting β the wave function has been multiplied by $(-1)^{m+n}$. On the other hand, our process corresponds to an exchange of the entire nuclei α and β. Thus the wave function is multiplied by $(-1)^{m+n}$ when the coordinates of our two nuclei are interchanged, which means that the wave function is symmetrical (antisymmetrical) in the coordinates of the two nuclei if the total number of particles $m+n$ in each nucleus is even (odd). This is the theorem which we wanted to prove.

The proof given above is however not quite rigorous, because it is not possible to construct a wave function which is antisymmetrical in all protons and neutrons and which at the same time assigns definite particles to a definite nucleus. A rigorous proof has been given by Ehrenfest and Oppenheimer (E1). It should be noted that at the time when Ehrenfest and Oppenheimer's paper was written, the neutron had not yet been discovered and it was therefore believed that nuclei consist of protons and *electrons*. Thus the word "electrons" in their paper should be replaced by "neutrons" throughout. (If this is done, the contradiction between theoretical and experimental results concerning the statistics of N^{14} disappears.)

The first use we make of our theorem is to deduce the statistics of neutrons from the experimental fact that the proton obeys Fermi and the deuteron Bose statistics. Assuming the deuteron to consist of one neutron and one proton, we must conclude from our theorem that the neutron follows the Fermi statistics. Knowing now the statistics of the neutron, we can predict that any nucleus of even atomic weight will obey Bose statistics because the total number of particles, protons and neutrons together, is equal to the atomic weight, and it is this total number which determines the statistics. Similarly, all nuclei with odd atomic weight must obey Fermi statistics. No exception has been found to this rule. The nuclei investigated for statistics are H^1, H^2, He^4, Li^7, N^{14}, O^{16}, Na^{23}, P^{31}, S^{32}, Cl^{35} and K^{39}.

The old nuclear theory assumed the nuclei to consist of protons and electrons. The number of protons then had to be assumed equal to the atomic weight A, because only the protons contributed to the weight, while $A-Z$ electrons had to be assumed in order to neutralize the charges of $A-Z$ protons and leave a resultant charge of only Z. The total number of particles was thus $2A-Z$ and was thus even or odd according to whether the nuclear charge was even or odd. Accordingly, all elements with odd Z, e.g., H and N, should have obeyed Fermi statistics, irrespective of their atomic weight. This was in direct contradiction to the experimental result for H^2 [13] and N^{14} (R7) which contradiction constituted another strong argument against the "electron theory" of nuclear constitution.

§5. Spin and Magnetic Moment (cf. chapter VIII)

The intrinsic angular momentum (spin) of nuclei can be determined from hyperfine structure, molecular ray analysis, depolarization of resonance radiation, and alternating intensities in rotational band spectra of molecules. These methods except for the last also determine the magnetic moment associated with the spin. A detailed description of methods and results will be given in chapter VIII.

The most important result for a general theory of the nuclei is that the spins of all nuclei of odd atomic weight seem to be half-integer multiples of $\hbar$, while all nuclei of even atomic weight have integer spin, most of them probably having spin zero. The total spin of a nucleus is the resultant of all the angular momenta of the orbital motions of all particles inside the nucleus, and of all the spins of the nuclear particles. The resultant has to be taken according to the rules of the vector model of quantum theory. Now the *orbital* angular momenta are always integers (in units $\hbar$). Thus the appearance of half-integer values for the total spin of some nuclei, must be attributed to half-integer values for the *spins* of the individual nuclear particles.

[13] Actually the statistics of deuterons were only determined after the discovery of the neutron.

The empirical rule connecting atomic weight and nuclear spin must thus be interpreted as showing that both the proton and the neutron have half-integer spins. In the case of the protons it can be shown experimentally that the spin is exactly $\frac{1}{2}$. The neutron spin might, from experimental evidence, be just as well $\frac{3}{2}$ as $\frac{1}{2}$. However, simplicity is a strong argument in favor of the value $\frac{1}{2}$ which we shall, therefore, assume throughout this article. It seems to be a general rule that this value of the spin is true for all elementary particles known, *viz.*, proton, neutron, electron (positive and negative) and neutrino (§39).

If both proton and neutron have spin $\frac{1}{2}$, then the resultant of the spins of A elementary particles, neutrons and protons, will be integer or half-integer according to whether the atomic weight A is even or odd. This conclusion from the vector model of quantum theory is in accord with all existing observations. It is analogous to the statement about the statistics of nuclei in the preceding paragraph, such that nuclei obeying Bose statistics have integer spins and such obeying Fermi statistics have half-integer spins. The two statements are, however, independent of each other—at least as long as we do not understand the connection between statistics and spin of a particle which seems to exist but has thus far not been explained.

It need hardly be pointed out that the old nuclear theory which assumed the nuclei to consist of protons and electrons, faced with respect to spin a difficulty analogous to that regarding the statistics. The experimental situation was even worse in the case of the nuclear spin, because a great number of spins of nuclei with even nuclear charge and odd atomic weight had been measured and were found to be half-integer in contradiction to the "electron theory" and conforming to the "neutron theory" of nuclei, whereas the statistics had actually been determined for only one nucleus with even A and odd Z.

The magnetic moment of the proton has been determined by Stern and Estermann (E4) and by Rabi, Kellogg, and Zacharias (R2). It does not have the value of one nuclear "magneton,"

$$\mu_0 = \hbar e/2Mc = 5.02 \cdot 10^{-24} \text{ gauss cm}^3 \quad (11)$$

(M = mass of the proton). μ_0 would be the magnetic moment which would be expected if Dirac's theory would hold for protons. The value actually observed is about $2.9\mu_0$,[14] i.e., much greater than the "theoretical" value μ_0.[15] Attempts to explain this discrepancy will be explained in §45. It has been proved recently by Kellogg, Rabi and Zacharias (K3) that the magnetic moment points in the *same* direction as the angular momentum of the proton spin, as would be expected for a positive charge (for electrons, the directions of magnetic moment and spin are opposite).

The magnetic moment of the neutron is hardly accessible to a direct measurement. It has to be inferred from the moments of other, more complex, nuclei. The simplest of these is the deuteron which consists of one proton and one neutron. Its spin has been measured and turns out to be unity. It can safely be assumed that the deuteron has no orbital angular momentum in its ground state (§12); thus its observed angular momentum 1 must be attributed to the *spins* of the proton and the neutron in the deuteron. This means that the spins of proton and neutron are parallel in the deuteron. Thus the magnetic moment of the deuteron is the sum of the magnetic moments of proton and neutron.

The most exact experimental determination of the magnetic moment of the deuteron is that of Kalckar and Teller (theory of the method, K1) and of Farkas, Farkas and Harteck (experiment, F1). Its basis is the measurement of the velocity of the conversion of orthohydrogen into parahydrogen by the action of paramagnetic gases. This velocity is, among other things, proportional to the square of the magnetic moment of the hydrogen nuclei. By measuring the velocity of conversion for light hydrogen (H_2) and heavy hydrogen (D_2), the ratio of the magnetic moments of H and D can be determined. The result is

$$\mu_D : \mu_H = 1 : 4 \quad (12)$$

[14] New measurements of Rabi, Kellogg and Zacharias (private communication).

[15] In spite of this fact, the magnetic moment of the proton is, of course, much smaller than the Bohr magneton of an electron, $\mu_1 = \hbar e/2mc = 1838\mu_0$ (m = electron mass).

with an accuracy of about 5 percent.* The magnetic moment of the deuteron is also directed in the same direction as the spin (R4), therefore, we find

$$\mu_{\text{neutron}}=\mu_D-\mu_H=-\tfrac{3}{4}\mu_H=-2.2\mu_0. \quad (13)$$

Thus the magnetic moment of the neutron has the direction which would be expected if the neutron had a negative charge. Its magnitude is of the same order as the magnetic moment of the proton.

II. Qualitative Arguments about Nuclear Forces

In the theory of *atomic* structure, we deal with electric particles, *viz.*, nucleus and electrons, and we therefore know the forces acting between them. The problem of atomic physics has therefore not been to determine the forces between atomic particles but to find out how electrons move if subjected to a known force. This problem has been solved by quantum theory.

In nuclear theory we can have confidence that the quantum theory holds for the motion of the neutrons and protons in the nucleus. This assumption is strongly supported by the relation between size and binding energy of nuclei (cf. §3), which is just what should be expected from quantum theory, and also by the success of actual calculations (chapters III, IV, etc.). Furthermore, we can safely assume that relativity corrections are small because the binding energies are small compared to the energy corresponding to the rest mass of proton and neutron (end of §2).

On the other hand, we do not know the forces between the nuclear particles, with the exception of the Coulomb repulsion between the protons in the nucleus, which, however, plays only the role of a correction (§8). The principal attractive forces are certainly not electric in nature because they act upon neutrons which bear no charge. What the nature of these forces is, how they depend on the distance of nuclear particles, on their spin and possibly other quantities, has to be inferred from experimental data. We shall do that in this chapter in a qualitative way, and in later chapters apply the knowledge thus obtained to special problems which will furnish more quantitative data on the nuclear forces.

§6. The Ratio of Atomic Weight to Nuclear Charge (H7)

When the periodic system was first discovered, nothing was known about nuclear charge. The atoms were ordered according to their atomic weight and this led to the discovery of the periodic system. This fact alone shows that the atomic weight is closely connected with the nuclear charge. Indeed, the known stable isotopes of any given element do not vary greatly in atomic weight, the variation being only 10 percent even for an element with so many isotopes as tin. Thus, to a first approximation, we may speak of a definite relation between atomic weight and nuclear charge.

For the light elements, up to about argon, this relation is very simple indeed. The atomic weights are very nearly twice as large as the nuclear charge, e.g., C^{12}, N^{14}, O^{16}, etc. Therefore the number of neutrons $N=A-Z$ in any of these light nuclei is approximately equal to the number of protons Z.

This experimental rule must be interpreted as showing that the largest attractive forces in the nucleus are forces between neutrons and protons. If this were not the case, e.g., if two neutrons would attract each other more strongly than a neutron and a proton, the most stable nuclei would obviously be composed exclusively of neutrons. We can, of course, not deduce from our empirical rule that there are no forces between a pair of neutrons or a pair of protons[16] at all, but if there are such forces, they must be smaller than the force between a proton and a neutron.

Our rule $N=Z$ tells us even more about the forces between like particles (two protons, or two neutrons). Provided such forces exist at all,

* *Note added in proof:* Recent experiments of Rabi, Kellogg and Zacharias seem to show that the ratio of the moments is smaller, about 1 : 3.5. This corresponds to a deuteron moment of $0.85\mu_0$ and a neutron moment of about $-2.0\mu_0$.

[16] We refer here to forces between two protons *beside* the Coulomb repulsion.

they must be very nearly equal, i.e., the force between two neutrons must be nearly equal to that between two protons, leaving out the electrostatic repulsion between the latter. For instance, if the attraction between two protons were larger than that between two neutrons, the maximum of stability would not occur for equal numbers of neutrons and protons but would be shifted towards a relatively larger number of protons in the nucleus. Explicitly, if (nn), (pp), (np) denote the binding energies between two neutrons, two protons, and one neutron and one proton, respectively, the energy of a nucleus containing N neutrons and Z protons would be, neglecting 1 compared to N and Z:

$$\tfrac{1}{2}N^2(nn)+\tfrac{1}{2}Z^2(pp)+NZ(pp), \qquad (14a)$$

which reaches its maximum, for a given atomic weight $A=N+Z$, when

$$N-Z(=A-2Z)=\tfrac{1}{2}A\frac{(nn)-(pp)}{(np)-\tfrac{1}{2}(nn)-\tfrac{1}{2}(pp)}. \qquad (14)$$

Experimentally, the difference $N-Z$ does, for the most stable light nuclei, certainly not exceed 10 percent of the nuclear charge $Z\approx\frac{1}{2}A$. According to (14), the difference $(nn)-(pp)$ must then certainly be smaller than one-tenth of $(np)-\frac{1}{2}(nn)-\frac{1}{2}(pp)$. The most satisfactory assumption from the standpoint of symmetry as well as from the experimental evidence, is that the force between two neutrons and that between two protons are *exactly* equal—disregarding, of course, the Coulomb energy between the protons, and provided there are any forces between like particles at all.

§7. Saturation of Nuclear Forces (H7, M6)

If every particle in the nucleus is supposed to interact with every other particle, the interaction energy, and therefore the binding energy holding the nucleus together, would be roughly proportional to the number of interacting pairs, i.e., to the *square* of the number of particles in the nucleus. If any deviation could be expected from this law, it would be in the direction of a more rapid increase of the binding energy with the number of particles, because with increasing interaction between them, the particles will draw closer together, and this will lead to an increase of the interaction even between a single pair of particles, making the total binding energy increase faster than A^2.[17]

Actually, it is found experimentally that the mass defects of nuclei, and therefore the binding energies, increase only linearly with increasing number of particles (§2). This fact may be compared to the behavior of a liquid or solid containing many atoms where the total chemical binding energy is sensibly proportional to the number of atoms present.

We therefore try to be guided by the chemical analogy. How does the proportionality of the chemical binding energy to the number of atoms arise? There are essentially three possible reasons for this, corresponding to the three possible types of chemical binding: polar binding, homopolar binding and van der Waals (polarization) binding.

The van der Waals type is most clearly realized in the case of rare gases in the liquid or solid state. Between any two rare gas atoms, we have an attractive force (van der Waals force) when the atoms are more than a certain distance r_0 apart. The attractive force falls rapidly, approximately as r^{-7}, when the distance r between the atoms increases. For distances smaller than r_0, a strong repulsive force begins to act which prevents any appreciable interpenetration of the two atoms. In the liquid, only near neighbors have any appreciable interaction, because of the rapid decrease of the attractive force with increasing atomic distance. Any given atom thus interacts only with a small number of neighbors, however large the total number of atoms may be. On the other hand, the repulsive forces prevent any increase in density which would allow more atoms to interact with any given atom. Thus, the repulsive forces which prevent the interpenetration of atoms are, in this case, primarily responsible for the binding energy being proportional to the number of atoms. However, it would seem very unsatisfactory to transfer such a mechanism to nuclei: it would involve the assumption of a force between *elementary* particles, *viz.*, protons and neutrons, which would be attractive at large distances and repulsive at small distances, an assumption which one would make only very reluctantly. (For particles with internal structure, such as atoms or the α-particle, the assumption of such a force is, of course, not objectionable but results directly from *simple* assumptions about the forces between elementary particles.)

The polar binding is realized in salts, e.g., NaCl. Two unlike atoms (Na and Cl) attract each other with a force

[17] The conditions outlined in this paragraph are actually found for the electrons in atoms. The total binding energy of all electrons in an atom increases roughly as the 7/3 power of the number of electrons, i.e., faster than Z^2 (cf. e.g., reference S23).

which decreases very slowly with increasing distance, two like particles repel each other with a similar force. The counteraction of these two forces keeps the binding energy from increasing quadratically with the number of particles, in spite of the fact that there is appreciable interaction between far distant ions. The assumption of repulsive forces between like particles in the nucleus is, however, impossible for other reasons (§10); this is *a fortiori* true for the assumption that these repulsive forces are equal in magnitude to the attractive forces between neutron and proton. Thus the analogy to the polar binding must also be rejected.

The homopolar binding is most clearly represented by elements like hydrogen. There is strong binding between two hydrogen atoms. A third atom, however, would not be strongly attached to the H_2 molecule. We say, the H_2 molecule is saturated. An assembly of many hydrogen atoms, e.g., in a drop of liquid hydrogen, therefore has an energy approximately equal to that of the corresponding number of hydrogen molecules, and therefore proportional to the number of atoms present. It is true that the binding energy of a hydrogen droplet will be slightly greater than that of separated molecules because of van der Waals forces between the H_2 molecules, but they again give an energy proportional to the first power of the number of atoms.

We thus see that we shall obtain the correct dependence of nuclear binding energies on the number of particles in the nucleus, if we assume forces between the nuclear particles which show saturation, in much the same way as the forces of homopolar chemical binding. It is at once clear that the association of nuclear particles which will correspond most nearly to a saturated molecule, is the α-particle. In fact, the binding energy of the α-particle, as deduced from its mass defect, is 28 MV, or 7 MV per elementary particle. The binding energy of the nuclei which have the highest packing fractions, *viz.*, those with Z round 30, is $8\frac{1}{2}$ MV per elementary particle. In our analogy, this means that 7 MV of these $8\frac{1}{2}$ are due to the chemical "binding energy of the molecule" He^4, while the remaining $1\frac{1}{2}$ MV are to be attributed to "van der Waals" forces between the α-particles.

The "van der Waals" forces between the α-particles can, without difficulty, be assumed analogous to those between rare gas atoms or H_2 molecules; i.e., an attraction at larger distances, falling off very rapidly with increasing distance, and a repulsion at small distances, giving something like mutual impenetrability of two α-particles. A force of this type has actually been deduced by Heisenberg (§31). The assumptions will, of course, lead to a binding energy between α-particles which is approximately proportional to the number of such particles.

The α-particle contains 2 neutrons and 2 protons. The forces between neutrons and protons must thus be such that they are saturated when 2 neutrons and 2 protons are brought together and are practically nil when a third proton or neutron is brought into the neighborhood of the first 4 particles. Now 2 neutrons and 2 protons can, on account of Pauli's principle, just be placed into the same quantum state as regards their motion in space, due to the possibility of two different states of spin. We shall therefore assume that *protons and neutrons exert strong forces upon each other only if they are in the same*, or approximately the same, quantum state *with regard to their motion in space*, i.e., if their wave functions depend in approximately the same way on the spacial coordinates.

However, we must not assume that the forces depend critically upon the relative spin of the two particles. If we did so, e.g., if we would assume that a proton and a neutron would only interact strongly when their spins are parallel, then the nuclear forces would already be saturated in the deuteron (proton plus neutron with parallel spins). In reality, the binding energy of the deuteron is only slightly over 2 MV, compared to 28 MV for the α-particle. This shows that the deuteron can certainly not be regarded as saturated. *The forces between proton and neutron can therefore depend only slightly, if at all, upon the relative spin directions of the two particles.*[18]

Turning to the mathematical representation of the interaction, we may again be guided by the chemical analogy. The forces of homopolar

[18] Heisenberg (H7) had originally assumed an interaction which was attractive for particles with parallel spins, repulsive for opposite spins. According to the foregoing consideration, such an interaction would make the deuteron a saturated structure. This was pointed out by Majorana (M6) and the assumed interaction changed accordingly.

binding are "*exchange*" forces, connected with the changing places of the electrons from one atom in the molecules to the other. We know from the chemical analogy that such exchange forces show saturation. *We therefore assume that the nuclear forces* also *have the character of exchange forces between neutron and proton.* Just as in the case of molecules, but of course not due to the same mechanism, an electron[19] passes from neutron to proton so that the former neutron is transformed into a proton and the former proton into a neutron. This can be considered as an exchange of the coordinates of neutron and proton: Thus our "exchange forces" mean that neutron and proton interchange their positions whenever they interact. The mathematical formulation of these ideas will be deferred to §11.

Since this paragraph contains the clue to all nuclear theory, we want to sum up. The proportionality of nuclear binding energy and number of particles in the nucleus requires the assumption of exchange forces between the nuclear particles which show saturation. The high binding energy of the α-particle, compared to the deuteron, requires these forces *not* to show saturation for the deuteron, and therefore not to depend to any considerable extent upon the relative spin directions of the interacting particles.

From our analogy between nuclear and chemical forces, we can draw a conclusion about the size of nuclei, as a function of the number of particles contained in them. We know that the volume of a droplet of a liquid, or of a solid, is proportional to the number of atoms contained in it, each atom occupying about the same volume. Since the nuclei are held together by forces similar to chemical forces, we may expect them also to have a volume proportional to the number of particles in the nucleus. This is in agreement with the experimental evidence referred to in §3.

[19] Together with electron, a neutrino must pass over. The "passage" is, of course, not meant literally in the sense that the electron and neutrino are bound in the neutron and then pass bodily to the proton. We might assume that electron and neutrino are just "created" for a very short time, of the order $e^2/mc^3 = 10^{-24}$ sec., and are then reabsorbed. For details see §44.

It remains to be said that no exact proportionality between binding energy and number of particles is to be expected. Not only will there be irregular variations depending on the special structure of any particular nucleus, but also a regular trend towards slightly increasing binding energy per particle, with increasing size of the nucleus. This effect is analogous to the surface tension of a droplet of a liquid. The atoms on the surface of the droplet do not receive the full attraction which atoms in the interior would receive, and do therefore not contribute their full share to the binding energy. The same is true for the particles at the surface of a nucleus. Since the number of the surface particles, as a fraction of the total number of particles, decreases when the total number increases, we expect a slight increase of the binding energy per particle with increasing atomic weight. This is actually shown by Fig. 1 which represents the packing fractions as function of A.

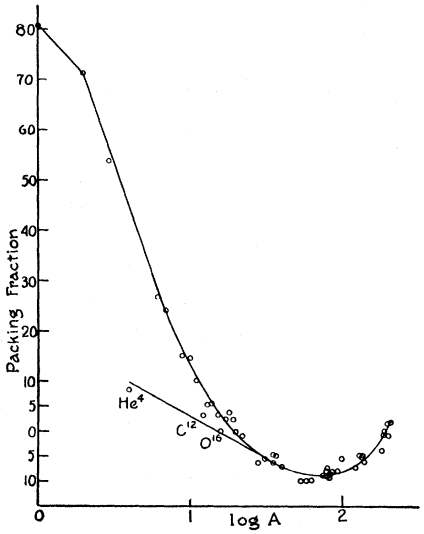

FIG. 1. Packing fraction as a function of atomic weight. In order to represent light and heavy atoms in the same diagram, a logarithmic scale has been chosen for the absicissa. It is seen that among the light nuclei the α-particle and its multiples (C^{12}, O^{16}) have much smaller packing fractions than all the other nuclei. (The packing fractions are expressed in ten-thousandths of a mass unit.)

§8. The Electrostatic Repulsion of the Protons. Stability Against α-Decay (H7, W2)

The preceding two paragraphs have led us to assume (1) that the binding energy of nuclei is approximately proportional to the number A of the particles in the nucleus and (2) that the binding energy, for given total number of particles, is a maximum when neutrons and protons are present in equal numbers. This can be expressed by the rough formula

$$E_n=(Z+N)\epsilon[1-\alpha((N-Z)/(N+Z))^2], \quad (15)$$

where ϵ is the binding energy per particle for a nucleus containing equally many protons and neutrons, and α a constant which measures the dependence of the binding energy on the "isotopic number" $I=N-Z$.

α might, in principle, still depend on the total number of particles, A. It would not do so, however, if we assume the binding energy to be proportional to (14a) *for a given value of* A. It is true that (14a) has been derived by assuming every particle in the nucleus to interact with every other particle which is not possible when the forces show saturation. On the other hand, the saturation will affect primarily the dependence of the binding energy on the total number of particles, rather than that on the ratio of the numbers of neutrons and protons. We therefore accept (15) as a preliminary expression, deferring a more accurate treatment to chapter V.

In our discussion thus far, we have neglected the electrostatic forces between the protons. Taking for the mutual distance of two protons r approximately the nuclear radius (cf. §3), i.e., about $5\cdot10^{-13}$ cm $\approx 2e^2/mc^2$ (m=electron mass), we find for their electrostatic potential energy $e^2/r\approx\frac{1}{2}mc^2=\frac{1}{4}$ MV. This is indeed negligible compared to the average binding energy per particle, *viz.*, $8\frac{1}{2}$ MV (cf. §2).

However, because of the saturation character of the specifically nuclear forces the Coulomb repulsion between the protons becomes important for heavy nuclei in spite of its smallness for a single pair of protons. For the Coulomb force shows, of course, no saturation. Therefore the total energy of the Coulomb interaction is actually equal to the number of pairs of protons in the nucleus, i.e., $\frac{1}{2}Z(Z-1)$, times the potential energy of a single proton pair. The latter is in the average $(6/5)e^2/R$ with R the radius of the nucleus, if the protons are considered as distributed uniformly over the nucleus.[20] Thus the total electrostatic energy of the protons is

$$E_{el}=\tfrac{3}{5}Z^2e^2/R \quad (16)$$

if 1 is neglected in comparison to Z. Since R is proportional to the cube root of the atomic weight (cf. §3, and end of §7), and Z proportional to the atomic weight itself, the electrostatic energy is proportional to $A^{5/3}$. On the other hand, the binding energy E_n due to the specifically nuclear forces is only proportional to the first power of the atomic weight A. Thus the relative importance of the electrostatic forces increases with increasing atomic weight, roughly as $A^{2/3}$.

The consequences of this are twofold: Firstly, we shall obtain for heavier nuclei a deviation from the rule $N=Z$, in the sense that stable nuclei contain fewer protons than neutrons, because the replacement of a proton by a neutron decreases the electrostatic repulsion and thus the total energy of the nucleus. This effect is well-known experimentally: The ratio N/Z, i.e., number of neutrons to number of protons, increases from 1 for light nuclei gradually to 1.6 for uranium.

Secondly, the binding energy per particle will decrease, on account of the electrostatic forces, with increasing atomic weight. This effect works in the opposite direction from the "surface tension" discussed at the end of §7. The surface tension will be more important as long as the nucleus is still small, therefore we get a decrease of the packing fraction with increasing atomic

[20] If $v=4\pi R^3/3$ is the volume of the nucleus, $1/v$ will be the charge density due to one proton distributed uniformly. The electrostatic potential due to this charge density, at a distance r from the center of the nucleus, is according to ordinary electrostatics,

$$V(r)=e\frac{4\pi}{v}\left(\int_0^r\frac{\rho^2d\rho}{r}+\int_r^R\frac{\rho^2d\rho}{\rho}\right)=\frac{4\pi e}{v}\left(\frac{1}{2}R^2-\frac{1}{6}r^2\right).$$

The energy of a second proton, uniformly distributed, in this potential is

$$w=\frac{e}{v}\int_0^R 4\pi r^2dr\,V(r)=\left(\frac{4\pi e}{v}\right)^2\left(\frac{1}{2}R^2\frac{1}{3}R^3-\frac{1}{6}\frac{1}{5}R^5\right)$$
$$=\left(\frac{3e}{R^3}\right)^2\cdot\frac{1}{6}\frac{4}{5}R^5=\frac{6}{5}\frac{e^2}{R}.$$

weight in that region. On the other hand, for heavy nuclei the electrostatic repulsion between the protons will be more important so that the packing fraction rises again towards the end of the periodic table. This has actually been observed (cf. Fig. 1). For very heavy nuclei, this rise becomes so pronounced that the nuclei become unstable against α-disintegration.

The total binding energy of a nucleus is the difference of the binding energy given in (15) and the Coulomb repulsion (16). To write the latter in a suitable form, we put

$$R=r_0A^{\frac{1}{3}}, \tag{17}$$

where r_0 can be determined from the known radii of radioactive nuclei: The average of these radii as determined from the lifetimes of α-decaying nuclei, is $9\cdot10^{-13}$ cm. The average atomic weight of the nuclei concerned is 222, therefore

$$r_0=9\cdot10^{-13}\cdot222^{-\frac{1}{3}}=1.48\cdot10^{-13}\text{ cm} \tag{17a}$$

and

$$\frac{3}{5}\frac{e^2}{r_0}=0.60mc^2\frac{e^2/mc^2}{r_0}$$

$$=0.307\cdot\frac{2.80\cdot10^{-13}}{1.48\cdot10^{-13}}\text{MV}=0.58\text{ MV}. \tag{17b}$$

We shall abbreviate this expression by the letter γ.

Inserting (17), (17b) into (15) and (16), we obtain for the binding energy

$$E=A\epsilon-\alpha\epsilon(A-2Z)^2/A-\gamma A^{-\frac{1}{3}}Z^2. \tag{18}$$

The maximum of this expression for given A is obtained when

$$2\alpha\epsilon(A-2Z)/A-\gamma A^{-\frac{1}{3}}Z=0$$

or

$$\frac{I}{Z}=\frac{N-Z}{Z}=\frac{A-2Z}{Z}=\frac{\gamma}{2\alpha\epsilon}A^{\frac{2}{3}}, \tag{18a}$$

$$I/A=\gamma A^{\frac{2}{3}}/(4\alpha\epsilon+\gamma A^{\frac{2}{3}}). \tag{18b}$$

The ratio of the isotopic number $I=N-Z$ to the nuclear charge Z, is thus proportional to the two-third power of the atomic weight. This relation is illustrated by Fig. 2, in which $(A-2Z)$ is plotted against A, for known stable nuclei. It can be seen that the observed points fall near, and on both sides of the solid line.[21] That line represents the relation (18b), with

$$\gamma/2\alpha\epsilon=0.0146. \tag{18c}$$

This value is so chosen that the line passes

[21] For deviations from that line (periodicities) cf. §34.

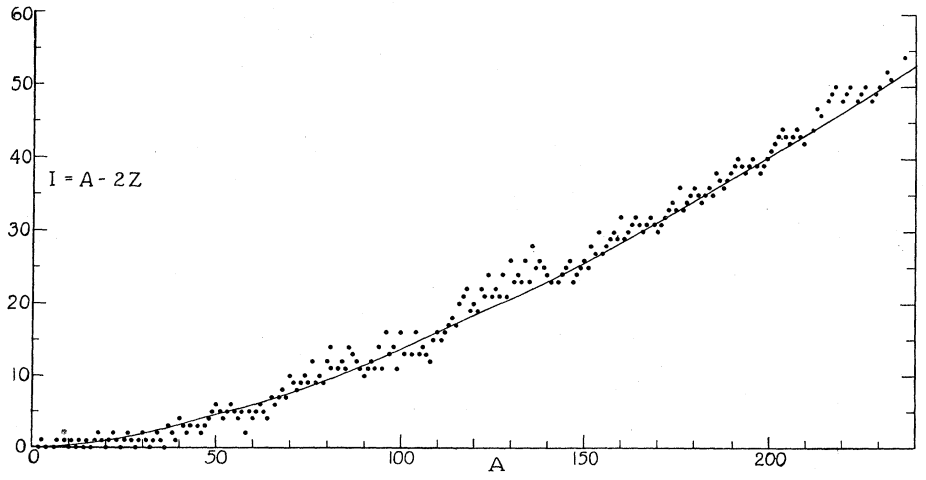

FIG. 2. Existing isotopes (isobars omitted). Abscissa mass number, ordinate isotopic number. Each dot represents a known isotope (cf. Table I). The line gives the empirical relation (18b) between average isotopic number and atomic weight. Fluctuations of the isotopic number of the existing isotopes around the solid line are clearly shown (cf. §34).

through Hg^{200}, for which $A=200$, $Z=80$, therefore $(A-2Z)/Z=0.50$ and

$$\gamma/2\alpha\epsilon=0.50/(200)^{\frac{2}{3}}=0.0146. \quad (18d)$$

We may now calculate the total binding energy of a nucleus whose charge has the "most favorable" value for the given mass of nucleus. To do this, we insert the value (18b) for $(A-2Z)/A$ into (18) and find

$$E_{\max}(A)=A\epsilon-A\alpha\epsilon[\gamma A^{\frac{2}{3}}/(4\alpha\epsilon+\gamma A^{\frac{2}{3}})]^2 -\gamma A^{5/3}[2\alpha\epsilon/(4\alpha\epsilon+\gamma A^{\frac{2}{3}})]^2, \quad (19)$$

$$E_{\max}(A)=A\epsilon[1-\alpha\gamma A^{\frac{2}{3}}/(4\alpha\epsilon+\gamma A^{\frac{2}{3}})],$$

or, denoting by Z_A the "most favorable charge" for the atomic weight A, we get from (18b)

$$E_{\max}(A)=A\epsilon-(A-2Z_A)\alpha\epsilon. \quad (19a)$$

Now the value of $\alpha\epsilon$ can be deduced from (17b) and (18c):

$$\alpha\epsilon=0.58/2\cdot 0.0146=20 \text{ MV}. \quad (19b)$$

Inserting this and the observed value of the binding energy for Hg^{200} into (19) we can deduce ϵ. The binding energy can be calculated from the atomic weight of Hg^{200}, 200.016, and the combined weight of 80 protons and $200-80=120$ neutrons, which gives (cf. 75a)

$$E(Hg^{200})=120\cdot 1.00846+80\cdot 1.00807 -200.016=1.645 \text{ mass units} \quad (19c)$$
$$=931\cdot 1.645 \text{ MV}=1530 \text{ MV}.$$

Therefore, from (19a) and (19b)

$$\epsilon=(1530+40\cdot 20)/200=11.6_5 \text{ MV}. \quad (19d)$$

This is rather higher than the binding energy per particle for medium sized nuclei (8.5 MV). The reason is that the actual binding energy is reduced due to surface tension as well as due to the electrostatic repulsion of the protons, and even for medium sized nuclei both these effects are quite appreciable and reduce the "naive" binding energy, which would be $11.6A$, by over 25 percent. For mercury, the observed binding energy per particle is only $1530/200=7.6_5$ MV, which is 35 percent less than the binding energy would be if there were no electrostatic forces. *Thus the electrostatic forces amount to 35 percent of the "specifically nuclear" forces for mercury.*

From (19b) and (19c) we find furthermore

$$\alpha=20/11.6_5=1.72,$$

which is quite reasonable (cf. §30, Eq. (185)).

The most interesting question to be answered approximately by our rough formulae is the probable energy of α-particles which might be emitted by radioactive atoms. Of course, we can only give an "average" value for this energy which depends smoothly on the atomic weight while the actual α-energies vary irregularly from one radioactive atom to the other, which variation could only be deduced from a more refined theory. Let us suppose the α-emitter has a nuclear charge "most favorable" for its atomic weight; its energy is thus given by (19a). The nucleus produced in the α-disintegration will have a charge slightly different (too small) from the most favorable charge for its weight. But since the binding energy has, for given A, a maximum at $Z=Z_A$, it varies only quadratically with the difference $Z-Z_A$. Thus we may assume that (19a) is very nearly true even for the product nucleus. Therefore we have:

(1) binding energy of the α-emitter (atomic weight A, nuclear charge Z_A)

$$E_1=A\epsilon-(A-2Z_A)\alpha\epsilon;$$

(2) binding energy of the product nucleus (atomic weight $A-4$, nuclear charge approximately Z_{A-4})

$$E_2=(A-4)\epsilon-(A-4-2Z_{A-4})\alpha\epsilon$$
$$=E_1-4\epsilon+4\alpha\epsilon d(A-2Z_A)/dA$$
$$=E_1-4\epsilon+4\alpha\epsilon\gamma A^{\frac{2}{3}}\frac{(20/3)\alpha\epsilon+\gamma A^{\frac{2}{3}}}{(4\alpha\epsilon+\gamma A^{\frac{2}{3}})^2}; \quad (20a)$$

(3) binding energy of the α-particle

$$E_3=4\epsilon', \quad \text{where} \quad \epsilon'=6.9 \text{ MV};$$

therefore energy of a would-be-emitted α-particle

$$E_\alpha=E_2+E_3-E_1=-4(\epsilon-\epsilon') +4\alpha\epsilon\gamma A^{\frac{2}{3}}\frac{(20/3)\alpha\epsilon+\gamma A^{\frac{2}{3}}}{(4\alpha\epsilon+\gamma A^{\frac{2}{3}})^2}. \quad (20)$$

For $A=222$ (radon; this atomic weight corresponds approximately to the average for all radioactive elements) we find, with the values (17b), (19b), (19d), (20a) for γ, α, ϵ, ϵ':

$$E_\alpha = -4\cdot 4.7_5 + 80\cdot 0.58\cdot 222^{\frac{2}{3}} \cdot \frac{133+0.58\cdot 222^{\frac{2}{3}}}{(80+0.58\cdot 222^{\frac{2}{3}})^2} = 6.6 \text{ MV.} \quad (20b)$$

This is indeed fairly close to, but slightly larger than, the average kinetic energy of the α-particles emitted by radioactive substances. The surface effect, which will be discussed in §29, 30, will decrease the theoretical value (20b) to 3.8 MV.

From (20) it is obvious that α-radioactivity will, in general, only be possible if the atomic weight exceeds a certain critical value A_0 which is determined by $E_\alpha(A_0)=0$, or

$$\frac{x(5/3+x)}{(1+x)^2} = \frac{\epsilon-\epsilon'}{\alpha\epsilon} = \frac{4.7_5}{20} = 0.275$$

with $x=\gamma A_0^{\frac{2}{3}}/4\alpha\epsilon$. The solution is $x=0.176$, or

$$A_0 = 119. \quad (20c)$$

Thus nuclei of higher atomic weight than 120 should, in the average, be unstable against α-decay. "In the average" means that the binding energy of α-particles in nuclei of atomic weight around 120 should be positive in about as many causes as it is negative. The stability limit will be shifted to slightly higher atomic weights if the "surface tension" is taken into account (§30), but only to 14.7 Why, then, has actual α-radioactivity only been found for much higher atomic weights (lowest observed: polonium, $A=210$)? The answer is that the lifetime of an α-radioactive nucleus becomes extremely long when the kinetic energy of the α-particle when emitted is small (chapter IX). Thus a nucleus is practically stable against α-decay, although not perfectly stable, if the decay energy is not very large. Indeed, no α-particles of kinetic energies less than 2 MV have actually been observed.

This explains why actually only the nuclei heavier than about 200 have an observable α-radioactivity. A notable exception is one samarium isotope, of atomic weight near 140: In this case, we obviously have a fairly large deviation of an individual binding energy from the "average" binding energy prevailing in that region of atomic weights.

§9. Deuteron and α-Particle: The Form of the Potential Function (W12)

It is known experimentally that the mass defect of the α-particle is about 13 times as large as that of the deuteron, *viz.*, 27.7 MV compared to 2.14 MV. On the other hand, we have proved in §6 that forces between like particles, if they exist at all, must be smaller than the forces between proton and neutron. (In §21 we shall show that the ratio of these two kinds of forces is about 2 : 3.) Thus we would, from a naive consideration, expect that the α-particle has only slightly more than 4 times, and certainly less than 6 times, the binding energy of the deuteron: For we have in the deuteron one pair of interacting particles, in the α-particle each of the two neutrons interacts with each of the two protons, which gives 4 times the deuteron interaction.

The solution of this problem has been given by Wigner (W12). We have to assume that the forces between neutron and proton are very strong when the two particles are close together, but fall off very rapidly when the distance between them becomes larger than a certain, small distance a. We thus assume a strong, short-range force between the two particles.

To make a *complete* use of such a short range force, the nuclear particles have to get very close, more accurately, inside the reach of the force. If their wave function is to be confined in that small region, its wave-length must be of the order of the range of the forces, i.e., very small. Accordingly, the momentum, and the kinetic energy of the particles must become very large; the larger, the smaller the range of the force. The kinetic energy of the particles in the deuteron may in this way become even larger than the potential energy at close distances; if so, the two particles *cannot* be confined within the range of the forces between them; the particles will actually travel over a larger region in space, in which way their kinetic energy may be kept down. But when this is **the**

case, also the time during which the particles actually are near enough to exert a strong attraction is reduced, and thus the binding energy will come out quite small compared to the potential energy between the particles.

If we now take the α-particle, four to five times the attractive forces are available, while the number of moving particles is only twice as large as for the deuteron. Thus it is well conceivable that now the attractive potential will suffice to overcome the kinetic energy, and to actually draw the particles into the range of their mutual forces. Then full use can be made of the large interaction potential, and the binding energy will be of the same order of magnitude as the interaction potential.

This shows that with a deep and narrow hole representing the potential energy between proton and neutron, the binding energy (mass defect) of the α-particle can be made very much larger than that of the deuteron. Thomas (T2) has actually shown that the ratio of the mass defects of H^3 and H^2 becomes infinitely large if the range of the forces is reduced to zero, and at the same time the magnitude of the potential energy increased in such a way that it yields the observed binding energy of the deuteron (§19). If the binding energy of H^3 tends to infinity, this is, of course, *a fortiori* true for that of the α-particle. Thus Thomas' calculation shows that any desired value may be obtained for the ratio of the mass defects of α-particle and deuteron, by a suitable choice of the range of the forces.

The actual determination of the range of the forces from the given mass defects requires, of course, the solution of the Schrödinger equation for the α-particle and the deuteron. A suitable form must be assumed for the potential energy between neutron and proton as a function of the distance, leaving two parameters free which determine the width and the depth of the potential hole (range and magnitude of the force). Then the Schrödinger equations for H^2 and He^4 have to be solved with this potential. The first equation, for the deuteron, can easily be solved rigorously (chapter III). That for the α-particle has to be treated by approximate methods, e.g., by the Ritz method based on the variation principle. Unfortunately, this method converges very slowly, so that the results thus far obtained are not very certain although much work has been put into the attempt of solving the problem, especially by Feenberg (F2, F3). The range of the forces resulting from his calculations is about $2 \cdot 10^{-13}$ cm, i.e., approximately the radius of a sphere whose volume is equal to the volume per particle of heavier nuclei, and considerably less than the radius of the deuteron ($4.36 \cdot 10^{-13}$ cm, cf. §12). Details of the calculation will be given in chapter IV.

The determination of the analytical *form* of the dependence of the nuclear forces upon the distance between the nuclear particles, is at present quite hopeless. Any rapidly decreasing function, whether $e^{-\alpha r^2}$, $e^{-\beta r}$, a rectangular potential hole or a more complicated function having the same characteristic behavior, will fit the experimental data equally well as long as no *very* accurate calculations of the binding energies expected for a given force, are available. It has been suggested (cf. §44) that the potential should be proportional to some high negative power of the distance for large r, and become more or less constant at small r. For the present, however, the potential can be represented by a function which is most convenient for the integration of the Schrödinger equation, without introducing any error comparable to that due to the insufficiency of our present methods for integrating that equation.

§10. Forces Between Like Particles. Odd and Even Isotopes (Y1)

The considerations of §§6–8 have given us an idea about the general dependence of the binding energy of nuclei upon the atomic weight and the nuclear charge. Experimental evidence about this general dependence was used to fix some constants in the assumed expression for the binding energy. However, no reference has been made to any details in the distribution of known isotopes.

There is one such detail which is very outstanding and which strikes one immediately if one glances at the table of known isotopes (Table I): While there are 154 isotopes known[22]

[22] β-emitting substances have been excluded. The figure 148 includes 16 radioactive α-emitters.

with even nuclear charge and even atomic weight, there are only 4 with *odd* nuclear charge and even weight, and all of these latter have atomic weights smaller than 14. In the remainder of the periodic system, from $A=14$ up to 238, there is not a single stable isotope with odd charge and even weight.—The isotopes of odd atomic weight occupy, as far as their number is concerned, an intermediate position: There are 106 stable isotopes of this kind well established, of which 55 have even nuclear charge and 52 odd charge (in addition, 7 α-emitters of odd atomic weight are known).

What is the reason for the striking difference between nuclei with even weight and even charge, and such with even weight and odd charge? To account for this difference, in fact, to make any theory of nuclear stability, it is necessary to know the condition for stability. If we would accept the, obviously necessary, condition that the removal of any neutron or proton from the nucleus must require energy, then practically any pair of values A, Z would lead to a stable nucleus. More stringent is the condition of stability against α-emission (cf. §9) but even this would allow a wide variety of nuclear charges for any given atomic weight. Actually, the most important condition is stability against β-transformation, i.e., against emission or absorption of electrons.

The emission of an electron by a nucleus leads to a new nucleus whose mass number is identical with that of the original nucleus while its charge is one unit higher. The β-emission can take place energetically, if the energy of the original nucleus is higher than that of the produced nucleus plus mc^2, where m is the mass of the electron; in other words, if the exact atomic weight of the original nucleus is higher than that of the nucleus produced in the β-decay (cf. §2). In stating this condition, the mass of the neutrino has been assumed equal to zero (cf. §39). Now the experimental evidence about β-decay seems to show that, whenever β-decay is energetically possible, the decay occurs almost always in a reasonably short time, ranging from fractions of a second up to a few years. Some notable exceptions, primarily radioactive potassium and rubidium, which have lifetimes of the order of 10^8 years, can be accounted for without serious difficulty (§43). We shall thus assume that in general any substance which is energetically unstable against β-decay, will disintegrate in a time very short compared to the life of the earth, and will thus not be found among the existing isotopes in nature. (For the explanation of exceptions, and the conditions therefore, see §43.)

From the standpoint of nuclear theory, the nucleus produced in β-disintegraton differs from the original nucleus by containing one proton more and one neutron less. So we get the rule: *A nucleus is unstable* against β-disintegration, *if the replacement of a neutron in the nucleus by a proton would make the energy of the corresponding atom smaller.*

A similar rule holds for the replacement of a proton by a neutron. This replacement is brought about when the nucleus absorbs an electron, e.g., one of the orbital electrons of its own atom. That such an absorption of external electrons by a nucleus is possible, can be inferred with practical certainty from the fact that positron-emitting radioactive nuclei are known in great number: The emission of a positron can, according to Dirac's theory, be considered as the absorption of an electron which has been in a state of negative energy. If this process is possible, there is no conceivable reason why electrons in states of positive energy could not be absorbed by nuclei. The energetical condition for such an absorption is obviously that the energy of the absorbing nucleus, plus the intrinsic energy mc^2 of the absorbed electron, is larger than the energy of the nucleus produced by the absorption. Thus *a nucleus is unstable against absorption of electrons if the atomic weight is decreased when a proton in the nucleus is replaced by a neutron.*

For complete stability, a nucleus must therefore *be lighter than both the two neighboring isobars*, i.e., the two nuclei whose mass number is the same and whose charge is by one unit less or greater than that of the given nucleus.

We thus conclude that *the energy of any nucleus with even atomic weight and odd nuclear charge is larger than* that of at least one of its neighboring isobars, which would have *even mass and even charge.* If this theorem can be proved, it follows at once that all stable nuclei with even mass *must* have even charge, in agreement

with experiment. (Of course, it must also be proved why the four light nuclei H^2, Li^6, B^{10} and N^{14} are exceptions from the rule.)

Nuclei having even mass and even charge, obviously contain even numbers of neutrons and protons. For even mass and odd charge, we would have an odd number of neutrons as well as protons. Thus we can express our empirical rule by saying that even numbers of neutrons and protons lead to a lower total energy of the nucleus than odd numbers, in other words that a pair of neutrons or protons has much the same function in nuclear physics as closed shells in atomic physics, insofar as it leads to a specially low energy of the system. That two neutrons (or protons) may form a "closed shell," is plausible because two particles with opposite spin may just be placed in exactly the same quantum state with respect to orbital motion. A third neutron would have to go into the next higher quantum state, and would therefore be less strongly bound. It is true that this rule differs appreciably from the rule valid in atomic physics where we find groups of $2(2l+1)$ electron states with the same n and l (principal and azimuthal quantum number) all having sensibly the same energy. The difference seems to be due to the fact that in an atom we have practically a central field which we have not in nuclei. This problem will be discussed in more detail in chapter VI, where we shall also show that there is evidence for other periodicities in the structure of nuclei, with longer periods than 2, which are more similar to the electron shells in atoms.

For our present discussion we simply accept that every state of orbital motion of a proton or neutron has its own energy, differing from the energy of all other states, so that two neutrons or two protons form a "closed shell." With this assumption, it is quite simple to prove that no nucleus containing an odd number of protons and neutrons can be stable, except in the very beginning of the periodic system.

Let us take any nucleus of even atomic weight A and even nuclear charge Z and call it the standard nucleus; e.g., we might choose Ni^{60} for this purpose. In the field of this standard nucleus, there will be certain energy levels for neutrons and other levels for protons. By "level" or "state" we refer, in this discussion, to the state of orbital motion only, so that each state can take two particles. The $\frac{1}{2}Z$ lowest proton states, and the $\frac{1}{2}N$ lowest neutron states are occupied in our standard nucleus. Of the empty states, either the lowest proton or the lowest neutron state will be lower. If we construct the nucleus of atomic weight $A+1$, this nucleus will have $Z+1$ protons and N neutrons if the proton state is the lower of the two, or Z protons and $N+1$ neutrons if the neutron state is the lower. Both cases will occur with about equal probability, therefore we expect that *for odd atomic weight nuclei with even charge are about as numerous as such with odd charge.* This is actually true, the statistics we have mentioned before showed 52 known nuclei with odd weight and odd charge and 55 stable nuclei with odd weight and even charge. In the case of our "standard" nucleus Ni^{60}, the addition of a neutron leads to the stable nucleus, Ni^{61}. Now let us add a second particle to our standard nucleus. If the first unoccupied neutron level in the standard nucleus lies lower than the first unoccupied proton level, the most stable nucleus of weight $A+2$ will be obtained by adding two neutrons to the standard nucleus so that it has nuclear charge Z. The addition of one neutron and one proton will lead to a less stable nucleus (charge $Z+1$), because the proton level lies higher. The addition of two protons will give us a nucleus (charge $Z+2$) which is even less stable. Conversely, if the proton level is the lower of the two, the most stable nucleus will be obtained by adding two protons, a less stable one by adding one proton and one neutron, and the least stable if we add two neutrons. Thus *in both cases the most stable resulting nucleus of atomic weight $A+2$,* has even nuclear charge, in one case the same charge Z as the "standard" nucleus, in the other case, the charge $Z+2$. In no case will the nucleus of charge $Z+1$ be the most stable. (In our case of Ni^{60}, we know from the first step of adding one particle that the neutron level lies lower. Thus we expect the stable nucleus of atomic weight 62 to be nickel again which actually is the case.)

Thus it seems that we have proved our theorem from a simple consideration about neutron and proton levels. However, we know that in many cases *isobars* exist, i.e., several stable nuclei having the same atomic weight, and nuclear

charges usually differing by two units. This means that, starting from our "standard nucleus" of weight A and charge Z it often happens that both the addition of two neutrons and the addition of two protons lead to stable nuclei, while the addition of one proton and one neutron never does. This is quite inexplicable from our previous considerations, which led us to expect that the energy of the nucleus of atomic weight $A+2$ and charge $Z+1$ always lies in between the energies of its isobars with charges Z and $Z+2$.

The fact that both the nuclei Z and $Z+2$ may have energies lower than the intermediate nucleus $Z+1$, thus requires a special explanation. We must obviously assume that there is some *attraction between* the *two neutrons or* the *two protons* which we added to the "standard" nucleus. With such an attraction, the energies of the nuclei Z and $Z+2$ become depressed below the value expected from our previous considerations which assumed the additional two particles to move independently from each other. The attraction between particles of equal kind therefore allows the existence of two stable isobars, both with even charges, differing by two units, while the intermediate nucleus of odd charge is unstable.

The objection might be raised that the neutron and the proton, which we have added to the standard nucleus in order to obtain the nucleus $(A+2, Z+1)$ should also show an interaction. In fact, we have even proved (§6) that the interaction between a neutron and a proton must be larger than that between a pair of neutrons or a pair of protons. This would mean the energy of the nucleus $(A+2, Z+1)$ would be more decreased by the interaction of the two additional particles than that of the nuclei $(A+2, Z)$ or $(A+2, Z+2)$. From such a reasoning, we would therefore conclude that the nucleus of *odd* charge and even weight $(A+2, Z+1)$ is stable and the nuclei of even charge and weight, $(A+2, Z)$ and $(A+2, Z+2)$ are not, in contradiction to experience.

The fallacy in this argument is due to the fact that a neutron and a proton interact strongly only if they are in approximately the same quantum state, because the forces are "saturation" forces (§7). We know, however, that the first empty neutron state in the field of our standard nucleus is quite different from the first empty proton state. If we add a neutron and a proton, they will therefore have practically no interaction. On the other hand, two added neutrons (or two protons) will move in the same orbit and therefore have full interaction, irrespective of what we assume about the forces between like particles—i.e., whether they also act only between particles in the same quantum state, or between any pair of light particles (cf. §24). Thus we have accounted for the rule that all stable isotopes with even weight have even charge, and also for the existence of isobars of this type.

Our argument shows us, however, also the reason for the exceptions from that rule: If the added proton and neutron would move in the same quantum state, they would have a strong interaction, and therefore the atom of even weight and *odd charge* would be stable. The condition for this is obviously that equally many proton states are occupied in the "standard" nucleus as neutron states, in other words, that the standard nucleus contains exactly equal numbers of neutrons and protons. This is then also true for the nucleus which we obtain by adding a neutron and a proton to the standard nucleus. Therefore *nuclei of even weight and odd charge will be stable if they contain exactly as many protons as neutrons*. This is true for all the four stable nuclei of the type, *viz.*, H^2, Li^6, B^{10}, N^{14}. All these nuclei are very light, in fact, they are the lightest possible nuclei of their type. Only for light nuclei, can the number of neutrons be exactly equal to that of the protons. As soon as the nuclei get heavier, the Coulomb repulsion between the protons begins to become appreciable and to make the number of neutrons greater than that of protons in any given nucleus. Then the existence of nuclei with even weight and odd charge becomes impossible. This is true already for the atomic weight 18, for which the nuclear charge 8 rather than 9 leads to a stable nucleus (O^{18}, not F^{18}).

A convenient way of visualizing the results of this paragraph is to plot the energy of isobars (stable and unstable) as a function of their nuclear charge. If we do this for odd atomic

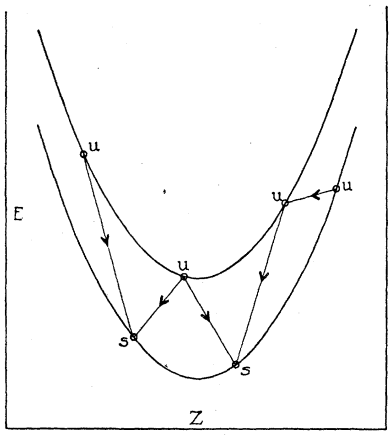

FIG. 3. Schematic graph representing the energy of isobars of even atomic weight as a function of nuclear charge Z. The upper parabola contains the nuclei of odd charge, the lower one those of even charge. The arrows represent possible β-disintegrations. The two "even" nuclei marked S are stable, all the others unstable.

weight, we shall get a smooth curve. The nucleus nearest to the minimum of that curve will be the stable nucleus for the given atomic weight. Chances are equal that this stable nucleus has even or odd charge.

For even atomic weight, on the other hand, we obtain two separate curves for nuclei with odd and such with even charge. We may assume that each of these curves is smooth. On the lower curve (even charge) we find always one, but in many cases several, stable nuclei, i.e., nuclei which have less energy than both their neighbors (Fig. 3).

We may try to determine the magnitude of the forces between like particles from the statistics of isobars. It is reasonable to assume that the two curves representing the energies of even charge and odd charge isobars, are parabolas just shifted vertically by an amount δ. The minimum of the parabolas will in general lie at a fractional value of the nuclear charge, Z_A. Let us denote by β the difference between Z_A and the nearest *odd* number; β obviously may be any number between -1 and $+1$. The curvature of the parabola may be determined from our general formula (18) for nuclear energies:

$$\kappa = -\tfrac{1}{2}\partial^2 E/\partial Z^2 = (4\alpha\epsilon + \gamma A^{2/3})/A. \tag{21}$$

Then we have for the energies of a nucleus of weight A and charge Z:

$$\begin{aligned} E &= E(Z_A) + \kappa(Z - Z_A)^2 && \text{if } Z \text{ is even,} \\ E &= E(Z_A) + \delta + \kappa(Z - Z_A)^2 && \text{if } Z \text{ is odd;} \end{aligned} \tag{21a}$$

e.g., for the odd nucleus whose charge is nearest to Z_A:

$$E_{\text{odd}} = E(Z_A) + \delta + \kappa\beta^2 \tag{21b}$$

for the two even nuclei nearest to Z_A:

$$\begin{aligned} E^{+}_{\text{even}} &= E(Z_A) + \kappa(1+\beta)^2, \\ E^{-}_{\text{even}} &= E(Z_A) + \kappa(1-\beta)^2. \end{aligned} \tag{21c}$$

The conditions that both the even nuclei are stable, read therefore

$$\delta > \kappa(1+2\beta) \quad \text{and} \quad \delta > \kappa(1-2\beta),$$

or, both conditions in one:

$$\delta > \kappa(1+2|\beta|). \tag{21d}$$

To find out the meaning of δ, we remember that we have proved that the energy of the "odd" nucleus must be midway between the two even ones if there are no forces between like particles. In this case, therefore, we would have

$$E'_{\text{odd}} = E(Z_A) + \kappa(1+\beta^2),$$

so that, in the absence of forces between like particles, $\delta' = \kappa$. Therefore the effect of the forces between like particles is given by

$$\delta'' = \delta - \delta' = \delta - \kappa \tag{21e}$$

and, according to (21d), we have two stable isobars if

$$\delta'' > 2\kappa|\beta| \tag{21f}$$

and only one stable element of atomic weight A if

$$\delta'' < 2\kappa|\beta|. \tag{21g}$$

To determine the critical value of $|\beta|$ above which the existence of isobars becomes impossible, we use the statistics of the known isotopes of atomic weight between 110 and 140. The reason for choosing this region is that the elements of higher atomic weight are not well explored as regards their isotopic structure, because they are rare earths, while among the elements of lower weight too few isobars are found. Of the 15 even mass numbers from 112 to

140, 10 are occupied by two isobars,[23] 5 by one nucleus only. We thus conclude that $\beta_0=10/15=2/3$ is the critical value for $|\beta|$ such that for $|\beta|>\beta_0$ only one nucleus of weight A is stable while for $|\beta|<\beta_0$ two stable isobars exist. Inserting $A=125$ into (21), we have (cf. 19b)

$$\kappa=\frac{4\cdot 20+0.58\cdot 25}{125}=\frac{80+15}{125}=0.76 \text{ MV}$$

and
$$\delta''=2\beta_0\kappa=1.01 \text{ MV}. \qquad (22)$$

From formula (21f) we can conclude that the existence of isobars is the more probable the smaller κ. According to (21), κ decreases with increasing atomic weight. Thus we expect the more isobars the heavier the nuclei. This is actually true, at least up to $A\approx 150$. In the region of the rare earths, there are probably a great number of isobaric pairs yet unknown. For still higher atomic weight, stability against α-decay probably also plays an important role so that our rule cannot be expected to hold.

There is another indication of forces between like particles from the scattering of protons by protons (§18). That scattering is not in agreement with the expectation from a purely electrostatic interaction between the protons. Strong evidence for the forces between like particles comes also from the quantitative calculations of the binding energies of H^3 and He^4 (§21).

III. Theory of the Deuteron

§11. The Wave Equations of Heisenberg, Wigner and Majorana (H7, W12, M6, B16)

We have shown in §7 that the force between neutrons and protons must be exchange forces and must not depend on the relative spin directions of the two particles. This type of force was first suggested by Majorana (M6). Earlier, Heisenberg had suggested saturation forces which did depend on the relative spins (H7), and Wigner ordinary forces which did not show saturation (W12).[24]

Although we have given good evidence for the Majorana type of force, we want to write down the wave equations for the two other suggested types as well. The reasons are on one side to facilitate comparisons, on the other hand (and this seems even more important) it seems probable that a small force of the Heisenberg type is superposed upon the main Majorana force (cf. §14).

To lead up to the wave equation, we consider again the chemical analogy. The system most nearly comparable to a system of a neutron and a proton, is the hydrogen molecular ion, H_2^+.[25] Its wave function is the product of an electronic wave function φ and a wave function ψ describing the motions of the two nuclei. φ depends on the distances ρ_α and ρ_β of the (single) electron of the H_2^+ from the two protons α and β. φ may be either symmetrical or antisymmetrical with respect to ρ_α and ρ_β, and therefore with respect to the coordinates of the two protons. Since the protons obey Fermi statistics, ψ must be antisymmetrical in the two protons if φ is symmetrical, and *vice versa*. We consider in particular those electron states of H_2^+ which go over into a hydrogen atom in the ground state, plus a proton. There are just two states of this type, one whose eigenfunction φ is symmetrical in the two proton coordinates and one antisymmetrical state. Only the former leads to binding. The electronic energy of the system may be written:

$$V(r)=-C(r)\mp A(r), \qquad (23)$$

where $C(r)$ is the Coulomb interaction between a hydrogen atom in the ground state and a proton at a distance r from the nucleus of the hydrogen atom, while $A(r)$ is the "exchange integral" which measures how often the electron changes its place, going over from one proton to the other. The upper sign in (23) is related to the symmetrical electron wave function φ, the lower sign to the antisymmetrical φ.

The electron energy for fixed nuclear distance r must, as is well known, be regarded as a potential energy for the motion of the nuclei in the molecule. That motion is described by the wave function ψ. For antisymmetrical ψ the upper sign in (23) has to be taken, for symmetrical ψ the lower. We have thus two different Schrödinger equations for ψ according to the symmetry of ψ. We can, however, formally write them into one, by using the fact that

$$\begin{aligned}&\psi(r_\beta r_\alpha)=+\psi(r_\alpha r_\beta) \text{ for symmetrical } \psi,\\ &\psi(r_\beta r_\alpha)=-\psi(r_\alpha r_\beta) \text{ for antisymmetrical } \psi,\end{aligned} \qquad (23a)$$

according to the definition of symmetry and antisymmetry

[23] One mass number (124) is actually occupied by 3 isobars.

[24] Recently, Bartlett (B10) has pointed out that there is still another type of force, *viz.*, "ordinary" forces depending on the relative spin. These forces would lead to both the difficulties of the Heisenberg and the Wigner theory, *viz.*, saturation at H^2 and too large binding energies for heavy nuclei.

[25] Cf. *Handbuch der Physik*, Vol. 24/1, p. 524.

($r_\alpha r_\beta$ are the coordinates of the two nuclei, including spin). We can thus write the Schrödinger equation:

$$(\hbar^2/2M)(\Delta_\alpha+\Delta_\beta)\psi(r_\alpha r_\beta)+E\psi(r_\alpha r_\beta) = -C(r)\psi(r_\alpha r_\beta)+A(r)\psi(r_\beta r_\alpha). \quad (23b)$$

In the second term on the right-hand side the coordinates of the two nuclei have been interchanged. The term is equivalent to $+A(r)\psi(r_\alpha r_\beta)$ if ψ is symmetrical, and $-A(r)\psi(r_\alpha r_\beta)$ if ψ is antisymmetrical. Any reference to the electron has disappeared from (23b), therefore we can take it over directly into nuclear theory.

It must be emphasized at this point that the analogy to the hydrogen molecular ion must in no way be regarded as a deduction or justification of the wave equation for the deuteron. The forces between neutron and proton are an entirely new phenomenon, not connected in any way with forces familiar in atomic physics. A particular form chosen for the interaction between neutron and proton can therefore only be justified by comparison of the results deduced from this interaction with experimental data about nuclei. The theory of the hydrogen molecular ion serves only to suggest a possible form of the interaction which leads to saturation of the forces. The analogy to the H_2^+ has not been introduced because we think the neutron is in any way comparable to a small hydrogen atom but only because we know from qualitative considerations that the nuclear forces show saturation.

We now write down the wave equation for a neutron and a proton, interacting with each other, in analogy to (23b):

$$(\hbar^2/2M)(\Delta_x+\Delta_\xi)\psi(xs, \xi\sigma) +E\psi(xs, \xi\sigma)=J(r)\psi(\xi\sigma, xs). \quad (24)$$

M is the mass of the proton which is sensibly the same as that of the neutron. The first two arguments in the wave function denote the position and the spin coordinate[26] of the proton, the last two refer to position and spin of the neutron. $|\psi(xs, \xi\sigma)|^2$ thus means the probability that the proton is found at the point x (x is, of course, a vector) with spin s, while the neutron is at the point ξ and has spin σ. Δ_x and Δ_ξ are the Laplacian operators with respect to the coordinates of neutron and proton, E the total energy and $J(r)$ the potential energy as a function of the distance $r=x-\xi$ between proton and neutron. (In comparing (24) to (23b), it may be noted that we have assumed no "ordinary" interaction $C(r)$ between proton and neutron but only "exchange" interaction $A(r)=J(r)$.)

[26] As such we may choose the spin component in a given direction z, which may have either of the two values $+\frac{1}{2}$ or $-\frac{1}{2}$.

Eq. (24) is that originally proposed by Heisenberg. It shows saturation effects (cf. §7) but it corresponds to an interaction which depends upon the relative spin directions of proton and neutron. To see this, we write the wave function ψ as a product of a function depending on the positions of the two particles, and a function depending on spin only:

$$\psi(xs, \xi\sigma)=\varphi(x\xi)\chi(s\sigma). \quad (24a)$$

If the spins of proton and neutron are parallel, the spin wave function χ is symmetrical[27] in the two spin coordinates s and σ, *viz.*,

$$\chi(\sigma s)=\chi(s\sigma) \quad \text{(parallel spins)} \quad (24b)$$

if the spins are antiparallel, s is antisymmetrical, *viz.*,

$$\chi(\sigma s)=-\chi(s\sigma) \quad \text{(antiparallel spins)}. \quad (24c)$$

Therefore (24) goes over into an equation involving the "spatial" wave function φ only:

$$(\hbar^2/2M)(\Delta_x+\Delta_\xi)\varphi(x\xi)+E\varphi(x\xi) = \pm J(r)\varphi(\xi x), \quad (25)$$

the upper sign holding for parallel spin of neutron and proton, the lower for antiparallel spin. Thus, if $J(r)$ is negative, we get an attraction between a neutron and a proton with parallel spins, but a repulsion for antiparallel spins. If $J(r)$ is positive, the reverse is true. In any case, the force between the two particles depends on the relative spin directions, and saturation is obtained when a single neutron is bound to a proton, the neutron spin being parallel or antiparallel to the proton spin according to the sign of J. A second neutron could not be bound to the proton but would even be repelled. The deuteron would be the "saturated" nucleus instead of the α-particle, in contradiction to experience.

This is avoided in the wave equation of Majorana (M6)[28]

$$(\hbar^2/2M)(\Delta_x+\Delta_\xi)\psi(xs, \xi\sigma)+E\psi(xs, \xi\sigma) = J(r)\psi(\xi s, x\sigma). \quad (26)$$

It differs from Heisenberg's equation in the interaction term on the right-hand side: While in Heisenberg's equation spatial coordinates as well

[27] Cf., e.g., *Handbuch der Physik*, Vol. 24/1, p. 325.

[28] For some difficulty connected with the difference in mass between proton and neutron, cf. references B27, P6.

as spin coordinates of neutron and proton are interchanged in the interaction term, only the spatial coordinates are interchanged in Majorana's equation. The spin of the proton is s in the interaction term of (26) as well as on the left-hand side of that equation, the spin of the neutron is σ on both sides of the equation. Therefore, if we again write the wave function in the form (24a), we have

$$\psi(\xi s, x\sigma) = \varphi(\xi x)\chi(s\sigma);$$
$$\psi(xs, \xi\sigma) = \varphi(x\xi)\chi(s\sigma) \quad (26a)$$

in other words, the *same* spin function occurs on both sides of Eq. (26). Irrespective of the relative spin directions, Eq. (26) therefore reduces to

$$(\hbar^2/2M)(\Delta_x+\Delta_\xi)\varphi(x\xi) + E\varphi(x\xi) = J(r)\varphi(\xi x). \quad (27)$$

A negative interaction potential J leads to binding, for any spin directions.

Both Majorana's and Heisenberg's equation assume "exchange" forces which is necessary to explain the observed "saturation" of nuclear forces (§7). Forces of the ordinary type (the term "ordinary forces" will be used in distinction from exchange forces) have been assumed by Wigner (W12). Wigner's wave equation for the deuteron thus reads

$$(\hbar^2/2M)(\Delta_x+\Delta_\xi)\psi(xs, \xi\sigma) + E\psi(xs, \xi\sigma) = J(r)\psi(xs, \xi\sigma), \quad (28)$$

or, after separation of spin

$$(\hbar^2/2M)(\Delta_x+\Delta_\xi)\varphi(x\xi) + E\varphi(x\xi) = J(r)\varphi(x\xi). \quad (28a)$$

The generalizations of the Eqs. (24), (26), (28) to more than two particles are obvious. On the right-hand side there appear interaction terms for each pair of particles, similar to those in (24), (26), (28). Explicitly, we have for a nucleus containing N neutrons and Z protons:

$$\frac{\hbar^2}{2M}\left(\sum_{i=1}^{Z}\Delta_{xi}+\sum_{k=1}^{N}\Delta_{\xi k}\right)\psi(x_1s_1\cdots x_is_i\cdots\xi_k\sigma_k\cdots\xi_n\sigma_n)+E\psi(\cdots x_is_i\cdots\xi_k\sigma_k\cdots)$$
$$=\sum_{i=1}^{Z}\sum_{k=1}^{N}J(r_{ik})\psi(x_1s_1\cdots\xi_k\sigma_k\cdots x_is_i\cdots\xi_n\sigma_n) \text{ (Heisenberg)}, \quad (29a)$$
$$=\sum_{i=1}^{Z}\sum_{k=1}^{N}J(r_{ik})\psi(x_1s_1\cdots\xi_ks_i\cdots x_i\sigma_k\cdots\xi_n\sigma_n) \text{ (Majorana)}, \quad (29b)$$
$$=\sum_{i=1}^{Z}\sum_{k=1}^{N}J(r_{ik})\psi(x_1s_1\cdots x_is_i\cdots\xi_k\sigma_k\cdots\xi_n\sigma_n) \text{ (Wigner)}. \quad (29c)$$

For the application to the deuteron, it is convenient to separate the motion of the center of gravity of the deuteron from the relative motion of the two particles in the deuteron. We are only interested in the latter. It will be described by a wave function u which depends upon the relative coordinate $\mathbf{r}=\mathbf{x}-\boldsymbol{\xi}$ of the proton with respect to the neutron. Interchanging the coordinates of the two particles means replacing $\mathbf{r}$ by $\boldsymbol{\xi}-\mathbf{x}=-\mathbf{r}$, in other words, changing the sign of the relative coordinates. Then we have for u the wave equation

$$\frac{\hbar^2}{M}\Delta u(\mathbf{r})+Eu(\mathbf{r}) =$$
$$\pm J(r)u(-\mathbf{r}) \text{ Heisenberg}, \quad (30a)$$
$$J(r)u(-\mathbf{r}) \text{ Majorana}, \quad (30b)$$
$$J(r)u(\mathbf{r}) \quad \text{Wigner}. \quad (30c)$$

u depends only on the three relative coordinates $\mathbf{r}$, and Δ is therefore the ordinary Laplacian operator in three dimensions. The proton mass M which appeared in (24) to (29) has been replaced

by the reduced mass $\frac{1}{2}M$ since we are dealing with relative motion. The signs in (30a) refer to parallel (+) and antiparallel (−) spins of the proton and neutron.

§12. Ground State of the Deuteron (B16)

The deuteron plays in nuclear physics the same role as the hydrogen atom in atomic physics. It consists of two elementary particles, one proton and one neutron. It is well known that any two-body problem can be integrated explicitly if the force between the two particles is a known function of the distance of the particles. Thus the theoretical results about the deuteron are free from approximations made to simplify the mathematical treatment. They are, as we shall see, also to a very large extent independent of the assumptions we make about details of the force between neutron and proton, i.e., of the function $J(r)$ in (30). The theory of the deuteron is thus more suitable for quantitative comparisons with experiment, and therefore for a check of the underlying ideas about nuclear structure, than any other part of nuclear theory.

Using the Majorana hypothesis about the forces between neutron and proton, we have obtained the wave equation

$$(\hbar^2/M)\Delta U(\mathbf{r})+EU(\mathbf{r})=J(r)\,U(-\mathbf{r}), \quad (30b)$$

where r is the relative coordinate of the proton with respect to the neutron, M the mass of proton or neutron and E the energy of the system. If E is negative, $\epsilon=-E$ is the binding energy of the deuteron.

The potential energy $J(r)$ is spherically symmetrical. Therefore (30b) can be separated in polar coordinates r, θ, φ by putting

$$U(\mathbf{r})=(u_l(r)/r)P_{lm}(\theta)e^{im\varphi}, \quad (31)$$

where P_{lm} is a spherical harmonic which we assume to be normalized unless otherwise stated. In (30b) the function $U(-\mathbf{r})$ enters as well as $U(\mathbf{r})$. If the polar coordinates of the point $\mathbf{r}=(x, y, z)$ are r, θ, φ, then those of the point $-\mathbf{r}=(-x, -y, -z)$ are r, $\pi-\theta$, $\pi+\varphi$. Now it is easy to show[29] that

$$P_{lm}(\pi-\theta)e^{im(\pi+\varphi)}=(-1)^lP_{lm}(\theta)e^{im\varphi}. \quad (31a)$$

Therefore the wave equation for u becomes[30]

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l\right)+Eu_l=(-1)^lJ(r)u_l. \quad (32)$$

If $J(r)$ is assumed to be negative, it is seen that the right-hand side of (32) corresponds to an attractive potential energy, if l is even, and to a repulsive potential, if l is odd. This alternation of the sign of the force for even and odd l, is a characteristic feature of exchange forces and could, in principle, be used to decide whether the forces acting between neutron and proton are exchange forces or ordinary ones (§14, reference W9).

The lowest quantum state will be obtained for $l=0$, if we assume $J(r)$ to be negative. Its eigenfunction obeys the wave equation

$$d^2u_0/dr^2=(M/\hbar^2)(J(r)-E)u_0. \quad (33)$$

u_0 is subject to the condition that it vanishes for small r as r itself, because otherwise $U(r)$ would, according to (31), become infinite at small distances r. Furthermore, u_0 must not become infinite for large r.

First we shall discuss the behavior of u_0 qualitatively, making very general assumptions about the interaction potential $J(r)$. We know from the ratio of the binding energies of α-particle and deuteron (§9) that $J(r)$ must be very large for small r, and must fall off very steeply at larger distances. We may therefore define a range a of the force such that $|J(r)|$ is negligible compared to E if $r>a$. If r is, by a sufficient amount, smaller than a, $|J(r)|$ will be large compared to $|E|$. The behavior of u_0 up to $r=a$ will therefore be determined almost exclusively by J and will not depend to a large extent upon E. On the other hand, beyond a the energy E alone will determine u_0.

Since $J(r)$ is negative and, for the most part of the region $r<a$, absolutely greater than E, the right-hand side of (33) is negative in this region and therefore u_0 is concave towards the r axis (cf. Fig. 4). By integrating (33) up to $r=a$, we

[29] $P_{lm}(\theta)$ is an even or odd function of $\cos\theta$, according to whether $l-m$ is even or odd (see, e.g., Jahnke-Emde, table of functions, p. 173). Since $\cos(\pi-\theta)=-\cos\theta$, we have $P_{lm}(\pi-\theta)=(-1)^{l-m}P_{lm}(\theta)$. Furthermore, $e^{im(\pi+\varphi)}=(-1)^me^{im\varphi}$.

[30] For the algebra involved in the separation of the Schrödinger equation in polar coordinates, cf. e.g., *Handbuch der Physik*, Vol. 24, 1, p. 275.

can determine u_0 and its first derivative u_0' at the point $r=a$. We put

$$(u_0'/u_0)_{r=a} = -\alpha. \tag{34}$$

α is a reciprocal length characteristic for the potential $J(r)$. According to the foregoing, α does not depend sensitively on the energy E. Of course, the sign of α as well as its absolute magnitude will depend on the strength of the forces. If, e.g., $J(r)$ is very small, the curvature of u_0 will be very small, and u_0 will still be increasing when we arrive at $r=a$ for any negative value of E. In that case, u_0'/u_0 would be positive, therefore α negative, and, as we shall see, no stable state ($E<0$) of the deuteron would be possible.

For $r>a$, the potential energy $J(r)$ is supposedly negligible. Therefore Eq. (33) can be solved immediately. Assuming E to be negative, *viz.*,

$$E = -\epsilon \tag{35a}$$

we have $$d^2u_0/dr^2 = (M\epsilon/\hbar^2)u_0; \tag{35}$$

therefore $$u_0 = ce^{-(M\epsilon)^{\frac{1}{2}}r/\hbar} \tag{36}$$

because the alternative solution, $e^{+(M\epsilon)^{\frac{1}{2}}r/\hbar}$, is to be excluded. c is a coefficient to be determined by normalization.

At $r=a$, the two solutions obtained by intetrating the Schrödinger equation "inside" ($r<a$) and "outside" ($r>a$) must join smoothly, so that

$$(u_0'/u_0)_{\text{outside}} = (u_0'/u_0)_{\text{inside}}. \tag{36a}$$

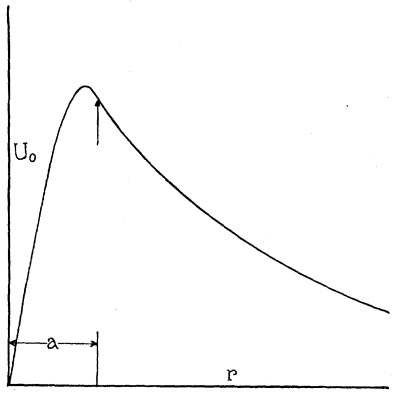

FIG. 4. Eigenfunction of the ground state of the deuteron. Ordinate: r times eigenfunction; abscissa: r. Simple potential hole of width a.

Now (34) gives us the "inside" value of u_0'/u_0 while the outside value follows by differentiating (36). We have therefore

$$(M\epsilon)^{\frac{1}{2}}/\hbar = \alpha. \tag{37}$$

The constant α which is directly connected to the given force field, thus determines the binding energy ϵ. Or, conversely, we may use *the observed value of ϵ to deduce α and thus to get some information about the forces between neutron and proton.*

If α were negative, i.e., if the force is too weak (see above), the inside wave function could not be joined smoothly to an exponentially decreasing wave function outside, but only to an exponentially increasing function. Such a function being excluded because of becoming infinite for $r=\infty$, we find that for negative α no solution with a positive binding energy ϵ can be found. (For positive energy E, there is, of course, always a solution which behaves, for $r>a$, like $\sin((ME)^{\frac{1}{2}}r/\hbar)$ or $\cos((ME)^{\frac{1}{2}}r/\hbar)$ instead of exponentially. From the mere existence of the deuteron we can therefore conclude that the forces between neutron and proton must be strong enough to make α positive, i.e., to make the curvature of $u_0(r)$ large enough so that u_0 *decreases* with increasing r, at $r=a$.

We can even deduce from this condition a quantitative estimate of $J(r)$ no matter what the particular dependence of J on r. We have

$$u_0'' = (M/\hbar^2)(J(r)u_0 + \epsilon u_0). \tag{37a}$$

Integrating over r from 0 to a, we find

$$(u_0')_{r=a} - (u_0')_{r=0} = M\hbar^{-2}\int_0^a J(r)u_0 dr + M\hbar^{-2}\epsilon\int_0^a u_0 dr. \tag{37b}$$

Now $(u_0')_{r=0}=1$, if u_0 is suitably normalized. u_0 is, for small r, proportional to r (cf. after Eq. (33)). This rule will no longer hold exactly for $r=a$, but it might serve as an approximation. Then $(u_0')_{r=a} \approx -\alpha a$, which is negligible compared to unity since the range of the forces, a, is supposed to be small compared to the (known) quantity $1/\alpha$ (cf. 44a). Furthermore, the last term in (37b) can be neglected compared to unity, because it is approximately $\frac{1}{2}M\epsilon a^2/\hbar^2 \approx \frac{1}{2}a^2\alpha^2 \ll 1$ (cf. (37)). Therefore, we obtain approximately

$$\int_0^a J(r)r dr = -\hbar^2/M. \tag{37c}$$

Actually, we should expect a somewhat larger absolute value for the integral on the left of (37c), because $u_0<r$. [In the special case of the rectangular potential hole, we have, according to (40), $\int_0^a J(r)r dr = -\frac{1}{2}V_0a^2 = -\pi^2\hbar^2/8M$, which is rather close to (37c).]

We now discuss the solutions for a few simple forms of the potential function $J(r)$:

(a) *Rectangular hole:*

$$\begin{aligned} J(r)&=-V_0 \quad \text{for} \quad r<a,\\ J(r)&=0 \quad \text{for} \quad r>a. \end{aligned} \tag{38}$$

Solution

$$\begin{aligned} u_0&=b\sin\left[M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}r/\hbar\right] \quad \text{for } r<a,\\ u_0&=c\exp\left[-(M\epsilon)^{\frac{1}{2}}(r-a)/\hbar\right] \quad \text{for } r>a. \end{aligned} \tag{38a}$$

Joining solutions:

$$(u_0'/u_0)_{\text{inside}}=M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}\hbar^{-1}\times\cot\left[M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a/\hbar\right]$$

$$(u_0'/u_0)_{\text{outside}}=-(M\epsilon)^{\frac{1}{2}}/\hbar.$$

Therefore

$$\cot\left[M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a/\hbar\right]=-(\epsilon/V_0-\epsilon)^{\frac{1}{2}}. \tag{39}$$

Since $V_0\gg\epsilon$ (cf. §9), the right-hand side of (39) is small. Therefore in first approximation

$$M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a/\hbar=\tfrac{1}{2}\pi+(\epsilon/V_0)^{\frac{1}{2}}+0(\epsilon/V_0)^{\frac{3}{2}}, \tag{39a}$$

$$V_0=\frac{\hbar^2}{Ma^2}\left(\frac{\pi}{2}+\left(\frac{\epsilon}{V_0}\right)^{\frac{1}{2}}\right)^2+\epsilon+0\left(\frac{\epsilon^{\frac{3}{2}}}{V_0^{\frac{1}{2}}}\right),$$

$$V_0=\frac{\pi^2}{4}\frac{\hbar^2}{Ma^2}+2\left(\frac{\hbar}{Ma^2}\epsilon\right)^{\frac{1}{2}}+\left(1-\frac{4}{\pi^2}\right)\epsilon+0\left(\frac{\epsilon^{\frac{3}{2}}}{(\hbar/Ma^2)^{\frac{1}{2}}}\right). \tag{39b}$$

In very rough approximation,

$$V_0a^2=\pi^2\hbar^2/4M. \tag{40}$$

This means that the product of the depth and the square of the breadth of the potential hole can be determined from the mere existence of the deuteron. For a separate determination of breadth and depth, it is necessary to consider the binding energies of other nuclei, e.g., the α-particle, as well. But whatever the value of a and V_0, the product V_0a^2 will not differ very much from the value (40) which is a universal constant. The smaller the range a of the forces, the more exactly will (40) be true.

Normalization of the wave function: From (38a) (39) we have

$$b=c/\sin\left(M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a/\hbar\right)=c(V_0/V_0-\epsilon)^{\frac{1}{2}}, \tag{41}$$

$$\int_0^\infty u_0^2dr=b^2\int_0^a\left[\tfrac{1}{2}-\tfrac{1}{2}\cos\left(2M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}r/\hbar\right)\right]dr+c^2\int_a^\infty dr\exp\left[-2M^{\frac{1}{2}}\epsilon^{\frac{1}{2}}(r-a)/\hbar\right]$$

$$=b^2\left(\tfrac{1}{2}a-\frac{\hbar}{4M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}}\sin 2\frac{M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a}{\hbar}\right)+c^2\hbar/2(M\epsilon)^{\frac{1}{2}}$$

$$=\tfrac{1}{2}c^2\frac{V_0}{V_0-\epsilon}\left[\frac{\hbar}{(M\epsilon)^{\frac{1}{2}}}+a\right]. \tag{41a}$$

Normalizing to unity, we have (cf. 37)

$$c=\left(\frac{2(V_0-\epsilon)}{V_0}\right)^{\frac{1}{2}}\left(\frac{\hbar}{(M\epsilon)^{\frac{1}{2}}}+a\right)^{-\frac{1}{2}}=\left(\frac{2(V_0-\epsilon)\alpha}{V_0(1+\alpha a)}\right)^{\frac{1}{2}} \tag{41b}$$

and, according to (31), (38a), (41)

$$U(\mathbf{r})=\frac{U_0}{(4\pi)^{\frac{1}{2}}r}=\begin{cases}\left(\dfrac{\alpha}{2\pi(1+a\alpha)}\right)^{\frac{1}{2}}\dfrac{\sin\left(M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}r/\hbar\right)}{r},\\ \left(\dfrac{\alpha}{2\pi(1+a\alpha)}\right)^{\frac{1}{2}}\left(\dfrac{V_0-\epsilon}{V_0}\right)^{\frac{1}{2}}e^{-\alpha(r-a)}.\end{cases} \tag{41c}$$

(b) *Exponential potential*

$$J(r)=-V_0e^{-r/a}. \tag{42}$$

We introduce the independent variable

$$x=e^{-r/a} \tag{42a}$$

so that $x=1$ for $r=0$, $x=1/e$ for $r=a$, $x=0$ for $r=\infty$

and $$d/dr=-(1/a)x(d/dx).$$

Then the Schrödinger Eq. (33) becomes

$$\begin{aligned} &x\frac{d}{dx}\left(x\frac{du_0}{dx}\right)-\frac{Ma^2}{\hbar^2}(-V_0x+\epsilon)u_0=0,\\ &\frac{d^2u_0}{dx^2}+\frac{1}{x}\frac{du_0}{dx}+\left(\frac{Ma^2V_0}{\hbar^2}\frac{1}{x}-\frac{Ma^2\epsilon}{\hbar^2}\frac{1}{x^2}\right)u_0=0. \end{aligned} \tag{42b}$$

This is the differential equation for a Bessel function (cf. Jahnke-Emde, p. 214), its solution is

$$u_0 = cJ_p(2M^{\frac{1}{2}}V_0^{\frac{1}{2}}a\hbar^{-1}x^{\frac{1}{2}}) = cJ_p(2M^{\frac{1}{2}}V_0^{\frac{1}{2}}a\hbar^{-1}e^{-r/2a}), \quad (42c)$$

where

$$p = 2a(M\epsilon)^{\frac{1}{2}}/\hbar \quad (42d)$$

is the order of the Bessel function. At large distances r, the argument of the Bessel function is small so that the first term of the ordinary expansion of J in a power series is sufficient. We have then:

$$u_0 = \frac{c}{p!}\left(\frac{2(MV_0)^{\frac{1}{2}}a}{\hbar}\right)^p \exp\left[-(M\epsilon)^{\frac{1}{2}}r/\hbar\right] \quad (r \gg a), \quad (42e)$$

which is, apart from the constant factor, identical with (38a). The eigenvalue ϵ is determined by the condition that $u_0(r=0)$ must vanish. For a given V_0, we have therefore to find p from the condition

$$J_p(2(MV_0)^{\frac{1}{2}}a/\hbar) = 0 \quad (42f)$$

and then to calculate ϵ from p with the help of (42d). In order that (42f) has a solution p at all, it is necessary that V_0 be greater than a certain limit. This limit follows readily from the fact that the first zero x_p of $J_p(x)$ moves towards smaller values of x when p decreases. Therefore certainly $x_p > x_0$. Now the first zero of the Bessel function of order zero is $x_0 = 2.4048$ (Jahnke-Emde, p. 237). Therefore (42f) has a solution p only if

$$2(MV_0)^{\frac{1}{2}}a/\hbar > 2.4048;$$

$$V_0 > (\hbar^2/Ma^2)\cdot 1.4457. \quad (42g)$$

If $V_0 = 1.4457\ \hbar^2/Ma^2$, the solution of (42f) will be $p=0$ and therefore (cf. 42d) $\epsilon = 0$, i.e., just no binding energy. If the binding energy remains small compared to V_0, which we have to assume (cf. §9), then V_0 must be only slightly larger than the value (42g).

Table II gives the values of $MV_0a^2\hbar^{-2}$ for different ranges a of the force.

(c) *"Error function" potential*

$$J(r) = -Be^{-r^2/a^2}. \quad (43)$$

The solution u_0 has to be obtained by numerical integration of the differential equation. As in case (a) and (b), and in the qualitative discussion, Ba^2 must be larger than a certain universal constant, in our case 2.65 $\hbar^2/M$, to give any binding for the deuteron at all. Table III gives the relation between $MBa^2/\hbar^2$ and the range a according to Feenberg (F2, F3). This table allows us to determine Ba^2 if a is known. Ba^2 changes only slightly with changing range of the forces a.

TABLE II. *Relation between width a and depth V_0 of exponential force $V_0e^{-r/a}$* (deuteron energy 2.15 MV).

$a=$ 0	0.5	1.0	1.5	2.0	2.5	3.0	$\cdot 10^{-13}$ cm
$p=$ 0	0.228	0.456	0.684	0.912	1.140	1.368	index of Bessel function
$MV_0a^2\hbar^{-2}=$ 1.446	1.888	2.370	2.890	3.466	4.039	4.664	
$V_0 = 59.5a^{-2}$ 310	97	53	35.4	26.5	21.3		MV

TABLE III. *Relation between width a and depth B of "Gaussian" potential Be^{-r^2/a^2}* (deuteron energy 2.15 MV).

a (in 10^{-13} cm)	0	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.25
$T=\hbar^2/Ma^2$ (MV)	∞	64.3	41	28.5	20.9	16.1	12.7	10.25	8.10
B/T (pure number)	2.70	3.22	3.37	3.53	3.68	3.84	4.01	4.18	4.40
B (MV)	∞	207	138	100.5	74.8	61.8	51.0	42.8	35.6
$\bar{B}/T$*	2.70	3.09	3.20	3.32	3.43	3.55	3.68	3.81	3.97
a	2.5	2.75	3	3.5	4	4.5	5	5.5	6
T	6.56	5.43	4.55	3.35	2.56	2.02	1.64	1.35	1.14
B/T	4.62	4.85	5.08	5.55	6.04	6.56	7.13	7.70	8.28
B	30.3	26.3	23.1	18.6	15.45	13.2	11.7	10.4	9.4
$\bar{B}/T$*	4.14	4.31	4.48	4.84	5.20	5.59	6.02	6.45	6.88

* $\bar{B}$ represents the Majorana force between proton and neutron, plus *half* the Heisenberg force (cf. §14, end). This combination enters the theory of H^3, He^4 and heavier nuclei (§20).

All our results about the ground state of the deuteron remain unchanged if we assume either the Wigner or the Heisenberg interaction between neutron and proton instead of the Majorana force. For the ground state, and indeed for any S state, $U(-\mathbf{r})=U(\mathbf{r})$ (cf. (31), (31a)) so that the Wigner equation (30c) and the Majorana equation (30b) become identical. It is only for odd azimuthal quantum number l that there exists any difference between the Wigner and the Majorana theory (cf. 32). The Heisenberg theory also becomes identical with the Majorana theory, if we restrict ourselves to states in which the spins of neutron and proton are parallel, as is the case for the ground state of the deuteron (experimental value of the deuteron spin = one unit).

In conclusion, we like to emphasize that the eigenfunction of the ground state outside of the range of the forces (i.e., for $r>a$) is completely determined by the binding energy ϵ of the deuteron, as is shown by Eq. (36). With the observed value of that binding energy, *viz:*

$$\epsilon=2.15 \text{ MV} \tag{44}$$

we find

$$\frac{1}{\alpha}=\frac{\hbar}{(M\epsilon)^{\frac{1}{2}}}=\frac{1.042\cdot10^{-27}}{(1.66_5\cdot10^{-24}\cdot2.15\cdot1.59\cdot10^{-6})^{\frac{1}{2}}}$$
$$=4.36\cdot10^{-13}\text{ cm}. \tag{44a}$$

$1/\alpha$ may be regarded as the "radius" of the deuteron (cf. the wave function (36)). The range of the forces a is probably about $2\cdot10^{-13}$ cm, which is considerably smaller than the radius of the deuteron. We can therefore say that the solution (36) represents the eigenfunction of the ground state of the deuteron over the greater part of space. For this reason, matrix elements, normalization integrals, etc., may be calculated to a good approximation by assuming (36) to be valid throughout. Then the normalization integral becomes

$$\int u_0^2 dr=c^2\int_0^\infty e^{-\alpha r}dr=c^2/2\alpha=1$$

or
$$u_0=(2\alpha)^{\frac{1}{2}}e^{-\alpha r}, \tag{44b}$$

$$U(r)=\frac{u_0}{(4\pi)^{\frac{1}{2}}r}=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{e^{-\alpha r}}{r} \tag{44c}$$

independent of the form of the potential $J(r)$.

In (41c) we have derived the exact normalization factor for the special case of a rectangular hole potential. It differs from that in (44c) by a factor

$$\left(\frac{V_0-\epsilon}{V_0}\right)^{\frac{1}{2}}\frac{e^{\alpha a}}{(1+\alpha a)^{\frac{1}{2}}}=1+\tfrac{1}{2}\alpha a-O(\alpha a)^2. \tag{44d}$$

(ϵ/V_0 would be of the order $(\alpha a)^2$.) Thus (44c) corresponds to putting the range of the forces equal to zero.

§13. Excited States of the Deuteron

It can easily be shown that no stable excited states of the deuteron exist, if we disregard the spin and make the same assumptions about the forces as in the preceding section, *viz.*, strong forces acting only over a limited range a. For simplicity, we choose the rectangular hole as representing the potential.

The following excited states might *a priori* be expected: p states ($l=1$), d states ($l=2$), etc., or higher s states ($l=0$).

(a) *p states:* In the Majorana theory there would be repulsion between proton and neutron, if the angular momentum of their relative motion is $l=1$ (cf. 32). For Wigner forces, and for Heisenberg forces in the case of antiparallel spins of the two particles, the forces are attractive. Thus in the Majorana theory which we have accepted, stable p states of the deuteron are entirely impossible. But even in the Wigner theory their impossibility can be concluded from the known binding energy of the ground state. We have the wave equation for p states

$$\frac{d^2u_1}{dr^2}-\frac{2}{r^2}u_1=\frac{M}{\hbar^2}\cdot\begin{cases}(-V_0+\epsilon)u_1 & \text{for } r<a,\\ \epsilon u_1 & \text{for } r>a,\end{cases} \tag{45}$$

with the solution

$$u_1=b\left(\frac{\sin kr}{kr}-\cos kr\right)$$
$$\text{with}\quad k=M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}/\hbar,\quad r<a,$$
$$u_1=ce^{-\alpha(r-a)}(1+1/\alpha r)\quad\text{with}\quad \alpha=(M\epsilon)^{\frac{1}{2}}/\hbar,\quad r>a. \tag{45a}$$

We equate the expressions for u_1'/u_1, obtained from these two expressions, for $r=a$:

$$k\frac{(\cos ka)/ka+\sin ka[1-(ka)^{-2}]}{(\sin ka)/ka-\cos ka}$$
$$=-\alpha\frac{1+(\alpha a)^{-1}+(\alpha a)^{-2}}{1+(\alpha a)^{-1}}. \tag{45b}$$

Now we certainly find the minimum V_0 necessary to give a bound p state, by putting the binding energy $\epsilon=0$ and therefore $\alpha=0$. When doing so, the right-hand side of

(45b) becomes $-1/a$. Therefore putting $ka=x$, we must have:

$$\cos x+\sin x(x-1/x)=-(\sin x/x-\cos x); \quad (45c)$$

$$ka=x=\pi; \quad V_0a^2=\pi^2\hbar^2/M. \quad (45d)$$

This value is irreconcilable with our previous conclusion (cf. (40)) that V_0a^2 is only slightly larger than $(\pi^2/4)\hbar^2/M$, which followed from the fact that V_0 is large compared to the binding energy of the deuteron. (From the theory of the α-particle, a value V_0a^2 of about $4\hbar^2/M$ can be deduced, §21.) We therefore have to conclude that there is no stable p state of the deuteron even if we assume Wigner forces.

(b) *d states.* For Wigner forces, s, p and d states form a monotonous sequence with increasing "centrifugal force" $l(l+1)/r^2$. Therefore, if p states do not exist for Wigner forces, this is *a fortiori* true for d states. Since the Majorana and Wigner equations are identical for d states ($l=2$), there is no stable d state in the case of Majorana forces either.

(c) *Higher s states.* From (39) it follows that one stable s state may be found for $M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a/\hbar$ between $\pi/2$ and π, another for $M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}a/\hbar$ between $3\pi/2$ and 2π, etc. If the second bound s state is to exist, we must certainly have

$$(MV_0)^{\frac{1}{2}}a/\hbar>3\pi/2; \quad V_0a^2>(9\pi^2/4)\hbar^2/M, \quad (45e)$$

which is again impossible for the same reason as excited p states.

Therefore *no excited states of the deuteron exist which differ from the ground state with respect to orbital motion.*

However, we should expect to find a second state of the deuteron which differs from the ground state with respect to the total spin. In the ground state the spin of the deuteron is 1 unit, i.e., the spins of proton and neutron are parallel. We should expect another state with antiparallel spins of the two particles and therefore total spin $s=0$. This second state would be a singlet state in spectroscopic nomenclature (nondegenerate state, statistical weight 1), while the ground state is a triplet state (triply degenerate because of three possible orientations of the deuteron spin in an external magnetic field).

If we assume the pure Majorana interaction, the energies of singlet and triplet state are, in first approximation, equal. Their difference arises only from the magnetic interaction between the two spins. We assume the classical interaction between the magnetic moments

$$W=\mu_n\mu_p[(\boldsymbol{\sigma}_n\boldsymbol{\sigma}_p)r^2-3(\boldsymbol{\sigma}_n\mathbf{r})(\boldsymbol{\sigma}_p\mathbf{r})]r^{-5}, \quad (46)$$

where μ_n and μ_p are the magnetic moments of neutron and proton, $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_p$ the respective spin operators and $\mathbf{r}$ the relative coordinate of the proton with respect to the neutron. We may write explicitly

$$W=\mu_n\mu_p[\sigma_{nz}\sigma_{pz}r^{-3}(1-3\cos^2\theta)+\cdots]. \quad (46a)$$

The diagonal matrix element of W with respect to an s state vanishes because of the dependence of W upon the angular coordinates θ, φ of the point $\mathbf{r}$. The splitting δE of an s state due to the magnetic spin interaction is therefore a second-order effect. By the ordinary Schrödinger perturbation theory it can be shown that the order of magnitude of the splitting is about

$$\delta E\approx W^2(a)/V_0 \quad (46c)$$

where $W(a)$ is some average value of the magnetic interaction if the particles are a distance a apart, *viz.*,

$$W(a)\approx\mu_n\mu_pa^{-3}=2.9\cdot2.0(e\hbar/2Mc)^2a^{-3}=1.5(\hbar/Mca)^2e^2/a.$$

With $\hbar/Mc=0.21\cdot10^{-13}$, $a=10^{-13}$ and $e^2/mc^2=2.80\cdot10^{-13}$ (m = electron mass), we have

$$W(a)=1.5mc^2\cdot(0.21)^2\cdot2.80=0.10 \text{ MV}, \quad (46d)$$

$$V_0=\pi^2\hbar^2/4Ma^2=100 \text{ MV},$$

$$\delta E=100 \text{ volts}. \quad (46e)$$

Actually, the range of the forces is rather larger than $1\cdot10^{-13}$ cm which would make δE even smaller.

We should therefore expect an energy difference of the order of 100 volts between the singlet and triplet state of a deuteron, if we assume it to be due only to magnetic interaction between the spins, and if we use the classical formula for this interaction. Experimentally, there is strong reason to believe that the singlet state lies about 2 million volts higher than the triplet state (§14). This can apparently not be explained by magnetic interaction. We may, however, assume that the nuclear forces themselves depend to a certain extent upon the relative spin directions of the interacting proton and neutron. In other words, we assume that *small Heisenberg forces exist*, after all, *besides the principal forces of the Majorana type.*

If $J(r)$ is the "Majorana" potential and $K(r)$ the "Heisenberg" potential, we have then, according to (30)

$$(\hbar^2/M)\Delta U(\mathbf{r})+EU(\mathbf{r})=(J(r)\pm K(r))U(-\mathbf{r}), \quad (47)$$

where the upper sign holds for a triplet, the lower for a singlet state. To explain that the triplet state of the deuteron lies lower than the singlet state, we have to assume that K is negative as well as J. The magnitude of K may be determined by assuming that K is represented

by a rectangular hole of the same width as J, such that

$$K(r)=-V_1 \quad \text{for} \quad r<a,$$
$$K(r)=0 \quad \text{for} \quad r>a. \tag{47a}$$

Then we have, analogous to (39)

$$\cot\left[M^{\frac{1}{2}}(V_0+V_1-\epsilon_t)^{\frac{1}{2}}a/\hbar\right] = -(\epsilon_t/V_0-V_1-\epsilon_t)^{\frac{1}{2}} \quad \text{(triplet state)}, \tag{47b}$$

$$\cot\left[M^{\frac{1}{2}}(V_0-V_1-\epsilon_s)^{\frac{1}{2}}a/\hbar\right] = -(\epsilon_s/V_0+V_1-\epsilon_s)^{\frac{1}{2}} \quad \text{(singlet state)}.$$

To deduce the numerical value of V_1, we use the result of the scattering experiments (§14) that ϵ_s is very nearly zero, *viz.*, $\epsilon_s\approx 40{,}000$ volts. Therefore we may neglect ϵ_s entirely compared to ϵ_t (2.15 MV). Then we have, in analogy to (39b)

$$V_0-V_1=(\pi^2/4)(\hbar^2/Ma^2),$$
$$V_0+V_1=(\pi^2/4)\hbar^2/Ma^2+2\epsilon_t^{\frac{1}{2}}(\hbar/Ma^2)^{\frac{1}{2}} \tag{47c}$$

or
$$V_1=(\epsilon_t\hbar/Ma^2)^{\frac{1}{2}}=(2/\pi)(V_0\epsilon)^{\frac{1}{2}}. \tag{48}$$

Since the most probable value for V_0 is about 30 MV (§21) and $\epsilon=2.15$ MV, we have

$$V_1\approx 5 \text{ MV}. \tag{48a}$$

The magnitude of the Heisenberg force is of the order of the geometric mean between the Majorana potential and the deuteron binding energy.

§14. Scattering of Neutrons by Protons. I: Cross Section (W13, B18, M8, T1)

Closely related to the deuteron problem is the scattering of neutrons by protons. Here again we have just two interacting particles, one proton and one neutron, the only difference being that the system has *positive* energy E. Since all our calculations refer to the relative motion of the two particles, E is the kinetic energy of the two particles in a coordinate system in which the center of gravity of the particles is at rest (C system). If v is the relative velocity of the particles, $\frac{1}{2}v$ will be the velocity of each particle in the C system, and therefore

$$E=\tfrac{1}{4}Mv^2. \tag{49}$$

In general, the experimental arrangement will be such that neutrons of a given velocity v are shot against protons at rest. The kinetic energy of the neutrons in a coordinate system which is at rest (R system) is then

$$E_0=\tfrac{1}{2}Mv^2=2E. \tag{49a}$$

The velocity of the center of gravity in the R system is $\frac{1}{2}v$.

In the scattering process, the neutron may be deflected by an angle θ in the C system. Then its velocity component in its original direction of motion will be $\frac{1}{2}v\cos\theta$ in the C system, or $\frac{1}{2}v(1+\cos\theta)$ in the R system. The velocity of the proton in the C system must be opposite and equal to that of the neutron, it has therefore the components $-\frac{1}{2}v\cos\theta$, $\frac{1}{2}v\sin\theta$ parallel and perpendicular to the direction of motion of the incident neutron, respectively. The velocity of the proton in the R system has therefore the components

$$\begin{aligned} v_{||}' &= \tfrac{1}{2}v(1-\cos\theta)=v\sin^2\tfrac{1}{2}\theta, \\ v_{\perp}' &= \tfrac{1}{2}v\sin\theta \quad\quad =v\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta. \end{aligned} \tag{49b}$$

The angle between the motion of the proton after collision and that of the neutron before collision is therefore given by

$$\varphi=\tfrac{1}{2}\pi-\tfrac{1}{2}\theta. \tag{49c}$$

If the neutron suffers a small deflection θ, the proton goes off at right angles. A proton emitted in the direction of the incident neutron corresponds to a reversal of the motion of the neutron in the C system ($\theta=180°$). The energy of the recoil proton is

$$E'=\tfrac{1}{2}Mv'^2=\tfrac{1}{2}Mv^2\sin^2\tfrac{1}{2}\theta=E_0\cos^2\varphi; \tag{49d}$$

therefore the energy of the neutron after collision

$$E''=E_0\sin^2\varphi. \tag{49e}$$

In a head-on collision ($\varphi=0°$, $\theta=180°$) all the energy is transferred to the proton. In a soft collision ($\varphi\approx 90°$, $\theta\approx 0°$) practically no energy is lost by the neutron.

The wave function of two particles interacting with a central force can always be expanded in a series of spherical harmonics, *viz*:

$$U(\mathbf{r})=\sum_l c_l[u_l(r)/r]P_l(\cos\theta). \tag{50}$$

We choose the direction of motion of the incident neutron as axis of our polar coordinate system. The wave function representing the scattering of the two particles will obviously have axial symmetry round that direction, therefore (50) contains only the ordinary Legendre polynomials $P_l(\theta)$, not any associated functions $P_{lm}(\theta)e^{im\varphi}$ The radial functions u_l satisfy the equation

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l\right)+(E-(-1)^lJ(r))u_l=0 \tag{32}$$

if the Majorana theory is accepted.

Asymptotically for large r, we may neglect the terms $l(l+1)/r^2$ and $J(r)$ in (32) so that the solution of (32) is

$$u_l = c \sin (kr - \tfrac{1}{2}l\pi + \delta_l) \tag{51}$$

with $k = (ME)^{\frac{1}{2}}/\hbar = Mv/2\hbar = (\tfrac{1}{2}ME_0)^{\frac{1}{2}}/\hbar$. (52)

The "phase" δ_l is a constant which has to be determined by integration of (32). If $J(r)=0$, i.e., if no force acts between the two particles, all δ_l's turn out to be zero. The knowledge of the δ_l's is sufficient to determine the scattering cross section for a given angle θ (cf. Mott and Massey, *Atomic Collisions*, p. 24), *viz.*:

$$d\sigma = (\pi/2k^2)\,|\sum_l (2l+1)P_l(\theta)(e^{2i\delta_l}-1)\,|^2 \times \sin\theta d\theta. \tag{53}$$

The cross section $d\sigma$ is defined as the number of neutrons scattered per unit time through an angle between θ and $\theta+d\theta$, if there is one neutron crossing unit area per unit time in the incident beam.

We know (§9, §12) that the forces between proton and neutron are restricted to a very small range a of the order of $2\cdot 10^{-13}$ cm. This is considerably less than the wave-length of all neutrons which have been used for scattering experiments: The fastest neutrons thus far used for such experiments have an energy of about $E_0=4$ MV. The neutron wave-length λ is given by

$$\frac{\lambda}{2\pi} = \frac{1}{k} = \frac{\hbar}{(\tfrac{1}{2}ME_0)^{\frac{1}{2}}} = \frac{1.04\cdot 10^{-27}}{(\tfrac{1}{2}\cdot 1.66\cdot 10^{-24}\cdot 1.59\cdot 10^{-6}E_0^{\mathrm{MV}})^{\frac{1}{2}}} = \frac{9.0_5\cdot 10^{-13}}{(E_0^{\mathrm{MV}})^{\frac{1}{2}}}\ \mathrm{cm} \tag{54}$$

in the system where the center of gravity of neutron and proton is at rest, E_0^{MV} being the kinetic energy of the neutron, expressed in MV. For $E_0=4$ MV, we have

$$\lambda\!\!\!^{-} = \lambda/2\pi = 4.5\cdot 10^{-13}\ \mathrm{cm}, \tag{54a}$$

which is more than twice the range of the forces, q.e.d. (We have purposely calculated $\lambda\!\!\!^{-}=\lambda/2\pi$ rather than λ itself, because $\lambda\!\!\!^{-}$ is the quantity which enters directly into the following considerations about the magnitude of the δ_l's.)

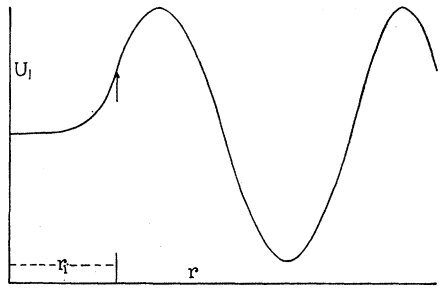

FIG. 5. The wave function of a particle of high angular momentum ($l=4$) in a potential field. The wave function is concave towards the line $U_l=0$ for $r>r_l$, convex and very small for $r<r_l$. (The axis $U_l=0$ has been omitted and should be the continuation of the beginning of the curve.)

From the fact that $a \ll \lambda\!\!\!^{-}$, one can easily deduce that *all phases δ_l will be small except δ_0*. To see this, we first prove that any wave function u_l becomes small if r is small compared to the "classical impact parameter" $r_l = \lambda\!\!\!^{-} l = l\hbar/(\tfrac{1}{2}Mv)$ which is the distance at which particles of the angular momentum $l\hbar$ and the linear momentum $\tfrac{1}{2}Mv$ would pass each other. ($\tfrac{1}{2}M$=reduced mass.) In quantum mechanics, r_l marks a change in the behavior of the wave function u_l. For $r>r_l$, u_l has the character of a wave, i.e., it is concave towards the r axis, while u_l is convex (has exponential character) for $r<r_l$ (see Fig. 5). This follows immediately from (32). We neglect $J(r)$ which we can do for $r>a$, and therefore certainly for $r=l\lambda\!\!\!^{-}$ if $l\neq 0$. Of the remaining terms, the term $-\hbar^2 l(l+1)u/Mr^2$ in (32) will be larger than Eu_l if

$$r^2 < l(l+1)\hbar^2/ME = l(l+1)\lambda\!\!\!^{-2}. \tag{54b}$$

This is certainly fulfilled if $r<r_l=l\lambda\!\!\!^{-}$. Neglecting accordingly Eu_l and Ju_l, we have the differential equation

$$d^2u_l/dr^2 - [l(l+1)/r^2]u_l = 0 \quad \text{for} \quad r<r_l=l\lambda\!\!\!^{-}, \tag{54c}$$

whose solution is $u_l = cr^{l+1}$ for $r<r_l$ (54d)

(c a constant). Thus u_l decreases rapidly with decreasing r when r becomes smaller than $l\lambda\!\!\!^{-}$. Since the range a of the potential energy $J(r)$ is supposed to be small compared to $\lambda\!\!\!^{-}$, these considerations apply certainly for $r=a$, if only l is different from zero; i.e., $u_l(l\neq 0)$ will be very small in the region where the force $J(r)$ is acting, which minimizes the effect of the potential $J(r)$ upon the wave function. Since δ_l is a measure of that effect, δ_l must be small. The case $l=0$ is an exception because here the "critical distance" r_l would be zero so that our argument does not apply. (For the quantitative proof of our argument, see §15.)

In the scattering cross section (53), only the term $l=0$ will therefore be important. From this result, which is based solely on the fact that the forces between proton and neutron have a

very small range a, smaller than the neutron wave-length λ, two important conclusions can be drawn.

(1) The scattering cross section will be independent of the angle θ, in other words, *the scattering will be spherically symmetrical* in the C system, i.e., *in the system in which the center of gravity of neutron and proton is at rest.*

(2) The total scattering cross section can be obtained by calculating only the phase δ_0. We have, putting all δ_l's in (53) except δ_0 equal to zero,

$$d\sigma = 2\pi k^{-2} \sin^2 \delta_0 \sin \theta d\theta \tag{55}$$

or for the total cross section

$$\sigma = \int d\sigma = 4\pi k^{-2} \sin^2 \delta_0. \tag{55a}$$

We defer the discussion of the angular distribution to §15, and begin with the calculation of δ_0, starting from our treatment of the ground state of the deuteron. We know that for the ground state, i.e., for the energy $-\epsilon$, the slope of the eigenfunction $l=0$ is given by (34), *viz.*:

$$[(1/u_0)(du_0/dr)]_{r=a} = -\alpha \quad \text{for} \quad E = -\epsilon. \tag{34}$$

We show first that this relation still holds approximately for not too large *positive* values of E, of the order ϵ, whatever the forces between neutron and proton may be. To show this, we write down the wave equation for the wave functions u_0^E and $u_0^{-\epsilon}$:

$$\begin{aligned} d^2u_0^E/dr^2 + (M/\hbar^2)(E - J(r))u_0^E &= 0, \\ d^2u_0^{-\epsilon}/dr^2 + (M/\hbar^2)(-\epsilon - J(r))u_0^{-\epsilon} &= 0. \end{aligned} \tag{55b}$$

Multiplying the first equation by $u_0^{-\epsilon}$, the second by u_0^E, and subtracting, we have

$$u_0^{-\epsilon}\frac{d^2u_0^E}{dr^2} - u_0^E\frac{d^2u_0^{-\epsilon}}{dr^2} = -\frac{M}{\hbar^2}(E+\epsilon)u_0^E u_0^{-\epsilon} \tag{55c}$$

or integrated from 0 to a:

$$\left| u_0^{-\epsilon}\frac{du_0^E}{dr} - u_0^E\frac{du_0^{-\epsilon}}{dr} \right|_0^{r=a} = -\frac{M}{\hbar^2}(E+\epsilon)\int_0^a u_0^E u_0^{-\epsilon} dr. \tag{55d}$$

The expression on the left-hand side vanishes for $r=0$, because $u_0^{-\epsilon}$ as well as u_0^E vanish as r itself for small r [cf. the remark after Eq. (33)]. Dividing by $u_0^{-\epsilon}(a)u_0^E(a)$, we obtain therefore

$$\left(\frac{1}{u_0^E}\frac{du_0^E}{dr}\right)_{r=a} - \left(\frac{1}{u_0^{-\epsilon}}\frac{du_0^{-\epsilon}}{dr}\right)_{r=a} = -\frac{M}{\hbar^2}\frac{E+\epsilon}{u_0^E(a)u_0^{-\epsilon}(a)}\int_0^a u_0^E u_0^{-\epsilon} dr. \tag{56}$$

The integral on the right-hand side is of the order $a \cdot u_0^E(a)u_0^{-\epsilon}(a)$. It actually is somewhat smaller because $u_0^{-\epsilon}$ and u_0^E vanish at $r=0$ and their value at $r=a$ is practically equal to their maximum value (cf. the explicit wave function 38a, for $r<a$). Therefore we put

$$\int_0^a u_0^E u_0^{-\epsilon} dr = \gamma a u_0^E(a) u_0^{-\epsilon}(a). \tag{56a}$$

For the special case of a rectangular potential hole, $\gamma = \frac{1}{2}$, see below. Inserting the value (34) for $du_0^{-\epsilon}/u_0^{-\epsilon}dr$, we have then

$$\left(\frac{1}{u_0^E}\frac{du_0^E}{dr}\right)_{r=a} = -\alpha - \frac{\gamma Ma}{\hbar^2}(E+\epsilon). \tag{57}$$

The rignt-hand side of (57) reaches the value -2α when E has the value

$$E' \approx \frac{\hbar^2\alpha}{Ma\gamma} = \frac{1}{\gamma}\left(\frac{\hbar^2}{Ma^2}\epsilon\right)^{\frac{1}{2}} \tag{58}$$

remembering (37). In the case of a rectangular potential hole we obtain, using (40) and $\gamma = \frac{1}{2}$:

$$E' = (4/\pi)(V_0\epsilon)^{\frac{1}{2}}. \tag{58a}$$

If we assume $V_0 = 30$ MV (§21) and take the observed value $\epsilon = 2.15$ MV, we find $E' = 10$ MV. For energies $E \ll E'$ the value of $(1/u_0)(du_0/dr)$ will be approximately equal to $-\alpha$.

Since E is only one-half of the kinetic energy E_0 of the neutron, we may put

$$(1/u_0)(du_0/dr) = -\alpha \tag{57a}$$

for all neutron energies E_0 small compared to $2E' = 20$ MV, i.e., for all neutron energies thus far available.

Assuming now (57a) to hold, we can easily calculate δ_0. For $r>a$, $J(r)$ is zero so that

$$d^2u_0/dr^2 = -(ME/\hbar^2)u_0 = -k^2u_0$$

(cf. (52)). Therefore

$$u_0 = c \sin(kr + \delta_0), \tag{51$_0$}$$

where c is a constant. Joining this solution to the solution for $r<a$, we have

$$[(1/u_0)(du_0/dr)]_{r=a} = k \cot(ka + \delta_0) = -\alpha, \tag{59}$$

which yields

$$\delta_0 = \tfrac{1}{2}\pi + \text{arc tan}(\alpha/k) - ka. \tag{59a}$$

Now we have assumed throughout this section that the range of the forces a is small compared to $\lambda = 1/k$. Therefore we neglect ka in (59a) and have

$$\cot \delta_0 = -\alpha/k. \tag{59b}$$

Inserting this into (55a), we obtain for the total cross section

$$\sigma = 4\pi/(\alpha^2 + k^2) \tag{60}$$

or, inserting the values of α and k from (37) and (52):

$$\sigma = 4\pi\hbar^2/M(\epsilon + \tfrac{1}{2}E_0), \tag{61}$$

or, with $\epsilon = 2.15$ MV:

$$\sigma = 2.39\cdot 10^{-24}\epsilon/(\epsilon + \tfrac{1}{2}E_0) \text{ cm}^2. \tag{61a}$$

This value for the cross section is in fair agreement with experimental determinations for fast neutrons, considering the difficulty of the experiment, which is chiefly that of obtaining a beam of neutrons of well-defined energy. For $E_0 = 4.3$ MV Chadwick (C6) found a cross section between 0.5 and $0.8\cdot 10^{-24}$ cm² compared to a theoretical value of $1.2\cdot 10^{-24}$; for $E_0 = 2.1$ MV the agreement is better, *viz.*, 1.1 to $1.5\cdot 10^{-24}$ experimental and $1.6\cdot 10^{-24}$ theoretical.[31]

However, for slow neutrons our formula fails completely. The experimental cross section, observed by Dunning, Pegram, Fink and Mitchell* (D4) is about $35\cdot 10^{-24}$ cm², i.e., more than 14 times as large as our theoretical cross section (61a) for $E_0 \ll \epsilon$. Since the theoretical value depends only upon the binding energy ϵ and not upon any details about the force between neutron and proton, this discrepancy looks at first sight very serious. This is the more true because our assumption that the range of the forces is small compared to the wave-length is much better fulfilled for slow neutrons than for fast ones.

To solve this difficulty, it has been pointed out by Wigner (private communication) that the observed binding energy of the deuteron refers only to the binding of a proton and a neutron with parallel spins while nothing can be deduced from it about the interaction of protons and neutrons with opposite spins. The binding energy ϵ' of a deuteron in a singlet state, i.e., with the spins of the two constituent particles antiparallel, may well be assumed to be very small. This assumption is sufficient to make the probability that a slow neutron is scattered by a proton with opposite spin extremely large, according to (61).

Accepting this explanation, we may use the experimental value of the cross section to determine the binding energy ϵ' of the deuteron in the singlet state. The cross section for the scattering of a neutron by a proton with spin parallel to that of the neutron, will be given by (61); the cross section for antiparallel spins will have the same form, only with ϵ' instead of ϵ. Now it is just 3 times as probable that the spins of a given neutron and a given proton are parallel, as that they are antiparallel.[32] Therefore the average cross section for the scattering of neutrons by protons, averaged over the possible directions of spin, is

$$\sigma^a = \frac{4\pi\hbar^2}{M}\left(\frac{1}{4}\frac{1}{\epsilon' + \tfrac{1}{2}E_0} + \frac{3}{4}\frac{1}{\epsilon + \tfrac{1}{2}E_0}\right). \tag{62}$$

From the observed cross section for $E_0 = 0$, *viz.*, $35\cdot 10^{-24}$ cm², and the known value of $\epsilon = 2.15$ MV, we can deduce

$$\epsilon' = 40{,}000 \text{ ev}. \tag{62a}$$

The binding energy of the singlet state of the deuteron must therefore be supposed to be very small, compared to that of the ground (triplet) state.

Fig. 6 shows the cross section (62) as a function of the neutron energy E_0. At high energies, the difference between (62) and (61) is hardly noticeable because the binding energies ϵ and ϵ' can be neglected compared to E_0. This explains why formula (61) was found to agree with

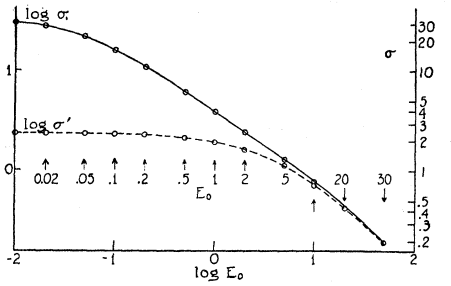

FIG. 6. Scattering of neutrons by protons. Abscissa: energy in MV, ordinate: cross section; both on a logarithmic scale. Solid curve: actual theoretical cross section. Dotted curve: cross section, if interaction is independent of the relative spin direction of the particles.

* *Note added in proof:* Recent experiments of Amaldi and Fermi give an even larger cross section, *viz.* about $80\cdot 10^{-24}$ cm². The reason for this discrepancy is not clear.

[31] Dunning, Pegram, Fink and Mitchell give $1.68\cdot 10^{-24}$ cm² as the cross section for "fast" neutrons from $Be + \alpha = C^{12} + n$ (mixed velocities).

[32] If the spins are parallel, the total spin of the system (proton+neutron) is 1 unit. This total spin can orient itself in three different ways with respect to a given direction, e.g., the direction of an external field, the three orientations corresponding to a spin component +1, 0, or −1 in the given direction. There are therefore three possible spin states if the two spins are parallel. In the case of antiparallel spins, the total spin is 0, and only one quantum state exists.

experiments about fast neutrons. At about $E_0=2$ MV, the actual cross section (62) will begin to become considerably larger than the cross section (61) which we would expect if singlet and triplet state had the same binding energy. The cross section then increases rapidly with decreasing neutron energy, reaching half the "slow neutron value" for E_0 about 100,000 volts.

The bearing of the large difference between the binding energies of singlet and triplet state of the deuteron, upon the problem of nuclear forces, has been discussed in §13. At this place, we want only to point out that it cannot be inferred from the scattering cross section whether there actually *is* a *stable* singlet state of the deuteron. For the formula (60) for the cross section contains only α^2 or rather the corresponding quantity for the singlet state which we may call β^2. We can therefore only infer the magnitude of β from the scattering but not its sign. Only for positive β will there be a stable singlet state, for negative β the singlet state would be just unstable. Which of the two alternatives is true, does not make much difference for the conclusions about the Heisenberg forces drawn in §13, because there we needed only to use the fact that $|\beta|\ll\alpha$. There will, however, be some difference in the probability for the capture of slow neutrons by protons (§17) and that effect may lead to a decision whether the singlet state is just stable or just unstable.

Another means to decide the sign of the energy of the singlet level, and at the same time to test the whole theory, has been pointed out by Teller (T1). We have assumed that the intensity of the scattering of neutrons by protons depends strongly on their relative spin orientations. This makes the scattering of slow neutrons by parahydrogen molecules quite different from that by orthohydrogen. In the latter case, both protons in the molecule have their spins *parallel* to each other: Therefore a neutron will either be scattered strongly by both the protons, or weakly by both of them; in either case, the scattered intensity will be the same for both protons. Consequently, there will be strong interference effects, if the wave-length of the neutrons is of the same order as the distance of the protons in the H_2 molecule, which is actually the case for neutrons of about thermal energy. The scattered neutrons will have an angular distribution identical to that of x-rays of the same wave-length scattered by a diatomic molecule.

The scattering by parahydrogen will be quite different: There we have in each molecule one proton whose spin is parallel to that of the incident neutron while the spin of the other proton is opposite. In first approximation, only the *second* proton will scatter so that we get no interference effects. The presence of interference in the scattering of neutrons by orthohydrogen and the absence in the scattering by parahydrogen would be a direct test of our assumption about the dependence of the scattering on the spin orientation.

Now we consider the case of parahydrogen in second approximation. There will be *some* scattering from the proton having its spin *parallel* to the neutron. The ratio of the amplitudes of the waves scattered by the "parallel" and by the "antiparallel" proton is for small energy of the neutron:

$$\alpha/\beta=(\epsilon'/\epsilon)^{\frac{1}{2}}=1/7, \quad \text{approximately.} \tag{62b}$$

Now if the singlet state is a real bound state, the phases of the waves scattered by the two protons will be the same, whereas they will be opposite if the singlet state is a virtual state. In the first case, the amplitude of the neutron wave scattered by a parahydrogen molecule in the *forward* direction (i.e., through a very small angle) will be 8/7 times that scattered by an isolated proton having its spin opposite to the neutron; in the second case, the amplitude will be only 6/7. The ratio of the scattered intensities in the two cases will therefore be

$$\frac{\sigma_{\text{real singlet state}}}{\sigma_{\text{virtual state}}}=\left(\frac{8}{6}\right)^2\approx 1.8. \tag{62c}$$

On the other hand, if we could observe at such an angle that the difference in path between the neutrons scattered by the two protons of the molecule, is equal to half a wave-length, the ratio of the scattering intensities would be reversed. Actually, such an angle cannot be found because the molecules rotate in space. The result is that the scattering through large angles should be almost equal in the two cases (real and virtual singlet state).

We must now discuss the accuracy of our formulae. We have made two neglections which both amount to assuming the range a of the forces to be zero. On one hand, we have neglected ka in (59a), on the other hand, $\gamma Ma(E+\epsilon)/\hbar^2\alpha$ in (57a). It seems in order, to calculate the necessary corrections to our formulae by using the simple rectangular hole potential. We have, then, the following expressions for the wave function

$$\begin{aligned} \psi &= b \sin \kappa r \quad \text{for } r<a, \\ \psi &= c \sin (kr+\delta_0) \quad \text{for } r>a \end{aligned} \tag{63}$$

with $$\kappa = M^{\frac{1}{2}}(V_0+E)^{\frac{1}{2}}/\hbar, \quad k=(ME)^{\frac{1}{2}}/\hbar. \tag{63a}$$

Joining at $r=a$ yields

$$\cot (ka+\delta_0) = (\kappa/k) \cot \kappa a. \tag{63b}$$

We now use the fact that for the ground state of the deuteron (cf. (39) (37))

$$\kappa_0 \cot \kappa_0 a = -\alpha \tag{63c}$$

with $$\kappa_0 = M^{\frac{1}{2}}(V_0-\epsilon)^{\frac{1}{2}}/\hbar, \quad \alpha = (M\epsilon)^{\frac{1}{2}}/\hbar. \tag{63d}$$

Expanding in powers of k, we have

$$\begin{aligned} \kappa &= \kappa_0 + (\alpha^2+k^2)/2\kappa_0, \\ \kappa \cot \kappa a &= \kappa_0 \cot \kappa_0 a + \tfrac{1}{2}(\alpha^2+k^2)\kappa_0^{-1} \\ &\quad \times (\cot \kappa_0 a - \kappa_0 a \sin^{-2} \kappa_0 a). \end{aligned} \tag{63e}$$

Now $\cot \kappa_0 a$ is very small compared to unity, and can therefore be neglected in the second term of (63e), while $\sin^2 \kappa_0 a$ is practically 1. Thus:

$$\kappa \cot \kappa a = -\alpha - \tfrac{1}{2}(\alpha^2+k^2)a. \tag{63f}$$

By using the definition of α and k, this is identical with (57), if γ in that equation is put equal to $\frac{1}{2}$. Inserting (63f) into (63b), and expanding the left-hand side, we have

$$\cot \delta_0 - ka(\sin \delta_0)^{-2} = -\alpha/k - (\alpha^2+k^2)a/2k. \tag{63g}$$

In the second term on the left, we may insert for $\cot \delta_0$ the approximate value $-\alpha/k$. Then we find

$$\begin{aligned} \cot \delta_0 &= -\frac{\alpha}{k} + ka\left(1+\frac{\alpha^2}{k^2}\right) - \frac{(\alpha^2+k^2)a}{2k} \\ &= -\frac{\alpha}{k} + \frac{(\alpha^2+k^2)a}{2k} \end{aligned} \tag{63h}$$

and for the cross section

$$\begin{aligned} \sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 &= \frac{4\pi}{k^2(1+\cot^2 \delta_0)} \\ &= \frac{4\pi}{k^2[1+(\alpha/k)^2 - \alpha a(\alpha^2+k^2)k^{-2}]} \approx \frac{4\pi(1+\alpha a)}{\alpha^2+k^2}, \end{aligned} \tag{63i}$$

neglecting higher powers of αa than the first. Thus we see that the cross section is simply multiplied by a constant factor $1+\alpha a$. In accord with the theory of H^3 and He^4 (§21) we assume the range a to be about $2\cdot10^{-13}$ cm. With the experimental value $1/\alpha = 4.36\cdot10^{-13}$ cm, we obtain then $1+\alpha a = 3/2$ for the scattering of neutrons by protons with parallel spin. For opposite spin, the correction factor would be only $1+\beta a$, where $\beta^2 = M\epsilon'/\hbar^2$. With $\epsilon' = 40{,}000$ volts, this gives

$$1/\beta = 32\cdot10^{-13} \text{ cm} \tag{63k}$$

and $1+\beta a = 1.06$. The correction is thus small for the larger part of the cross section.

For extremely high energies, the expansion in powers of E/V_0 is, of course, no longer legitimate. If $E \gg V_0$, we can expand in powers of V_0/E instead. With

$$\gamma = (MV_0)^{\frac{1}{2}}/\hbar \tag{64a}$$

we have then

$$\begin{aligned} &\kappa = k + \gamma^2/2k; \\ &(\kappa/k) \cot \kappa a = \cot ka + \tfrac{1}{2}(\gamma/k)^2(\cot ka - ka \sin^{-2} ka). \end{aligned} \tag{64b}$$

Therefore, from (63b):

$$\begin{aligned} \delta_0 &= -\tfrac{1}{2} \sin^2 ka (\gamma/k)^2 (\cot ka - ka \sin^{-2} ka) \\ &= (\gamma/2k)\gamma a (1 - \sin 2ka/2ka). \end{aligned} \tag{64c}$$

The last term may be neglected for large ka, and γa be put equal to $\pi/2$, according to (40). Thus we find

$$\delta_0 = (\pi/4)(\gamma/k) = (\pi/4)(V_0/E)^{\frac{1}{2}}; \tag{64d}$$

therefore $$\sigma_0 = \frac{\pi^3}{4}\frac{\gamma^2}{k^4} = \frac{\pi^3}{4}\frac{\hbar^2}{M}\frac{V_0}{E^2}. \tag{64}$$

However, in this case also the contribution of the partial waves with nonvanishing angular momentum contribute appreciably to the scattering (§15, end).

§15. Scattering of Neutrons by Protons. II: Angular Distribution (W12, W9, B18)

We have already shown in the preceding section that the angular distribution of the neutrons scattered by protons should be practically spherically symmetrical in a coordinate system in which the center of gravity of the two particles is at rest. We shall now discuss the deviations from spherical symmetry. To do this, we have to calculate the phases δ_l in the scattering formula (53), for $l \neq 0$.

Since we know already these phases to be small, we may calculate them by a perturbation method. We do this for the particular case $l=1$. Let v_1 denote the wave function for $l=1$ in the case of vanishing potential energy $J(r)$, i.e., the solution of the equation

$$\frac{\hbar^2}{M}\left(\frac{d^2v_1}{dr^2} - \frac{2}{r^2}v_1\right) + Ev_1 = 0. \tag{65}$$

Multiplying (65) by u_1, i.e., the wave function for nonvanishing J, and the Eq. (32) for $l=1$ by v_1 and subtracting, we have

$$v_1\frac{d^2u_1}{dr^2} - u_1\frac{d^2v_1}{dr^2} = -\frac{M}{\hbar^2}J(r)u_1v_1. \tag{65a}$$

We integrate from zero to a very large radius R, remembering that at the lower limit $r=0$ both the functions u_1 and v_1 vanish (cf. (55d)), and divide by $u_1(R)v_1(R)$:

$$\left(\frac{1}{u_1}\frac{du_1}{dr}\right)_{r=R} - \left(\frac{1}{v_1}\frac{dv_1}{dr}\right)_{r=R} = -\frac{M}{\hbar^2}\frac{\int_0^R J(r)u_1v_1dr}{u_1(R)v_1(R)}. \tag{65b}$$

Now at large distances u_1 as well as v_1 behave like sine waves; we put (cf. 51)

$$v_1=\sin(kr-\tfrac{1}{2}\pi)=-\cos kr,\qquad u_1=-\cos(kr+\delta). \tag{65c}$$

The amplitudes have been assumed unity in both cases, which can always be achieved by suitable normalization of u_1 and v_1. We apply (65b) to a point where v_1 has one of its maxima, i.e., kR is a multiple of 2π. Then

$$[(1/u_1)(du_1/dr)]_{r=R}=-k\tan(kR+\delta)=-k\tan\delta;\quad [(1/v_1)(dv_1/dr)]_{r=R}=0;\quad u_1v_1\approx 1. \tag{65d}$$

Then from (65b)

$$\tan\delta=M\hbar^{-2}k^{-1}\int_0^R J(r)u_1v_1dr. \tag{66}$$

The potential J is only large for $r<a$. In that region we know, however, the wave functions u_1 and v_1 to be small [cf. the discussion following Eq. (54)]. This shows that δ must be small. A quantitative estimate may be obtained by putting u_1 in the integral equal to v_1. The latter function, i.e., the solution of the wave equation (65) for a free particle with unit angular momentum, is well known. It is:

$$v_1=-\cos kr+\sin kr/kr. \tag{66a}$$

For $kr\ll 1$ we find by expansion:

$$v_1=\tfrac{1}{3}(kr)^2. \tag{66b}$$

Inserting this into (66), we have

$$\delta_1=(1/9)M\hbar^{-2}k^3\int_0^a J(r)r^4dr. \tag{66e}$$

The integral can be estimated with the help of Eq. (37c) which is based upon the fact that a stable state of the deuteron with comparatively low binding energy exists. We may estimate

$$\int_0^a J(r)r^4dr=\tfrac{1}{2}\mu a^3\int_0^a J(r)rdr=-\tfrac{1}{2}\mu a^3\hbar^2/M, \tag{66d}$$

where μ is a constant of the order of magnitude unity. For the particular case of a rectangular hole potential, μ may be determined by explicit solution of the wave equation, it turns out to be

$$\mu=6\left(1+\frac{12}{\pi^2}\right)-\frac{36}{\pi}\frac{1+e^{-\pi}}{1-e^{-\pi}}=0.89. \tag{66c}$$

Inserting (66d) into (66c) we find

$$\delta_1=-(1/18)\mu(ka)^3. \tag{67}$$

δ_1 is therefore very small compared to unity, as long as the wave-length $1/k$ of the neutron is large compared to the range of the forces a, i.e., as long as the neutron energy is small compared to the depth V_0 of the potential hole (cf. (40), (52)). This is true for any neutrons thus far available since V_0 is of the order 30 MV. The higher δ_l's (for $l>1$) are, of course, even smaller than δ_1. A calculation similar to the above yields

$$\delta_l\approx\frac{(-1)^l}{1^2\cdot 3^2\cdot\cdots\cdot(2l+1)^2}\mu_l(ka)^{2l+1} \tag{67a}$$

with μ_l a constant rather smaller than unity.

In calculating the angular distribution of scattered neutrons for available velocities, we may therefore neglect the contributions of all l's larger than one, and also neglect δ_1^2. Then we find from (53)

$$\begin{aligned}d\sigma&=(\pi/2k^2)\,|\cos 2\delta_0-1+i\sin 2\delta_0\\&\qquad+6i\delta_1\cos\theta|^2\sin\theta d\theta\\&=(\pi/2k^2)(4\sin^2\delta_0\\&\qquad+12\delta_1\sin 2\delta_0\cos\theta)\sin\theta d\theta,\end{aligned} \tag{68}$$

$$d\sigma=(2\pi/k^2)\sin^2\delta_0(1+6\delta_1\cot\delta_0\cos\theta)\sin\theta d\theta.$$

The parenthesis determines the deviation of the scattering from spherical symmetry. According to (67), δ_1 is negative. Thus we shall find the scattering backwards ($\theta>90°$) greater than the scattering forwards ($\theta<90°$) provided cot δ_0 is positive. If cot δ_0 is negative, the reverse will be the case. Now cot δ_0 has been calculated in (63h). It is obviously negative for small neutron energies, and positive for high energies. Cot δ_0 vanishes when

$$\alpha^2+k^2=2\alpha/a, \tag{69}$$

or, introducing energies according to (37), (40), (52)[33]

$$\tfrac{1}{2}E_s^0+\epsilon=2(\epsilon\hbar^2/Ma^2)^{\frac{1}{2}}=(4/\pi)(\epsilon V_0)^{\frac{1}{2}}. \tag{69a}$$

If neutron and proton have parallel spin, we have to insert $\epsilon=2.15$ MV, and V_0 about 30 MV, so that

$$E_s^0=20\text{ MV}. \tag{69b}$$

For opposite spins, the asymmetry is negligible. Neutrons of energy less than about 20 MV have, therefore, to be considered as "slow" as regards the sign of the deviation from spherically symmetrical scattering.

To calculate the deviation from spherical symmetry explicitly, we have to add the cross sections for parallel and antiparallel spin. From (60), (62), (68) and (63h) we have

[33] We introduce the actual kinetic energy E_0 of the neutron, rather than the kinetic energy in the system where the center of gravity of neutron and proton is at rest.

$$d\sigma=\frac{3}{4}\frac{2\pi}{k^2+\alpha^2}\left(1-\frac{6\delta_1\cos\theta}{k}\right.$$
$$\left.\times[\alpha-\tfrac{1}{2}a(\alpha^2+k^2)]\right)\sin\theta d\theta$$
$$+\frac{1}{4}\frac{2\pi}{k^2+\beta^2}\left(1-\frac{6\delta_1\cos\theta}{k}\right.$$
$$\left.\times[\beta-\tfrac{1}{2}a(\beta^2+k^2)]\right)\sin\theta d\theta. \quad (70)$$

Inserting δ_1 from (67), we find that the asymmetry can only be appreciable for comparatively high energies. We therefore neglect in (70) terms of the relative order $(\alpha/k)^4$. Neglecting also β compared to α_1, we find that the relative asymmetry

$$A=[\sigma(\theta=0)-\sigma(\theta=\pi)]/\sigma(\theta=\tfrac{1}{2}\pi) \quad (70a)$$

has a maximum for

$$k_0^2=(3\alpha/4a)(1-\tfrac{1}{2}a\alpha) \quad (70b)$$

corresponding to an energy of about 3 MV. For this energy, the asymmetry becomes

$$A_{\max}=\frac{9}{48}\mu(a\alpha)^2\left(1-\frac{5}{3}a\alpha\right)=0.85 \text{ percent} \quad (71)$$

if we put $a=2\cdot10^{-13}$ cm and $1/\alpha=4.4\cdot10^{-13}$ cm. The asymmetry of scattering should therefore, even at the maximum, not exceed the minute amount of one percent. Such a small asymmetry is, with the present methods, quite unobservable.

An appreciable asymmetry in the scattering should only be found for neutron energies higher than 20 MV (cf. (69b)), which are at present unavailable. These high energy neutrons should preferentially be scattered backwards by protons (cf. Eq. (68)), quite in contrast to other scattering processes. This unusual behavior is due to the exchange type of the forces between neutron and proton. The scattering process can be interpreted by saying that the incident particle actually is only deflected by a small angle but has, in the process of scattering, changed roles with the scattering particle so that it goes off as a proton if it was a neutron before the collision.

It was first pointed out by Wick that by observing the asymmetry of the scattering of neutrons by protons, one could decide whether the forces between the particles are of the "ordinary" or the "exchange" type; e.g., at high energies one would expect to find the neutrons to be scattered preferentially in the forward direction, if ordinary forces act, while we have seen that they are scattered preferentially backwards by exchange forces. At low energies, the reverse would be true, because of the negative sign of cot δ_0 in (68). Unfortunately, the asymmetry of the scattering should be much too small for the available neutron energies to allow any decision about the forces.

Our main result is thus that the scattering of neutrons by protons should, theoretically, be spherically symmetrical in the coordinate system, in which the center of gravity of neutron and proton is at rest. In the "ordinary" coordinate system, in which the proton is initially at rest, the distribution should be

$$d\sigma=\text{const}\cdot\sin\theta d\theta=2\text{ const}\cdot\sin\varphi\cos\varphi d\varphi, \quad (72)$$

i.e., the number of scattered protons per unit solid angle $\sin\varphi d\varphi$ should have a flat maximum in the forward direction $\varphi=0$. The experimental results are highly contradictory. While Chadwick (C6), Monod-Herzen (M15), Kurie (K15) and Barton, Mueller and Lampson (unpublished[34] preliminary results in photographic emulsion) found spherical symmetry within the limits of their experimental error in accord with theory, Harkins, Gans, Kamen and Newson (H3) found many more protons at small angles φ with respect to the incident neutron, than at large angles. This would mean that large deflections θ of the neutrons are much more probable than small ones. This is quite irreconcilable with our considerations. Thus far, the possible experimental error is still very large. Should, however, the deviation from spherical symmetry as observed by Harkins and others, be confirmed by more extensive experiments, it would make our assumption about the short range of the forces between neutron and proton quite untenable. There would then arise a very grave difficulty in how to explain the large mass defect of the α-particle as compared to the deuteron.

[34] We are indebted to Mr. Mueller for communicating to us these preliminary results, based on the observation of 105 tracks.

Before concluding this section, we want to calculate the cross section for very high neutron energies, $E \gg V_0$. In this case, we may apply Born's approximate method for calculating the scattering cross section which gives

$$d\sigma = (\tfrac{1}{2}M)^2/2\pi\hbar^4 \cdot |\int U'^*(\mathbf{r})J(r)U(-\mathbf{r})d\tau|^2 \sin\theta d\theta, \quad (73)$$

where $U(\mathbf{r})$ and $U'(\mathbf{r})$ are the wave functions of incident and scattered neutron, in zero approximation. It should be noted that $U(-\mathbf{r})$ enters rather than $U(\mathbf{r})$ because of the exchange nature of the forces. Denoting the wave vectors of incident and scattered neutron by $\mathbf{k}$ and $\mathbf{k}'$, we have

$$U(\mathbf{r}) = e^{i(\mathbf{k}\cdot\mathbf{r})}, \quad U'(\mathbf{r}) = e^{i(\mathbf{k}'\cdot\mathbf{r})},$$

$$d\sigma = (M^2/8\pi\hbar^4)\,|\int J(r)e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}d\tau|^2 \sin\theta d\theta. \quad (73a)$$

We assume $J(r)$ to depend exponentially[35] upon r, *viz.*:

$$J(r) = -V_0 e^{-r/a} \quad (73b)$$

and obtain by an elementary calculation

$$d\sigma = \frac{2\pi M^2 V_0^2 \sin\theta d\theta}{\hbar^4 a^2[(\mathbf{k}+\mathbf{k}')^2 + a^{-2}]^4}. \quad (73c)$$

We consider now that (cf. Table II)

$$(\mathbf{k}+\mathbf{k}')^2 = 2k^2(1+\cos\theta), \quad 2\hbar^2k^2/M = E_0, \quad (73d)$$

$$V_0 = A\hbar^2/Ma^2, \quad \text{where} \quad A \approx \sqrt{8}.$$

We have, except for very small values of $1+\cos\theta$ (of the order V_0/E_0):

$$d\sigma \approx (\pi/4)a^2(V_0/E_0)^4(1+\cos\theta)^{-4}\sin\theta d\theta, \quad (74)$$

or, using the angle φ between the recoil proton and the incident neutron ($\varphi = \frac{1}{2}(\pi-\theta)$)

$$d\sigma = (\pi/64)a^2(V_0/E_0)^4 \sin^{-8}\varphi \cos\varphi \sin\varphi d\varphi. \quad (74a)$$

The total cross section is, neglecting higher powers of V_0/E_0:

$$\sigma = \frac{\pi}{3k^2}\left(\frac{MV_0a^2}{\hbar^2}\right)^2 \approx \frac{\pi A^2}{3k^2} = \frac{16\pi}{3}\frac{\hbar^2}{ME_0}. \quad (74b)$$

For neutrons of 10^9 volts energy, this gives a cross section of $0.68\cdot10^{-26}$ cm^2 which is rather large.

[35] The rectangular potential hole gives for $d\sigma$ an expression which is a rapidly oscillating function of $(\mathbf{k}+\mathbf{k}')$.

It was first pointed out by Bhabha (B19) that the collisions between protons and neutrons of very high energy have an important application for cosmic rays. Suppose there are protons of about 10^9 volt energy in cosmic rays. Such protons have a fair chance of knocking a neutron out of a nucleus they encounter. This neutron would, due to the exchange character of the forces, carry away almost all the energy of the proton. The cross section for the occurrence of this process in a collision between a proton and, say, a nitrogen nucleus, may safely be assumed to be 7 times the cross section for a proton and a free neutron, because the nuclear binding energies are small compared to the kinetic energy of the proton. The cross section becomes thus $\frac{1}{2}\cdot10^{-25}$ cm^2 for N which means that the process should happen about once per 5 meter water equivalent of the atmosphere. Cosmic-ray protons would in this way "become" neutrons after traversing a comparatively small amount of air, and become protons again after going through more air. An incident proton radiation would therefore be about half protons half neutrons when reaching sea level. Moreover, the thickness of matter required to stop protons of a given energy, would be doubled by this process, since neutrons suffer no appreciable energy losses.

§16. Photoelectric Disintegration of the Deuteron (C7, C8, B16, F8, H1)

Chadwick and Goldhaber (C7, C8) have observed that the deuteron can be disintegrated into a neutron and a proton by the γ-rays from Th C', of energy $h\nu = 2.62$ MV. This experiment, besides being of high interest in itself, gives at the same time the most exact determination of the binding energy of the deuteron. For this purpose, Chadwick and Goldhaber have determined the number of ions formed by the proton which is produced in the disintegration. This number turns out to be about 7200, with an accuracy of, say, ± 20 percent. The average energy spent by the proton in producing one ion in air is certainly close to 33 volts, which figure holds for the ionization by α-particles as well as electrons. The kinetic energy of the proton is therefore $33\cdot7200 = 240{,}000$ volts± 20 percent. The neutron formed in the disintegration must, because of momentum

conservation, receive the same energy and go off in the opposite direction from the proton, since the momentum of the γ-ray quantum is negligible compared to that of neutron and proton.[36] The kinetic energy of neutron plus proton is thus $2 \cdot 0.24$ MV $= 0.48$ MV, with an accuracy of ± 20 percent $= \pm 0.10$ MV.* Therefore the binding energy of the deuteron is

$$\epsilon = 2.62 - 0.48 \pm 0.10 = 2.14 \pm 0.10 \text{ MV} \quad (75)$$

a value which we have made use of repeatedly.

From this binding energy, we can also determine the mass of the neutron, since the masses of both deuteron and proton are known from Bainbridge's mass-spectroscopic determinations. Choosing the values suggested by Bethe (B13), we have

$$H^2 = 2.01423 \pm 0.0002,$$

$$H^1 = 1.00807,$$

$$\epsilon = (2.14 \pm 0.10)/931 = 0.00230 \pm 0.00010 \text{ mass unit}, \quad (75a)$$

$$n^1 = H^2 + \epsilon - H^1 = 1.00846 \pm 0.0002.$$

The mass of the neutron thus turns out to be 0.00039 mass units $= 0.36 \pm 0.20$ MV larger than that of the hydrogen atom. This result is only based upon the ratio of the atomic weights of heavy and light hydrogen, as determined by Bainbridge (B5); it is independent of the absolute value of the atomic weight of, say, deuterium compared to oxygen. An error in ϵ as large as 0.36 MV can almost certainly be excluded; thus the only error which could materially influence the difference between the atomic weights of neutron and proton would be an error in Bainbridge mass determinations which would have to be much larger than the probable error deduced from the internal consistency of his results.[37] At present, it seems more likely that the result (75a) is correct i.e., that the neutron is really considerably heavier than the hydrogen atom. A free neutron must then disintegrate spontaneously into a proton plus an electron (§43), the lifetime being about a month.

Returning to the disintegration of the deuteron by γ-rays, it is obvious that this effect is closely analogous to the photoelectric effect in atoms. The electric field of the γ-ray produces an optical transition of the deuteron from the ground state to a state of positive energy

$$E = h\nu - \epsilon, \quad (76)$$

E being the sum of the kinetic energies of proton and neutron produced in the process. The cross section for the photoelectric effect is given by the well-known formula

$$\sigma = 8\pi^3 \nu |M_{0E^{el}}|^2/c, \quad (77)$$

where M_{0E} is the matrix element of the electric moment of the deuteron relative to its center of gravity and in the direction of polarization of the γ-ray, the matrix element referring to the transition from the ground state to the state of energy E. (The transition can also be produced by a *magnetic* moment, this "magnetic dipole" photoelectric effect is, however, small compared to the "electric dipole" effect discussed here, except for very low energies E. Cf. the end of this §, and §17.) Since only the proton has a charge e, and since its coordinate relative to the center of gravity of the deuteron is $\frac{1}{2}\mathbf{r}$, we have

$$M_{0E^{el}} = \tfrac{1}{2} e \int U_0 z U_E d\tau \quad (77a)$$

if the γ-ray is polarized in the z direction. Here U_0 is the wave function of the ground state, as given in (44c), and U_E is the wave function of the final state, normalized per unit energy.

From the familiar selection rule for the angular

[36] A simple momentum consideration shows that the energy of the proton should vary from 0.21 MV, if it is emitted in the direction opposite to the incident γ-ray, to 0.27 MV in the forward direction, 0.24 MV being assumed as the average proton energy.

* *Note added in proof:* Feather (Nature **136**, 467 (1935)) measured the range of the protons and deduced from it a kinetic energy of 0.18 MV for the protons. The range-energy relation in present use is likely to give too low values for the energy (Chapter XV) so that the correct energy may be about 0.20 MV. This would raise ϵ to 2.22 MV, raise the weight of the neutron to 1.00855, lower $1/\alpha$ (cf. 44a) to $4.29 \cdot 10^{-13}$ cm, lower the numerical factor in (61a) to $2.31 \cdot 10^{-24}$ and the factor in (80) to $1.12 \cdot 10^{-26}$. Furthermore, in (80b) $\gamma - 1$ should be 0.178 and $\sigma = 5.3 \cdot 10^{-28}$ cm^2, in perfect agreement with the observed value $5 \cdot 10^{-28}$ cm^2. However, the finite range of the forces (paragraph below (80b)) and the addition of the photomagnetic effect (81) would again raise the theoretical cross section for the disintegration of the deuteron to nearly twice the experimental value, which is just within experimental error.

[37] With Aston's recent "preliminary" determination of the mass ratio $D : H$ (A5), the neutron mass would come out as high as 1.0090, i.e., 0.9 MV larger than that of the hydrogen atom.

momentum we infer that the final state U_E must have the angular momentum $l=1$, i.e., in spectroscopic notation, it must be a P state.[38] We know, however, from the discussion in §13 and §15 that P states are practically uninfluenced by the force between neutron and proton, provided the range of the force a is small compared to the wavelength $\lambda\!\!\!^{-}=\hbar/(ME)^{\frac{1}{2}}$ corresponding to the state E. This condition is well fulfilled for our case ($E=0.5$ MV, $\lambda\!\!\!^{-}=9\cdot10^{-13}$ cm$=4a$). Therefore the wave function for our P state will have the same form as if the neutron and proton were free, *viz.*:

$$U_E=(3/4\pi)^{\frac{1}{2}}\cos\theta(2/\pi)^{\frac{1}{2}}(dk/dE)^{\frac{1}{2}} \times r^{-1}(-\cos kr+\sin kr/kr). \quad (77b)$$

Here $(3/4\pi)^{\frac{1}{2}}\cos\theta$ represents the (normalized) first spherical harmonic. The bracket is the radial wave function as given in (66a), while the factor $1/r$ has been introduced in (50). The factor $(2/\pi)^{\frac{1}{2}}$ normalizes the radial wave function "per unit wave number dk"[39] while the factor

$$(dk/dE)^{\frac{1}{2}}=(M/2\hbar^2k)^{\frac{1}{2}} \quad (77c)$$

transforms to normalization per unit energy. We rewrite (77b):

$$U_E=\frac{3^{\frac{1}{2}}}{2\pi\hbar}\cos\theta\left(\frac{M}{k}\right)^{\frac{1}{2}}\frac{1}{kr^2}\,\mathrm{Re}\,[e^{ikr}(-i-kr)], \quad (77d)$$

"Re" denoting the real part. We assume the axis of our polar coordinate system to be parallel z, so that

$$z=r\cos\theta. \quad (77e)$$

Furthermore, we use the fact the wave function of the ground state is, for the larger part of the space, represented by (44c). Then we obtain from (77a)

$$M_{0E}^{\mathrm{el}}=\tfrac{1}{2}e\frac{3^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}\hbar}\left(\frac{\alpha M}{k}\right)^{\frac{1}{2}}\int 4\pi r^3dr\,\overline{\cos^2\theta}\frac{1}{kr^3} \times\mathrm{Re}\,[(-i-kr)e^{-\alpha r+ikr}]. \quad (78)$$

$\cos^2\theta$ is the average of $\cos^2\theta$ over all directions in space, *viz.*, $\frac{1}{3}$. We carry out the integration:

$$M_{0E}^{\mathrm{el}}=\frac{e}{(6\pi)^{\frac{1}{2}}\hbar}\left(\frac{\alpha M}{k}\right)^{\frac{1}{2}} \times\mathrm{Re}\left(-\frac{i}{k(\alpha-ik)}-\frac{1}{(\alpha-ik)^2}\right) =\frac{e}{(6\pi)^{\frac{1}{2}}\hbar}\left(\frac{\alpha M}{k}\right)^{\frac{1}{2}}\left(\frac{1}{\alpha^2+k^2}-\frac{\alpha^2-k^2}{(\alpha^2+k^2)^2}\right) =\left(\frac{2}{3\pi}\right)^{\frac{1}{2}}\frac{e}{\hbar}(\alpha M)^{\frac{1}{2}}\frac{k^{\frac{1}{2}}}{(\alpha^2+k^2)^2}. \quad (78a)$$

Inserting this into (77), we find for the cross section

$$\sigma=\frac{8\pi}{3}\frac{2\pi\nu}{c}\frac{e^2M}{\hbar^2}\frac{\alpha k^3}{(\alpha^2+k^2)^4}. \quad (78b)$$

Here we use (76):

$$\hbar\omega=2\pi\hbar\nu=E+\epsilon=\hbar^2(\alpha^2+k^2)/M \quad (78c)$$

and express α and k in terms of ϵ and E (cf. (37), (52)):

$$\sigma=\frac{8\pi}{3}\frac{e^2}{\hbar c}\frac{\hbar^2}{M}\frac{\epsilon^{\frac{1}{2}}E^{\frac{1}{2}}}{(E+\epsilon)^3}. \quad (79)$$

Introducing the ratio of the energy of the γ-ray to the binding energy of the deuteron, *viz.*,

$$\gamma=h\nu/\epsilon=(E+\epsilon)/\epsilon \quad (79a)$$

we have, using (44a)

$$\sigma=\frac{8\pi}{3}\frac{1}{137}\frac{1}{\alpha^2}\frac{(\gamma-1)^{\frac{1}{2}}}{\gamma^3} =1.16\cdot10^{-26}(\gamma-1)^{\frac{1}{2}}\gamma^{-3}\ \mathrm{cm}^2. \quad (80)$$

This cross section vanishes for $\gamma=1$, i.e., if the energy of the γ-ray is just sufficient to produce disintegration. σ then increases slowly with increasing γ-ray energy, and reaches a maximum for $\gamma=2$, i.e., when the kinetic energy of the disintegration products E is just as large as the binding energy ϵ of the deuteron. This maximum is, according to (80),

$$\sigma_{\max}=14.5\cdot10^{-28}\ \mathrm{cm}^2. \quad (80a)$$

For still more energetic γ-rays, the cross section decreases again.

[38] More accurately, we may say it to be a ^3P state, since the spin remains unchanged in the photoelectric transition, and since the ground state is a triplet state.

[39] Cf., e.g., *Handbuch der Physik*, Vol. 24, p. 292, Eq. (4.18).

For the γ-rays of Th C′, we have (see above) $E=0.48$ MV, therefore

$$\gamma-1=E/\epsilon=0.224$$

and
$$\sigma=6.7\cdot10^{-28}\text{ cm}^2. \qquad (80b)$$

The experiments of Chadwick and Goldhaber give a cross section of $5\cdot10^{-28}$ cm^2 with an uncertainty of a factor of 2 in either direction. The agreement between theory and experiment is therefore satisfactory.

It must again be emphasized that the theoretical formula contains no assumptions except our usual one that the range of the forces between neutron and proton is small compared to the radius of the deuteron. The error introduced by this assumption is mainly due to the normalizing factor of the eigenfunction of the ground state, which should, according to (44d), be multiplied by the factor $1+\frac{1}{2}\,\alpha a$, giving a factor $1+\alpha a\approx1.4$ to the cross section (cf. H1). (The correction necessary in the calculation of the matrix element M_{0E} itself, apart from the change in normalization, is very small because the contribution of the region $r<a$, for which the expressions (44c) and (77d) fail to hold, to the matrix element is only of the relative order of magnitude $(ka)^3$, i.e., about $\frac{1}{2}$ percent.) The theoretical cross section should thus be somewhat greater than (80).

Furthermore, we have neglected the possible "photomagnetic" effect, i.e., the transition due to the magnetic dipole moment (F8). In the following §, we shall calculate the corresponding dipole moment M_{0E}^{magn} (cf. 34). The ratio is (cf. 78a, 37, 52, 78c, 94)

$$\tau=\frac{1}{3}\left(\frac{M_{0E}^{\text{magn}}}{M_{0E}^{\text{el}}}\right)^2 =\frac{(\mu_p-\mu_n)^2}{4}\,\frac{(E+\epsilon)^2}{E+\epsilon'}\,\frac{(\epsilon^{\frac{1}{2}}\mp\epsilon'^{\frac{1}{2}})^2}{EMc^2}. \qquad (81)$$

The factor $\frac{1}{3}$ arises from the fact that each of the three magnetic substates of the ground state of the deuteron may be disintegrated by a γ-ray of given polarization in the photo*electric* effect while only *one* of the three substates may be disintegrated photo*magnetic*ally (cf. the remarks after (92b)). For $\mu_p-\mu_n=4.9$ (cf. §5); $E=0.48$, $\epsilon=2.14$, $\epsilon'=0.040$ and $Mc^2=931$ MV, (81) gives 0.31 or 0.56, according to whether the negative or the positive sign holds. (The negative sign stands if there is a stable singlet state of the deuteron, cf. §13, §15, §17; the positive sign if there is no such state.) For Th C′ γ-rays, we should therefore add about 31 percent (or 56 percent) to the cross section (80b) for the photoelectric disintegration. For γ-rays of smaller energy which are just capable of producing disintegration, the ratio (81) would be higher, becoming $\tau=1$ for $E_c=210{,}000$ or 300,000 volts respectively, according to the sign standing in (81). If the disintegration products get less energy than E_c, the magnetic effect will predominate and will cause the cross section to tend less rapidly to zero with decreasing energy E. The complete formula for the photoelectric cross section, including electric and magnetic effect, is

$$\sigma=\frac{8\pi}{3}\,\frac{e^2}{\hbar c}\,\frac{\hbar^2}{M}\left(\frac{\epsilon^{\frac{1}{2}}E^{\frac{3}{2}}}{(E+\epsilon)^3} +\frac{(\mu_p-\mu_n)^2}{4}\,\frac{E^{\frac{1}{2}}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{2}}\pm\epsilon'^{\frac{1}{2}})^2}{(E+\epsilon)(E+\epsilon')Mc^2}\right), \qquad (81a)$$

the minus and plus sign standing according to whether a stable singlet state of the deuteron exists or not.

The most important influence of the "photomagnetic" effect is that upon the *angular distribution* of the protons and neutrons produced in the disintegration of the deuteron. As we have seen, the photo*electric* effect leads to a P state in the continuous spectrum, more exactly to *that* P state which has no angular momentum around the direction of polarization of the incident γ-ray. In this case, the number of protons (or neutrons) emitted per unit solid angle should be proportional to $\cos^2\theta$, θ being the angle between the direction of the proton and the polarization of the γ-ray. Averaging over the directions of polarization of the γ-ray, we obtain a distribution proportional to $\sin^2\Theta$, where Θ is the angle between the direction of *propagation* of the γ-ray and the motion of the proton.

On the other hand, the magnetic effect leads (cf. §17) to a ^1S state in the continuous spectrum of the deuteron, i.e., to a uniform angular distribution of the protons. If τ is the ratio of probabilities of magnetic to electric effect, we shall therefore expect the number of protons emitted

into the solid angle $\sin\Theta d\Theta$ to be proportional to

$$\sigma(\Theta)\sin\Theta d\Theta=(\sin^2\Theta+\tfrac{2}{3}\tau)\sin\Theta d\Theta. \quad (82)$$

The magnetic effect can therefore be verified by observing the number of protons (or neutrons) projected in the direction of propagation of the γ-ray. This number should vanish if the magnetic effect were absent. By actually measuring the number in the forward direction ($\Theta=0$) as well as perpendicularly to the γ-ray ($\Theta=90°$) one might determine the coefficient τ quantitatively. Inserting τ into (81), one might then decide which sign in (81) corresponds to reality, in other words, whether a stable singlet state of the deuteron exists or not.

The disintegration of the deuteron can also be brought about by electron bombardment of deuterium. It can be shown (B16) that the electric field of an electron is approximately as effective as 1/137 of a light quantum. If the electron energy W is large compared to the binding energy ϵ of the deuteron, the cross section is (B16, Eq. (28))

$$\sigma_E=\frac{2\pi}{3\alpha^2}\left(\frac{e^2}{\hbar c}\right)^2\left(\log\frac{W^2}{\epsilon mc^2}-1.432\right) \quad (83)$$

$$=2.1\cdot10^{-29}\left(\log\frac{W^2}{\epsilon mc^2}-1.432\right)\text{ cm}^2, \quad (83a)$$

where m is the electron mass and log denotes the natural logarithm. It seems not impossible to observe this effect. In fact, the disintegration of the beryllium nucleus was first carried out by electron bombardment (B21). However, it must be borne in mind that fast electrons produce a great number of x-rays (continuous x-ray spectrum). These secondary x-rays may be more effective in producing nuclear disintegrations than the primary electrons.

The scattering of light by a deuteron has been calculated (B16). It is found to be smaller than that by a free proton and is thus hardly observable.

§17. Capture of Neutrons by Protons (F8, W3, D4)

Neutrons may be captured by protons with the emission of a γ-ray which carries away the surplus energy

$$h\nu=E+\epsilon=\tfrac{1}{2}E_0+\epsilon. \quad (76a)$$

[E_0=kinetic energy of the neutron in the "ordinary" coordinate system (R system of §14), E=kinetic energy in the system in which the center of gravity of neutron and proton is at rest (C system of §14).] This process is the inverse of the photoelectric disintegration discussed in the preceding section, and the probabilities of the two processes are therefore connected by thermodynamic relations.

In each of the two processes we have in the initial state an incident particle which is in one case the neutron in the other case the light quantum. The cross section of any process is, quite generally, given by the number of the processes occurring per second, divided by the incident current. The number of processes per second is proportional to the square of the matrix element, and to the number of states of the final system per unit energy. Now we have in the final state an outgoing particle, either a light quantum or a neutron. The number of possible states of a particle of momentum p and energy E, per dE and per unit volume, is

$$4\pi p^2dp/h^3dE. \quad (84)$$

By using the relation

$$E^2=c^2p^2+m^2c^4, \quad (84a)$$

where E is the energy including the "rest" energy mc^2, (84) becomes

$$4\pi pE/h^3c^2. \quad (84b)$$

If therefore the indices 1 refer to the incident particle, 2 to the particle produced, the cross section becomes

$$\sigma_{12}\propto(p_2E_2/c^2v_1)\,|M|^2, \quad (85)$$

M being the matrix element for unit density of both particles and v_1 the velocity of the first particle. The cross section for the reverse process is

$$\sigma_{21}\propto(p_1E_1/c^2v_2)\,|M|^2. \quad (85a)$$

The matrix elements being the same in both cases, we have *cet. par.*

$$\sigma_{21}/\sigma_{12}=p_1E_1v_1/p_2E_2v_2. \quad (86)$$

Using the relativistic formula

$$Ev = pc^2 \tag{86a}$$

we have $$\sigma_{21}/\sigma_{12} = p_1^2/p_2^2. \tag{87}$$

The cross sections are proportional to the square of the momenta of the particles produced in the respective reactions.

However, formula (84) gives only the number of possible states of *motion* of the particle produced. In the case of light quanta, we must multiply this by a factor 2 because of the two possible directions of polarization. Furthermore, we must multiply the cross section of the capture process by the probability that the spin of the incident neutron is suitable for capture. Denoting this probability by g_σ we have

$$\frac{\sigma_{\text{capture}}}{\sigma_{\text{disintegr.}}} = 2g_\sigma\left(\frac{p_{\text{quantum}}}{p_{\text{neutron}}}\right)^2 = 2g_\sigma\left(\frac{h\nu/c}{\hbar k}\right)^2 = 2g_\sigma\left(\frac{\omega}{kc}\right)^2, \tag{88}$$

where $\omega = 2\pi\nu$. Using expression (77) for $\sigma_{\text{disintegr.}}$, we have

$$\sigma_{\text{capt.}} = 8\pi^2 g_\sigma \omega^3 k^{-2} c^{-3} |M_{0E}|^2. \tag{89}$$

(It should be noted that in (89) M_{0E} is the matrix element of the electric moment in *one* direction, in accord with (77).) Now the reverse of the photoelectric effect of the deuteron can occur if the spin of incident neutron is parallel to the spin of the capturing proton, because this is the case for the ground state. Therefore $g_\sigma = \frac{3}{4}$ and, using (78b)

$$\sigma_c = \frac{3}{2}\left(\frac{\omega}{kc}\right)^2 \sigma = 4\pi \frac{\omega^3}{c^3}\frac{Me^2}{\hbar^2}\frac{\alpha k}{(\alpha^2+k^2)^4}. \tag{89a}$$

With (78c) this reduces to

$$\sigma_c = 4\pi \frac{e^2}{Mc^2}\frac{\hbar}{Mc}\frac{(E\epsilon)^{\frac{1}{2}}}{E+\epsilon}. \tag{90}$$

The formula (90) for the capture cross section contains the Compton wave-length of the proton,

$$\hbar/Mc = 2.09 \cdot 10^{-14} \text{ cm} \tag{90a}$$

and the "classical proton radius"

$$e^2/Mc^2 = (1/137)(\hbar/Mc) = 1.52_5 \cdot 10^{-16} \text{ cm}. \tag{90b}$$

As a function of the energy E, the maximum of the cross section occurs for $E = \epsilon$ and has the value

$$\sigma_c^{\max} = 2\pi(e^2/Mc^2)(\hbar/Mc) = 2.00 \cdot 10^{-29} \text{ cm}^2, \tag{90c}$$

which is extremely small. For slow neutrons ($E \ll \epsilon$) the cross section would be even smaller.

On the other hand, it has been observed experimentally by various authors (W3, D4) that slow neutrons are absorbed by hydrogen-containing substances such as water and paraffin. Therefore there must be another mechanism of capture which is more efficient than that due to the electric dipole transition, and particularly so at low velocity.

The latter fact gives us a valuable clue. The decrease of the capture cross section (90) with decreasing neutron energy E is due to the small amplitude of the p wave function of a slow neutron at small distances r between neutron and proton. In other words, a slow neutron with angular momentum $l = 1$ has little chance to get sufficiently near the proton to be captured. This would be different if the incident neutron had zero angular momentum (cf. §§14, 15), i.e., for "*s* neutrons." In this case, particularly if the spins of neutron and proton are opposite, we know that the chance of the neutron coming sufficiently near to the proton is very large, resulting in a very large scattering cross section (§14, end). We shall therefore expect a very large capture cross section as well, if there is any process leading to the capture of a neutron with no angular momentum by a proton having opposite spin. Such a capture would correspond to a transition of the system consisting of neutron and proton, from a ^1S state to a ^3S state (ground state of the deuteron), while the capture process considered in (90) corresponds to a transition from a ^3P to a ^3S state.

The required mechanism giving rise to the $^1S \rightarrow {}^3S$ transition is found in the *magnetic dipole radiation*. In the transition from a singlet to a triplet state, the spin of either the proton or the neutron must change its direction; the magnetic moments of neutron and proton will therefore

have matrix elements referring to that transition, and this magnetic dipole moment causes the emission of radiation.

To calculate the transition probability, we have simply to replace the matrix element of the electric dipole moment, M_{0E}^{e1} (cf. 78a) by the matrix element of the magnetic dipole moment. We denote by μ_n and μ_p the magnetic moments of neutron and proton, in units of the "nuclear magneton"

$$\mu_0 = e\hbar/2Mc; \tag{91}$$

then, according to §5, $\mu_p = 2.9$ and $\mu_n = -2.0$. Furthermore, we introduce the spin operators $\boldsymbol{\sigma}_p$ and σ_n of proton and neutron so that $\mu_0\mu_p\sigma_p$ would be, as to magnitude as well as direction, the magnetic moment of the proton. Finally, we introduce the spin wave functions χ_0 and χ_E for ground state and excited state, respectively, then we have

$$M_{0E}^{\text{magn}} = \mu_0 \int \sum_\sigma U_0 \chi_0 (\mu_p \boldsymbol{\sigma}_p + \mu_n \boldsymbol{\sigma}_n) U_E \chi_E d\tau, \tag{92}$$

the $\sum_\sigma$ denoting summation over the spin coordinates of proton and neutron.

The excited state being a singlet state, we have for its spin function[40]

$$\chi_E = 2^{-\frac{1}{2}}[\alpha(p)\beta(n) - \alpha(n)\beta(p)], \tag{92a}$$

p and n denoting the spin coordinates of neutron and proton, respectively. (α means spin parallel to a given direction z, β = spin opposite to z.) The ground state has three substates (triplet) corresponding to spin components in the z direction of $m = +1$, 0 and -1. The z components of the spin operators $\boldsymbol{\sigma}_p$ and $\boldsymbol{\sigma}_n$ in (92) will cause transitions from the state U_E to the substate $M = 0$ of the ground state, the x and y components of the magnetic moment transitions to the substates $M = \pm 1$. We calculate the transition produced by $\sigma_p{}^z$ and $\sigma_n{}^z$ and have therefore for the spin wave function of the ground state

$$\chi_0 = 2^{-\frac{1}{2}}[\alpha(p)\beta(n) + \alpha(n)\beta(p)]. \tag{92b}$$

According to the remark after Eq. (89), we must only calculate the transition probability due to the magnetic moment in *one* direction.

From the definition of the spin wave functions it follows that

$$\sigma_p{}^z\alpha(p) = \alpha(p) \quad \sigma_n{}^z\beta(n) = -\beta(n), \quad \text{etc.}, \tag{92c}$$

therefore

$$\chi_0(\mu_p\boldsymbol{\sigma}_p + \mu_n\boldsymbol{\sigma}_n)\chi_E = \tfrac{1}{2}[\alpha(p)\beta(n) + \alpha(n)\beta(p)] \cdot (\mu_p - \mu_n)[\alpha(p)\beta(n) + \alpha(n)\beta(p)]. \tag{92d}$$

[40] Cf., e.g., *Handbuch der Physik*, Vol. 24, 1, p. 372.

For the summation over the spin coordinates we have to remember that

$$\sum_\sigma \alpha^2(p)\beta^2(n) = \sum \beta^2(p)\alpha^2(n) = 1; \quad \sum_\sigma \alpha(p)\beta(p)\alpha(n)\beta(n) = 0 \tag{92e}$$

and obtain

$$\chi_0(\mu_p\boldsymbol{\sigma}_p + \mu_n\boldsymbol{\sigma}_n)\chi_E = \mu_p - \mu_n \tag{92f}$$

$$M_{0E}^{\text{magn}} = \frac{e\hbar}{2Mc}(\mu_p - \mu_n)\int U_0 U_E d\tau. \tag{93}$$

For the eigenfunction of the ground state we take, as in §16, the expression (44c) which is valid for $r > a$. For U_E we have to insert the wave function corresponding to energy E, angular momentum $l = 0$ and opposite spin of proton and neutron, normalized to unit energy. This function is, according to (50), (51)

$$U_E = u_0/r = (M/k)^{\frac{1}{2}} \sin(kr + \delta_0')/2\pi\hbar r \tag{93a}$$

where δ_0' is to be calculated from a formula analogous to (59a), only with α replaced by β since we are dealing with a singlet state (cf. end of §14). The normalization is identical to (77d). We have now

$$\begin{aligned}\int U_0 U_E d\tau &= \frac{(\alpha M)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\hbar k^{\frac{1}{2}}} \int 4\pi r^2 dr \frac{e^{-\alpha r}}{r} \frac{1}{r} \sin(kr + \delta_0') \\ &= \frac{(2\alpha M)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\hbar k^{\frac{1}{2}}} \operatorname{Im}\left(\frac{e^{i\delta_0'}}{\alpha - ik}\right) \\ &= \frac{(2\alpha M)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\hbar k^{\frac{1}{2}}} \frac{k\cos\delta_0' + \alpha\sin\delta_0'}{\alpha^2 + k^2},\end{aligned} \tag{93b}$$

where Im denotes the imaginary part. δ_0' is given by (59b) where, however, α has to be replaced by β (singlet state!). Therefore we have

$$\int U_0 U_E d\tau = \frac{(2\alpha M k)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\hbar} \frac{\alpha - \beta}{(\alpha^2 + k^2)(\beta^2 + k^2)^{\frac{1}{2}}}. \tag{93c}$$

It should be noted that this expression is only different from zero because $\alpha \neq \beta$, i.e., because the forces between proton and neutron are considerably different according to whether the spins of the particles are parallel or opposite (cf. §13, §14). This fact makes the Schrödinger equations for U_0 and U_E different; were this not the case, U_0 and U_E would be orthogonal and the integral $\int U_0 U_E d\tau$ would vanish. The capture of neutrons by protons by our mechanism is therefore only due to the dependence of the neutron-proton force upon spin.

Inserting (93c) into (93), we have

$$M_{0E}^{\text{magn}} = -\frac{e}{c}\left(\frac{\alpha k}{2\pi M}\right)^{\frac{1}{2}} \frac{\alpha - \beta}{(\alpha^2 + k^2)(\beta^2 + k^2)^{\frac{1}{2}}}(\mu_p - \mu_n). \tag{94}$$

We insert this into (89) and consider that only one in every four neutrons has its spin opposite to a given proton so that $g_\sigma = \frac{1}{4}$. Then we find

$$\sigma_c = \pi \frac{e^2\omega^3}{Mc^5} \frac{\alpha}{k} \frac{(\alpha - \beta)^2(\mu_p - \mu_n)^2}{(\alpha^2 + k^2)^2(\beta^2 + k^2)}. \tag{94a}$$

By using again (78c) and expressing the α, β, k in terms of the energies ϵ, ϵ', $E_0=\frac{1}{2}E$, this reduces to

$$\sigma_c=\pi\frac{e^2}{Mc^2}\frac{\hbar}{Mc}\left(\frac{2\epsilon}{E_0}\right)^{\frac{1}{2}} \times\frac{(\epsilon^{\frac{1}{2}}\mp\epsilon'^{\frac{1}{2}})^2(\epsilon+\frac{1}{2}E_0)}{(\epsilon'+\frac{1}{2}E_0)Mc^2}(\mu_p-\mu_n)^2. \quad (95)$$

The $\mp$ sign stands according to whether the singlet state of the deuteron is stable or virtual.

It will be noticed that the capture cross section *increases* with decreasing kinetic energy E_0 of the neutron. We are therefore particularly interested in the cross section for very small E_0, *viz.*, $E_0\ll\epsilon'$. We compare, for this case, the capture cross section (95) to the cross section for elastic scattering given in (62). We have

$$\kappa=\left(\frac{\sigma_c}{\sigma_{sl}}\right)_{E_0\ll\epsilon'}=\left(\frac{\epsilon}{8E_0}\right)^{\frac{1}{2}}\frac{e^2}{\hbar c}\frac{\epsilon^2}{\frac{1}{4}\epsilon+\frac{3}{4}\epsilon'} \times\frac{(\epsilon^{\frac{1}{2}}\mp\epsilon'^{\frac{1}{2}})^2}{(Mc^2)^2}(\mu_p-\mu_n)^2. \quad (95a)$$

We insert the numerical values $\mu_p=2.9$, $\mu_n=2.0$, $e^2/\hbar c=1/137$, $Mc^2=931$ MV, $\epsilon=2.15$ MV, $\epsilon'=0.040$ MV. Then, if we express E_0 in volts, we find

$$\kappa=\frac{\sigma_c}{\sigma_a}=\left(\frac{2.15\cdot10^6}{8E_0}\right)^{\frac{1}{2}}\frac{4.9^2}{137} \times\frac{2.15^2(\sqrt{2.15}\mp\sqrt{0.040})^2}{(\frac{1}{4}\cdot2.15+\frac{3}{4}\cdot0.040)\cdot931^2}=\begin{cases}0.00138E_0^{-\frac{1}{2}},\\ 0.00227E_0^{-\frac{1}{2}},\end{cases} \quad (96)$$

the upper value holding if there is a stable singlet state of the deuteron, the lower, if there is none. In particular, we may calculate the ratio κ for the case that the neutrons are in thermal equilibrium with the protons which can be achieved by multiple scattering (chapter XII, reference C12). Then in the average, $E_0=kT$ which is equal to 1/40 volt for room temperature ($T=290°$). In this case

$$\kappa=\begin{cases}1/118 \text{ if there is a stable}\\ \qquad\text{singlet state of the deuteron}\\ 1/71 \text{ if there is no stable}\\ \qquad\text{singlet state of the deuteron.}\end{cases} \quad (96a)$$

This means that neutrons of thermal velocity will make, in the average, 71 or 118 elastic collisions with protons before they are captured. The probability of capture is thus fairly large. The cross section is 0.5 or $0.3\cdot10^{-24}$ cm^2 in the two cases, respectively.

The lifetime τ of a neutron in a substance containing hydrogen, such as water, can easily be calculated. The "number of captures per second" is, by definition

$$1/\tau=N\sigma_c v, \quad (97)$$

$v=(2E_0/M)^{\frac{1}{2}}$ being the velocity of the neutrons and N the number of hydrogen atoms per cubic centimeter, which is

$$N=6.73\cdot10^{22} \quad \text{for } H_2O.$$

We have from (95), (97)

$$\tau=\begin{cases}2.84\cdot10^{-4} \text{ sec.},\\ \qquad\text{if stable singlet state exists,}\\ 1.63\cdot10^{-4} \text{ sec.},\\ \text{if stable singlet state does not exist.}\end{cases} \quad (97a)$$

The theory seems to be in sufficient agreement with experiments of Amaldi and Fermi (F10). These authors investigated the diffusion of slow neutrons (absorbable by Cd, energy probably of the order kT) in paraffin. They inserted Cd absorbers into a paraffin block, at various distances from a Rh detector, and observed the decrease in activity of the detector as a function of this distance. In this way, Fermi and Amaldi could determine the average distance which the neutrons would have traveled had they not been absorbed in the Cd. The distance was found to be 2 cm. Theoretically, this distance is (F10) equal to $\lambda(3\kappa)^{-\frac{1}{2}}$ where λ is the mean free path for elastic collisions. With the value of the collision cross section as determined by Dunning and others (D4), *viz.* $\sigma=35\cdot10^{-24}$ cm^2, we find $\lambda=0.35$ cm and therefore $\kappa=100$.

The correction to our results for the finite range of the forces between neutron and proton amounts to an increase in the capture cross section proportional to the increase in the elastic scattering (end of §14) and to a very slight change of the ratio of the capture cross section to the elastic cross section, changing the values in (96a) to 1/113 and 1/75, respectively.

In conclusion, it might be noted that there would also be capture of neutrons by protons with *parallel* spin, if the singlet state of the deuteron is stable. The cross section for this capture which is a ${}^3S \rightarrow {}^1S$ transition, is however very small: It is obtained by interchanging ϵ and ϵ' in (95), and is therefore, at low energies E_0, smaller by a factor $(\epsilon'/\epsilon)^{5/2} = 1/15{,}000$ than the capture into the ground state discussed here.

§18. Scattering of Protons by Protons (W8, T11, P7)

If there is no other interaction between a pair of protons but the Coulomb repulsion, the scattering cross section is given by the Rutherford formula as modified by Mott to take account of the possibility of exchange of the two protons. The cross section for a deflection by an angle between θ and $\theta+d\theta$ is[41]

$$2\pi\sigma_0(\theta)\sin\theta d\theta = \frac{e^4}{M^2v^4}\left(\frac{1}{\sin^4\theta}+\frac{1}{\cos^4\theta} - \frac{\cos((e^2/\hbar v)\log\tan^2\theta)}{\sin^2\theta\cos^2\theta}\right)2\pi\sin 2\theta d(2\theta). \quad (99)$$

2θ is the deflection of the incident proton in the coordinate system in which the center of gravity of the two protons is at rest (C system, cf. the beginning of §14), therefore the element of solid angle in that system is $2\pi\sin 2\theta d(2\theta) = 8\pi\cos\theta\sin\theta d\theta$. The first term in the bracket in (99) gives the number of incident protons deflected by an angle θ (i.e., $\frac{1}{2}\theta$ in the C system) according to the Rutherford formula. (Cf. Mott and Massey, p. 36. Note that in that formula the reduced mass $\frac{1}{2}M$ has to be inserted!) The second term gives the number of incident protons deflected by an angle $\pi/2-\theta$; each of these is accompanied by a recoil proton at right angles to its own motion (cf. §14), i.e., making an angle θ with the incident beam; these recoil protons are also counted among the protons scattered through θ. The last term in the bracket is the effect of exchange between incident and scattering proton. We shall be particularly interested in fast protons, having velocities of the order one-twentieth of the velocity of light (energy ≈ 1 MV); for these, $e^2/\hbar v \ll 1$ and

$$\cos((e^2/\hbar v)\log\tan^2\theta)\approx 1 \quad (99a)$$

unless θ is *very* near or $\pi/2$. We therefore have, with $E_0 = \frac{1}{2}Mv^2$

$$\sigma_0(\theta) = \pi\frac{e^4}{E_0^2}\left(\frac{1}{\sin^4\theta}+\frac{1}{\cos^4\theta} - \frac{1}{\sin^2\theta\cos^2\theta}\right)\cos\theta. \quad (99b)$$

Actually, experiments of White (W8) and of Tuve, Heydenburg, and Hafstad do not agree at all with this formula, but indicate that there are actually considerably more protons at 45° than according to (99b). This proves that there is another force acting besides the Coulomb force. We have already deduced the existence of a force between two protons, and more exactly of an attractive force, from the existence of isobars of even atomic weight and charge (§10). It seems reasonable to assume that this force has the same characteristics as that between neutron and proton, since they no doubt originate from the same reason, i.e., we assume the force between two protons to be also restricted to a small range a and to be large inside that range. Then we can, similarly to the neutron scattering, conclude that only the partial wave function $l=0$ will be materially influenced by the "nuclear forces" between the two protons.

To deduce the scattering including such a force, we have to calculate the wave function of the relative motion of the two protons. There are two cases to distinguish. Either the spins of the protons are parallel or antiparallel. In the first case, the wave function of the system will be symmetrical in the two spins; it must then necessarily be antisymmetrical in the spatial coordinates of the proton, since protons obey the Fermi statistics, i.e., the wave function of a system of two protons changes sign when both the spatial and the spin coordinates of the protons are interchanged. If the spins of the two protons are opposite, the spatial wave function will be symmetrical.

Now we have seen in §14, Eq. (32) that a wave function describing the relative motion of two

[41] Cf. Mott and Massey, *Atomic Collisions*, p. 75, 76, Eqs. (25), (26).

particles, of the form

$$\psi = u_l(r) P_{lm}(\cos\theta) e^{im\varphi}$$

is symmetrical in the coordinates of the two particles, if l is even, and antisymmetrical if l is odd. Therefore, for parallel spins of the protons, the wave function contains only odd azimuthal quantum numbers l, and will therefore not be materially influenced by short range forces. Only for opposite spins of the two protons, i.e., only in 1/4 of all cases, will there be an appreciable influence of the "nuclear" forces between the protons, since in this case the wave function contains the term $l=0$, which is the only term strongly influenced by the nuclear forces. We can calculate the total wave function including the nuclear forces by simply calculating the total wave function without these forces, subtracting from it the part corresponding to $l=0$ without forces and adding the wave function for $l=0$ including the nuclear forces.

We first have to write down, for two protons with Coulomb interaction between them, a wave function symmetrical in the spatial coordinates of the protons. According to Mott and Massey, p. 35, Eq. (16) and p. 72, an *un*symmetrical wave function would be

$$\psi = e^{ikz+i\alpha \log k(r-z)} + e^2/(Mv^2 r \sin^2 \tfrac{1}{2}\Theta) \cdot e^{ikr-i\alpha \log 2kr - i\alpha \log \sin^2 \frac{1}{2}\Theta + i\pi + 2i\eta_0}, \quad (100)$$

where $$\alpha = e^2/\hbar v, \quad k = Mv/2\hbar, \quad (100a)$$

r = distance of the two protons, z = difference of their z coordinates, $\cos\Theta = z/r$, $e^{i\eta_0} = \Gamma(1+i\alpha)/|\Gamma(1+i\alpha)|$. The first term in (100) represents the incident wave, a plane wave moving in the z direction corrected by a small phase shift due to the Coulomb field. The second term is the scattered spherical wave. A *symmetrical* wave function is

$$\Psi(x, y, z) = \psi(x, y, z) + \psi(-x, -y, -z), \quad (100b)$$

because changing the sign of all the relative coordinates $x = x_1 - x_2$, $y = y_1 - y_2$, $z = z_1 - z_2$ of the first proton relative to the second, corresponds to an interchange of the coordinates of the two protons. Now in (100) only the z coordinate occurs explicitly, therefore we have to replace z by $-z$, $\cos\Theta$ by $-\cos\Theta$, and therefore $\sin^2\frac{1}{2}\Theta = \frac{1}{2}(1-\cos\Theta)$ by $\frac{1}{2}(1+\cos\Theta) = \cos^2\frac{1}{2}\Theta$, and then to add the function thus obtained to (100). The resulting function will represent two beams of protons proceeding in the positive and the negative z direction, each containing one proton per cm³.

Next we consider the part of the wave function (100b) corresponding to $l=0$. If we put, as usual, $\psi_0 = v_0/r$, the function v_0 will be a solution of the radial Schrödinger equation

$$d^2v_0/dr^2 + (M/\hbar^2)(\tfrac{1}{2}E_0 - e^2/r)v_0 = 0, \quad (101)$$

which has, for large distances r, the asymptotic form (cf. Mott and Massey, p. 33, Eq. (7) for $n=0$).

$$v_0 = 2e^{i\eta_0} \sin(kr + \eta_0 - \alpha \log 2kr)/kr = (i/kr)(e^{-i(kr-\alpha \log 2kr)} + e^{i(kr+\pi+2\eta_0-\alpha \log 2kr)}). \quad (101a)$$

v_0 is, according to (86), normalized correctly, i.e., in such a way that the first term in (101a) is the first term (the term not depending on the angle) of the expansion of the function (100b) in spherical harmonics.

If a force acts between the two protons at close distances, the phase of the wave function will be shifted by a certain amount δ_0, analogous to the case of scattering of neutrons by protons. Thus the wave function v_0 has to be replaced by

$$u_0 = (2c/kr)e^{i\eta_0} \sin(kr + \eta_0 - \alpha \log 2kr + \delta_0). \quad (101b)$$

The constant c has to be fixed in such a way that the term containing e^{-ikr} is identical with (101a), so that $u_0 - v_0$ contains only an *outgoing* spherical wave, i.e., one proportional to e^{+ikr}, which represents the scattering. This is achieved by putting $c = e^{i\delta_0}$ so that

$$u_0 = (i/kr)[e^{-i(kr-\alpha \log 2kr)} + e^{i(kr+\pi+2\eta_0+2\delta_0-\alpha \log 2kr)}]. \quad (101c)$$

The effect of the force between the two protons is now simply to replace the function v_0 (valid for no force) by u_0 (which takes the force into account), i.e., to add $u_0 - v_0$ to the "unperturbed" wave function (100b):

$$\Psi' = \Psi + u_0 - v_0 = e^{ikz+i\alpha \log k(r-z)} + e^{-ikz-i\alpha \log k(r-z)} + \frac{1}{r} e^{ikr-i\alpha \log 2kr + i\pi + 2i\eta_0}\left[\frac{e^2}{Mv^2}\left(\frac{e^{-i\alpha \log \sin^2 \frac{1}{2}\Theta}}{\sin^2 \frac{1}{2}\Theta} + \frac{e^{-i\alpha \log \cos^2 \frac{1}{2}\Theta}}{\cos^2 \frac{1}{2}\Theta}\right) + \frac{i}{k}(e^{2i\delta_0} - 1)\right]. \quad (102)$$

The scattering cross section per unit solid angle (in the C system) for a given angle Θ is equal to the absolute square of the square bracket. Inserting $k = Mv/2\hbar$ and neglecting $\alpha \log \sin^2 \frac{1}{2}\Theta$ because $\alpha = e^2/\hbar v \ll 1$, we have

$$\sigma(\Theta) = \left(\frac{e^2}{Mv^2}\right)^2 \left[\left(\frac{1}{\sin^2 \frac{1}{2}\Theta} + \frac{1}{\cos^2 \frac{1}{2}\Theta} - \frac{2\hbar v}{e^2} \sin 2\delta_0\right)^2 + \left(\frac{2\hbar v}{e^2}\right)^2 (1 - \cos 2\delta_0)^2\right]. \quad (103a)$$

If there were only the Coulomb field, the cross section $\sigma_0(\Theta)$ would be obtained by putting $\delta_0 = 0$. We have therefore, with $\frac{1}{2}\Theta = \theta$,

$$\sigma(\Theta) - \sigma_0(\Theta) = \left(\frac{e^2}{Mv^2}\right)^2 \left[\left(\frac{4\hbar v}{e^2}\right)^2 \sin^2 \delta_0 - \frac{8\hbar v}{e^2} \sin \delta_0 \cos \delta_0 \left(\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta}\right)\right]. \quad (103b)$$

We now consider the fact that only one-quarter of all proton pairs have spin opposite, therefore the additional cross section (103b) will only be present in 1/4 of all cases. In the average, we have to add 1/4 of (103b). Furthermore, we calculate the cross section per unit solid angle in the *ordinary* coordinate system, by multiplying (103b) with $4 \cos\theta$ (see after (99)).

Thus we find for the complete cross section

$$\sigma(\theta)=\sigma_0(\theta)+\cos\theta(\sigma(\Theta)-\sigma_0(\Theta)),$$

$$\sigma(\theta)=\frac{e^4}{E^2}\cos\theta\left[\frac{1}{\sin^4\theta}+\frac{1}{\cos^4\theta}-\frac{1}{\sin^2\theta\cos^2\theta}\right. \quad (103)$$
$$\left.-\frac{2\hbar v}{e^2}\frac{\sin\delta_0\cos\delta_0}{\sin^2\theta\cos^2\theta}+\left(\frac{2\hbar v}{e^2}\right)^2\sin^2\delta_0\right].$$

The actual magnitude of this expression depends, of course, upon δ_0. This phase shift δ_0 might, of course, be quite small. But even then it will contribute appreciably to (103), since $\sin^2\delta_0$ is multiplied by the very large factor $(2\hbar v/e^2)^2$ which is 160 for protons of 1 MV energy. Therefore the scattering of protons by protons is extremely sensitive to the existence of a force between the protons other than the Coulomb force. This is particularly true for angles θ near 45° where $\cos\theta$ as well as $\sin\theta$ are comparatively large. A good measure of the effect of the "nuclear" proton-proton forces is therefore the ratio of the actual scattering observed at 45° and the scattering following from the Mott formula (99). This ratio is, for $\sin^2\theta=\cos^2\theta=\frac{1}{2}$:

$$\frac{\sigma(\theta)}{\sigma_0(\theta)}=\left(\frac{\hbar v}{e^2}\sin\delta_0\right)^2-2\frac{\hbar v}{e^2}\sin\delta_0\cos\delta_0+1. \quad (104)$$

The first experiments on proton-proton scattering were carried out by White (W8). The protons were accelerated in a cyclotron and their tracks observed in a hydrogen-filled cloud chamber. Since the scattering through large angles is a rare event, and the experimental arrangement does not yield high intensities of the incident proton beam, the number of large deflections observed by White is very small; e.g., only 5 protons were observed which had been deflected by more than 40°. Even so, the experiments seem to show that the scattering through 45° is many times as probable than would be expected from (99b). The ratio of observed scattering to theoretical scattering in a Coulomb field at 45° is 9 in White's experiments, for an average proton energy of, probably, about 750 kv.[42] The angular distribution does not agree well with (104) (cf. 97, and calculations of Serber mentioned by White) which is probably due to the very small number of data taken by White.

Much more extensive experiments have been performed by Tuve, Heydenburg and Hafstad.[43] At 920 kv, they find a ratio of observed scattering to Coulomb scattering of 4.65 at 40°. An energy of 920 kv corresponds to a velocity $v=0.0445c$ (c = velocity of light) so that $\hbar v/e^2=137\cdot0.0445=6.09$. Inserting this and the observed ratio 4.65 into (103), we find

$$\delta_0=30.8° \quad (104a) \quad \text{or} \quad \delta_0=-12.2°. \quad (104b)$$

Inserting these values into (103), the angular distributions may be found. They are given in curves A and R in Fig. 7. Curve A corresponds to the positive value of δ_0 and therefore (see below) to an *attractive* potential between the protons, R corresponds to the negative δ_0

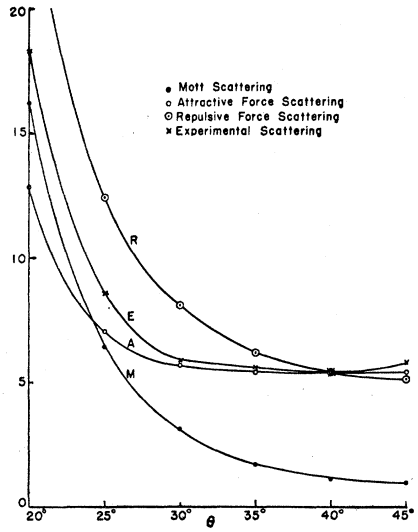

FIG. 7. Scattering of protons by protons. Abscissa: angle of deflection, ordinate: Number of particles scattered through an angle between θ and $\theta+d\theta$, divided by $\sin 2\theta d\theta$. The scale of the ordinate has been chosen so that the Mott scattering in the Coulomb field alone has the value unity at 45°. The theoretical curves are made to agree with the experiments at 40°. Proton energy = 920 kv.

[42] White gives 675 kv, this figure being apparently deduced from the range of protons with the help of the range-energy relation of the Cavendish laboratory. This relation gives too low values for the proton energy.

[43] We are indebted to Messrs. Tuve, Heydenburg and Hafstad for communicating their results to us before publication.

(*repulsive* potential). Curve A is seen to be very flat between 30 and 40°, while R decreases continuously with increasing angle θ. Curve R lies higher than A throughout, except at 45° where the two curves are made to give the experimental value, *viz.*, 4.65 times the Coulomb scattering. The scattering in the pure Coulomb field, according to the Mott formula (99b), is represented in curve M; it lies between the scattering for attractive and repulsive fields for small θ, but falls, of course, below both of them for larger deflections. The experimental points of Tuve, Heydenburg and Hafstad (marked by crosses) follow between 30° and 45° rather closely the curve for the attractive potential. At small angles, the agreement between experimental points and curve A is not perfect; the experimental scattering being too large. However, it is unmistakable that curve A agrees with the experimental data very much better than R; and that the Mott scattering curve M is entirely out of question.

Thus the scattering of protons by protons shows conclusively:

(1) There must be a force between two protons besides the Coulomb force.

(2) This force must be attractive.

(3) Reasonable agreement with experiments is obtained by assuming the force to have a short range.

We have already come to the conclusions (1) and (2) in §10 when discussing the even-odd rule of isotopes. Moreover, the existence of an attractive force between two protons is shown by the calculations of the binding energies of H^3, He^3 and He^4 (§21). We shall even find that the quantitative results for the magnitude of this force as calculated from the binding energies of H^3, He^3, He^4, and from the proton-proton scattering, agree quite well [see (107), and (128)].

The existence of a force between two protons necessitates the assumption of a force between two *neutrons* of practically the same strength. This is shown by the fact that the numbers of neutrons and protons in light nuclei are equal (§6) and, even more accurately, by the difference of the binding energies of H^3 and He^3 (§22).

From (104a), we may calculate a quantity γ which is the analog of the quantities α and β for the neutron-proton scattering (cf. 34). In other words, γ is defined by

$$\gamma = -((1/u_0)(du_0/dr))_{r=a}, \tag{105a}$$

u_0 being the partial wave function $l=0$, and a the range of the forces. We have (cf. 59b)

$$k \cot (kr_0+\delta_0) = -\gamma. \tag{105b}$$

Assuming the range r_0 to be very small, we may neglect kr_0 compared to δ_0. For $E_0=920$ kv, we have (cf. 54) $k=1.06\cdot 10^{12}$ cm^{-1}. With δ_0 given by (104a), we have therefore

$$\gamma = -1.06\cdot 10^{12} \cot 30.8° = -1.78\cdot 10^{12} \text{ cm}^{-1} \tag{105}$$

or
$$1/\gamma = 5.6\cdot 10^{-13} \text{ cm}, \tag{105c}$$

$$T=\hbar^2\gamma^2/M = 1.30 \text{ MV}. \tag{105d}$$

γ turns out to be negative which shows that there is no stable energy level of a "di-proton." (From symmetry arguments, we can then conclude that there exists no stable "di-neutron" either.) In contrast to the scattering of *neutrons* by protons, the sign of γ can be determined from the proton-proton scattering. The reason is that there is interference between the scattering due to the Coulomb field and that due to the specifically nuclear forces. The resultant angular distribution depends therefore on the sign of δ_0 (cf. the curves A and R in Fig. 7).

The "virtual energy level" of the system of two protons lies, according to (105d), at 1.15 MV kinetic energy, i.e., appreciably higher than the singlet level of the deuteron (near zero, cf. 62a) but in the same general region. The force between two protons is therefore smaller, but not very much smaller than the force between neutron and proton (cf. §44).

From (105), (105b), we may deduce the value of δ_0 for other values of the energy. We find, neglecting kr_0:

$$\cot \delta_0 = \gamma/k. \tag{105e}$$

Inserting this value into (104), we find

$$R=\frac{\sigma(45°)-\sigma_0(45°)}{\sigma_0(45°)}=\frac{E_0}{C(T+\frac{1}{2}E_0)} \times (E_0-2(TC)^{\frac{1}{2}}) \tag{106}$$

with
$$C=Me^4/\hbar^2 = 49{,}800 \text{ volts}. \tag{106a}$$

It can be seen from (106) that the ratio R of "excess scattering" to Coulomb scattering at 45° increases rapidly with increasing energy E_0, being about proportional to E_0 for high energies ($E_0 \gg T$) and to E_0^2 for $E_0 \ll T$. This increase of R is due to the fact that the Coulomb field gives, especially for high energies, a particularly low scattering at 45°, which will be increased by almost any perturbation of the Coulomb field. At very *low* energies, (106) will be negative, corresponding to a *smaller* scattering at 45° than the Coulomb scattering: The reversal of sign occurs for

$$E_0{}^r = 2(TC)^{\frac{1}{2}} = 510{,}000 \text{ volts} \qquad (106b)$$

with our values (105d), (106a) for T and C. The smallest value of R is obtained for 240,000 volts, in this case,

$$(\sigma(45°)/\sigma_0(45°))_{E_0=0.23 \text{ MV}} = 0.09, \qquad (106c)$$

i.e., the scattering at 45° should be only one-eleventh of the Coulomb scattering. For $E_0 = 0.74$ MV, we find

$$R = 2.0, \qquad \sigma(45°)/\sigma_0(45°) = 3.0, \qquad (106d)$$

which is considerably larger than the results of Tuve, Heydenburg and Hafstad ($R = 0.65$).

The calculations above should be corrected for two reasons: Firstly, the wave function in the Coulomb field is not a plane wave. Therefore (105b), (105e) hold only approximately. This is particularly true for small energy of the proton. The effect of this correction is to make the ratio R decrease even faster with decreasing energy E_0.

Secondly, the range of the nuclear forces is not zero. This fact has the opposite effect on the change of R with decreasing energy. If we take for the range of the forces the value derived in (128), *viz.*,

$$a = 2.3 \cdot 10^{-13} \text{ cm} \qquad (107a)$$

we have $kr_0 = 0.244 = 14.0°$, and therefore instead of (105):

$$\gamma = -1.06 \cdot 10^{12} \cot 44.8° = -1.09 \cdot 10^{12} \text{ cm}^{-1} \qquad (107b)$$

$$T = 0.49 \text{ MV}, \qquad (107c)$$

i.e., the virtual level lies much nearer to zero. If we now assume a simple potential hole of width a and depth

$$V_0 = \hbar^2\kappa^2/M \qquad (107d)$$

we may determine κ from the condition (cf. 39)

$$(\kappa^2+k^2)^{\frac{1}{2}} \cot ([\kappa^2+k^2]^{\frac{1}{2}} r_0) = -\gamma \qquad (107e)$$

which gives

$$\kappa = 5.94 \cdot 10^{12} \text{ cm}^{-1}, \qquad (107f)$$

$$V_0 = 14.5 \text{ MV}. \qquad (107)$$

This depth may be compared to the depth of the hole for a proton and a neutron with opposite spin, *viz.*, $V_0 = 18.7$ MV. (This depth is necessary to give a quantum state at zero energy.) The potential energy between two protons is therefore about 80 percent of that between neutron and proton of opposite spin, or about 55 percent of the interaction between neutron and proton of parallel spin. This agrees satisfactorily with the result of the theory of H^3 and He^4 (Eq. 128).

IV. Theory of H^3, He^3 and He^4

§19. Thomas' Proof of the Finite Range of Nuclear Forces (T2)

We have shown in our discussion of the deuteron that the binding energy and the physical properties of that nucleus depend essentially only on a certain combination of the depth V_0 and the width a of the potential hole representing the interaction between neutron and proton. The combination involved is approximately V_0a^2, for details cf. §12. The same properties of the deuteron can be obtained with a deep and narrow hole, or with a shallow and wide hole. To fix depth and width separately, we have therefore to use the properties of other nuclei. The most suited for the purpose appear to be the nuclei immediately following upon the deuteron in complication, i.e., the nuclei of mass three H^3 and He^3, and the α-particle He^4.

We have already pointed out in §9 that the mass defects of these nuclei are very much larger than that of the deuteron, and that this fact may be explained by assuming the potential hole to be deep and narrow (Wigner, W12). However, no indication could be given at that

time as to *how* deep and narrow the hole must be chosen in order to explain the observed mass defects. In fact, we cannot even say whether *any* finite depth and width of the hole will suffice to explain the very large mass defect of the α-particle which is as much as 14 times the mass defect of the deuteron. It might be necessary to assume an infinitely deep and narrow hole for that purpose. This point has been cleared up by Thomas (T1) in favor of the *finite* depth and width of the hole. Thomas has shown that an infinite binding energy would be obtained for H^3 if the hole were assumed to be infinitely deep and narrow, and if at the same time the product V_0a^2 were kept constant so that the binding energy of the deuteron would retain its observed value. From his calculations, Thomas estimates that the range a of the forces cannot be less than $1\cdot10^{-13}$ cm. Thomas' proof is the more gratifying since we have up to the present no method for the explicit calculation of the binding energies of H^3 and He^4 which is at the same time rigorous and practical for obtaining numerical results.

The assumptions of Thomas are very general. No forces are assumed to act between two neutrons. This makes the conclusion hold *a fortiori* if there are attractive forces between like particles, as suggested by the odd-even-rule of isotopes (§10) and the scattering of protons by protons (§18) because such forces will lower the energy of H^3 even further. Repulsive forces between like particles can be considered as ruled out with certainty by the considerations of §10; but even small forces of this kind would not alter Thomas' conclusions. Finally, the force between protons and neutrons may be either an ordinary (Wigner) or an exchange (Majorana) force.

The method adopted by Thomas is based on the Schrödinger variation principle. The eigenfunctions inserted into the variational integral in order to calculate the energy are chosen symmetrical in proton and neutron whenever these particles are close enough to interact. Under these circumstances it does not make any difference whether the Wigner or Majorana type of forces is chosen. We shall therefore speak in the following in terms of ordinary forces, for simplicity.

The interaction between proton and neutron is supposed to vanish if the distance between them is larger than a certain value a. For $r<a$, we assume a potential energy

$$V(r)=-a^{-2}f(r/a)+O(a^{-1}) \qquad r<a. \quad (108)$$

$f(r/a)$ is an arbitrary function of its argument except that it shall give the correct binding energy of the deuteron. (108) has been chosen so that Va^2 stays approximately constant when a changes, which is necessary to make the binding energy of the deuteron independent of a (cf. theory of the deuteron, §12).

The solution of the wave equation for the deuteron in the potential (108) will be (cf. §12)

$$\psi=\varphi(r), \quad (108a)$$

where $$\varphi(r)=Ae^{-\alpha r}/r \quad \text{for} \quad r>a \quad (108b)$$

with $$\alpha^2=M\epsilon\hbar^{-2}, \quad (108c)$$

ϵ being the binding energy of the deuteron. The eigenfunction φ is of course supposed to be normalized.

We shall later on need the contribution of the inner region ($r<a$) to the "kinetic energy" of the deuteron, *viz.*,

$$T=4\pi\int_0^a(d\varphi/dr)^2r^2dr. \quad (108d)$$

When the range of the forces decreases, the normalizing factor A in (108b) stays almost constant because the normalization is determined chiefly by the "outside" part of the wave function (cf. §12). Therefore $\varphi(a)$ increases about as $1/a$, and so will the wave function for $r<a$. From our assumption that V retains its shape (cf. 108) becoming only proportionately larger with decreasing range a, the same follows for φ so that we may put

$$\varphi(r)=a^{-1}\chi(r/a) \qquad (r<a), \quad (108e)$$

which makes $$T=K/a, \quad (108f)$$

where $$K=4\pi\int_0^1(d\chi/dx)x^2dx \quad (108g)$$

is a function depending on the shape of the potential hole. For the rectangular hole,

$$K=\alpha\pi^2/4. \quad (108h)$$

The H^3 nucleus consists of one proton (coordinate $\mathbf{r}_3$) and two neutrons (coordinates $\mathbf{r}_1\mathbf{r}_2$). The wave equation of H^3 is

$$(\hbar^2/2M)(\Delta_1+\Delta_2+\Delta_3)\psi + (E-V(r_{13})-V(r_{23}))\psi=0. \quad (109)$$

We introduce the relative coordinates

$$\mathbf{s}_1=\mathbf{r}_{13}=\mathbf{r}_1-\mathbf{r}_3; \qquad \mathbf{s}_2=\mathbf{r}_2-\mathbf{r}_3 \quad (109a)$$

and suppose that ψ depends only on these relative coordinates; i.e., we leave out the motion of the center of gravity. Then (109) transforms into

$$(\hbar^2/M)(\Delta_1\psi+(\mathrm{div}_1\cdot\mathrm{grad}_2)\psi+\Delta_2\psi) + (E-V(s_1)-V(s_2))\psi=0, \quad (109b)$$

where Δ_1 and Δ_2 now refer to differentiations with respect to $\mathbf{s}_1$ and $\mathbf{s}_2$ rather than $\mathbf{r}_1$ and $\mathbf{r}_2$. The Schrödinger variation principle equivalent to (109b) is

$$E<[\int d\tau_1 d\tau_2\psi^2]^{-1}\int d\tau_1 d\tau_2\{(\hbar^2/M)[(\mathrm{grad}_1\psi)^2+(\mathrm{grad}_1\psi\cdot\mathrm{grad}_2\psi)+(\mathrm{grad}_2\psi)^2] +[V(s_1)+V(s_2)]\psi^2\}.$$

This unequality is true if ψ is *any* continuous function of $\mathbf{s}_1$, $\mathbf{s}_2$. If ψ happens to be the correct eigenfunction, the "equal" sign stands instead of the "$<$" sign.

The success of the variational method depends on a good choice of the approximate eigenfunction ψ in (110.) First of all, we shall find the *exact* solution of the wave equation (109b) for the regions in which the potential energy is zero, i.e., for large distances between the particles ("outer wave function," region I). Then we shall set up a wave function for *small* distances s_1 between first neutron and proton and large distances s_2 of the second neutron (region II), which resembles closely the wave function of the deuteron, and join it on to the "outer" wave function. A corresponding function will hold for small s_2 and large s_1 (region III). In region IV both neutrons are near the proton.

Region I: We assume both s_1 and s_2 to be larger than the range a of the forces. Then the potential energies $V(s_1)$ and $V(s_2)$ vanish and (109b) reduces to

$$\Delta_1\psi+\mathrm{div}_1\,\mathrm{grad}_2\,\psi+\Delta_2\psi=\mu^2\psi \quad (111)$$

with

$$\mu^2=-ME/\hbar^2. \quad (111a)$$

The solution of (111) is

$$\psi=3^{\frac{1}{2}}K_0(\mu s)\left[\frac{\arccos(s_1/s)}{s_1(s^2-s_1^2)^{\frac{1}{2}}}+\frac{\arccos(s_2/s)}{s_2(s^2-s_2^2)^{\frac{1}{2}}}\right], \quad (111b)$$

where

$$s^2=\tfrac{2}{3}(r_{12}^2+r_{23}^2+r_{31}^2)=\tfrac{4}{3}(s_1^2-(\mathbf{s}_1\cdot\mathbf{s}_2)+s_2^2) \quad (111c)$$

is, except for the factor 2/3, the sum of the squares of the three sides of the triangle formed by the three particles, and $K_0(x)$ is the Hankel function of imaginary argument and zero order which vanishes exponentially for large arguments, *viz.*[44]:

$$K_0(x)=\tfrac{1}{2}\pi i H_0^{(1)}(ix). \quad (111d)$$

The proof that (111b) is the solution of (111) is found in Thomas' paper. We might use (111b) everywhere provided only $s_1>a$ and $s_2>a$. The region thus defined would, however, not be convenient for the integration of the variational integral (110). We shall therefore use (111b) only if, besides the conditions $s_1>a$ and $s_2>a$, the further condition $s>l$ is fulfilled where l is an auxiliary length large compared to the range a of the forces but small compared to $1/\mu$, the "radius of H^3" (definition of region I).

Region II: We assume $s_1<a$, but $s>l$. In this region we put

$$\psi=K_0(\mu s)\left[c(s)\varphi(s_1)+3^{\frac{1}{2}}\frac{\arccos(s_2/s)}{s_2(s^2-s_2^2)^{\frac{1}{2}}}-\frac{2\pi}{3^{\frac{1}{2}}s^2}\right], \quad (112)$$

where $c(s)$ is defined by

$$c(s)\varphi(a)=3^{\frac{1}{2}}\frac{\arccos(a/s)}{a(s^2-a^2)^{\frac{1}{2}}}+\frac{2\pi}{3^{\frac{1}{2}}s^2} \quad (112a)$$

and φ is the solution of the wave equation of the deuteron (cf. 108a). This choice ensures that ψ is approximately a solution of the wave equation (109b) in region II. Moreover, (112) is chosen so that it joins smoothly to solution (111b) at $s_1=a$.

[44] Cf. Jahnke-Emde, Tables of Functions, p. 199. The function K_0 is also known as Macdonald's Bessel function: Watson, *Theory of Bessel Functions*, p. 78.

Region III: $s_2<a$, $s>l$: Analogous to region II.
Region IV: $s<l$. We put

$$\psi(\mathbf{s}_1\mathbf{s}_2)=(s/l)\psi[(l/s)\mathbf{s}_1,\ (l/s)\mathbf{s}_2]. \quad (112b)$$

This assumption serves merely to keep the wave function small in region IV.

The integral (110) can then be carried out. The rather involved calculation is found in Thomas' paper. The contribution of region I (outside) may be reduced to an integral over the *surfaces* separating region I from II, III and IV. The contribution of region IV involves the kinetic energy of the particles when they are close together, and is therefore proportional to K, (108f). Regions II and III each contribute, of course, the same amount. These contributions partly cancel the surface integrals over the surface between II or III and I, which arise from the outside region I. The net result for the ratio of the binding energies of H^3 and H^2 is

$$\frac{|E|}{\epsilon}>\frac{24(2\pi\cdot 3^{-\frac{1}{2}}-1)}{\pi}\frac{\mu^2}{\alpha^2}|\log \mu l|^3 \times\left[1+0\left(\frac{K}{(a/l)|\log \mu l|}\right)+0\left(\frac{a}{l}\right)\right]. \quad (113)$$

Sign and magnitude of the second and third term are unknown so that it is necessary to make them small. This is always possible. We have only to choose

$$a/l=\delta\ll 1; \qquad \mu l=e^{-1/K\delta^2}. \quad (113a)$$

Then

$$|E|/\epsilon>1.6(a\alpha)^{-2}e^{-2/K\delta^2}K^{-3}\delta^{-4}(1+0(\delta)). \quad (113b)$$

The factor $e^{-2/K\delta^2}K^{-3}\delta^{-4}$ may be rather small if the second and third term in (113) are large and necessitate the choice of a small δ. However, δ and K are both *independent* of a. Therefore it is always possible to make (113b) as large as one wishes by choosing a very small range a. This shows that an infinitely short range of the forces would lead to an infinite binding energy of the H^3 nucleus. Therefore the *range of the forces between neutron and proton must be finite* though possibly very small. Furthermore to any (experimentally given) finite ratio of the binding energies of H^3 and H^2, there must correspond a definite range a of the forces, when we assume the shape of the potential function $V(r)$.

Thomas has also shown that it is impossible to assume a "δ-function" for the potential energy. Such an assumption would mean that the wave function of a neutron and a proton would have a singularity when the two particles coincide, and would behave about in the following way:

$$\psi(\mathbf{r}_1\mathbf{r}_2)=A(\mathbf{r}_1+\mathbf{r}_2)(1/r_{12}-\lambda)+0(r_{12}). \quad (114)$$

The proof is to a large extent analogous to that for a potential energy with finite but short range.

The argument is not changed if part of the binding energy of the deuteron is attributed to a Heisenberg force (§14). As we shall show in the next section, only *half* of such a Heisenberg force would be active in the binding of the H^3 nucleus, while the full force is active in the deuteron. This will make the H^3 nucleus relatively less stable. However, the only assumption necessary for Thomas' proof is that a stable state of the deuteron exists if the force between the neutron and proton in the deuteron is the same as the average force between the proton and each neutron in H^3. This is actually the case: It follows from the scattering of slow neutrons by protons (§15) that the deuteron would have practically *no* binding energy if the force acting were the Majorana *minus* the Heisenberg force, while Majorana *plus* Heisenberg force lead to the experimental binding energy. The Majorana force plus half the Heisenberg force would therefore give some binding energy in between, *viz.*, about 50 to 65 percent of the observed binding energy of the deuteron. This is quite sufficient for the validity of Thomas' arguments given in this section.

§20. Calculation of the Energy of H^3, He^3 and He^4 from the Variation Principle (F2, F3, P8, M9, W1)

The binding energies of the light nuclei H^2, H^3, He^3, He^4 are the most suitable quantities to deduce the nuclear forces from. The procedure must of course be indirect. A law of force is assumed, in which one or more parameters are left arbitrary. The binding energies of the light nuclei are calculated as functions of the parameters in the interaction force, and these parameters

are then so determined that the calculated energies agree with the observed ones.

The binding energies of H^2, H^3, He^3, He^4 are sufficient to determine *three* parameters in the law of nuclear forces. It might seem four parameters could be determined from the four nuclear binding energies. However, the binding energies of the two nuclei H^3 and He^3 are closely related to each other. H^3 consists of two neutrons and one proton, He^3 of two protons and one neutron. Now we have assumed throughout this article that the nuclear forces are symmetrical in neutrons and protons, i.e., the force between two protons is exactly the same as between two neutrons[45] except for the Coulomb repulsion. This means that the energy of He^3 must differ from that of H^3 just by the Coulomb repulsion between the two protons. This prediction seems indeed to be true, although the experimental evidence is somewhat contradictory (see §22). Thus He^3 provides a check of our assumption of the symmetry of nuclear forces in protons and neutrons.

There remain then the three nuclei H^2, H^3 and He^4 to determine the nuclear forces. It would therefore be useless to assume a force containing more than three parameters. Accordingly, we assume that the range of the forces between like particles is the same as between neutrons and protons, but that the strengths of the forces are different. In formulae, we assume a neutron-proton interaction potential

$$J(r) = -Be^{-r^2/a^2} \tag{115}$$

and an interaction potential between like particles

$$K(r) = -Ce^{-r^2/a^2}. \tag{115a}$$

We have then to determine the common range a of the forces, and the strengths B and C, respectively, of the two kinds of interactions.

The neutron-proton interaction is assumed to be of the Majorana exchange type. The nature of the interaction between the like particles will be discussed in more detail in §24. For the present, it may be assumed to be an "ordinary" force; the force suggested in §24 will give the same result for all nuclei up to He^4.

With the potentials (115) (115a), the energy of any nucleus could in *principle* be calculated exactly from the Schrödinger equation. Actually the calculation can be carried out *explicitly* only for the deuteron. (§12.) Since the deuteron consists of one neutron and one proton, its binding energy gives us just one relation between the two constants B and a determining the neutron-proton force. This relation is given in Table III for two cases: (a) if the whole binding energy of the deuteron is due to the "Majorana force" (115), and (b) if part of the binding energy is due to a "Heisenberg force" depending on the relative spins of proton and neutron, as it seems to follow from the scattering of slow neutrons by protons (§14).

In the latter case, there is given an "effective" force $\bar{B}$, which is the strength B of the Majorana force plus one-half the strength of the Heisenberg force. It is this $\bar{B}$ which enters the energy of nuclei containing an even number of neutrons or an even number of protons or both, at least in first approximation, i.e., if the Heisenberg force is small compared to the Majorana force.

In that case, the complete eigenfunction of the H^3 nucleus may be written

$$\varphi(1,2,3) = \psi(x_1,x_2,x_3)\alpha(1)\cdot 2^{-\frac{1}{2}}[\alpha(2)\beta(3)-\beta(2)\alpha(3)]. \tag{116}$$

Here the first factor is the positional wave function. The second is the spin wave function of the proton (particle 1), whose spin is supposed to be parallel to the z axis. (The spin function α indicates a spin parallel to z, β a spin antiparallel to z, cf. 92a.) The last factor is the spin wave function of the two neutrons (particles 2, 3) whose spins are of course opposite to each other. The factor $2^{-\frac{1}{2}}$ stands for normalization.

Now consider a Majorana potential V_M and a Heisenberg potential V_H to act between the proton and the neutron 2. The potential energy operator operated on (116) gives then

$$\begin{aligned}(V_H+V_M)\varphi &= 2^{-\frac{1}{2}}\psi(x_2x_1x_3)\{J_M(r_{12})\alpha(1)[\alpha(2)\beta(3)-\beta(2)\alpha(3)]\\ &\quad +J_H(r_{12})\alpha(2)[\alpha(1)\beta(3)-\beta(1)\alpha(3)]\},\end{aligned} \tag{116a}$$

where J_H and J_M are the Heisenberg and Majorana potentials as functions of the distance r_{12}. The average value of the potential energy is then

$$\begin{aligned}\sum_\sigma \varphi(V_M+V_H)\varphi d\tau &= \int d\tau\psi(x_1x_2x_3)\psi(x_2x_1x_3)J_M(r_{12})\\ &\quad\times\tfrac{1}{2}\sum_\sigma\alpha^2(1)[\alpha(2)\beta(3)-\beta(2)\alpha(3)]^2\\ &\quad+\int d\tau\psi(x_1x_2x_3)\psi(x_2x_1x_3)J_H(r_{12})\\ &\quad\times\tfrac{1}{2}\sum_\sigma[\alpha^2(1)\alpha^2(2)\beta^2(3)-\alpha^2(1)\alpha\beta(2)\alpha\beta(3)\\ &\quad-\alpha\beta(1)\alpha^2(2)\alpha\beta(3)-\alpha\beta(1)\alpha\beta(2)\alpha^2(3)].\end{aligned} \tag{116b}$$

When we carry out the summation over the spins of all

[45] It would be in accord with this assumption if the forces between like particles vanish identically.

three particles, we find two terms contributing unity to the first sum but only one term in the second, all other terms vanishing (cf. 92e). Therefore (116b) reduces to

$$\sum_\sigma \int \varphi(V_M + V_H)\varphi d\tau = \int d\tau \psi(x_1x_2x_3)\psi(x_2x_1x_3)(J_M + \tfrac{1}{2}J_H), \quad (116c)$$

which shows that half the Heisenberg interaction is to be added to the full Majorana term.

It might have been expected that the Heisenberg force would not contribute at all to the energy of nuclei like H^3, containing only pairs of neutrons. For one might think that of the two neutrons *one* would have spin parallel to the given spin of the proton, the other antiparallel so that the contributions of the Heisenberg forces due to the two neutrons would cancel each other. The fallacy is due to the use of the word antiparallel: Two spins are antiparallel only if the wave function is antisymmetrical in the respective spin coordinates. In the wave function $\alpha(1)\beta(2)$, however, merely the z components of the spins 1 and 2 are antiparallel, the spins themselves might still be just as well parallel as antiparallel, because we can form either of the two wave functions

$$\begin{aligned} &2^{-\frac{1}{2}}(\alpha(1)\beta(2)+\alpha(2)\beta(1)), \\ &2^{-\frac{1}{2}}(\alpha(1)\beta(2)-\alpha(2)\beta(1)), \end{aligned} \quad (116d)$$

the first corresponding to parallel, the second to antiparallel spins. The correct reasoning is therefore as follows: If the components of two spins in a given direction are parallel, the spins are certainly parallel; if the components are opposite, it is just as likely that the spins are parallel as that they are antiparallel. Thus the neutron which has its spin *component* parallel to the proton will have just once the Heisenberg interaction with the latter, while the neutron whose spin is opposite the proton spin, has no Heisenberg interaction at all with the proton. Therefore we get two Majorana interactions and one Heisenberg interaction in the H^3 nucleus, or per neutron one Majorana and one-half Heisenberg term, as we found in (116c).

Accordingly, the value $\overline{B}$ of Table III has to be used for computations of the energies of H^3 and He^4.

We return to the problem of determining the binding energy from the given force. While the Schrödinger equation can be integrated explicitly for the deuteron, this is not the case for the nuclei H^3 and He^4. Approximate methods must be used. The method used most frequently for similar problems is the Ritz variational method.[46] It gives a definite upper bound for the energy of any quantum mechanical system, but this upper bound sometimes converges very slowly towards the correct energy. The energy of nuclei is, unfortunately, one of the cases in which this is true. The reason for this is the very short range of the nuclear forces. We know from the theory of the deuteron (§12) that the correct wave function (cf. 38a) changes very rapidly when the two particles have a distance smaller than the range a of their mutual forces, while for large distances $r > a$ the wave function varies very slowly. The same will be true for the correct wave function of more complicated nuclei: We shall find a comparatively *slow* variation of the wave function *in general*, and superposed on it a *rapid* change *whenever* two nuclear *particles* come close *together*. Such a combination of rapidly and slowly varying functions cannot be easily represented analytically. In fact, a rather complicated trial wave function had to be chosen by Thomas (§19) in order to prove that the binding energy of H^3 tends to infinity when the range of the forces goes to zero. The use of a wave function of similar complication for *quantitative* computations of the binding energy for a finite range of the forces would be prohibitively laborious, especially for more complicated nuclei than H^3. On the other hand, any reasonably *simple* analytical expression will be a poor approximation to the correct wave function, because the analytical expression will change too rapidly when the nuclear particles are far away from each other, and too slowly when they are close together. This will be particularly true if the range of the forces is very small. Therefore we cannot expect any too good agreement between the results of the Ritz variational method and the experimental binding energies, and we must expect the discrepancy to become worse with decreasing range of the forces. (See following, especially Table IV.)

The first thing to do in applying the variational method is to choose an approximate wave function for the system. A sufficiently simple function for the H^3 nucleus is

$$\psi = \exp\left[-\tfrac{1}{2}\nu(r_{12}^2 + r_{13}^2) - \tfrac{1}{2}\mu r_{23}^2\right], \quad (117)$$

where the subscript 1 denotes the proton, 2 and 3 the two neutrons so that r_{12} is the distance of the first neutron from the proton and r_{23} the distance between the two neutrons. The function $e^{-\frac{1}{2}\nu r^2}$ "ties together" two unlike particles, $e^{-\frac{1}{2}\mu r^2}$ two like particles. Since the forces between unlike particles are larger, we expect ν to be larger than μ. For the α-particle, we choose analogously:

[46] Cf., e.g., *Handbuch der Physik*, Vol. 24, 1, p. 353.

$$\psi=\exp\left[-\tfrac{1}{2}\nu(r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2)-\tfrac{1}{2}\mu(r_{12}^2+r_{34}^2)\right], \quad (117a)$$

1 and 2 referring to the protons, 3 and 4 to the neutrons.

With the wave functions (117), (117a) all integrals occurring in the variational method can be carried out elementarily; e.g., for the H^3 nucleus we obtain for the kinetic energy

$$E_{\text{kin}}=-\frac{\hbar^2}{2M}\frac{\int\psi(\Delta_1+\Delta_2+\Delta_3)\psi d\tau_2 d\tau_3}{\int\psi^2 d\tau_2 d\tau_3}=3\hbar^2/2M(\mu+2\nu), \quad (117b)$$

where Δ_1 is the Laplacian with respect to the coordinates of the first particle, and the integrals have to be extended over all positions of the second and third particle relative to the first. The potential energy becomes

$$E_{\text{pot}}=-2\bar{B}\frac{\int e^{-r_{12}^2/a^2-\nu r_{12}^2-\frac{1}{2}(\mu+\nu)(r_{13}^2+r_{23}^2)}d\tau_2 d\tau_3}{\int e^{-\nu(r_{12}^2+r_{13}^2)-\mu r_{23}^2}d\tau_2 d\tau_3}-C\frac{\int e^{-r_{23}^2/a^2-\mu r_{23}^2-\nu(r_{12}^2+r_{13}^2)}d\tau_2 d\tau_3}{\int e^{-\nu(r_{12}^2+r_{13}^2)-\mu r_{23}^2}d\tau_2 d\tau_3} \quad (117c)$$

$$=-16\bar{B}\left[\frac{\nu(\nu+2\mu)}{(\nu+\mu)(5\nu+\mu+4a^{-2})}\right]^{\frac{3}{2}}-C\left[\frac{\nu+2\mu}{\nu+2\mu+2a^{-2}}\right]^{\frac{3}{2}}. \quad (117d)$$

The form of the integral in the first numerator in (117c) is due to the Majorana type of the forces: The particles 1 and 2 are supposed to interact, we have therefore to multiply the wave function (117) with the function obtained by interchanging particles 1 and 2 in (117) instead of taking the square of (117). The factor 2 in the first term of (117c) arises from the fact that we have *two* neutrons interacting with the proton.

The expressions (117b) (117d) may be simplified by introducing the abbreviations

$$\sigma=\tfrac{1}{4}(5\nu+\mu)a^2 \qquad p=(2\nu+4\mu)/(5\nu+\mu) \qquad T=\hbar^2/Ma^2. \quad (118a)$$

Then the upper bound for the energy of H^3 becomes

$$E^0(H^3)=(2+p)\sigma T-2\bar{B}[p(4-p)/(1+2p)]^{3/2}(\sigma/\sigma+1)^{3/2}-C(p\sigma/p\sigma+1)^{3/2}. \quad (118)$$

Similarly, we find for the α-particle

$$E^0(He^4)=\tfrac{3}{2}(2+p)\sigma T-4\bar{B}[p(2-p)]^{3/2}/(\sigma/\sigma+1)^{3/2}-2C(p\sigma/p\sigma+1)^{3/2} \quad (119)$$

with the abbreviations

$$\sigma=\tfrac{1}{2}(3\nu+\mu)a^2, \qquad p=(2\nu+2\mu)/(3\nu+\mu) \quad (119a)$$

(118) and (119) have to be minimized by varying the parameters p and σ.

We want to carry out the calculations first without assuming a force between like particles, in order to obtain an idea about the degree of approximation afforded by (118), (119). We know from the preceding section that the eigenvalue of H^3 must tend to minus infinity if we let the range a of the forces go to zero, i.e., if B and T in (118) go to infinity, and if at the same time the relation between B and T is kept such that the binding energy of the deuteron remains correct. This must hold *a fortiori* for He^4. We may reasonably expect the binding energy of both nuclei to decrease monotonously with decreasing a.—For very *long* range of the forces, the binding energy of H^3 will be just twice, that of He^4 just four times the binding energy of the deuteron.

Actually, the minimum of (118) and, to a lesser extent, that of (119), behave very differently from these expectations. While for long range of the forces the result is rather satisfactory, the minimum of $E^0(He^4)$ decreases only very slowly with decreasing range of the forces, and that of $E^0(H^3)$ even *increases* with decreasing a. This shows that the variational method becomes increasingly worse with decreasing range of the forces, which is to be expected (see above).

The procedure to determine the minimum of (118), (119) is the following. First of all, the minimum with respect to p is determined. The position of that minimum depends only slightly on the value of σ, and the dependence of the minimum energy on p is quite negligible.* The minimum lies at

* When the ratio of the potential energy of the H^3 nucleus to its kinetic changes from 1.0 to 1.2, p_0 changes from 0.735 to 0.765; the coefficient of T in (117c) from 2.735 to 2.765, i.e., about 1.1 percent; the coefficient of B increases by 1.0 percent; the ratio of the coefficients, which is the most important quantity determining the binding energy, rises by 0.12 percent which is quite negligible.

TABLE IV. *Energies of H^3 and He^4 as functions of the range of the neutron-proton forces.* (All energies in MV, a in 10^{-13} cm.)

			H^3 ($E_{obs}=-8.3$)			He^4($E_{obs}=-27.6$)				
T	a	$\bar{B}$	σ_0	E^0	E_{Feen}	σ_0	E_{kin}	E_{pot}	E^0	E_{Feen}
5	2.86	22.05	0.740	− 1.60	——	1.080	23.65	32.7	− 9.05	——
10	2.02	38.4	0.551	− 0.44	−2.8	0.891	38.95	49.15	−10.2	−15.6
20	1.43	69.4	0.389	+ 1.67	+1.0	0.760	66.5	77.9	−11.4	−17.4
40	1.01	129.2	(0.25)*	+ 5.30	+4	0.666	116.5	129.5	−13.0	−19.3
80	0.715	245	(0.25)*	+12.9	——	0.593	207.5	220.0	−12.5	——

* The expression (118c) has, for these values of T, no minimum. 0.25 is the position of the minimum when it just disappears.

$p_0=0.75$, corresponding to $\mu=0.54\nu$ for H^3, (118b)

$p_0=0.92$, corresponding to $\mu=0.7\nu$ for He^4. (119b)

Inserting these values, (118), (119) reduce to

$$E^0(H^3) = 2.750T\sigma - 1.925_5\bar{B}(\sigma/\sigma+1)^{3/2}, \quad (118c)$$

$$E^0(He^4) = 4.378T\sigma - 3.961\,\bar{B}(\sigma/\sigma+1)^{3/2}. \quad (119c)$$

The minimum of these expressions as functions of σ is found for σ_0, which is given by

$$(\sigma_0+1)^{5/2}\sigma_0^{-\frac{1}{2}} = \begin{cases} (3/2)(1.925_5/2.750)\bar{B}/T & \text{for } H^3, \quad (118d) \\ (3/2)(3.961/4.378)\bar{B}/T & \text{for } He^4. \quad (119d) \end{cases}$$

From these equations, σ_0 can easily be determined for any given ratio

$$\lambda=\bar{B}/T=M\bar{B}a^2/\hbar^2. \quad (118e)$$

σ_0 is then inserted in (118c, 119c) and E^0 calculated.

The result is shown in Table IV. For 5 different ranges of the forces (column 2), the strength of the force, i.e., the constant $\bar{B}$ (column 3), is computed from the theory of the deuteron (Table III, §12), due account being taken that part of the binding of the deuteron arises from Heisenberg forces (cf. above, and §14). For each pair a, $\bar{B}$ (or T, $\bar{B}$), σ_0 and the total energy is calculated for H^3 and He^4. In addition, the values of the energy derived by Feenberg (F2) are given; they differ from ours because he assumed the deuteron binding energy to be entirely due to Majorana forces.[47] Moreover, for the α-particle the kinetic and potential energy are given separately, in order to show that both these quantities increase rapidly when the forces become stronger and of shorter range.

[47] Feenberg corrected for the Heisenberg forces at a later stage of his calculations.

The result is by no means satisfactory. Even for the α-particle, the lowest value of the energy obtainable is −13.0 MV, i.e., only one-half of the observed binding energy of 28 MV. (Feenberg's values are somewhat better, because he assumed a stronger force for any given value of the range, for the reason mentioned.) For H^3, the binding energy disappears completely for ranges below $2.6\cdot10^{-13}$ cm. The reason for the unsatisfactory result is, of course, that our wave functions (117), (117a) represent a very poor approximation to the exact eigenfunction if the range of the force is short.

Feenberg has tried to get a better approximation by choosing better wave functions: The best approximation was obtained with the wave function

$$\psi=\psi_0+\lambda H\psi_0, \quad (120)$$

where λ is a parameter to be varied, ψ_0 the wave function (117) and H the Hamiltonian operator, i.e., the sum of the operators of kinetic and potential energy. Wave functions of the type (120) were first introduced for use with the variational method by Hassé (H6). They often correct the original wave function ψ_0 to something very well approximating the correct one. In our case, (120) gives an energy of about $E^0+0.6(E'-E^0)$ where E^0 is the energy calculated from (117c) and E' that following from Feenberg's "method of the equivalent two-body-problem" (cf. §21).

A still better result for H^3 with a variational method was obtained by Present (P8). He chose a simple exponential potential $V_0e^{-r/a}$ between neutrons and protons, with $a=10^{-13}$ cm and $V_0=97$ MV. This value of V_0 follows from the theory of the deuteron (Table III), it is not corrected for Heisenberg forces. No forces are assumed to act between the two neutrons. Present assumed furthermore that the inter-

action was of the "ordinary" (Wigner) rather than the exchange (Majorana) type (this simplifies calculations and should give a somewhat too high result for the binding energy). For the wave function, Present chose a simple exponential times a power series in the distances between the particles, varying the coefficients of the terms in the series. With eight terms in the power series, he obtained a binding energy of 4.85 MV for the H^3 nucleus. The results for the binding energy obtained with increasing number of terms in the eigenfunction showed rapid convergence. He therefore concludes that the correct binding energy is 4.9 ± 0.05 MV. A similar value (about 4.7 MV for $a=10^{-13}$ cm, or, in their notation, $\mu=0.5\cdot10^{13}$ cm^{-1}) was obtained by Massey and Mohr (M9) by a variation calculation using a different expression for the approximate wave function. Feenberg's method of the equivalent two-body problem which will be described in the following section, gives, according to Present, 4.5 MV, i.e., again almost the same binding energy.

In judging these figures, it should be borne in mind that a binding energy of twice the binding energy of the deuteron, i.e., 4.3 MV, would be expected for H^3 even in the most unfavorable case, *viz.*, for infinitely long range of the forces. The actual results found from the calculations are only very slightly larger than this value, at the best, by about 15 percent. On the other hand, the *observed* binding energy of 8.3 MV is nearly twice as large as our "elementary" figure 4.3 MV. This seems to show that a *very much shorter range of the forces would be required* in order to explain the observed mass defect of H^3, if we assume only forces between neutrons and protons and none between two neutrons. On the other hand, no range of the forces shorter than $1\cdot10^{-13}$ cm seems to be reconcilable with the binding energies of α-particle and H^2 (Table V, §21). We must therefore *conclude that attractive forces between two neutrons must exist in order to explain the observed mass defects of H^2, H^3, He^4.*

This conclusion is strengthened by the fact that Present has purposely made some neglections in his calculations which will tend to make his result for the binding energy too big: Firstly, he assumed Wigner instead of Majorana forces. Secondly, he did not take into account the fact that part of the binding energy of H^2 is due to Heisenberg forces which will contribute relatively less to the binding energy of H^3 (see above). Thirdly, a range of the forces of $1\cdot10^{-13}$ cm seems already somewhat shorter than can be reconciled with the theory of the α-particle (Table V). Present's conclusion that forces between neutrons must exist, is also strongly confirmed by the more extensive calculations presented in the next section.

The opposite conclusion has been reached by Dolch (quoted by Weizsäcker, W1) also on the grounds of a variational calculation. The details of Dolch's calculation are not yet available, but only some curves representing the values of B and a in the expression (115) which are necessary to obtain the correct binding energies of H^2, H^3 and He^4, respectively (Fig. 1 in Weizsäcker's paper). Apparently, the data for H^2 have not been obtained by an *exact* solution of the Schrödinger equation for that nucleus but also by a variational method. What the particular method was, seems rather doubtful. It seems from the data published that the wave function used was less good than the simple wave function (121); e.g., for $a=1.4\cdot10^{-13}$ cm, the wave function (121) gives, according to (121a), the correct binding energy for the deuteron if B is chosen to be 87 MV. The value obtained from Dolch's calculation for $a=1.4\cdot10^{-13}$ cm ($b=2.0$ in Weizsäcker's notation) is 0.100 mass unit = 93 MV, i.e., *more* than from (121).[48] Since the B required to give the observed value for the binding energy of the deuteron is larger in Dolch's calculations, his wave function represents a poorer approximation to the wave function of the deuteron than even the simple function (121). On the other hand, his results for H^3 and He^4 seem to be better approximations than those obtained from (117), (117a) and listed in Table IV. It does not seem consistent to us to compare a fairly good approximation for H^3 and He^4 with a very poor approximation for H^2. For these reasons, we cannot follow the conclusions drawn by Weizsäcker and Dolch from Dolch's calculations, *viz.*, that no forces between like particles need to be assumed to explain the mass defects of H^2, H^3, He^4. (Moreover, if Dolch's calculations are corrected for Heisenberg forces it becomes even more necessary to assume forces between like particles.)

From the variational calculations of Present and of Massey and Mohr we can also conclude that Feenberg's method of the equivalent two-body problem (§21) gives an almost correct result for the binding energy of H^3 (4.5 MV, compared to 4.9 MV according to the best variational calculation of Present). We shall therefore apply this method with some confidence to H^3 and He^4 in the next section.

[48] We are indebted to Dr. Feenberg for drawing our attention to this point.

§21. Feenberg's "Equivalent Two-Body Problem" (F2, F3)

Since the variational method gives, at least with simple choices for the approximate wave function, very poor results for the binding energy of H^3 and He^4, that method cannot be used for a determination of the nuclear force constants from the observed binding energies. A better method has to be devised. The method used thus far most extensively is Feenberg's method of the equivalent two-body problem. A similar method has been used by Wigner in his first calculation of the binding energies of He^4 (W12).

The method of the equivalent two-body problem cannot be founded rigorously. But it is about as likely that it gives too low a result for the binding energy for a given force, as too high one. Moreover, Present has shown that for H^3 the use of a better approximate wave function with the variation method, gives a binding energy very nearly equal to that obtained by the "equivalent two-body problem."[49]

The method proceeds as follows: The variation method is applied to the *deuteron*, with the same type of wave function which was used for H^3 and He^4 (cf. 121). The energy of the deuteron (cf. 121a) resulting from the variation method has a form exactly analogous to that found for energies of the nuclei H^3 and He^4 (cf. 118c, 119c), only the coefficients being different. The "variational energy" of H^2 would therefore have exactly the same value as that for H^3, if the constants T and B determining the force were replaced by some other constants T' and B' which are multiples of T and B chosen in such a way as to make up for the difference in the coefficients in formulae (118c) and (121a). Next the exact wave equation of the deuteron is solved with the new constants T' and B' which may be called the "equivalent force constants" for the H^3 nucleus. The result of the solution of the deuteron equation with the force constants $T'B'$ ("equivalent two-body problem") is assumed as the true energy of the H^3 nucleus.

The wave function chosen for the deuteron is

$$\psi = e^{-\frac{1}{2}\nu r^2} \tag{121}$$

with r the distance of proton and neutron. With this wave function, the energy of the deuteron resulting from the variation principle is

$$E^0(H^2) = \tfrac{3}{2}T\sigma - B(\sigma/\sigma+1)^{\frac{1}{2}}, \tag{121a}$$

where

$$\sigma = \nu a^2. \tag{121b}$$

We may write the energy (117c) of H^3 in a form analogous to (121a), *viz.*,

$$E^0(H^3) = \tfrac{3}{2}T'\sigma - B'(\sigma/\sigma+1)^{\frac{1}{2}} \tag{122}$$

if we put

$$T' = 1.833T, \qquad B' = 1.925\bar{B}. \tag{122a}$$

The method of the equivalent two-body problem assumes that the energy of the H^3 nucleus is equal to the eigenvalue E' in the wave equation (equivalent two-body equation)

$$(\hbar^2/M)\Delta\psi + (E' + B'e^{-r^2/a'^2})\psi = 0 \tag{122b}$$

where (cf. 118a)

$$a'^2 = \hbar^2/MT'. \tag{122c}$$

The eigenvalue E' in (122b) may be obtained, without any further calculation, from Table III in §12. Suppose we have two equations of the type (122b), one with the constants E', B', a', T' and another with the constants E'', B'', a'', T''. Suppose furthermore that

$$B'/T' = B''/T'' \tag{123}$$

or in other words

$$B'a'^2 = B''a''^2. \tag{123a}$$

Then the ratio of the energies is the same as the ratio of the potentials, *viz.*:

$$E'/B' = E''/B''. \tag{123b}$$

To see this, introduce into (122b) new coordinates which are a'/a'' times the old ones, thus:

$$r'' = ra''/a', \quad \Delta'' = (a'/a'')^2\Delta, \tag{123c}$$

where Δ'' denotes the Laplacian with respect to the new coordinates. Multiplying the resulting equation by $(a'/a'')^2$, we find

$$(\hbar^2/M)\Delta''\psi + (E'(a'/a'')^2 + B'(a'/a'')^2 e^{-r''^2/a''^2})\psi = 0 \tag{123d}$$

which, with (123a) and (123b), is equivalent to

$$(\hbar^2/M)\Delta''\psi + (E'' + B''e^{-r''^2/a''^2})\psi = 0, \tag{123e}$$

showing that $E'' = E'B''/B'$ is the eigenvalue corresponding to the force field $B''T''$ if E' is the eigenvalue corresponding to the force $B'T'$. Now Table III gives the pairs of values $B''T''$ which lead to the binding energy $E'' = -2.14$ MV. From these, we can easily deduce the eigenvalue corresponding to a given pair of force constants $B'T'$ by use of (123b).

As an illustration, we want to calculate the energy of H^3 for the case $T = 10$ MV. Table III gives us as the value $\bar{B}$

[49] The deviation of the result of the equivalent two-body problem from that of Present's refined variational calculation is only about 6 percent of the difference between the simple variation method and Present's calculation.

corresponding to this T according to the theory of the deuteron, $\bar{B}=38.4$ MV. From (121a), we find for the force constants of the two-body problem equivalent to the H^3 problem, $T'=18.33$ MV, $B'=73.9$ MV. The ratio of the two is $B'/T'=4.03$. We seek in Table III this particular ratio B''/T'' and find listed above it the value $T''=12.5$ MV. This value of T'' would lead to a binding energy $-E''=2.14$ MV. The binding energy of H^3 therefore becomes $-E'=-E''T'/T''=2.14\cdot 18.3/12.5=3.13$ MV.

The analogous procedure can be carried out for He^4, only the relation between the equivalent force constants $B'T'$ and the given constants of the actual force, $\bar{B}\,T$, being changed to (cf. 118c, 121a)

$$T'(\mathrm{He}^4)=2.919T,\quad B'(\mathrm{He}^4)=3.961\bar{B}. \tag{124}$$

TABLE V. *Energies of H^3 and He^4 from the method of the equivalent two-body problem, as functions of the range of the forces.* (No forces between like particles.) (All energies in MV, range a in 10^{-13} cm. Values in parentheses estimated.)

			H^3 obs. $E=-8.3$		He^4 obs. $E=-27.6$			
T	a	$\bar{B}$	$-E'$	$-E_{\mathrm{Feen}}$	$-E'$	C	$-E$	$-E_{\mathrm{Feen}}$
5	2.84	22.05	2.82	(4.2)	11.05	0.56	10.5	(18.2)
10	2.02	38.4	3.00	4.5	14.05	0.72	13.3	24.3
15	1.65	54.0	3.12	4.7	16.7	0.83	15.9	26.8
20	1.43	69.4	3.29	4.9	19.15	0.93	18.2	29.0
30	1.16_5	99.3	3.46	5.2	23.65	1.10	22.55	(33.2)
40	1.01	129.2	3.70	5.4	28.25	1.24	27.0	——
50	0.905	158.3	3.86	—	32.55	1.35	31.2	——
60	0.825	187.	4.02	—	36.5	1.45	35.05	——
80	0.715	245.	4.30	—	44.3	1.63	42.7	——

Table V gives the result of the calculations for H^3 and He^4 for various ranges a of the force. The forces between like particles are still assumed to be zero. E' is the energy derived from the "equivalent two-body problem." C is the correction for the Coulomb repulsion of the two protons in the α-particle (cf. §22), $E=E'+C$ is the total calculated energy. In the table the energies calculated by Feenberg are included, the difference being again that Feenberg determined the force constant B for every range a so that the total binding energy of the deuteron is accounted for by the Majorana force while we considered part of it as due to a Heisenberg force. The differences are seen to be rather large, Feenberg's binding energies being, or course, much larger for any given range a of the forces.

We observe that the binding energy of the α-particle increases rapidly with decreasing range of the forces, the experimental value of 27.6 MV being obtained for

$$T=41.5\text{ MV},\quad a=0.99\cdot 10^{-13}\text{ cm},\quad \bar{B}=133\text{ MV}. \tag{125}$$

With Feenberg's assumptions about the origin of the binding energy of the deuteron, the potential hole would be much less deep and narrow, *viz.*,

$$T=16.8\text{ MV},\quad a=1.56\cdot 10^{-13}\text{ cm},\quad \bar{B}=60\text{ MV}. \tag{125F}$$

The binding energy of the H^3 nucleus turns out to be much smaller than the observed one (8.3 MV) and to increase only very slowly with decreasing range. For the range which yields the experimental value for the binding energy of the α-particle ($T=41.5$ MV) the binding energy of H^3 is only 3.7 MV, i.e., 45 percent of the observed value. (With Feenberg's assumptions, it would be markedly better, *viz.*, 4.8 MV at $T=17$ MV, which, however, is still unsatisfactory.) We must therefore conclude that either (a) the method of the equivalent two-body problem, while satisfactory for the α-particle, gives much too small binding energies for the H^3 nucleus, or (b) there are additional forces which depress the energy of H^3 relatively much more than that of the α-particle.

The first possibility can be excluded almost with certainty, in view of the variational calculations of Present (cf. end of §20). Therefore we adopt alternative (b). The additional forces which we assume are attractive forces between like particles. This assumption is the more preferable over assumption (a) since we have already found other evidence for the forces between like particles (§10, §18).

The energy of H^3 and He^4 resulting from the variation method and including forces between like particles, is given in (118), (119). These two expressions do not have exactly the same form as the "variational energy" of H^2 given in (121a). For the last term in (118) as well as (119) contains $(p\sigma/p\sigma+1)^{\frac{1}{2}}$ instead of $(\sigma/\sigma+1)^{\frac{1}{2}}$. Therefore the method of the equivalent two-body problem cannot be applied immediately.

Two procedures suggest themselves: (a) We put simply $p=1$. In this way, we lose one parameter in the variation principle and therefore impair the result somewhat. The error will, however, not be serious because the introduction of forces between like particles tends to equalize the constants μ and ν in the wave function and therefore to bring p closer to one. If the forces

between like particles were equal to those between unlike ones, we would have exactly $\mu=\nu$ and $p=1$. For the ratio of the forces derived below (Eq. (128)), one finds $p=0.94$ for H^3 and $p=0.985$ for He^4 as the values giving the minimum of the energy.

(b) We introduce a new parameter σ' by putting

$$\sigma=\sigma'(1+c(1-p)). \tag{126}$$

c can then be fixed so that the *sum* of the second and third term in (118) has the form $(2Bf(p)+C)(\sigma'/\sigma'+1)^{3/2}$. This is achieved by putting approximately

$$c=C/(2B+C). \tag{126a}$$

The minimum of (118) with respect to p occurs then approximately for

$$p=1-(B-C)/2(2B+C) \tag{126b}$$

and (118) reduces to

$$E(H^3)=\left[3-\left(\frac{B-C}{2B+C}\right)^2\right]T\sigma' + \left[2B+C-\tfrac{1}{4}B\left(\frac{B-C}{2B+C}\right)^2\right](\sigma'/\sigma'+1)^{3/2}. \tag{126c}$$

The results obtained from this equation are almost the same as from method (a). The constant C, determined in the way described below from the binding energy of H^3 turns out 0.15 MV smaller by method (b) than by method (a) if we take the range of the forces finally chosen (cf. $T=8$ MV). For shorter range, the correction is larger, being 0.9 MV for $T=20$ and 1.7 MV for $T=30$. But these differences are, of course, very small compared to the accuracy of the method. For the α-particle the effect of applying method (b) would be still smaller. We have applied method (b) in the final calculations, but we shall now describe the procedure according to method (a).

Putting $p=1$ in (118), (119) and comparing the result to (121a), we obtain for the parameters in the equivalent two-body problems:

$$\text{for } H^3: \quad T'=2T, \qquad B'=2B+C, \tag{127}$$

$$\text{for } He^4: \quad T'=3T, \qquad B'=2(2B+C). \tag{127a}$$

Since we want now to fix *three* constants B, C and T (T is equivalent to the range of the forces a), we must use the exact binding energies of all three nuclei H^2, H^3 and He^4. Therefore we choose the following procedure: We take a given value of T. The value of $\bar{B}$ corresponding to it can be read directly from Table III, which is based on the theory of the deuteron. C can then be determined from the observed binding energy of the H^3 nucleus, *viz.*, $E'=8.28$ MV $=3.87$ times the binding energy of the deuteron. The method of determination is that described in Eqs. (123) to (123e): We know that the value $T'=2T$, together with the unknown value of B', must give the observed binding energy $E'=3.87E_0$. Therefore the force constants $T''=T'/3.87$, and $B''=B'/3.87$, would give exactly the binding energy of the deuteron. Therefore we have only to look in Table III for the value $T''=T'/3.87 =2T/3.87$, and to find the corresponding value of B''/T'' from the table: This value of B''/T'', multiplied by the given $T'=2T$, gives the required value B' for the H^3 nucleus. Subtracting $2\bar{B}$ from B', we find *that* value of C which gives the correct binding energy of H^3 (and H^2) for the given value of T; we may call it $C(H^3, T)$.

The same procedure is then carried out for the α-particle. In this case, a correction has first to be applied to the observed binding energy to allow for the Coulomb repulsion of the two protons (for its determination, see (129b)). The result of the calculation is another value for C, *viz.*, $C(He^4, T)$ which is the value necessary to give the correct binding energy of He^4 and H^2.

TABLE VI. *Strength of the force between like particles necessary to explain the binding energies of H^3 and He^4.* (Equivalent two-body problem. Energies in MV, a in 10^{-13} cm.)

			H^3		He^4		
T	a	$\bar{B}$	B'	C	Coul.*	$1/2\,B'$	C
5	2.86	22.05	62.15	18.05	0.70	64.7	20.6
$7\frac{1}{2}$	2.33	30.4	81.75	20.95	0.79	81.85	21.05
10	2.02	38.4	100.3	23.5	0.86	97.5	20.7
15	1.65	54.0	135.8	27.8	0.96	127.7	19.7
20	1.43	69.4	169.3	30.5	1.05	156.7	17.9
30	1.165	99.3	233.8	35.2	1.18	211.4	12.8

* Coulomb energy of the two protons.

It is seen from Table VI that $C(H^3, T)$ increases rapidly with decreasing range of the forces while $C(He^4, T)$ stays constant down to about $1.5\cdot10^{-13}$ cm range and then *decreases*. This is due to the fact that the binding energy of the α-particle may be explained without having recourse to like-particle forces if we only assume $a=1.00\cdot10^{-13}$ cm while the corresponding calculations for H^3 (Table V) do not give a satisfactory result.

We now choose the point where the curves $C(H^3, T)$ and $C(He^4, T)$ cross. The constants of the force for this point are

$$T=7.6 \text{ MV}, \qquad \bar{B}=30.7 \text{ MV}, \qquad C=21.0 \text{ MV}, \qquad a=2.32\cdot10^{-13} \text{ cm}. \tag{128}$$

These constants give, according to our method, the correct binding energies of the three light nuclei H^2, H^3 and He^4.

Feenberg obtained slightly different constants by assuming in the first part of his calculation that there are no Heisenberg forces and then correcting for them afterwards. His values are

$$\bar{B}=34 \text{ MV}, \qquad C=21 \text{ MV}, \qquad a=2.17\cdot 10^{-13} \text{ cm}, \tag{128F}$$

the difference arising from slightly different experimental values for the binding energies of $H^2H^3He^4$.

It must be emphasized that the result (128) is by no means accurate. There is no way of estimating the error of the method of the equivalent two-body problem. It seems that the value of C (strength of the forces between like particles) is not as sensitive to errors as the values of B (force between proton and neutron) and a (range of the forces). E.g., let us suppose that the C values for the α-particle are too low by 2 MV, and those for H^3 too high by the same amount. Then the "crossing point" would be shifted to $T=11.2$ MV, $\bar{B}=42$ MV, $C=22.5$ MV, $a=1.91\cdot 10^{-13}$ cm, i.e., $\bar{B}$ would increase by almost 40 percent, a would show a corresponding decrease by almost 20 percent, while C would change by only 7 percent. The present determination can therefore not be regarded as final. However, at least the existence of forces between like particles seems almost certain, and the order of magnitude should not be very different from (128).

Moreover, it is very gratifying that the values for the force between like particles obtained from our theory *agree almost perfectly with those following from the scattering of protons by protons* (§18). This agreement is very satisfactory and represents at present the *only real check on our fundamental assumptions about nuclear forces.* That this is so, can be seen as follows: We have five accurate experimental data at our disposal in order to fix the constants in the theory of nuclei, *viz.*, the binding energies of $H^2H^3He^4$, the scattering of slow neutrons by protons, and the scattering of protons by protons. The first four are needed to fix the four constants: range of the forces, strength of the Majorana and Heisenberg forces between neutron and proton, and strength of the proton-proton forces. Only the fifth experimental result is therefore available for a check of the consistency of our assumptions.

§22. Comparison of H^3 and He^3

We have assumed throughout that the nuclear forces are symmetrical in neutrons and protons (§6); i.e., the force between two neutrons is assumed to be the same as that between two protons, disregarding the Coulomb force between the latter. From this point of view, the difference between the binding energies of H^3 and He^3 should be due entirely to the Coulomb repulsion between the two protons in He^3. A computation of this effect will give the most direct check of the assumption of symmetry of the nuclear forces in neutrons and protons. At the same time, we may compute the Coulomb energy for the α-particle, which was needed as a correction to the calculations in the preceding section (Tables V and VI).

The Coulomb energy may be calculated very easily from the wave function (117); the result is for He^3

$$\text{C.E.}=e^2[2(\nu+2\mu)/\pi]^{\frac{1}{2}}. \tag{129}$$

Similarly, we obtain for the α-particle

$$\text{C.E.}=e^2[4(\nu+\mu)/\pi]^{\frac{1}{2}}. \tag{129a}$$

In order to compute these expressions, we have to calculate ν and μ. It seems sufficiently accurate to put $\nu=\mu$ which is very nearly true for our case (see above). Then we may use (118a) (119a) to replace $\nu=\mu$ by σ, giving

$$\text{C.E.}=2(e^2/a)(\sigma/\pi)^{\frac{1}{2}} \tag{129b}$$

for *both* H^3 and He^4. Now σ may be determined approximately by making the *variational* energy of H^3 and He^4 a minimum. For H^3 and $\mu=\nu$ this variational energy is (cf. 118, put $p=1$):

$$E=3T\sigma-(2B+C)(\sigma/\sigma+1)^{\frac{1}{2}}. \tag{129c}$$

The minimum occurs for

$$(\sigma+1)^5/\sigma=[(2B+C)/2T]^2. \tag{129d}$$

For our values of the constants (128), this

expression has the value 29.0. This gives

$$\sigma = 0.934, \tag{129e}$$

$$\text{C.E.} = 0.68 \text{ MV}. \tag{129f}$$

A somewhat larger value of the Coulomb energy may be expected in view of the fact that the two protons shall be more frequently close together than would be expected from the wave function (117), owing to their strong mutual attraction at close distances. However, it can be expected that this correction is not very large.

Therefore (129f) should be approximately equal to the observed difference between the binding energies of H^3 and He^3, He^3 being the heavier nucleus. This difference is given experimentally directly by the difference between the energies evolved in the two reactions

$$H^2+H^2=H^3+H^1+Q_1, \tag{130}$$

$$H^2+H^2=He^3+n^1+Q_2. \tag{130a}$$

For we may write:

Binding energy of H^3 = mass of two neutrons + one proton − H^3

$$= 2n^1+2H^1-(H^3+H^1).$$

Binding energy of He^3

$$= n^1+2H^1-He^3 = 2n^1+2H^1-(He^3+n^1).$$

Therefore difference of binding energies equals

$$(He^3+n^1)-(H^3+H^1)=Q_1-Q_2. \tag{130b}$$

The reaction energy Q_1 is very well known from experiments of Oliphant, Kempton and Rutherford (O1), it is

$$Q_1 = 3.97 \pm 0.01_5 \text{ MV}. \tag{130c}$$

Q_2 has been measured by Bonner and Brubaker (B20) and by Dee and Walton (D1). The latter authors used the range of recoil He^4 nuclei set into motion by the neutrons to determine the neutron energy, the value of Q_2 derived from their measurements is about 2.95 MV (chapter XVI). This value is rather uncertain because the ranges of the recoil α-particles are very short, and lie in a region where the range-energy relation is not very well established. Bonner and Brubaker used recoil *protons* whose range is much longer and falls into the region where the range-energy relation is best known. They find

$$Q_2 = 3.21 \pm 0.13 \text{ MV}. \tag{130d}$$

If we decide for this value, the difference of the binding energies of He^3 and H^3 turns out to be

$$(He^3)-(H^3)=Q_2-Q_1=0.76\pm0.14 \text{ MV} \tag{130e}$$

in close agreement with the calculated value (129f).

§23. Excited States of the α-Particle (F4, C17)

Experimental evidence has been obtained by Crane and Lauritsen (C17) that the α-particle possesses excited states. The evidence is based on the fact that γ-rays are emitted when protons fall on Li^7 nuclei. The reaction taking place is probably (cf. chapter XVI)

$$Li^7+H^1=He^4+He^{4*};\quad He^{4*}\rightarrow He^4+\gamma, \tag{131}$$

where He^{4*} denotes an excited α-particle which, after the nuclear transmutation is completed, radiates its excitation energy as a γ-ray. (Cf. chapter XIII for arguments against the γ-ray being emitted *during* the nuclear transmutation itself.)

The γ-ray spectrum from reaction (131) appears to be complex. The maximum γ-energy observed is 16 MV, i.e., nearly the total energy available in the reaction (17 MV). In addition, there seem to be some γ-rays of smaller energy. A satisfactory interpretation would be to assume one excited level of the α-particle at 16 MV above the ground state, and one or two more excited states at lower energies, the lowest being perhaps 10 MV above ground. Then the low energy γ-rays would come from transitions between the high levels, while the energetic γ-rays would correspond to transitions from one of the high levels to the ground state.

α-particles with an excitation energy of 16 MV would be perfectly stable against disintegration. For the disintegration which would require least energy would be into H^3+H^1, and this process would require 19.4 MV energy, as calculated from the masses of H^1H^3 and He^4.

The problem is now whether (one or more) stable excited states of the α-particle can reasonably be expected. Feenberg (F4) has shown that the answer is affirmative provided the values for the force constants derived in the preceding section are anywhere near correct. There are

probably two or three stable excited states not very far apart.

One could try to calculate the energy of the excited states by the variation method. This method is, however, very troublesome for excited states[50] because their wave functions must be chosen orthogonal to that of the ground state. Moreover, the variation method did not give very good results even for the ground state. Feenberg therefore chose a different method for the treatment of the excited levels of the α-particle.[51]

The method is based on the so-called *sum rules* for the matrix elements of the coordinates. We introduce the following coordinates

$$\begin{aligned}&\mathbf{u}=\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4;\quad \mathbf{v}=\mathbf{r}_1-\mathbf{r}_2;\\&\mathbf{w}=\mathbf{r}_3-\mathbf{r}_4;\quad \mathbf{S}=\tfrac{1}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4),\end{aligned}\tag{132}$$

where $\mathbf{r}_1\mathbf{r}_2$ are the positions of the protons and $\mathbf{r}_3\mathbf{r}_4$ those of the neutrons. $\frac{1}{2}\mathbf{u}$ is the vector from the center of gravity of the two neutrons to that of the two protons, $\mathbf{v}$ the vector from one proton to the other, $\mathbf{w}$ the corresponding vector for the neutrons, and $\mathbf{S}$ the position of the center of gravity of the α-particle. The 12 Cartesian components of these four vectors form an orthogonal set of coordinates in the configuration space of the four particles. Accordingly, the Laplacian operator (kinetic energy) transforms into a sum of Laplacian operators with respect to the coordinates (132) without any cross terms, thus:

$$\Delta_1+\Delta_2+\Delta_3+\Delta_4=4\Delta_u+2\Delta_v+2\Delta_w+\tfrac{1}{4}\Delta_S.\tag{132a}$$

We insert this expression into the wave equation, and then deduce the sum rule for the oscillator strength in the usual way.* If u' denotes any Cartesian component of the vector $\mathbf{u}$, and u_{mn}' its matrix element with respect to the two states mn of the α-particle, we have for any state m

$$\sum_n(E_n-E_m)\,|\,u_{mn}'\,|^2=2\hbar^2/M,\tag{133}$$

$$\sum_n(E_n-E_m)\,|\,v_{mn}'\,|^2=\sum_n(E_n-E_m)\,|\,w_{mn}'\,|^2=\hbar^2/M,\tag{133a}$$

the sums including, of course, integrations over the continuous spectrum. We apply (133) particularly to the ground state $m=0$. We denote by E_1 the energy of the lowest excited state for which the matrix element u_{0n}' does not vanish. Then we have certainly from (133)

$$(E_1-E_0)\sum_n|\,u_{0n}'\,|^2<2\hbar^2/M.\tag{133b}$$

The sum occurring here can be evaluated at once:

$$\sum|\,u_{0n}'\,|^2=(u'^2)_{00},\tag{133c}$$

where $(u'^2)_{00}$ is the average of u'^2 over the eigenfunction of the ground state. We obtain therefore

$$E_1-E_0<2\hbar^2/M(u'^2)_{00}.\tag{134}$$

(134) gives an upper bound for the energy of the first excited state which involves only the knowledge of the wave function of the ground state. We take the wave function (117a) which may be rewritten

$$\psi_0=e^{-\frac{1}{2}\nu u^2-\frac{1}{2}(\mu+\nu)(v^2+w^2)},\tag{134a}$$

where u is the length of the vector $\mathbf{u}$. Then we find easily

$$(u'^2)_{00}=\int e^{-\nu u'^2}u'^2du'/\int e^{-\nu u'^2}du'=1/(2\nu).\tag{134b}$$

From (119a) we find $\nu=\frac{1}{2}\sigma/a^2$ if we put $p=1$ [actual value $p=0.985$, cf. §21, above Eq. (126)] so that

$$E_1-E_0<2\hbar^2\sigma/Ma^2=2T\sigma\tag{134c}$$

(cf. 118a). σ may be determined by making the variational energy of the ground state of the α-particle, *viz.* (cf. (119) with $p=1$)

$$4\tfrac{1}{2}T\sigma-2(2B+C)(\sigma/\sigma+1)^{\frac{3}{2}}\tag{134d}$$

a minimum. Inserting the values of the constants (128), this gives

$$\sigma=1.32,\tag{134e}$$

$$E_1-E_0<20.1\text{ MV}.\tag{135}$$

Since the condition for stability against disintegration into H^3+H^1 is

$$E_1-E_0<19.4\text{ MV},\tag{135a}$$

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[50] Cf. the corresponding calculations for the helium *atom*, *Handbuch der Physik*, Vol. 24/1, p. 364.

[51] Actually, Feenberg has also carried out variation calculations for the excited levels. However, the calculations were only performed for an ordinary rather than an exchange interaction. Moreover, the resulting energy levels lay not appreciably lower than the upper limit derived from the sum rule.

* *Note added in proof:* Feenberg has pointed out that the sum rules (133), (133a) hold only for *ordinary* forces. For Majorana forces, it seems that the upper bound for the energy turns out much lower than (135). This would make the existence of stable excited states a certainty.

the level E_1 needs only to be slightly below the upper limit (135) in order to be stable. Thus the stability of the state E_1 seems almost certain.[52]

The properties of the level E_1 may be deduced from the fact that the coordinate u has a non-vanishing matrix element between the ground state (completely symmetrical wave function) and E_1. This has two consequences: Firstly, the eigenfunction of E_1 must be symmetrical in the space coordinates of the protons and in the space coordinates of the neutrons; this makes the state a singlet state. Secondly, the eigenfunction must change sign if the two neutrons are replaced by the two protons and *vice versa* which shows that the function has a nodal plane; it is therefore a P function. Consequently, the state E_1 is a ^1P state.

Two ^3P levels are obtained by making the eigenfunction antisymmetrical in the space coordinates of the two protons, or of the two neutrons. Calling these ^3P levels E_2 and E_3, it follows that the matrix elements v_{02}' and w_{03}', respectively, are different from zero. Upper bounds for the energies E_2 and E_3 can be obtained in a way analogous to E_1, *viz.*,

$$E_2-E_0<\hbar^2/M(v'^2)_{00}, \tag{135b}$$

$$(v'^2)_{00}=1/(2\mu+2\nu). \tag{135c}$$

With $\mu=\nu$, the upper bounds for E_2 and E_3 become identical to that for E_1. From this fact, we might conclude that both the levels E_2 and E_3 might be stable. However, since the two levels have the same symmetry properties, there will be some "interaction" between them which will depress one level and raise the other possibly above the stability limit.

With all reservation, we may therefore form the following tentative picture of the excited states of the α-particle: There will be a ^1P state, odd in the u coordinate, even in the v and w coordinates, less than 20 MV above the normal state. Furthermore, there will be a ^3P_s state even in u, odd in v or w, which we might expect to lie lower than the ^1P state because of interaction with the other ^3P state. This other ^3P state will, because of the same interaction, probably lie *higher* than the ^1P state and probably it will not be stable at all.

We propose to identify the ^1P state with the observed level at 16 MV, the lower ^3P state with a lower level at, perhaps, 10 MV. This choice seems plausible from the standpoint of selection rules: Only the coordinate u has an electric dipole moment attached to it. Allowed optical transitions will therefore lead from a state odd in u to a state even in u or *vice versa*. Both the transitions from ^1P to ^1S and to ^3P are therefore allowed. The fact that the latter transition is an "intercombination" should only moderately decrease the intensity, because the rather strong Heisenberg forces (§14) prevent the spin of the α-particle from being a true quantum number. The transition $^3P-{}^1S$ must then also occur, because it is the only way in which α-particles in the ^3P state can get rid of their energy. We therefore expect three lines, corresponding to the transitions $^1P-{}^1S$ (about 16 MV),[53] $^1P-{}^3P$ ($\approx$6 MV) and $^3P-{}^1S$ ($\approx$ 10 MV). This seems compatible with experiments, in view of the small number of observations made and the large statistical fluctuations to be expected accordingly.

V. Statistical Theory of Nuclei

§24. The Hartree Method

In this and the following chapter, we shall use an approximation to the nuclear problem in which each particle is, in first approximation, supposed to move independently of the others. This method has been introduced into *atomic* physics by Hartree (H4), and has been used in that domain with great success.

In the Hartree approximation, we assume certain wave functions ψ_1, $\psi_2\cdots\psi_Z$ for each individual proton, and $\varphi_1\cdots\varphi_N$ for the neutrons. Each of these wave functions is supposed to be a function of the position as well as of the spin

[52] Feenberg has pointed out that the eigenfunction (134a) falls off too rapidly for large values of u, the correct dependence being $e^{-\alpha u}$ rather than $e^{-\frac{1}{2}\mu u^2}$. This makes the calculated value of $(u'^2)_{00}$ too small. A correct wave function would therefore give a *lower* value for the upper bound (135).

[53] There is, of course, no theoretical foundation for the figures. They are chosen to fit the experiments as good as possible.

of the respective particles; p_i and n_i shall stand for all coordinates (positional and spin) of the ith proton and neutron, respectively. Then a wave function of the nucleus as a whole which satisfies Pauli's principle, is (cf., e.g., reference S22)

$$\Psi=\begin{vmatrix}\psi_1(p_1) & \psi_2(p_1) & \cdots & \psi_Z(p_1)\\ \psi_1(p_2) & \psi_2(p_2) & \cdots & \psi_Z(p_2)\\ \cdots & \cdots & \cdots & \cdots\\ \psi_1(p_Z) & \psi_2(p_Z) & \cdots & \psi_Z(p_Z)\end{vmatrix}\cdot\begin{vmatrix}\varphi_1(n_1) & \varphi_2(n_1) & \cdots & \varphi_N(n_1)\\ \varphi_1(n_2) & \varphi_2(n_2) & \cdots & \varphi_N(n_2)\\ \cdots & \cdots & \cdots & \cdots\\ \varphi_1(n_N) & \varphi_2(n_N) & \cdots & \varphi_N(n_N)\end{vmatrix}. \tag{136}$$

The total energy of the nucleus is given, in first approximation, by

$$E=[\int\Psi^*[V-(\hbar^2/2M)(\sum_{i=1}^{N}\Delta_{ni}+\sum_{i=1}^{Z}\Delta_{pi})]\Psi d\tau]/\int|\Psi|^2 d\tau, \tag{137}$$

where V is the total potential energy as a function of the coordinates, Δ_{ni} and Δ_{pi} are the Laplacian operators with respect to the coordinates of the neutron i and proton i, and the integral has to be extended over all coordinates of all particles.

The expression (137) actually represents an upper bound for the energy of the system since it is well known that the right-hand side of (137) becomes an absolute minimum if we insert the correct wave function Ψ_0 instead of Ψ (Schrödinger's variation principle). Of course, if Ψ_0 is inserted in (137), that equation will give the *correct* energy E_0. Thus the correct energy is always lower than the E calculated from (137) with an *approximate* wave function Ψ. This has a very important consequence: If Ψ contains one or more parameters which are left arbitrary in the early stages of the calculation, the values of the parameters should finally be determined in such a way as to make (137) as small as possible. This will then make Ψ and E as close approximations to the correct wave function and energy as possible with the assumed form of the wave function (Ritz method). The parameter which we shall usually introduce into the wave function and then fix by this minimum condition, is the nuclear radius, but occasionally more parameters will be introduced.

Before we evaluate (137), we want to say a few words about the Hartree approximation in nuclear physics. It can be said at once that this approximation will not be as successful in nuclear theory as in the theory of atoms. The main reason for this is the saturation type of the nuclear forces: Any given nuclear particle interacts essentially only with two particles of the other kind (§7). Therefore the force between a given pair of particles will be of the same order of magnitude as the force exerted by the whole nucleus on one particle. This is contrary to the assumptions of the Hartree theory. These are that in first approximation the total action of the nucleus on one particle may be represented by an *average* field, corresponding to the average distribution of all other particles over the nucleus. The "correlations" between the different particles, i.e., the fact that the motion of one particle is influenced by the instantaneous position of the others, is supposed to cause only small perturbations in the Hartree theory. These assumptions of the Hartree theory are well satisfied in the atomic problem because the force due to the nucleus, and the force corresponding to the *average* charge distribution of the electrons, are very much stronger than the fluctuations of the force caused by, say, a close approach of one other electron to the electron considered. In nuclear physics, the force on one neutron changes by 100 percent or more according to whether a proton happens to be near the neutron or not. Therefore the correlations between the nuclear particles will be of extreme importance for any satisfactory calculation of nuclear energies, and the Hartree method will afford only a poor approximation. In spite of these serious objections against the Hartree method, we are forced to use it because no better method seems practicable at the moment.

Proceeding now to the evaluation of the energy (137), it is useful to assume the eigenfunctions of the individual particles to be orthogonal and

normalized. This is most conveniently achieved by assuming the ψ's and φ's to be the solutions of certain one-particle wave equations, thus:

$$\begin{aligned}(\hbar^2/2M)\Delta\psi_i+(E_i-V_p)\psi_i=0,\\(\hbar^2/2M)\Delta\varphi_i+(W_i-V_n)\varphi_i=0,\end{aligned} \tag{138}$$

where V_p and V_n are certain "auxiliary potentials" which may be chosen suitably so as to make the energy (137) a minimum.

We assume now the total potential energy V in (137) to consist of a number of terms corresponding to the interaction of pairs of particles:

$$V=\sum_{i=1}^{Z}\sum_{k=1}^{N}V_{ik}(p_i, n_k)+\tfrac{1}{2}\sum_{i=1}^{Z}\sum_{k\neq i}P_{ik}(p_i, p_k)+\tfrac{1}{2}\sum_{i=1}^{N}\sum_{k\neq i}N_{ik}(n_i, n_k), \tag{139}$$

where V_{ik} represents the interaction of a proton and a neutron, P_{ik} that between two protons and N_{ik} that between two neutrons. The factors 1/2 stand in order to count each pair only once.

The evaluation of (137) is then straightforward and similar to the theory of complex atoms (C13). The result is

$$\begin{aligned}E=&\sum_{i=1}^{Z}E_i+\sum_{i=1}^{N}W_i-\sum_{i=1}^{Z}\int|\psi_i|^2V_p d\tau-\sum_{i=1}^{N}\int|\varphi_i|^2V_n d\tau\\&+\sum_{i=1}^{Z}\sum_{k=1}^{N}\int\psi_i^*(p)\varphi_k^*(n)V(p,n)\psi_i(p)\varphi_k(n)d\tau_p d\tau_n\\&+\tfrac{1}{2}\sum_{i=1}^{Z}\sum_{k\neq i}^{Z}\Big[\int|\psi_i(1)|^2|\psi_k(2)|^2P(1,2)d\tau_1 d\tau_2\\&-\int\psi_k^*(1)\psi_i^*(2)\psi_i(1)\psi_k(2)P(1,2)d\tau_1 d\tau_2\Big]\\&+\tfrac{1}{2}\sum_{i=1}^{N}\sum_{k\neq i}^{N}\Big[\int|\varphi_i(1)|^2|\varphi_k(2)|^2N(1,2)d\tau_1 d\tau_2\\&-\int\varphi_k^*(1)\varphi_i^*(2)\varphi_i(1)\varphi_k(2)N(1,2)d\tau_1 d\tau_2\Big].\end{aligned} \tag{140}$$

The first line contains the kinetic energy of the particles. The second line represents the effect of the interaction between unlike particles, the next two lines the proton and the last two the neutron interaction. It is seen that exchange terms appear only in the four last lines which refer to the interaction between like particles, because the terms arise from the antisymmetry of the wave function with respect to particles of the same kind.

We now assume that $V(p, n)$ is an interaction of the Majorana type. In other words, if xs denote position and spin of the proton, $\xi\sigma$ the same quantities for the neutron, then

$$V(p,n)\psi_i(xs)\varphi_k(\xi\sigma)=J(x-\xi)\psi_i(\xi s)\varphi_k(x\sigma), \tag{141}$$

where J is an ordinary function of the distance between the particles, as treated in chapter III.

The integral signs in (140) imply, of course, a summation over the spin coordinates as well. This sum can be carried out if we assume that each of the functions ψ and φ is the product of a function which depends only on the space coordinates and one depending only upon spin, which assumption is always justified as long as the "auxiliary" potentials V_p and V_n (cf. 139) do not depend upon spin. Then the contribution of the interaction between proton i and neutron k to (140) becomes

$$V_{ik}=\int\psi_i^*(x)\varphi_k^*(\xi)\psi_i(\xi)\varphi_k(x)J(x-\xi)dx d\xi, \tag{141a}$$

the integral now extending over the space coordinates only.

The interaction between protons contains the Coulomb interaction and the specific interaction between like particles discussed in §§10, 18 and 21. Nothing definite is known about the type of these forces. Two types suggest themselves: The interaction may or may not depend on the relative orientation of the spins of the particles.[54] We decide in favor of the former choice because an interaction independent of the relative spins would be essentially equivalent to a Wigner type force and would therefore lead to difficulties, giving excessive binding energies for heavy nuclei[55] (§28, V1). We therefore assume

$$\begin{aligned}N(1,2)&=-\tfrac{1}{3}K(r_{12})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),\\P(1,2)&=-\tfrac{1}{3}K(r_{12})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+e^2/r_{12}.\end{aligned} \tag{142}$$

The forces neutron-neutron and proton-proton have been assumed equal, except for the Coulomb repulsion (cf. §6, 22). $\boldsymbol{\sigma}$ denotes the Pauli spin

[54] Majorana and Heisenberg forces may be expressed in terms of these two kinds of forces, owing to the antisymmetry of the wave function with respect to like particles, reference V1.

[55] A linear combination of a large force depending on spin, and a small force independent of spin, cannot be excluded.

operator. The factor $-\frac{1}{3}$ has been inserted in order to make the interaction of two neutrons (or protons) with opposite spins equal to $+K$; namely, for two such neutrons, we have (with $\boldsymbol{\sigma}$ denoting the resultant $\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2$, which is zero):[56]

$$2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=\boldsymbol{\sigma}^2-\boldsymbol{\sigma}_1^2-\boldsymbol{\sigma}_2^2=-6. \quad (142a)$$

K is thus identical with the quantity denoted by K in (115a)

To evaluate the neutron-neutron interaction in (140), we first add formally a term $k=i$ referring to the interaction of a neutron with itself. This does not change the expression, because the "ordinary" and the "exchange" term cancel exactly for $k=i$. Then we write each wave function φ_i as the product of a spatial wave function u_i and a spin wave function, which is α or β according to whether the spin component in a given direction z is $+\frac{1}{2}$ or $-\frac{1}{2}$. We then take a particular state $\varphi_i=u_i\alpha$, and consider its interaction with the *two* states $\varphi_{k1}=u_k\alpha$ and $\varphi_{k2}=u_k\beta$. (u_k may or may not be equal to u_i.) Then we have for the "direct" part of the interaction (first integral in last line of (140)):

$$-\tfrac{1}{3}\int|u_i(1)|^2|u_k(2)|^2K(r_{12})d\tau_1d\tau_2 \sum_{s_1s_2}\alpha(s_1)\alpha(s_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\alpha(s_1)\alpha(s_2) \quad (142b)$$

and

$$-\tfrac{1}{3}\int|u_i(1)|^2|u_k(2)|^2K(r_{12})d\tau_1d\tau_2 \sum_{s_1s_2}\alpha(s_1)\beta(s_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\alpha(s_1)\beta(s_2) \quad (142c)$$

corresponding to the two states φ_{k1} and φ_{k2}. Now the spin wave functions obey the relations:

$$\sigma_x\alpha=\alpha,\quad \sigma_z\beta=-\beta,\quad \sigma_x\alpha=\beta,\quad \sigma_x\beta=\alpha, \quad \sigma_y\alpha=i\beta,\quad \sigma_y\beta=-i\alpha. \quad (142d)$$

Therefore

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\alpha(1)\alpha(2)=\alpha(1)\alpha(2),$$
$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\alpha(1)\beta(2)=-\alpha(1)\beta(2)+2\beta(1)\alpha(2). \quad (142e)$$

Thus the sum over the spin coordinates s_1s_2 in (142b) gives $+1$, that in (142c) -1. The two expressions (142b) (142c) are thus equal and opposite; therefore the ordinary part of the interaction between the neutrons vanishes, if we take the interaction potential (142).[57]

The exchange part (second integral in last line of (140)) becomes

$$+\tfrac{1}{3}K_{ik}\sum_{s_1s_2}\alpha(s_1)\alpha(s_2)\cdot\alpha(s_1)\alpha(s_2) \quad (143)$$

and

$$+\tfrac{1}{3}K_{ik}\sum_{s_1s_2}\beta(s_1)\alpha(s_2) \cdot[-\alpha(s_1)\beta(s_2)+2\beta(s_1)\alpha(s_2)] \quad (143a)$$

for the interaction of φ_i with φ_{k1} and φ_{k2}, respectively. K_{ik} is the integral

$$K_{ik}=\int u_k^*(r_1)u_i^*(r_2)u_i(r_1)u_k(r_2) \times K(r_{12})d\tau_1d\tau_2. \quad (143b)$$

The summation over s_1s_2 gives 1 and 2, respectively, in (143) and (143a), so that both contributions together are just equal to K_{ik}. Thus K_{ik} is the interaction between one neutron i and a pair of neutrons k of opposite spin; or $\frac{1}{2}K_{ik}$ is the average interaction between two neutrons ik. Therefore the total interaction of all neutrons is $\frac{1}{4}\sum_{i=1}^{N}\sum_{k=1}^{N}K_{ik}$, since another factor $\frac{1}{2}$ comes from (140).

For the protons, the same result holds, but there is to be added the Coulomb repulsion which gives rise to a "direct" and to an "exchange" term.

Collecting our terms, we may now write for (140),

$$E=T+V+C-A+F \quad (144)$$

where T is the kinetic energy, V the contribution of the forces between protons and neutrons, C the direct Coulomb interaction between the protons, A the correction to the Coulomb interaction because of proton exchange, and F the contribution of the nuclear forces between like particles. Explicitly, we have

[56] $\boldsymbol{\sigma}_1$ is *twice* the spin angular momentum of particle 1 in units $\hbar$. The eigenvalue of the square of the angular momentum is $s(s+1)$ where $s=\frac{1}{2}$. Therefore $\boldsymbol{\sigma}_1^2=4\cdot\frac{3}{2}\cdot\frac{1}{2}=3$.

[57] This fact prevents the neutron-neutron interaction from giving excessive contributions to the binding energy of heavier nuclei. This is actually the reason for choosing the particular form (142) for the interaction potential.

$$T=\sum_{i=1}^{Z}(E_i-\int|\psi_i|^2V_pd\tau)+\sum_{i=1}^{N}(W_i-\int|\varphi_i|^2V_nd\tau), \tag{144a}$$

$$V=\sum_{i=1}^{Z}\sum_{k=1}^{N}\int\psi_i^*(x_1)\psi_i(x_2)\varphi_k^*(x_2)\varphi_k(x_1)J(r_{12})d\tau_1d\tau_2, \tag{144b}$$

$$C=\sum_{i=1}^{Z}\sum_{k=1}^{Z}\int|\psi_i(x_1)|^2|\psi_k(x_2)|^2(e^2/r_{12})d\tau_1d\tau_2, \tag{144c}$$

$$A=\sum_{i=1}^{Z}\sum_{k=1}^{Z}\int\psi_i^*(x_1)\psi_i(x_2)\psi_k^*(x_2)\psi_k(x_1)(e^2/r_{12})d\tau_1d\tau_2, \tag{144d}$$

$$F=\tfrac{1}{4}\sum_{i=1}^{Z}\sum_{k=1}^{Z}\int\psi_i^*(x_1)\psi_i(x_2)\psi_k^*(x_2)\psi_k(x_1)K(r_{12})d\tau_1d\tau_2$$

$$+\tfrac{1}{4}\sum_{i=1}^{N}\sum_{k=1}^{N}\int\varphi_i^*(x_1)\varphi_i(x_2)\varphi_k^*(x_2)\varphi_k(x_1)K(r_{12})d\tau_1d\tau_2. \tag{144e}$$

A formal simplification of these equations can be achieved by introducing the total density of protons and neutrons, *viz.*,

$$\rho_p(x)=\sum_{i=1}^{Z}|\psi_i(x)|^2,\quad \rho_n(x)=\sum_{i=1}^{N}|\varphi_i(x)|^2, \tag{145}$$

and Dirac's "mixed densities"

$$\rho_p(x_1,x_2)=\sum_{i=1}^{Z}\psi_i^*(x_1)\psi_i(x_2),\qquad \rho_n(x_1,x_2)=\sum_{i=1}^{N}\varphi_i^*(x_1)\varphi_i(x_2). \tag{145a}$$

Obviously,

$$\rho_p(x_2,x_1)=\rho_p^*(x_1,x_2),\quad \rho_p(x)=\rho_p(x,x).$$

Interchanging the order of summations and integrations in (144a) to (144e), we have

$$T=\sum_{i=1}^{Z}E_i+\sum_{i=1}^{N}W_i-\int(V_p\rho_p+V_n\rho_n)d\tau, \tag{146a}$$

$$V=\int J(r_{12})\rho_p(x_1x_2)\rho_n^*(x_1x_2)d\tau_1d\tau_2, \tag{146b}$$

$$C=\tfrac{1}{2}\int(e^2/r_{12})\rho_p(x_1)\rho_p(x_2)d\tau_1d\tau_2, \tag{146c}$$

$$A=\tfrac{1}{4}\int(e^2/r_{12})|\rho_p(x_1x_2)|^2d\tau_1d\tau_2, \tag{146d}$$

$$F=\tfrac{1}{4}\int K(r_{12})(|\rho_p(x_1x_2)|^2+|\rho_n(x_1x_2)|^2)d\tau_1d\tau_2. \tag{146e}$$

§25. The Statistical Model. Qualitative Conclusions (M6)

In the statistical model the eigenfunction of each individual proton or neutron is supposed to be that of a free particle *viz.*, a plane wave. These eigenfunctions are inserted in (146) and the energy of the nucleus calculated.

Accordingly, the auxiliary potentials V_p and V_n are assumed to have certain constant, negative values inside a sphere of radius R (nuclear radius) and to be zero outside that sphere. The radius R is an arbitrary parameter which has to be determined in such a way as to make the nuclear energy a minimum (cf. §24, beginning). The value of the auxiliary potentials V_p and V_n inside the "nucleus" does not affect the nuclear energy materially because only the kinetic energy of the particles and the form of the wave functions enter the formula (146) neither of which quantities depend upon the value of the auxiliary potentials except for surface effects (§29).

For the sake of the simplicity of the eigenfunctions it may be allowable to replace the spherical "box" for the nuclear particles by a cubical one having the same volume. Denoting the length of the cube by l we have

$$l^3=4\pi R^3/3,\quad l=R(4\pi/3)^{\frac{1}{3}}. \tag{147}$$

The wave functions in such a box are plane waves

$$\psi_i=l^{-\frac{3}{2}}e^{i(\mathbf{k}_i\cdot\mathbf{r})},\quad \varphi_i=l^{-\frac{3}{2}}e^{i(\mathbf{k}_i\cdot\mathbf{r})}. \tag{147a}$$

The factor $l^{-\frac{3}{2}}$ serves to normalize the functions.

The components of the wave vectors $\mathbf{k}_i$ are determined by the boundary condition for the eigenfunctions at the boundary of the cube. Neglecting the details of surface effects we may impose the usual condition

$$k_{ix}=2\pi\kappa_x/l, \quad k_{iy}=2\pi\kappa_y/l, \quad k_{iz}=2\pi\kappa_z/l, \tag{148}$$

$\kappa_x\kappa_y\kappa_z$ being three integers. For each triple of integers $\kappa_x\kappa_y\kappa_z$ there is just one proton state and one neutron state of either spin direction, so that the total number of neutron (or proton) states having wave vector components between k_x and k_x+dk_x, k_y and k_y+dk_y, k_z and k_z+dk_z is

$$ndk_xdk_ydk_z=2(l/2\pi)^3dk_xdk_ydk_z, \tag{148a}$$

the factor 2 arising from the two possible spin directions. The number of states for which the absolute value of the wave vector is between k and $k+dk$, is (cf. 147)

$$2(l/2\pi)^3 4\pi k^2dk=(4/3\pi)R^3k^2dk. \tag{148b}$$

The kinetic energy of a particle of wave number k is

$$T(k)=\hbar^2k^2/2M, \tag{149}$$

so that the number of neutrons with kinetic energy between T and $T+dT$ becomes

$$N(T)dT=(2^{5/2}/3\pi)M^{\frac{3}{2}}\hbar^{-3}R^3T^{\frac{1}{2}}dT. \tag{149a}$$

The states of lowest energy will be those for which the kinetic energy, and therefore the wave number, is smallest. The number of neutron states having wave numbers below k_0, or kinetic energies below T_0, is

$$N=(4/3\pi)R^3\textstyle\int_0^{k_0}k^2dk=(4/9\pi)R^3k_0^3, \tag{150}$$

$$N=(2^{7/2}/9\pi)M^{\frac{3}{2}}\hbar^{-3}R^3T^{\frac{3}{2}}. \tag{150a}$$

From these equations we may find the maximum wave number and kinetic energy of the neutrons in the nucleus if the total number of neutrons is N, *viz.*,

$$k_0=(9\pi N/4)^{\frac{1}{3}}R^{-1}, \tag{150b}$$

$$T_0=(9\pi N/4)^{\frac{2}{3}}\hbar^2/2MR^2. \tag{150c}$$

The average kinetic energy per neutron follows from (149a)

$$\overline{T}=\textstyle\int N(T)TdT/\int N(T)dT=\frac{3}{5}T_0. \tag{150d}$$

$\overline{T}$ is thus proportional to the 2/3 power of the number of neutrons, just as T_0. The total kinetic energy is therefore proportional to $N\overline{T}\propto N^{5/3}R^{-2}$. All these formulae are familiar from Fermi statistics. They apply to protons as well as to neutrons, with only N to be replaced by Z.

We may now proceed to calculate the interaction of neutrons and protons (146b), using our free particle eigenfunctions (147a). We have first to calculate the mixed density

$$\rho(\mathbf{r}_1\mathbf{r}_2)=\sum_{i=1}^{N}\psi_i^*(\mathbf{r}_1)\psi_i(\mathbf{r}_2)=l^{-3}\sum_{i=1}^{N}e^{i\mathbf{k}_i\cdot(\mathbf{r}_2-\mathbf{r}_1)}$$

$$=l^{-3}(l^3/4\pi^3)\int_0^{k_0}dk_xdk_ydk_ze^{i\mathbf{k}\cdot\mathbf{r}}, \tag{151}$$

where $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$. Introducing a polar coordinate system in k space with the polar axis parallel to $\mathbf{r}$, and using (147) again, we have

$$\rho(\mathbf{r}_1\mathbf{r}_2)=(1/4\pi^3)\textstyle\int_0^{k_0}2\pi\sin\theta d\theta k^2dke^{ikr\cos\theta}$$

$$=(1/\pi^2r)\textstyle\int_0^{k_0}kdk\sin kr$$

$$=(\sin k_0r-k_0r\cos k_0r)/\pi^2r^3. \tag{151a}$$

This function has a pronounced maximum for $r=0$, *viz.*,

$$\rho(\mathbf{r}_1\mathbf{r}_1)=k_0^3/3\pi^2=N/(4\pi R^3/3) \tag{151b}$$

[cf. (150)]. $\rho(\mathbf{r}_1\mathbf{r}_2)$ falls off rapidly when k_0r becomes larger than unity. This means that there is practically no correlation between the wave functions at two points whose distance is considerably larger than $1/k_0=\lambda_0/2\pi$, where λ_0 is the shortest wave-length of any neutron in the nucleus. Because of (150), $1/k_0\approx\frac{1}{2}RN^{-\frac{1}{3}}$ which is small compared to the radius of the nucleus if N is large. The mixed density is therefore only large if the two points $\mathbf{r}_1$, $\mathbf{r}_2$ are very close together compared to the radius of the nucleus. This is very essential for the validity of the statistical model: One may reasonably hope that the actual wave functions of the nuclear particles resemble plane waves at least over a limited region of the nucleus even if they differ widely from plane waves if considered over the whole nucleus.

A number of important conclusions can be drawn by inserting (151a) into (146b), even without assuming a special form of the interaction potential $J(r_{12})$. We only introduce for convenience a certain length a which determines the width of the potential hole J, and we put

$$J(r)=-V_0f(r/a), \tag{152}$$

where V_0 is a constant determining the strength of the interaction. f may be any function; e.g., we may take any of the following:

rectangular hole: $f(x)=1$, for $x<1$,
$f(x)=0$ for $x>1$,

exponential function: $f(x)=e^{-x}$,

Gaussian function: $f(x)=e^{-x^2}$,

or a more complicated function. The only assumptions we make are

(1) $J(r)$ vanishes at infinity sufficiently rapidly
(2) $J(r)$ does not become infinite at $r=0$ more strongly than $1/r$
(3) a is chosen in such a way that the main drop of the function $f(x)$ occurs near $x=1$.

The total potential energy of the neutron-proton interaction becomes now, according to (146b) and (151a)

$$V=-\frac{4\pi}{3}R^3V_0\int_0^\infty 4\pi r_{12}^2dr_{12}f(r_{12}/a)\times\frac{\sin k_Nr_{12}-k_Nr_{12}\cos k_Nr_{12}}{\pi^2r_{12}^3}\times\frac{\sin k_Pr_{12}-k_Pr_{12}\cos k_Pr_{12}}{\pi^2r_{12}^3}, \quad (152a)$$

where k_N and k_P are the maximum wave numbers for neutrons and protons, as given by (150b). (In case of k_P, we have to replace N in (150b) by Z.) The integral over $d\tau_1d\tau_2$ in (146b) has been replaced by an integral over $d\tau_1d\tau_{12}$, i.e., we integrate first over the coordinates of particle 2, keeping the position of the first constant, which is equivalent to integrating over the relative coordinate $\mathbf{r}_{12}$ (volume element $4\pi r_{12}^2dr_{12}$). Since the integrand falls off rapidly for large distances r_{12}, the result of this first integration will not be materially changed if we extend the range of integration with respect to r_{12} to infinity. This extension corresponds to the neglection of surface effects. Then the result will obviously not depend upon the position of the first particle $\mathbf{r}_1$ so that the integration over $d\tau_1$ gives simply a factor equal to the volume of the nucleus, *viz.*, $(4\pi/3)R^3$.

It is convenient to introduce the radius of a sphere corresponding to the volume occupied by one particle, *viz.*,

$$r_0=R(N+Z)^{-\frac{1}{3}} \quad (153)$$

and to put accordingly

$$r_{12}=r_0\rho;\quad r_0/a=\alpha, \quad (153a)$$

$$k_Nr_{12}=\kappa_N\rho,$$

$$k_Pr_{12}=\kappa_P\rho, \quad (153b)$$

with

$$\kappa_P=[9\pi Z/4(N+Z)]^{\frac{1}{3}};$$
$$\kappa_N=[9\pi N/4(N+Z)]^{\frac{1}{3}}. \quad (153c)$$

Then ρ is our new integration variable while α is a parameter determining the radius of the nucleus. α has to be varied such that the total energy becomes a minimum. (152a) may now be rewritten:

$$V=-\frac{16}{3}(N+Z)V_0\pi^{-2}\int_0^\infty f(\alpha\rho)\times(\sin\kappa_N\rho-\kappa_P\rho\cos\kappa_N\rho)\times(\sin\kappa_P\rho-\kappa_P\rho\cos\kappa_P\rho)\rho^{-4}d\rho. \quad (154)$$

Since κ_N and κ_P depend only upon the ratio N/Z but not upon the magnitude of either N or Z, the integral is a function of α and N/Z only. Similarly, if we introduce the notations (153), etc., into (150c, d), we have

$$T=T_N+T_P=\frac{3}{5}\frac{\hbar^2}{2M}(Nk_N^2+Zk_P^2)=\frac{3}{5}\frac{\hbar^2}{2Ma^2}\left(\frac{9\pi}{4}\right)^{2/3}(N+Z)\frac{N^{5/3}+Z^{5/3}}{(N+Z)^{5/3}\alpha^2}. \quad (154a)$$

In this formula, the kinetic energy has been written as the number of particles, $N+Z$, times a function of N/Z and α only. Therefore, from (154) and (154a)

$$E=T+V=(N+Z)F(N/Z,\alpha), \quad (155)$$

where the function F can be determined when the interaction potential J, i.e., the function f, is given.

Making (155) a minimum by varying α, we see: (1) The value of α corresponding to the minimum will only depend upon the *ratio* N/Z but not upon the number of particles $N+Z$. This means, according to (153, 153a) that the radius of the nucleus R is, for a given ratio N/Z, proportional to $(N+Z)^{\frac{1}{3}}$; in other words that *the volume of the nucleus* is, again for given N/Z, *proportional to the number of particles* contained in it. (2) *The binding energy* $-E$ of the nucleus is, for a given ratio N/Z, proportional to the number of *particles* contained in the nucleus.

These two conclusions are in accord with experiment. In fact, we have assumed exchange forces to act between neutrons and protons just in order to account for the two experimental facts mentioned. (§7.) Thus our calculation

merely shows that *exchange forces are really suitable for obtaining the desired result.*

In making this statement we must bear in mind that actually we have only shown that the *statistical model* applied to a nucleus held together by exchange forces, leads to a binding energy proportional to the number of particles. Apart from the neglection of surface effects, we have made two assumptions:

(a) the "auxiliary potential energy" V_N and V_P was supposed to be constant over the nucleus. This led to a constant density of the particles (cf. (151b)).

(b) the wave functions were supposed to be representable by plane waves over a region of the order of one wave-length (cf. the remarks after (151b)).

The first assumption is very plausible indeed: There is no force establishing a correlation between the positions of the particles and a *fixed* point, in contrast, e.g., to the case of atomic electrons. There also is no long-distance force between the particles which might establish differences between the density at different points. The force upon one particle actually depends only on the density in its immediate neighborhood; one given value of the density will lead to minimum energy; any fluctuations of the density from point to point in the nucleus will obviously lead to an increase in energy. A mathematical proof of the constancy of the density for a particular case was given by Majorana (M6).

The second point will, of course, not actually be fulfilled because of correlations between different particles (§24). However, it can be shown by a group theoretical argument that, even with quite general assumptions about the eigenfunctions, the total energy of a nucleus is proportional to the number of particles if exchange forces act between them.[58]

We now discuss the quantitative implications of (154), (154a), (155). The behavior of the integral in (154) can be easily found for the two limiting cases $\alpha \ll 1$ and $\alpha \gg 1$. In the first case, we may replace $f(\alpha\rho)$ by $f(0)$. Then the integral may be carried out elementarily and gives

$$V = \begin{cases} -2J(0)Z & \text{if } Z<N \\ -2J(0)N & \text{if } Z>N \end{cases} \qquad \alpha \ll 1. \qquad (156)$$

If, therefore, the radius of the nucleus is so small that the average distance of neighboring particles, r_0, is small compared to the range of the forces, a, then the total potential energy can be thought of as due to the interaction of each particle of the sort of which there are fewer in the nucleus, with two particles of the other sort (one of each spin direction).[59] The total potential energy (156) is independent of α while the kinetic energy (154a) increases as α^{-2}. Therefore the total energy is certainly positive for sufficiently small α, and the most favorable value of α is certainly not $\alpha=0$.

Similarly, for α very large, we may replace all factors *except* $f(\alpha\rho)$, by their value for very small ρ, and obtain

$$V = -3NZV_0(N+Z)^{-1}\int f(\alpha\rho)\rho^2 d\rho$$

$$= -\frac{3NZ}{N+Z}\frac{V_0}{\alpha^3}\int f\left(\frac{r}{a}\right)\left(\frac{r}{a}\right)^2 d\left(\frac{r}{a}\right)$$

$$= \text{const}\cdot\alpha^{-3}. \quad \alpha \gg 1. \qquad (156a)$$

Thus, for large α, the potential energy decreases more rapidly than the kinetic. Therefore the total energy must be positive for very large α. A negative total energy, i.e., a resultant binding energy, can only be obtained for intermediate values of α, and since all constants are of the order of magnitude unity, the minimum may be expected to lie at a value of α near unity. This means that r_0 is of the same order as a. Now r_0 is the radius of a sphere whose volume is equal to the volume occupied by one nuclear particle. r_0 is thus of the same order as the distance between neighboring particles in the nucleus. We conclude that this distance is about as large as the radius of action of the nuclear forces. This conclusion corresponds to reality: The radius of radioactive nuclei is about 8 to $9\cdot10^{-13}$ cm (chapter IX), their atomic weight about 220, therefore $r_0 = 9\cdot10^{-13}\cdot220^{-\frac{1}{3}} \approx 1.5\cdot10^{-13}$ cm which is indeed of the same order of magnitude as the range of nuclear forces (§21).

[58] We are indebted to Professor Wigner for this communication.

[59] Formula (156) has, of course, only a meaning if $J(0)$ is finite. If this is not the case, $J(\alpha\rho)$ cannot be replaced by $J(0)$ however small α.

§26. Quantitative Results and Limitations of the Statistical Model (H9, W2, W10)

In order actually to calculate the binding energy and the nuclear radius, we must make a special assumption about the form of the potential $J(r)$. We take (cf. 115)

$$J(r) = -Be^{-r^2/a^2}. \tag{157}$$

This potential has been assumed in the extensive calculations of Feenberg about light nuclei; therefore the constants B and a are well known. We have to insert $f(\alpha\rho) = e^{-\alpha^2\rho^2}$ into (154). The integrations can then be carried out explicitly and elementarily, giving

$$\begin{aligned} V = -B(N+Z)\pi^{-\frac{1}{2}}x^{-3}\{&[2-x^2(n^{\frac{2}{3}}+n^{\frac{1}{3}}(2-n)^{\frac{1}{3}} \\ &+(2-n)^{\frac{2}{3}})]\cdot e^{-\frac{1}{4}x^2[n^{\frac{1}{3}}-(2-n)^{\frac{1}{3}}]^2} \\ &-[2-x^2(n^{\frac{2}{3}}-n^{\frac{1}{3}}(2-n)^{\frac{1}{3}}+(2-n)^{\frac{2}{3}})] \\ &\times e^{-\frac{1}{4}x^2[n^{\frac{1}{3}}+(2-n)^{\frac{1}{3}}]^2}+\pi^{\frac{1}{2}}x^3\Phi(\tfrac{1}{2}x[n^{\frac{1}{3}}+(2-n)^{\frac{1}{3}}]) \\ &-\pi^{\frac{1}{2}}(n-1)x^3\Phi(\tfrac{1}{2}x[n^{\frac{1}{3}}-(2-n)^{\frac{1}{3}}])\}, \end{aligned} \tag{158}$$

where[60]

$$n = 2N/(N+Z), \quad 2-n = 2Z/(N+Z), \tag{158a}$$

$$x = \frac{3}{2}\left(\frac{\pi}{3}\right)^{\frac{1}{3}}\frac{a}{r_0} = \frac{3}{2}\left(\frac{\pi}{3}\right)^{\frac{1}{3}}\frac{1}{\alpha}, \tag{158b}$$

and $\Phi(y)$ is the Gaussian error integral (cf., e.g., Jahnke-Emde, Tables of Functions, p. 97).

If the nucleus contains equally many neutrons and protons, we have $n=1$ so that (158) reduces to[61]

$$\begin{aligned} V = -\pi^{-\frac{1}{2}}x^{-3}B(N+Z)\{&2-3x^2 \\ &+(x^2-2)e^{-x^2}+\pi^{\frac{1}{2}}x^3\Phi(x)\}, \end{aligned} \tag{159}$$

while the kinetic energy (154a) may be rewritten:

$$T = \frac{3}{10}\frac{\hbar^2}{Ma^2}x^2(N+Z). \tag{159a}$$

The most satisfactory procedure would now be to insert the values for B and a derived from the theory of light nuclei, and to determine the minimum of $V+T$ and the corresponding value of x. Unfortunately, the binding energy obtained in this way is much too small; it is, indeed, entirely wiped out when due corrections are made for the Coulomb repulsion of the protons and the "surface effect" (§29).

The values for B and a derived from the theory of light nuclei are:

$$\begin{aligned} &\text{(a)}\quad B = 133 \text{ MV}, \qquad a = 0.99\cdot 10^{-13} \text{ cm}, \\ &\qquad\qquad T = \hbar^2/Ma^2 = 42 \text{ MV}, \end{aligned} \tag{160}$$

if no interaction between like particles is assumed (cf. 125),

$$\begin{aligned} &\text{and (b)}\quad B = 41 \text{ MV},[62] \quad a = 2.32\cdot 10^{-13} \text{ cm}, \\ &\qquad\qquad T = 7.6 \text{ MV}, \end{aligned} \tag{160a}$$

if the interaction between like particles has the value derived in (128). According to Table VII, we get with these constants the following results in the statistical model:

(a) without interaction between like particles we have $D = MBa^2/\hbar^2 = 3.17$. For this value of D, the energy $V+T$ has *no minimum at all* but is positive for any value of x.

(b) with interaction between like particles: $D = MBa^2/\hbar^2 = 5.3_5$. Minimum of energy for $x = 1.99$ $\qquad r_0 = 0.79a = 1.82\cdot 10^{-13}$ cm.

$$\begin{aligned} \text{Minimum energy} &= -0.067B(N+Z) \\ &= -2.7(N+Z) \text{ MV}. \end{aligned} \tag{160b}$$

With the constants under (a), we thus obtain no binding energy at all, even without correcting for Coulomb repulsion and surface effect. With the constants under (b), we find a small binding energy, about one-quarter of the observed binding energy for medium weight nuclei. However, even this small binding disappears when we correct for Coulomb repulsion and surface effect (see below).

We must therefore conclude that *the statistical model is quite inadequate for the treatment of nuclear binding energies*. This is not very surprising in view of the objections raised in §24 against the application of the Hartree method to

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[60] The old constants κ_N and κ_P are related to n by

$$\kappa_N = (9\pi/8)^{\frac{1}{3}}(2N/N+Z)^{\frac{1}{3}} = (3/2)\cdot(\pi n/3)^{\frac{1}{3}}$$
$$\kappa_P = \tfrac{3}{2}(\pi(2-n)/3)^{\frac{1}{3}}.$$

[61] This formula was first calculated by Heisenberg (quoted by Flügge, F12).

[62] The figure given is the force between a proton and a neutron, plus one-half the force between like particles ($B+\frac{1}{2}C$, cf. §28). This combination enters the energy of a heavy nucleus containing approximately equal numbers of protons and neutrons (cf. 174).

the nuclear problem which hold *a fortiori* for the statistical method. In view of these objections we would even go so far as to say that any expression for the nuclear forces which *would* make the nuclear binding energy following from the statistical model equal to the observed binding, must certainly give too large values for the nuclear forces.

It is of some interest to note that the statistical model becomes the worse the shorter the range of the forces. This is due to the fact that the wave functions will change very rapidly whenever the distance between two particles is smaller than the range of the nuclear forces (cf. §20). The fact is shown by a comparison of our cases (a) and (b). We may add that for *zero* range of the forces the potential energy (159) does never exceed 0.73 times the kinetic energy (159a), if the constant $D=MBa^2/\hbar^2$ is taken from the theory of the deuteron (2.70, cf. Table III).

The value of the statistical model can, according to the foregoing, not lie in quantitative calculations of nuclear binding energies, but lies, in our opinion, in qualitative results such as those obtained in the preceding section, and in the possibility of setting *upper* limits to the nuclear forces (§27). Furthermore, we might try to deduce the dependence of the binding energy on the atomic weight, on the ratio of the numbers of neutrons and protons, etc.; but all these conclusions should be considered as tentative only.

Obviously, if we want to make use of the statistical model, we must fix the constants B and a *ad hoc* and must not take the values derived from the theory of light nuclei. The constants B and a derived from the statistical model are, then, of course not correct; but it may seem more consistent in drawing conclusions from that model to fix the constants entering it also from the statistical model.

To fix the constants B and a, we need *two* experimental data. Various data have been used by different authors; e.g., Heisenberg (H9) used the mass defects of medium-sized and heavy nuclei, Wick (W10) used mass defect and radius of the oxygen nucleus, v. Weizsäcker (W2) the mass defects of O^{16} and Hg^{200}. Wick's method seems preferable to us because, to a first approximation, the mass defect per particle should be the same for all nuclei, according to the statistical model. Only the Coulomb repulsion of the protons and the surface effects should produce some differences in the mass defects per particle. Since the surface effect cannot be treated as satisfactorily as the "volume" energy, one should not base the determination of the constants on quantities which depend sensitively on the surface effect, as does the difference of the mass defects of O^{16} and Hg^{200}. The only objection which could be raised against Wick's procedure is that the radius of O^{16} is not well known experimentally. This can be avoided by taking radius and binding energy of a heavy atom as standards. In doing so, we get the further advantage of minimizing the surface effect.

We therefore take radius and binding energy of Hg^{200} as standards. Since this nucleus has an atomic weight just below that of radioactive nuclei, it seems reasonable to assume a radius slightly smaller than theirs. We take

$$R_{Hg^{200}}=8\cdot 10^{-13}\text{ cm}. \tag{161}$$

The binding energy follows from Aston's determination of the atomic weight of Hg, *viz.*, 200.016, and from the atomic weights of proton and neutron which are 1.00807 and 1.00846, respectively (cf. (75a)). This gives

$$\begin{aligned} E_{Hg^{200}}&=200.016-(80\cdot 1.00807+120\cdot 1.00846)\\ &=200.016-201.661\\ &=-1.645\text{ mass units}=-1530\text{ MV}. \end{aligned} \tag{161a}$$

Per nuclear particle, the binding energy is -7.6_5 MV.

From the nuclear radius (161), we can immediately calculate the kinetic energy of all neutrons and protons together, which becomes, according to (150c), (150d), with $N=120$ and $Z=80$:

$$T=\frac{3}{5}(ZT_Z+NT_N)=\frac{3}{5}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{\hbar^2}{2MR^2}\times(Z^{5/3}+N^{5/3})=2980\text{ MV}. \tag{161b}$$

Here T_Z and T_N denote the maximum kinetic energies for protons and neutrons, which are, for Hg, 22.2 and 28.9 MV, respectively. The average kinetic energy of each of the 200 particles is, according to (161b), 14.9 MV.

The potential energy of the nuclear forces, V, is obtained from

$$E=T+V+C+S, \tag{162}$$

where C is the potential energy of the Coulomb repulsion of the protons which is (Eq. (16))

$$C=\tfrac{3}{5}Z^2e^2/R=670 \text{ MV} \tag{162a}$$

and S is the surface energy which we estimate (cf. 184)

$$S=450 \text{ MV}. \tag{162b}$$

Therefore $V=-(1530+2980+670+450)$
$$=-5630 \text{ MV}, \tag{163}$$

$$|V|/T=1.89. \tag{163a}$$

The procedure for determining B and a is then the following: For any value of $D=MBa^2/\hbar^2$, we may determine the minimum of $E'=V+T$ as a function of x, using (159) and (159a). The value $x_{\min}$ for which that minimum occurs, is given as a function of D in Table VII. (n has been put equal to 1, corresponding to an equal number of protons and neutrons.) Inserting $x_{\min}$ into (159), (159a) we may find the ratio V/T as a function of D (fourth line of Table VII). That value of D which gives the observed ratio V/T, is the "correct" value of D for the statistical model; for $|V|/T=1.89$ we find

$$D=MBa^2/\hbar^2=9.6. \tag{164}$$

The corresponding value of $x_{\min}$ is 2.70 corresponding to (cf. 158b)

$$a=0.64\cdot 2.70 r_0=1.73 r_0 \tag{164a}$$

and with the observed value $R=8\cdot 10^{-13}$ cm:

$$r_0=8\cdot 10^{-13}\cdot 200^{-\frac{1}{3}}=1.37\cdot 10^{-13} \text{ cm}, \tag{164b}$$

$$a=2.37\cdot 10^{-13} \text{ cm}, \tag{164c}$$

$$B=\hbar^2D/Ma^2=70 \text{ MV}. \tag{164d}$$

Thus the range of the nuclear forces turns out almost identical with that derived by Feenberg from light nuclei, whereas the depth of the potential hole must be chosen about 70 percent larger than the actual depth in order to bring the "statistical energy" into agreement with the observed nuclear binding energy.

From the table we note that the depth of the potential hole (fifth line) depends practically only on the nuclear radius and not on the observed binding energy. The latter determines only the range a of the forces.

We may now use our constants to deduce the dependence of the "volume energy" $V+T$ on the ratio of the numbers of neutrons and protons. Expanding (158), (158a) in a power series in

$$n-1=(N-Z)/(N+Z) \tag{165}$$

we have

$$V(n)=V(1)+\frac{B}{3(\pi)^{\frac{1}{2}}}\frac{(N-Z)^2}{N+Z}\frac{1}{x}\times(x^2-1+e^{-x^2}), \tag{165a}$$

$$T(n)=T(1)+\frac{\hbar^2}{6Ma^2}\frac{(N-Z)^2}{N+Z}x^2, \tag{165b}$$

where $V(1)$ and $T(1)$ are the values of potential and kinetic energy for $n=1$, i.e., $N=Z$. Adding (165a) and (165b), we have for the total energy

$$E(n)=E(1)+\beta(N-Z)^2/(N+Z). \tag{166}$$

Inserting the values for x, B and a derived above for the statistical model, we find

$$\beta=39 \text{ MV} \quad \text{(statistical)}. \tag{166a}$$

Inserting, on the other hand, the constants B and a derived from light nuclei (cf. 160a), and determining x from that value of a and the observed r_0 (cf. 164b) with the help of (158b), we have

$$\beta=26 \text{ MV}$$
(force constants from light nuclei). (166b)

In both cases, the increase of the kinetic energy

TABLE VII. *Relation of constants in the statistical model.* Potential $J(r)=-Be^{-r^2/a^2}$.

$D=MBa^2/\hbar^2$	3.5	4	5	6	7	8	9	10	11
$x_{\min}$	1.22	1.56	1.92	2.14	2.34	2.48	2.62	2.76	2.90
a/r_0	0.78	0.99	1.22	1.36	1.48	1.57	1.66	1.75	1.84
$\|V\|/T$	0.90	1.04	1.25	1.40	1.54	1.67	1.81	1.94	2.08
$B/[\hbar^2/Mr_0^2]$	5.8	4.1	3.35	3.25	3.20	3.23	3.25	3.26	3.27
$E_{\min}/B(N+Z)$	+0.013	−0.008	−0.053	−0.093	−0.127	−0.159	−0.187	−0.213	**−0.234**

contributes about 8 MV to the constant β while the main contribution arises from a decrease in magnitude of the potential energy when the numbers of neutrons and protons become different.

The value of β has to be compared to the value derived in §8 from the observed ratio of the numbers of neutrons and protons in heavy nuclei (then denoted by $\alpha\epsilon$, cf. (15), (19b)), *viz.*,

$$\beta=20 \text{ MV} \quad \text{(semiempirical)}, \qquad (166c)$$

which is somewhat less than the two theoretical values (166a), (166b). The agreement is improved by introducing forces between like particles (§28).

The nuclear radius is also somewhat altered when the numbers of protons and neutrons are unequal, because the minimum of $V(n)+T(n)$ (cf. 165a, b) occurs for a slightly different value of x than the minimum of $V(1)+T(1)$. The nuclear radius becomes

$$R=r_0(N+Z)^{\frac{1}{3}}[1+0.665(N-Z/N+Z)^2]. \qquad (167)$$

The correction term containing $N-Z$ is very small, amounting to only $3\frac{1}{2}$ percent even in the case of uranium ($Z=92$, $N=146$).

In concluding this section, we want to mention that quite analogous results are obtained for the simple exponential law of force

$$J(r)=-Be^{-r/a}, \qquad (168)$$

which has been the basis of previous investigations of the statistical model for nuclei by Heisenberg (H9), Wick (W10) and v. Weizsäcker (W2). Instead of (158), we find

$$V=-\pi^{-1}x^{-3}B(N+Z)\{n^{\frac{1}{2}}(2-n)^{\frac{1}{2}}x^2 -(\tfrac{1}{4}+\tfrac{3}{4}x^2[n^{\frac{1}{2}}+(2-n)^{\frac{1}{2}}])\log\frac{1+x^2[n^{\frac{1}{2}}+(2-n)^{\frac{1}{2}}]^2}{1+x^2[n^{\frac{1}{2}}-(2-n)^{\frac{1}{2}}]^2} +2\arctan(x[n^{\frac{1}{2}}+(2-n)^{\frac{1}{2}}]) -2(n-1)\arctan(x[n^{\frac{1}{2}}-(2-n)^{\frac{1}{2}}])\}, \qquad (168a)$$

which for $n=1$ ($N=Z$) reduces to

$$V(n=1)=-\pi^{-1}x^{-3}B(N+Z) \times\{x^2-(\tfrac{1}{4}+\tfrac{3}{4}x^2)\log(1+4x^2) +\arctan 2x\} \qquad (168b)$$

a formula first derived by Heisenberg.[63] Again, we obtain practically no binding energy if we insert the constants B and a derived from the theory of light nuclei. The constants necessary to give the observed binding energy and radius of the Hg^{200} atom, are

$$D=MBa^2/\hbar^2=6.7, \qquad (169)$$

$$x_{\min}=1.75; \quad a=0.64\cdot 1.75r_0=1.53\cdot 10^{-13}\text{ cm}, \qquad (169a)$$

$$B=118 \text{ MV}. \qquad (169b)$$

[63] Heisenberg (H9) gives the energy per unit volume, i.e. (168b) divided by the nuclear volume $(4\pi/3)R^3\cdot(N+Z)$.

These constants are not very different from those obtained by Wick ($B=88$ MV, $a=1.47\cdot 10^{-13}$ cm), as is to be expected since the experimental data used are similar. (The difference in B arises from the fact that Wick did not correct for the surface energy which is rather large for his standard nucleus O^{16}.) Weizsäcker found $B=184$ MV, $a=1.03\cdot 10^{-13}$ cm, $D=4.7_5$, i.e., a very short range and very deep hole. Heisenberg's results deviate from ours to the other side, *viz.*, $B=25$ MV, $a=8\cdot 10^{-13}$ cm.

The dependence of the energy on the "isotopic number" $I=N-Z$ becomes with the exponential potential:

$$V=V(1)+\frac{B}{6\pi x}\frac{(N-Z)^2}{N+Z}(4x^2-\log(1+4x^2)), \qquad (168c)$$

giving for the constant β in (166) a value 42 MV, i.e., almost the same as that obtained from the e^{-r^2/a^2}-potential.

§27. Disproof of Ordinary Forces

In this section we want to prove that the assumption of ordinary forces between neutrons and protons would lead to binding energies of heavy nuclei far larger than those observed. To prove this, we use the statistical model which certainly gives a lower bound for the binding energy.

If we replace the "exchange" (Majorana) by "ordinary" (Wigner) forces the binding energy of the ground state of the deuteron would not be influenced at all, and also that of the α-particle would remain almost unchanged. Thus we obtain practically the same force constants B and a as for Majorana forces.

The expressions (144b), (146b) have to be replaced by

$$V'=\sum_{i=1}^{Z}\sum_{k=1}^{N}\int|\psi_i(x_1)|^2|\varphi_k(x_2)|^2J(r_{12})d\tau_1d\tau_2 \qquad (170)$$

$$=\int J(r_{12})\rho_p(x_1)\rho_n(x_2)d\tau_1d\tau_2, \qquad (170a)$$

where ρ_p is the *ordinary* proton density as defined in (145). With the plane wave functions used in the statistical model, we have obviously

$$\rho_p(r)=\text{const.}=Z/(4\pi R^3/3) \quad \rho_n=N/(4\pi R^3/3) \quad (\text{for } r<R),$$
$$\rho_p=\rho_n=0 \quad (\text{for } r>R). \qquad (170b)$$

We must carry out the integral in (170a) exactly, i.e., we must not consider R to be large compared to the range a of the forces, because this assumption will be contradicted by the result.

With
$$J(r)=-B'e^{-r^2/a^2} \qquad (171)$$

the elementary calculation gives

$$V'=-B'ZN(3/4\pi)^2 8\pi^2R^{-6} \times\int_0^R r_1dr_1\int_0^R r_2dr_2\int_{|r_1-r_2|}^{r_1+r_2}r_{12}dr_{12}e^{-r_{12}^2/a^2} \qquad (171a)$$
$$=-6B'ZNy^{-6}\{\pi^{\frac{1}{2}}y^3\Phi(y)+2-3y^2+e^{-y^2}(y^2-2)\},$$

where
$$y=2R/a. \qquad (171b)$$

For simplicity, we assume an equal number of neutrons and protons, $N=Z$. We divide (171a) by the total kinetic energy $T=(6/5)NT_{0N}$ where T_{0N} is the maximum kinetic energy per particle as given by (150c), and obtain

$$|V'|/T=(5/3\pi)(\pi/3)^{\frac{1}{2}}A^{\frac{1}{3}}(Ma^2B'/\hbar^2)y^{-4} \times\{\pi^{\frac{1}{2}}y^3\Phi(y)+\cdots\}. \quad (171c)$$

Inserting for $Ma^2B'/\hbar^2$ the smallest possible value, *viz.*, 2.70 (cf. Table III, §12) we have

$$|V'|/T>1.50A^{\frac{1}{3}}[\pi^{\frac{1}{2}}y^{-1}\Phi(y)+2y^{-4} -3y^{-2}+e^{-y^2}(y^{-2}-2y^{-4})]. \quad (172)$$

Now we remember that R, and therefore y, is an arbitrary parameter which has to be chosen so that the total energy is a minimum. If we therefore choose *that* value for y which makes the square bracket of (172) a minimum, we can only obtain too *low* a value for the binding energy. The maximum of the square bracket is obtained for $y=2.80$ and has the value 0.288. Thus

$$|V'|/T>0.433A^{\frac{1}{3}}. \quad (172a)$$

Thus for $A>12$ the potential energy becomes larger than the kinetic, even in the poor approximation obtained by the statistical model. If A is much larger than 12, the kinetic energy is only a small fraction of the potential. But the potential energy (171a) is of the order of magnitude A^2B'. In other words, the binding energy of nuclei of atomic weight greater than, say, 30, would increase as the *square* of the atomic weight, in contradiction to experience (§7). Moreover, the total binding energy would attain huge values; e.g., if we insert $a=2.3\cdot10^{-13}$ cm (cf. 128) and $B'=41$ MV (cf. 160a), the minimum of the total energy for uranium is obtained for $y=0.62$ and has the value

$$E_U=(V'+T)_U=-366{,}000 \text{ MV}. \quad (172b)$$

This would correspond to a mass defect of the uranium nucleus of about 380 mass units, i.e., about one and a half times the mass of uranium itself! Since our theoretical value is a lower limit, the impossibility of the assumption of "ordinary" forces which are attractive over their whole range, has been established strikingly enough.

In addition to the very large binding energies, the model gives much too small nuclear radii. For y is approximately constant for all values of A; indeed it even decreases somewhat with increasing A. The nuclear radius becomes therefore independent of A, and is of the same order as the range of the forces; e.g., for uranium the radius would turn out to be about $1.3\cdot10^{-13}$ cm, one-seventh of the observed radius.

The Coulomb forces are, of course, unimportant compared to the huge binding energies resulting from our model and can therefore not alter the conclusions.

Our statement that ordinary forces are impossible, must, however, be qualified in two respects. Firstly, "ordinary" forces which are repulsive at very small distances and attractive at somewhat larger distances can, of course, not be disproved. Plausibility is the only argument against the assumption of such forces between *elementary* particles. (Cf. §31 where such forces are *derived* for the (complex) α-particle.)

Secondly, a *small* "ordinary" force in addition to a large "exchange" force cannot be excluded. We denote by the "strength" of the "ordinary" force in MV and assume the range to be the same for ordinary and exchange force. If we accept the values B and a given in (164c, d) for the exchange force, we find:

Change of nuclear properties caused by a small additional ordinary force

Atomic weight $A=$	10	30	100	238
Relative change of nuclear radius (percent)	$-1.70B'$	$-2.85B'$	$-4.0\ B'$	$-4.6\ B'$
Change of binding energy per particle (MV)	$0.63B'$	$0.87B'$	$1.07B'$	$1.19B'$

An "ordinary" force of $B'=5$ MV might be just tolerable. It would correspond to a difference between the binding energies per particle for U^{238} and Si^{30} of $0.32B'=1.6$ MV, the uranium nucleus having the stronger binding. Such a difference seems about the highest reconcilable with the experimental facts. B' might, of course, be negative and of about the same magnitude. In any case the "ordinary" forces must be very small (not more than about 7 percent) compared to the exchange forces.

§28. Forces Between Like Particles

The total interaction energy due to the forces between like particles has, according to (146e), practically the same form as that due to the interaction between protons and neutrons (146b). As in §21, we assume the shape of the interaction potential K for like particles to be the same as for unlike ones, and the range of the forces to be the same in both cases, so that

$$K(r)=-Ce^{-r^2/a^2}. \quad (173)$$

Then we find in analogy to (158):

$$F=-\tfrac{1}{4}\pi^{-\frac{1}{2}}x^{-3}C(N+Z)\{[4-3x^2(n^{\frac{2}{3}}+(2-n)^{\frac{2}{3}})] -[2-x^2n^{\frac{2}{3}}]e^{-x^2n^{\frac{2}{3}}}-[2-x^2(2-n)^{\frac{2}{3}}] \times e^{-x^2(2-n)^{\frac{2}{3}}}+\pi^{\frac{1}{2}}x^3[n\Phi(xn^{\frac{1}{2}}) +(2-n)\Phi(x(2-n)^{\frac{1}{2}})]\}, \quad (173a)$$

where x has the same meaning as in (158b).

The value of x which makes $T+V+F$ a minimum, depends only on $n=2N/(N+Z)$, and upon the force constants $MBa^2/\hbar^2$ and $MCa^2/\hbar^2$, but *not* upon the total number of particles. This makes, as in §26, the volume of the nucleus and the total binding energy proportional to the number of particles, in agreement with experience. Quantitatively, for $n=\frac{1}{2}$,

$$F/V=C/2B. \tag{174}$$

The total interaction, due to interaction between like *and* unlike particles, is therefore proportional[62] to $B+\frac{1}{2}C$. From the theory of light nuclei (cf. 128), we obtain

$$C/2B=0.35. \tag{174a}$$

Using the values for B, C and a derived in (128), we found the result (160b) for the binding energy which is much too small, although the addition of the interaction between like particles helps somewhat to increase the binding energy following from the statistical model (cf. the result obtained without forces between like particles, above Eq. (160b)).

We again determine the constants *ad hoc* from the statistical model, as in §26, Eq. (164). The only alteration necessary is that we must now make $B+\frac{1}{2}C$ as large as B was in §26. Keeping the ratio $B:C$ as given by the theory of light nuclei, *viz.* $\frac{1}{2}C=0.35B$, the B of §26, Eq. (164d) should be reduced by a factor 1/1.35, giving

$$\begin{aligned} B&=70/1.35=52 \text{ MV}\\ C&=0.70\cdot 52=36 \text{ MV}\\ a&=2.37\cdot 10^{-13} \text{ cm} \end{aligned} \tag{175}$$

(from statistical model, with forces between like particles equal to 0.70 times the neutron-proton forces).

The interaction of like particles has some influence on the dependence of the nuclear energy on $N-Z$. As might be expected, the potential energy of this interaction decreases (i.e., its absolute magnitude becomes *larger*) when the numbers of neutrons and protons become different. The constant β defined in (166) consists now of three parts, arising respectively from the forces between protons and neutrons, the forces between like particles, and the kinetic energy. For the values of the constants B, a, C chosen in (175) we have

$$\begin{aligned} \beta&=0.44B+0.39\hbar^2/Mr_0^2-0.070C\\ &=23+8-1=30 \text{ MV}, \end{aligned} \tag{176}$$

while the constants (128) would give

$$\beta=22 \text{ MV} \tag{176a}$$

as compared to 39 and 28 MV, respectively, when no forces between like particles are assumed (cf. 166a, 166b). The semiempirical value of §8, *viz.*, 20, is almost identical with (176a).

In addition to the "specific nuclear forces" between like particles we have also the electrostatic repulsion between the protons which we found responsible for the increase of the ratio N/Z with increasing atomic weight (§8). We assume the protons to be uniformly distributed over the nucleus so that their density is $Z/(4\pi R^3/3)$; then their interaction, without taking into account exchange, is (cf. 146c)

$$\begin{aligned} C&=\tfrac{1}{2}Z^2\int(e^2/r_{12})(4\pi R^3/3)^{-2}d\tau_1 d\tau_2\\ &=\tfrac{3}{5}e^2Z^2/R, \end{aligned} \tag{177}$$

each of the space integrals $d\tau_1 d\tau_2$ extending over a sphere of radius R. The result has to be corrected for exchange (term A, cf. (146d)). With the expression (151a) for the mixed density, and with the assumption $k_0R\gg 1$ (i.e., large atomic weight), we obtain[64]

$$\begin{aligned} A&=\frac{1}{4}\frac{4\pi}{3}R^3\int_0^\infty \frac{e^2}{r_{12}}\frac{4\pi r_{12}^2 dr_{12}}{(\pi^2 r_{12}^3)^2}\\ &\quad\times(\sin k_P r_{12}-k_P r_{12}\cos k_P r_{12})^2\\ &=e^2R^3k_P^4/3\pi^2=3^{5/3}2^{-8/3}\pi^{-2/3}e^2Z^{4/3}R^{-1}. \end{aligned} \tag{177a}$$

Expressing R in terms of r_0 (cf. 153) we have for the total effect of the electrostatic forces

$$\begin{aligned} C-A&=(e^2/r_0)(Z/N+Z)^{1/3}\\ &\quad\times(\tfrac{3}{5}Z^{5/3}-3^{5/3}\pi^{-2/3}2^{-8/3}Z) \end{aligned} \tag{177b}$$

$$=(e^2/r_0)(Z/N+Z)^{\frac{1}{3}}(0.600Z^{5/3}-0.460Z). \tag{177c}$$

With our value $r_0=1.37\cdot 10^{-13}$ cm, deduced from experiment, we have

$$e^2/r_0=1.08 \text{ MV}. \tag{177d}$$

[64] Weizsäcker's expression (W2, Eq. (50)) is too large by a factor $(4\pi/3)^{\frac{1}{3}}$.

The exchange term is proportional to the nuclear charge, and amounts to only about 0.18 MV per nuclear particle. The first term, which increases as $Z^{5/3}$, reaches the value 890 MV for uranium, i.e., $3\frac{3}{4}$ MV per particle in the U nucleus, which reduces the binding energy of that nucleus by about one-third (cf. the deduction of the nuclear forces from the mass defect of Hg in §26).

The Coulomb forces have also some effect on the nuclear radius. The relative change of the latter due to Coulomb forces is, using (159), (159a) and putting $B=70$ MV:

$$\delta R/R=+0.60(e^2/Br_0)Z^2A^{-4/3}=0.0092Z^2A^{-4/3}. \quad (177e)$$

For uranium, this amounts to 5.3 percent, for Fe^{56} to 2.8 percent. We found in (167) that the radius of uranium should be increased by 3.5 percent because the numbers of neutrons and protons are not equal. Altogether, we should thus expect that the radius of U is (3.5+5.3−2.8) percent =6 percent larger than would be expected from the radius of the Fe nucleus, assuming the nuclear volume to be proportional to the number of particles. However, it must be borne in mind that the nuclear radius cannot easily be defined to such an accuracy.

§29. The Surface Effect (W2, W10)

Wick (W10) has first pointed out that the nuclear binding energy will be reduced, especially for small nuclei, by the existence of a surface of the nucleus. Particles at the surface interact, in the average, only with half as many other particles as do particles in the interior of the nucleus. The situation is, of course, quite analogous to the surface tension of liquids.

Weizsäcker (W2) has calculated the surface effect quantitatively. He has shown that the effect consists of a "classical" and a "quantum mechanical" part. Classically, we may assume the nucleus to have a sharp boundary at a certain distance R from the center. Those particles which are nearer to the boundary than the range of the forces a, will then not contribute their full share to the binding energy. This leads to an increase of the total energy by an amount of the relative order a/R (classical surface effect).

In quantum theory, the boundary can never be sharp, because this would mean an infinite derivative of all particle wave functions and consequently an infinite kinetic energy. The surface layer must thus be spread out over a certain region of the order of magnitude of one particle wave-length. The narrower the surface region, the larger will be the additional kinetic energy; the broader the region, the more will the total potential energy be reduced in magnitude. Therefore, there will be an optimum breadth of the surface layer.

Weizsäcker (W2) has calculated the surface effect by an extension of the statistical model. If we use the statistical model in its ordinary form we find no surface effect at all. To see this, we assume that there is a certain "auxiliary potential energy" U which has a given negative value $-U_0$ inside a sphere of radius R (nuclear radius) and then rises gradually to zero outside that sphere. We assume the rise of the potential to be gradual enough so that the statistical method is applicable; i.e., we suppose that we may choose volume elements $d\tau$ so small that the potential energy U is sufficiently nearly constant inside $d\tau$, and on the other hand so large that we may apply the considerations of §25 to each volume element. This condition means essentially that the thickness of the surface layer, i.e., the region in which the auxiliary potential U changes from $-U_0$ to 0, must be *large compared to the wave-length of the particles in the nucleus.*

If this condition is fulfilled, the particle density at any point $\mathbf{r}$ (or in any volume element $d\tau$), as well as the maximum and the average kinetic energy of the particles at that point, and the contribution of $d\tau$ to the total potential energy are all completely determined by the value of the "auxiliary potential" U at that point. We have (cf. (150))

$$\rho(\mathbf{r})=N/(4\pi R^3/3)=k_{\max}{}^3(\mathbf{r})/3\pi^2 =(2M[E_0-U(\mathbf{r})]\hbar^{-2})^{\frac{3}{2}}/3\pi^2, \quad (178)$$

where
$$(\hbar^2/2M)k_{\max}{}^2=E_0-U(\mathbf{r}) \quad (178a)$$

is the maximum kinetic energy of any particle at the point $\mathbf{r}$, E_0 being the total energy of the most energetic particle. The average kinetic energy of the particles at $\mathbf{r}$ is

$$\frac{3}{5}\frac{\hbar^2}{2M}k_{\max}{}^2=\frac{3^{5/3}\pi^{4/3}}{10}\frac{\hbar^2}{M}\rho^{2/3}. \quad (178b)$$

Multiplying this expression by ρ, we obtain the kinetic energy per unit volume.

The mixed density is given by (151a), with only k_0 to be replaced by $k_{\max}$, provided again

k_{max} does not change appreciably between the two points $\mathbf{r}_1$, $\mathbf{r}_2$, i.e., provided the auxiliary potential changes sufficiently slowly. The contribution to the potential energy per unit volume can then be calculated similarly to (158) or (159) if we assume the density of neutrons and protons to be equal at every point. We have only to divide (159) by the total nuclear volume $(4\pi/3)(N+Z)r_0^3$, and to express, in the definition (158b) of x, the quantity r_0 by the density ρ of protons or neutrons. Since r_0 was defined as the radius of a sphere containing one particle, i.e., in the average one-half neutron and one-half proton, we have

$$(4\pi/3)r_0^3=\tfrac{1}{2}\rho^{-1}, \tag{178c}$$

$$x=(3\pi^2\rho)^{\frac{1}{3}}a. \tag{178d}$$

The *total* energy per volume element $d\tau$ becomes

$$dE=-\tfrac{2}{3}\pi^{-5/2}a^{-3}B\{2-3x^2-(2-x^2)e^{-x^2} +\pi^{1/2}x^3\Phi(x)\}d\tau+3^{5/3}\pi^{4/3}(\hbar^2/10M)\rho^{5/3}d\tau. \tag{178e}$$

The total energy *per particle* in the volume element $d\tau$, *viz.*, $dE/(\rho d\tau)$, is, as we know, a minimum if x has the value x_{min} derived in §26 (Table VII). The value of ρ corresponding to x_{min} will be called the standard density ρ_0; it will be the density in the interior of nuclei. Any region of the nucleus in which the density ρ falls short of its standard value ρ_0 will increase the total energy of the nucleus over its value derived in §26.

From this point of view, the total energy of the nucleus would attain its minimum value if the density is ρ_0 throughout the nucleus, falling to zero suddenly at the boundary. This minimum value would be exactly equal to the energy derived in §26; thus there would be no surface effect at all, as we mentioned before. However, in order to make the statistical method applicable, we had to assume that the density varies *slowly* at the boundary of the nucleus, more accurately that is does not drop from ρ_0 to 0 in a distance shorter than the wave-length of the nuclear particles, which is of the order of r_0. If the density changes too rapidly, the kinetic energy becomes much larger than its value in the statistical model.

We therefore assume that the *drop of the density from ρ_0 to 0* occurs in a layer of a thickness λ, of the order of magnitude r_0. Furthermore, we assume the "auxiliary potential" U to fall off linearly in the surface region. This means, according to (178), that ρ varies as the 3/2-power of the distance from the surface. Denoting by y the coordinate perpendicular to the surface, we thus assume

$$\begin{aligned}&\rho=0 && \text{for } y<0,\\ &\rho=\rho_0(y/\lambda)^{3/2} && \text{for } 0<y<\lambda,\\ &\rho=\rho_0 && \text{for } \lambda<y.\end{aligned} \tag{179}$$

It is then easy to calculate the change of the nuclear energy due to the surface effect. We denote by S the total surface of the nucleus, by Ω its volume, where Ω is defined by

$$\Omega\rho_0=N, \tag{179a}$$

N being the total number of neutrons. T_0 is the total kinetic energy without surface effect, given in §26. Then we obtain a decrease of the kinetic energy[65] by

$$\delta T=-(4/35)T_0S\lambda/\Omega \tag{179b}$$

and an increase of the potential energy by

$$\delta V=\tfrac{3}{5}\pi^{-\frac{1}{2}}B(N+Z)(S\lambda/\Omega)x_0^{-3}(\tfrac{1}{2}x_0^2-2 +3x_0^{-2}-(3x_0^{-2}+1)e^{-x_0^2}), \tag{179c}$$

where x_0 is the value of x making the nuclear energy a minimum. We insert (cf. 164d, 161b)

$$S=4\pi R^2, \quad \Omega=4\pi R^3/3, \tag{179d}$$

$$B=70 \text{ MV}, \quad T_0=15(N+Z) \text{ MV}, \quad x_0=2.70 \tag{179e}$$

and obtain

$$\begin{aligned}\delta T&=-5.1(N+Z)\lambda/R \text{ MV},\\ \delta V&=+8.3(N+Z)\lambda/R \text{ MV},\\ \delta E&=+3.2(N+Z)\lambda/R \text{ MV},\end{aligned} \tag{180}$$

or, using (153)

$$\delta E=3.2(N+Z)^{\frac{1}{3}}\lambda/r_0 \text{ MV}. \tag{180a}$$

λ/r_0 should, according to our assumptions, be of the order of magnitude unity. A considerably larger value seems necessary to obtain agreement with experiment (cf. §30, especially (185a)).

Weizsäcker has tried to determine λ theoretically. He supposes that each individual wave function contains an exponentially decreasing factor near the surface of the nucleus, but behaves otherwise in the same way as the

[65] This is due to the reduced *density* in the surface layer.

statistical method assumes. The exponential decay introduces an additional term in the kinetic energy. It is then assumed that the density corresponding to each individual state decreases in the same way as the total density. Obviously the additional term in the kinetic energy will then be the larger the thinner the surface layer. On the other hand, the term (179c) is the larger the thicker the surface layer. The condition that the sum of the two terms shall be a minimum, leads immediately to a determination of the thickness of the surface layer and of the additional surface energy. However, the basic assumption that the decrease in density at the surface is due to a uniform decrease of the density due to each individual state, does not seem to be justified: In reality when we approach the surface one state after the other "dies out" because its total energy becomes less than the potential energy at the given point; and this dying out accounts for the decrease in density without any exponential decay of the individual wave functions being necessary. (The exponential tail of each wave function can be neglected in the statistical method.)

Flügge (F12) has used Weizsäcker's method to treat light nuclei, for which the surface layer cannot be considered as thin compared to the nuclear radius so that the distinction of the "interior" of the nucleus and the "surface layer" is no longer justified. He finds that the mass defects of all light nuclei from He^4 to Si^{28} can be well represented by Weizsäcker's extension of the statistical method described above, the potential energy of the interaction between and proton being assumed as

$$J(r) = -Be^{-r^2/a^2}$$

with $$B = 85 \text{ MV}, \quad a = 1.46 \cdot 10^{-13} \text{ cm}. \tag{181}$$

The density of neutrons and protons is supposed to depend like a Gaussian function on the distance r from the center of the nucleus, *viz.*,

$$\rho(r) = e^{-2r^2/R^2}, \tag{181a}$$

where R is the "nuclear radius." R is not exactly proportional to the cube root of the atomic weight, as it is for heavy nuclei, but is somewhat larger for light nuclei, *viz.*,

$$R = (0.67A^{\frac{1}{3}} + 0.93A^{-\frac{1}{3}}) \cdot 10^{-13} \text{ cm}. \tag{181b}$$

Accordingly, the volume of light nuclei is larger than would be expected if the volume were exactly proportional to the number of particles.

Flügge has also carried through calculations using a simple exponential potential, with similar results.

§30. Weizsäcker's Semiempirical Formulae (W2)

Since the statistical model does not give satisfactory results, Weizsäcker has proposed a semiempirical method for calculating nuclear energies. The nuclear energy is assumed to have a *form* such as is indicated by the statistical method, and, indeed, by very simple qualitative considerations. However, in the formula certain *constants* are left arbitrary and are determined from experimental data.

We choose the following form for the *total mass* (energy) of an *atom* which is slightly simpler than Weizsäcker's:

$$M = NM_n + ZM_p - \alpha A + \beta(N-Z)^2/A + \gamma A^{\frac{2}{3}} + \tfrac{3}{5}(e^2/r_0)Z^2A^{-\frac{1}{3}}, \tag{182}$$

where A is the atomic weight, N and Z the numbers of neutrons and protons, $r_0A^{\frac{1}{3}}$ the nuclear radius, M_n and M_p the masses of neutron and hydrogen atom and $\alpha\beta\gamma r_0$ empirical constants. The first two terms in (182) are self-evident. The third represents the main binding energy which we know to be proportional to the number of particles in the nucleus A, the constant α to be determined empirically.

The fourth term in (182) represents the decrease of the binding energy when the numbers of protons and neutrons become different; it has the form derived in §26 from the statistical model. This form holds, of course, only if $N-Z \ll A$; but this condition is fulfilled for all existing nuclei.

The fifth term is the surface effect, the last term the Coulomb repulsion of the protons. Both these terms have again the form suggested by the statistical model. The exchange correction to the electrostatic repulsion (177a) may be considered as contained in the first term since it is proportional to Z.

To determine the constants, we proceed in the following way:

1. We determine r_0 from the empirical radii of radioactive nuclei this gives (cf. 17a, b)

$$r_0 = 1.48 \cdot 10^{-13} \text{ cm} \tag{182a}$$

$$\tfrac{3}{5}e^2/r_0 = 0.58 \text{ MV}. \tag{182b}$$

2. β is determined so that the most stable nucleus of atomic weight 200 has the nuclear charge 80 (Hg^{200}). This gives

$$\beta = \left[\frac{3}{5}\frac{e^2}{r_0}\frac{Z}{A^{\frac{1}{3}}} + \frac{1}{2}(M_p - M_n)\right]\frac{A}{2(N-Z)}$$

$$= \left(0.58 \cdot \frac{80}{200^{\frac{1}{3}}} - 0.20\right)\frac{200}{80} \text{ MV},$$

$$\beta = 19.5 \text{ MV}. \tag{183}$$

With this value for β, the most stable nucleus of atomic weight A has the "isotopic number"

$$I_A=(N-Z)_A=A\frac{0.3(e^2/r_0)A^{\frac{2}{3}}+\frac{1}{2}(M_p-M_n)}{2\beta+0.3(e^2/r_0)A^{\frac{2}{3}}},$$
$$I_A=A(A^{\frac{2}{3}}-0.7)/(134+A^{\frac{2}{3}}), \tag{183a}$$

and the mass

$$M_{\min}(A)=\tfrac{1}{2}A(M_n+M_p)-\alpha A+\gamma A^{\frac{2}{3}}$$
$$+\frac{3}{20}(e^2/r_0)A^{5/3}-A\frac{[0.3(e^2/r_0)A^{\frac{2}{3}}-\frac{1}{2}(M_n-M_p)]^2}{2(2\beta+0.3(e^2/r_0)A^{\frac{2}{3}})} \tag{183b}$$
$$=\tfrac{1}{2}A(M_n+M_p)-\alpha A+\gamma A^{\frac{2}{3}}$$
$$+0.145A^{5/3}\frac{135}{134+A^{\frac{2}{3}}}.$$

In the last transformation, some very small terms involving the difference of the masses of neutron and hydrogen atom have been neglected. The figures represent energies in MV.

3 and 4. We determine the coefficients α and γ so that the masses of O^{16} and Hg^{200} are correct. We have for O^{16}: atomic weight 16.0000; $8M_n+8M_p=16.1322$.

Difference	-0.1322 mass unit $=-123.1$ MV,
Coulomb energy	$0.58\cdot 8^2\cdot 16^{-\frac{1}{3}}=14.7$ MV,
The term	$\beta(N-Z)^2/A$ vanishes.

Therefore

$$-16\alpha+16^{\frac{2}{3}}\gamma=-(123.1+14.7)=-137.8 \text{ MV}. \tag{184a}$$

For Hg^{200}: atomic weight 200.016; $120M_n+80M_p=201.661$.

Difference	-1.645 mass units $=-1532$ MV,
Coulomb energy	$0.58\cdot 80^2\cdot 200^{-\frac{1}{3}}=633$ MV,
	$\beta(N-Z)^2/A=19.5\cdot 40^2/200=156$ MV.

Therefore

$$-200\alpha+200^{\frac{2}{3}}\gamma=-(1532+633+156)$$
$$=-2321 \text{ MV}. \tag{184b}$$

From (184a), (184b) we find

$$\alpha=13.86 \text{ MV}, \quad \gamma=13.2 \text{ MV}. \tag{184}$$

We convert all energies into thousandths of a mass unit, and insert the constants into (182). Then the excess of the exact atomic weight of an atom (A, Z) over the "mass number" A is, in thousandths of a mass unit:

$$E=1000(M-A)=8.0_5Z+8.4_5N-14.9A$$
$$+21(N-Z)^2/A+14.2A^{\frac{2}{3}}+0.625Z^2A^{-\frac{1}{3}} \tag{185}$$

or

$$E=-6.6_5A+0.4I+21I^2/A+14.2A^{\frac{2}{3}}$$
$$+0.625Z^2A^{-\frac{1}{3}}, \tag{185a}$$

where $I=N-Z=A-2Z$ is the isotopic number.

As Weizsäcker has pointed out, this formula is immediately applicable only to nuclei with even numbers of neutrons and protons. Nuclei containing an odd number of either neutrons or protons have higher mass (less binding energy). This can be seen by the following argument. The energy (185a) is, for a given number of protons Z, very nearly a quadratic function of the number of neutrons N. Therefore, any further neutron which may be bound to a given nucleus, would be bound less strongly than the preceding neutron. Actually, however, if we have a nucleus containing an even number of neutrons and protons, it will always bind *two* additional neutrons with the *same* binding energy, because they both can be bound in the same state. Therefore, the energy of a nucleus containing an *odd* number of neutrons is to be computed by taking the arithmetic mean between the energies of the two adjacent nuclei with even numbers of neutrons. The same is true for nuclei containing an odd number of protons. The energy of nuclei with both N and Z odd are to be obtained by double interpolation.

Table VIII gives the mass excess for some nuclei throughout the periodic table in thousandths of mass units. It shows the relative importance of main energy, surface energy and Coulomb repulsion. The agreement with experiment is rather satisfactory. The incomplete agreement for the standard O^{16} is due to rounding off in (185).

Table IX gives the mass excess of the known light nuclei, calculated and observed. The values derived by Weizsäcker from his semiempirical formula are also given. They agree somewhat better with the observed values than ours. The reason is that Weizsäcker adjusted his constants so that the masses of light nuclei are represented as well as possible. Weizsäcker's formula is

$$E=[-(\alpha^2+\beta^2)^{1/2}+(\alpha^2+\beta^2((Z-N)/A)^2)^{1/2}]\times[A-1-\gamma(A-1)^{2/3}]+(3e^2/r_0A^{1/3})(\tfrac{1}{5}Z^2-2^{-4/3}Z^{4/3}) \quad (185b)$$

with $\alpha=1.6$, $\beta=13.9$, $\gamma=0.6$ thousandths of a mass unit, and $r_0=1.26\cdot10^{-13}$ cm.

The general trend of the nuclear masses seems to be represented fairly well by the theoretical formulae. Notable exceptions are the lightest nuclei, for which the formulae cannot be expected to hold, and a marked break near oxygen. While the experimental energies are in the average about equal to the theoretical ones up to O^{16}, and in some instances (Be^8 C^{12}) even lower, they are very much higher than the theoretical energies for the nuclei between O and Al. The differences reach 6 to 8 milli-mass units. For still heavier nuclei, the agreement improves again, and is almost complete for sulphur. The reason for all these facts seems to be the completion of a "closed neutron shell" at O^{16} (cf. §33).

The differences between the energies of isobars such as $C^{13}N^{13}$, $N^{15}O^{15}$, $O^{17}F^{17}$, etc., seem to agree well with the experimental values deduced from the upper limit of β-ray spectra (§39). In agreement with experiment, B^{10} turns out to be more stable than Be^{10}, N^{14} more stable than C^{14}, but F^{18} less stable than O^{18}.[66] However, there are also notable discrepancies, especially for heavier nuclei of even atomic weight; e.g., the nuclei Si^{28} and Al^{28} should be equally heavy according to our semiempirical formula, while actually Al^{28} is 5 units heavier, P^{30} should be 4 units heavier than Si^{30} and is actually 6 units heavier, P^{32} should be 1 unit lighter than S^{32} and is 2 units heavier. This shows that our method does not give a big enough difference between nuclei of even mass and odd charge and such of even mass and even charge especially for heavier nuclei. It therefore points to the necessity of introducing forces between like particles (§10, 18, 21, 28).

For the lightest nuclei maximum stability is found for those which contain an integral number of α-particles (Be^8, C^{12}, O^{16}). For higher atomic weight these differences in stability gradually disappear, and the maximum of stability is shifted towards nuclei containing more neutrons than protons (compare Be^8, Be^9, Be^{10} to S^{32}, S^{33}, S^{34}). It should be noted that the exceptional stability of light nuclei containing an integral number of α-particles is obtained without the assumption that there are actually α-particles as secondary units in the nucleus. We have only made the rather obvious assumption that neutrons and protons form shells of two, each shell containing two particles of opposite spin and equal spatial wave function.

TABLE VIII. *Masses of some nuclei calculated from the semi-empirical formula (185), in thousandths of a mass unit.*

Atom	Mass Excess of Consti.*	Main Energy	Term in $(N-Z)^2$	Surface Energy	Coulomb Energy	Total Theor.	Total Exp.
O^{26}	+132.0	−238.4	0	90.1	15.8	−0.5	0.0
A^{40}	329	−596	8	166	59	−33	−29
K^{82}	329	−1222	25	268	186	−78	−73
Xe^{134}	1101	−1997	106	372	354	−62	−71
Hg^{200}	1658	−2980	168	486	680	+16	+16
U^{238}	1953	−3547	257	545	838	+52	+99**

* $8.0_6Z+8.4_8N$, i.e., the excess of the masses of the neutrons and protons contained in the nucleus over the "mass number" $Z+N$.
** Computed from the energies of the α-particles emitted in the uranium series, and an assumed mass of Pb^{206} of 206.020.

We want to use our empirical formula for a redetermination of the limit of stability against α-disintegration (cf. §8). We take the most stable nucleus of given atomic weight A, whose energy is given by (183b). Inserting the values of α and γ from (185), we find for the mass excess of the most stable nucleus of atomic weight A in thousandths of a mass unit

$$E_{\min}(A)=-6.65A+14.2A^{2/3}+0.156A^{5/3}135/(134+A^{2/3}). \quad (186)$$

The condition for α-instability is

$$E_{\min}(A)-E_{\min}(A-4)>3.35 \quad (186a)$$

because 3.35 thousandths of a mass unit is the mass excess of the helium atom. This gives

$$-26.6+37.9A^{-\frac{1}{3}}+0.625A^{\frac{1}{3}}\times\frac{135\cdot((5/3)\cdot134+A^{\frac{1}{3}})}{(134+A^{\frac{1}{3}})^2}>3.35. \quad (186b)$$

This condition is fulfilled for

$$A>147, \quad (186c)$$

[66] This seems to show that the break in the isotope pattern near O (cf. §34) is *not* connected with the completion of a closed shell at O^{16}. For the latter fact is *not* represented in our calculation, while the former comes out automatically since the greater stability of O^{18} compared to F^{18} is sufficient to explain the change in the isotope pattern.

TABLE IX. *Mass excesses of light nuclei calculated from the semiempirical formula (185), in thousandths of a mass unit.*

Nucleus	Theor.	Weizsäcker	Exp.	Nucl.	Theor.	Weiz.	Exp.	Nucl.	Theor.	Weiz.	Exp.	Nucl.	Theor.	Weiz.	Exp.
He^3	29.3	13.3	16.4	He^4	10.8	9.6	3.4	He^5	16.8	14.1	—	He^6	22.8	18.6	—
Li^5	17.5	15.3	—	Li^6	16.5	13.9	16.1	Li^7	15.5	12.7	16.9	Li^8	25.1	17.6	18.3
Be^7	16.5	13.7	—	Be^8	8.3	7.5	7.0	Be^9	10.4	10.1	13.9	Be^{10}	12.5	13.6	15.4
B^9	12.5	12.1	15.5	B^{10}	10.5	10.4	14.6	B^{11}	8.5	8.5	11.1	B^{12}	13.2	12.8	16.6
C^{11}	10.6	10.3	14.2	C^{12}	4.6	3.5	3.7	C^{13}	4.7	6.0	6.9	C^{14}	4.9	8.5	7.8
N^{13}	7.5	8.7	10.0	N^{14}	4.8	6.5	7.6	N^{15}	2.2	4.3	5.3	N^{16}	4.6	7.9	7.5
O^{15}	5.0	7.4	8.6	O^{16}	−0.4	0.4	0.0	O^{17}	−1.0	1.8	4.0	O^{18}	−1.7	3.4	4.5
F^{17}	2.4	5.5	7.8	F^{18}	−0.5	3.1	>5.5	F^{19}	−3.4	0.4	+4.0	F^{20}	−2.6		5.4
Ne^{19}	0.0	—	—	Ne^{20}	−5.2	−2.3	−0.2	Ne^{21}	−6.7		−0.5	Ne^{22}	−8.3		−2.2
				Na^{22}	−5.9		−0.3	Na^{23}	−9.2		−2	Na^{24}	−9.2		−0.5
				Mg^{24}	−10.2		−6.1	Mg^{25}	−12.2		−6.5	Mg^{26}	−14.2		−7.5
				Al^{26}	−10.8		−3.5	Al^{27}	−14.5		−10.5	Al^{28}	−15.2		−10.0
				Si^{28}	−14.8		−15.	Si^{29}	−17.2		−16	Si^{30}	−19.6		−17.5
				P^{30}	−15.7		−11.5	P^{31}	−19.5		−20	P^{32}	−21.0		−20.5
				S^{32}	−19.5		−22.5	S^{33}	−22.4		−23.5	S^{34}	−25.3		−25.5

Note: All experimental data on nuclei of atomic weight greater than 17, with the exception of F^{19} and Ne^{22}, are tentative only. They are based on the scarce scattered data about transmutations of these heavier elements, and partly only on interpolation. Errors up to about 3 units in the difference between neighboring elements, and maybe 10 units in the absolute values, seem possible.

i.e., we obtain practically the same condition for α-instability as in §8 when we neglected the surface effect. An estimate of the average kinetic energy of α-particles emitted by radioactive nuclei is obtained by inserting into the left-hand side of (186a) an average atomic weight for radioactive nuclei, say, 226 (radium). Then we have in satisfactory agreement with the experimental average energy of radioactive α-particles

$$Q_\alpha = E_{\min}(226) - E_{\min}(222)$$
$$-3.35 = 3.8 \text{ MV}. \quad (186d)$$

VI. More Detailed Theory of Heavier Nuclei

Not many definite results concerning the details of the structure of heavier nuclei have yet been obtained. We shall discuss in this chapter only a few of the attempts to obtain such a theory, and only those which we consider likely to become starting points for future development.

§31. α-Particles as Subunits of Heavier Nuclei

The following arguments have been given for the assumption that α-particles exist in heavier nuclei as subunits:

1. The mass defect per particle is for all heavier nuclei of the same order of magnitude as for the α-particle. In other words, when a heavy nucleus (atomic weight between 20 and 200) is built up, most of the binding energy is set free when groups of two neutrons and two protons are combined into α-particles (27 MV per α-particle) and only a relatively small additional energy (about 7 MV per α-particle) is gained when the α-particles are put together in the heavy nucleus.

2. The assumption of α-particles as subunits therefore seems to offer a straightforward method for a theoretical calculation of the binding energy of heavier nuclei: Already in "zero approximation," the binding energy of the heavy nucleus would be the sum of the binding energies of the α-particles contained in it; and if it can be shown that α-particles attract each other, there will be justified hope to arrive at a theoretical binding energy reasonably near the observed one in the next approximation. In contrast to this, the "Hartree" approximation which assumes the elementary particles to move independently in the nucleus (§§32 to 35) fails to lead to satisfactory results for the binding energy, whether it is used in the crude form of the statistical method (chapter V) or in the more elaborate one described in this chapter (§§35, 36).

3. Among the light nuclei, those which can be regarded as consisting exclusively of α-particles, i.e., Be^8, C^{12}, O^{16}, Ne^{20}, etc., have higher binding energies per particle than any of their neighbors.

4. Radioactive nuclei emit α-particles.

However, the last two arguments can easily be refuted, and in the course of refuting argument 3 we shall come across some strong arguments against the existence of α-particles as subunits.

Ad 4: This argument is not at all conclusive, because it can be shown by very simple considerations involving energy and probability only, that *α-particles are the only particles which can be emitted from heavy radioactive nuclei.* Firstly, the internal energy of the α-particle is much lower than that of the preceding nuclei H^1, H^2, H^3, He^3. Therefore a given nucleus Z^A (Z=nuclear charge, A=atomic weight) may have higher energy than the nuclei $(Z-2)^{A-4}$ and He^4 together, but will in general have lower energy than, say, $(Z-1)^{A-3}$ plus H^3. Thus it is energetically unstable against emission of an α-particle, but stable against emission of any of the lighter particles. Of course, the nucleus Z^A will in general be energetically unstable against the breaking-up into $(Z-6)^{A-12}$ and C^{12}, or $(Z-8)^{A-16}$ and O^{16} if it is unstable against α-emission. But here the second point, *viz.*, the probability considerations, set in: It is almost impossible that such a heavy particle as C^{12} "leaks through" the high and broad potential barrier existing between it and the residual nucleus (chapter IX), while the comparatively light α-particle with its comparatively small charge may leak through quite easily.

Ad 3: This rule may be explained without assuming α-particles as subunits. For there are two main principles governing the structure of nuclei: Firstly, the overwhelming strength of the neutron-proton interaction which, for small atomic weight, makes those nuclei most stable which contain equally many neutrons and protons (§6). Secondly, the "even-odd rule" (§10) stating that nuclei are most stable if they contain even numbers of neutrons and protons, the reason being the Pauli principle (§30) and probably in addition attractive forces between like particles (§10, 18, 21). The nuclei containing exclusively α-particles are favored by both these points which explains their particular stability. In fact, we could even account quantitatively for the difference between the binding energies of these nuclei and their neighbors without assuming α-particles as subunits (§30).

After having disposed of arguments 4 and 3, we shall give a more general argument against α-particles as nuclear subunits. For nuclei heavier than about 30, the preference for nuclei composed exclusively of α-particles ceases to exist. The reason is of course the Coulomb repulsion of the α-particles; this repulsion makes it necessary that stable heavy nuclei contain some extra neutrons as "mortar" keeping the α-particles together.

This fact in itself does not speak against the existence of α-particles as subunits. However, as far as the rather scarce experimental evidence goes, it seems that the binding energy of these additional neutrons is materially the same as the interior binding energy of an α-particle per particle, i.e., 7 to 8 MV. If the concept of α-particles as nuclear subunits were a good approximation, we would expect that all interactions between α-particles and additional neutrons, or between pairs of α-particles, must be small compared to the internal binding energy of the α-particle. A binding energy of 8 MV per additional neutron must correspond to a large perturbation of the α-particles, so that it becomes very doubtful to what extent one may speak of their existence as subunits in nuclei at all. This holds at least for nuclei which contain a large number of extra neutrons.

Another argument which points in the same direction may be drawn from Heisenberg's attempt to calculate the interaction between two α-particles. This interaction will, similarly to chemical interactions, consist of two parts, the exchange interaction and the van der Waals interaction. The exchange interaction is obtained by averaging the mutual potential energy of all the individual particles over the unperturbed eigenfunction of the interacting systems (molecules or α-particles), taking due account of the Pauli principle. The van der Waals interaction is connected with a mutual polarization of the two interacting systems. Now the α-particle is a "closed-shell" system, analogous to the rare gas atoms in chemistry [i.e., protons and neutrons in the α-particle fill all places in the lowest quantum state ($1s$ state)]. Accordingly, the exchange interaction between two α-particles, or between an α-particle and an elementary particle, must be repulsive, just as between a rare gas atom and

another atom (rare gas or otherwise). The reason is that the eigenfunction must be antisymmetric with respect to interchange of a neutron or proton in the α-particle, and one in the system interacting with it, because of the Pauli principle. This introduces nodes into the wave function which lead to increased total energy. The van der Waals forces are always attractive.

In molecular theory, the repulsive exchange forces are very much stronger than the van der Waals forces at close distances, making molecules practically impenetrable for each other. On the other hand, the van der Waals forces extend to much larger distances. For the exchange forces exist only if the wave functions of the two interacting molecules overlap, while the van der Waals forces are not subject to this condition. The exchange forces between molecules therefore fall off exponentially with increasing distance of the molecules, whereas the van der Waals forces behave as a *power* of the distance r, usually r^{-6}. The reason for the slow decrease of the van der Waals forces is the very slow decrease of the force between two individual electrons, *viz.*, e^2/r.

In the case of nuclei, the force between elementary particles falls off very rapidly. We shall therefore expect the van der Waals forces to have a range only slightly larger than the exchange forces. Roughly, we may expect the range of the exchange forces between two α-particles to be about equal to the diameter of the α-particle, while the van der Waals forces will extend over a distance about equal to that diameter *plus* the range of the forces between neutron and proton. Since the latter is certainly not larger and probably considerably *smaller* than the diameter of the α-particle, the difference will not be great.

We shall thus obtain an interaction potential between two α-particles which has the following characteristics: There will be a strong repulsion at close distances, a not quite so strong attraction at slightly larger distances, and the Coulomb repulsion at great distances. Such a shape of the potential seems to agree qualitatively with the scattering of α-particles by α-particles (chapter X).

Heisenberg has computed the interaction between two α-particles, assuming a potential Be^{-r^2/a^2} between a neutron and a proton and no interaction between like particles. For each α-particle a wave function of the form (117a) was assumed. The result of Heisenberg's calculation is very surprising: The exchange force acts only if the two α-particles coincide exactly,[67] and the van der Waals force has the same range as the neutron-proton force. It does not seem clear whether this peculiar result is due to the particular form of interaction potential and wave functions (Gaussian functions!) or to the approximations made in Heisenberg's derivation. Detailed investigation will be needed to clear up this point.

However, it seems at least certain from Heisenberg's calculations that both exchange and van der Waals forces become of the same order of magnitude as the binding energy of the α-particle when the two α-particles are close together. This again seems to be a serious objection against the α-particle approximation.

It may be asked why the binding between α-particles is so small if the forces between them are large. The reason is probably the small region of space in which there is a large attraction between them.

It may be mentioned that this peculiar shape of the interaction between two α-particles as a function of their distance, would make the structure of nuclei composed of α-particles very simple provided the approximation from α-particles has any sense: Two neighboring α-particles in a nucleus would in general have a mutual distance falling inside the "trough" of the interaction potential, i.e., very near a given value. This means that the structure of nuclei containing exclusively α-particles could be considered from a purely geometrical point of view: The 3 α-particles of C^{12} would be arranged in an equilateral triangle, the 4 in O^{16} in a tetrahedron, etc. The binding energy (energy of the respective nucleus compared to the energy of the corresponding number of free α-particles) would then in first approximation be proportional to the number of pairs of neighboring α-particles which is 1 for Be^8, 3 for C^{12} (triangle), 6 for O^{16} (tetrahedron), and 3 more for each additional α-particle. Experimentally, the mass of Be^8 is almost exactly that of two α-particles; thus the mutual attraction of one pair of α's is not sufficient to overcome the kinetic energy associated with their relative motion, the situation being similar to that found in the deuteron

[67] Heisenberg assumes that the centers of gravity of the α-particles are not exactly localized. The interaction between two α-particles at a fixed distance s is obtained from his formulae (14), (15), (17) by letting η go to infinity. Then the exchange interaction (15) becomes the Dirac δ-function, while the van der Waals interaction is proportional to e^{-2r^2/a^2}.

(cf. Massey and Mohr, M10). The binding energy of the next α-particle is 0.0067 mass unit (mass of $Be^8 = 8.0070$, $He^4 = 4.0034$, $C^{12} = 12.0037$, binding energy $Be^8 + He^4 - C^{12}$), of the following 0.0071 mass unit ($C^{12} + He^4 - O^{16}$). From our simple picture, we would expect the binding energy of the fourth α-particle (leading to O^{16}) to be about 50 percent larger than that of the third, because three new "bonds" are created when the fourth, and only two when the third α-particle is added. On the other hand, it seems to be correct that the addition of every further α-particle sets about the same energy free as the addition of the fourth, in agreement with expectations from our simple approximation.

Summarizing, we must say that it can at present not be decided how much truth is in the assumption of α-particles as nuclear subunits. Certainly, this assumption must not be taken literally, and the α-particles undergo considerable deformations (polarizations) in the nucleus. On the other hand, the approximation assuming the elementary particles to move independently (Hartree approximation) is certainly not correct either, but must be supplemented by introducing correlations between the particles (end of §36). Such correlations would lead at least in the direction towards the α-particle approximation. The truth will therefore probably lie between the two extremes, as Heisenberg (H10) has pointed out. However, it seems to us that at present the Hartree approximation offers more prospects for being perfected.

§32. Quantum States of Individual Particles (Neutron and Proton "Shells") (H10, B9, E3, G13)

The opposite extreme to the assumption of α-particles as nuclear subunits is that of independent motion of the individual protons and neutrons. This assumption can certainly not claim more than moderate success as regards the calculation of nuclear binding energies (§35). However, it is the basis for a prediction of certain periodicities in nuclear structure for which there is considerable experimental evidence (§33, 34). Also, the individual-particle-approximation seems to offer some hope for the development of a rational theory of nuclear spins in the future (§36).

The procedure in the individual particle-scheme is very simple: To start with, we assume a certain "auxiliary potential" which we suppose to act on each proton and neutron. (The auxiliary potentials may be chosen different for protons and neutrons, to account roughly for the electrostatic repulsion between the protons.) We calculate the wave functions of the individual particles in the auxiliary potential. Then we compute the total kinetic and potential energy of the nucleus from the wave functions, with the help of (144), (146).

In "zero-order" approximation, the energy of the nucleus will be given by the sum of the eigenvalues of the individual particles in the auxiliary potential, provided the latter has been chosen suitably. This zero-order approximation will be studied in this section and will lead us to the prediction of periodicities in nuclear structure. In the following two sections we shall discuss the experimental evidence concerning the periodicity. In §35 and 36, we shall then proceed to the "first approximation," in which the energy of the nucleus is calculated as the average of kinetic plus potential energy, averaged over the wave function of the nucleus. We shall also indicate the probable influence of a second approximation (§36).

For the zero-order approximation, we need only know the eigenvalues of the individual particles in a given auxiliary potential. The first problem is therefore the suitable choice of an auxiliary potential. The potential suggesting itself immediately is the simple potential hole: The potential is assumed to be $-U_0$ inside a sphere of radius R (nuclear radius) and zero outside. Such a potential will represent the actual state of affairs fairly accurately since we know that the density of nuclear matter is practically constant inside heavy nuclei; thus the potential energy of one particle in the field of the nucleus will also be practically constant.[68] A still better approximation may be obtained by letting the potential go to zero gradually at the edge of the nucleus, but such refinements seem hardly worth while at present. Only for light nuclei the simple potential hole will be unsatisfactory because the thickness of the "surface layer" (§29) in which the potential goes gradually to zero, is of the same order as the nuclear radius (Flügge, F12). Therefore it seems more appropriate actually to represent the gradual change in the auxiliary

[68] We shall take, in this section, the *exact* solutions of the Schrödinger equation in a spherically symmetrical potential hole. The result will differ considerably from the result of the statistical method, in which the eigenfunctions were approximated by plane waves (chapter V).

potential, which can be done by choosing, e.g., an "oscillator" potential

$$U=-U_0+\tfrac{1}{2}Cr^2=-U_0+\tfrac{1}{2}M\omega^2r^2 \quad (187)$$

as the auxiliary potential (Heisenberg, H10). (ω is the frequency of a "classical oscillator" of mass M in the potential U.) The oscillator potential has the additional advantage of giving very simple wave functions (Hermitian functions).

(a) We shall first discuss the quantum states of the individual particles in the *oscillator potential* (187). The quantum states may be described by three quantum numbers n_1, n_2, n_3, which are connected to the energies of the vibrations along the x, y, and z axis, respectively. The total energy of a particle is

$$E=-U_0+\hbar\omega(N+\tfrac{1}{2}) \quad (187a)$$

with $$N=n_1+n_2+n_3+1. \quad (187b)$$

The eigenfunction of the state n_1, n_2, n_3 is

$$\psi=e^{-\frac{1}{2}\rho^2}(x^{n_1}+\cdots)(y^{n_2}+\cdots)(z^{n_3}+\cdots) \quad (187c)$$

with $$\rho^2=M\omega r^2/\hbar, \quad (187d)$$

the dots denoting lower powers in x, y and z, respectively. In order to compare the results for the "oscillator" potential with those for other central fields, e.g., the potential hole, the wave function must be written as a function of r times a spherical harmonic. This is always possible by suitable linear combinations of the wave functions (187c); e.g., we have:

$N=1:\psi=\psi_{000}=e^{-\frac{1}{2}\rho^2}$, $l=0$, $1s$ level,

$N=2:\psi=\psi_{001}=e^{-\frac{1}{2}\rho^2}\rho\cos\theta$, $l=1$, $2p$ level,

and two similar functions ψ_{010} and ψ_{100}

$N=3:\psi_1=\psi_{200}+\psi_{020}+\psi_{002}$
$=e^{-\frac{1}{2}\rho^2}(\rho^2-(3/2))$, $l=0$, $2s$ level,

$\psi_2=\psi_{002}-\frac{1}{2}(\psi_{200}+\psi_{020})$
$=e^{-\frac{1}{2}\rho^2}\rho^2((3/2)\cos^2\theta-\frac{1}{2})$,
$\psi_3=\psi_{110}=e^{-\frac{1}{2}\rho^2}\rho^2\sin^2\theta$
$\times\sin 2\varphi$, } $l=2$, $3d$ level,

and their similar functions $\psi_{011}\psi_{101}$ and $\psi_{200}-\psi_{020}$.

$N=4:\psi_{003}+\psi_{201}+\psi_{021}$
$=e^{-\frac{1}{2}\rho^2}(\rho^3-(5/2)\rho)\cos\theta)$ } $l=1$, $3p$ level,

and two similar functions

$\psi_{003}-(3/2)(\psi_{201}+\psi_{021})$
$=e^{-\frac{1}{2}\rho^2}\rho^3((5/2)\cos^3\theta$
$-(3/2)\cos\theta)$ } $l=3$, $4f$ level.

and six similar functions.

We see first of all, that oscillator levels with even N correspond to odd azimuthal quantum numbers l because the wave functions are odd functions of the coordinates xyz, and *vice versa*. There is considerable degeneracy of levels, the second s level coinciding with the first d level, etc. The levels are designated by the usual spectroscopic notation, giving the lowest level of azimuthal quantum number l the principal quantum number $n=l+1$ and numbering consecutive levels of the same l by successive values of n. Then $n-1$ is the total number of nodes of the wave function, radial and angular together.[69]

The general relation between the principal quantum number n in polar coordinates, the azimuthal quantum number l and the "energy quantum number" N is

$$N=2n-l-1. \quad (187e)$$

This follows from the examples given above, and can be shown generally.—For a given energy (given N), we have $\frac{1}{2}N$ or $\frac{1}{2}(N+1)$ different quantum levels in the nl scheme, according to whether N is even or odd. These levels have the azimuthal quantum numbers $l=N-1$, $N-3$, $N-5$, etc., and the principal quantum numbers $n=N$, $N-1$, $N-2$, etc., respectively. The total statistical weight of the energy level N is $N(N+1)$, taking account of the spin (factor 2 in statistical weight). Thus the weight of the levels $N=1$, 2, 3, 4, 5, 6, 7, is 2, 6, 12, 20, 30, 42, 56, respectively. The total number of quantum states having an N smaller or equal to N_0, is 2, 8, 20, 40, 70, 112, 168 for $N_0=1$ to 7.

[69] In some theoretical papers on nuclei the lowest level of *any* l has been given the principal quantum number 1. This seems an unhappy choice, in view of the analogy to atomic spectroscopy.

(*b*) If we take a *simple potential hole* as auxiliary potential, the wave functions are spherical harmonics, multiplied by Bessel functions of order $l+\frac{1}{2}$ of the radius, l being the order of the spherical harmonics (azimuthal quantum number). If the walls of the hole are infinitely high, the Bessel functions must vanish for $r=R$ (nuclear radius). If the height of the walls is finite, this boundary condition has to be relaced by a more complicated one, involving the wave function and its derivative.

The order of the energy levels has been worked out by Elsasser (E3) for infinitely high walls, by Margenau (M7) for finite walls of a certain height. The arrangement of the energy levels is in both cases similar to that for the oscillator potential, but the "accidental degeneracy" of levels with different l and the same N which we found for the oscillator potential, is of course removed. The oscillator level N splits into levels with given l and n in such a way that the level of highest l lies lowest. The arrangement of the levels is shown in Fig. 8 for the oscillator potential, the potential hole with infinitely high walls, and the potential hole of finite depth, just sufficient to take 58 particles (this is the case considered by Margenau). The figure shows all levels below $100\hbar^2/MR^2$ in a potential hole of radius R with infinitely high walls. These levels are also given in Table X. According to our

TABLE X. *Energy levels in potential hole with infinite walls.* Energy in units $\hbar^2/MR^2$ where R = radius of hole.

	1ST LEVEL OF AZIMUTHAL QUANTUM NUMBER l									1ST LEVEL	
l	Des.*	En.**	Des.	En.	Des.	En.	Des.	En.	l	Des.	En.
0	1*s*	4.93	2*s*	19.74	3*s*	44.42	4*s*	78.96	6	7*i*	55.27
1	2*p*	10.12	3*p*	29.85	4*p*	59.45	5*p*	98.92	7	8*j*	67.98
2	3*d*	16.61	4*d*	41.35	5*d*	75.96			8	9*k*	81.79
3	4*f*	24.40	5*f*	54.25	6*f*	93.83			9	10*l*	96.74
4	5*g*	33.51	6*g*	68.49							
5	6*h*	43.76	7*h*	83.98							

* Spectroscopic designation.
** Energy.

scheme, we should expect a successive filling-up of the quantum states with neutrons and protons. The first two neutrons (or protons) will go into the 1*s* shell, the next six into the 2*p* shell, etc. The shells are tabulated in Table XI in the order of their energy; below each shell the number of quantum states in it is given (n_i); in the third line the n_i's of shells with nearly identical energy are added (N_i); in the last line the N_i's of all shells up to the one considered are added: The figures in the last line (S_i) therefore represent the numbers of neutrons (or protons) for which we would expect a shell (or group of shells of nearly identical energy) to be completed.

TABLE XI. *Successive filling of neutron (or proton) shells in potential hole with infinitely high walls.*

SHELL	1*s*	2*p*	3*d*	2*s*	4*f*	3*p*	5*g*	4*d*	6*h*	3*s*	5*f*	7*i*	4*p*	8*j*	6*g*
n_i	2	6	10	2	14	6	18	10	22	2	14	26	6	30	18
N_i	2	6		12	14	6	18		34		40		6		48
$S_i=\sum_{k=1}^{i} N_k$	2	8		20	34	40	58		92		132		138		186

Whenever a shell is completed, we should expect a nucleus of particular stability. When a new shell is begun, the binding energy of the newly added particles should be less than that of the preceding particles which served to complete the preceding shell. We should thus expect that the 3rd, 9th, 21st, etc., neutron or proton is less strongly bound than the 2nd, 8th, 20th.

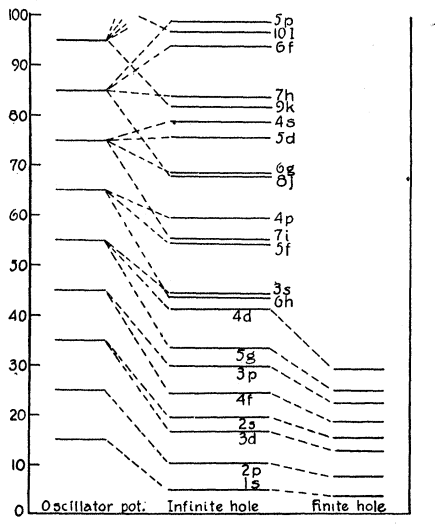

FIG. 8. Energy levels in an oscillator potential, in a potential hole with infinitely high walls, and in a hole with finite walls. The levels in the infinite hole are drawn to scale.

§33. Evidence for Periodicities from the Energy of Nuclei

We have seen in the preceding section that the concept of individual quantum states for the elementary particles leads to a particular stability of nuclei containing 2, 8, and 20 neutrons (or protons), because these numbers of neutrons are just sufficient to fill the $1s$, $2p$ and $(3d+2s)$-shell. The first number mentioned (2 neutrons, 2 protons) corresponds to the α-particle whose particular stability is well known but cannot be used as an argument for our scheme because it follows from any approximation whatever. The last number (20 neutrons, 20 protons) leads to the nucleus Ca^{40}. Unfortunately no exact data about nuclear masses are available for such high atomic weights so that no direct check is possible concerning the special stability of Ca^{40}. Some indirect evidence will be mentioned in §34. There remains thus the nucleus containing 8 neutrons and 8 protons, i.e., O^{16}, to test the "shell structure" hypothesis by means of nuclear energies. It seems in fact that there is ample evidence for a particular stability of O^{16}, and thus for the individual-particle approximation.

To be free from other fluctuations of the binding energy, we shall compare the analogous nuclei He^4, Be^8, C^{12}, O^{16}, Ne^{20}, etc., all of which can be considered as containing exclusively α-particles, and the nuclei which are produced by adding neutrons and protons to these "standard" nuclei. In Table XII, the masses of analogous nuclei are given (part (a)), together with the increase in mass connected with the addition of one or more particles to the standard nucleus (part (b)).

By comparing the figures in any one column of Table XIIb, it is seen immediately that the *figures decrease steadily as we go down the column, with the sole exception of a marked increase when going from the C^{12} to the O^{16} line.* This feature repeats itself in each column of the table. It means that a *given particle* or group of particles is *bound* the *more strongly* to a "standard nucleus" *the heavier* that *standard nucleus* is, *but* that the binding *to* the nucleus O^{16} is *less strong than to* C^{12}. This is exactly what we must expect if O^{16} marks the completion of a neutron and proton "shell": *All particles added to O^{16} must go into the next outer shell ($3d$) and will therefore be bound much less strongly than the preceding particles.* The fact that the decrease in binding energy from C^{12} to O^{16} occurs whatever particle or particles are added to the "standard nucleus," constitutes very strong evidence for the shell-structure indeed.

It seems worth while to discuss the reliability of the data underlying the Table XII, from which we drew our conclusion. The most reliable data are those for the addition of 2 neutrons and one proton to the standard nucleus. The general downward trend of the mass increase connected with that addition is unmistakable. The increase from C^{12} to O^{16} depends on the mass differences $N^{15}-C^{12}$ and $F^{19}-O^{16}$. These differences are very well established from transmutation data, and if there is any error, it can only be in the direction that $N^{15}-C^{12}$ is actually smaller, or $F^{19}-O^{16}$ actually larger than the figures given in our table, changes which would strengthen our point. The mass differences are based on the reactions (L4, L5)

$$N^{14}+H^2=C^{12}+He^4 \qquad (A)$$

together with $$N^{14}+H^2=N^{15}+H^1 \qquad (B)$$

and on $$F^{19}+H^1=O^{16}+He^4. \qquad (C)$$

It is extremely unlikely that a γ-ray is connected with (A). On one hand, C^{12} probably has no excited states less than 5.5 MV above the ground state;[70] on the other hand, the assumption that the α-particles observed in reaction (A) are associated with a γ-ray of as much as 5.5 MV energy, would conflict severely with a great number of other transmutation, and with mass spectroscopic data. Thus γ-ray emission, if any, can only be associated with processes (B) and (C). In case of (B), this would *lower* the mass of N^{15}, in case of (C), it would *raise* the mass of F^{19} The figures given in Table XII are therefore an upper limit for $N^{15}-C^{12}$, and a lower limit for $F^{19}-O^{16}$, which makes our conclusion valid *a fortiori*.

The next most reliable column in Table XII is that referring to the addition of one neutron. The

[70] No lower excited state has been observed. Also it is very unlikely theoretically that a nucleus of so high binding energy as C^{12} should have any low-lying excited states (cf. excited states of the α-particle, §23).

situation is similar to the foregoing case. The crucial mass differences are $C^{13}-C^{12}$ and $O^{17}-O^{16}$. Of these, the latter is derived independently from three different reactions: (C9, N2)

$$O^{16}+H^2=O^{17}+H^1, \quad (D)$$

$O^{16}+H^2=F^{17}+n^1$, together with

$$F^{17}\rightarrow O^{17}+\epsilon^+, \quad (E)$$

$O^{16}+H^2=N^{14}+He^4$, together with

$$N^{14}+He^4\rightarrow O^{17}+H^1. \quad (F)$$

All three figures check closely, at least if Haxel's data are used for the second reaction in (F). It is therefore almost impossible that the data can be invalidated by γ-emission because it is exceedingly improbable that all three reactions (D), (E), (F) lead to the same excited state of O^{17}. (Reaction (D) was used for the actual determination of the mass difference.)

The difference $C^{13}-C^{12}$ may be obtained in two ways, *viz.*; (C9, C10, T10)

$$C^{12}+H^2=C^{13}+H^1, \quad (G)$$

$C^{12}+H^2=N^{13}+n^1$, together with

$$N^{13}\rightarrow C^{13}+\epsilon^+. \quad (H)$$

(G) leads to the figure given in the table, assuming no γ-rays. (H) leads to a value about 1 MV *less* for the difference $C^{13}-C^{12}$, indicating that there may be a γ-ray of 1 MV associated with the proton group in (G). This would again be more favorable to our argument than the values given in Table XII.

It may thus be said safely that the completion

TABLE XII. (*a*) *Masses of analogous light nuclei.*

Standard Nucleus	1 Neutron	Nucleus Obtained by Addition to the Standard Nucleus of 1 Proton	2 Neutrons	1 Neu.+1 Pro.	2 Neu.+1 Pro.
$He^4=4.0034$	$He^5>5.013\ J$	(Li^5)	(He^6)	$Li^6=6.0161$	$Li^7=7.0170$
$Be^8=8.0070$	$Be^9=9.0139$	$B^9=9.0155F$	$Be^{10}=10.0154$	$B^{10}=10.0152$	$B^{11}=11.0117$
$C^{12}=12.0037$	$C^{13}=13.0069$	$N^{13}=13.0100$	$C^{14}=14.0077E$	$N^{14}=14.0076$	$N^{15}=15.0053$
$O^{16}=16.0000$	$O^{17}=17.0040$	$F^{17}=17.0078$	$O^{18}=18.0065H$	$F^{18}>18.0065$	$F^{19}=19.0040$
$Ne^{20}=19.9994G$	$Ne^{21}=20.999\ J$	(Na^{21})	$Ne^{22}=21.9977$	$Na^{22}=21.9996F$	$Na^{23}=22.9980E$

(*b*) *Mass increase connected with the addition to the standard nucleus of*

Standard	1 Neutron	1 Proton	2 Neutrons	1 Neu.+1 Pro.	2 Neu.+1 Pro.	α-Particle
He^4	$1.009D$	——	——	$2.0127A$	$3.0136A$	$4.0036A$
Be^8	$1.0069A$	$1.0085C$	$2.0084B$	$2.0082A$	$3.0047A$	$3.9967A$
C^{12}	$1.0032A$	$1.0063B$	$2.0040B$	$2.0039A$	$3.0016A$	$3.9963A$
O^{16}	$1.0040A$	$1.0078C$	$2.0065C$	$>2.0065C$	$3.0040A$	3.9994C
Ne^{20}	$1.000D$	——	$1.998C$	$2.000D$	$2.999C$	$3.995D$

Explanation of signs used in Table XII: A, B, C, D denote decreasing grades of certainty of the values given. Figures denoted by *A* are deduced from reliable transmutation data, *B* from transmutations whose interpretation is not absolutely certain. *C* means that at least one of the masses used is based on mass spectroscopic or band spectroscopic data or such β-disintegrations for which the upper limit of the electron energy is not exactly known. *D* data are based on estimates. *E, F, G, H, J* refer to various ways of obtaining nuclear masses: *E*=transmutation whose interpretation is not quite certain, *F*=β-disintegration with unaccurately known upper limit, *G*=mass spectroscopic, *H*=band spectroscopic data, *J*=estimate from analogous nuclei. Masses without letter in the first table are well established by transmutation data.

Special remarks: He^5 is estimated according to Atkinson (A6), by comparison with analogous nuclei. C^{14} is obtained from the reaction N^{14}+slow neutrons=C^{14}+protons, Na^{23} from the reaction $Ne^{20}+He^4=Na^{23}+H^1$. The last reaction is subject to some doubt, firstly because the measurements are very old, secondly because Ne consists of three isotopes and it is not known from which isotope the observed protons arise. But firstly Ne^{20} is the most abundant isotope, and secondly there are reasons to believe that the longest range protons are emitted from Ne^{20}, rather than Ne^{21} or Ne^{22}.—The mass of Ne^{20} itself is obtained by assuming the ratio of the masses of Ne^{20} and Ne^{22} as determined mass spectroscopically by Bainbridge, to be correct. The mass of Ne^{21} is then estimated by assuming the difference $Ne^{21}-Ne^{20}$ to be about 0.001 mass unit larger than $Ne^{22}-Ne^{21}$, in analogy to $O^{16,\,17,\,18}$ and $C^{12,\,13,\,14}$. For details of the determination of nuclear masses and references to experimental papers, cf. chapter XVII.

Example of calculation: The increase in mass when one neutron and one proton are added to C^{12}, is equal to the mass of N^{14}, i.e., of the nucleus which is formed by that addition, minus the mass of C^{12}. Thus increase=14.0076 $-12.0037=2.0039$.

of a neutron-proton shell at O^{16} is established beyond doubt from the data about nuclear masses.

§34. Periodicities in the Existing Isotopes (M11, E3, G13, G3)

If all the isotopes found in nature are represented in a diagram, preferably plotting the "isotopic number" $I=N-Z$ against the atomic weight A (Fig. 2), unmistakable breaks are apparent; e.g., the maximum isotopic number I does not increase quite smoothly with increasing A, but seems to "refrain from increasing" up to a certain A, and then to jump suddenly by several units. Bartlett (B9) has first suggested a connection between these irregularities and the neutron and proton shells discussed in the preceding section, while Elsasser (E3) and Guggenheimer (G13) have worked out some details.

Before discussing the experimental results, it is necessary to give a strong warning against taking the neutron and proton shells too literally. This has been done very frequently in the past with the effect of discrediting the whole concept of neutron and proton shells among physicists.

First of all, the concept of quantum states of the individual particles is to be regarded as a zero-order approximation only, which has to be completed by a consideration of at least one, preferably two more approximations (§36). This fact alone shows that the effects connected with the completion of a shell cannot be too well marked, and it seems reasonable to expect them to be the less well marked the greater the number of particles already in the nucleus. Therefore, apparent *deviations from* the *simple shell structure* expected should of course be *attributed* to the *crude approximation* used. *Under no circumstances* do such deviations justify far reaching *ad-hoc assumptions* such as the introduction of *negative protons* as building stones of nuclei. There is at present not a single piece of reasonably well-founded evidence for the existence of negative protons in nuclei, but several grave reasons against this existence.

Secondly, it should be borne in mind that the filling of individual quantum states is not the only thing determining nuclear energies. In fact, other points have a much stronger influence on nuclear energies and stability. These points are:

(a) The general trend of the isotopic number as a function of the nuclear charge (§8).

(b) The odd-even rule, i.e., the rule that there are no stable nuclei of odd nuclear charge greater than 7 and even atomic weight (§10), and that nuclei are most stable when they contain even numbers of neutrons and protons.

(c) The isobar rule, stating that there exist almost no pairs of neighboring isobars (§43).

(d) The general trend of the average number of isotopes per element to increase from light to medium atomic weight (cf. end of §10) and to decrease again for the heaviest elements (because of instability against α-decay).

Thirdly, great caution should be applied in drawing conclusions from the mere existence or nonexistence of isotopes because there may be some rare isotopes yet undiscovered, and on the other hand, some spurious ones among the reported isotopes (cf. Mattauch, M11).

We shall now discuss the experimental evidence.

(a) *Nuclei of odd atomic weight*

These nuclei seem to be more suitable for the detection of irregularities in the increase of the isotopic number than nuclei of even weight, because there exists in general only one stable nucleus for a given odd atomic weight, so that we may give a definite isobaric number $I=A-2Z$ for any value of A. Moreover, there is no theoretical reason for any preference for even or odd nuclear charge for these nuclei, and no such preference seems actually to exist.

Table XIII gives, for each isotopic number from 1 to 43, the nucleus of maximum atomic weight observed for the given I. Column 2 gives the chemical symbol of the nucleus, column 3 its atomic weight $A(I)$. In column 4 we have calculated the atomic weight $A_0(I+1)$ which would correspond to the isotopic number $I+1$ according to formula (18b). If the increase of the isotopic number were quite regular, we should expect that all nuclei of odd atomic weight *smaller* than A_0 should have the isotopic number I, all nuclei of odd weight *larger* than A_0 the isotopic number $I+2$. Therefore, the greatest odd number smaller than A_0, let us say (A_0), should be the heaviest nucleus of isotopic number I. The difference $\delta A=A(I)-(A_0)$ (column 5) measures the deviation of the isotope scheme

from regularity: A *positive* value δA indicates that relatively too many protons, a negative value that too many neutrons are contained in the respective nucleus.

Leaving out small fluctuations ($\delta A = \pm 2$), we observe two major and two minor deviations from a regular increase. The first major fluctuation is an excess of protons (positive δA) in all nuclei between $A = 75$ and about 110, the second an excess of neutrons (negative δA) in the nuclei immediately following, *viz.*, from 110 to 140. The minor deviations are the single nucleus K^{39} (too many protons) and the group from 150 to 180 (slightly too many neutrons, the difference δA being scarcely significant).

The theoretical sequence of levels of the individual particles is given in Table XI (§32). According to that table, we should expect "closed shell structures" for nuclei containing 2, 8, 20, 34, 40, 58, 92, 132 neutrons or protons.

Of all the fluctuations found experimentally in the isotope scheme, there is only one which can readily be explained on the grounds of these "closed shells," *viz.*, the case of K^{39}. This nucleus constitutes actually quite a strong piece of evidence for the completion of a neutron shell with 20 neutrons. For the isotopic number moves actually *against* the general trend, being 3 for the atomic weight 37 (nucleus Cl^{37}) and dropping back to 1 for K^{39}. The explanation in the neutron-shell scheme is that for $A = 39$ the nucleus with isotopic number 3 (A^{39}) would contain 18 protons and 21 neutrons, i.e., one neutron outside the closed shell, which makes the nucleus less stable than K^{39} which contains 19 protons and 20 neutrons, all of them *inside* the (3*d*, 2*s*) shell. For $A = 37$ there is no influence of the completion of the shell; both the nuclei of isotopic number 1 (A^{37}) and 3 (Cl^{37}) would contain only neutrons in the inner shell, *viz.*, 19 and 20, respectively. Therefore $A = 37$ can be considered as "regular" showing that for "regular" nuclei of this atomic weight the isotopic number should be 3, and that the stability of K^{39} rather than A^{39} really is to be attributed to an irregularity, *viz.*, the completion of a neutron shell.

On the other hand, the two "long periods" in the isotope scheme do not fit at all to the simple shell concept. For $A = 110$, i.e., the end of the first period, we have about 48 protons and 62 neutrons. The latter number is near to 58, so that we may expect a closed *neutron* (5*g*) shell. The number of protons is midway in the 5*g* shell. Thus we would expect, from the naive standpoint, to find an excess of *neutrons* in the nuclei below $A = 110$, exactly the contrary of the experimental result.

Similarly, the end of the experimental period of an excessive number of neutrons ends with La^{139}, i.e., $Z = 57$ and $N = 82$. This corresponds to a closed shell of protons, but does not represent any particular point with regard to the neutrons.[72]

Therefore it seems that the naive theory of neutron-proton shells fails for higher atomic number. The reason may be the following: Because of the interaction of the particles, there will be a great number of energy levels of the nucleus as a whole, for a given distribution of the protons and neutrons over the individual quantum states. Let us call such a distribution of the particles a "configuration" and the levels of the nucleus, corresponding to a given configuration, a "level system." Then the lowest level of a system will lie much lower than the average energy of the level system, the difference being largest when the outermost shells of neutrons and protons are just half-filled, because this state of affairs corresponds

TABLE XIII. *The isotopic number of nuclei of odd atomic weight.* (Explanation in text.)

I	Elem.	$A(I)$	$A_0(I+1)$	δA	I	Elem.	$A(I)$	A_0	δA
1	K	39	31	+8	25	La	139**	151	−12
3	Ti	47	46.5	+2	23(!)	Sm	147**	143.5	+4
5	Cu	63	60	+4	25	Eu	151**	151	0
7	Ga	69	72	−2	27	Gd	155	158.5	−2
9	Br	79	77.5	+2	29	Dy	161	166.5	−4
11	Ru	99	92	+8	31	Yb	171	173	−2
13	Pd	105	101.5	+4	33	Hf	177	179	−2
15	Sn	115*	110	+6	35	Re	185	186	0
17	Sn	117	119.5	−2	37	Os	189	193	−4
19	Te	123	128.5	−4	39	Hg	199	200	0
21	Kr	129	136	−6	41	Tl	203	206	−2
23	Ba	135**	143.5	−8	43	Bi	209	212	−2

* *Remarks:* There are two isobars for $A = 115$, *viz.*, Sn and In. Assuming In to have the smaller energy (it is more abundant!), Sn^{115} would only exist because the transition $Sn^{115} + \epsilon^- = In^{115} + n^0$ is forbidden. For $A = 113$ there are also two isobars, In and Cd. Thus 111 (Cd) is the heaviest nucleus of isotopic number 15 which is certainly energetically stable. This would make $\delta A = 2$ for $I = 15$, more in line with the general trend of δA.

** There is a pronounced irregularity for $I = 23$ and 25: The isotopic number 25 appears already in Ba^{137} and La^{139}. Then the isotopic number drops back to 23 in Pr^{141}, Nd^{143} and Sm^{147}, to reach 25 again in Nd^{145}, Sm^{149} and Eu^{151}. We had therefore to include the values $I = 23$ and 25 *twice* in our table.

[72] Bartlett, Elsasser and Guggenheimer have left out the shells 2*s*, 3*p* and 3*s*, without giving any reason for such a procedure. According to them, the 5*g* shell should be filled when there are 50 particles, the (4*d* 6*h*) shell with 82 particles. These numbers would agree with the experimental result, but they lack theoretical foundation.

to the largest number of levels in the system. Accordingly, we shall have two effects counteracting each other: The energy of zero approximation (average of the energies of a system of levels) decreases when a shell approaches completion, but the difference between the average energy and the lowest level of a system decreases as well. The minimum energy will therefore lie between the middle and the end of a shell. The larger the number of places in the shell, the larger will be the number of levels in a "system," and the more will the minimum energy be shifted towards half-complete shells. Thus it may happen that 48 rather than 58 protons, and 82 rather than 92 neutrons, correspond to minimum energy.

This explanation of the discrepancy between the observed isotope scheme and the naive theory is tentative only and somewhat *ad hoc*. However, it seems certainly necessary to include higher approximations to the energy than the zero approximation, in nuclear physics much more than in atomic physics, and the direction of the deviation from the naive theory due to higher approximations is correctly given by our argument.

(*b*) *Even atomic weight*

For even atomic weight, even nuclear charge Z corresponds to greater stability than odd charge (§10, and rule 2 above). This means that stable nuclei of atomic weight $4n$ (n an integer) have isotopic numbers $I=A-2Z$ divisible by 4, while nuclei of atomic weight $4n+2$ have in general isotopic numbers of the form $4m+2$.

The first rule holds without exception, the second rule for all nuclei of the type except the very lightest ones (H^2, Li^6, B^{10}, N^{14}). For these four nuclei, the "even-odd rule" conflicts too severely with the rule that among light nuclei those with isotopic number 0 are by far the most stable, because the forces between neutrons and protons are the strongest forces in the nucleus. It seems significant that this conflict is decided in favor of the even-odd rule (isotopic number 2) as early as for $A=18$, for which atomic weight we have the stable nucleus O^{18} containing 8 protons and 10 neutrons while F^{18}, which would contain an equal number of neutrons and protons, is unstable. We do not consider it significant that the change from isotopic number 0 to 2 occurs just after the completion of the $2p$ shells of neutrons and protons (cf. §30).

The change of the isotopic number of the nuclei $4n+2$ from 0 to 2, which is such a natural consequence of general principles, is quite sufficient to explain the striking change of the isotope pattern at $Z=8$: For $Z\leq 7$, each element has two isotopes of isotopic number 0 and 1, whether its nuclear charge Z is even or odd [*Exceptions:* For $Z=1$, there is in addition the proton, with isotopic number -1. For $Z=2$, the isotope He^5 is unknown, and there are very strong reasons that it does not exist at all (Atkinson, A6), being unstable against disintegration into an α-particle and a neutron. For $Z=8$, Be^8 is unknown because it can disintegrate into two α-particles. The exceptions are not serious for our point: We are now interested in the most stable nucleus for any given atomic weight; thus it suffices that He^5 is more stable than Li^5, and Be^8 more so than Li^8 which is certainly true]. For $Z\geq 8$, each element of even Z has three isotopes, of isotopic number 0, 1, 2; while the elements of odd Z have only one isotope each ($I=1$). This change of the pattern means only that the nuclei of weight $4n+2$ have odd nuclear charge for $A\leq 14$, even charge for $A\geq 18$. There is therefore no profound reason behind the change of the isotope pattern at oxygen.

Turning now to the nuclei of atomic weight $4n$, we remark that for low atomic weight these nuclei have isotopic number 0 throughout. This fact, together with the fact that they contain even numbers of neutrons and protons, makes them the most stable nuclei in the region (§30). (The special stability follows also from the picture of α-particles as subunits. §31.) The heaviest nucleus of this kind is Ca^{40}. It may be significant that this nucleus contains just 20 neutrons and protons, corresponding to complete $1s$, $2p$, and $(3d, 2s)$ shells.

The behavior of heavier nuclei of even weight shows fluctuations analogous to those of the nuclei of odd weight. In fact, the nuclei of odd atomic weight lie always in the center of the broad band filled by nuclei of even weight (Fig. 2). There are therefore the same difficulties in explaining the observed periodicities as for the odd nuclei.

We shall now mention a few other points connected with the shell-structure of nuclei.

(a) *Radioactivity.* A feature giving some support to our ideas about shell-structure is the start of α-radioactivity. The lightest α-radioactive atom (except for the odd case of Sm) is Po^{210}. For this element, $Z=84$ and $N=126$. This is fairly near the completion of the group of proton shells ($4d$, $6h$, $3s$) and the group of neutron shells ($5f$, $6i$). The shells would be complete for $Z=92$ and $N=132$, but we

expect minimum energy for somewhat smaller values of Z and N.

(b) *Nuclear spins and magnetic moments.* Nothing definite can be said in this respect until a better method of attack has been found, or at least until the first and second approximation mentioned in §36 have been calculated. However, it seems significant that large spins appear for the first time after the $4f$ shell is begun (scandium, $Z=21$, spin 7/2).

With all reserve, we may be allowed to infer from our considerations the probable spin of K^{40} which is supposedly the radioactive isotope of K (Klemperer, K4). K^{40} contains 19 protons and 21 neutrons, i.e., complete 20-shells minus a proton in the $2s$ shell plus a neutron in the $4f$ shell. The orbital momentum of K^{40} is therefore most probably 3, while we cannot say whether the spins of proton and neutron are parallel or antiparallel to each other and to the orbital momentum. In any case, a total momentum of 2, 3 or 4 would result. 4 would be amply sufficient, 3 just sufficient to explain the long life of radioactive K^{40} (§43).

(c) *Another type of shell-structure* was suggested by Landé (L2). He assumed that as many neutrons and protons as possible are combined in α-particles while the remaining neutrons are arranged in shells. It was necessary to assume that some inner neutron shells are left incomplete while outer shells are being built up, and that the inner shells are completed afterwards. The capacity of the successive neutron shells was assumed to be 2, 6, 8, 12. It seems hard to attach any theoretical significance to these numbers.

In conclusion, we want to emphasize again that *reliable* conclusions about the shell-structure of nuclei can only be drawn when atomic weight determinations will be available which are guaranteed to be accurate to at least three decimals, i.e., 1 part in 100,000 for atomic weights of the order 100.

§35. Energy of O^{16} and Ca^{40} in the Hartree Approximation (H10)

Heisenberg (H10) has calculated the energy of the closed-shell nuclei He^4, O^{16}, Ca^{40}, using the individual-particle approximation and assuming oscillator wave functions (cf. §32). He assumes the "Gaussian" potential

$$J(r_{12}) = -Be^{-r^2/a^2} \tag{188}$$

to act between a neutron and a proton, and no force between like particles. The wave function of the neutrons and protons in the $1s$ shell is

$$\psi_{1s} = (\mu/\pi)^{\frac{3}{4}} e^{-\mu r^2}, \tag{188a}$$

where r is the distance from the center of the nucleus and μ a constant characteristic for the auxiliary oscillator potential used (cf. §32). If this potential is

$$U = \tfrac{1}{2}Cr^2 = \tfrac{1}{2}M\omega^2 r^2, \tag{188b}$$

where M is the mass of the proton (or neutron) and $\omega/2\pi$ its "classical" frequency in the oscillator potential, then

$$\mu = (MC)^{\frac{1}{2}}/\hbar = M\omega/\hbar. \tag{188c}$$

The wave functions of the other states ($2p$, $3d$, etc.) are similar to (188a). C (or ω, or μ) has to be regarded as an arbitrary parameter which must be fixed in such a way as to make the total energy a minimum.

For the details of the calculation we refer to Heisenberg's paper. The procedure is the following: The kinetic energy is equal to half the eigenvalues of the particles in the oscillator potential, i.e., $(3/4)\hbar\omega$, $(5/4)\hbar\omega$ and $(7/4)\hbar\omega$, respectively, for particles in the $1s$, $2p$, and ($3d$, $2s$) shell. The potential energy consists of the neutron-proton interaction and the Coulomb repulsion between the protons: Both terms can be calculated from the wave functions, but the final expressions are somewhat complicated. Potential and kinetic energy are added, and the sum minimized as a function of $x=\mu a^2$ (cf. (188), (188c)). The result represents the binding energy as a function of the force constants B and a. Instead of calculating the mass defect from the constants derived from the theory of light nuclei (chapter IV), Heisenberg uses the inverse procedure, i.e., to calculate the force constants from the observed mass defect. (This procedure has been used by us in connection with the statistical model, chapter V.) These constants may be compared to those necessary to give the correct eigenvalue of the α-particle, with the *variation principle* and the eigenfunction

$$\psi = e^{-\lambda(r_{12}^2+r_{34}^2)-\nu(r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2)}. \tag{188d}$$

For any given range a of the forces, the constant B must be chosen about 25 percent larger to obtain the correct binding energy of He^4 with the oscillator wave functions, than with the wave function (188d). This means that the oscillator wave functions are less good approximations than (188d). The reason is, of course,

that in (188d) the interacting particles are linked directly to each other while in the oscillator approximation they are linked to the center of the nucleus, and only indirectly to each other.

The approximation to the energy of the nuclei O^{16} and Ca^{40} afforded by the oscillator potential is slightly worse than for He^4, the values of B necessary to obtain agreement with the observed binding energy being about 8 percent and 15 percent larger for O^{16} and Ca^{40}, than for He^4.

Nothing is changed if forces between like particles are assumed, provided the force between two like particles is of the form (cf. 142)

$$K(r_{12})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \tag{189}$$

For in this case the total interaction between all neutrons will be of the form

$$\tfrac{1}{4}\int K(r_{12})\,|\,\rho_n(\mathbf{r}_1\mathbf{r}_2)\,|^2 d\tau_1 d\tau_2 \tag{189a}$$

(cf. 146e), while the interaction between neutrons and protons is

$$\int J(r_{12})\rho_n{}^*(\mathbf{r}_1\mathbf{r}_2)\rho_p(\mathbf{r}_1\mathbf{r}_2)d\tau_1 d\tau_2, \tag{189b}$$

where J is the interaction potential between neutrons and protons, and $\rho_n\rho_p$ the mixed densities of neutrons and protons (§24). Since the numbers of neutrons and protons and their wave functions are equal, we have $\rho_n=\rho_p$ and all interactions together give an integral of the form (189b), with only J being replaced by $J+\frac{1}{2}K$. Thus the result for the binding energy will be the same as without like-particle forces, only B means now the sum of the interaction between two unlike particles, and half the interaction between two like particles. This will not change the comparison between the results for He^4, O^{16}, Ca^{40}, with each other and with the result of the variational method applied to He^4. (It would change the comparison with the theory of the deuteron.)

§36. The Coupling Scheme in Nuclei

For nuclei with incomplete shells, it is of great interest to find out how the orbital momenta and the spins of the individual particles are coupled to the resultant nuclear momentum I. Up to the present, practically no calculations concerning this problem have been carried out. But it seems as if the Hartree approximation discussed in the preceding sections, might in the future lead to a rational theory of the nuclear spins I and the associated magnetic moments μ, at least for light nuclei. One must, however, be prepared that higher order perturbations may seriously affect the picture, at least as regards the magnetic moments.

It seems reasonable to assume Russell-Saunders coupling to hold at least approximately in the nucleus, the Heisenberg forces being small compared to the Majorana forces (§13, 14). We shall thus introduce a total orbital momentum Λ of the nucleus and a total spin momentum Σ whose resultant is the "nuclear spin" (total angular momentum of the nucleus) I. The momenta Λ and Σ are the resultants of the λ's and σ's of the individual particles. A level of the nucleus shall be denoted similarly to the usual spectroscopic way, giving first the configuration of the protons, then that of the neutrons, then the characteristics of the level of the nucleus as a whole; e.g., $(1s^2\,2p)\,(1s^2\,2p^2)^2P_{3/2}$ means that there are two protons in the $1s$ shell, one in the $2p$ shell, 2 neutrons in $1s$, two in $2p$, and that the resultant orbital momentum of the nucleus as a whole is 1 (P term), the resultant spin 1/2 (doublet term), the total nuclear moment 3/2.

The energy of the various levels corresponding to a given configuration of neutrons and protons, can be calculated by a method similar to that used in the theory of optical spectra. The calculation is, however, much more involved because there are twice as many particles in each shell (neutrons and protons!) and the particles of different kind are not equivalent. This makes the number of levels extremely high. The terms expected from all possible neutron and proton configurations in the p shell are listed in Table XIV. ($6\,{}^3P$ means that the configuration leads, among others, to 6 different ${}^3P$ terms of the nucleus.)

The calculation of the energy levels is simple only in the case of one neutron and one proton in the p shell. Since there are no restrictions due to the Pauli principle in this case, we may disregard the spins entirely in first approximation. We denote by 1 the coordinates of the proton, by 2 those of the neutron, and by M the

component of the total orbital momentum in a given direction (axis of the polar coordinate system). We leave out the wave functions of the particles in the closed $1s$ shell. Putting furthermore

$$f=(8/3)\pi^{-\frac{1}{2}}e^{-(\rho_1^2+\rho_2^2)}\rho_1\rho_2, \tag{190}$$

where ρ is defined as in (187d), we have:

For $M=2$ one wave function

$$\psi_2=(3/8\pi)f\sin\theta_1e^{i\varphi_1}\sin\theta_2e^{i\varphi_2}. \tag{190a}$$

For $M=1$ two wave functions

$$\begin{aligned}\psi_{11}&=(3/8\pi)\sqrt{2}f\sin\theta_1e^{i\varphi_1}\cos\theta_2,\\ \psi_{12}&=(3/8\pi)\sqrt{2}f\cos\theta_1\sin\theta_2e^{i\varphi_2}.\end{aligned} \tag{190b}$$

For $M=0$ three wave functions

$$\begin{aligned}\psi_{01}&=(3/8\pi)f\sin\theta_1e^{i\varphi_1}\sin\theta_2e^{-i\varphi_2},\\ \psi_{02}&=(3/8\pi)f\sin\theta_1e^{-i\varphi_1}\sin\theta_2e^{i\varphi_2},\\ \psi_{03}&=(3/4\pi)f\cos\theta_1\cos\theta_2.\end{aligned} \tag{190c}$$

Following Slater's "method of sums,"[73] the energy of the D term is given by the diagonal matrix element of the interaction between neutron and proton, corresponding to the wave function (190a) for $M=2$. The P term is found by adding the diagonal matrix elements corresponding to the two functions (190b), and subtracting the D term from the sum. The S term is equal to the sum of the three diagonal elements corresponding to (190c), *minus* the sum of the two diagonal elements corresponding to (190b). In every case, an additional constant has been left out containing the interaction of the $1s$ neutrons and protons among themselves and with the $2p$ neutrons and protons, and also the kinetic energy of the particles. Since this constant is the same for all three terms *SPD*, it is irrelevant for the question of which term is the lowest.

The interaction between neutron and proton may be expanded in spherical harmonics of the angle Θ between the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of the two particles, *viz.*,

$$V(r_{12})=-V_0-3V_1P_1(\cos\Theta)-5V_2P_2(\cos\Theta)-\cdots, \tag{191}$$

where the minus sign has been chosen in order to make V_0, V_1, etc., *positive*. If we take

$$V(r_{12})=-Be^{-r_{12}^2/a^2}, \tag{191a}$$

then

$$V_l=(-i)^l(\pi a/4ir_1r_2)^{\frac{1}{2}}B\times J_{l+\frac{1}{2}}(2ir_1r_2/a^2)e^{-(r_1^2+r_2^2)/a^2}, \tag{191b}$$

where J is the Bessel function. In particular,

$$V_0=(B/2x)(e^x-e^{-x})e^{-(r_1^2+r_2^2)/a^2}, \tag{191c}$$

$$V_1=(B/2x^2)[e^x(x-1)-e^{-x}(x+1)]e^{-(r_1^2+r_2^2)/a^2}, \tag{191d}$$

$$V_2=(B/2x^3)[e^x(x^2-3x+3)-e^{-x}(x^2+3x+3)]e^{-(r_1^2+r_2^2)/a^2} \tag{191e}$$

with

$$x=2r_1r_2/a^2. \tag{191f}$$

Every V_l is positive for any positive value of x.

Considering that the interaction is of the Majorana type, we have now, e.g., for the

[73] J. C. Slater, Phys. Rev. **34**, 1293 (1929).

TABLE XIV. *Nuclear levels expected from various neutron and proton configurations in the p shell.*

CONFIGURATIONS	TERMS	TOTAL NUMBER
$(p^1)\ (p^1)$; $(p^1)\ (p^5)$; $(p^5)\ (p^5)$	$^3D\ ^3P\ ^3S\ ^1D\ ^1P\ ^1S$	6
$(p^2)\ (p^1)$; $(p^4)\ (p^1)$; $(p^2)\ (p^5)$; $(p^4)\ (p^5)$	$^4D\ ^4P\ ^4S\ ^2F\ 2^2D\ 3^2P\ ^2S$	10
$(p^2)\ (p^2)$; $(p^2)\ (p^4)$; $(p^4)\ (p^4)$	$^5D\ ^5P\ ^5S\ 2^3F\ 3^3D\ 5^3P\ ^3S\ ^1G\ ^1F\ 4^1D\ 2^1P\ 3^1S$	25
$(p^3)\ (p^2)$; $(p^3)\ (p^4)$	$^6P\ ^4F\ 3^4D\ 3^4P\ 2^4S\ ^2G\ 3^2F\ 5^2D\ 6^2P\ 2^2S$	27
$(p^3)\ (p^3)$	$^7S\ 2^5D\ 2^5P\ ^5S\ ^3G\ 3^3F\ 6^3D\ 6^3P\ 3^3S\ ^1G\ 3^1F\ 4^1D\ 4^1P\ ^1S$	38

average value of the interaction over the wave function ψ_{11} (190b):

$$V^{(11)} = \int \psi_{11}^*(1,2)\, V(r_{12})\psi_{11}(2,1)\, d\tau_1 d\tau_2$$
$$= (9/32\pi^2)\int f^2(r_1 r_2)\, V(r_{12}) \sin\theta_1 e^{-i\varphi_1} \cos\theta_2 \sin\theta_2 e^{i\varphi_2} \cos\theta_1 d\tau_1 d\tau_2. \quad (192)$$

We see that the Majorana force makes the diagonal matrix element very different from its usual form for "ordinary forces." To evaluate (192), we put[74]

$$\sin\theta_2 \cos\theta_2 e^{i\varphi_2} = (8\pi/15)^{\frac{1}{2}} Y_{21}(\theta_2\varphi_2), \quad (192a)$$

where Y is a normalized spherical harmonic. Moreover, we insert (191) for $V(r_{12})$ and expand the spherical harmonics of Θ according to the addition theorem[75]

$$(2l+1)P_l(\cos\Theta) = 4\pi \sum_{m=-l}^{l} Y_{lm}(\theta_1\varphi_1)\, Y_{lm}^*(\theta_2\varphi_2). \quad (192b)$$

Only the term $Y_{21}(\theta_1\varphi_1)\, Y_{21}^*(\theta_2\varphi_2)$ in the expansion (191), (192b) contributes to (192), since all other terms vanish upon integration over the angles. We thus have

$$V^{(11)} = -(9/32\pi^2)(8\pi/15)4\pi \int f^2(r_1 r_2)\, V_2(r_1 r_2) r_1^2 dr_1 r_2^2 dr_2 = -\tfrac{3}{5}L, \quad (192c)$$

where L denotes the integral. With the value (190) for f, and (191e) for V_2, the integral can be evaluated; the result is

$$L = \int f^2 V_2 r_1^2 dr_1 r_2^2 dr_2 = (5/3)\alpha^{3/2}(2+\alpha)^{-7/2}. \quad (193)$$

In a similar manner, we may evaluate the other diagonal elements of V; e.g., for the state ψ_2 we obtain

$$V^{(2)} = (9/64\pi^2)\int f^2 V \sin^2\theta_1 \sin^2\theta_2 d\tau_1 d\tau_2$$
$$= (1/16\pi^2)\int f^2 V[1-P_2(\theta_1)][1-P_2(\theta_2)] \times d\tau_1 d\tau_2 = -(K+\tfrac{1}{5}L), \quad (192d)$$

where

$$K = \int f^2 V_0 r_1^2 dr_1 r_2^2 dr_2 = \tfrac{1}{3}\alpha^{3/2}(2+\alpha)^{-7/2}(5+6\alpha+3\alpha^2). \quad (193a)$$

The diagonal matrix elements of V are:

$$V^{(2)} = -(K+\tfrac{1}{5}L), \qquad V^{(11)} = V^{(12)} = -(3/5)L, \quad (194)$$
$$V^{(01)} = V^{(02)} = -(6/5)L, \quad V^{(03)} = -(K+(4/5)L).$$

Therefore the energy levels:

$$D = V^{(2)} = -K - \tfrac{1}{5}L, \quad (195)$$
$$P = 2V^{(11)} - V^{(2)} = +K - L, \quad (195a)$$
$$S = 2V^{(01)} + V^{(03)} - 2V^{(11)} = -K - 2L. \quad (195b)$$

Since both K and L are positive and $K > L$ (cf. 193, 193a), the lowest level is the S level, the next D, and the highest P. This order is opposite to the order of levels in atomic spectra. The reason for this reversion of the order is that the forces between the particles are attractive in nuclei, repulsive in atoms (Coulomb force between the electrons).

The ratio of the intervals $SD : DP$ *cannot* be predicted on general grounds as in atomic spectra, but depends on the radial wave functions and on the form of the interaction potential. The reason is the Majorana type of the forces: This makes the integral K appear with *different sign* in the expressions for the energy of the different levels, while K would appear with the same sign throughout in atomic theory. The interval (DP) therefore depends on K as well as L, while in atomic spectra both intervals (DP) and (SD) would only depend on L.

Now we have to consider the spin interaction. We know (§14) that there are Heisenberg forces between neutron and proton which increase the Majorana forces if the two particles have parallel spin, and decrease them if the spins are antiparallel. Thus the spins of the neutron and the proton in the $2p$ shell will be *parallel* in the lowest state of the nucleus Li^6. This lowest state is thus a ^3S state, i.e., the total angular momentum of the nucleus must be 1 unit, and the magnetic moment must be the same as for the deuteron, save for perturbations. Both these predictions seem to agree with the experiments of Fox and Rabi, (F13).

In view of the extreme complication of the level scheme for more than 2 particles in the $2p$ shell (Table XIV), and of the fundamental difference between atomic and nuclear theory

[74] Cf. *Handbuch der Physik*, Vol. 24/1, p. 275, Eq. (1.8).
[75] *Handbuch der Physik*, Vol. 24/1, p. 559, (65.59), also p. 554, (65.21), (65.22).

due to the Majorana forces, it does not seem possible to make any safe predictions regarding the spins of other nuclei, without actually carrying the calculations through. However, it seems as if the order of levels is roughly opposite to that in atoms: Thus we might expect to find always a level of *low* orbital momentum Λ and comparatively low resultant spin[76] Σ as the lowest level of nuclei. It is conceivable that one might explain in this way the fact that all nuclei containing even numbers of neutrons and protons seem to have zero total momentum I.

The calculations carried out in this section represent only the first approximation to the energy. This will probably be insufficient in many cases. A second approximation may be obtained as follows: We first determine the configuration of neutrons and protons which has lowest energy in zero approximation. Then we determine which of the energy levels corresponding to the given configuration lies lowest. (Thus far the procedure corresponds to the procedure in this section.) Then we look for the next higher neutron-proton configurations and take those terms arising from them which have the same symmetry as the lowest level which we have determined just before. We may calculate the perturbation of the ground level due to these higher levels of the same symmetry according to the usual methods, with regard to energy as well as magnetic moment, etc. In some cases, it may happen that the lowest level does not have the angular momentum deduced from the first approximation but a different angular momentum which is more favored by the interaction with the higher levels arising from other proton-neutron-configurations.

§37. Van Vleck's Potential (V1)

It is of great interest to investigate whether the forces exerted by a nucleus on one of its particles, can adequately be represented by a potential, i.e., whether the wave equation for a given particle can be written in the form

$$(\hbar^2/2M)\Delta\psi+(W-U(r))\psi=0. \tag{196}$$

This is not obvious in the case of Majorana forces, in fact, the wave equation which one obtains at first does not have the form (196) at all.

To find the appropriate one-particle wave equation in the Majorana theory, we start from the general Majorana equation (29b), leave out the spins,[77] and write the positional wave function as a product of wave functions of all individual particles,[77] *viz.*,

$$\Psi(x_1x_2\cdots x_Z\xi_1\cdots\xi_N)=\prod_{i=1}^{Z}\psi_i(x_i)\prod_{k=1}^{N}\varphi_k(\xi_k). \tag{196a}$$

We then integrate over the positions of all protons and neutrons except one proton, let us say the jth, after having multiplied Eq. (29b) by

$$\prod_{i\neq j}^{Z}\psi_i^*(x_i)\prod_{k=1}^{N}\varphi_k(\xi_k):$$

The wave function and coordinate of the proton j shall simply be denoted by ψ and x. Then we find

$$(\hbar^2/2M)\Delta\psi+W\psi=\sum_k\int d\xi\varphi_k^*(\xi)J(\mathbf{x}-\xi)\psi(\xi)\varphi_k(\mathbf{x}). \tag{196b}$$

W is a constant connected to the total energy of the nucleus E and certain integrals over the wave functions. The right-hand side contains a sum over all proton wave functions and has not at all the familiar form $U(r)\psi(x)$.

Van Vleck has shown that (196b) nevertheless can be reduced to the form (196) with a suitably chosen potential function $U(r)$. This is at least true if the wave functions φ and ψ are solutions of a wave equation of the type (196) with a simple-hole potential; i.e., U is supposed to be constant and equal to $-U_0$ inside a sphere of radius R, and to be zero outside. From this assumption it cannot only be derived that Eq. (196b) can be reduced to the form (196) but moreover that the potential U acting on the neutron is again a simple potential hole of radius R. Thus the scheme is consistent.

In the simple potential hole, we may represent the wave functions φ_k and ψ_i by plane waves (statistical method, chapter V) or by spherical waves (§32) which can be considered as superpositions of plane waves. Since $\sum_k\varphi_k^*(\xi)\varphi_k(x)$ is the mixed density (§24), we have (cf. 151)

$$\rho(\xi\mathbf{x})=\sum_k\varphi_k^*(\xi)\varphi_k(x)$$
$$=(2/h^3)\int_0^P d\mathbf{p}\ \exp\left[i(\mathbf{x}-\xi)\cdot\mathbf{p}/\hbar\right], \tag{197}$$

where $P=\hbar k_0$ is the maximum momentum of the neutrons (cf. 150b). For ψ, we write

$$\psi(\xi)=\exp\ (i\mathbf{p}_0\cdot\xi/\hbar). \tag{197a}$$

J may be expanded in a Fourier series

$$J(\mathbf{x}-\xi)=\int d\mathbf{q}a(\mathbf{q})\ \exp\ (i\mathbf{q}\cdot(\mathbf{x}-\xi)/\hbar), \tag{197b}$$

where the coefficients $a(\mathbf{q})$ are given by

$$a(\mathbf{q})=h^{-3}\int d\mathbf{x}J(x)\ \exp\ (-i\mathbf{q}\cdot\mathbf{x}/\hbar). \tag{197c}$$

[76] Two factors governing the spin should be distinguished: The Pauli principle, and the actual forces acting on the spin (Heisenberg forces). The Pauli principle will probably require *low* resultant spin of all neutrons, and low resultant spin of all protons, as far as these terms have any meaning. Inasmuch as the Pauli principle does not yet determine the resultant spin, the Heisenberg forces will tend to make it as *large* as possible.

[77] Spin and symmetry of the wave function do not enter in first approximation.

Then the right-hand side of (196b) becomes

$$Q=2h^{-3}\int_\Omega d\xi\int_0^\infty dq\int_0^P dp\,a(q)\,\exp\,[i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}/\hbar + i(\mathbf{p}_0-\mathbf{p}-\mathbf{q})\cdot\xi/\hbar]. \quad (198)$$

In this equation, we invert the order of integration, integrating first over ξ (i.e., the volume of the nucleus), then over $\mathbf{q}$, finally over $\mathbf{p}$. If the nucleus is very large compared to the wave-lengths $\hbar/p$ of the particles and the range of the forces (which is approximately $\hbar/q$), we have

$$\int d\xi\,\exp\,[i(\mathbf{p}_0-\mathbf{p}-\mathbf{q})\cdot\xi/\hbar]=\hbar^3\delta(\mathbf{p}_0-\mathbf{p}-\mathbf{q}), \quad (198a)$$

δ being Dirac's δ-function. Then the integration over $\mathbf{q}$ yields (cf. 197a)

$$\int a(\mathbf{q})d\mathbf{q}\,\exp\,[i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}/\hbar]\delta(\mathbf{p}_0-\mathbf{p}-\mathbf{q}) = a(\mathbf{p}_0-\mathbf{p})\,\exp\,(i(\mathbf{p}_0\cdot\mathbf{x})/\hbar)=a(\mathbf{p}_0-\mathbf{p})\psi(\mathbf{x}) \quad (198b)$$

since the δ-function makes the integrand vanish except for $\mathbf{q}=\mathbf{p}_0-\mathbf{p}$. On the right-hand side of (198b) we have now the wave function of the proton $\psi(\mathbf{x})$, as required. Inserting into (198), we find

$$Q=2\psi(\mathbf{x})\int_0^P dp\,a(\mathbf{p}_0-\mathbf{p}). \quad (198c)$$

Thus (196b) reduces to the form (196) with

$$U=2\int_0^P dp\,a(\mathbf{p}_0-\mathbf{p}). \quad (199)$$

Thus we may indeed use an ordinary Schrödinger equation for each individual particle. Moreover, U does not depend on the direction of p_0, since $a(\mathbf{p}_0-\mathbf{p})$ depends only on $|\mathbf{p}_0-\mathbf{p}|$ and the integral in (199) goes over a sphere in p space. The form (196) is thus independent of our assumption in (197a) that ψ is a plane wave: It is valid for any linear combination of plane waves of given wave number $p_0/\hbar$ (or given energy W), e.g., for spherical waves, etc. (Van Vleck treated first the case of spherically symmetrical waves.)

However, there is still a serious flaw in our considerations: U obviously depends on the value of p_0, i.e., on the energy of the particle (it decreases with increasing energy). Thus we do not obtain the same wave equation for individual-particle states of different energy. Therefore the solutions of (196), each taken with the potential U appropriate for its particular energy parameter W, do not form an orthogonal set. Thus this method is not applicable for the construction of a set of individual wave functions to be used for the calculation of nuclear energies according to the scheme of §§35, 36. All wave functions of such a set must be derived from one and the same "auxiliary potential" which may be chosen as some average of the U's derived from (199).

The scheme of this section may, however, be useful in deriving an approximate wave function for one particular particle, and for this purpose (neutron scattering) the method was originally devised by Van Vleck.

The value of U can easily be obtained from (199) if the range of the nuclear forces is *long* compared to the wave-length of the particles ($\hbar/P$ or $\hbar/p_0$). In this case, $a(q)$ will only be large for small q's so that

$$U\approx\begin{cases}2\int_0^\infty dp\,a(\mathbf{p}_0-\mathbf{p})=2J(0) & \text{if } p_0<P\\ 0 & \text{if } p_0>P.\end{cases} \quad (199a)$$

This would mean that only such neutrons would be bound whose momentum is less than the maximum momentum of the protons.

Actually, the range of the forces is of the same order as the wave-lengths. In this case, U will decrease slowly with increasing p_0. It may be expressed in terms of the mixed density of the protons (197), with the help of (197c), (199):

$$U=2h^{-3}\int_0^\infty d\mathbf{x}\int d\mathbf{p}J(x)\,\exp\,[i(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{x}/\hbar]$$

$$=\int_0^\infty d\mathbf{x}J(x)e^{-ip_0\cdot\mathbf{x}/\hbar}\rho(\mathbf{x}) \quad (199b)$$

$$=(4\pi\hbar/p_0)\int_0^\infty rdr\rho(r)J(r)\,\sin\,(p_0r/\hbar). \quad (199c)$$

With the "Gaussian potential"

$$J(r)=-Be^{-r^2/a^2}, \quad (199d)$$

U can easily be evaluated, the result being

$$U=-B[\Phi(x-y)+\Phi(x+y) + \pi^{-\frac{1}{2}}y^{-1}(e^{-(x+y)^2}-e^{-(x-y)^2})] \quad (199e)$$

with

$$x=Pa/2\hbar,\quad y=p_0a/2\hbar, \quad (199f)$$

and Φ the error integral (cf. Jahnke-Emde, Table of Functions).

VII. β-Disintegration and Nuclear Forces

§38. Disproof of the Existence of Electrons in Nuclei

It is now generally believed that no electrons exist inside nuclei. The main reasons are the following:

1. *The statistics of nuclei.* Nuclei of even atomic weight generally obey Bose, such of odd weight Fermi statistics. This is to be expected (§4) if the nucleus contains only neutrons and protons. If we would, however, replace one neutron by a proton plus an electron, there would be an increase of the number of elementary particles by one, and therefore a change from Bose to Fermi statistics and *vice versa*.

2. *The nuclear spin.* The corresponding argument holds for the nuclear spins which are integer or half-integer according to whether the atomic weight is even or odd (§5).

3. *The nuclear magnetic moments.* They are all

of the order of the proton magneton $e\hbar/2Mc$, while they should be of the order of the Bohr magneton $e\hbar/2mc$ if electrons existed in the nucleus.

4. *The size of the electron wave function.* The wave-length of an electron with a kinetic energy of the order of a few MV (energy of most β-rays!) is much larger than the nuclear radius (§3, end).

5. *The impossibility of a potential barrier sufficient to keep the electrons inside the nucleus.* This argument is the strongest of all, and we shall therefore discuss it in detail.

The nuclei emitting β-particles have mean lifetimes from 1/50 of a second up to about 10^8 years. There must, therefore, be some force keeping the β-particles inside the nucleus for that length of time, in spite of the fact that they have amply enough energy to escape. It might be tried to assume a potential barrier keeping the electrons from leaving the nucleus, in analogy to α-particles (chapter IX). There are three grave reasons in the way of such an assumption:

(a) To all our knowledge, a nucleus *attracts* an electron at any distance. This is certainly true at large distances (Coulomb force) and at very small distance (owing to the very assumption that there are electrons bound in the nucleus). In order to provide a potential barrier for the electrons, there would have to be a strong repulsion at intermediate distances (a few times nuclear radius, say).

(b) In relativistic theory it is nearly impossible to devise *any* potential barrier which would keep high energy electrons inside the nucleus. To see this, it is sufficient to consider the relativistic *Schrödinger* equation (without spin), *viz.*,

$$\hbar^2c^2\Delta\psi+[(E-V)^2-m^2c^4]\psi=0. \tag{200}$$

E is the total energy of the electron, i.e., its kinetic energy at infinite distance from the nucleus plus mc^2. This equation has a solution of exponential character only if

$$|E-V|<mc^2. \tag{200a}$$

However, it is necessary that the solution is of exponential type in the region of the potential barrier, because only in this case the potential barrier prevents the particles from immediate escape. This means that the potential energy V inside the potential barrier must not differ from the total energy E of the electron by more than mc^2. It is obvious that such a requirement is very unlikely to be fulfilled by a given potential barrier, especially for such nuclei for which the energy E of the β-particles is very large. There is one case (B^{12}) in which $E=24\ mc^2$. Then V would have to be between 23 and 25 mc^2, a very improbable assumption indeed.[78]

(c) Granted that V has really a value satisfying (200a), there would be extremely large perturbations of the optical electrons due to that potential barrier which absolutely contradict experiment. The most favorable assumption is that $V=E$ inside the barrier, let us say for r between R and $R+b$ (R=nuclear radius, b=breadth of barrier, r=distance between nucleus and electron). In this case, the solution of (200) is

$$\psi=Ae^{-mcr/\hbar} \quad \text{for } R<r<R+b, \tag{200b}$$

where A is a constant. The lifetime is then, similar to that of nuclei emitting α-particles (cf. chapter IX)

$$\tau=(R/c)e^{2mcb/\hbar}. \tag{200c}$$

Putting $R=8\cdot10^{-13}$ cm, $c=3\cdot10^{10}$ cm/sec., and assuming a lifetime $\tau=1$ sec., we have

$$\begin{aligned} 2mcb/\hbar&=\log(4\cdot10^{22})=52,\\ b&=26\hbar/mc; \end{aligned} \tag{200d}$$

i.e., the breadth of the potential barrier would have to be much larger than the Compton wave-length $\hbar/mc$. Since the radius of the K shell of heavy atoms is less than twice the Compton wave-length, the potential barrier assumed would change the potential acting on the K electrons, and even on more distant electrons, beyond recognition, and would have tremendous effects on the energies of all these electrons. This disproves completely the assumption of a potential barrier keeping the electrons in the nucleus.

Therefore we are forced to assume that the *electrons observed in β-disintegration did not pre-exist in the emitting nucleus. We suppose that they are formed in the same moment when they are actually emitted*, and that it is this process of formation which is so improbable that it accounts for the long lifetime of β-emitting nuclei.

The process of β-disintegration should therefore be compared not to α-disintegration, but to the emission of light by atoms (or nuclei). This comparison to light seems quite advantageous to explain what is meant by non-pre-existence and formation in the moment of emission: Nobody would say that a hydrogen atom in the third state contains the light quanta corresponding to the spectral lines it may emit, *viz.*, the $H\alpha$-line and the two first lines of the Lyman series. (The second Lyman line may be emitted immediately, or the first after the emission of $H\alpha$.) Still, the hydrogen atom is capable of emitting these light quanta, and it is generally accepted that the

[78] The fact that the β-particles coming from a given nucleus have a continuous energy spectrum, would make the situation quite untenable, because we would have to assume a different height of the barrier for different nuclei of the same species, corresponding to the energy of the β-particle emitted. But the continuous energy distribution of the β-rays cannot be properly understood without the neutrino hypothesis anyway, so we prefer not to use it here as an argument.

quanta are produced in the moment of their emission. The emission of electrons by nuclei is entirely analogous, we have just to substitute "electron" throughout for "light quantum."

§39. The Neutrino (F7, B17, N1, C14, C15, C16)

The assumption that the β-particles which are emitted by radioactive nuclei, did not exist in the nucleus before the emission but are "created" in the moment of emission, solves the difficulties 3, 4 and 5 pointed out in §38. However, there remain the difficulties about statistics and spin, and the still graver difficulty of the continuous β-spectra. These difficulties can only be solved by introducing a new, hypothetical particle, having no charge, very small mass (electron mass or less), spin $\frac{1}{2}\hbar$, and Fermi statistics. This particle is called the neutrino.

The main evidence for the neutrino is the continuous character of the β-spectra. The β-particles emitted by radioactive nuclei do not all have the same energy, but have energies distributed over the whole range from zero to a certain upper limit which we shall denote by E_0. This is in violent contradiction to the fact that the parent nucleus before the emission of the β-ray, and the product nucleus after the emission, have quite exactly determined energies. This follows for natural radioactive nuclei from the fact that the α-particles emitted have definite energies for each transformation. For the artificial radioactive nuclei the proof is even more conclusive, since the masses of the radioactive nucleus as well as of the product nucleus can be determined with very great accuracy, by means of the energy balance of nuclear transmutations involving only heavy particles, or by mass-spectrographic measurements.

We are thus confronted with the following situation: A parent nucleus which is in a quantized state of definite energy, emits an electron and leaves a residual product nucleus, again in a quantized state of definite energy. However, the energy of the emitted electron is not equal to the difference ΔE between the energies of the nucleus before and after emission, but may have *any* amount between 0 and E_0, the energy in each particular case being apparently determined by chance.

There are only two ways of accounting for this situation: either (a) we have to give up the conservation of energy for β-disintegration or (b) we have to assume that, simultaneously with the electron, another particle is emitted which ordinarily is not detected. Such an assumption would immediately account for the experimental facts: The total available energy ΔE will be distributed among the electron and the second, unobservable, particle (neutrino). The electron would therefore only receive a part of ΔE which will vary from case to case. The maximum kinetic energy the electron may receive is

$$E_0=\Delta E-(m+\mu)c^2, \tag{201}$$

where m and μ are the masses of electron and neutrino. E_0 will thus be the upper limit of the β-spectrum.

It seems that the hypothesis (a) i.e., nonconservation of energy, should not be made if it can possibly be avoided. Not only in classical physics and in all branches of atomic physics has the principle of conservation of energy proved extremely successful, but also in the transmutations of nuclei it holds perfectly as long as only heavy particles (of at least proton mass) are involved. This success seems to justify the retention of the principle by all means.

Moreover, there seems to be direct experimental proof against the nonconservation hypothesis, at least if one accepts that energy is conserved *statistically*, in the average over a great number of β-processes. Such an assumption seems necessary; if it were not made, it would be possible to construct a machine for perpetual motion, either using β-processes or their inverse. Assuming now statistical conservation of energy, we have

$$\bar{E}=\Delta E-mc^2, \tag{201a}$$

$\bar{E}$ being the average kinetic energy of the β-particles. This equation is contradicted by experiments, most violently for the artificial radioactive nuclei Li^8 and B^{12} (C14, 15, 16).

The nucleus Li^8 disintegrates into $Be^8+\epsilon^-$, the β-particles having an average energy (C16)

$$\bar{E}_{Li}=3.8 \text{ MV}. \tag{201b}$$

The difference between the masses of Li^8 and Be^8 can be derived from the data

$$Li^7+H^2=Li^8+H^1+Q, \quad (201c)$$

$$Li^7+H^1=2He^4+17.2 \text{ MV}, \quad (201d)$$

$$Be^8=2He^4+0.3 \text{ MV}, \quad (201e)$$

$$H^2=2H^1-1.2 \text{ MV}. \quad (201f)$$

The first transmutation is the one used to produce the radioactive Li^8. The energy Q evolved in it is not known with any certainty. However, it is known that ordinary lithium (mixture of Li^6 and Li^7) bombarded by deuterons yields one and only one proton group of a range of about 30 cm, corresponding to an energy evolution around 5 MV. This group has been attributed to the reaction

$$Li^6+H^2=Li^7+H^1 \quad (201g)$$

since at the time the existence of Li^8 was not suspected, and since the separated isotope Li^6 showed the group. However, no search for the proton group was made with pure Li^7 as target. Therefore we consider it most plausible that *both* isotopes Li^6 and Li^7 contribute to the group, the energies evolved in the two reactions being accidentally the same. In any case, it is absolutely certain that the energy evolved in (201c) is *not more* than 5 MV, it might be equal to that figure or less.—The energy evolved in (201d) is very accurately measured; the difference of the energies of Be^8 and 2α-particles is deduced from the transformation of B^{11} by protons and is certainly accurate to ± 0.5 MV, the difference between the deuteron and two protons is based on the mass spectrographic determination of Bainbridge and also certainly correct to ± 0.5 MV. Thus we find

$$\Delta E=17.2-0.3-1.2-Q \geqslant 10.7 \text{ MV}. \quad (201h)$$

This is more than 6 MV greater than the average energy of the β-particles emitted by Li^8 which is irreconcilable with the assumption of statistical conservation of energy in β-disintegration.

On the other hand, the figure in (201h) agrees perfectly with the upper limit of the β-spectrum emitted by Li^8 which is

$$E_0=10.5 \text{ MV}. \quad (201i)$$

Therefore Eq. (201) holds, which gives strong support to the neutrino hypothesis, designated as alternative (b) above.

The evidence from B^{12} (C15) is similar: B^{12} is formed in the reaction

$$B^{11}+H^2=B^{12}+H^1. \quad (202)$$

From the bombardment of boron by deuterons several proton groups arise, the longest having a range of 92 cm. This bombardment may, besides (202), lead to the reaction

$$B^{10}+H^2=B^{11}+H^1. \quad (202a)$$

The decision between (202) and (202a) is possible by observing the number of the protons of each group and the number of the β-particles emitted by B^{12}. It is found that the β-particles are much more numerous (20 times) than the protons in any group of range longer than 10 cm. Therefrom it follows that the energy evolved in (202) is less than 2.5 MV. With the masses $B^{11}=11.0111$, $H^2=2.0142$, $H^1=1.0081$, we have thus

$$B^{12} \geqslant 11.0111+2.0142-1.0081 \\ -0.0027=12.0145 \quad (202b)$$

and, with the mass 12.0037 for C^{12}, we find

$$\Delta E-mc^2 \geqslant 0.0108 \text{ mass unit}=10.1 \text{ MV}. \quad (202c)$$

This is to be compared to the observed average energy of the β-particles

$$\bar{E}=4 \text{ MV} \quad (202d)$$

and to the maximum energy

$$E_0=11.5 \text{ MV}. \quad (202e)$$

Again, the experiments definitely contradict the assumption of only statistical conservation of energy, and are in good agreement with the neutrino hypothesis.

Further examples are the artificial radioactive nuclei F^{17}, P^{30}, etc. We can thus say that the proof against the only statistical conservation of energy is conclusive, and that the idea of non-conservation of energy has to be abandoned altogether, in favor of the neutrino hypothesis.

Further support for the neutrino hypothesis is derived from the difficulty about the statistics and the spin of nuclei (1 and 2, §38). The assumption that only neutrons and protons are present in nuclei, solves this difficulty for stationary states of nuclei. However, there remains

a difficulty for the β-transformation. For the atomic weight of the nucleus remains unchanged in the β-transformation, therefore its statistics does not change and its spin remains integer (half-integer) if it was integer (half-integer) before. On the other hand, the emitted electron has Fermi statistics and spin $\frac{1}{2}$: Consequently, the total spin of the system cannot be conserved in the β-disintegration, being, for a nucleus of even weight, integral for the parent nucleus, and half-integral for the product nucleus and the electron together. Such a nonconservation of total spin, and a similar nonconservation of the statistics of the system, is almost as bad a contradiction against very well-established laws of nature as the nonconservation of energy would be. Therefore we are again forced to assume the emission of a second particle (neutrino) in the β-disintegration. Doing this, the difficulty is removed if we assume the neutrino to have the spin $\frac{1}{2}\hbar$ and Fermi statistics, as every other elementary particle (electron, positron, proton, neutron). Then the resultant of the spins of electron and neutrino is integral (1 or 0), and the resultant of the spins of all particles left after the β-disintegration, *viz.*, product nucleus, electron and neutrino, is integral or half-integral according to whether the parent nucleus has integral or half-integral spin which allows the total angular momentum to be conserved. Similarly, the statistics remains conserved because now the total number of particles (protons and neutrons in the nucleus, electron and neutrino) increases by 2 in the β-transformation which leaves the statistics of the system unchanged (§4).

A further point to support the neutrino hypothesis is the success of the theory of the β-decay, especially as regards the energy distribution of the electrons (K5, and §40), and the dependence of the lifetime on the maximum energy (§41).

There is thus considerable evidence for the neutrino hypothesis. Unfortunately, all this evidence is indirect; and more unfortunately, there seems at present to be no way of getting any direct evidence. At least, it seems practically impossible to detect neutrinos in the *free state*, i.e., *after* they have been emitted by the radioactive atom. There is only *one* process which neutrinos can *certainly* cause. That is the inverse β-process, consisting of the capture of a neutrino by a nucleus together with the emission of an electron (or positron). This process is, however, so extremely rare (§42) that a neutrino has to go, in the average, through 10^{16} km of solid matter before it causes such a process. The present methods of detection must be improved at least by a factor 10^{13} in sensitivity before such a process could be detected.

Whether there are other processes by which a free neutrino may be detected, depends entirely on its properties. We know for certain that the neutrino has no charge, because the charge of the electron alone accounts for the change of the charge of the radioactive nucleus in β-emission (increase by one unit). The absence of charge precludes any strong ionization due to neutrinos. However, it is theoretically quite conceivable that the neutrino might have a magnetic moment associated with its spin. The ionization due to such a magnetic moment has been calculated (B14) and was found to be about $100n^2$ ions per km path in air, n being the magnetic moment expressed in Bohr magnetons. Nahmias (N1) has searched for ionization produced by neutrinos, using strong radioactive sources shielded by large amounts (about 1 meter) of Pb in order to absorb α-, β- and γ-rays and leave only the neutrinos. No ionization was found larger than the fluctuations of the ionization due to cosmic rays, in spite of the latter's intensity having been cut down by performing the experiment in an underground railway of London. The evaluation shows that neutrinos cannot form more than 1 ion in about 500,000 km path in air, which means that their magnetic moment, if any, must be smaller than 1/7000 Bohr magneton. It seems therefore probable that the neutrino does not have any magnetic moment at all. This makes it futile to search for ionization produced by neutrinos.

Therefore the only hope of getting more direct evidence for the neutrino is from the radioactive decay itself. The recoil of the product nucleus, which can be observed in principle, will decide definitely between the hypothesis of nonconservation of energy and the neutrino hypothesis. According to the neutrino hypothesis, the momentum of the recoil nucleus should be equal and opposite to the resultant of the momenta of the

electron and the neutrino. Therefore, if the momentum of the recoil nucleus and the emitted electron can be observed simultaneously as to magnitude and relative direction, the momentum of the neutrino can be inferred. On the other hand, the energy of the neutrino is directly given as the difference between the upper limit of the β-spectrum and the energy of the β-particle actually observed in a particular experiment. Now the neutrino momentum p and its kinetic energy E must be related by an equation of the form

$$(E+\mu c^2)^2=p^2c^2+\mu^2c^4, \tag{203}$$

μ being the neutrino mass. All observations must be representable by the same value of μ—a severe test to the neutrino hypothesis if the experiments can be performed. It is seen that such experiments would lead to a direct determination of the neutrino mass as well as to a more direct proof for its existence.

The difficulty of the experiments lies in the smallness of the kinetic energy of the recoil nucleus. If we assume that all the energy available (E_0) is given to the electron, the recoil energy of a nucleus of mass M is easily found to be

$$E_r=E_0(E_0+2mc^2)/2Mc^2 \\ =540E_0^{\text{MV}}(E_0^{\text{MV}}+1)/A \text{ volts}, \tag{203a}$$

where E_0^{MV} is the upper limit of the β-spectrum in MV, and A the atomic weight of the radioactive nucleus. For $E_0=2$ MV, which is about average for artificial radioactive nuclei, and $A=20$, we have $E_r=160$ volts. The most favorable case would be Li^8, with $E_0=10.5$ MV, $A=8$ and therefore $E_r=8000$ volts; unfortunately, this element has a very short life ($\frac{1}{2}$ sec.).

The present evidence about neutrinos can be summarized as follows:

No charge
Very small mass, probably zero, at least small compared to electron mass (from β-spectra, §40)
Spin $\frac{1}{2}\hbar$
Fermi statistics
Magnetic moment less than 1/7000 Bohr magneton, if any
No detectable effects in free state

In concluding this section, a word must be said about antineutrinos. It seems probable that neutrinos obey a wave equation similar to the Dirac equation, only the charge (and possibly also the mass) being zero. This wave equation will allow solutions with both positive and negative energy. Just as in the case of electrons, it must be assumed that all states of negative energy are ordinarily filled, in order to avoid the serious difficulties connected with the possibility of transitions from positive to negative energy. A state of negative energy which happens to be empty, is equivalent to a particle analogous to the positron. This particle is called an antineutrino. Since the neutrino has no charge and probably no magnetic moment, the antineutrino cannot be distinguished from the neutrino in any way. There is thus no need of distinguishing neutrinos and antineutrinos, except for the mathematical formalism.

§40. Theory of β-Disintegration

If a nucleus emits a β-particle, its charge increases by one unit while its weight remains unchanged. In other words, the number of protons in the nucleus increases by one, while the number of neutrons decreases by one. Thus the β-transformation can be considered as consisting of the transformation of one neutron into one proton, one electron and one neutrino:

$$n^1\rightarrow H^1+\epsilon^-+n^0. \tag{204}$$

Similarly, a radioactive process in which a positron is emitted, is to be considered as

$$H^1\rightarrow n^1+\epsilon^++n^0. \tag{204a}$$

It need hardly be mentioned after the discussions of §38 that the *neutron* should *not* be considered as *composed of a proton, an electron and a neutrino, but is only able of transforming into these three particles*, and similarly for the proton.

The problem of the theory of β-disintegration is to calculate the *probability* of the processes (204), (204a). Of course, this cannot be done on the grounds of any existing theory, but an entirely new "force" has to be introduced which produces just the transitions (204), (204a), i.e., which converts a neutron into a proton (or *vice versa*) and at the same time produces a (negative or positive) electron and a neutrino. Such a force has been introduced by Fermi (F7), using the analogy to the emission of light discussed at the end of §38.

The probability that a charged particle emits light and at the same time goes over from the state m to the state n, is given by the well-known formula (relativistic theory)

$$w = C|\int u_n^*(\mathbf{r})\mathbf{A}(\mathbf{r})\cdot\boldsymbol{\alpha}u_m(\mathbf{r})d\tau|^2, \tag{205}$$

where u_n and u_m are the wave functions of the particle, $\boldsymbol{\alpha}$ the Dirac operator, C a certain constant and $\mathbf{A}(\mathbf{r})$ the vector potential of a light wave of the correct frequency $(E_m-E_n)/h$ and unit intensity, at the place of the charged particle. (205) can also be expressed by saying that there is a certain term in the Hamiltonian of the charged particle which corresponds to the spontaneous emission of radiation and which has the form

$$H = C'\mathbf{A}(\mathbf{r})\cdot\boldsymbol{\alpha}, \tag{205a}$$

where the transitions of the particle under the influence of this Hamiltonian have to be calculated according to the ordinary methods of the perturbation theory. The vector potential $\mathbf{A}$ may be regarded as a sort of wave function of the emitted light quantum: Thus in the Hamiltonian there appears the wave function of the emitted particle at the place of the emitting particle.

It is reasonable to assume a similar expression for the interaction between a heavy particle, an electron and a neutrino. There are only two differences. Firstly, two particles are produced rather than one, therefore both the wave functions of electron and neutrino have to appear in the Hamiltonian. Secondly, the emission of the two particles changes the character of the heavy particle, converting a neutron into a proton and *vice versa*. Let us introduce an operator Q which corresponds to the conversion of a neutron into a proton, and Q^* corresponding to the opposite conversion. Then a plausible expression for the Hamiltonian of β-emission would be

$$H = g(\psi^*\varphi^*Q + \psi\varphi Q^*), \tag{206}$$

where ψ is the wave function of the electron, φ that of the neutrino, both taken at the place of the heavy particle. The first term corresponds to the creation of an electron and a neutrino, together with the conversion of a neutron into a proton, the second term to the absorption of an electron and a neutrino, or the emission of a positron and an antineutrino, together with the conversion of a proton into a neutron. g is a constant to be derived from experiment.

The mathematical treatment is simplified, and the physical ideas and results not changed (K5) if we let one particle be created and one absorbed in each process, rather than two created or two absorbed. This can be done by assuming that the emission of a negative electron is associated with that of an antineutrino (or with the absorption of a neutrino), while the emission of a positron (or absorption of an electron) is accompanied by the emission of a neutrino. This is equivalent to our previous assumptions, because of the equivalence of neutrino and antineutrino. The Hamiltonian (206) is then to be replaced by

$$H = g(\psi^*\varphi Q + \psi\varphi^*Q^*). \tag{206a}$$

The probability of a β-transformation is given by the ordinary nonstationary perturbation theory. If u_m and u_n are the eigenfunctions of the heavy particle before and after emission, and G_eG_n the number of states of electron and neutrino per unit energy interval, the probability of a β-emission in which the electron receives an energy between E and $E+dE$, is per unit time:

$$w = (2\pi/\hbar)g^2|\int d\tau u_n^*(\mathbf{r})u_m(\mathbf{r})\psi^*(\mathbf{r})\varphi(\mathbf{r})|^2 \times G_nG_e dE. \tag{206b}$$

Thus far, we have not considered relativity. The introduction of relativistic wave functions for the light particles is absolutely essential because their energies are much larger than their "rest" energy, mass$\cdot c^2$. The introduction of relativity for the heavy particles would not be necessary, except for the calculation of forbidden transitions (§§41 to 43) and for symmetry.

To set up the relativistic analog to (206b), we start from the requirement that the integrand in (206b) is relativistically invariant (F8, K5). From two functions ψ and φ, we may build up five quantities, which behave under Lorentz transformations, respectively, like a scalar, a vector, a tensor, a pseudovector and a pseudoscalar,[79] *viz.*,

Scalar: $i(\psi\dagger\varphi) = (\psi^*\beta\varphi)$, (207a)

Four vector:

$$-i(\psi\dagger\boldsymbol{\gamma}\varphi) = \begin{cases} i(\psi^*\boldsymbol{\alpha}\varphi) & \text{(space components),} \\ (\psi^*\varphi) & \text{(time component),} \end{cases} \tag{207b}$$

Tensor:

$$(\psi\dagger\gamma_i\gamma_k\varphi) = \begin{cases} (\psi^*\beta\boldsymbol{\sigma}\varphi) & \text{(if } i \text{ and } k=1,\ 2,\ 3), \\ (\psi^*\beta\boldsymbol{\alpha}\varphi) & \text{(if either } i \text{ or } k=4), \end{cases} \tag{207c}$$

Pseudovector:

$$(\psi\dagger\gamma_i\gamma_k\gamma_l\varphi) = \begin{cases} (\psi^*\boldsymbol{\sigma}\varphi) & \text{("space" components, } i=4,\ kl=1,\ 2,\ 3), \\ i(\psi^*\gamma_5\varphi) & \text{("time" component, } ikl=1,\ 2,\ 3), \end{cases} \tag{207d}$$

Pseudoscalar: $(\psi\dagger\gamma_1\gamma_2\gamma_3\gamma_4\varphi) = (\psi^*\beta\gamma_5\varphi)$. (207e)

[79] Pauli, *Handbuch der Physik*, Vol. 24/1, p. 220, etc.

Here $\boldsymbol{\alpha}=(\alpha_x, \alpha_y, \alpha_z)$ and β are the ordinary Dirac matrices, $\boldsymbol{\gamma}$ is the "matrix vector" with the components

$$\gamma_k=-i\beta\alpha_k \quad \text{for } k=1, 2, 3, \quad \gamma_4=-\beta. \tag{207f}$$

$\boldsymbol{\sigma}$ is the Pauli spin operator, *viz.*,

$\sigma_l=-i\alpha_i\alpha_k$, where the indices ikl follow cyclically upon each other, each being one of the numbers 1, 2, 3 (207g)

$\psi\dagger$ is the "conjugate" to the Dirac wave function, *viz.*,

$$\psi\dagger=-i\psi^*\beta \tag{207h}$$

and finally

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4. \tag{207i}$$

The factors i and $-i$ on the left-hand sides of (207a) to (207e) are chosen so as to make the main components on the right-hand side real.

Five quantities analogous to (207a) to (207e) can be formed from the wave functions of the heavy particles u_mu_n. Multiplying any of the quantities (207a) to (207e) with the corresponding quantity formed from u_mu_n, we obtain an invariant. Thus we have five different possibilities to replace the integrand in (206b) in relativity theory:

Scalar: $(u_n^*\beta u_m)(\psi^*\beta\varphi)$, (208a)

Vector: $(u_n^*u_m)(\psi^*\varphi)-(u_n^*\boldsymbol{\alpha}u_m)\cdot(\psi^*\boldsymbol{\alpha}\varphi)$, (208b)

Tensor: $(u_n^*\beta\boldsymbol{\sigma}u_m)\cdot(\psi^*\beta\boldsymbol{\sigma}\varphi)+(u_n^*\beta\boldsymbol{\alpha}u_m)\cdot(\psi^*\beta\boldsymbol{\alpha}\varphi)$, (208c)

Pseudovector: $(u_n^*\boldsymbol{\sigma}u_m)\cdot(\psi^*\boldsymbol{\sigma}\varphi)-(u_n^*\gamma_5u_m)(\psi^*\gamma_5\varphi)$, (208d)

Pseudoscalar: $(u_n^*\beta\gamma_5u_m)(\psi^*\beta\gamma_5\varphi)$. (208e)

Fermi chose originally an expression similar to (208b). From the standpoint of the general theory of nuclear forces, the "tensor" or the "pseudovector" expression (208c), (208d) are preferable (§44). (The dot means scalar product.)

For the heavy particles, the operator β practically does not change the wave functions, the operator $\boldsymbol{\sigma}$ acts on the spin part of the wave function but leaves the order of magnitude practically unchanged, while $(u_n^*\boldsymbol{\alpha}u_m)$ and $(u_n^*\gamma_5u_m)$ are small compared to $(u_n^*u_m)$, *viz.*, of the relative order v/c where v is the velocity of the heavy particles.[80] Therefore the second terms in (208b, c, d) can practically be neglected (except for forbidden transitions, §§41 to 43). This makes the results from expressions (208a) and (208b), and from (208c) and (208d), very nearly identical.

The energy distribution of the β-particles can easily be calculated from (206b) after a definite one of the expressions (208a) to (208e) has been chosen to replace the integrand in (206b). One must simply insert a plane wave for the neutrino wave function, while the electron wave function ψ is to be taken in the Coulomb field of the disintegrated nucleus. For light nuclei, it is allowable to neglect the effect of the Coulomb field and thus to replace the electron wave functions by plane waves as well. Since the wave-lengths of electron and neutrino are large compared to the nuclear radius for all known β-transformations, ψ and φ may be regarded as constant and taken outside the integral. With these approximations, the energy distribution of the β-particles turns out almost identical whichever of the expressions (208a) to (208e) is accepted. The result for w is

$$w=\frac{1}{2\pi^3}\frac{mc^2}{\hbar}\left(\frac{g}{mc^2(\hbar/mc)^3}\right)^2\frac{E_e}{mc^2}\frac{p_e}{mc}\times\frac{E_n}{mc^2}\frac{p_n}{mc}\frac{dE}{mc^2}|G|^2, \tag{209}$$

where $E_eE_np_ep_n$ are energy and momentum of electron and neutrino. (The energies are supposed to include the terms mc^2 and μc^2, resp.) G is the matrix element

$$G=\int d\tau u_n^*u_m \quad \text{for (208a) or (208b)}, \tag{209a}$$

$$G=\int d\tau u_n^*\boldsymbol{\sigma}u_m \quad \text{for (208c) or (208d)}. \tag{209d}$$

A small term of the relative order $mc^2\mu c^2/E_eE_n$ has been neglected in (209).

From the shape of the β-spectrum near the maximum energy of the electrons, the mass of the neutrino can be deduced (F8). The experimental evidence points to a mass very small compared to the electron mass, probably zero. This conclusion is reached as follows: If E_e is near its upper limit E_0, the factors E_e and p_e in (209) may be regarded as constant. If the neutrino mass is not zero, E_n may also be considered constant, *viz.*, equal to μc^2, as long as $E_0-E_e\ll\mu c^2$. Under the same condition, we may insert for p_n the nonrelativistic expression $p_n=(2\mu)^{\frac{1}{2}}(E_n-\mu c^2)^{\frac{1}{2}}=(2\mu)^{\frac{1}{2}}(E_0-E_e)^{\frac{1}{2}}$. Thus at the upper limit of the energy spectrum (209) would go to zero as $(E_0-E_e)^{\frac{1}{2}}$, i.e., with *vertical tangent*. Actually, the observations show that the number of β-particles per unit energy goes to zero with *horizontal* tangent near the upper limit of the β-spectrum. This can only be understood if μ is assumed to be zero: Then $E_n=cp_n=E_0-E_e$, and (209) becomes proportional to $(E_0-E_e)^2$, if the electron energy E_e is near the upper limit E_0. A very

[80] Cf. *Handbuch der Physik*, Vol. 24/1, p. 301, etc.

small mass of the neutrino, up to about $\frac{1}{5}$ of the electron mass, would however seem tolerable in the light of the present evidence.

In the following, we shall put the neutrino mass equal to zero. Then (209) can be written

$$w=\frac{mc^2}{2\pi^3\hbar}\left(\frac{gm^2c}{\hbar^3}\right)^2|G|^2\epsilon(\epsilon^2-1)^{\frac{1}{2}}(\epsilon_0-\epsilon)^2d\epsilon \quad (210)$$

with the abbreviations

$$\epsilon=E_e/mc^2, \quad \epsilon_0=E_0/mc^2. \quad (210a)$$

If $\epsilon_0\gg 1$, which is the case for many of the radioactive substances, the 1 in $(\epsilon^2-1)^{\frac{1}{2}}$ may be neglected compared to ϵ over the larger part of the energy spectrum. Then w becomes proportional to $\epsilon^2(\epsilon_0-\epsilon)^2$, i.e., there is a maximum of the probability for equal distribution of the energy among electron and neutrino ($\epsilon=\frac{1}{2}\epsilon_0$) and the distribution is symmetrical with respect to the two particles (P5). This is in contradiction to experiments: It is found generally that the electron receives, in the average, much less than half the maximum energy E_0. In other words, the neutrino energy is in the average larger than the electron energy.[81]

This shows that the theory in the form hitherto used does not account for the experimental facts. It is necessary to correct it in such a way that the emission of neutrinos of high energy becomes theoretically more probable. This has been done by Konopinski and Uhlenbeck (K5), by introducing the derivative of the neutrino wave function with respect to time instead of the wave function itself. There are three possible expressions involving the first derivative of φ which correspond to the expressions (208b, c, d), *viz.*,

Vector: $(u_n^*u_m)(\psi^*\beta\partial\varphi/\partial t)$

$$-c(u_n^*\boldsymbol{\alpha}u_m)(\psi^*\beta\ \mathrm{grad}\ \varphi), \quad (211b)$$

Tensor: $(u_n^*\beta\boldsymbol{\sigma}u_m)\cdot(\psi^*[\alpha\times\mathrm{grad}\ \varphi])$

$$+(u_n^*\beta\boldsymbol{\alpha}u_m)(\psi^*\boldsymbol{\alpha}(\partial\varphi/c\partial t)-\psi^*\ \mathrm{grad}\ \varphi), \quad (211c)$$

Pseudovector:

$$(u_n^*\boldsymbol{\sigma}u_m)\cdot(\psi^*\beta\{[\alpha\times\mathrm{grad}\ \varphi]-i\boldsymbol{\sigma}(\partial\varphi/c\partial t)\})$$
$$+i(u_n^*\gamma_5u_m)(\psi^*\beta(\boldsymbol{\sigma}\ \mathrm{grad}\ \varphi)). \quad (211d)$$

(211b) is the expression chosen by Uhlenbeck, while a linear combination of (211b) and (211c) or (211d) must be taken if one wants to connect β-emission and general nuclear forces (§44). Making the same assumptions as when deriving (209), *viz.*, small nuclear charge and zero neutrino mass, we obtain from (211)

$$wd\epsilon=\frac{mc^2}{2\pi^3\hbar}\left(\frac{g}{mc^2(\hbar/mc)^4}\right)^2|G|^2\epsilon(\epsilon^2-1)^{\frac{1}{2}}(\epsilon_0-\epsilon)^4d\epsilon, \quad (212)$$

where G has the same meaning as in (209a) if the expression (211b) is chosen, while it is $(\frac{2}{3})^{\frac{1}{2}}$ and $(\frac{1}{3})^{\frac{1}{2}}$ times (209b), respectively, if (211c) or (211d) is accepted for the interaction between heavy particle, electron and neutrino. (212) differs from (210) by containing the *fourth* power of the neutrino energy $\epsilon_0-\epsilon$, rather than the second. This difference arises from the *derivatives* contained in (211): Since $\partial\varphi/\partial t=-iE_n\varphi/\hbar$, the introduction of the derivative introduces a factor E_n in the integral in (206b), and therefore a factor $E_n^2=(E_0-E_e)^2$ in the transition probability w. This additional factor $(E_0-E_e)^2$ is just what is required to bring about agreement with the experimental energy distribution in β-spectra: The factor makes for an increase in the probability of emission of slow electrons and fast neutrinos, compared to that of fast electrons and slow neutrinos. The most probable electron energy is shifted to $\frac{1}{3}E_0$ for large E_0.

A more quantitative comparison between formula (212) and the experimental energy distributions for all well-investigated β-spectra, has been carried out by Konopinski and Uhlenbeck (K5). The result is very satisfactory. Moreover, the *total* disintegration probability (integral of (212) over ϵ), i.e., the reciprocal lifetime of the radioactive nucleus, is also well represented by (212) in its dependence on the maximum energy (§41). It seems therefore that one of the expressions (211) must be very nearly correct. We shall therefore accept the Konopinski-Uhlenbeck theory as the basis of our future discussions.

[81] This is true of the light radioactive nuclei as well as of the heavy ones. The discrepancy can therefore not be attributed to the neglection of the electrostatic action of the nucleus on the electrons.

§41. Lifetime *vs.* Maximum Energy in β-Disintegration (F8, S1, K16)

The total probability of β-decay can be obtained easily by integrating (212) over all electron energies from $\epsilon=1$ to $\epsilon=\epsilon_0$. The result is

$$(\log 2)/\tau=\int_1^{\epsilon_0} w d\epsilon=|G|^2 f(\epsilon_0)/\tau_0, \quad (213)$$

where τ is the half-life of the β-disintegrating nucleus,

$$\tau_0=(2\pi^3\hbar/mc^2)(mc^2)^2(\hbar/mc)^3 g^{-2} \quad (213a)$$

is a time characteristic for β-decay, G is the matrix element referring to the transition of the heavy particles (cf. 209a, b), and

$$f(\epsilon_0)=(\epsilon_0^2-1)^{\frac{1}{2}}\left(\frac{1}{105}\epsilon_0^6-\frac{2}{21}\epsilon_0^4-\frac{247}{420}\epsilon_0^2-\frac{8}{105}\right)$$
$$+\tfrac{1}{2}\epsilon_0(\epsilon_0^2+\tfrac{1}{2})\log(\epsilon_0+(\epsilon_0^2-1)^{\frac{1}{2}}), \quad (213b)$$

is a function of the upper limit $E_0=mc^2\epsilon_0$ of the β-spectrum. If the kinetic energy of the β-particles is small, i.e., if ϵ_0-1 is small compared to unity, (213b) can conveniently be expanded in powers of ϵ_0-1, with the result

$$f(\epsilon_0)=(256/5\cdot 7\cdot 9\cdot 11)\sqrt{2}(\epsilon_0-1)^{5\frac{1}{2}}$$
$$+0(\epsilon_0-1)^{6\frac{1}{2}}. \quad (213c)$$

The lifetime τ of β-disintegrating nuclei is, according to (213), inversely proportional to $f(\epsilon_0)$, and therefore decreases rapidly with increasing kinetic energy of the β-particles. For small kinetic energy, $\tau\sim(\epsilon_0-1)^{-5\frac{1}{2}}$ (cf. 213c), for large kinetic energy, $\tau\sim\epsilon_0^{-7}$ (cf. 213b). This behavior agrees, qualitatively and quantitatively, with experiment. This can be seen from Table XV, in which the product $f(\epsilon_0)\tau$ is listed for a number of radioactive nuclei, for which the upper limit of the β-spectrum ϵ_0 is well known.[82] Now according to (213),

$$\tau f(\epsilon_0)=\tau_0\log 2\,|G|^{-2}. \quad (213d)$$

Here τ_0 is a universal constant. The matrix element G will be nearly unity if the neutron before the disintegration is in almost the same state as the proton after the disintegration. This will be true at least for a large number of light radioactive nuclei. If there is considerable difference between the states of neutron and proton, G will be smaller than unity. G may even vanish, in this case we have a "forbidden transition" which will be discussed below.

Since we expect the matrix element G to be nearly unity for a considerable number of radioactive nuclei, the product τf should have approximately the same value for all these nuclei. This is actually true for the first group of nuclei in Table XV (group $0A$), for all of which τf has a value between $0.4\cdot 10^5$ and $3\cdot 10^5$, in spite of considerable differences between the lifetimes τ of the various nuclei. Absolute agreement of the values τf is of course not to be expected because of the differences in the matrix element G. However, the agreement is good enough to allow the determination of a rough value of the universal constant τ_0. This constant must lie at least in the neighborhood of

$$\tau_0=0.7\cdot 10^5 \text{ sec.}, \quad (214)$$

i.e., approximately one day. Using this value, and the value of the "characteristic electronic time"

$$\hbar/mc^2=1.3\cdot 10^{-21} \text{ sec.} \quad (214a)$$

we find from (213a) for the constant g of the β-decay

$$g=1.1\cdot 10^{-13} mc^2(\hbar/mc)^4 \quad (214b)$$
$$=6.5\cdot 10^{-3} Mc^2(\hbar/Mc)^4 \quad (214c)$$
$$=1.9\cdot 10^{-60} \text{ erg cm}^4. \quad (214d)$$

Corresponding to the fact that the lifetime of β-disintegrating nuclei (order of some seconds) is extremely long compared to nuclear times ($\sim 10^{-21}$ sec.), the constant g turns out to be extremely small if mc^2 and $\hbar/mc$ are chosen as the unit of energy and length, respectively (cf. 214b). Since the β-decay is a property of heavy particles, it may seem more appropriate to choose Mc^2 and $\hbar/Mc$ as units: In these units, g is about 1/150. The smallness of g causes some difficulties if one tries to connect the neutron-proton forces and the β-decay (§44).

Formula (213), (213b) is only true for light nuclei for which the influence of the nuclear field

[82] A very useful method for determining this upper limit has been suggested by Kurie, Richardson and Paxton (K16).

on the wave function of the β-particle may be neglected. For heavy nuclei the wave functions of the electron in the Coulomb field must be used. Then (212) must be replaced by (cf. K5)

$$wd\epsilon = \frac{mc^2}{2\pi^3\hbar}\frac{g^2}{(mc^2)^2(\hbar/mc)^8}|G|^2 \times\epsilon(\epsilon^2-1)^{\frac{1}{2}}(\epsilon_0-\epsilon)^4 e^{\pi\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}} \times|\Gamma(s+i\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}})/\Gamma(2s+1)|^2 \times(2pR/\hbar)^{2s-2}d\epsilon, \quad (215)$$

where

$$\gamma = Z/137, \qquad s=(1-\gamma^2)^{\frac{1}{2}},$$
$$p = \text{electron momentum}, \quad (215a)$$
$$R = \text{nuclear radius}, \quad \Gamma = \Gamma\text{-function}.$$

(215) may be approximated as follows

$$wd\epsilon = \frac{|G|^2}{\tau_0}2\pi\gamma\epsilon(\epsilon_0-\epsilon)^4\frac{\epsilon}{1-e^{-2\pi\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}}} \times[\epsilon^2(1+4\gamma^2)-1]^{s-1}\left(\frac{Rmc}{\hbar}\right)^{2(s-1)}\frac{4}{(2s)!^2}. \quad (215b)$$

The expression $e^{-2\pi\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}}$ may be neglected, even for very high energy and medium large nuclear charge. The main differences between (212) and (215b) are: (1) an additional factor $2\pi\gamma$, which, for $Z=88$ (radium), is equal to 4, and (2) the four last factors in (215b) which stand instead of $(\epsilon^2-1)^5$, giving, for $Z=88$, $R=9\cdot10^{-13}$ and $\epsilon=3$, an increase by a factor 6. The first factor would be present in non-relativistic wave mechanics as well; it is due to the influence of the Coulomb field which increases the probability of the electron being near the nucleus. The other factors are characteristic of the relativistic wave mechanics. Both (1) and (2) tend to increase the probability of β-disintegration for heavy nuclei, altogether by a factor of about 24. *The lifetime of a heavy β-radioactive nucleus should therefore be considerably shorter than that of a light radioactive nucleus, if the upper limit of the β-spectrum is the same in both cases.*[83]

[83] It is therefore not correct to plot lifetimes *vs.* energy in the *same* diagram for heavy and light nuclei (Sargent curves). If heavy and light radioactive nuclei are to be compared, the factors due to the Coulomb field and to relativity have to be taken into account.

From (215b), the reciprocal half-life may easily be found by integrating over ϵ. The square bracket may be regarded as constant for this purpose. We find a formula of the type (213), with

$$f_0(\epsilon_0) = 2\pi\gamma[\bar\epsilon^2(1+4\gamma^2)-1]^{s-1}\left(\frac{2}{5}\frac{Rmc}{\hbar}\right)^{2(s-1)} \times\frac{4}{(2s!)^2}\left[\frac{1}{105}\epsilon_0^7-\left(\frac{1}{5}\epsilon_0^5-\frac{1}{3}\epsilon_0^3+\frac{1}{7}\epsilon_0\right)\right]. \quad (216)$$

Here $\bar\epsilon$ is a suitably chosen average energy of the electrons. The dependence on the maximum energy is mainly contained in the last bracket; in this bracket the first term is much larger than the others unless ϵ_0 is very small. For large energies ϵ_0 the dependence on energy is the same for light (cf. 213b) and heavy nuclei (216).

The values of $\tau f(\epsilon_0)$ for some heavy radioactive nuclei are tabulated in Table XV under $0B$. While they agree fairly well among themselves, they are appreciably higher than the values for light nuclei (group $0A$ in table). This may indicate that the matrix elements G for heavy nuclei are in the average smaller than for light ones, which would be quite plausible because neutron and proton wave functions are certainly very different in heavy, and very similar in light nuclei. Probably, a β-disintegration of a heavy nucleus is always connected with a complete rearrangement, which should reduce the value of G. (Changes in the fundamental expression for the β-disintegration do not affect the ratio of the disintegration probability of light and heavy nuclei appreciably.)

We shall now discuss the *forbidden transitions*. We call a β-disintegration forbidden if the matrix element G vanishes for the transition. The most common cause for this will be a change of the total angular momentum I of the nucleus. We may distinguish forbidden transitions of the first, second, third . . . kind according to changes of I by $L=1, 2, 3\cdots$. The "forbidden" transitions will, of course, not be completely forbidden but will only be much less probable than the "allowed" transitions with $L=0$.

In order to calculate the probability of forbidden transitions, we must not make certain approximations which we made thus far (cf.

paragraph above 209). The dependence of the electron and neutrino wave function on the coordinates inside the nucleus, and the "small" second term in the fundamental expression (211b) for the interaction, must be taken into account. The small term in the interaction does in no case yield a bigger transition probability than the large term, and can therefore be neglected even now, provided we want only to know the order of magnitude of the effects.

If the nuclear moment changes by L, the product of the eigenfunctions u_n and u_m will contain a spherical harmonic of order L,[84] besides a factor depending on the radius r. The transition probability will then depend on the integral (cf. (211b), $\partial\varphi/\partial t = -iE_n\varphi/\hbar = \text{const}\cdot\varphi$)

$$\int P_L(\theta)F(r)[\psi_1^*(r\theta)-\psi_3^*(r\theta)]e^{-iE_{ns}l/\hbar c}d\tau, \quad (217)$$

where $F(r)P_L(\theta)$ represents the product $u_m^*u_n$, the exponential is the neutrino wave function, and ψ_1 and ψ_3 are the first and third Dirac component of the electron wave function.[85] The direction of motion of the neutrino is assumed parallel to z, and without loss of generality z may be chosen as axis of the polar coordinate system, so that $z=r\cos\theta$. The exponential in (217) may be expanded; any term in the expansion is much smaller than the preceding, the ratio being about $E_nR/\hbar c$ (R=nuclear radius), i.e., about 1/20 for neutrino energies of about 1 MV. The nth term in the expansion contains a factor $(\cos\theta)^n$. Of the electron wave functions ψ_1 and ψ_3, *one* contains a spherical harmonic $P_k(\theta)$, the other $P_{k-1}(\theta)$ if $k-\frac{1}{2}=j$ is the angular momentum of the electron. The function containing P_k is somewhat smaller than the other, but only by a factor of the order γ, i.e., about 1/2 for heavy nuclei and still 1/7 for a nucleus as light as K.

In order that (217) does not vanish we must take such a term in the expansion of the neutrino wave function that its product with ψ_1 or ψ_3, contains $(\cos\theta)^L$. For this purpose, we have to take the $L-k$th term, multiplied by the smaller electron function, or the $L-k+1$st term, multiplied by the larger electron function. According to the foregoing, the former choice will give much the larger contribution to the integral. The contribution from different values of k is about the same, that from $k=1$ (s and $p_{\frac{1}{2}}$ electrons) being possibly slightly larger than that from higher values of $k(2, 3, \cdots, L)$. The contribution of $k=1$ to the transition probability will therefore be of the same order as the total transition probability.[86] The lifetime in the case of forbidden transitions then becomes again of the form (213), but now with $f(\epsilon_0)$ being replaced by

$$f_L(\epsilon_0)=\frac{4\pi}{9}\gamma^3\epsilon_0^{2L+5}\left(\frac{mcR}{\hbar}\right)^{2L-4+2s} \times\frac{L}{(2L+3)(2L+4)(2L+5)}\cdot\frac{1}{1^2\cdot3^2\cdot5^2\cdots(2L-1)^2} \times\frac{4}{(2s!)^2}[\epsilon^2(1+4\gamma^2)-1]^{s-1} \quad (218)$$

and G being replaced by

$$G_L=\int u_n^*u_mY_{LM}(r/R)^Ld\tau, \quad (218a)$$

where Y_{LM} is a spherical harmonic and M is the difference between the magnetic quantum numbers of the states u_m and u_n. G_L will again be of the order of magnitude unity, but rather smaller.

[84] There may be terms containing higher spherical harmonics, but their contribution to the transition probability is negligible.

[85] The neutrino spin has been assumed parallel to z. If it is opposite, ψ_2 and ψ_4 appear instead of $\psi_1\psi_3$. This does not change the result.

[86] The addition of the contributions of higher k's will increase the transition probability, the consideration of the second term in the interaction expression (211b) will decrease it somewhat. The result seems to be about within a factor 2 equal to (218).

TABLE XV. *Lifetimes of β-radioactive nuclei.**

NUCLEUS	τ(sec.)	ϵ_0-1	$f(\epsilon_0)$	$\tau f(\epsilon_0)$
GROUP 0A				
C^{11}	1200	2.5	30	$0.4\cdot10^5$
N^{13}	660	2.83	60	$0.4\cdot10^5$
O^{15}	150	3.9	450	$0.7\cdot10^5$
F^{17}	70	4.9	1900	$1.3\cdot10^5$
Si^{27}	150	3.9	450	$0.7\cdot10^5$
Mg^{27}	620	(3.9)	450	$2.8\cdot10^5$
GROUP 0B			f_0	τf_0
UX_2	94	5.5	19700	$18\cdot10^5$
Ra B	2300	2.27	21.9	$5\cdot10^5$
Th B	55000	1.70	1.1	$6\cdot10^5$
Th C″	275	4.5	5700	$16\cdot10^5$
Ac C″	410	3.73	1410	$6\cdot10^5$
GROUP 2 OR HIGHER				
P^{32}	$1.3\cdot10^6$	4.1	600	$0.8\cdot10^9$
K^{40}	$3\cdot10^{15}$	1.4	1.0	$3\cdot10^{15}$

NUCLEUS	τ	ϵ_0-1	$f(\epsilon_0)$	$\tau f(\epsilon_0)$
GROUP 1A				
Li^8	1/2	21.8	$3\cdot10^7$	$1.5\cdot10^7$
B^{12}	1/50	25.8	$12\cdot10^7$	$0.6\cdot10^7$
N^{16}	9	(13)	10^6	$0.9\cdot10^7$
F^{20}	40	(10)	$2\cdot10^5$	$0.8\cdot10^7$
Na^{22}	$1.5\cdot10^7$	(1.2)	0.4	$0.6\cdot10^7$
Na^{24}	54000	4.1	600	$3.4\cdot10^7$
Al^{28}	180	7.8	$3.6\cdot10^4$	$0.7\cdot10^7$
P^{30}	195	9.6	$1.3\cdot10^5$	$2.5\cdot10^7$
Si^{31}	9600	4.1	600	$0.6\cdot10^7$
Cl^{38}	2200	12.5	10^4	$2.2\cdot10^7$
K^{42}	4850	9.0	10^5	$4.5\cdot10^7$
GROUP 1B			f_0	τf_0
Ra C	1700	7.14	140000	$2.4\cdot10^8$
Ra E	$6\cdot10^6$	3.38	650	$4\cdot10^9$
Th C	8500	5.29	18000	$1.5\cdot10^8$
$MsTh_2$	32000	6.21	55000	$1.7\cdot10^9$

* Data on maximum energy mostly from Kurie, Richardson and Paxton (K16) and from Fowler, Delsasso and Lauritsen (F15).

Comparing (218) to (216), we find

$$f_L(\epsilon_0)/f_0(\epsilon_0)=\frac{\gamma^2}{9}\left(\frac{E_0R}{\hbar c}\right)^{2L-2} \times\frac{7\cdot6\cdot5}{(2L+5)(2L+4)(2L+3)}\frac{L}{1^2\cdot3^2\cdot5^2\cdot\cdots(2L-1)^2}. \quad (218b)$$

Thus the probabilities of the first forbidden ($L=1$) and the allowed transitions are in the ratio $\gamma^2/9$, which is about 1 : 40 for heavy nuclei and considerably less for light nuclei. However, for very light nuclei and high energy E_0 this does not hold, the ratio being then $(E_0R/\hbar c)^2$ rather than $\gamma^2/9$ whenever the first quantity is larger than the second. Since the forbidden transitions of the first order have, for light nuclei, been observed mainly in cases of high maximum energy E_0, a ratio 1 : 100 is in the average to be expected for the probabilities of first forbidden and allowed transitions. In Table XV, we have listed a number of nuclei for which the β-disintegration is apparently of the first forbidden type: Group 1A contains light nuclei, 1B heavy nuclei of this type. For simplicity, we have again calculated $\tau f_0(\epsilon_0)$ for each nucleus. Since actually $\tau f_1(\epsilon_0)$ should be equal to $\tau_0(\log 2)|G|^{-2}$, we expect τf_0 to be about 100 times as large. Indeed, the values of τf_0 are about 100 times larger for group 1 than for group 0. The difference between heavy and light nuclei is again found for the forbidden transitions.

The probability of forbidden transitions of higher order decreases, according to (218b), by a factor $(E_0R/\hbar c)^2$ per order. Besides, there is another factor (last factor in (218b)) which also decreases rapidly with increasing L. Thus the lifetime for β-active nuclei becomes very long if the β-disintegration corresponds to a forbidden transition. There are three β-disintegrations known which have exceedingly long lifetimes: K, Sr and Nd. The isotope concerned in the case of K is probably K^{40}, according to a suggestion of Klemperer (K4). The maximum energy of the β-rays of K is about 0.7 MV $=1.4mc^2$, the nuclear radius is probably about $4.5\cdot10^{-13}$ cm, therefore $E_0R/\hbar c=0.016$. *A change of the angular momentum by $L=3$ will be amply sufficient to account* for the observed lifetime. We have for $L=3$ from (218b)

$$f_3/f_0=\frac{1}{9}\cdot\frac{1}{7^2}\cdot0.016^4\cdot\frac{7\cdot6\cdot5}{11\cdot10\cdot9}\cdot\frac{3}{3^2\cdot5^2}=4\cdot10^{-13} \quad (218c)$$

corresponding to a life $2.5\cdot10^{12}$ times longer than for an allowed transition. Actually, the quantity $\tau f(\epsilon_0)$ is only $3\cdot10^{10}$ times larger for K than for most allowed disintegrations. Thus the assumption of a change of the nuclear moment by 3 units is more than sufficient to account for the long life of K^{40}. Such a change seems likely from general considerations (§34).

§42. The Inverse β-Processes: Capture of Orbital Electrons by Nuclei, Disintegration of Nuclei by Electrons and Neutrinos

The following three "inversions" of the ordinary β-process seem of interest

(1) The capture of an orbital electron of the atom by a nucleus, with the emission of a neutrino.

(2) The capture of an incident free electron by a nucleus, with the emission of a neutrino.

(3) The capture of an incident neutrino by a nucleus, with the emission of a (positive or negative) electron.

Processes (1) and (2) lead to a decrease of the nuclear charge by one unit, (3) to an increase, if a negative, a decrease, if a positive electron is emitted.

All three processes are, of course, only possible if the necessary energy is available; e.g., the condition for process (1) is, if the binding energy of the orbital electron is neglected compared to the nuclear energies:

$$Z^A+\epsilon^->(Z-1)^A+n^0. \quad (219)$$

Here Z^A denotes the mass of the original nucleus, of charge Z and mass number A, $(Z-1)^A$ is the mass of the product nucleus, ϵ^- that of the captured electron and n^0 that of the emitted neutrino. The condition is certainly fulfilled for all positron-emitters; in fact, for these the more stringent condition

$$Z^A>(Z-1)^A+\epsilon^++n^0 \quad (219a)$$

must be fulfilled. However, the process of capture of an electron is of no great interest for positron emitters, because for these nuclei the emission of

a positron will in general be much more probable than the capture of an orbital electron.

There will, however, certainly be some nuclei Z^A for which (219) but not (219a) is fulfilled. One of these nuclei is He^3, provided the mass of the neutrino is zero (or very small), as we have assumed in §40. F^{18} might be another, judging from the general trend of the masses of analogous nuclei in its neighborhood. Finally, it is probable that In^{113}, Sn^{115} and Te^{123} (or perhaps one of them) belong in this category, being isobaric with the "neighboring" nuclei Cd^{113}, In^{115} and Sb^{123}, respectively (cf. §43).

The probability of capture is, of course, greatest for the K electrons, since they are most frequently inside the nucleus. The probability can easily be calculated from the general theory of β-decay. It is a function of the kinetic energy $E_0=\epsilon_0 mc^2$ given to the neutrino in the process, i.e., of the excess of the energy on the left-hand side of (219) over than on the right-hand side. The lifetime may again be expressed in the form (213), with G having the form (213b) and (218a) for allowed and forbidden transitions, respectively, and with

$$f_K(\epsilon_0)=2\pi\gamma^3\epsilon_0^4\left(\frac{2R}{a_K}\right)^{2s-2}\frac{1+s}{2s!} \tag{220}$$

for allowed transitions ($L=0$) and

$$f_K{}^L(\epsilon_0)=\frac{2\pi}{9}\gamma^5\frac{\epsilon_0{}^2(L-1+\epsilon_0)^2}{1^2\cdot 3^2\cdot\cdots(2L-1)^2L}\times\left(\frac{E_0R}{\hbar c}\right)^{2L-2}\left(\frac{2R}{a_K}\right)^{2s-2}\frac{1+s}{2s!} \tag{220a}$$

for forbidden transitions in which the nuclear moment changes by L. a_K denotes the Bohr radius of the K shell, for s see (215a).

For the nucleus He^3 we have $\gamma=2/137$, and therefore $s\approx 1$. Thus

$$f_K=2\cdot 10^{-5}(E_0/mc^2)^4 \tag{221}$$

and, with $G=1$ and (213):

$$\tau=2.5\cdot 10^9(mc^2/E_0)^4 \text{ sec.}=100(mc^2/E_0)^4 \text{ years.} \tag{221a}$$

The energy set free in the capture of an electron by He^3 is equal to the difference of the masses of He^3 and H^3. This difference seems to be somewhere in the neighborhood of 0.0002 mass unit $=0.4mc^2$ (cf. chapter XVII, or §22). With this value, He^3 would have a lifetime of about 5,000 years. This would mean that He^3 cannot be found in nature since it would have decayed long ago; but artificially produced He^3 would not change over into H^3 in any time allowing easy observations. It is to be noted that the capture of an electron by a nucleus is not observable as a β-process but could only be deduced from the fact that the product substance (in our case H^3) slowly accumulates in a material which originally contained only the parent substance (He^3).

For $Z=50$, i.e., in the region in which pairs of neighboring isobars are found (§43), we have $\gamma=0.365$, $s=0.93$, $R=7\cdot 10^{-13}$ cm, $a_K=1.05\cdot 10^{-10}$ cm, and therefore

$$f_K{}^0=0.62\epsilon_0{}^4, \tag{222}$$

$$f_K{}^L=0.009\epsilon_0{}^2\frac{(L-1+\epsilon_0)^2}{1^2\cdot 3^2\cdot\cdots(2L-1)^2L}\left(\frac{E_0R}{\hbar c}\right)^{2L-2}. \tag{222a}$$

Therefore the lifetime becomes

$$\tau=0.8\cdot 10^5(mc^2/E_0)^4 \text{ sec.}=1(mc^2/E_0)^4 \text{ days} \quad \text{for } L=0, \tag{222b}$$

$$\tau=0.15(mc^2/E_0)^4 \text{ years} \quad \text{for } L=1, \tag{222c}$$

$$\tau=0.8\cdot 10^4(mc^2/E_0)^4[mc^2/(E_0+mc^2)]^2 \text{ years} \quad \text{for } L=2, \tag{222d}$$

$$\tau=0.9\cdot 10^9(mc^2/E_0)^6[mc^2/(E_0+2mc^2)]^2 \text{ years} \quad \text{for } L=3. \tag{222e}$$

The lifetime thus increases very rapidly with increasing order of the forbidden transition, a fact which is very important for the question of stability of isobars (§43).

We shall now turn to the processes (2) and (3) mentioned at the beginning of this section, i.e., the disintegration of nuclei by free electrons or neutrinos. Both these processes are about equally probable for equal energy of the incident particle, because the β-theory is almost symmetrical in electrons and neutrinos. Both processes are exceedingly rare, because of the small value of the characteristic constant g. Their probability can be estimated very easily from the probability

of capture of a β-particle: The probability of "allowed" capture processes is proportional to the probability of the incident particle being at the nucleus. For the capture from the K shell, this probability is equal to the nuclear volume divided by the volume of the K shell, which, for small nuclear charge, is πa_K^3. For a free electron moving through a material containing per cm^3 N atoms whose nuclei can be disintegrated, the probability of being in the nucleus is the nuclear volume divided by the volume per atom, the latter being equal to $1/N$. The ratio of the probabilities is thus $\pi N a_K^3$. Since $a_K=\hbar/mc\gamma$, the function $f(\epsilon_0)$ becomes for the capture of a free electron by light nuclei (cf. 220)

$$f_e(\epsilon_0)=2\pi^2 N(\hbar/mc)^3\epsilon_0^4. \tag{223}$$

The time until a free electron is captured, is thus (cf. 213) $\tau_0 \log 2/|G|^2 f_e(\epsilon_0)$. Assuming that the electrons travel with the velocity of light, the path traveled before causing a disintegration, would be c times the "lifetime," i.e.,

$$l_e=c\tau_0 \log 2 \frac{1}{2\pi^2|G|^2}\left(\frac{mc^2}{E_0}\right)^4\left(\frac{mc}{\hbar}\right)^3 N^{-1}. \tag{223a}$$

For a solid material all of whose atoms may be disintegrated by the electron, the number N may be estimated as $6\cdot 10^{22}$. Putting $G=1$, we thus obtain

$$l_e=2\cdot 10^{22}(mc^2/E_0)^4 \text{ cm}. \tag{223b}$$

Thus, even if the disintegration of the nuclei is energetically possible and corresponds to an "allowed" transition, and even if the energy E_0 given to the neutrino is very big, the probability of the process if entirely negligible. In other words, the disintegration of nuclei by the capture of free electrons and emission of neutrinos is practically unobservable.[87]

The probability of disintegration of nuclei by free neutrinos is almost the same as that by electrons. The path which a neutrino of energy E_0 has to travel in a solid material containing $6\cdot 10^{22}$ nuclei per cm^3, is

$$l_n=2\cdot 10^{22}(mc^2/E_0)^2(mc^2/E)(mc/p) \text{ cm} \tag{223c}$$

if E and p are energy and momentum of the electron which would be emitted in the nuclear disintegration, and if every nucleus in the material can be disintegrated by the neutrinos of the given energy. It is indeed very unfortunate that the probability of the disintegration of nuclei by neutrinos is so unobservably small, because this disintegration is the only action of free neutrinos which can be predicted with certainty.

§43. Stability of Isobars and Forbidden β-Processes

We have found in §10 that two nuclei of the same mass number and nuclear charge differing by one unit (neighboring isobars) cannot both be stable. In fact, pairs of neighboring isobars practically do not occur at all in nature. However, a few such pairs seem well established experimentally. These are Cd^{113} In^{113}; In^{115} Sn^{115}; Sb^{123} Te^{123}; K^{40} Ca^{40}; and Rb^{87} Sr^{87}. All the isotopes in the first three pairs have been confirmed recently by Bainbridge[88] using hydrogen-free sources, thus excluding spurious "isotopes" due to hydrides. Other rare isotopes which would be isobaric with "neighboring" well-established nuclei could not be confirmed by Bainbridge. These are Cd^{115} (would be isobaric to In^{115}), $Sn^{121}(Sb^{121})$, $Hg^{197}(Au^{197})$, $Hg^{203}(Tl^{203})$, $Pb^{203}(Tl^{203})$, $Pb^{205}(Tl^{205})$, $Pb^{209}(Bi^{209})$, Pb^{210} (this nucleus can be said to be spurious almost with certainty, because the radioactive element Ra D is identical with Pb^{210}, thus Pb^{210} cannot occur in nature as a stable isotope). All these isotopes seem to be, at least, much rarer than was originally claimed; however, Bainbridge believes that his results need rechecking in order to be sure that the doubtful isotopes really do not exist.

The last two of the well-established pairs of isobars do not enter our present discussion, because K and Rb are known to be radioactive. In the case of K, it seems highly probable

[87] Electrons can, however, disintegrate nuclei by virtue of their electric field. This process is analogous to the disintegration by γ-rays (cf. §16, and B17) and might be observable; the disintegrating electron is *not* captured in the process.—The disintegration by *capture* of electrons may also become observable if the expression for the probability of β-disintegration is modified in such a way as is necessary to explain the nuclear forces (§44) and if at the same time the energy of the incident electron and the emitted neutrino are of the order 137 mc^2.

[88] We are indebted to Dr. Bainbridge for communicating his results to us before publication.

that K^{40} is responsible for the radioactivity. In fact, K^{40} was discovered by Nier (N3) after Klemperer (K4) had given good reasons for its being the radioactive isotope of K (cf. §33, 40). In the case of Rb, it seems reasonable to assume that Rb^{87} is the radioactive isotope, just because it is isobaric with Sr^{87}. This view is confirmed by the fact that no other isotopes of Rb than Rb^{85} and Rb^{87} could be detected (N4).

There remain three pairs of well-established neighboring isobars, none of which shows any observable β-activity. The number of these pairs is very small indeed, compared to the number of "allowed" isobaric pairs with nuclear charges differing by two units, which is over 50 (cf. Table I). However, the fact that three "forbidden" pairs of isobars exist, is significant enough and must be explained.

There appear to be two possible explanations. Either (a) the mass of the neutrino is not zero or (b) the β-transformation of one isobar into the other is highly forbidden.

If we accept alternative (a), the conditions for the energetic stability of two isobars Z^A and $(Z-1)^A$ are the following: The nucleus $(Z-1)^A$ must not be capable of β-disintegration. This is certainly the case if its mass is smaller than the sum of the masses of the particles which would be formed in such a disintegration, i.e., the nucleus Z^A, an electron and a neutrino. We thus have the condition for the masses

$$(Z-1)^A < Z^A + \epsilon^- + n^0. \tag{224}$$

Similarly, Z^A must be incapable of capturing one of the orbital electrons attached to it (§42) and emitting a neutrino. This condition will be fulfilled if the sum of the masses of the original particles is smaller than the masses of the produced particles, *viz.*,

$$Z^A + \epsilon^- < (Z-1)^A + n^0. \tag{224a}$$

The nuclei Z^A and $(Z-1)^A$ will therefore *both* be energetically stable if

$$(Z-1)^A - n^0 < Z^A + \epsilon^- < (Z-1)^A + n^0. \tag{224b}$$

In other words, if the mass of the nucleus Z^A, plus an electron is identical with the mass of the nucleus $(Z-1)^A$ within an accuracy of one neutrino mass, then both isobars Z^A and $(Z-1)^A$ will be energetically stable against β-transformations.

We may try to obtain an estimate of the neutrino mass from this condition, assuming that there are just 3 pairs of neighboring isobars of atomic weight below 150. We have shown in §9 and 10 (cf. (21)) that, in a very rough approximation, the exact weight of the atoms of mass number A can be represented as a function of the charge Z as follows:

$$E(Z) = B + \kappa(Z - Z_A)^2 = B + C(Z - Z_A)^2/A, \tag{225}$$

where B and C are certain constants and Z_A is the "most favorable" charge for the atomic weight (cf. (19a)) A. C depends only slightly on the atomic weight A, and has, for A in the neighborhood of 120, the value 100 MV (cf. (21), (22)). If we have two isobaric nuclei of charges Z and $Z-1$, we define a quantity β by putting

$$Z_A = Z - \tfrac{1}{2} + \beta. \tag{225a}$$

β is supposed to lie between $-\frac{1}{2}$ and $+\frac{1}{2}$. We assume now that any value of β in this range is equally probable, i.e., that the values of β are distributed perfectly at random. The weights of the two isobaric atoms, as functions of β, are

$$\begin{aligned} &B + (C/A)(\tfrac{1}{2} - \beta)^2 \\ &\qquad \text{for the nucleus of charge } Z, \\ &B + (C/A)(\tfrac{1}{2} + \beta)^2 \\ &\qquad \text{for the nucleus of charge } Z-1, \end{aligned} \tag{225b}$$

so that the difference is

$$\Delta E = 2C\beta/A. \tag{225c}$$

This difference is supposed to be smaller than the neutrino mass n_0. Thus the two isobars will both be stable if

$$|\beta| < n_0 A/2C. \tag{225d}$$

The probability for this is, for random distribution of the β's:

$$p = n_0 A/C. \tag{225e}$$

The total number of isobaric pairs of odd atomic weight smaller than 150 should therefore be

$$P = \sum_{\text{all odd } A\text{'s}}^{150} n_0 A/C = 150^2 n_0/4C = 5600 n_0/C. \tag{225f}$$

Putting this equal to three, and inserting $C = 100$

MV, we find

$$n_0 = 0.05 \text{ MV}. \qquad (225g)$$

Thus the mass of the neutrino must be one-tenth of the electron mass in order to explain the observed number of pairs of isobars on the basis of energetic stability.[89] Such a mass would seem just reconcilable with the data about the energy distribution in β-spectra. However, it does not seem very plausible to assume a neutrino mass which is so small, and still not zero. The assumption of a zero mass would seem much more satisfactory.

We are thus led to alternative (b) (see above) which assumes that one of the neighboring isobars may be energetically unstable, but does not transform into the other because the corresponding β-transformation is highly forbidden.

We must distinguish two cases: Either the isobar of larger nuclear charge has higher energy, or that of smaller charge. In the first case, there will be an "inverse" β-process, i.e., a capture of K electrons by the unstable nucleus; in the second case, there will be an ordinary β-process. Only in the second case there will be a radioactivity which is observable in principle; whereas the "inverse" β-process would manifest itself only in the gradual disappearance of the more energetic isobar. Accordingly, we have to make the following requirements regarding the lifetime of the unstable isobar:

(a) If the nucleus of higher charge is the unstable one, its lifetime must be of the order of the age of the earth (about 10^9 years). Otherwise, this nucleus could no longer be found on the earth.

(b) If the nucleus of *lower* charge has higher energy, its lifetime must be such that radioactivity becomes unobservable.[90] In order to compute the necessary lifetime, we may assume that a radioactivity of one β-particle per hour from one cm^2 area of a solid target could just be observed. Furthermore, let us assume an upper limit of the β-spectrum of 200,000 volts, corresponding to an average energy slightly below 100,000 volts. If τ is the half-life in years, the number of β-particles observed per cm^2 per hour is

$$\beta = N x_0 \log 2/24 \cdot 365 \tau, \qquad (226)$$

where N is the number of atoms of the disintegrating isotope per cm^3 in the material and x_0 the average depth from which β-particles will escape. The latter is influenced by scattering and stopping and can be calculated (B14, formulae (32), (30), (29)). If the disintegrating isotope constitutes almost 100 percent of the material, which would be the case for In^{115}, one finds $Nx_0 = 5 \cdot 10^{19}$ cm^{-2} for $Z = 50$ and an electron energy of 100,000 volts. Then (226) becomes

$$\beta = 4 \cdot 10^{15}/\tau \quad \text{particles/hour}. \qquad (226a)$$

Thus a lifetime of 10^{15} years would be required in order to make the radioactivity unobservable.

When we find a pair of neighboring isobars in nature, without observing any β-radioactivity in the isobar of smaller nuclear charge, it is *more likely* that the *isobar of larger nuclear charge is the energetically unstable* one. We shall therefore assume in the following that case (a) is realized in all three observed isobaric pairs, so that In^{113}, Sn^{115} and Te^{123} are the energetically unstable isobars.

For these isobars, the lifetime is given by (222b, c, d, e). It is seen that a change of the nuclear moment by $L = 2$ units would lead to the required lifetime of 10^9 years or more, only if the energy E_0 of the emitted neutrino is less than about 25,000 volts. It is rather improbable that the energies of the two isobars coincide to that accuracy. However, a change of the nuclear moment by 3 units will lead to a lifetime of 10^9 years even if E_0 is as large as 500,000 volts which is certainly a conservative estimate of the energy difference between the isobars. Thus the existence of pairs of neighboring isobars is possible, if

1. The difference of the nuclear spins of the two isobars is at least 3 units.

2. The isobar of the larger charge has the higher energy.

[89] This figure is based on the assumption that the dependence of the energy of isobaric nuclei on the nuclear charge is perfectly regular, as given by (225). If there are irregularities, a larger value would be required for the neutrino mass. Also, it should be pointed out that it is not very satisfactory to base statistics on only 3 pairs.

[90] In making this statement, it is assumed that no radioactivity can be observed from the elements Cd, In and Sb. It would, of course, be very interesting to search for such radioactivity. The β-particles may be expected to have very small energy.

Differences of 3 units in total spin seem very plausible, in view of the high angular momenta of the individual particles in the nucleus (§32), and the high total momenta observed for a number of nuclei (§48). Thus the explanation of isobaric pairs on the grounds of forbidden β-transitions seems satisfactory. There is, then, no objection to assuming the neutrino mass to be zero.

In concluding this section, we want to mention the most fundamental pair of isobars, *viz.*, neutron and proton. Of these particles, the neutron must be unstable, its weight being about $E_0=350{,}000$ volts higher than the weight of a proton and an electron together. This figure is based on the binding energy of the deuteron (2.14 MV, §16) and Bainbridge's determination of the masses of proton and deuteron ($H^2=2.01423$, if $H^1=1.00807$). There seems to be some evidence (A5) that the deuteron is actually heavier than 2.01423; this would make the neutron even more unstable.

The transition $n^1\rightarrow H^1+\epsilon^-+n^0$ is certainly "allowed," therefore its probability is given by (213), (213b). Inserting $\epsilon_0=1.7$ into (213b), we obtain $f(\epsilon_0)=0.02$, and with (213), (214), and $G=1$:

$$\tau_{\text{Neutron}}=2.5\cdot 10^6 \text{ sec.}=1 \text{ month.} \quad (227)$$

A similar value has been deduced by Motz and Schwinger (M17). The lifetime (227) is too long to allow observation of the β-decay of neutrons. This would even be true if we take Aston's value for the mass of H^2 (2.0148) which would lead to an upper limit of the β-spectrum of the neutron of about 0.9 MV, therefore $\epsilon_0=2.8$, $f(\epsilon_0)=4$, and

$$\tau=1.2\cdot 10^4 \text{ sec.}=3 \text{ hours.} \quad (227a)$$

It might also be worth while to estimate the lifetime of the nuclei Be^{10} and C^{14} which are formed in certain transmutations. Since the nuclei B^{10} and N^{14} are known to be stable, their isobaric neighbors Be^{10} and C^{14} must be unstable, if we assume the neutrino mass to be zero. The difference in energy seems to be very small in both cases, probably about 100,000 to 200,000 volts. Assuming the β-transitions to be allowed, the lifetimes would be between $\frac{1}{2}$ and 20 years.

§44. Nuclear Forces and β-Disintegration

It was first suggested by Heisenberg[91] that there may be a connection between the "Fermi field" corresponding to β-disintegration, and the forces between neutrons and protons. This connection may be thought of as analogous to the connection between the emission of light (electromagnetic field) and the Coulomb interaction between charged particles. In quantum electrodynamics, the Coulomb interaction between two particles is not introduced as a separate assumption, but each particle is only assumed to interact with the electromagnetic field. Only because *both* charged particles interact with the field, there is also some interaction between them. The Coulomb interaction is thus regarded as a second approximation of the interaction between field and individual particles.

The same program may be carried out for Fermi's β-field and the nuclear forces. Let us suppose we have two particles 1 and 2 and want to investigate the interaction energy corresponding to the transformation of the first particle from a neutron into a proton, simultaneously with the inverse transformation of the second particle. Given is the Hamiltonian describing the interaction of both particles with the Fermi field, which we assume in the Konopinski-Uhlenbeck form (206a), (211b), leaving out the small term containing the α operator of the heavy particles:

$$\begin{aligned} &H=H_1+H_1^*+H_2+H_2^*, \\ &H_1=(gE_n/\hbar c)(\psi^*(\mathbf{r}_1)\beta\varphi(\mathbf{r}_1))Q_1. \end{aligned} \quad (228)$$

Here E_n is the neutrino energy, β the Dirac operator, $\varphi(\mathbf{r}_1)$ and $\psi(\mathbf{r}_1)$ neutrino and electron wave function at the place of the heavy particle, Q the operator transmuting a neutron into a proton and Q^* the inverse operator. The indices 1 and 2 refer to first and second heavy particle, and H_1^* is the complex conjugate of H_1.

The Hamiltonian (228) will, in second approximation, lead automatically to a simultaneous transformation of particle 1 from a neutron into a proton, and of 2 from a proton into a neutron. The Hamiltonian connected with this transformation is, according to the ordinary Schrödinger

[91] Heisenberg, Lectures at the Cavendish Laboratory, Cambridge, 1934. Unpublished.

perturbation theory

$$W=-\sum_b \frac{H_2^{*bd}H_1^{ab}}{E_b-E_a}-\sum_c \frac{H_1^{cd}H_2^{*ac}}{E_c-E_a}. \quad (228a)$$

Here a, b, c, d denote initial, two intermediate and final state, H_1^{ab}, etc., the respective matrix elements and E_aE_b the total energies. The states are describable as follows:

Initial state (a): Particle 1 = Neutron, 2 = Proton, all electron and neutrino states of negative energy occupied, all states of positive energy empty.

Intermediate state (b): Particle 1 = Proton, 2 = Proton, one neutrino state of negative energy $-E_n$ empty, one electron state of positive energy E_e occupied

Intermediate state (c): Particle 1 = Neutron, 2 = Neutron, one electron state of negative energy $-E_e$ empty, one neutrino state of positive energy E_n occupied

Final state (d): Particle 1 = Proton, 2 = Neutron, all negative states occupied, all positive states empty.

Thus in both cases

$$E_b-E_a=E_c-E_a=E_n+E_e. \quad (228b)$$

We now insert for φ and ψ plane waves, normalized per unit momentum:

$$\varphi(\mathbf{r}_1)=h^{-3}\bar{\varphi}(\mathbf{p}_n)\exp[i\mathbf{p}_n\cdot\mathbf{r}_1/\hbar], \quad (228c)$$

etc., where $\mathbf{p}_n$ is the momentum of the neutrino and $\bar{\varphi}$ a constant spinor. After a simple calculation involving these spinors, (228a) reduces to

$$W=-2\left(\frac{g}{\hbar c}\right)^2 h^{-6}\int \frac{d\mathbf{p}_n d\mathbf{p}_e}{E_n+E_e}E_n^2\exp[i(\mathbf{p}_n-\mathbf{p}_e)\cdot\mathbf{r}/\hbar] \quad (228d)$$

if the neutrino mass is assumed to be zero. $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ is the distance of the two particles, and the integral extends over all momentum space of electron and neutrino. Introducing polar coordinates in $\mathbf{p}_n$ and $\mathbf{p}_e$-space, and integrating over the angles, we obtain

$$W=-32\pi^2\left(\frac{g}{\hbar c}\right)^2(2\pi\hbar)^{-6}\left(\frac{\hbar}{r}\right)^2 \times\int \frac{p_n\,dp_n p_e dp_e E_n^2}{E_n+E_e}\sin(p_nr/\hbar)\sin(p_er/\hbar). \quad (228e)$$

The main contribution to the integral clearly comes from high energies of electron and neutrino. We therefore put

$$p_n=E_n/c, \quad p_e=E_e/c. \quad (228f)$$

Furthermore, we introduce the abbreviations

$$x=E_nr/\hbar c, \quad y=E_er/\hbar c. \quad (228g)$$

Then (228e) becomes

$$W=-\tfrac{1}{2}\pi^{-4}\frac{g^2}{\hbar c}r^{-7}\int_0^\infty \frac{x^3dxydy}{x+y}\sin x\sin y. \quad (228h)$$

The integral diverges. It can, however, be calculated as

$$\left(\frac{\partial^4 F(p,q)}{\partial p^3\partial q}\right)_{p=q=1} \quad (228i)$$

with

$$F=\int_0^\infty\int^\infty \frac{dxdy}{x+y}\cos px\cos qy. \quad (228j)$$

F has the value[92] $\pi/2(p+q)$; therefore (228i) becomes $4!\pi/2(1+1)^5=3\pi/8$.

Thus $$W=-(3/16)\pi^{-3}(g^2/\hbar c)r^{-7}. \quad (229)$$

The interaction of a neutron and a proton is thus proportional to the inverse seventh power of their distance, i.e., it increases very rapidly with decreasing distance, as we have always assumed. The interaction would become infinite for $r=0$, and the binding energy of all nuclei would become infinite, if (229) held down to $r=0$. We must therefore assume that, for some reason as yet unknown, (229) breaks down at small distances. To define the breakdown radius a more accurately, we put

$$\begin{aligned} W&=-W_0=-(3/16)\pi^{-3}(g^2/\hbar c)a^{-7} && \text{for } r<a,\\ W&=\text{expression (229)} && \text{for } r>a. \end{aligned} \quad (229a)$$

This would correspond practically to a potential hole of depth W_0 and radius a. By choosing a suitable value for the breakdown distance a, we could make the interaction (229a), and the corresponding binding energies, as large as we please; e.g., if we want to obtain the correct

[92] To calculate F, we introduce $s=x+y$, $t=x-y$, $\alpha=(p+q)/2$, $\beta=(p-q)/2$. Then

$$F=\frac{1}{4}\int_0^\infty\frac{ds}{s}\int_{-s}^{s}dt[\cos(\alpha s+\beta t)+\cos(\beta s+\alpha t)] \quad (A)$$

$$=\frac{1}{4}\int_0^\infty\frac{ds}{s}\frac{1}{\beta}[\sin(\alpha+\beta)s-\sin(\alpha-\beta)s] +\frac{1}{4}\int_0^\infty\frac{ds}{s}\frac{1}{\alpha}[\sin(\beta+\alpha)s-\sin(\beta-\alpha)s]. \quad (B)$$

Now $$\int_0^\infty\frac{ds}{s}\sin\kappa s=\begin{cases}+\pi/2, & \text{if } \kappa>0,\\ -\pi/2, & \text{if } \kappa<0.\end{cases} \quad (C)$$

Since $\alpha+\beta=p>0$, $\alpha-\beta=q>0$, the first integral in (B) is zero, the second is equal to $\pi/4\alpha=\pi/2(p+q)$.

value for the binding energy of the deuteron, we must choose (cf. (40))

$$W_0 a^2 = (\pi^2/4)\hbar^2/M, \tag{229b}$$

$$\text{i.e., } (\pi a)^5 = \frac{3}{4} g^2 M/\hbar^3 c = \frac{0.75 \cdot 3.6 \cdot 10^{-120} \cdot 1.7 \cdot 10^{-24}}{1.1 \cdot 10^{-81} \cdot 3 \cdot 10^{10}} \times 1.3 \cdot 10^{-73} \text{ cm}^5,$$

$$a = 0.85 \cdot 10^{-15} \text{ cm}, \quad W_0 = 1.3 \cdot 10^{12} \text{ volts}. \tag{230}$$

However, the assumption of such a short range and such a large magnitude of the forces between neutron and proton is quite unsatisfactory. It would lead to an extremely big binding energy of the α-particle (chapter IV). In order to obtain agreement with the empirical facts about nuclear forces, we must assume that the "cut-off" of the interaction (229) occurs in the neighborhood of $r = 2 \cdot 10^{-13}$ cm (§21). This, however, leads to an interaction energy which is negligibly small compared to the empirical value. The quantity $W_0 a^2$, which determines the binding energy of the deuteron, is proportional to a^{-5}, and is therefore decreased by a factor $250^{-5} = 10^{-12}$ if the range of the forces is increased by a factor 250, from $0.8 \cdot 10^{-15}$ to $2 \cdot 10^{-13}$ cm. *Thus the interaction comes out to be too weak by a factor 10^{12}, if we cut the potential (229) off at the observed range of the nuclear forces.*

This highly unsatisfactory result is, of course, due to the extremely small value of the constant g which governs the β-emission. However, the general idea of a connection between β-emission and nuclear forces is so attractive that one would be very reluctant to give it up. In principle, several ways seem open:

(a) The interaction leading to β-emission is only part of a more general interaction. The other "components" of that general interaction are larger. This hypothesis was suggested by Heisenberg, in analogy to electrodynamics, where also the Hamiltonian leading to emission and absorption of light (transverse electromagnetic waves) is small compared to that connected with "longitudinal" electromagnetic waves. The latter cause most of the Coulomb interaction. How such a modification could be introduced into the β-theory, is of course not clear.

(b) The fundamental expression for the β-emission contains actually higher derivatives of the electron and neutrino wave function. This would mean a more rapid increase of the β-disintegration probability with increasing energy. Now the constant g is derived from the given lifetimes of nuclei emitting β-rays of a few MV energy. On the other hand, the energies which contribute most to the interaction of a neutron and a proton at a distance a, are those for which the electron and neutrino wave-length is of the order a. The energy corresponding to $a = 2 \cdot 10^{-13}$ cm, is $E = \hbar c/a \approx 100$ MV, i.e., about 100 times the energy of most β-particles. Each additional derivative introduced into the expression (211), multiplies the Hamiltonian corresponding to β-decay by a factor E_e or E_n, according to whether it is introduced into the electron or the neutrino wave function. Since W contains (cf. 228a) the square of the β-Hamiltonian, a factor E_e^{2k} (or E_n^{2k}) is introduced if k more derivatives are introduced in (211). Since the constant g must be redetermined in such a way as to make the lifetimes of β-decaying nuclei agree with experiment, W is multiplied by the $2k$th power of the ratio of the energies occurring in (228a), and in β-decay. This ratio being about 100, W is multiplied by 10^{4k}. The introduction of *three* more derivatives (i.e., altogether four) into (211) would, therefore, bring about agreement with the observed nuclear forces.

However, such a change of the fundamental assumptions of the theory would also lead to a considerable decrease of the lifetime of nuclei emitting high energy β-rays, compared to those emitting less energetic ones. This would destroy the agreement obtained in §41, Table XV. The only way out would be to assume that all the observed high energy β-transformations are forbidden at least of the second order, which does not seem plausible at all.

(c) It may be that the behavior of electrons of wave-length near $e^2/mc^2 (= 2.8 \cdot 10^{-13}$ cm) is so completely different from the usual one, that these electrons contribute much more to (228a) than we anticipate, without the β-interaction differing appreciably from the Konopinski-Uhlenbeck expression for lower energies. This idea is sufficiently vague to make it hard to disprove it.

Assuming for the present that the problem of the magnitude of the nuclear forces, as compared to the β-interaction, will be solved in some way or other in the future, we may inquire into the *nature* of the forces between neutron and proton following from the β-theory. The force is clearly an exchange force, being connected with a change of roles of neutron and proton (see above). This is in agreement with our empirical knowledge about neutron-proton forces. However, it turns out to be a *Heisenberg* force, because the spins remain unchanged in the process (228) if we consider a particle at a given *point*, $\mathbf{r}_1$ or $\mathbf{r}_2$. Only the "charge passes over" from point $\mathbf{r}_2$ to $\mathbf{r}_1$, so that the neutron formed in the process, which is situated at $\mathbf{r}_2$, has the same spin as the proton previously situated at that place. This is the characteristic of a Heisenberg force, which is not admissible for the fundamental force of nuclear physics (§7, §11).

However, it is easy to change the fundamental expression for the β-disintegration so as to obtain a Majorana rather than a Heisenberg force. It is only necessary to accept one of the interactions (211c), (211d), rather than the Konopinski-Uhlenbeck interaction (211b). Besides irrelevant changes in the magnitude of the forces, this will introduce a factor $\boldsymbol{\sigma}_{\text{heavy}}$ in the Hamiltonians (228), and therefore a factor $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ in the expression (228a) for the interaction energy of neutron and proton. The product $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ is positive when proton and neutron spin are parallel, in this case we obtain the same result as before. $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ is negative if the spins are antiparallel; in this case we therefore find the sign of the interaction reversed. This is exactly the difference between a Heisenberg and a Majorana force (Eq. (30a, b)). Their signs are *equal* if the spins of proton and neutron are parallel, opposite for opposite spins.

However, the numerical factor is not correct. $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ has the value 1 for parallel, -3 for antiparallel spins. The force between neutron and proton would thus be 3 times as large for antiparallel than for parallel spins, whereas actually (§14) the forces are almost equal, and the force for *parallel* spins is even somewhat bigger. Thus we ought to take a suitable linear combination of the forces (211b) and (211c, d) to obtain a neutron-proton interaction of the desired form. This again is somewhat unsatisfactory.

The replacement of (211b) by (211c, d) does not change our discussion about β-spectra materially. The only difference is that changes of the *spin* angular momentum of the nucleus by one unit are now allowed transitions. If the coupling between spin and orbital momentum is weak, we have, then, just to refer to changes of the orbital momentum of the nucleus rather than to its total momentum, in discussing forbidden β-transitions. If the spin-orbit coupling in the nucleus is strong, which is probably the case, changes of the total momentum by $L+1$ will be about as probable as changes by L were in our previous discussion. Then we must require a change by 4 units for radioactive K^{40}, and for the isobaric pairs (§43).

The "β-hypothesis of nuclear forces" gives, in first approximation, only forces between neutrons and protons. In second approximation forces between like particles would appear, the mechanism being about as follows: Each of two neutrons emits "virtually" (intermediate state, cf. 228a) an electron and a neutrino, and then absorbs the particles emitted by the other neutron. It should be expected that this second approximation is *not* small compared to the first. For if the β-theory is to lead to the observed magnitude and range of neutron-proton forces, we must assume that for electron and neutrino energies of about 100 MV the β-interaction H (228) is also of the order 100 MV: Electrons of energy $\sim$100 MV should give the main contribution to W, because their wave-length is of the order of the range of the nuclear forces. On the other hand, W is of the order $H^2/(E_e+E_n)$, according to (228a, d), and is about 30 MV empirically (§21). This requires the matrix elements of H, corresponding to the emission of electrons and neutrinos of 100 MV, to be also of the order 100 MV. But if this is true, we should expect the second approximation to be nearly as big as the first, i.e., the forces between like particles should be not much smaller than those between neutron and proton, in agreement with the conclusions from nuclear binding energies (§21).

§45. The Magnetic Moments of Proton and Neutron (W11)

It was first suggested by Wick (W11) that the anomalous values of the magnetic moments of proton and neutron (§5) may be explained on the grounds of the β-theory. According to that theory, a neutron can never be regarded as entirely isolated, but is always associated with a "β-particle field" surrounding it. In other words, if we observe a neutron at a given moment, we shall not always find a neutron, but sometimes we shall find a proton, an electron and a neutrino instead. During the short intervals of time when the neutron is replaced by a proton, an electron and a neutrino, an external magnetic field will find the spin of the electron to act upon. The interaction energy between a magnetic field and a neutron, will therefore be equal to the interaction energy between the field and an electron, times the probability of finding the neutron temporarily "dissolved" into proton, electron and neutrino at any given instant, times the probability that the spin of the electron is parallel to the spin of the neutron rather than antiparallel.

The same argument holds for the proton, with the only difference that "positron" should be inserted instead of "electron." Moreover, the *proton* will interact with the magnetic field even if it is not "dissolved"; during these times, its interaction will correspond to its "normal" magnetic moment $he/2Mc$, which follows from the Dirac theory.

The probability that a neutron is found to be temporarily dissolved into a proton, an electron of energy E_e and a neutrino of energy E_n is, according to the Schrödinger perturbation theory,

$$|H(E_e, E_n)|^2/(E_e+E_n)^2, \qquad (231)$$

where H is the matrix element of the β-interaction (211) which refers to the emission of an electron and a neutrino of the respective energies E_e, E_n.

If we insert in (231) the ordinary interaction derived from the probability of β-disintegration itself, we are faced with the same difficulties as in the preceding section when trying to account for the nuclear forces: If we accept the β-interaction as it stands, the expression for the magnetic moment of the neutron will diverge. If we avoid the divergence by "cutting off" the β-interaction for high energies of electron and neutrino, we shall obtain much too small a value for the magnetic moment.

We shall therefore assume, just as in the preceding section, that the present β-ray theory is not adequate in the region of high electron and neutrino energies, but that a future correct theory will give a higher disintegration probability for electron energies of the order 100 MV. We shall furthermore assume that this probability will turn out to be such as to give the correct magnitude of the nuclear forces. This means (end of §44) that the matrix elements H in (231) are nearly as large as the denominator E_e+E_n, if E_e and E_n are of the order 100 MV. Then (231) becomes almost unity; may be of the order 1/10. In other words, during a considerable fraction of the time, the neutron will be found dissolved into proton, electron and neutrino.

The magnetic moment of a high energy electron is one-third Bohr magneton.[93] The correlation between the spin directions of the neutron and the emitted electron is not easy to estimate; it depends on the particular linear combination of the expressions (211b, c, d) which represents the correct β-ray interaction.[94] A correlation of a few percent would be sufficient to account for the observed magnetic moment of the neutron, *viz.*, $2.0\hbar/2Mc = 1/900$ Bohr magneton. According to the foregoing, the probability that an

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[93] This can be seen easily from the Dirac wave function of a free electron with spin parallel to the Z axis, whose components are

$$\psi_1 = -Acp_z \quad \psi_2 = -Ac(p_x+ip_y) \quad \psi_3 = -A(E+mc^2) \quad \psi_4 = 0,$$
$$A = [2E(E+mc^2)]^{-\frac{1}{2}} \exp[i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar].$$

The magnetic moment, in Bohr magnetons, is given by

$$(\psi^*\sigma_z\psi) = \psi_1^*\psi_1 - \psi_2^*\psi_2 + \psi_3^*\psi_3 - \psi_4^*\psi_4$$
$$= [c^2p_z^2 - c^2(p_x^2+p_y^2) + (E+mc^2)^2]/2E(E+mc^2).$$

Averaging all over directions of motion of the electron, we obtain

$$p_x^2 = p_y^2 = p_z^2 = \tfrac{1}{3}p^2 = (E+mc^2)(E-mc^2)/3c^2.$$

We have therefore

$$(\psi^*\sigma_z\psi) = [E+mc^2-\tfrac{1}{3}(E-mc^2)]/2E = \tfrac{1}{3}+\tfrac{2}{3}mc^2/E.$$

For small E, of the order of mc^2, this expression has the value 1; for large E ($\gg mc^2$), it is only one-third.

[94] If the β-ray interaction is represented by one of the expressions (211b, c, d) *alone*, there will be no correlation between the spins of neutron and electron.

electron is present, is of the order 1/10; the magnetic moment of the electron, *if* present, is 1/3 Bohr magneton; thus the magnetic moment of the neutron would be 1/30 Bohr magneton if the spin of the emitted electron were *always* parallel to the neutron spin.

It appears thus that our theory gives rather *too large* a value for the magnetic moment of the neutron if we deduce the β-interaction from the forces between neutron and proton.

The magnetic moment of the proton can be calculated in the same way. Since the β-ray theory is perfectly symmetrical in neutrons and protons, the *additional* moment of the proton should be equal and opposite to the magnetic moment of the neutron. By additional moment we understand the excess of the actual magnetic moment of the proton over its "elementary" moment $\hbar/2Mc$ which follows from the Dirac theory. We should thus conclude that the *sum* of the magnetic moments of neutron and proton, i.e., the magnetic moment of the deuteron, is equal to the elementary magnetic moment of the proton, since the additional moment of the proton, and the magnetic moment of the neutron, cancel each other. The observed value of the magnetic moment of the deuteron is about $0.85\hbar/2Mc$. This is nearly, but not exactly, equal to the value following from our considerations. The difference may either (a) be connected with the difference in mass between neutron and proton, or (b) with the fact that the proton is actually dissolved into a neutron, a positron and a neutrino during a considerable fraction of the time, and does not possess its "elementary moment" during that time.

VIII. Nuclear Moments

The essential features of atomic and molecular structure can be accounted for by the quantum theory on the assumption of an atom consisting of a small massive nucleus surrounded by electrons which are held in the nuclear field. It is quite satisfactory for most of the purposes of atomic and molecular structure to consider the field only at distances from the nucleus sufficiently large that its field is a Coulomb field. As far as the main features of atomic and molecular spectra are concerned, therefore, the nucleus does not enter except as the center of this Coulomb field. There are, however, certain facts of both atomic and molecular spectra which lead directly to information concerning the atomic nucleus.

For atoms it is well known that the totality of energy states found from the usual analyses of spectra can be accounted for both as to number and, with some difficulty, as to position by the quantum theory treatment of the electrons moving in the central Coulomb field. These states are characterized by quantum numbers of the electrons, by the total angular momentum of the electrons and usually by the spin and orbital angular momenta as well. The most closely adjacent states are those (at least for the case of Russell-Saunders coupling) which have only different total angular momenta. Such states compose a "multiplet" and are referred to as fine structure because of their frequent close spacing.

A more detailed examination of the spectral lines involved frequently has indicated that states considered above are not themselves single but are actually composed of a group of states. This multiplicity is called hyperfine structure. The hyperfine structure of spectral lines cannot be accounted for on the basis of the assumptions mentioned above. It was first suggested by Pauli (P4) that hyperfine structure is due to the action of the electrons in the field of a nuclear magnetic dipole. That the interaction is essentially a magnetic one is immediately seen by a comparison of hyperfine structure groups with ordinary fine-structure multiplets. Such a comparison shows striking similarities. The nuclear origin of hyperfine structure is confirmed on many sides but it seems sufficient to mention only one such confirmation at this time. Those states which have electrons with a higher probability of being near the nucleus show the structure while those with a very low probability show no structure. It is now completely certain that the hyperfine structure of spectral lines is for the greatest part due to the interaction

of a nuclear magnetic moment with the electrons not appearing in closed shells. The origin of the nuclear magnetic dipole whose presence gives rise to the hyperfine structure is probably due to the motion of electrical charges and it is to be expected that the nucleus possesses a certain angular momentum (spin). This same conclusion is of course attained since the nucleus is considered to be built of protons and neutrons which themselves have intrinsic angular momenta and magnetic moments. A detailed study of the hyperfine structure for a particular element allows us to determine the angular momentum of the nucleus in question and, with somewhat more difficulty in interpretation, the magnitude of the nuclear magnetic moment as well. Complications arise when the element has several isotopes but these can frequently be overcome.

For diatomic molecules composed of like atoms it is found that the presence of a nuclear angular momentum changes the statistical weight of the rotational states. For zero nuclear angular momentum we find that alternate lines of the bands are missing. With a nuclear angular momentum it is found that successive lines of the bands have an intensity ratio which depends on the magnitude of the nuclear spin. It is thus possible by studying the intensities in such molecular spectra to determine the nuclear angular momentum.

There are other indications of the presence of nuclear spins and magnetic moments. In hydrogen for example, since the proton is known to have a spin, the hydrogen molecule may consist of two hydrogen atoms with their nuclear spins in the same (orthohydrogen) or opposite directions (parahydrogen). The presence of these two sorts of hydrogen is known theoretically from the behavior of the specific heat at low temperatures (D2) and indeed the two sorts of hydrogen have been separated experimentally. In the presence of a paramagnetic gas such as oxygen the rate of conversion from para- into orthohydrogen depends on the size of the nuclear magnetic moment. It has thus been possible (F1) to determine the ratios of the nuclear magnetic moments for hydrogen and deuterium from this dependence. A more detailed account of these atomic and molecular effects will be found in the following paragraphs.

§46. The Interaction of the Nuclear Moment with the Electrons

If the interaction which gives rise to the hyperfine structure is due to the presence of a nuclear magnetic dipole in the field of the electrons it should be possible to deduce certain properties of it without calculation. Let us denote by $\mathbf{I}$ and $\mathbf{J}$ the angular momenta of the nucleus and the electrons, respectively, in units of $\hbar$. The interaction term which must be added to the potential energy of the system will be proportional to the cosine of the angle between $\mathbf{I}$ and $\mathbf{J}$, and may be written

$$V=A'(\mathbf{I}\cdot\mathbf{J})=\tfrac{1}{2}A'(\mathbf{F}^2-\mathbf{I}^2-\mathbf{J}^2), \qquad (232)$$

where $\mathbf{F}$ is the vector sum of $\mathbf{I}$ and $\mathbf{J}$ or the total angular momentum of the system in units $\hbar$. The addition to the energy which such a perturbing potential will give can be found immediately from the characteristic values for the squares of angular momenta.

$$W=\tfrac{1}{2}A\{F(F+1)-I(I+1)-J(J+1)\}. \qquad (233)$$

All the states of the hyperfine structure group have the same values of the quantum numbers I and J. F takes on a series of values from $I+J$ down to $|I-J|$ as determined by the quantum theory treatment of vectors. W then has a series of values $A[IJ]$; $A[IJ-(I+J)]$; $A[(IJ+1)-2(I+J)]$; $A[(IJ+3)-3(I+J)]$ . . . and the energy differences of successive states are $A(I+J)$; $A(I+J-1)$; $A(I+J-2)$. . . . For the energy differences the part in brackets is just the larger F value for the two states considered. This regularity is called the interval rule and the factor A the interval factor. (233) above leads directly to this regularity and may therefore be regarded as a statement of the interval rule. Since (233) follows directly from the cosine form of the interaction it is expected that the interval rule will hold exactly (G9). The only exception to this should occur when two atomic states with different J are separated by an amount which is not large compared to the hyperfine structure separations. The hyperfine structure separations are rarely more than a few cm^{-1} so this exception occurs very infrequently. Recently deviations from the interval rule have been found which do not come under this exception and which there-

fore mean that the form of the interaction term must be slightly modified. This can be done by the assumption that the nucleus has a small electric quadrupole moment and will be discussed somewhat further in §50.

It is apparent from (233) that the relative spacing and number of the hyperfine structure states is fixed as soon as the quantum numbers F, I and J are known. In case $I<J$, then I can be determined directly from the number of hyperfine states which would be $2I+1$. For $I>J$ however, the number of states is determined by J. In either case the hyperfine quantum number F can be determined from exact measurements of the relative separations by the use of the interval rule (except for $J=\frac{1}{2}$). With F thus determined and J known, the value of I can be found. It is thus possible to determine the nuclear angular momentum without any further knowledge of the nature of the interaction.

The interaction constant A will contain the nuclear magnetic moment μ and factors which relate to the electrons and the probability of their being near the nucleus. The constant μ is related to the absolute size of the hyperfine structure separations. In order to determine μ it will be necessary to make a determination of the other factors in A and then use the experimental size to find μ. Let us consider the case of a single electron in the field of the nuclear magnetic dipole with charge Ze. In order to have the calculation apply to s electrons as well as others it is necessary to use the Dirac equations. We wish then to write the interaction term in terms of the vector potential

$$A=[\mathfrak{y}\mathbf{r}]/r^3=g\mu_0[\mathbf{I}\mathbf{r}]/r^3, \tag{234}$$

where $\mathfrak{y}=g\mu_0\mathbf{I}$ is the nuclear magnetic moment and $\mu_0=e\hbar/2mc$. If this vector potential is introduced into the Dirac equations and if the two "small" wave functions are eliminated to obtain an equation in the two "large" wave functions, the term representing the interaction of the nucleus with the electron can be written (B23)(F5)(B12).

$$H_1=g\mu_0(\mathbf{A}\cdot\mathbf{I}), \tag{235}$$

where

$$\mathbf{A}=2\mu_0[r^{-3}\mathbf{L}-r^{-3}\boldsymbol{\sigma}+3r^{-5}(\mathbf{r}\cdot\boldsymbol{\sigma})\mathbf{r}]\times(1+eA_0/2mc^2)^{-1}-2\mu_0[r^{-2}\boldsymbol{\sigma}-r^{-4}(\mathbf{r}\cdot\boldsymbol{\sigma})\mathbf{r}]\times\frac{d}{dr}\left(1+\frac{eA_0}{2mc^2}\right)^{-1}. \tag{235a}$$

Here $\mathbf{L}\hbar$ is the angular momentum vector, $\boldsymbol{\sigma}\hbar$ is the spin angular momentum and $\boldsymbol{\sigma}$ is $\frac{1}{2}$ times the Pauli spin matrix, A_0 is the scalar potential of the nuclear electrostatic field and E has been replaced by mc^2. With (235) for the interaction and using the properties of angular momenta and the fact that $\mathbf{A}$ contains only electron variables the perturbed energy can be found (B23) to be

$$W_1=g\mu_0(\mathbf{A}\mathbf{J})_J[F(F+1)-I(I+1)-J(J+1)][2J(J+1)]^{-1}. \tag{236}$$

$(\mathbf{A}\,\mathbf{J})_J$ is the diagonal element in the matrix $(\mathbf{A}\,\mathbf{J})$ for state J. $(\mathbf{A}\,\mathbf{J})$ can be found from (235a) using $\mathbf{J}=\mathbf{L}+\boldsymbol{\sigma}$. If the second term of (235a) is neglected in comparison to the first and if $eA_0/2mc^2$ is neglected compared to 1, we obtain

$$(\mathbf{A}\,\mathbf{J})=2\mu_0[r^{-3}\mathbf{L}^2-r^{-3}\boldsymbol{\sigma}^2+3r^{-5}(r\cdot\boldsymbol{\sigma})^2+3r^{-5}(\mathbf{r}\cdot\boldsymbol{\sigma})(\mathbf{r}\cdot\mathbf{L})]. \tag{237}$$

Here $(\mathbf{r}\cdot\mathbf{L})=0$ since $\mathbf{L}=\hbar[\mathbf{r}\mathbf{p}]$ and $3r^{-5}(\mathbf{r}\cdot\boldsymbol{\sigma})^2-r^{-3}\boldsymbol{\sigma}^2=0$ due to the properties of the Pauli spin matrix $2\boldsymbol{\sigma}$.

Using (237) which holds for electrons other than s electrons (236) now becomes

$$W_1=g\mu_0{}^2\frac{L(L+1)}{J(J+1)}\overline{(r^{-3})}\times[F(F+1)-I(I+1)-J(J+1)]. \tag{238}$$

For s electrons the first term of (235a) vanishes and we have

$$(\mathbf{A}\,\mathbf{J})^s=-2\mu_0[r^{-2}\boldsymbol{\sigma}^2-r^{-4}(\mathbf{r}\cdot\boldsymbol{\sigma})^2]\times\frac{d}{dr}\left(1+\frac{eA_0}{2mc^2}\right)^{-1}=-\mu_0 r^{-2}\frac{d}{dr}\left(1+\frac{eA_0}{2mc^2}\right)^{-1}.$$

The diagonal element for J becomes

$$(\mathbf{A}\,\mathbf{J})_{J^s} = -\mu_0 \int_0^\infty R^2(r) r^{-2} \frac{d}{dr}\left(1+\frac{eA_0}{2mc^2}\right)^{-1} r^2 dr$$

$$= \mu_0 \int_0^\infty \left(1+\frac{eA_0}{2mc^2}\right)^{-1} \frac{d}{dr} R^2(r) dr$$

$$\leqq \mu_0 R^2(0). \quad (239)$$

$R(r)$ is the radial function and $R(0)$ its value at the origin. $eA_0/2mc^2$ is neglected in comparison to 1. From (238), (236) and (239) it is possible by comparison with (233) to obtain expressions for the separation factors. For single electrons these will be called a_s or a_l and the quantum numbers l, j, etc.

$$a_s = (4/3) g(I) \mu_0^2 R^2(0), \quad (240)$$

$$a_l = 2 g(I) \mu_0^2 \overline{r^{-3}} (l(l+1)/j(j+1). \quad (241)$$

Since $\mu = g\mu_0 I$, equations (240) or (241) determine the size of the nuclear magnetic moment[95] if the separation factors as well as the quantum numbers are known from experiment and if it is possible to determine $R^2(0)$ or $\overline{r^{-3}}$. For any particular atom a knowledge of the wave functions is necessary therefore to determine the magnetic moment. Furthermore it is necessary to have wave functions which are quite accurate in the neighborhood of the nucleus if μ is to be determined accurately. At present such wave functions are not available. Calculations by Wills and Breit (W15) and by Shoupp, Bartlett and Dunn (S21) for Na $3s$, $3p_{3/2}$ and $4p_{3/2}$ states using Hartree functions give $\mu = 5.85$, 22 and 10.4 nuclear magnetons, i.e., $\mu_0/1838$, respectively, while $\mu = 2.5$, 5.1, 3.1 n.m., respectively, using Fock functions. Both of these sorts of wave functions give the spin doublet separations too high. Empirical correction assuming that the wave functions give the correct ratio of doublet separation to hyperfine structure separation leads to values for μ of 2.5–3.0 n.m. While Na may not be a favorable example, it is apparent that calculations using present wave functions are rather dangerous. It is very desirable, therefore, to have some approximation method which leads to consistent results and which could be applied even where wave functions are not available.

Let us consider the case of hydrogenic wave functions. For this case $R^2(0) = 4Z^3/n^3 a_0^3$ and we find that (240) can be rewritten in the form

$$a_s = \frac{8}{3} \frac{R\alpha^2 Z^3}{n^3} g(I) \text{ cm}^{-1}, \quad (242)$$

where $R = me^4/4\pi\hbar^3 c$ and $\alpha = e^2/\hbar c$. For hydrogenic functions $\overline{r^{-3}} = Z^3/n^3 a_0^3 (l+1)(l+\frac{1}{2})l$ and using this value in (241) we find

$$a_l = g(I) \frac{R\alpha^2 Z^3}{n^3 j(j+1)(l+\frac{1}{2})} \text{ cm}^{-1}. \quad (243)$$

For this case (243) reduces to (242) for $j = \frac{1}{2}$ and $l = 0$.

For atoms consisting of a single electron outside a closed shell (243) would be a poor approximation if Z denotes the atomic number, since the screening of the other electrons which plays an important role, would be neglected. A way in which the screening can be taken into account is indicated by the fact that (243) is similar to the expression for the spin doublet separation.

$$a(\Delta\nu) = \frac{\Delta\nu}{l+\frac{1}{2}} = \frac{R\alpha^2 Z^4}{n^3 (l+\frac{1}{2}) l(l+1)} \text{ cm}^{-1}.$$

Approximations to the spin doublet separation which replace Z^4 by $Z_i^2(1+z)^2$ and n by n^* have been found to hold for alkali type atoms. Here z is the degree of ionization, n^* the effective total quantum number and $Z_i = Z$ for s electrons while $Z_i = (Z-4)$ seems to fit the data for p electrons. For higher l the difference $Z - Z_i$ becomes greater and the necessary approximation more difficult to obtain. This approximation for deeply penetrating states was derived first by Landé (L1) using the Bohr theory and can now be justified on the basis of the wave mechanics.

By a similar argument Goudsmit (G6) and Fermi and Segrè (F11) have obtained an approximation for the hyperfine structure using the observed doublet separation

$$a_l = g(I) \frac{\Delta\nu l(l+1)}{j(j+1)(l+\frac{1}{2}) Z_i} \text{ cm}^{-1}. \quad (244)$$

This approximate expression can be used with

[95] It is customary to give μ/μ_0, i.e., to measure μ in units of μ_0, the Bohr magneton. Nuclear moments are of the order $10^{-3}\mu_0$ and it is customary to express them in the nuclear magneton (n.m) unit $\mu_0/1838$.

$\Delta\nu$ and Z_i to calculate $g(I)$ from the observed separation factor. For s electrons, an approximation[96] similar to that made in the doublet separation gives

$$a_s = g(I)\frac{R\alpha^2 Z_i(1+z)^2}{n^{*3}j(j+1)(l+\frac{1}{2})}\ \text{cm}^{-1}. \quad (245)$$

With Z_i fixed as above (244) and (245) become semiempirical formulas whose main virtues are that they apply to a considerable range of examples and lead to values of μ which, although determined from different states of the electron, are nearly the same.

Certain approximations ($E \cong mc^2$; $eA_0 \ll 2mc^2$) were made to obtain (240) and (241) which hold for light elements but are not valid for heavier ones. More exact calculations by Breit (B22) and Racah (R5) which do not use these approximations show that (244) or (245) should be multiplied by a factor

$$\kappa(j, Z) = 4j(j+1)(j+\tfrac{1}{2})/(4\rho^2-1)\rho \quad (246)$$

where $$\rho = [(j+\tfrac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}}.$$

This factor becomes important (~ 1.20) for $Z=40$ for $j=\frac{1}{2}$ and for $Z=80$ for $j=3/2$. This relativistic correction factor can be carried over directly to the approximations (244) and (245) using Z_i as determined before for Z above. The doublet separation appearing in (244) is subject to the same sort of correction. In this case it is found (B22, R5) that the doublet separation should be multiplied by a factor

$$\lambda(l, Z) = [2l(l+1)/Z^2\alpha^2][\{(l+1)^2 - Z^2\alpha^2\}^{\frac{1}{2}} - 1 - (l^2 - Z^2\alpha^2)^{\frac{1}{2}}]. \quad (247)$$

Using these correction factors (245) and (244) become

$$\mu' = \frac{aIn^{*3}j(j+1)(l+\frac{1}{2})}{R\alpha^2 Z_i(1+z)^2\kappa(l, Z_i)}1838, \quad (248)$$

$$\mu' = \frac{aIj(j+1)(l+\frac{1}{2})Z_i\lambda(l, Z_i)}{\Delta\nu\, l(l+1)\kappa(l, Z_i)}1838. \quad (249)$$

Here μ' is the nuclear magnetic moment in nuclear magnetons ($\mu_0/1838$) and $\Delta\nu$ is the observed doublet separation. Table XVI gives the values (G6) of κ and λ for several Z_i. If the doublet separation is large as it is for the heavy elements a further correction must be made (B24).

A comparison with (238) for example shows that for g (or μ) positive the hyperfine structure levels will be arranged with that state with smallest F as lowest. In such a case the levels are said to be regular, the a factor positive and μ positive. Similarly for the case where the state with largest F is lowest the levels are said to be inverted, the a factor negative and μ negative.

TABLE XVI.

Z_i	κ $j=1/2$	κ $j=3/2$	λ $l=1$	Z_i	κ $j=1/2$	κ $j=3/2$	λ $l=1$
10	1.01	1.00	1.00	70	1.78	1.10	1.12
20	1.04	1.01	1.00	80	2.25	1.15	1.17
30	1.09	1.02	1.01	85	2.61	1.17	1.20
40	1.18	1.03	1.03	90	3.10	1.20	1.24
50	1.30	1.05	1.05	92	3.36	1.21	1.27
60	1.49	1.07	1.08				

Table XVI gives the correction factor κ for the hyperfine separation and the correction factor λ for the doublet separation for various values of Z_i.

If the atom considered has more than one valence electron the above relations cannot be applied directly. It is frequently the case that the interaction is due to the presence of one penetrating s electron in the group of valence electrons. In such a case the separation for a given state can be obtained simply in terms of the separation constant of the s electron (G7). For many configurations all of the valence electrons will have a considerable interaction with the nucleus. It is possible in such cases to find relations which give the hyperfine structure size in terms of the separation constants of the various electrons involved.[97] Having thus determined the one-electron separation factors, the nuclear magnetic moments can be found from (248) and (249). If these equations are to yield

[96] It has been pointed out by Fermi and Segrè that (255) should be multiplied by a factor $(1-ds/dn)$ where $n-s=n^*$. This factor would increase the values of μ calculated from the low s states of Na I, Cs I and Tl III by 2 percent, 5 percent and 10 percent, respectively. This factor is certainly negligible for light elements and for heavy elements is probably smaller than errors due to perturbation effects.

[97] The hyperfine structure for several valence electrons is discussed in the following papers: Goudsmit (G5); Guttinger and Pauli (G16); Racah (R5); Breit and Wills (B28) (sp, sd, sf, p^2, $p\cdot p$, p^3, p^2s); Crawford (C18) (d^2s); Crawford and Wills (C22) (p^3s). The more recent work of Breit, Crawford and Wills give the separations for the indicated electron configuration in intermediate coupling as well, in terms of the one electron separation factors.

constant values of μ from different atomic states, we see that the hyperfine separation must differ greatly for the various states. This is indeed found to be the case.

It is well known that perturbations between states are very prevalent in atomic spectra; i.e., the wave function representing a particular atomic state contains the pure wave function not only of that state but of others which are thus said to perturb it. These perturbations frequently become very large when the states involved are close together, since the real wave function can be written

$$\psi_0' = \psi_0 + \sum_i \psi_i V_{0i}/(E_0 - E_i), \qquad (250)$$

where V_{0i} is the matrix element of the electrostatic energy between the two states. The factor $(E_0 - E_i)^{-1}$ insures that the correction to the wave function usually will not be great. A case of very small perturbation would be if $V_{01} = 1$ ev and $(E_0 - E_1) = 10$ ev. The perturbing state might well be above the ionization potential with such a value of $E_0 - E_1$. ψ_0' would contain $0.1\psi_1$, and the ψ_1^2 would make only a 1-percent contribution[98] in $\psi_0'^2$. It has been pointed out by Fermi and Segrè (F11) that even in such unfavorable cases the hyperfine structure perturbation may be large. Suppose that the hyperfine structure of the perturbing state is 50 times that of state 0, which might well be the case if the former had an unpaired s electron and the latter did not. The hyperfine structure for the latter state would be increased by 50·1 percent due to the perturbation and it would thus be half again as large as without the perturbation. It is very difficult to calculate such perturbations exactly and since they may have a great influence in the determination of μ it is desirable to avoid using states in the determination of μ which are subject to great change. States with small hyperfine structure will have the greatest percentage change. It is therefore desirable in the determination of μ to use states which have large hyperfine structure and are not subject to violent perturbation. States which perturb each other are those with the same value of J and the same parity. For strict L–S coupling there is the additional restriction that the states must have the same resultant L and S.

[98] In some cases it is also possible to have cross terms contributing. If the two electron configurations for ψ_0 and ψ_1 differ in only a single electron and are furthermore of the same parity, there will be a term $\int \psi_0 H' \psi_1 d\tau$ if H' is the hyperfine interaction term. The most frequent perturbations are between configurations which differ in two electrons but this effect might be expected between *ps* and *pd* for example.

§47. Methods Used to Determine the Nuclear Angular Momentum and the Hyperfine Structure Separations

I. Direct observation of the hyperfine structure

There are a considerable number of methods which are used to determine the nuclear spin and the hyperfine separations. For the greater part the methods are best applicable to different cases; i.e., they are very largely supplementary in scope. The greatest amount of information has come from the direct study of the hyperfine structure of the spectral lines. It is possible from such a study to get information from both normal and excited states and for almost any atom whose spectrum can be excited. Information about excited states is important in order to obtain independent determinations of μ. The main limitations are those arising from the complexity of the patterns and the smallness of the separations.

A. Number of components. If each of the two states between which a radiative transition takes place, has hyperfine structure then the resulting spectral line will show structure. This line structure will be more complex than the state structure because the selection rules for the hyperfine quantum number F may be shown to be the same as for the total electronic angular momentum J, namely, $F \to F$, $F \pm 1$, $0 \to 0$ being forbidden. It is thus not infrequent for line patterns to consist of fifteen or twenty components which, because of their close spacing, overtax the present possibilities of resolution. There are many cases, however, where the experimental possibilities are sufficient. If the hyperfine levels of one state are very close and are considerably larger for the other state, the resulting pattern will show essentially only the larger separations. Because of the interval rule such a line has a characteristic "flag" appearance, i.e., the separations decrease uniformly across the

pattern. It has been pointed out in §46 that the number of hyperfine states into which an atomic level is split by the presence of a nuclear moment is $2I+1$ for $J>I$ or $2J+1$ for $I>J$. It is apparent that the complete analysis of the hyperfine structure for any state which is split into a number of components less than $2J+1$ is a conclusive determination of I. For this purpose the presence of flag-type patterns is very useful. For small values of the nuclear spin the hyperfine structure of the spectral line is very frequently resolved completely. Exact values of the hyperfine structure separations will come for the most part from those patterns which can be completely resolved.

B. Relative separations. It is frequently the case in simple spectra of the one-electron type that no state of sufficiently large J can be found which has any appreciable hyperfine structure. In such a case the interval rule (233) can be used to determine the F values for the hyperfine states and thus the value of the nuclear spin I if J is known. The relative separations are not sensitive to I if I is large so that in such a case extremely accurate values of the hyperfine separations must be used. A determination of I by the use of the interval rule relies absolutely on its validity and therefore upon the cosine law of interaction. In view of deviations from this law which have been found recently and which seem to be due to the presence of an electric quadrupole moment, it can be considered as safe to use the interval rule to determine I only for those cases which would show no quadrupole effects (B15) namely, those states involving only s or $p_{\frac{1}{2}}$ electrons. Since the interval rule cannot be applied to a state with $J=\frac{1}{2}$, it can now be used with safety only for certain states involving more than a single electron.

C. Relative intensities. The relative intensities of the components of a hyperfine structure multiplet have been shown to obey the intensity relations (H14) which hold for multiplets in Russell-Saunders coupling when the quantum numbers L, J and S are replaced by J, F, and I, respectively. With J known for both initial and final state it is possible from an accurate knowledge of the relative intensities to deduce I since the F value can be written in terms of I and J. This method is very useful where the spin cannot be determined directly from the number of components.

D. Zeeman effect. There is one further method for determination of the nuclear spin from the direct observation of the hyperfine structure and that comes in the study of the Zeeman effect. The Zeeman effect of an atomic level leads to a displacement $M_J g(J)\mu_0 H$ where H is the magnetic field strength, M_J a magnetic quantum number for the state J and $Jg(J)\mu_0$ is the magnetic moment of the atom for the state J. If a nuclear moment is included two additional terms must be added to the energy (B3).

$$W_1 = M_J g(J)\mu_0 H + M_I g(I)\mu_0 H + A M_I M_J.$$

M_I refers to one of the magnetic substates of I and, $g(I)\mu_0 I$ is the nuclear magnetic moment and hence the second term is negligible compared to the first. The third term represents the interaction between the nuclear magnetic moment and the outside electrons, A being the separation constants. The above expression for the energy holds only for strong fields, i.e., for those fields in which the Zeeman-effect separation is large compared to the hyperfine separations. The usual field strengths which are used in Zeeman effects give separations which generally fulfill this condition and we therefore usually have a Paschen-Back effect of the hyperfine structure multiplet. Due to the third term above, each M_J state is split into $2I+1$ (the possible number of values of M_I) "hyperfine" states. In a spectral line each transition $M_J \rightarrow M_J'$ will consist of $2I+1$ components since in the strong field case changes in M_I will not be allowed. It is thus possible to determine the nuclear spin directly from the number of components in the Zeeman effect. The hyperfine Zeeman effect is also known experimentally and theoretically for field strengths which are not "strong."

II. Polarization of resonance radiation

If we have atoms in a weak magnetic field and excite them with their own radiation incident in the direction of the field, it is known that the radiation produced will show polarization effects. It is found that the percentage polarization changes with the strength of the magnetic field. This change with field strength comes about because the intensities of the Zeeman components

may be shown for particular states to depend on H/A where H is the field strength and A is the hyperfine separation constant. Different components of the radiation are thus emitted by atoms in the field with varying intensities and even if the components cannot be resolved this change in intensity is evident in the polarization of the emitted radiation. The polarization for a particular field also depends on the nuclear spin. The important application of this method as it has been developed by Ellet and Heydenburg (E2)(H12)(H13)(L3) is not in the determination of the nuclear spin but in the determination of very small hyperfine structure separations which cannot be studied directly. It has been possible by this method to determine hyperfine separations which are less than 10^{-3} cm^{-1}. The small separations of the excited states of Na and Cs have been obtained in this way.

III. Molecular and atomic beams

The principle of the Stern-Gerlach effect has been applied directly to a beam of hydrogen molecules by Stern Estermann and Frisch (F14)(E4). The beam of hydrogen molecules is passed through a strong magnetic field which has a steep gradient at right angles to the direction of the beam. The beam is separated into a number of components depending on the number of magnetic substates and the magnetic moment can be determined from the amount of the splitting. For hydrogen molecules there is, in addition to the magnetic moment which may come from the two nuclei, a magnetic moment arising from the molecular rotation. For parahydrogen the two nuclei have their nuclear moments in opposite directions so that for pure parahydrogen any observed magnetic splitting is entirely of rotational origin. It is possible to determine the rotational magnetic moment from observations on parahydrogen and then to make the necessary correction on the moment observed for orthohydrogen when ordinary hydrogen is used for the beam. The orthohydrogen will be split into three components (total spin 1) each of which will consist of three components due to the rotational moment. The central one of these coincides with the parahydrogen position but the magnetic moment can be calculated from the separation of the outer two components, corrections being made for the rotational moment. Measurements made by this method constitute the only direct determination of the nuclear magnetic moment.

A beam of atoms passed through a strong magnetic field with gradient perpendicular to the beam shows a separation into $(2J+1)$ components (Stern-Gerlach). If a nuclear moment is present each of these components consists of $2I+1$ individual components which for strong fields all fall together. Breit and Rabi (B26) have pointed out that as the field strength approaches zero these individual components no longer all fall together. In the case of the hydrogen atom ($I=\frac{1}{2}$) in its normal state there are four magnetic substates which for weak fields are all separate in the deflection pattern. Furthermore the spacing of the components changes as the field decreases until the two central ones come together at zero field. The deflection in the field will depend on the component of the magnetic moment in the field direction. For the two states with $M=M_I+M_S=0$ this component will depend on H/A, where H is the magnetic field strength and A is the hyperfine separation factor. The components of the magnetic moment in the field direction are

$$\mu_z(M_S, M_I)=\mu_z(\tfrac{1}{2}, \tfrac{1}{2})=\mu_0;$$

$$\mu_z(\tfrac{1}{2}, -\tfrac{1}{2})=\frac{x\mu_0}{(1+x^2)^{\frac{1}{2}}}; \quad \mu_z(-\tfrac{1}{2}, \tfrac{1}{2})=\frac{-x\mu_0}{(1+x^2)^{\frac{1}{2}}};$$

$$\mu_z(-\tfrac{1}{2}, -\tfrac{1}{2})=-\mu_0,$$

where $x=\mu_0 H/\pi\hbar c\Delta\nu$. An accurate measurement of the deflection pattern, together with knowledge of the field gradient, allows the determination of $\mu_z(\frac{1}{2}, -\frac{1}{2})$ and $\mu_z(-\frac{1}{2}, \frac{1}{2})$ and hence a determination of $\Delta\nu$ the hyperfine separation. The nuclear spin is obtained directly from the number of components $(2I+1)$ which have the same M_S and which thus fall together in strong fields. The method of atomic beams has been extensively developed by Rabi and his co-workers (R2)(R3)(R1)(M13)(F13) and has recently been extended by these workers (R4)(K3) so that the regularity or inversion of the hyperfine structure can also be detected. This allows one to say whether the magnetic moment is positive or negative. It has been applied to hydrogen and the various alkalis, its limitations coming chiefly

from the fact that a beam of atoms must be produced and detected. This method gives the nuclear spin and the hyperfine structure separation of the normal state only. It thus gives just the information which cannot be obtained from the study of the polarization of resonance radiation. The atomic beams method can be applied in cases where the hyperfine structure separation is too small to be measured directly. It has been applied successfully to H, D, Li, Na, K, Rb and Cs.

IV. Band spectra

The presence of a nuclear spin causes a change in the statistical weight associated with a given rotational state of a homonuclear diatomic molecule and thus causes a change in the expected intensities found in the band spectrum. If there were no nuclear spin, the statistical weight of any state for which the total angular momentum is J, would be $2J+1$. The presence of a nuclear spin further increases the degeneracy and changes the statistical weight of the states. It is found that the states whose wave functions are symmetrical in the position coordinates of the two nuclei are not affected in the same way as those whose wave functions are antisymmetrical. Let us suppose that a wave function symmetrical in the position coordinates is multiplied by a weight factor g_s while one antisymmetrical is multiplied by a factor g_a. We know (see paragraph 4) that a given molecule will have only those states which have wave functions which are totally antisymmetric (symmetric) if the nuclei obey the Fermi (Bose) statistics. If the particular nucleus has even atomic weight it will follow the Bose statistics, for odd atomic weight the Fermi statistics. For nuclear spin zero therefore we see that either g_s or g_a must be zero since the symmetry is entirely determined by the position coordinates.

If I is the nuclear spin of each of the two nuclei there will be $2I+1$ spin orientations and thus $(2I+1)^2$ spin functions representing the components of the two nuclear spins. Of these, $(2I+1)$ have the same component for both spins and will thus be symmetrical. For the remaining $2I(2I+1)$ functions, half can be built up as symmetrical combinations and half as antisymmetrical combinations. There are thus $(I+1)(2I+1)$ symmetrical and $I(2I+1)$ antisymmetrical spin functions. For a molecule containing nuclei which obey the Fermi statistics we can make the total wave function antisymmetrical from either a symmetrical or antisymmetrical position function by making the spin function either antisymmetrical or symmetrical. Because of the unequal weighting of the states corresponding to these latter, those states for which the position functions are antisymmetrical will have the greater weight. The ratio will be

$$g_a/g_s=(I+1)/I.$$

For a molecule containing nuclei which obey the Bose statistics the total wave function can be made symmetrical from either a symmetrical or antisymmetrical position function with a spin function which is either symmetrical or antisymmetrical. This leads to a ratio of the weight factors

$$g_s/g_a=(I+1)/I.$$

Successive rotational states will show alternate symmetry in the position coordinates and thus will have different statistical weights. This leads to a band structure in which successive lines have an intensity ratio $(I+1)/I$, no matter whether the Fermi or the Bose statistics are obeyed. A determination of the positional symmetry characteristics for the rotational states allows one to decide the symmetry of those states of greater weight and hence whether the nuclei follow the Fermi or the Bose statistics. H^1, Li^7, Na^{23}, P^{31}, Cl^{35}, and K^{39} are found to obey the Fermi statistics while H^2, He^4, N^{14}, O^{16} and S^{32} are found to obey the Bose statistics.

The above method is very useful in the determination of nuclear spins, particularly for light nuclei. The main advantage of the band spectrum determination is that since it depends only on the nuclear angular momentum, it is possible to obtain the spin of nuclei even in case the spin is zero. Methods which determine the hyperfine structure separation are unable to distinguish between zero magnetic moment and zero spin. On the other hand, no information about the nuclear magnetic moment can be obtained by this method of alternating intensities. Nuclei whose angular momenta have been determined from band spectra include H^1, H^2, He^4, Li^7, C^{12}, N^{14}, O^{16}, F^{19}, Na^{23}, P^{31}, S^{32} and Cl^{35}.

V. Other methods

The specific heat of a gas will depend on the distribution of molecules over the various low states. At ordinary temperatures, kT is large compared to the distance between the rotational states and any possible weighting of these states is irrelevant. For very low temperatures this is not the case and the specific heat of hydrogen, for example is well known at temperatures where kT is not large. For the hydrogen molecule the alternate rotational states are symmetrical in the nuclei beginning with the lowest with $J=0$, while those with J odd are antisymmetrical. From above this means that since the nuclei are expected to obey the Fermi statistics, $g_s/g_a = I/(I+1)$. Under ordinary conditions there will be no transitions from the symmetrical to the antisymmetrical states so that if the gas were run down to a very low temperature the molecules would not concentrate in the lowest state. This lack of transitions means that hydrogen must be treated effectively as a mixture of two gases and the specific heat determined accordingly. It was only after this had been done by Dennison (D2), that it was possible to get an explanation of the behavior of the specific heat at low temperatures. Furthermore it was found necessary to weight the even and odd states in the ratio $\frac{1}{3}$ in order to fit the specific heat and hence, from above, $I=\frac{1}{2}$.

While there are no transitions between symmetrical and antisymmetrical states under ordinary conditions, it is well known that it is possible to separate parahydrogen and also orthodeuterium by adsorption of the hydrogen or deuterium on charcoal at the temperature of liquid hydrogen. These preparations can be made with considerable purity and are very stable. It is, however, possible to induce *para-ortho* transitions by the presence of an inhomogeneous magnetic field such as that due to the paramagnetic oxygen molecule. A. Farkas and L. Farkas (F1) have found that both parahydrogen and orthodeuterium will slowly reach the equilibrium condition in the presence of oxygen. It is possible to compare the rates of conversion of parahydrogen and orthodeuterium under suitable conditions and Kalckar and Teller (K1) have found that the relative speed of conversion depends only on the spins and magnetic moments and on the equilibrium concentrations. It is possible by measuring the relative speeds of conversion to make an accurate determination of the ratio of magnetic moments for hydrogen and deuterium. Farkas and Farkas find $\mu_P/\mu_D=3.96 \pm 0.11$.

§48. Values of Nuclear Spins and Magnetic Moments

There are a considerable number of elements which have been investigated by one or more of the methods described in §46 and nuclear spins determined. These values of the nuclear spins are gathered in Tables XVII, XVIII and XIX. The first of these contains those nuclear spins which it is believed are known with certainty. Those known with somewhat less certainty are given in Table XVIII where a grade of A, B or C has been appended to indicate decreasing certainty. Such a division into groups of this sort is necessarily somewhat arbitrary because there are no sharp divisions in the degree of certainty of the nuclear spin. It is apparent from §47, however, that not all methods used to determine the nuclear spin are equally sure and it is because of this that the above division has been made. The

Table XVII. *Nuclear spins.*

Element	Source	Z	A	I	Element	Source	Z	A	I
H	*	1	1	1/2	Rb	*	37	85	5/2
H	*	1	2	1				87	3/2
He	B	2	4	0	Cd	*	48	111	1/2
Li	*	3	7	3/2				113	1/2
C	B	6	12	0	In	*	49	115	9/2
N	*	7	14	1	Sn	*	50	117	1/2
O	B	8	16	0				119	1/2
F	*	9	19	1/2	Sb	*	51	121	5/2
Na	*	11	23	3/2				123	7/2
Al	*	13	27	1/2	Cs	*	55	133	7/2
P	B	15	31	1/2	Pr	H(W5)	59	141	5/2
S	B	16	32	0	Eu	*	63	151	5/2
K	*	19	39	3/2				153	5/2
	*	19	41	3/2	Ta	H(M5)	73	181	7/2
Mn	*	25	55	5/2	Re	H(Z1)	75	185	5/2
Cu	*	29	63	3/2		(G14)(M12)		187	5/2
			65	3/2	Hg	*	80	199	1/2
Ga	*	31	69	3/2				201	3/2
			71	3/2	Tl	*	81	203	1/2
As	*	33	75	3/2				205	1/2
Br	*	35	79	3/2	Pb	*	82	207	1/2
			81	3/2	Bi	*	83	209	9/2

Table XVII contains those nuclear spins which are believed to be certain. In the "Source" column, B means determined from band spectra and H from hyperfine structure, while an asterisk indicates that it is discussed in the text.

TABLE XVIII. *Additional probable nuclear spins.*

ELEMENT	SOURCE	Z	A	I	ELEMENT	SOURCE	Z	A	I
Li	*	3	6	1C	Xe	*	54	129	1/2A
Cl	B	17	35	5/2C				131	3/2B
K	*	19	41	>1/2	Ba	*	56	135	5/2C
Sc	*	21	45	7/2A				137	5/2C
V	*	23	51	7/2B	La	*	57	139	7/2A
Co	*	27	59	7/2A	Tb	H(S6)	65	159	3/2B
Zn	*	30	67	3/2B	Ho	H(S14)	67	165	7/2A
Kr	*	36	83	9/2C	Tu	H(S13)	69	169	1/2B
Sr	*	38	87	≧3/2C	Lu	*	71	175	7/2B
Cb	*	41	93	9/2B	Hf	H(R8)	72	177	≦3/2C
Ag	*	47	107	3/2C				179	≦3/2C
			109	3/2C	Ir	H(S15)	77	191	1/2C
I	*	53	127	5/2A	Pt	H(V2)	78	195	1/2C
					Au	*	79	197	3/2C
					Pa	H(S5)	91	231	3/2A

Table XVIII contains additional nuclear spins which are considered probable. Decreasing probability is indicated by the letters A, B, C, with those marked A being nearly certain.

tables give in the second column an indication of the source of the information in some cases and in other cases simply an asterisk to indicate that the particular element is discussed briefly below.

The nuclear spins are in every case assigned to particular isotopes. In general this can be done simply, because the element is either single or consists of only two isotopes whose relative abundances are well known. In such cases there is no confusion in assigning the nuclear spins. In other cases where there are a large number of isotopes some further interpretation is necessary. It is first noticed that the majority of the isotopes represented in the tables of spins have odd mass numbers and that all of these isotopes with odd mass numbers show half-integer spins. The isotopes with even mass numbers show integer spins, mostly zero. Furthermore it should be stated that no measurable hyperfine structure has been found for any isotope with even mass number A, and even atomic number Z (meaning either small magnetic moment or zero spin). Now the cases mentioned above for which there exist a considerable number of isotopes have all been studied by the direct observation of the hyperfine structure. In the interpretation of the resulting patterns it has been assumed that the even isotopes show no structure. This assumption is very reasonable in view of the above regularity and indeed has a striking confirmation in the cases of Pb and Hg. For the first of these Kopfermann (K7) has been able to designate the isotopes by using uranium and thorium lead. For Hg the evidence that the strong central component is formed by the even isotopes which do not have hyperfine structure is already well indicated by the intensities but is confirmed by the presence of one line for which the central

TABLE XIX. *Nuclear magnetic moments.*

ELEMENT	Z	A	I	μ	CLASS	OBS. RATIO	ELEMENT	Z	A	I	μ	CLASS	OBS. RATIO
H	1	1	1/2	2.9	I	3.96	Cd	48	111	1/2	−0.65	II	1.00
		2	1	0.85	III				113	1/2	−0.65	II	
Li	3	6	1C	∼0.8	III		In	49	115	9/2	5.7	I	
		7	3/2	3.2	I		Sn	50	117	1/2	−0.89	II	1.00
N	7	14	1	∼0.2	III				119	1/2	−0.89	II	
F	9	19	1/2	3	I		Sb	51	121	5/2	3.7	I	1.32
Na	11	23	3/2	2.0	I				123	7/2	2.8	I	
Al	13	27	1/2	2.2	I		Xe	54	129	1/2A	−0.9	II	−1.11
K	19	39	3/2	0.40	I	1.81			131	3/2B	0.8	II	
		41	3/2	±0.22	I		Cs	55	133	7/2	2.5	I	
Sc	21	45	7/2A	3.6	I		Ba	56	135	5/2C	1.0	II	
Cu	29	63	3/2	2.5	I	1.00			137	5/2C	1.0	II	
		65	3/2	2.5	I		La	57	139	7/2A	2.8	I	
Zn	30	67	3/2B	−1.7	II		Eu	63	151	5/2		I	2.2
Ga	31	69	3/2	2.1	I	0.79			153	5/2		I	
		71	3/2	2.7	I		Au	79	197	3/2C	0.3	I	
As	33	75	3/2	1.5	I		Hg	80	199	1/2	0.5	II	−0.90
Kr	36	83	9/2C	−1	II				201	3/2	−0.6	II	
Rb	37	85	5/2	1.4	I	0.494	Tl	81	203	1/2	1.4	I	0.98
		87	3/2	2.8	I				205	1/2	1.4	I	
Sr	38	87	≧3/2C	∼−0.8	II		Pb	82	207	1/2	0.6	II	
Ag	47	107	3/2C	0.2	I		Bi	83	209	9/2	4.0	I	
		109	3/2C	0.2	I								

Table XIX contains the magnetic moments (μ) of the various elements in units $eh/2Mc$. A number given in the ratio column is the ratio of the magnetic moment of the isotope with smaller A (mass number) to that with larger A. All elements given here are discussed briefly in the text.

components are absent (G8). This line is ordinarily forbidden but arises because two atomic states are not separated by an amount large compared to the hyperfine structure. The "forbidden" line should thus occur only for those isotopes possessing structure. The other cases in which a number of even isotopes exist in addition to the odd ones mentioned here are Zn, Kr, Sr, Cd, Sn, Xe, Ba and Hf. These even isotopes either have spin zero or very small magnetic moments.

Most of the known nuclear spins are of isotopes with Z and A both odd (class I) and these spins are half-integer. There are fewer with Z even and A odd (class II) and these are also half-integer. There are only three (this type of nucleus is very rare) with Z odd and A even (class III) and they have spin unity. There are four with Z and A both even (class IV) and these spins are zero.

The nuclear magnetic moments are gathered in Table XIX. These are mostly derived from the hyperfine structure separation by the use of Goudsmit's equations (248) and (249). The magnetic moment is given in the column headed μ and is in units $\mu_0/1838$. All elements for which magnetic moments are given, are discussed briefly in the text. The nuclear magnetic moments roughly follow certain regularities. In general those nuclei of class I have magnetic moments which are large (>1) and positive. The nuclei of class II all seem to have small magnetic moments and are mostly negative. The three of class III are all small, that of H^2 being positive. No magnetic moments are known for elements of class IV, since none of them show any hyperfine structure. As mentioned before this may mean zero angular momentum or small magnetic moment. Since the few nuclei of this type for which spins are known all have $I=0$, perhaps the former possibility is the more probable.

In view of the approximations which are necessary in order to determine nuclear magnetic moments, there is one sort of information which is of great interest because it does not depend on these approximations. In cases where there are two isotopes the ratio of the magnetic moments can be determined directly from the hyperfine separations and the values of the spins by making use of the fact that wave functions of the two isotopes are the same in the neighborhood of the nucleus. There are twelve such ratios which are known and they are listed in Table XIX, giving the ratio of the magnetic moment of the isotope with smaller A to that of the isotope with larger A. Since they should have an accuracy which is limited only by the accuracy of the hyperfine measurements these ratios constitute the most accurate information about nuclear magnetic moments.

There is really no satisfactory way of ascertaining the errors in the values of the nuclear magnetic moments themselves. Calculations using wave functions have been found to give widely varying results in the case of Na (see p. 209). The better the wave functions were corrected, the more nearly did the values of the magnetic moment obtained agree with that obtained by the use of the Goudsmit-Fermi-Segrè approximate formulas. Probably the best indication of the accuracy comes from the consistency of the values obtained from different atomic states, i.e., by using different individual electron hyperfine separation constants. In determining the magnetic moment care must be exercised not to use any atomic states which are subject to large perturbations (see p. 211). It is nearly impossible to avoid such perturbations in atoms which have complicated configurations of valence electrons, though it is sometimes possible to find cases where they are not important. In general the most accurate information will come from atoms or ions which have simple valence configurations. There are numerous nuclei whose external electron structure is too complicated to allow a determination of the nuclear moments at present (for example Eu).

H

The nuclei H^1 and H^2 are known to have spin $\frac{1}{2}$ and 1, respectively. These values have been determined from band spectra (H16)(K2)(M19) and in the first case also by the atomic beams method (unpublished) and from the specific heat at low temperatures (D2). The value of the magnetic moment ($\mu=2.9$) of H^1 is that determined by Rabi, Kellogg and Zacharias from the determination of the hyperfine structure separation of the normal state (unpublished and (R2)(R4)). The magnetic moment of H^2 is also the value given by Rabi, Kellogg and Zacharias

(R3). For hydrogen the calculations of the magnetic moment use the exact wave functions of course; the errors arise from the difficulties of measurement. The value 0.7 for H^2 is in agreement with the ratio $\mu(H^1)/\mu(H^2)=3.96$ determined by Farkas and Farkas (F1) which is more accurately known than either of the magnetic moments.*

Li

The work of Fox and Rabi (F13) using the atomic beam method indicates that Li^6 has a spin $\geqq 1$. It seems probable that the spin is 1 (see §36). The ratio of the magnetic moments can be determined directly and gives $\mu(Li^6)/\mu(Li^7) \leqq 0.25$. This means that the magnetic moment of the Li^6 is about the same as for H^2. For Li^7 the nuclear spin has been determined from band spectra by Harvey and Jenkins (H5), from hyperfine structure by the work of Guttinger (G15), Schuler (S2); Guttinger and Pauli (G16) and Granath (G10) and by atomic beams by Fox and Rabi (F13). The magnetic moment has been determined from the hyperfine structure separations of Li II $1s2s\,^3S_1$ (G10) and from the hyperfine separation of the ground state of Li I (F13). The two values thus obtained are practically identical.

N

The N^{14} nucleus is known to have a spin of 1 from the alternations in intensity in band spectra as observed by Ornstein and Van Wijk (O2). From the study of lines in the N I spectrum expected to have large hyperfine structure by reason of the large nuclear interaction, it has been concluded that the magnetic moment must be in the neighborhood of 0.2 or smaller (B1). Because of the importance of the magnetic moments of light elements it would be very desirable to have a measurement on the normal state by the atomic beams method. The nitrogen nucleus follows the Bose statistics (H11)(R7).

F

The fluorine nucleus has a spin of $\frac{1}{2}$, determined from band spectra by Gale and Monk (G1) and confirmed from hyperfine structure measurements by Campbell (C2). The nuclear magnetic moment has been obtained from these latter measurements by Brown and Bartlett (B30). They have carried out calculations using Hartree functions and have obtained values which are in the neighborhood of 3. There is quite a variation in the values obtained from different levels, at least part of which is due to the fact that the hyperfine structure patterns are not completely resolved.

Na

The nuclear spin of Na^{23} has been determined from atomic beam deflection (R1), from band spectra (J5), from hyperfine structure intensities (G12), and from the polarization of resonance radiation (E2)(L3), and the values thus found are all $I=3/2$. The hyperfine structure separations of several states have been measured (R1)(E2)(L3)(G12)(J3)(F13) and the magnetic moments have been calculated (W15)(S21) by using the G.F.S. relations and with various sorts of wave functions (see p. 209). The value given in these tables is that obtained from the G.F.S. relations for the states $3s\,^2S_{1/2}$ and $4p\,^2P_{3/2}$. The values for $3p\,^2P_{3/2}$ are between 2.25 and 2.6 depending on the value of the separation used.

Al

The nuclear spin $I=\frac{1}{2}$ has been determined by Ritschl (R11) from observations on the number of components in the hyperfine structure. The measurements give only a single one-electron separation factor $a(3s)$ since the separation observed in the Al II and Al I terms are attributed to this electron (R11)(P3)(G8). The value of the magnetic moment is determined from this single separation. Brown and Cook (B31) have obtained magnetic moment 2.4 from the same separation by using Hartree functions.

K

The nuclear spin of K^{39} has been determined by the atomic beam method by Millman (M13) and is 3/2. The magnetic moment is determined from the separation of the normal state $4s\,^2S_{1/2}$ which has been accurately determined by the deflection method (F13) as well as by the hyperfine structure measurements of Jackson and

* *Note added to proof:* More recently Rabi, Kellogg and Zacharias have found that $\mu(H^2)=0.85\pm0.03$. This is not in agreement with the ratio determined by Farkas and Farkas and this latter is therefore subject to doubt.

Kuhn (J3)(J4). Recent measurements (Rabi and co-workers, unpublished) with atomic beams indicate that the hyperfine structure of the normal state of K^{39} is regular and the magnetic moment therefore positive. This result is in disagreement with the observed intensities (J4) of the hyperfine structure, but this may be due to reversal in the hyperfine lines. The value -0.40 for the magnetic moment is given from the G.F.S. relation while Gibbons and Bartlett (G4) get $\mu=1.2$ using Hartree functions. This is a wide discrepancy but in view of similar difficulties with Hartree functions in Na, the former value is taken. The nuclear spin of K^{41} has been found by Manley (unpublished) to be 3/2 using atomic beams. The magnetic moment determined from the separation of the normal state is ± 0.22.

Sc

Schuler and Schmidt (S11) and Kopfermann and Rasmussen (K10) have found the nuclear spin of Sc^{45} to be 7/2, and the first workers have determined it from the number of components. The magnetic moment has been determined by the second workers from the $ds^2\ {}^2D_{5/2}$ and ${}^2D_{3/2}$ separations. Since these separations are not resolved directly but are inferred from an unresolved pattern and since it is doubtful what the correct value of Z_i should be (they use $Z_i=8$), the resulting magnetic moment is very approximate. The two separations give the same μ, however.

V

The hyperfine structure of the V I spectrum has been investigated by Kopfermann and Rasmussen (K12) who assign a nuclear spin of 7/2. Because of the extreme complexity of the unresolved patterns this value cannot be considered as certain. The interactions are too complicated to determine a magnetic moment from the observed separations.

Mn

The nuclear spin of 5/2 for Mn^{55} has been determined by White and Ritschl (W7). The complexity of the electron configurations do not allow a determination of a value for μ.

Co

From the work of More (M16) and Kopfermann and Rasmussen (K11) the nuclear spin 7/2 of Co^{59} is practically certain. No value of the magnetic moment can be obtained from the measured separations because of the complexity of the electron configurations.

Cu

The hyperfine structure of Cu I has been studied by Ritschl (R10) who found the spins of both Cu^{63} and Cu^{65} to be 3/2. Though the lines in the resultant patterns are not coincident for the two isotopes they both have the same spin and the same hyperfine separations. This means that they both have the same magnetic moment. The magnetic moment 2.5 is determined from the $d^{10}s\ {}^2S_{1/2}$ and the $d^9s^2\ {}^2D_{3/2,\ 5/2}$. These give, respectively, 2.5, 2.5 and 2.1 with the value $Z_i=19.6$ determined by Fermi and Segrè for the d electron. These terms are selected because they are expected to show the smallest perturbation effects from terms with much larger structure. Perturbation effects may, however, be present for both of these terms and this adds to the uncertainty of the magnetic moment. The approximate agreement of the values given above indicates that these perturbations are probably not serious.

Zn

Hyperfine structure has been found by Schuler and Westmeyer (S19) for Zn II. The observed components are very weak compared to the strong lines assigned to the even isotopes in accordance with expectations for Zn^{67}. They conclude that the spin is 3/2 though this conclusion is not certain because the pattern is not completely resolved. By using the observed separation for $d^9s^2\ {}^2D_{5/2}$ the magnetic moment would be about -1.7. This value has no other confirmation.

Ga

The nuclear spins of Ga^{69} and Ga^{71} have been determined by Jackson (J1) and Campbell (C1) from a study of the hyperfine structure. The two nuclei have the same spin $I=3/2$ but different magnetic moments as found by Campbell. The ratio $\mu^{69}/\mu^{71}=0.79$ is quite exact. The individual

magnetic moments are determined from the separation constants for the $5s\ ^2S_{1/2}$ and $4p\ ^2P_{1/2}$ states in Ga I and from the separation constant $a(4s)=0.43$ cm^{-1} as determined from numerous separations in Ga II. These give 2.24, 2.36 and 2.07 for μ^{69}, respectively. Perhaps none of these values is very accurate.

As

The nuclear spin of As^{75} has been determined by Tolansky (T4) and by Crawford and Crooker (C20) from interval measurements on the hyperfine structure. Interval measurements may be influenced by the presence of an electric quadrupole moment for the nucleus. Crawford and Crooker's determination on the intervals of $4s5s\ ^3S_1$ of As IV is free from this possibility, however. The magnetic moment $\mu=1.5$ is calculated from the separation factor $a(4s)$ determined from the As IV measurements. The separations in As II are subject to large perturbations so cannot be used directly to determine μ.

Br

The nuclear spins of Br^{79} and Br^{81} seem to be quite certainly 3/2 from the work of Tolansky (T3). Though none of the spectral lines investigated were completely resolved, the appearance of almost identical structures having four components for several lines having $4p^45s\ ^4P_{5/2}$ as a final state indicated $I=3/2$. Both isotopes have the same abundance and the structures are superimposed. There are several separations known but the interaction with the nucleus is through all the five electrons and perturbations are also very probable. No magnetic moment is determined.

Kr

Krypton has a number of even isotopes and one odd one, Kr^{83}. The spectral lines have been studied by Kopfermann and Wieth-Knudsen (K13) and are found to have a very strong central component corresponding to the even isotopes and a weak structure attributed to Kr^{83}. It is possible to conclude that $I>\frac{1}{2}$ and the value 9/2 makes the weighted center of the fine structure coincide with the even isotopes. This value is uncertain. By using Goudsmit's sum relations (G5) it is possible to obtain a value for μ from $a(5s)$. This value is negative and roughly unity.

Rb

The spins of Rb^{85} and Rb^{87} have been determined by Kopfermann (K8) from the hyperfine structure of the Rb II lines and also by Fox and Millman (unpublished) using the atomic beams method. Both methods give $I(85)=5/2$ and $I(87)=3/2$. The magnetic moments have been determined from the separations for $5s\ ^2S$, the normal state of Rb I. The ratio of the magnetic moments (0.494) is believed to be quite accurate ($\sim$1 percent).

Sr

Strontium is known to have hyperfine structure (M18)(S17)(S3)(W4) and it is attributed to Sr^{87}. The spin is very uncertain but using the value $I=3/2$ a value for the magnetic moment can be obtained from the separation of the normal state of Sr II. ($\Delta\nu=-0.15$ cm^{-1}.)

Cb

The spin of Cb^{93} has been determined by Ballard (B7) from the hyperfine structure. The lines are not completely resolved but a careful study has yielded the value $I=9/2$. The value of the magnetic moment (B7), $\mu=3.7$ is very rough at best.

Ag

A doubling of the resonance lines of silver has been observed by Hill (H15). Because of the intensity ratio of the components it is concluded that the structure is hyperfine structure and not isotope shift, since the two isotopes have nearly the same abundances and the observed structure shows a weak and a stronger component. A tentative value of 3/2 is suggested for the spins of both isotopes. If this is correct then the two isotopes would have the same magnetic moments. They would be about $+0.2$.

Cd

Cadmium has six isotopes, 4 even and 2 odd whose mass numbers are 111 and 113. The lines of the spectrum have been studied by Schuler and co-workers (S4)(S9) and are found to have a strong central component attributed to the

even isotopes and a simple hyperfine structure attributed to 111 and 113 together. The spins of the two odd isotopes are certain under these assumptions which are made almost mandatory by the detection of isotope shift (S20) in Cd II. The magnetic moments for the odd isotopes are determined from the separation factor $a(5s)$ for the $5s$ electron, which gives rise to most of the structure in Cd I, and that of the $6s$ electron (J6). These give $\mu=-0.66$ and -0.63, respectively, for both isotopes.

In

The nuclear spin of In^{115} has been determined by Jackson (J2) and Paschen and Campbell (P1)(P2). Though no lines have been found where the large spin could be determined directly from the number of components the accurate measurement of the separations of $5s6s\ ^3S_1$ which would not be expected to show any quadrupole effect, makes $I=9/2$ quite certain. The unusually large magnetic moment is determined from the separations observed in In I, $a(6s)=0.056$ cm^{-1} and $a(6p_1)=0.076$ cm^{-1} and deduced from In II, $a(5s)=0.70$ cm^{-1}. These give $\mu=5.2$, 5.9 and 6.0, respectively, for the magnetic moment.

Sn

Tin has a large number of isotopes but only two of these Sn^{117} and Sn^{119} are odd and present in any considerable amount. A study of the hyperfine structure by Tolansky (T5) and by Schuler and Westmeyer (S18) shows that the lines can be interpreted in a manner similar to Cd, assigning the weak structure to the odd isotopes. If this structure is due to both odd isotopes then it is quite certain that the spin of each is 1/2. From the hyperfine separations of $6s\ ^2S_{1/2}$ and $6p\ ^2P_{1/2}$ in Sn II the magnetic moments are $\mu=-0.90$, -0.87, respectively.

Sb

The nuclear spins of the two antimony isotopes Sb^{121} and Sb^{123} have been determined by the work of Crawford and Bateson (C19) Badami (B4) and Tolansky (T6). Crawford and Bateson have completely resolved the line $5s6p\ ^3P_0 \rightarrow 5s6s\ ^3S_1$ of Sb IV and interval measurements lead uniquely to the spins $I(121)=5/2$ and $I(123)=7/2$. The interval rule is expected to hold very well here as a possible quadrupole moment would show no effect for these states. The magnetic moments are determined from the separation factor $a(5s)=1.4$ cm^{-1} for Sb^{121} determined from these measurements. A determination of the same $a(5s)$ from the sp^3 configuration of Sb II leads to a lower value of $a(5s)=1.04$ but this might be expected since this configuration is undoubtedly perturbed and its hyperfine structure made smaller thereby. By using the former value, $\mu(121)=3.7$. The ratio $\mu(121)/\mu(123)=1.32$ can be determined directly from the observed separations.

I

The hyperfine structure of the iodine spectrum has been studied by Tolansky (T7)(T8)(T9) who has observed a large number of lines. Interval measurements on lines showing structure predominantly from one level indicate that the spin is very probably 5/2. The large number of possible energy states and the existence of large perturbations do not permit an evaluation of the magnetic moment of I^{127} at present.

Xe

There are a considerable number of Xe isotopes but only two, Xe^{129} and Xe^{131} with odd mass numbers. Kopfermann and Kindal (K9) and Jones (J7) find that the observed hyperfine structures can be accounted for by assigning nuclear moments to these odd isotopes. The spin of Xe^{129} is almost surely 1/2 and that of Xe^{131} probably 3/2. With the sum relations of Goudsmit (G5) it is possible to find a separation factor $a(6s)=-0.164$ cm^{-1} for Xe^{129}. This gives a rough value for the magnetic moment $\mu=-0.9$. The ratio of the moments is known from the observed separations, and the spins $\mu(129)/\mu(131)=-1.11$. In this case the two magnetic moments have opposite signs.

Cs

The nuclear spin of Cs^{133} has been determined from hyperfine structure by Kopfermann (K6) and by the method of atomic beams by Cohen (C11). The value $I=7/2$ is quite certain. Accurate measurements of the hyperfine separations of several states have been made by Granath and Stranathan (G11) and Heydenburg (H13).

The separation factors and magnetic moments obtained from them are as follows: $a(6^2S_{1/2}) = 0.0767$ cm^{-1}, $\mu = 2.70$; $a(6^2P_{1/2}) = 0.00925$ cm^{-1}, $\mu = 2.45$; $a(6^2P_{3/2}) = 0.00142$ cm^{-1}, $\mu = 2.37$; $a(7^2P_{1/2}) = 0.00329$ cm^{-1}, $\mu = 2.67$; $a(7^2P_{3/2}) = 0.000486$ cm^{-1}, $\mu = 2.48$. The values $Z_i = Z$ and $Z_i = Z - 4$ have been used for s and p electrons as usual. Slightly different values of Z_i determined either from the doublet separation or from the observed ratio of the hyperfine separations from the two members of the doublet lead to small changes in the above values (G11)(H13).

Ba

Kruger, Gibbs and Williams (K14) have concluded from a study of the intensities of the Ba II hyperfine structure that the spin of Ba^{135} and Ba^{137} is probably 5/2. The presence of several isotopes makes this value rather uncertain. All the lines show strong central components corresponding to the even isotopes. Using the separations (K14)(R12) of the $6\,^2S$ and $6\,^2P_{1/2}$ states the magnetic moments are respectively $\mu = 1.06$ and 0.82 if $I = 5/2$.

La

The angular momentum of La^{139} is very probably $7/2\ \hbar$ from the hyperfine structure measurements of Anderson (A1)(A2). The magnetic moment has been determined by Crawford and Grace (C21) from the separations observed in La III. They find for the states $6s\,^2S$ and $6p\,^2P_{1/2}$ the values $\mu = 2.84$ and 2.87. From a study of the d^2s configuration Crawford (C18) gets for the separation factor a somewhat different value which gives $\mu = 2.5$.

Eu

The two europium isotopes Eu^{151} and Eu^{153} have been found by Schuler and Schmidt (S15) to have angular momentum $5/2\ \hbar$ for each type of nucleus. These values are established directly by the number of components observed. The individual magnetic moments cannot be found easily due to the complicated electronic structure, but their ratio is found directly. There is evidence in Eu that the interval rule is not followed equally well for both isotopes. This is a deviation which cannot be explained by a perturbation effect (see §50).

Lu(Cp)

From the relative intensities of the hyperfine components, Schuler and Schmidt (S16) find a value $7/2\ \hbar$ for the angular momentum of the Lu^{175} nucleus. Evidence of deviations from interval rule are also found.

Au

Hyperfine structure found for gold by Ritschl (R9) and Wulff (W16) indicates that Au^{197} has a nuclear spin which is probably 3/2. By using this value and the observed separation of the normal state, $\mu = 0.2$.

Hg

Mercury has several isotopes of even mass number and two odd ones Hg^{199} and Hg^{201} to which the observed hyperfine structure is attributed. These two isotopes have been found by Schuler, Keyston and Jones (S8)(S7) to have spins 1/2 and 3/2, respectively. Numerous lines of the Hg I spectrum have been investigated and the separations are well known. These separations, however, are due mainly to the $6s$ electron which has a large separation factor. The values of $a(6s)$ for Hg^{199} determined from various configurations do not agree particularly well (from $6s\,9s$, $a(6s) = 1.37$ neglecting $a(9s)$; from $6s\,6p$, $a(6s) = 1.15$). These give $\mu(199) = 0.52$, 0.43, respectively. The ratio of the magnetic moments gives $\mu(199)/\mu(201) = -0.90$.

Tl

Hyperfine structure has been found in the Tl I, II and III spectra (S10)(M3)(M2). All of these patterns show conclusively that $I = \frac{1}{2}$ for Tl^{203} and Tl^{205}. The separations in Tl II are due almost entirely to the $6s$ electron so that while it is possible to make several estimations of this separation constant this leads to only one determination of μ. By using (248) and (249) the following values of μ are obtained from the various separations: Tl III, $a(7s) = 1.37$ cm^{-1}, $\mu = 0.9$; $a(8s) = 0.606$ cm^{-1}, $\mu = 1.5$; $a(6p_{1/2}) = 1.21$ cm^{-1}, $\mu = 1.8$; $a(7p_{1/2}) = 0.375$, $\mu = 1.4$; Tl II $a(6s) = 5.8$ cm^{-1}, $\mu = 1.7$; Tl I $a(7s) = 0.402$ cm^{-1}, $\mu = 1.1$; $a(6p_{1/2}) = 0.710$ cm^{-1}, $\mu = 2.0$. These values vary rather widely and are probably not very trustworthy. Calculations by Breit (B24) and Wills (W14) give $\mu(6p_{1/2}) = 1.45$ and $\mu(7s)$

$=1.35$ from the Tl I separations. There are rather large perturbations in the thallium spectra which make some of the separations very irregular and those expected to be particularly bad are not included above. Though the components due to the two isotopes are usually not separated there is evidence (S10) that Tl^{205} may have slightly larger separations, though this small effect may be due to the presence of a quadrupole moment for one of the isotopes.

Pb

The lead isotopes, with $A=204$, 206, 207, 208, are all evident in the hyperfine structure. Kopfermann (K7) has shown, by using samples of uranium lead (206) and thorium lead (208), just which single components should be assigned to these isotopes. Pb^{207} shows a structure which is due to a nuclear spin $I=\frac{1}{2}$. The presence of perturbations makes the determination of μ somewhat unreliable. Using the separation (M4) of $6s\ 7s\ ^3S_1$ from Pb III and neglecting the value $a(7s)$ we find from (248), $\mu=0.64$. This value would be smaller had $a(7s)$ not been neglected. Using $a(6s)=1.89$ as determined by Rose (R13) from Pb II sp^2 we find $\mu=0.40$. This is probably too small due to perturbations. Breit and Wills (B28) find $\mu=0.75$ and 0.67 from considering the $6p^2$ and $6p\ 7s$ configurations of Pb I.

Bi

For Bi^{209}, $I=9/2$. This value has been determined from the study of the Zeeman effect for the hyperfine structure by Back and Goudsmit (B3). Hyperfine structure has been observed and studied in the spectra of Bi I–V. Of these Bi II and III may be expected to show large perturbation effects due to excitation of inner electrons. For Bi IV, there are also perturbations but it is possible from some of the unperturbed levels to find (M1), $a(6s)=2.3$ cm^{-1}. This value is practically the same as that determined directly from $6s6p_{3/2}$ which should not be greatly perturbed. From Bi V the separation of the normal state (A3) gives $a(6s)=2.6$ cm^{-1}. By using this value in (248), $\mu=3.5$. It is possible that the p^3 configuration of Bi I is not badly perturbed and we may obtain the separation constant, $a(6p_{1/2})=0.375$ cm^{-1}. By using (249) this gives $\mu=3.5$. This agreement does not mean much since the second value should certainly be larger due to the screening of the other p electrons. Breit and Wills find $\mu=5.4$ from p^3 of Bi I. The value 3.5 given in Table XIX, probably is not very reliable.

§49. Isotope Shift in Atomic Spectra

The spectral lines which are due to the different isotopes of an element usually do not have the same wave-length and the energy states of the various isotopes must therefore be spaced differently. If this effect occurs alone, there are as many components of a given spectral line as there are isotopes and their intensities are proportional to the relative abundances. If we consider the energy states for a particular isotope and compare them to those of another, we cannot say from the isotope shift alone how the energy levels of one isotope are placed with respect to those of another; we can detect only differences in the separations of the energy levels. Accordingly we expect such an isotope effect to be detectable only by means which study the transitions between energy states.

In the preceding paragraphs the presence of several isotopes was a complication which made the interpretation of the hyperfine separations more difficult. We found that isotopes having even mass number A show no hyperfine separations except those few peculiar nuclei of class III (H^2 Li^6 B^{10} N^{14}). Those nuclei with A odd generally show hyperfine structure. For elements which have a number of isotopes we may expect isotope shift *alone* for those with even A and isotope shift plus hyperfine structure for those with odd A. In the first case we can determine the relative displacement of the levels directly from the observed shifts. In the second case it is necessary to allow for the hyperfine structure and find the position of the hypothetical level without hyperfine structure. Because of the regular spacing and known weighting of the levels, this can be done quite easily and it is thus possible to determine the isotope shift for all the isotopes whether they show hyperfine structure or not though the interpretation in case they do is much more involved.

In order to ascertain the cause of isotope shift it is perhaps best to examine first of all the effect

of different mass for the different isotopes. It is well known that a correction to the energy must be made in a hydrogen-like atom if the nucleus has been regarded as fixed. It is also well known that the final energy depends upon the nuclear mass and that the H^1 and H^2 spectral lines are not coincident but are displaced from each other by an amount which is just that expected by their mass difference.

In the case of an atom with several electrons it is found that the shift in the energy levels consists of two parts (H17)(B11), one of which is just like the hydrogen case and is called the normal effect and a new part which is characteristic of the many electron problem and is called the specific effect. If the center of mass is regarded as fixed the Schrödinger equation for N electrons of mass m and a nucleus of mass M can be written

$$-\hbar^2\left[\frac{1}{2\mu}\sum_{\kappa=1}^{N}\nabla_\kappa^2+\frac{1}{M}\sum_{\kappa<j}\nabla_\kappa\cdot\nabla_j\right]+(V(x)-W)\psi=0, \quad (251)$$

where $\mu=\dfrac{mM}{M+m}$, $\nabla_\kappa=\mathbf{i}\dfrac{\partial}{\partial x_\kappa}+\mathbf{j}\dfrac{\partial}{\partial y_\kappa}+\mathbf{k}\dfrac{\partial}{\partial z_\kappa}$,

and x_κ, y_κ and z_κ are the coordinates of the κth electron with respect to the nucleus. Let us consider that the characteristic values $W(m)$ and the solutions $\psi(m)$ are known for the case of the stationary nucleus. In order to compare with these values let us neglect the second term in (251) momentarily. It may then be seen by introducing coordinates with respect to the center of mass that the energy W' is related to $W(m)$, the energy for a stationary nucleus simply by

$$W'=\alpha W(m), \quad \text{where} \quad \alpha=\mu/m. \quad (251a)$$

Next if we consider the second term in (251) as a perturbation, it gives a contribution to the energy

$$\Delta W=-\frac{\hbar^2}{M}\int\psi^*(m)\Big[\sum_{\kappa<j}\nabla_\kappa\cdot\nabla_j\Big]\psi(m)d\tau. \quad (251b)$$

We now have the difference between the energy in (251) and that for the problem of the stationary nucleus, separated into two parts (251a) the normal effect and (251b) the specific effect. In order to evaluate (251b) it is necessary to use wave functions for the particular atom in question. Hughes and Eckart (H17) carried this out for the case of Li and found agreement with experimental data of Li II and Li I. Bartlett and Gibbons (B11) have made the calculation for the case of Ne. In this case the agreement is not satisfactory since some of the lines show much larger specific shifts than expected.

The shift due to the specific effect (251b) is inversely proportional to the mass M or to A if we replace M by A the mass number of the particular isotope considered. We have then for the normal effect.

$$W'=\frac{A}{A+m}W(m)=\left(1-\frac{m}{A}+\cdots\right)W(m), \quad (252a)$$

$$W'-W(m)=-(m/A)W(m).$$

Since the specific effect is also inversely proportional to the mass number, the total energy displacement $D=W'-W(m)+\Delta W$ is proportional to $1/A$. If we consider an element with several isotopes with mass numbers A, $A+\delta_1$, $A+\delta_2$, etc., the relative displacement for these various isotopes can be written

$$D_0-D_1=C/A-C/(A+\delta_1)\cong C\delta_1/A^2, \quad D_0-D_2\cong C\delta_2/A^2. \quad (252b)$$

Since δ_1, δ_2, etc., are practically always small compared to A this is a good approximation. The relative displacement is thus proportional to the differences of the masses of the two isotopes. For several isotopes with successive mass numbers we expect the components due to the various isotopes to be equally spaced. This affords a means of examining the isotope shift without making the calculations in each case. The first element in the periodic table which has more than two isotopes all of which have sufficient abundance, is Mg. There are three isotopes with $A=24$, 25, and 26, present with relative abundances 7, 1, 1. According to (252b) we should expect any energy state to have relative displacements proportional to the mass differences and hence the same must be true of the relative shift in the spectral lines. Some of

the spectral lines of Mg I do show isotope shift (B2) but instead of showing three equally spaced components, show only two components.* This means that the observed shift cannot be explained by mass effect alone. For other elements it has been found that the elements with even A are usually spaced approximately according to (252b) but the odd isotopes do not occupy the required positions.[99] Schuler and Schmidt (S12) have found in Sm that the even isotopes also do not seem to be regularly spaced. It must be concluded that mass effect alone is not sufficient to explain the observed shifts for any but the light elements.

If the isotope displacement of a state for which the outer electrons are seldom in the neighborhood of the nucleus (large l) is arbitrarily called zero, then it is found that states with electrons having small l and particularly with s electrons, have large isotope displacement. The amount of the displacement increases with the number of penetrating electrons and the degree of penetration. This means that perturbation effects are very important for isotope displacement as well as for hyperfine structure. A simple example (B2) of this is found in the case of Mg where most of the observed shifts are directly due to perturbation effects and are found to be quantitatively related to the amount of the perturbation. The importance of states showing penetration indicates that calculations (D3) particularly for heavy atoms, which assume Russell-Saunders coupling and neglect perturbations must be considered as unreliable.

The presence of large displacements where penetrating electrons are involved indicates that isotope shift may be due to some difference in the field in the neighborhood of the nucleus. It has been pointed out by Bartlett (B8) that such a difference is to be expected if one assumes constant nuclear density. The heavier isotopes have greater radii since $R\sim A^{\frac{1}{3}}$, and will bind a penetrating electron more loosely than the lighter nuclei. Calculations have been carried out by Breit and Rosenthal (B25)(R14) and Racah (R6) who find that the change in the energy of an s state due to a change in nuclear radius Δy_0 is

$$\delta W=\frac{Ra_0^2}{Z}R^2(0)\frac{1+s}{[\Gamma(2s+1)]^2}\left[y_0^{2s}+\frac{v}{2\gamma^2}y_0^{2s+1}\right]\frac{\Delta y_0}{y_0},$$

where $s=(1-\gamma^2)^{\frac{1}{2}}$; $y_0=\dfrac{2Zr_0}{a_0}$; $v=\dfrac{V}{mc^2}$; $\gamma=\dfrac{Z}{137}$.

r_0 is the nuclear radius in cm inside of which V, the potential energy is considered constant; a_0 is the Bohr radius and R the Rydberg constant; $R(0)$ is the value of the Schrödinger radial function at $r=0$ and it may be obtained in terms of the observed hyperfine separation from (240) if the magnetic moment has been previously determined from (248) or (249). Breit has found that, with $R^2(0)$ determined in this way, there is general agreement for Hg, Tl and Pb. Since it is expected that r_0 will be larger than normal for nuclei containing an odd neutron or proton, the shift for a nucleus with Z even and A odd will be more nearly like that of nucleus Z, $(A+1)$ than like Z, $(A-1)$. For heavy nuclei, however, it is known that the displacement for nucleus Z, A is more nearly like Z, $A-1$. It is possible that the extra neutron may interact with the electrons to cause this effect.[99] Detailed comparisons have not yet been made for other elements, but the light elements are in disagreement with the "radius" effect since for Ne, Mg, Cu and Zn it would be necessary to ascribe smaller radii to the nuclei of larger mass number. Although no entirely satisfactory explanation of the observed isotope shifts has been found, the variation for different atomic states indicates a difference in the fields of the various isotopes in the neighborhood of the nucleus.

§50. Quadrupole Moments

It has been found by Schuler and Schmidt (S15) that certain elements show deviations from the interval rule. Such deviations are to be expected when two atomic states are very close together but otherwise the interval rule should

* The possibility that Mg^{25} shows hyperfine structure can be eliminated because of the nature of the states involved.

[99] It was suggested by Breit and Condon at the New York meeting of the American Physical Society, February 1936, that a small neutron-electron interaction might, though insignificant elsewhere, show itself in the isotope shift. If the odd neutron which nuclei of even Z and odd A contain, (Mg^{25} for example) would show such interaction it might explain why these isotopes do not appear centrally between those of mass $A-1$ and $A+1$.

be obeyed if the interaction between the electrons and the nuclear moment follows a cosine law. The deviations found by Schuler and Schmidt (S15) for Eu and by Mintz and Granath (M14) for Bi cannot be explained by perturbation effects. In the first case the two isotopes Eu^{151} and Eu^{153} show different deviations from the interval rule and in the second case deviations are found for the $p^3\,{}^2D_{3/2}$ state of Bi which is not sufficiently close to any other state to allow an explanation on the basis of perturbations.

Schuler and Schmidt observed that the deviations could be satisfactorily accounted for by the presence of an interaction term which is proportional to the square of the cosine of the angle between the nuclear spin and the extra-nuclear angular momentum. Such an interaction, they pointed out, would be expected if the nucleus has an electric quadrupole moment. It is indeed reasonable that nuclei should have small electric quadrupole moments. Such a moment arises if the protons are not distributed over the nucleus on the average with spherical symmetry.

The calculation of the contribution to the energy due to the presence of an electric quadrupole moment for the nucleus has been carried out by Bethe (B15) and Casimir (C3). It is found that for the case of a single electron outside closed shells the change in the energy of a hyperfine state F associated with the atomic state $j=l-\frac{1}{2}$, can be written

$$W_F=-\tfrac{1}{2}\overline{R^2}e^2\overline{r^{-3}}\times\frac{3C(C+1)-4I(I+1)j(j+1)}{(2l+1)(2l+3)(2I-1)(2I+3)},\quad(253)$$

where $C=F(F+1)-I(I+1)-j(j+1)$. $\overline{R^2}$ is a measure of the nuclear quadrupole moment since R is the coordinate of a nuclear proton with respect to the center of gravity of the nucleus. It is possible to determine $\overline{r^{-3}}$ from (241). It is also possible to replace $\overline{r^{-3}}$ by $\overline{Z(r)r^{-3}}/Z_i$ and to determine this quantity from the fine structure separation since $\overline{Z(r)r^{-3}}=\Delta\nu/R\alpha^2a_0^3(l+\frac{1}{2})$. To find the corresponding expression for W_F when $j=l+\frac{1}{2}$, (253) should be multiplied by $(2l+3)/(2l-1)$.

From (253) it is found that for $j=\frac{1}{2}$, W_F vanishes. There is, therefore, no quadrupole effect for s and $p_{1/2}$ electrons. Other states will be expected to show effects which are roughly proportional to the fine structure doublet separation so that large effects will be expected for low $p_{3/2}$ and d electrons for the heavy elements.

For the case of several electrons the situation is somewhat more complicated. In the case of Eu, Casimir found that the observed deviations could be accounted for by quadrupole moments $\overline{R^2}=5.1,\ 2.4\cdot10^{-24}$ cm^2 for Eu^{151} and Eu^{153}, respectively. For Bi, Bethe found a quadrupole moment $\overline{R^2}=0.61\cdot10^{-24}$ cm^2. This moment for Bi is about the size which would be expected if a single proton were unsymmetrically distributed in the nucleus.

References

The following list of references is not intended to be a complete list It does contain those papers referred to explicitly in the text. Certain papers from which experimental facts have been used are not referred to here but will be given in Part B where the results of experiments on nuclear disintegrations will be presented.

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