

Reference frames, superselection rules, and quantum information

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(Published 5 April 2007)

Recently, there has been much interest in a new kind of “unspeakable” quantum information that stands to regular quantum information in the same way that a direction in space or a moment in time stands to a classical bit string: the former can only be encoded using particular degrees of freedom while the latter are indifferent to the physical nature of the information carriers. The problem of correlating distant reference frames, of which aligning Cartesian axes and synchronizing clocks are important instances, is an example of a task that requires the exchange of unspeakable information and for which it is interesting to determine the fundamental quantum limit of efficiency. There have also been many investigations into the information theory that is appropriate for parties that lack reference frames or that lack correlation between their reference frames, restrictions that result in global and local superselection rules. In the presence of these, quantum unspeakable information becomes a new kind of resource that can be manipulated, depleted, quantified, etc. Methods have also been developed to contend with these restrictions using relational encodings, particularly in the context of computation, cryptography, communication, and the manipulation of entanglement. This paper reviews the role of reference frames and superselection rules in the theory of quantum-information processing.

DOI: [10.1103/RevModPhys.79.555](https://doi.org/10.1103/RevModPhys.79.555)

PACS number(s): 03.67.–a, 11.30.–j, 03.65.Ta, 03.65.Ud

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I. INTRODUCTION: WHY CONSIDER REFERENCE FRAMES IN QUANTUM INFORMATION?

Classical information theory is typically concerned with *fungible information*, that is, information for which the means of encoding is not important. Shannon's coding theorems, for instance, are indifferent to whether the two values 0 and 1 of a classical bit correspond to two values of magnetization on a tape, two voltages on a transmission line, or two positions of a bead on an abacus. Most information-processing tasks of interest to computer scientists and information theorists are of this sort, whether they be communication tasks such as data compression, cryptographic tasks such as key distribution, or computational tasks such as factoring. Nonetheless, there are many tasks that cannot be achieved with fungible information but that are also aptly described as information processing tasks. Examples include the synchronization of distant clocks, the alignment of distant Cartesian frames, and the determination of one's global position. Imagine, for instance, that Alice and Bob are

in separate spaceships with no shared Cartesian frame (in particular, no access to the fixed stars). There is clearly no way for Alice to describe a direction in space to Bob abstractly, that is, using nothing more than a string of classical bits. Rather, she must send to Bob a system that can point in some direction, a token of one of the axes of her own Cartesian frame. This token cannot be spherically symmetric; it must have a degree of freedom that can encode directional information. On the other hand, if she wishes to synchronize her clock with Bob's by sending him a token system, she will need to make use of a system that has a natural oscillation. The information that is communicated in these sorts of tasks is said to be *nonfungible*. These two sorts of information, fungible and nonfungible, have also been referred to as *speakeable* and *unspeakeable* (Peres and Scudo, 2002b).

The relatively young field of quantum-information theory has been primarily concerned with developing a quantum theory of speakable information. Investigators have sought to determine the degree of success with which various abstract information-processing tasks can be achieved assuming that the systems used to implement these tasks obey the laws of quantum theory. Nonetheless, there has also been progress in developing a quantum theory of unspeakeable information, outlining, for instance, the success with which tasks such as clock synchronization and Cartesian frame alignment can be achieved in a quantum world.

That one must look to physics to answer questions of interest to computer scientists is a fact that has not always been obvious. [Landauer (1993) summarized this point in the slogan "Information is physical."] That one must look to physics to answer questions about the processing of unspeakeable information, on the other hand, comes as no surprise. Nonetheless, the quantum theory of unspeakeable information is only just beginning to be explored.

It is critical to note that when one has a system encoding directional information, such as a spin-1/2 particle in a pure state, the direction is not defined with respect to any purported absolute Newtonian space, but rather with respect to another system, for instance, a set of gyroscopes in the laboratory. Similarly, a system that contains phase information, such as a two-level atom in a coherent superposition of ground and excited states, is not defined relative to any purported absolute time, but rather relative to a clock. We refer to the systems with respect to which unspeakeable/nonfungible information is defined, clocks, gyroscopes, meter sticks, and so forth, as reference frames. The tasks we have highlighted thus far can all be described as the alignment of reference frames. Nonfungible information is nonfungible precisely because it can only be defined with respect to a particular type of reference frame.

Even a quantum-information theorist who is uninterested in tasks such as clock synchronization and Cartesian frame alignment must necessarily consider physical systems which make use of reference frames. The reason is that although fungible information can be encoded

into *any* degree of freedom, and thus defined with respect to any reference frame, it is still the case that *some* degree of freedom must be chosen, and consequently some reference frame is required. For instance, if a two-level atomic qubit is being used for some task, one still requires a clock in the background in order to implement arbitrary preparations and measurements on this qubit even if the task is to perform abstract quantum-information processing rather than as a means of distributing phase information. In this example, one can change the relative phase between the ground and excited states of a two-level atom by a specified amount by turning on a static electric field for a specific time interval, but this requires a suitably precise clock as well as alignment of the field with the atomic dipole moment.

It follows that to lack a reference frame for a particular degree of freedom has an impact on the success with which one can perform certain quantum-information processing tasks. On several occasions there has been considerable controversy over the performance of certain tasks because this impact was ignored or not treated properly. As we will see, the lack of a reference frame can be treated within the quantum formalism as a form of *decoherence*—quantum noise. As opposed to the typical source of decoherence, which is due to correlation with an environment to which one does not have access, this decoherence can be viewed as resulting from correlation with a (possibly hypothetical) reference frame to which one does not have access. This is a powerful result, because if the lack of a reference frame can be viewed as a form of decoherence, the now-standard techniques of combating decoherence in quantum-information theory (in particular, the use of decoherence-free subsystems) can be applied.

As it turns out, the restriction of lacking a reference frame is mathematically equivalent to that of so-called *superselection rules*—postulated rules forbidding the preparation of quantum states that exhibit coherence between eigenstates of certain observables. Originally, superselection rules were introduced to enforce additional constraints to quantum theory beyond the well-studied constraints of selection rules (conservation laws). They were considered to be axiomatic restrictions, applying to only certain degrees of freedom. For instance, a superselection rule for electric charge asserts the impossibility of preparing a coherent superposition of different charge eigenstates. As we shall see, however, for superselection rules associated with compact symmetry groups, the presence of appropriate reference frames can actually allow for the preparation of such superposition states, thereby obviating the superselection rules in practice. This shows that there is an intimate connection between the restriction of lacking a reference frame and that of a superselection rule.

As [Schumacher \(2003\)](#) has emphasized, interesting restrictions on experimental operations yield interesting information theories. For instance, the fact that classical channels and local operations are a cheap resource compared to quantum channels leads us to study what can be achieved with local operations and classical commu-

nication (LOCC). The resulting information theory is the theory of entanglement. As another example, the relative ease with which one can implement Gaussian operations in quantum optics leads one to consider the information theory that results from the restriction to these operations. By comparing and contrasting the information theories that result from various different restrictions we are led to a much broader perspective on all of them. In particular, analogies between the resulting theories allow us to apply the insights gained in the context of one to solve problems arising in the context of another. In this sense, studying the restriction of a superselection rule—or equivalently, as we shall demonstrate, the restriction of lacking a reference frame—may yield lessons for the rest of quantum-information theory.

In some cases, it is difficult to imagine lacking a reference frame. For example, Cartesian frames with precision on the order of fractions of a degree and clocks with precision on the order of fractions of a second are sufficiently ubiquitous that their presence typically does not even warrant mention. However, these same reference frames become quite difficult to prepare and maintain if one requires very high precision or very good stability. Furthermore, there are certain kinds of reference frames that are difficult to prepare even if one requires only low precision and poor stability. For instance, a Bose-Einstein condensate of alkali atoms can act as a reference frame for the phase that is conjugate to atom number, and the reliable preparation of these has only been achieved in the past decade. In addition, it is straightforward to imagine two parties with reference frames that are uncorrelated (such as the example of the space-faring Alice and Bob provided earlier). In this case we say that they lack a shared reference frame. All of these facts demonstrate that reference frames must be considered as resources.

Regardless of the degree of freedom in question, a reference frame is always associated with some physical system. As such, it may be treated within the formalism of quantum mechanics. In this case, we speak of quantum reference frames. Indeed, one can imagine an extreme case wherein the only system in a party's possession that plays the role of a reference frame (or plays the role of a shared reference frame with another party) is of bounded size. For instance, one can imagine a quantum clock consisting of an oscillator with a small maximum number of excitations, or a quantum gyroscope consisting of a handful of spin-1/2 systems. It is then natural to ask how well such a bounded-size reference frame approximates one that is of unbounded size.

The ability of a bounded reference frame to stand in for an unbounded reference frame is analogous to the ability of an entangled state to stand in for the possibility of implementing nonlocal operations. Recall that the teleportation protocol permits entanglement and classical communication to substitute for a nonlocal operation. More generally, when one lacks the ability to perform nonlocal operations (such as when qubits are remotely separated), entanglement becomes a quantifiable resource. Similarly, when one is subject to a super-

selection rule (i.e., when one lacks a reference frame for some degree of freedom) bounded reference frames become a quantifiable resource about which we can ask the same sorts of questions as we do for entanglement. For instance, we may ask the following questions: Which states are interconvertible? How many states of a standard form can be distilled from a given state and how many are required to form a given state? How much of the resource is required for a given task? How quickly is it used up? etc.

Finally, because it is all too easy to forget about the presence of reference frames, these are at the root of various conceptual confusions. These include the interpretation of quantum states exhibiting coherence between number states in a single mode (a subject of controversy in quantum optics, Bose-Einstein condensation, and superconductivity); the quantification of entanglement in systems of bosons or fermions, or in situations when operations are restricted; the efficiency with which frames may be aligned, clocks synchronized, etc.; and the significance of superselection rules on the possibility of implementing various quantum-information processing tasks.

In this paper, we provide a review of the recent investigations into these and related issues. In Sec. II, we introduce the formalism for treating the lack of a general reference frame in quantum theory, and show how this is equivalent to a superselection rule. Section III considers quantum-information processing without a shared reference frame. Section IV considers how to treat reference frames within the quantum formalism, which provides the starting point for a theory of distributing quantum reference frames—the topic of Sec. V. The effect of bounding the size of reference frames for quantum-information processing is considered in Sec. VI. Finally, we provide an outlook to the future of this field in Sec. VII.

II. FORMALIZING REFERENCE FRAMES AND SUPERSELECTION RULES

A. Reference frames in quantum theory

Reference frames (RFs) are implicit in the definition of quantum states. For example, in the position representation of the wave function of a quantum particle, $\psi(x)$, x parametrizes the position of the particle relative to a spatial reference frame. More generally, the quantum state of a system is a description of the system relative to a suitable reference frame.

Consider a quantum system with Hilbert space \mathcal{H} , prepared in a state $|\psi_0\rangle$ relative to a reference frame. We can now consider a transformation that changes this relation. Such a transformation can be *active*, changing the system such that it subsequently holds a different relation to the reference frame, or *passive*, in which case the system is unchanged but is now described relative to a new reference frame. In both situations, the transformation can be represented by a unitary operator $T(g)$, where g denotes the transformation; the transformed

system is then described by the state $T(g)|\psi_0\rangle$. Note that these operations can be composed, so that $T(g'g) \equiv T(g')T(g)$ is a transformation if both $T(g')$ and $T(g)$ are, and this composition is associative [i.e., $T(g''g')T(g) = T(g'')T(g'g)$]. Also, there exists an inverse transformation $T(g^{-1})$ to every transformation $T(g)$, such that $T(g^{-1})T(g) = I$, the identity. If this inverse is unique,¹ the set of all transformations form a group G . We use $g \in G$ to denote an abstract transformation within the group, and say that T is the unitary representation of this group on the quantum system (or, equivalently, on the Hilbert space \mathcal{H}).

In this review, we will often use two common examples of a reference frame to illustrate the concepts and ideas we cover. The first example is a *phase reference*, for which the relevant group of transformations is $U(1)$, the group of real numbers modulo 2π under addition. A representation of $U(1)$ on a quantum system determines how that system transforms under phase shifts. The second example that we use extensively in this review is a Cartesian frame specifying three orthogonal spatial directions; the group of transformations of orientation relative to a Cartesian frame is the group of rotations $SO(3)$. An element $\Omega \in SO(3)$ can be given, say, by a set of three Euler angles. The representation of $SO(3)$ on a quantum system then determines how that system transforms under rotations; for example, a spin- j particle transforms according to the unitary representation R_j (a Wigner rotation matrix). We will often extend the group of rotations $SO(3)$ to the group $SU(2)$ to allow for spinor representations.

Because group theory provides a powerful mathematical tool for analyzing the role of reference frames in quantum systems, we make frequent use of group-theoretic techniques throughout this review. We present a short introduction to the relevant techniques in this section, but the reader may consult a standard group theory text, such as [Fulton and Harris \(1991\)](#) or [Sternberg \(1994\)](#), for further details. Also, for an introduction to the standard mathematical techniques of quantum information, we direct the reader to [Nielsen and Chuang \(2000\)](#).

We begin by exploring an illustrative example.

B. Lacking a phase reference implies a photon-number superselection rule

In this section, we investigate an explicit example of a reference frame—a phase reference—and demonstrate that if one lacks a phase reference then the resulting quantum theory is equivalent to one in which there is a superselection rule for photon number.

¹If the inverse is not unique, then the RF is instead associated with a coset space. This situation occurs when the RF itself is invariant under some transformations. We consider an example of such an RF—a direction (as opposed to a full Cartesian frame)—in Sec. V.

In quantum optical experiments, states of an optical mode are always referred to some phase reference. Consider K optical modes as described by some party, Alice, relative to a phase reference in her possession—for example, a high intensity laser. Let $|n_1, \dots, n_K\rangle$ be the Fock state basis for the Hilbert space $\mathcal{H}^{(K)}$ describing these modes, with n_i the number of photons in the mode i , and \hat{N}_i the number operator for this mode.

Consider another party, Charlie, who has a different phase reference. Let ϕ be the angle that relates Charlie's phase reference to Alice's. Alice can perform an active transformation on her system of optical modes by allowing them to evolve under a Hamiltonian proportional to $\hat{N}_{\text{tot}} \equiv \sum_{i=1}^K \hat{N}_i$, the total photon-number operator. Specifically, the unitary transformation $U(\phi) = \exp(i\phi\hat{N}_{\text{tot}})$ will actively advance her system by an angle ϕ . Using the equivalence between the representations for active and passive transformations, we thus conclude that states prepared by Alice are represented by Charlie relative to his phase reference by performing a passive transformation of ϕ , using the representation U of $U(1)$ on K modes given by $U(\phi) = \exp(i\phi\hat{N}_{\text{tot}})$. If $|\psi\rangle$ is the state relative to Alice's phase reference, then this same state relative to Charlie's phase reference is given by the transformed state

$$U(\phi)|\psi\rangle = e^{i\phi\hat{N}_{\text{tot}}}|\psi\rangle. \quad (2.1)$$

For example, let Alice prepare the single-mode coherent state

$$|\alpha\rangle \equiv \sum_{n=0}^{\infty} c_n |n\rangle, \quad c_n \equiv e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}, \quad (2.2)$$

with $\alpha \in \mathbb{C}$; this state has a phase $\arg(\alpha)$ relative to Alice's phase reference. Charlie would describe this same state relative to his phase reference by a coherent state with the same amplitude but with phase $\arg(\alpha) + \phi$. This passive transformation agrees with that of Eq. (2.1) because

$$|e^{i\phi}\alpha\rangle = e^{i\phi\hat{N}}|\alpha\rangle, \quad (2.3)$$

where \hat{N} is the number operator on this single mode.

As another example, let Alice prepare the two-mode state $(|01\rangle + |10\rangle)/\sqrt{2}$. Because this state is an eigenstate of \hat{N}_{tot} , the transformation $U(\phi)$ induces only an unobservable overall phase when acting on this state. Thus Charlie also represents the state of the system as $(|01\rangle + |10\rangle)/\sqrt{2}$ relative to his phase reference. This two-mode state is an example of an invariant state; it is defined independently of any phase reference.

It will be useful for us to decompose the Hilbert space $\mathcal{H}^{(K)}$ of K modes into subspaces that transform in a simple way under the action of the group $U(1)$ as follows. Defining \mathcal{H}_n to be the Hilbert space consisting of states of K modes with precisely n total photons, i.e., eigenspaces of \hat{N}_{tot} with eigenvalue n , we can express the Hilbert space $\mathcal{H}^{(K)}$ as a direct sum

$$\mathcal{H}^{(K)} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \quad (2.4)$$

Any state $|\psi_n\rangle \in \mathcal{H}_n$ transforms under phase shifts, i.e., under the representation U of $U(1)$, as

$$U(\phi)|\psi_n\rangle = e^{in\phi}|\psi_n\rangle, \quad |\psi_n\rangle \in \mathcal{H}_n. \quad (2.5)$$

Define Π_n to be the projector onto \mathcal{H}_n . Then an arbitrary state $|\psi\rangle \in \mathcal{H}^{(K)}$ transforms as

$$U(\phi)|\psi\rangle = \sum_n e^{in\phi} \Pi_n |\psi\rangle. \quad (2.6)$$

Now consider the situation where Charlie has no knowledge of the angle ϕ that relates his phase references to Alice's, i.e., the laser serving as his phase reference is not phase locked to hers.² Let Alice prepare a quantum state $|\psi\rangle$ of K modes relative to her phase reference. Given that ϕ is completely unknown, one must average over its possible values to obtain the state relative to Charlie. This averaging yields the mixed state

$$\mathcal{U}[|\psi\rangle\langle\psi|] \equiv \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi)|\psi\rangle\langle\psi|U(\phi)^\dagger. \quad (2.7)$$

Using Eq. (2.6) yields

$$\begin{aligned} \mathcal{U}[|\psi\rangle\langle\psi|] &= \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{n,n'} e^{in\phi} \Pi_n |\psi\rangle\langle\psi| \Pi_{n'} e^{-in'\phi} \\ &= \sum_{n,n'} \Pi_n |\psi\rangle\langle\psi| \Pi_{n'} \left(\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n-n')\phi} \right) \\ &= \sum_n \Pi_n |\psi\rangle\langle\psi| \Pi_n. \end{aligned} \quad (2.8)$$

Because this result applies to any state $|\psi\rangle$, we can express the action of \mathcal{U} on an arbitrary density operator ρ as

$$\mathcal{U}[\rho] = \sum_n \Pi_n \rho \Pi_n. \quad (2.9)$$

The map \mathcal{U} removes all coherence between states of differing total photon number on Alice's systems. It follows in particular that $\mathcal{U}[\rho]$ is invariant under phase shifts,

$$[\mathcal{U}[\rho], U(\phi)] = 0, \quad \forall \phi. \quad (2.10)$$

Thus if states are described relative to Charlie's phase reference, Alice faces a restriction in what she can pre-

²It should be noted that if the phase between Alice and Charlie's references is changing in time in a known manner, then the transformation relating their descriptions is still of the form of Eq. (2.1) but with ϕ a function of time. Given that this function is known, the parties can compensate for this effect. However, a lack of knowledge of how this phase is changing, for instance, an unknown drift, can eventually lead to Alice and Charlie having no information about the relative phase between their references. In such a situation, the time scale of the drift relative to the operations they perform is critical; a slow drift may have negligible effect on a quick protocol.

pare. This restriction is characterized by the quantum operation \mathcal{U} , which ensures that Charlie will describe any state prepared by Alice as block diagonal in total photon number, or equivalently as invariant under phase shifts. We note in particular that the only pure states that Alice can prepare are those which lie entirely within a single eigenspace \mathcal{H}_n .

Now consider the following related question for operations: If a unitary operation V is performed by Alice relative to her phase reference, how is this operation described by Charlie relative to his phase reference? Let σ be the state of the system relative to Charlie's phase reference. To describe the action of V on this state if the angle ϕ that relates Charlie's phase reference to Alice's is known, Charlie could transform this state into Alice's frame, then apply the unitary V , then transform back to his frame; the resulting state is

$$U(\phi)VU(\phi)^\dagger\sigma U(\phi)V^\dagger U(\phi)^\dagger \quad (2.11)$$

relative to Charlie. Thus the operation is described by Charlie by the unitary $V_\phi = U(\phi)VU(\phi)^\dagger$. If the phase ϕ is unknown, then Charlie would instead describe the operation by an incoherent mixture of unitaries of this form, i.e., by the map

$$\tilde{\mathcal{V}}[\sigma] \equiv \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi)VU(\phi)^\dagger\sigma U(\phi)V^\dagger U(\phi)^\dagger. \quad (2.12)$$

A notable special case is if the system was prepared by Alice, so that the state σ relative to Charlie's RF is of the form $\sigma = \mathcal{U}[\rho]$, as in Eq. (2.9). In this case,

$$\tilde{\mathcal{V}}[\sigma] = \mathcal{U}[V\sigma V^\dagger], \quad (2.13)$$

so that $\tilde{\mathcal{V}}[\sigma]$ is also block diagonal in total photon number. Thus if operations are described relative to Charlie's phase reference, then Alice experiences a restriction on what operations she can perform.

We note that a restriction that requires states to be block diagonal in the eigenspaces of some operator is common in quantum theory: it is formally equivalent to a superselection rule (SSR) (Giulini, 1996). Many superselection rules in nonrelativistic quantum theory, such as the superselection rule for charge (Wick *et al.*, 1952) are characterized by an inability to prepare states with coherence between eigenspaces of some charge operator corresponding to different eigenvalues. Thus we can refer to the restriction described above as a superselection rule for photon number (Sanders *et al.*, 2003). Alice cannot prepare, say, a coherent state $|\alpha\rangle$ relative to Charlie's phase reference, but she can prepare a phase-invariant state such as $(|01\rangle + |10\rangle)/\sqrt{2}$. In addition, she cannot perform the unitary displacement operation that takes the vacuum $|0\rangle$ to a coherent state $|\alpha\rangle$, but she can perform any unitary operation on the two-dimensional subspace spanned by $|01\rangle$ and $|10\rangle$.

We note that in the present context the SSR only restricts preparations and operations by Alice (or any party who does not share Charlie's phase reference). The SSR does not forbid states with coherence between different total photon-number eigenstates from existing within the theory, and, in particular, Charlie (or any party who shares Charlie's phase reference) experiences no such restriction on what states he can prepare. Thus it makes sense within this context to consider what manipulations Alice can perform under the restriction of an SSR on general (possibly coherent) states. For example, Alice can perform the relative phase shift which takes the state $(|0\rangle + |1\rangle)/\sqrt{2}$ to $(|0\rangle - |1\rangle)/\sqrt{2}$. Also, we note that Alice is able to (incoherently) change the total photon number, i.e., she can perform an operation that maps the vacuum $|0\rangle$ to the single-photon state $|1\rangle$. Thus this restriction is not equivalent to a conservation law for photon number.

C. A general framework for reference frames and superselection rules

In this section, we consider how to generalize the basic idea of the previous section—that lacking a reference frame leads to a superselection rule—beyond the case of a phase reference. We present some formal mathematical tools, in particular, tools from group theory and linear algebra, that we will use throughout this review paper.

Suppose two parties, Alice and Charlie, are considering a single quantum system described by a Hilbert space \mathcal{H} . Let this system transform via a group G relative to some reference frame. Throughout this review, we consider both finite groups and continuous (Lie) groups. For the latter, we will restrict our attention to Lie groups that (i) are compact, so that they possess a group-invariant (Haar) measure dg ; and (ii) act on \mathcal{H} via a unitary representation T , ensuring that they are completely reducible (Sternberg, 1994). Many of the techniques in this review can be applied to other groups with some modification, but there are many technical difficulties which are beyond its scope.

Let $g \in G$ be the group element relating Charlie's reference frame to Alice's, i.e., the element in G that describes the passive transformation from Alice's to Charlie's reference frame. Furthermore, suppose that g is completely unknown, i.e., that Alice's reference frame and Charlie's are uncorrelated. It follows that if Alice prepares a state ρ on \mathcal{H} relative to her frame, the state of the system is represented relative to Charlie's frame by the state³

³The invariant measure is chosen using the maximum entropy principle: because Charlie has no prior knowledge about Alice's reference frame, he should assume a uniform measure over all possibilities.

$$\tilde{\rho} = \int_G dg T(g) \rho T^\dagger(g) \equiv \mathcal{G}[\rho], \quad (2.14)$$

with $T(g)$ a unitary representation of g on \mathcal{H} , and dg the group-invariant (Haar) measure.⁴ We call the operation \mathcal{G} the “ G -twirling” operation. If we choose to always represent preparations by Alice relative to the reference frame of Charlie, then all states are of the form $\tilde{\rho} = \mathcal{G}[\rho]$.

Any $\tilde{\rho}$ of this form satisfies

$$[\tilde{\rho}, T(g)] = 0, \quad \forall g \in G \quad (2.15)$$

and thus is said to be G -invariant. The proof follows from the fact that $T(g)\tilde{\rho}T^\dagger(g) = \int_G dg' T(gg')\rho T^\dagger(gg') = \tilde{\rho}$.

Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded operators on \mathcal{H} . Given that $\mathcal{B}(\mathcal{H})$ forms a Hilbert space under the Hilbert-Schmidt inner product $(\sigma, \tau) = \text{Tr}(\sigma^\dagger \tau)$, linear maps can be regarded as operators acting on $\mathcal{B}(\mathcal{H})$. These are called *superoperators* to distinguish them from operators acting on \mathcal{H} . It is useful to define the superoperator $\mathcal{T}(g)$ by $\mathcal{T}(g)[\rho] = T(g)\rho T^\dagger(g)$, which is the unitary representation of G on $\mathcal{B}(\mathcal{H})$. We may then express \mathcal{G} simply as $\mathcal{G} = \int_G dg \mathcal{T}(g)$.

We now consider the representation of transformations. The most general transformation upon a quantum system, i.e., the most general quantum operation, is represented by a completely positivity-preserving superoperator $\mathcal{E}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. [See [Nielsen and Chuang \(2000\)](#) for the definition and properties of these superoperators.] The question of interest to us is the following: If an operation is represented by the superoperator \mathcal{E} relative to Alice’s frame, how is this same operation represented relative to Charlie’s frame? Generalizing the justification given for Eq. (2.12) in the case of a phase reference, we conclude that relative to Charlie’s frame the operation is represented by the superoperator $\tilde{\mathcal{E}}$, where

$$\tilde{\mathcal{E}}[\rho] = \int_G dg T(g) \mathcal{E}[T^\dagger(g)\rho T(g)] T^\dagger(g), \quad (2.16)$$

or, equivalently,

$$\tilde{\mathcal{E}} = \int_G dg \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}(g^{-1}), \quad (2.17)$$

where $\mathcal{A} \circ \mathcal{B}[\rho] = \mathcal{A}[\mathcal{B}[\rho]]$. Given that $\mathcal{T}(g)$ is a representation of G on $\mathcal{B}(\mathcal{H})$, Eq. (2.17) has the form of Eq. (2.14) except with operators replaced by superoperators. We therefore refer to the map taking \mathcal{E} to $\tilde{\mathcal{E}}$ as super- G -twirling. Any superoperator of the form of $\tilde{\mathcal{E}}$ satisfies

⁴If the group G is instead a finite group, this expression is $\mathcal{G}_{\text{finite}}[\rho] \equiv |G|^{-1} \sum_{g \in G} T(g)\rho T^\dagger(g)$. In the following, we use the Lie group notation exclusively; however, all results apply equally well to finite groups.

$$[\tilde{\mathcal{E}}, \mathcal{T}(g)] = 0, \quad \forall g \in G, \quad (2.18)$$

where $[\mathcal{A}, \mathcal{B}] = \mathcal{A} \circ \mathcal{B} - \mathcal{B} \circ \mathcal{A}$ is the superoperator commutator. Thus $\tilde{\mathcal{E}}$ is invariant under the action of G ; it is a G -invariant operation.

The superoperator $\tilde{\mathcal{E}}$ acts on a G -invariant operator \tilde{A} as

$$\begin{aligned} \tilde{\mathcal{E}}[\tilde{A}] &= \int_G dg \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}(g^{-1})[\mathcal{G}[\tilde{A}]] \\ &= \int_G dg \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{G}[\tilde{A}] = \mathcal{G} \circ \mathcal{E} \circ \mathcal{G}[\tilde{A}], \end{aligned} \quad (2.19)$$

where we have used the fact that $\tilde{A} = \mathcal{G}[\tilde{A}]$ and $\mathcal{T}(g^{-1}) \circ \mathcal{G} = \mathcal{G}$.

Every completely positivity-preserving superoperator admits an operator-sum decomposition of the form $\mathcal{E}[\rho] = \sum_k A_k \rho A_k^\dagger$ where the A_k are called Kraus operators. Clearly, a sufficient condition for an operation \mathcal{E} to be a G -invariant operation is for all of its Kraus operators A_k to be G -invariant operators. In general, however, this is not a necessary condition. Note, in particular, that if V is a unitary operator that is G -invariant, then $\mathcal{V}[\cdot] = V(\cdot)V^\dagger$ is a G -invariant superoperator and the associated unitary transformation can be implemented without an RF. Nonetheless, there may exist G -invariant superoperators arising from G -noninvariant unitary operators.

Finally, we consider the representation of measurements. The most general measurement on a quantum system is represented by a set of completely positivity-preserving superoperators $\{\mathcal{E}_k\}$, the sum of which is trace-preserving. The probability of outcome k for the measurement is $p_k = \text{Tr}(\mathcal{E}_k[\rho])$ and upon obtaining this outcome, ρ is updated to $\mathcal{E}_k[\rho]/p_k$. The probability of outcome k may also be specified by $p_k = \text{Tr}(E_k \rho)$ where the set $\{E_k\}$ is a positive operator valued measure (POVM) (defined by the conditions $E_k \geq 0$ and $\sum_k E_k = I$). The POVM $\{E_k\}$ that is associated with a measurement is obtained from the set of superoperators $\{\mathcal{E}_k\}$ associated with it by $E_k = \mathcal{E}_k^\dagger[I]$, where the adjoint of a superoperator is defined relative to the Hilbert-Schmidt inner product on the operator space, $\text{Tr}(\mathcal{E}^\dagger(\sigma)\tau) = \text{Tr}(\sigma\mathcal{E}(\tau))$.

Recalling how operations transform under a change of reference frame, if a measurement is represented by the set of superoperators $\{\mathcal{E}_k\}$ relative to Alice’s frame, then it is represented by the set of superoperators $\{\tilde{\mathcal{E}}_k\}$ relative to Charlie’s, where $\tilde{\mathcal{E}}_k$ is given by Eq. (2.17). Taking the superoperator adjoint of Eq. (2.17), and using the fact that $E_k = \mathcal{E}_k^\dagger[I]$, it follows that the POVM $\{E_k\}$ relative to Alice’s frame is represented by the POVM $\{\tilde{E}_k\}$ relative to Charlie’s frame where

$$\tilde{E}_k = \mathcal{G}[E_k]. \quad (2.20)$$

It follows that

$$[\tilde{E}_k, T(g)] = 0, \quad \forall g \in G, \quad (2.21)$$

that is, the POVM $\{\tilde{E}_k\}$ is G -invariant.

Thus relative to Charlie's reference frame, the preparations, operations, and measurements that Alice can implement are represented by states, superoperators, and POVMs of the form of Eqs. (2.14), (2.17), and (2.20), respectively. We now demonstrate that this restriction has the same mathematical characterization as that of a superselection rule for a (possibly non-Abelian) group G .

First, we note that the representation T of the group G allows for a decomposition of the Hilbert space into charge sectors \mathcal{H}_q , labeled by an index q , as

$$\mathcal{H} = \bigoplus_q \mathcal{H}_q, \quad (2.22)$$

where each charge sector carries an inequivalent representation T_q of G . In the U(1) phase reference example presented above, the charge sectors corresponded to eigenspaces of total photon number. Each sector can be further decomposed into a tensor product,

$$\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q, \quad (2.23)$$

of a subsystem \mathcal{M}_q carrying an irreducible representation (irrep) T_q of G and a subsystem \mathcal{N}_q carrying a trivial representation of G . (Recall that a representation acts irreducibly on a space if there are no invariant subspaces.) Note that this tensor product does not correspond to the standard tensor product obtained by combining multiple qubits: it is virtual (Zanardi, 2001). The spaces \mathcal{M}_q and \mathcal{N}_q are therefore virtual subsystems. The \mathcal{M}_q and \mathcal{N}_q are sometimes referred to as gauge spaces and multiplicity spaces, respectively.⁵ For the U(1) phase reference example, the subsystems \mathcal{M}_q are one dimensional, and so the additional tensor product structure within the irreps is not required; for a general superselection rule corresponding to a non-Abelian group G , however, they can be nontrivial.

Expressed in terms of this decomposition of the Hilbert space, the map \mathcal{G} takes a particularly simple form. Because of the broad utility of this form, we present it as a theorem.

Theorem. The action of \mathcal{G} in terms of the decomposition

$$\mathcal{H} = \bigoplus_q \mathcal{M}_q \otimes \mathcal{N}_q, \quad (2.24)$$

is given by

$$\mathcal{G} = \sum_q (\mathcal{D}_{\mathcal{M}_q} \otimes \mathcal{I}_{\mathcal{N}_q}) \circ \mathcal{P}_q, \quad (2.25)$$

where \mathcal{P}_q is the superoperator associated with projection into the charge sector q , that is, $\mathcal{P}_q[\rho] = \Pi_q \rho \Pi_q$ with Π_q the projection onto $\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q$, $\mathcal{D}_{\mathcal{M}}$ denotes the trace-preserving operation that takes every operator on

the Hilbert space \mathcal{M} to a constant times the identity operator on that space, and $\mathcal{I}_{\mathcal{N}}$ denotes the identity map over operators in the space \mathcal{N} .

We provide a short proof of this theorem at the end of this section.

Note that the operation \mathcal{G} has the general form of decoherence. Whereas decoherence typically describes correlation with an environment to which one does not have access, in this case the decoherence describes correlation to a reference frame to which one does not have access. Given that \mathcal{G} acts as identity on subsystems \mathcal{N}_q , these subsystems are called decoherence-free subsystems (also known as noiseless subsystems) (Zanardi and Rasetti, 1997; Knill et al., 2000). In stark contrast, \mathcal{G} acts as the completely depolarizing operation on the subsystems \mathcal{M}_q ; these are called decoherence-full subsystems (Bartlett et al., 2004a).

It follows, in particular, that a G -invariant operator $\tilde{A} = \mathcal{G}(A)$ must have the form

$$\tilde{A} = \bigoplus_q I_{\mathcal{M}_q} \otimes A_{\mathcal{N}_q}, \quad (2.26)$$

where $I_{\mathcal{M}_q}$ are identity operators on the subsystems \mathcal{M}_q and $A_{\mathcal{N}_q}$ are arbitrary operators on the subsystems \mathcal{N}_q .

We are now in a position to see how the restriction of lacking a reference frame for the group G is equivalent to the standard notion of a superselection rule associated with this group. Superselection rules are most commonly discussed in the context of Abelian groups where they can be described simply as a restriction of the physical states and observables to those that are block diagonal with respect to the inequivalent representations of G (Giulini, 1996). (Occasionally, this restriction is argued to hold for the observables alone, but in this case every state that is not restricted in this way is operationally indistinguishable from a state that is, so one may as well assume this restriction for the states also.) The standard notion of a superselection rule for an arbitrary (possibly non-Abelian) group G is a restriction of the physical states and observables to those that commute with every element of G (Giulini, 1996). This restriction on states is precisely what is asserted in Eq. (2.15), and the restriction on observables is simply Eq. (2.21) applied to the special case of a projective measurement. The restriction on transformations has traditionally only been articulated for unitary transformations and asserts that only G -invariant Hamiltonians are physical. This is equivalent to asserting that the unitary itself be G invariant, and such unitaries were identified above as the only ones that can be achieved when lacking an RF for the group G . Equation (2.18) is a generalization of this restriction to irreversible transformations. Thus one can view the restrictions of Eqs. (2.15), (2.18), and (2.21) as formalizing the restrictions of a superselection rule associated with the group G in the language of quantum-information theory. We shall say that the restriction due to the lack of a reference frame for G is equivalent to a superselection rule associated with the group G .

⁵In high-energy physics, \mathcal{M}_q are called color spaces and \mathcal{N}_q are called flavor spaces.

We note that although the term “superselection rule” was initially introduced to describe an *axiomatic* restriction on quantum states, observables, and operations (Wick *et al.*, 1952), it has been emphasized by Aharonov and Susskind (1997) that whether or not coherent superpositions of a particular observable are possible is a practical matter, depending on the availability of a suitable reference system. We return to this issue in Sec. IV.

Finally, although we have thus far mentioned only the two limiting possibilities for the correlations that might hold between Alice and Charlie’s reference frames—completely correlated or completely uncorrelated—in general one might consider the intermediate scenario wherein they are partially correlated. To model this, one replaces the uniform Haar measure appearing in Eq. (2.14) with the nonuniform measure that characterizes Charlie’s partial knowledge of the group element g in order to obtain a weighted G -twirling operation. Like G -twirling, this operation is noiseless on the multiplicity spaces, but unlike G -twirling, which is completely decohering on the gauge spaces, the weighted G -twirling operation is only partially decohering on these spaces.

Proof of Theorem 1. Our proof, which follows Nielsen (2003), makes use of two central theorems of group representation theory known as Schur’s lemmas. We state these lemmas here without proof.

Lemma (Schur’s first). If $T(g)$ is an irreducible representation of the group G on the Hilbert space \mathcal{H} , then any operator A satisfying $T(g)AT^\dagger(g)=A$ for all $g \in G$ is a multiple of the identity on \mathcal{H} .

Lemma (Schur’s second). If $T_1(g)$ and $T_2(g)$ are inequivalent representations of G , then $T_1(g)AT_2^\dagger(g)=A$ for all $g \in G$ implies $A=0$.

We begin by decomposing the representation $T(g)$ appearing in Eq. (2.14) into a sum of irreducible representations $T(g)=\oplus_{q,\lambda}T_{q,\lambda}(g)$, where q labels inequivalent irreps and λ is a multiplicity index. It follows that

$$\mathcal{G}[A]=\bigoplus_{q,q',\lambda,\lambda'}\int dgT_{q,\lambda}(g)AT_{q',\lambda'}^\dagger(g). \quad (2.27)$$

Define $A_{q,q',\lambda,\lambda'}=\int dgT_{q,\lambda}(g)AT_{q',\lambda'}^\dagger(g)$. Because of the invariance of the measure dg , it follows that

$$T_{q,\lambda}(g)A_{q,q',\lambda,\lambda'}T_{q',\lambda'}^\dagger(g)=A_{q,q',\lambda,\lambda'}, \quad \forall g \in G. \quad (2.28)$$

Thus, by Schur’s second lemma, $A_{q,q',\lambda,\lambda'}=0$ for $q \neq q'$. Equation (2.27) can then be expressed as

$$\mathcal{G}[A]=\bigoplus_{q,\lambda,\lambda'}\int dgT_{q,\lambda}(g)AT_{q,\lambda'}^\dagger(g). \quad (2.29)$$

Let $\Pi_{q,\lambda}$ be the projection of \mathcal{H} onto the carrier space of $T_{q,\lambda}$, and let $\Pi_q=\sum_\lambda\Pi_{q,\lambda}$. Then the above equation can be expressed as

$$\mathcal{G}[A]=\sum_{q,\lambda,\lambda'}\int dgT_{q,\lambda}(g)\Pi_qA\Pi_qT_{q,\lambda'}^\dagger(g), \quad (2.30)$$

and thus we can express \mathcal{G} as

$$\mathcal{G}=\sum_q\mathcal{G}_q\circ\mathcal{P}_q, \quad (2.31)$$

where $\mathcal{G}_q[A_q]=\sum_{\lambda,\lambda'}\int dgT_{q,\lambda}(g)A_qT_{q,\lambda'}^\dagger(g)$ is a superoperator on \mathcal{H}_q , and recall that $\mathcal{P}_q[A]=\Pi_qA\Pi_q$.

We now determine the form of \mathcal{G}_q in terms of the tensor product structure $\mathcal{H}_q=\mathcal{M}_q\otimes\mathcal{N}_q$. The projector $\Pi_{q,\lambda}$ can be expressed in terms of this tensor product as $\Pi_{q,\lambda}=\Pi_{\mathcal{M}_q}\otimes\Pi_\lambda$, where $\Pi_{\mathcal{M}_q}$ is the projector onto \mathcal{M}_q , and Π_λ is the rank-1 projector on \mathcal{N}_q that picks out the representation λ of G . The rank-1 projectors Π_λ form a basis for \mathcal{N}_q , so that $\sum_\lambda\Pi_\lambda$ is the identity on \mathcal{N}_q . Given that $T_q(g)$ acts nontrivially only on \mathcal{H}_q , we can write $T_{q,\lambda}(g)=T_q(g)\otimes\Pi_\lambda$. It follows that

$$\begin{aligned} \mathcal{G}[A] &= \sum_{q,\lambda,\lambda'}\int dg[T_q(g)\otimes\Pi_\lambda]\Pi_qA\Pi_q[T_q^\dagger(g)\otimes\Pi_{\lambda'}] \\ &= \sum_q\int dg\left(T_q(g)\otimes\sum_\lambda\Pi_\lambda\right)\Pi_qA\Pi_q\left(T_q^\dagger(g)\right. \\ &\quad \left.\otimes\sum_{\lambda'}\Pi_{\lambda'}\right) \\ &= \sum_q(\mathcal{G}_{\mathcal{M}_q}\otimes\mathcal{I}_{\mathcal{N}_q})\circ\mathcal{P}_q[A], \end{aligned} \quad (2.32)$$

where the superoperator $\mathcal{G}_{\mathcal{M}_q}$ takes an operator B on \mathcal{M}_q to $\mathcal{G}_{\mathcal{M}_q}[B]=\int dgT_q(g)BT_q^\dagger(g)$. By Schur’s first lemma, $\mathcal{G}_{\mathcal{M}_q}[B]$ is a multiple of identity on \mathcal{M}_q . Therefore because the map \mathcal{G} is trace preserving $\mathcal{G}_{\mathcal{M}_q}=\mathcal{D}_{\mathcal{M}_q}$, the trace-preserving map that takes every operator on \mathcal{M}_q to a constant times the identity on \mathcal{M}_q . ■

III. QUANTUM INFORMATION WITHOUT A SHARED REFERENCE FRAME

In implementing multipartite cryptographic and communication tasks using quantum systems, it is generally presumed, at least implicitly, that all parties share perfect reference frames for all relevant degrees of freedom. Moreover, one might think that in order to achieve some or all of these tasks, they *must* share such reference frames; for instance, one might think that if they wish to achieve quantum communication using the Fock space of an optical mode, they must share a phase reference, and if they wish to do so using spin-1/2 systems, they must share a reference frame for spatial orientation. This impression is mistaken; quantum-information processing tasks can be achieved without first establishing a shared reference frame by using entangled states of multiple systems, that is, relational encodings.

A classical analog is elucidating. If two parties do not share a Cartesian frame, then they cannot communicate any classical information to one another through encodings in the directional degree of freedom of a system.

For instance, if Alice encodes information into the orientation, relative to her frame, of a physical arrow or gyroscope, Bob cannot access this information because he can compare the system with his frame only. Nonetheless, they can still communicate by encoding information in the relative orientations of two or more such systems. We shall be concerned with the quantum analog of such relational encodings.

The essential idea is to use the result, presented in Sec. II, that the effect of lacking a shared RF can be expressed as a form of decoherence. We then make use of the techniques of decoherence-free subspaces and subsystems (Zanardi and Rasetti, 1997; Knill *et al.*, 2000; Kempe *et al.*, 2001) to find quantum states that are protected from the noise. These techniques (and variants thereof) can be interpreted as yielding relational encodings. They are in fact ideally suited to the problem of overcoming the lack of a shared RF because the existence of decoherence-free subspaces and subsystems relies on there being nontrivial symmetries in the noise, something that may not occur for a realistic noise model, but which is guaranteed to occur in the present context. For instance, in order to redescribe, relative to one RF, a qubit state that is defined relative to a second, uncorrelated RF, one must apply to it an unknown unitary. To redescribe, relative to this RF, many qubits that were all prepared relative to the same RF, one must apply precisely the same unitary to each.

We begin in Sec. III.A by applying these techniques and others to determine the efficiency with which classical and quantum communication can be performed in the presence of such noise. The implications for quantum key distribution are discussed in Sec. III.B. We also discuss the important issue of sharing entanglement between two parties who lack a shared RF; we demonstrate in Sec. III.C that a rich structure emerges in bipartite entanglement of pure states when this restriction applies. Finally, in Sec. III.D, we investigate the cryptographic power of private shared RFs, where it is assumed that it is an eavesdropper Eve who fails to have a sample of Alice and Bob's RF.

A. Communication without a shared reference frame

1. Communication using photons without a shared phase reference

Consider the following problem: Alice wants to communicate some amount of classical or quantum information to Bob using an optical channel, i.e., using quantum states of some number of optical modes, when they do not share a phase reference. Using the formalism of Sec. II.B, a state ρ prepared by Alice is represented by Bob as the (generally mixed) state $\mathcal{U}[\rho] = \sum_n \Pi_n \rho \Pi_n$. This problem thus takes the form of a more standard one from quantum communication: how to communicate quantum or classical information through a noisy channel described by a decoherence map \mathcal{U} .

The communication may be constrained in some additional way, such as by a limit on the number of usable

optical modes, or by an energy limit that bounds the maximum number of photons that can be transmitted, or both. Because of these constraints, Alice and Bob wish to use a communication protocol that makes optimal use of these resources.

Let us first consider classical communication. The simplest possible problem is the one wherein Alice is restricted to sending at most one photon to Bob, using a single optical mode. Clearly, using such a channel, Alice can communicate a single classical bit to Bob by sending either a single photon $|1\rangle$ or no photon (the vacuum) $|0\rangle$.⁶ This protocol does not rely on Alice and Bob sharing a phase reference, because both the states $|0\rangle$ and $|1\rangle$ are invariant under the superoperator \mathcal{U} . Generalizing this result, if Alice can send at most N photons in a single mode, she can communicate $N+1$ classical messages [equivalently, $\log_2(N+1)$ classical bits] to Bob. With $K > 1$ modes, one has to consider all possible ways of distributing N photons among K modes. The dimension of the resulting Hilbert space is $(N+K)!/N!K!$, and specifies the number of classical messages Alice can communicate using eigenstates of photon number.

What about quantum communication? Again, consider a situation wherein Alice is restricted to sending at most a single photon to Bob using a single optical mode. Any state ρ_1 Alice prepares must then have support on the qubit Hilbert space spanned by $\{|0\rangle, |1\rangle\}$, and any such state is represented by Bob as $\mathcal{U}[\rho_1] = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$ for $p_i = \langle i|\rho_1|i\rangle$, i.e., as an incoherent mixture of the zero- and one-photon states. Any qubit state is completely depolarized according to Bob. Thus quantum communication cannot be performed by using only a single mode with at most one photon. This negative result is one of many disadvantages to this encoding of a qubit into states spanned by $|0\rangle$ and $|1\rangle$, known as the “single-rail” encoding (Kok *et al.*, 2007). Clearly, no quantum communication can be performed using any number of photons in a single mode, because Bob represents all states prepared by Alice as being diagonal in the photon-number basis.

Now consider the case where Alice can make use of two modes in her communication to Bob. Noting that Bob will represent any preparation by Alice as block diagonal in the eigenspaces of total photon number, Alice should prepare states lying in just one of these eigenspaces if she wishes to communicate quantum information. For example, the one-photon eigenspace of two modes (labeled a and b) is two dimensional, and a general pure state on this eigenspace has the form

$$|\psi\rangle_{n=1} = \alpha|1\rangle_a|0\rangle_b + \beta|0\rangle_a|1\rangle_b, \quad (3.1)$$

for $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. Any such state satisfies $\mathcal{U}[|\psi\rangle_1\langle\psi|] = |\psi\rangle_1\langle\psi|$; this two-dimensional subspace is

⁶Such an encoding is not feasible in practice, because all photon-based communication schemes rely on obtaining a detector event (a “click”) for every message. Specifically, the detection of the vacuum cannot be discriminated from an event where the photon is lost, or missed by the detector.

a decoherence-free subspace of \mathcal{U} . Using states of this form, Alice can communicate a single qubit to Bob without requiring a shared phase reference. We note that this encoding is the commonly used “dual-rail” encoding of optical quantum computing (Kok *et al.*, 2007). Evidently, to communicate quantum information using at most N photons in M modes without a shared phase reference, Alice and Bob should make use of the eigenspace of total photon number N' ($N' \leq N$) that has the largest dimension. This eigenspace is the one corresponding to $N' = N$, and has dimension $(N+K-1)!/N!(K-1)!$.

Using multiple modes of the optical field raises addition issues regarding the use of reference frames, depending on how these modes are identified, and this can lead to a much richer structure. For example, in the dual-rail encoding of Eq. (3.1), the modes a and b could represent different spatial or temporal modes, in which case Alice and Bob would require a shared Cartesian frame or a clock in order to identify these modes. Another common implementation for this encoding is for a and b to represent the two polarization modes of the single photon (for example, horizontal and vertical polarization)—a so-called “polarization encoding” (Kok *et al.*, 2007). For Alice and Bob to share quantum information using such an encoding, although they do not need to share a phase reference, they do need to share a reference frame for polarization, i.e., to agree on an axis for their polarizing materials that are used to prepare, manipulate, and measure such states. The efficiencies of general schemes for transmitting quantum information via the polarization and phase of optical modes when parties do not share a reference frame for polarization have been fully characterized (Ball and Banaszek, 2005, 2006).

Recently, optical quantum-information experiments have made use of the spatial mode structure of light (Mair *et al.*, 2001; Vaziri *et al.*, 2003; Langford *et al.*, 2004); use of this degree of freedom requires a shared reference frame for both position and orientation. Using spatial modes, it is possible to restrict attention to states of a single photon with a fixed orbital angular momentum [the standard basis for which is the Laguerre-Gauss-Vortex modes (Siegman, 1986)]. Encodings into a subspace of fixed orbital angular momentum will be invariant under rotations about the direction of propagation, and thus will not require a shared reference frame for orientation about this direction. These encodings do require a shared reference frame for the direction of propagation, and also a precise determination of the separation between parties in order to compensate for the relative phase (Gouy shift) acquired between different states of fixed orbital angular momentum during propagation (Spedalieri, 2004).

2. Communication without a shared Cartesian frame

We now turn our attention to the problem of how Alice and Bob can perform both classical and quantum communication through the exchange of spin-1/2 systems (qubits) when they lack a shared Cartesian frame

(Bartlett *et al.*, 2003). This problem has a much richer structure than the phase-reference case investigated above, due to the existence of decoherence-free subsystems (rather than subspaces). For simplicity, we consider a noiseless channel that transmits these spin-1/2 systems from Alice to Bob; these results can be extended to noisy channels or higher-dimensional (spin $> 1/2$) systems (Byrd, 2006; van Enk, 2006).

The group of transformations of orientation relative to a Cartesian frame is $\text{SO}(3)$, which we will extend to $\text{SU}(2)$ to allow for spinor representations. We will denote an element of $\text{SU}(2)$ by Ω , which might represent, for instance, a set of three Euler angles. In the case of a single spin-1/2 system, the Wigner rotation operators $R(\Omega)$ provide an irreducible representation of $\text{SU}(2)$. If Alice sends N spin-1/2 systems to Bob, and she describes these, relative to her Cartesian frame, by ρ then Bob describes these same spins relative to his Cartesian frame by

$$\mathcal{E}_N[\rho] = \int d\Omega R(\Omega)^{\otimes N} \rho R^\dagger(\Omega)^{\otimes N}. \quad (3.2)$$

That is, he averages over all passive rotations that might relate his frame to Alice’s, and every rotation acts on each of the N spins identically as $R(\Omega)^{\otimes N}$ because each spin experiences the same rotation by virtue of the fact that each is prepared relative to the same Cartesian frame. We refer to this representation of $\text{SO}(3)$ as *collective*. Thus Bob’s lack of Alice’s Cartesian frame has the same effect as collective noise on the channel. It is still possible for Alice and Bob to communicate by encoding in the relational degrees of freedom of the qubits, as we shall see.

The problem of determining the communication capacities in the presence of this restriction is simple if we decompose the Hilbert space in the manner dictated by Eqs. (2.22) and (2.23). We begin with some simple examples, illustrating the basic techniques and some few-qubit schemes for classical and quantum communication, before presenting the general results.

a. One transmitted qubit

Given that $R(\Omega)$ is an irreducible representation on $\mathcal{H}_{1/2}$, by Schur’s lemma, the $\text{SU}(2)$ twirling on one qubit is equivalent to the completely depolarizing operation,

$$\mathcal{E}_1 = \mathcal{D}_{\mathcal{H}_{1/2}}. \quad (3.3)$$

Thus if Alice prepares a single qubit in the state ρ and transmits it to Bob, he represents the state of this received qubit as the completely mixed state

$$\mathcal{E}_1[\rho] = \frac{1}{2}I. \quad (3.4)$$

Consequently, Bob can infer nothing about ρ from the outcome of any measurement. So, without a shared RF, Alice cannot communicate any information to Bob using only a single qubit.

b. Two transmitted qubits: A classical channel

The unitary representation $R(\Omega)^{\otimes 2}$ of $SU(2)$ is reducible. To decompose it into irreducible representations, we briefly review the representation theory of $SU(2)$.

The inequivalent representations of $SU(2)$ are labeled by the total angular momentum J^2 quantum number j . The carrier spaces of these representations are the charge sectors \mathcal{H}_j . The carrier spaces of the irreducible representations, the gauge spaces, are denoted as \mathcal{M}_j . Such spaces have dimensionality $2j+1$, and may be decomposed into a basis $|j, m\rangle$ of eigenstates of J_z with eigenvalues $\hbar m$ where $m \in \{-j, -j+1, \dots, j\}$. The multiplicity spaces \mathcal{N}_j arise when there are different ways of coupling multiple systems to a given total angular momentum. A pair of spins with angular momentum numbers j_1 and j_2 couple to any total angular momenta j satisfying $|j_1 - j_2| \leq j \leq j_1 + j_2$. We summarize this as $j_1 \otimes j_2 = |j_1 - j_2| \oplus \dots \oplus (j_1 + j_2)$.

It follows that for a pair of spin-1/2 systems, we have $(\frac{1}{2})^{\otimes 2} = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. The possible total angular momenta are $j=0$ and $j=1$ and each has multiplicity 1. The joint eigenstates of total angular momentum operators J^2 and J_z , denoted $|j, m\rangle$, form a basis of the Hilbert space (the coupled representation). We can relate this coupled basis to the joint eigenstates of $J_1^2, J_{1z}, J_2^2, J_{2z}$, denoted by $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ (the uncoupled representation) by

$$|1, 1\rangle = |00\rangle, \quad (3.5)$$

$$|1, 0\rangle = (|01\rangle + |10\rangle)/\sqrt{2}, \quad (3.6)$$

$$|1, -1\rangle = |11\rangle, \quad (3.7)$$

$$|0, 0\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \equiv |\psi^-\rangle, \quad (3.8)$$

where $|0\rangle$ ($|1\rangle$) is the quantum-information-theoretic shorthand for $|1/2, \pm 1/2\rangle$, and $|01\rangle \equiv |0\rangle \otimes |1\rangle$, etc. These are the $j=1$ (symmetric) triplet states and the $j=0$ (antisymmetric) singlet state.

Suppressing multiplicity spaces when they are one dimensional (because $\mathcal{H}_j = \mathcal{M}_j \otimes \mathbb{C} = \mathcal{M}_j$), we have

$$(\mathcal{H}_{1/2})_4^{\otimes 2} = \mathcal{H}_{j=1} \oplus \mathcal{H}_{j=0}, \quad (3.9)$$

where the dimensionality of each space is expressed in bold underneath each subspace. Writing $R(\Omega)^{\otimes 2} = R_{j=1}(\Omega) \oplus R_{j=0}(\Omega)$, and applying Schur's lemma, we infer that

$$\mathcal{E}_2 = (\mathcal{D}_{\mathcal{M}_{j=1}} \circ \mathcal{P}_{j=1}) + \mathcal{P}_{j=0}, \quad (3.10)$$

where $\mathcal{P}_j[\rho] = \Pi_j \rho \Pi_j$ and Π_j is the projector onto the subspace \mathcal{H}_j . Equation (3.10) asserts that the coherence between the singlet and triplet spaces is eliminated and the triplet space is depolarized.

Thus if Alice transmits two qubits and she assigns the state ρ to the pair, Bob describes the pair by

$$\mathcal{E}_2[\rho] = p_{j=1} \left(\frac{1}{3} \Pi_{j=1} \right) + p_{j=0} |\psi^-\rangle \langle \psi^-|, \quad (3.11)$$

where $p_j = \text{Tr}(\rho \Pi_j)$. Note that Bob can distinguish perfectly between the antisymmetric state $|\psi^-\rangle \langle \psi^-|$ and a state ρ_S which lies in the symmetric subspace because $\mathcal{E}_2[|\psi^-\rangle \langle \psi^-|] = |\psi^-\rangle \langle \psi^-|$ and $\mathcal{E}_2[\rho_S] = \frac{1}{3} \Pi_{j=1}$, and these two images are orthogonal.

Thus Alice can communicate one classical bit to Bob with every two transmitted qubits by implementing the following protocol: Alice sends Bob the antisymmetric state $|\psi^-\rangle$ to communicate $b=0$ and any state in the symmetric subspace (for example, the state $|00\rangle$) for $b=1$. Bob then performs a projective measurement onto the antisymmetric and symmetric subspaces and recovers b with certainty.

c. Three transmitted qubits: A quantum channel

We must determine how $R(\Omega)^{\otimes 3}$ is decomposed into irreducible representations. To see how three spin-1/2 systems couple to total spin, imagine coupling the first pair to a spin j_1 and then coupling this to the third: $(\frac{1}{2})^{\otimes 3} = (0 \oplus 1) \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$. Note that because the third spin 1/2 can couple to either $j_1=0$ or $j_1=1$ to yield $j=1/2$, the latter representation has multiplicity 2. We let $|1/2, \pm 1/2, \lambda\rangle$ denote a basis of $\mathcal{H}_{j=1/2}$ in the coupled representation, where λ is a degeneracy index which by convention we take to be 0 if the coupling was to $j_1=0$ and 1 if the coupling was to $j_1=1$. These states can be given explicitly in terms of the three spin-1/2 systems as

$$|\frac{1}{2}, \frac{1}{2}, 0\rangle = \frac{1}{\sqrt{2}} (|011\rangle - |101\rangle), \quad (3.12)$$

$$|\frac{1}{2}, -\frac{1}{2}, 0\rangle = \frac{1}{\sqrt{2}} (|010\rangle - |100\rangle), \quad (3.13)$$

$$|\frac{1}{2}, \frac{1}{2}, 1\rangle = \frac{1}{\sqrt{6}} (2|110\rangle - |101\rangle - |011\rangle), \quad (3.14)$$

$$|\frac{1}{2}, -\frac{1}{2}, 1\rangle = \frac{1}{\sqrt{6}} (-2|001\rangle + |010\rangle + |100\rangle). \quad (3.15)$$

We can then define an isomorphism $\mathcal{H}_{j=1/2} = \mathcal{M}_{j=1/2} \otimes \mathcal{N}_{j=1/2}$ through $|m\rangle \otimes |\lambda\rangle \equiv |\frac{1}{2}, m, \lambda\rangle$ with $|m\rangle$ a basis of $\mathcal{M}_{j=1/2}$ and $|\lambda\rangle$ a basis of the multiplicity space $\mathcal{N}_{j=1/2}$. Thus the total Hilbert space decomposes as

$$(\mathcal{H}_{1/2})_8^{\otimes 3} = \mathcal{H}_{j=3/2} \oplus (\mathcal{M}_{j=1/2} \otimes \mathcal{N}_{j=1/2}), \quad (3.16)$$

where again we have included the dimensions of each subsystem.

An application of Schur's lemma along the lines presented in Sec. II.C implies that

$$\mathcal{E}_3 = \mathcal{D}_{\mathcal{M}_{j=3/2}} \circ \mathcal{P}_{j=3/2} + (\mathcal{D}_{\mathcal{M}_{j=1/2}} \otimes \mathcal{I}_{\mathcal{N}_{j=1/2}}) \circ \mathcal{P}_{j=1/2}, \quad (3.17)$$

where \mathcal{I} is the identity map. Note that the operation $\mathcal{D}_{\mathcal{M}_j} \otimes \mathcal{I}_{\mathcal{N}_j}$ is only defined on the space of operators acting on $\mathcal{M}_j \otimes \mathcal{N}_j = \mathcal{H}_j$, but it is always preceded by \mathcal{P}_j , which projects into this space. If Alice prepares three qubits in the state ρ , then Bob assigns to them the state

$$\mathcal{E}_3[\rho] = p_{3/2} \left(\frac{1}{4} \Pi_{j=3/2} \right) + p_{1/2} \left(\frac{1}{2} \mathcal{I}_{\mathcal{M}_{j=1/2}} \otimes \rho_{\mathcal{N}_{j=1/2}} \right), \quad (3.18)$$

where $p_j = \text{Tr}(\rho \Pi_j)$ and

$$\rho_{\mathcal{N}_{j=1/2}} = p_{1/2}^{-1} \text{Tr}_{\mathcal{M}_{j=1/2}} (\Pi_{j=1/2} \rho \Pi_{j=1/2}). \quad (3.19)$$

We note that the subsystem $\mathcal{N}_{j=1/2}$ is unaffected by the decohering superoperator \mathcal{E}_3 ; i.e., it is a decoherence-free subsystem. Thus Alice can encode a logical qubit into this subsystem (Kempe *et al.*, 2001). That is, she can prepare states of the form $\sigma \otimes \rho$ on $\mathcal{M}_{j=1/2} \otimes \mathcal{N}_{j=1/2}$, where ρ is the logical qubit state she wishes to transmit to Bob, and Bob can access this decoherence-free subsystem and retrieve the quantum information without a shared RF. Thus one logical qubit can be transmitted using three physical qubits without a shared RF.

d. Asymptotic behavior

The above two schemes demonstrate that classical and quantum communication are possible without a shared RF. The efficiency of the above schemes can be increased through the use of more qubits, because the sizes of the decoherence-free subsystems can grow exponentially with increasing number of qubits.

For simplicity, we consider only the case where N is even. The collective (tensor) representation of $\text{SU}(2)$ on N spin-1/2 systems, $R(\Omega)^{\otimes N}$, can again be decomposed into a direct sum of $\text{SU}(2)$ irreps, each with angular momentum quantum number j ranging from 0 to $N/2$. That is, we can decompose the Hilbert space as

$$(\mathcal{H}_{1/2})^{\otimes N} = \bigoplus_{j=0}^{N/2} \mathcal{M}_j \otimes \mathcal{N}_j, \quad (3.20)$$

where we have indicated the dimensions of the various spaces. The multiplicity of the irrep j , which is the dimension of \mathcal{N}_j , is found from representation theory to be

$$c_j^{(N)} = \binom{N}{N/2 - j} \frac{2j + 1}{N/2 + j + 1}. \quad (3.21)$$

Relative to this decomposition, the $\text{SU}(2)$ -twirling operation \mathcal{E}_N has the form

$$\mathcal{E}_N = \sum_j (\mathcal{D}_{\mathcal{M}_j} \otimes \mathcal{I}_{\mathcal{N}_j}) \circ \mathcal{P}_j, \quad (3.22)$$

as can be inferred from the result for arbitrary groups in Sec. II.C. The carrier spaces for the irreducible representations of $\text{SU}(2)$, \mathcal{M}_j , are the decoherence-full subsystems for \mathcal{E}_N , while the multiplicity spaces \mathcal{N}_j , which carry the trivial representation of $\text{SU}(2)$, are the decoherence-free subsystems for \mathcal{E}_N .

Alice can choose to transmit classical messages by preparing orthogonal states as follows: For each irrep j , she can choose one arbitrary state from each multiplicity. Thus it is possible to transmit, without a shared RF, a number of classical messages equal to the number $C^{(N)}$ of $\text{SU}(2)$ irreps in the direct sum decomposition of the tensor representation of $\text{SU}(2)$ on N qubits, which is given by

$$C^{(N)} = \sum_{j=0}^{N/2} c_j^{(N)} = \binom{N}{N/2}. \quad (3.23)$$

In fact, this is the maximum number of classical messages that can be sent; for a proof, see Bartlett *et al.* (2003). Thus the number of classical bits that can be transmitted per qubit using the above scheme is $N^{-1} \log_2 C^{(N)}$, which tends asymptotically to $1 - (2N)^{-1} \log_2 N$; in the large N limit, one classical bit can be transmitted for every qubit sent. Remarkably, this rate is equivalent to what can be accomplished if Alice and Bob *do* possess a shared RF.

To determine the optimal scheme for transmitting quantum (rather than classical) information, again using N qubits and under the restriction of no shared RF, we identify the largest decoherence-free subsystem for \mathcal{E}_N . This is the subsystem \mathcal{N}_j with the greatest multiplicity $c_j^{(N)}$. Asymptotically, this is found to occur at $j_{\max} = \sqrt{N}/2$, and the number $N^{-1} \log_2 c_{j_{\max}}^{(N)}$ of logical qubits encoded per physical qubit in N physical qubits behaves as $1 - N^{-1} \log_2(N)$, approaching unity for large N . Full details can be found in Kempe *et al.* (2001). This remarkable result proves that quantum communication without a shared RF is asymptotically as efficient as quantum communication with a shared RF, and is the communication analog of ‘‘asymptotic universality’’ (Knill *et al.*, 2000). In addition, we note that the algorithm for encoding/decoding quantum information into decoherence-free subsystems can be done efficiently (Bacon *et al.*, 2006a, 2006b).

e. Relativistic considerations

The ability to perform quantum-information processing in a relativistic setting has also been of recent interest [see Peres and Terno (2004) for a review], and in this context it is natural to consider whether parties who do not share an inertial frame (i.e., a reference frame for the Poincaré group) can still perform quantum communication, and at what efficiency. It has been shown that classical and quantum communication can be performed at the same rate as demonstrated above using indistinguishable massive spin-1/2 particles, or using photons, if appropriately localized wave packets for these particles are used (Bartlett and Terno, 2005). In addition, continuous-variable quantum information can be shared using related methods (Kok *et al.*, 2005).

3. Consequences for quantum-information processing

The communication schemes presented above imply that Alice and Bob can share entangled states in the

absence of any particular shared RF. Consider the case of lacking a shared Cartesian frame as an example. Denoting the logical qubit that can be encoded using three physical qubits in Alice's (Bob's) possession by $\{|0_L\rangle_{A(B)}, |1_L\rangle_{A(B)}\}$, a triple of physical qubits in Alice's possession can be maximally entangled with a triple in Bob's possession using the state $\frac{1}{\sqrt{2}}(|0_L\rangle_A |0_L\rangle_B + |1_L\rangle_A |1_L\rangle_B)$. Because Alice and Bob can perform any measurement in their respective logical qubit Hilbert spaces, they can demonstrate quantum nonlocality (Bell's theorem) despite having no shared Cartesian RF (Bartlett *et al.*, 2003; Cabello, 2003). It also follows that such entangled states can be used for quantum teleportation of logical qubits, which implies that the latter does not rely upon the existence of a shared Cartesian RF either, contrary to some expectations (van Enk, 2001). In fact, for any quantum-information task that assumes some shared RF, it is possible to make use of logical encodings to perform the task without this shared RF. (Any task, that is, which deals with speakable rather than unspeakable quantum information; the alignment of RFs, for instance, obviously cannot be achieved in this way.) It should be noted, however, that although one can achieve quantum-information tasks without any particular kind of shared RF, some form of shared RF is always required. For instance, in the example just described, Alice and Bob must agree on the ordering of the three physical qubits, and this agreement constitutes a kind of shared RF (Bartlett *et al.*, 2004a).

Several recent experiments have demonstrated the key techniques required for quantum-information processing without a shared Cartesian frame. These experiments make use of single-photon polarization qubits. Lacking a shared RF for polarization means that Bob's polarizing elements (such as calcite crystals) are uncorrelated with Alice's. The relevant group is also $SU(2)$, and thus the analysis presented above applies to this scenario as well. Banaszek *et al.* (2004) have demonstrated that two orthogonal entangled states of two single-photon polarization qubits remain perfectly distinguishable between two parties who do not share a reference frame for polarization, thereby demonstrating the classical communication protocol in Sec. III.A.2.b. In addition, Bourennane *et al.* (2004) have demonstrated nonorthogonal entangled states—states of a logical qubit encoded in four single-photon polarization qubits—that are identical in any reference frame; see also Zou *et al.* (2006). These states demonstrate the basic principles of a decoherence-free subsystem that are needed for quantum communication without a shared RF.

B. Quantum key distribution without a shared reference frame

The possibility of performing secure communication through the use of quantum key distribution (QKD) is one of the most celebrated applications of quantum-information science (Gisin *et al.*, 2002). Because of its advanced state of development, it is also one of the first quantum protocols to require explicit consideration of shared reference frames, or the lack thereof, between

communicating parties. All practical QKD protocols are based on the exchange of quantum states of light, and as discussed in Sec. III.A.1, essentially any identification of a mode structure (either spatial, time bin, or polarization) requires a reference frame of some sort. For example, in all single-photon implementations of QKD, a shared clock is necessary in order to agree upon a short time window for communication; otherwise, dark counts from the photodetectors can greatly reduce security and efficiency (Brassard *et al.*, 2000).

Quantum key distribution schemes that obviate the need for certain shared reference frames (and that are robust against other forms of noise) have recently been developed, and make use of the techniques of decoherence-free subspaces and subsystems (Walton *et al.*, 2003). Consider the following proposal of Boileau *et al.* (2004). Alice (the sender) and Bob (the receiver) wish to perform QKD using the polarization states of single photons. This choice avoids stabilization problems inherent in phase-based schemes, but presents a problem of its own: If an optical fiber is used as the quantum channel, the polarization of a transmitted photon is rotated by a random amount due to optical birefringence. Although this random rotation fluctuates with time, it can be considered constant on a short time scale so that all photons in a pulse are subject to the same rotation. Thus the problem becomes equivalent to one in which Alice communicates to Bob using a noiseless channel, but in which they do not share a reference frame for polarization. The communication scenario then becomes equivalent to that analyzed in Sec. III.A.2.

Alice can perform quantum communication with Bob without a shared RF for polarization through the use of decoherence-free subspaces or subsystems. We now briefly outline two straightforward and experimentally accessible QKD protocols using these techniques; the first protocol makes use of four-photon decoherence-free subspace, and the second makes use of three-photon decoherence-free subsystems.

The smallest nontrivial decoherence-free subspace for the superoperator \mathcal{E}_N of Eq. (3.2) occurs for $N=4$. It is the two-dimensional $j=0$ (singlet) subspace. A simple QKD scheme using this subspace is as follows. Define the state $|\psi^-\rangle_{\mu\nu} = (|0\rangle_\mu |1\rangle_\nu - |1\rangle_\mu |0\rangle_\nu) / \sqrt{2}$ to be the two-photon singlet state of photons μ and ν ($\mu, \nu \in \{1, 2, 3, 4\}$). Define three four-photon states as products of singlet states of differing photons, i.e.,

$$\begin{aligned} |\Psi_1\rangle &= |\psi^-\rangle_{12} |\psi^-\rangle_{34}, \\ |\Psi_2\rangle &= |\psi^-\rangle_{13} |\psi^-\rangle_{24}, \\ |\Psi_3\rangle &= |\psi^-\rangle_{14} |\psi^-\rangle_{23}. \end{aligned} \tag{3.24}$$

Clearly, all three states are $j=0$ (singlet) states in the $N=4$ decoherence-free subsystem. Thus each of the states $|\Psi_a\rangle$ prepared by Alice is represented the same way by Bob, even though they do not share a reference frame for polarization.

Note that these states are also nonorthogonal, satisfying $|\langle\Psi_a|\Psi_b\rangle|=1/2$ for $a\neq b$. Thus if Alice restricts her transmitted states to a pair of these, then they can implement a B92-type QKD protocol (Bennett, 1992). In addition, this protocol can be defined in such a way that Bob need only perform single-photon measurements in some fixed polarization basis (i.e., without the need for entangling measurements); see Boileau *et al.* (2004) for details.

As noted in Sec. III.A.2, there exists a two-dimensional decoherence-free subsystem with $N=3$. There is a simple modification of the above QKD protocol which makes use of this subsystem. Define the following three mixed states, obtained from the three pure states of Eq. (3.24) by discarding the last photon, i.e.,

$$\rho_a = \text{Tr}_4[|\Psi_a\rangle\langle\Psi_a|]. \quad (3.25)$$

In terms of the decomposition of the three-qubit Hilbert space of Sec. III.A.2.c, all three of these states lie on the $\mathcal{H}_{j=1/2}$ subspace, and in terms of the tensor product structure $\mathcal{H}_{j=1/2} = \mathcal{M}_{j=1/2} \otimes \mathcal{N}_{j=1/2}$ these states are products of the completely mixed state $\frac{1}{2}I$ on $\mathcal{M}_{j=1/2}$ and one of three pure nonorthogonal states on $\mathcal{N}_{j=1/2}$. Again, if Alice restricts her transmitted states to a pair of these, then they can implement a B92-type QKD protocol without the need for a shared RF for polarization.

The unconditional security of the QKD schemes of Boileau *et al.* (2004), which are based on using the above states, has been proven (Boileau *et al.*, 2005). In addition, a BB84 version of this QKD scheme, which does not require a shared reference frame for polarization, has been demonstrated experimentally (Chen *et al.*, 2006). We note that the essential concept of this scheme—to use the techniques of decoherence-free subspaces or subsystems to obviate the necessity for a shared reference frame in QKD—can be applied to any system and RF. In particular, it has been proposed to use spatial encodings of optical modes discussed at the end of Sec. III.A.1 to perform QKD (Spedalieri, 2004).

C. Entanglement without a shared reference frame

Entanglement is often considered the key resource in quantum-information processing, and so it is valuable to consider the role of shared reference frames in both qualitative and quantitative properties of bipartite entanglement. As we demonstrate in this section, the very meaning of entanglement between parties who do not share a reference frame must be reassessed, with some surprising results.

1. Entanglement without a shared phase reference

As an example, we again consider a number of optical modes shared between two parties, Alice and Bob, who do not share a common phase reference. We consider all states and operations to be described relative to the phase reference of a third party, Charlie, which is

assumed to be uncorrelated with both Alice's and Bob's local phase references. (For many of the issues considered, we could dispense with Charlie and describe everything relative to either Alice or Bob, but this introduces an artificial asymmetry into the formalism which easily leads to confusion. We therefore opt to describe all states relative to Charlie, whether he participates in the protocol or not.) As such, Alice redescribes states prepared relative to Charlie's phase reference by mixing over all possible phase shifts. Bob does the same, and because Alice and Bob's phase references are uncorrelated, the phases over which they mix are independent. Recalling the results of Sec. II.B, the mixing over phases yields a photon-number superselection rule, and the independence implies that Alice and Bob are subject to local photon-number superselection rules. In this case, all of Alice's operations commute with the local map \mathcal{U}_A , defined as in Eq. (2.9) as

$$\mathcal{U}_A[\rho_A] \equiv \sum_n \Pi_n^A \rho_A \Pi_n^A, \quad (3.26)$$

where Π_n^A is the projector onto the eigenspace of total photon number n on Alice's local modes. All of Bob's operations commute with the local map \mathcal{U}_B , defined similarly.

In such situations, there has been considerable debate over the entanglement properties of certain types of states, such as the two-mode single-photon state (Tan *et al.*, 1991; Hardy, 1994, 1995; Greenberger *et al.*, 1995; van Enk, 2005b),

$$(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}. \quad (3.27)$$

There is a temptation to say that this state is entangled simply because of its nonproduct form. However, it is far more useful to consider whether or not this state satisfies certain operational notions of entanglement. One such notion is whether a state can be used to violate a Bell inequality. Another is whether it is useful as a resource for quantum-information processing, for instance, to teleport qubits or implement a dense coding protocol. In the context of a local photon-number superselection rule, this two-mode single-photon state fails to satisfy either of these notions of entanglement, because all such tasks would require Alice and Bob to violate the local photon-number superselection rule. A different but equally common notion of entanglement is that a state is entangled if it cannot be prepared by LOCC. The two-mode single-photon state certainly does fit this notion because pure nonproduct states cannot be prepared by LOCC. Thus we see that operational notions of entanglement that coincided for pure states under unrestricted LOCC, namely, being not locally preparable and being useful as a resource for tasks such as teleportation or violating a Bell inequality, do not coincide under a local photon-number superselection rule, and the state

in question is judged entangled by one notion and not the other.⁷

Another class of states whose entanglement properties have been discussed in the quantum optics literature is those that are separable but not locally preparable under a local photon-number superselection rule (Rudolph and Sanders, 2001b; Verstraete and Cirac, 2003). Examples of such states are

$$|+\rangle_A|+\rangle_B, \quad |-\rangle_A|-\rangle_B, \quad (3.28)$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. [Rudolph and Sanders (2001b) and Verstraete and Cirac (2003) considered states such as the equal mixture of $|+\rangle_A|+\rangle_B$ and $|-\rangle_A|-\rangle_B$. For simplicity, we restrict our attention to pure states.] Because of the superselection rule, these states cannot be prepared locally. However, because they are product states, they clearly cannot be used for tasks such as teleportation or violating a Bell inequality. We will return our attention to states such as these in Sec. IV.

In contrast, consider a state of the form

$$(|01\rangle_A|10\rangle_B + |10\rangle_A|01\rangle_B)/\sqrt{2}. \quad (3.29)$$

This state is certainly not locally preparable. In addition, it can be used to violate a Bell inequality, implement dense coding, and so on, despite the superselection rule. This is because Alice and Bob can still implement any measurements they please in the two-dimensional subspaces spanned by $|01\rangle$ and $|10\rangle$. Thus this state is unambiguously entangled by any reasonable notion.

We then see that the remarkable and often confusing entanglement properties of states when parties do not share a reference frame can be understood by recognizing that different operational notions of entanglement do not coincide in this case. Specifically, for pure quantum-optical states in a situation where Alice and Bob do not share a phase reference, there exists a proper gap between states that are locally preparable under LOCC, and states that are useful for performing quantum-information tasks such as teleportation and violating a Bell inequality. The existence of this proper gap is reminiscent of a similar situation for mixed quantum states: that of bound entanglement (Horodecki *et al.*, 1998). This analogy can be extended further; in the following section, we demonstrate that some of the strange phenomena from mixed-state entanglement—activation and multicopy entanglement distillation—are present as well in pure-state quantum optics with a local photon-number SSR. This analogy is pursued in Bartlett, Doherty, *et al.* (2006).

2. Activation and entanglement distillation

In this section, we demonstrate that there exist analogous processes of activation (Horodecki *et al.*, 1999) and

multicopy entanglement distillation (Watrous, 2004) using pure bipartite quantum-optical states when Alice and Bob do not share a phase reference. An understanding of these processes and their relation to the above-mentioned gap between two commonly used notions of entanglement is key to resolving several recent controversies regarding the entanglement of quantum-optical states (van Enk, 2005a; Bartlett, Doherty, *et al.*, 2006).

We now demonstrate that to achieve a Bell inequality violation with the state

$$(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}, \quad (3.30)$$

it is necessary to use a process that is analogous to activation. Understanding the necessity of an additional resource for this process resolves the controversy over the use of the state to demonstrate quantum nonlocality (Tan *et al.* 1991; Hardy, 1994, 1995; Greenberger *et al.*, 1995).

As we have shown, this state cannot be used for tasks such as violating a Bell inequality when Alice and Bob do not share a phase reference, i.e., when a local photon-number superselection rule applies. However, combining $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ with $|+\rangle_A|+\rangle_B$, one obtains a state that is useful for such tasks. The state $|+\rangle_A|+\rangle_B$ is said to activate the entanglement of $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$. This is seen as follows. Let Alice and Bob both perform a quantum nondemolition measurement of local photon number on both of their local modes, and postselect the case where they both find a local photon number of 1. The resulting state is

$$\begin{aligned} \Pi_1^A \otimes \Pi_1^B \left[\frac{1}{\sqrt{2}} (|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) |+\rangle_A|+\rangle_B \right] \\ \propto \frac{1}{\sqrt{2}} (|01\rangle_A|10\rangle_B + |10\rangle_A|01\rangle_B). \end{aligned} \quad (3.31)$$

Violations of a Bell inequality have recently been demonstrated experimentally using the state (3.30) by Babichev *et al.* (2004) and Hessmo *et al.* (2004). One can take two different perspectives on such an experiment. It is illustrative to consider them both.

In the work of Hessmo *et al.* (2004), in addition to the state (3.30), a correlated pair of coherent states $|\alpha\rangle_A|\alpha\rangle_B$, where $|\alpha\rangle \equiv \sum_n (e^{-|\alpha|^2/2} \alpha^n / \sqrt{n!}) |n\rangle$, are assumed to be shared between Alice and Bob. These modes are used as local oscillators in the homodyne detections at each site. Noting that neither $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ nor $|\alpha\rangle_A|\alpha\rangle_B$ can be used individually for violating a Bell inequality, it is unclear how it is possible to do so using such resources. The resolution of the puzzle is that a pair of correlated coherent states $|\alpha\rangle_A|\alpha\rangle_B$, much like the state $|+\rangle_A|+\rangle_B$ discussed above, activates the entanglement of the two-mode single-photon state.

An experimental demonstration of nonlocality using the two-mode single-photon state has also been described by Babichev *et al.* (2004). Rather than treating local oscillators as coherent states, they are treated as correlated classical phase references. In this case, they constitute an additional resource that “lifts” the restric-

⁷Of course, if there is no local photon-number superselection rule, this state would satisfy all of these notions of entanglement, as emphasized by van Enk (2005b). In particular, no such superselection rule would apply if all parties share a common phase reference.

tion of the local photon-number superselection rule, and the state $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ becomes unambiguously entangled. These two alternative descriptions are equally valid; see [Bartlett, Rudolph, and Spekkens \(2006\)](#).

The existence of such activation processes also resolves a controversy concerning the source of entanglement in the experimental realization of [Furusawa *et al.* \(1998\)](#) of continuous-variable quantum teleportation. Again, it is illustrative to consider two different perspectives of this experiment.

The first perspective is a variant of the one presented by [Rudolph and Sanders \(2001b\)](#). In our language, it can be synopsised as follows. Alice and Bob are presumed to be restricted in the operations they can perform by a local photon-number superselection rule. They share a two-mode squeezed state $|\gamma\rangle = \sqrt{1-\gamma^2} \sum_{n=0}^{\infty} \gamma^n |n, n\rangle$, where $0 \leq \gamma \leq 1$. In addition, they share two other modes prepared in a product of correlated coherent states $|\alpha\rangle|\alpha\rangle$.⁸ The former is the purported entanglement resource in the teleportation protocol, while the latter is a quantum version of a shared phase reference. These states are analogous to $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ and $|+\rangle_A|+\rangle_B$, respectively—neither can be used as a resource for teleportation when considered on its own. So the question arises as to how teleportation could possibly have been achieved. The answer is that the product of coherent states activates the entanglement in the two-mode squeezed state.⁹

The second perspective is one wherein the shared phase reference is treated classically; this perspective was taken by [Furusawa *et al.* \(1998\)](#). As described above, this classical shared phase reference acts as a resource that lifts the superselection rule, and causes the two-mode squeezed state to become unambiguously entangled.

An analog of multicopy entanglement distillation can also be demonstrated in our quantum optical example. Two copies of the state $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ can be used to obtain free entanglement (i.e., not bound) in the presence of the SSR, whereas only one copy cannot. The protocol, introduced by [Wiseman \(2003\)](#) and discussed in greater detail by [Vaccaro *et al.* \(2003\)](#), is as follows. As in the activation example above, Alice and Bob both perform a quantum nondemolition measurement of local photon number (on both local modes) and postselect the case where they both find a local photon number of 1. The resulting state is

$$\begin{aligned} & \Pi_1^A \otimes \Pi_1^B \left[\frac{1}{\sqrt{2}} (|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) \right]^{\otimes 2} \\ & \propto \frac{1}{\sqrt{2}} (|01\rangle_A|10\rangle_B + |10\rangle_A|01\rangle_B), \end{aligned} \quad (3.32)$$

where $|\psi\rangle^{\otimes 2} = |\psi\rangle|\psi\rangle$. A process very similar to this two-copy entanglement distillation has been demonstrated in quantum optics experiments [cf. [Ou and Mandel \(1988\)](#); [Shih and Alley \(1988\)](#)], where correlated but unentangled photon pairs from parametric down-conversion were made incident on the two input modes of a beam splitter, so each photon transforms to a state of the form $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$. Subsequently, measurements on the two output modes are postselected for one-photon detection at each output mode. The fact that their postselected results are consistent with a description of an entangled state demonstrates that the entanglement of the state $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ has been distilled by making use of two copies.

Finally, we consider the analog of multicopy entanglement distillation from two copies of the two-mode squeezed state $|\gamma\rangle = \sqrt{1-\gamma^2} \sum_{n=0}^{\infty} \gamma^n |n, n\rangle$. Homodyne measurements by Alice and Bob (relative to their uncorrelated local oscillators) can be performed on one copy of this state to establish a shared phase reference, which then lifts the superselection rule and causes the second copy to become unambiguously entangled.

3. Quantifying bipartite entanglement without a shared reference frame

As we have seen above, operational notions of entanglement for a bipartite pure state no longer coincide when parties do not share a reference frame. How then does one quantify the amount of entanglement of a bipartite state in such a situation? Entanglement measures can be defined in the presence of such a restriction again by being operational. In the following, we discuss one such operational measure which quantifies the distillable entanglement under a local Abelian superselection rule. [This measure is directly related to the entanglement of particles ([Wiseman and Vaccaro, 2003](#)).] We note that these results apply directly to a general (possibly non-Abelian) SSR, with local operations restricted as in Eq. (2.17) ([Bartlett and Wiseman, 2003](#)); however, for simplicity, we focus here on the Abelian case.

We continue with the scenario of the previous section. Consider a bipartite state ρ_{AB} shared by Alice and Bob and defined relative to Charlie's phase reference. We assume that in addition to this bipartite system, Alice and Bob each possess a number of quantum registers, not subject to any SSR, with total Hilbert space dimension equal to or greater than that of their respective systems. (For example, these registers could be standard qubits over which Alice and Bob have complete control.) These registers are initiated in a pure product state ρ_{AB} .

The entanglement in the presence of an SSR of the state ρ_{AB} is quantified through a measure E_{SSR} , which is defined by the maximum amount of entanglement that

⁸The state assigned by [Rudolph and Sanders \(2001b\)](#) is simply a mixed version (mixed over the phase of the pump beam) of $|\gamma\rangle|\alpha\rangle|\alpha\rangle$.

⁹[van Enk and Fuchs \(2002a\)](#) suggest a similar protocol to the one we describe here, for the mixed states discussed in the previous footnote. Homodyne measurement is performed on the pump beam with respect to an external phase reference, and the measurement result will yield a two-mode squeezed state that is unambiguously entangled with respect to this external RF.

Alice and Bob can produce between their quantum registers using local $U(1)$ -invariant operations and classical communication [U(1)-LOCC]. The latter can be quantified by an appropriate standard measure E ; it seems most appropriate to use the entanglement of distillation.

We now prove that the entanglement in the presence of an SSR, $E_{\text{SSR}}(\rho_{AB})$, is given by the entanglement $E(\mathcal{U}_{\text{loc}}[\rho_{AB}])$ that they can produce from the state $\mathcal{U}_{\text{loc}}[\rho_{AB}]$ by unconstrained LOCC, where $\mathcal{U}_{\text{loc}} \equiv \mathcal{U}_A \otimes \mathcal{U}_B$.

The proof is illustrative, so we present it here. Let $O = \{\mathcal{O}\}$ be the set of all LOCC operations by Alice and Bob that commute with \mathcal{U}_{loc} . Note that, for any quantum operation \mathcal{E} , the composite operation $\bar{\mathcal{E}} \equiv \mathcal{U}_{\text{loc}} \circ \mathcal{E} \circ \mathcal{U}_{\text{loc}}$ is in the set O . Let $\mathcal{O} \in O$ be some operation on the initial state $\rho_{AB} \otimes \varrho_{AB}$. The final state of the registers is given by $\varrho'_{AB} = \text{Tr}_{\text{sys}}(\mathcal{O}[\rho_{AB} \otimes \varrho_{AB}])$, where the trace is over the shared system. The maximum entanglement produced between registers is given by maximizing $E(\varrho'_{AB})$ over all operations in O . Thus

$$\begin{aligned} E_{\text{SSR}}(\rho_{AB}) &= \max_{\mathcal{O}} E(\text{Tr}_{\text{sys}}(\mathcal{O}[\rho_{AB} \otimes \varrho_{AB}])) \\ &= \max_{\mathcal{O}} E(\text{Tr}_{\text{sys}}((\mathcal{U}_{\text{loc}} \circ \mathcal{O} \circ \mathcal{U}_{\text{loc}})[\rho_{AB} \otimes \varrho_{AB}])) \\ &= \max_{\mathcal{E}} E(\text{Tr}_{\text{sys}}((\mathcal{U}_{\text{loc}} \circ \mathcal{E} \circ \mathcal{U}_{\text{loc}})[\rho_{AB} \otimes \varrho_{AB}])) \\ &= \max_{\mathcal{E}} E(\text{Tr}_{\text{sys}}(\mathcal{E}[\mathcal{U}_{\text{loc}}[\rho_{AB}] \otimes \varrho_{AB}])), \end{aligned} \quad (3.33)$$

where the second line follows from the properties of the trace and by applying the definition (2.7) to \mathcal{U}_A and \mathcal{U}_B , and the last line follows from the properties of the trace. The latter maximization is over all LOCC (not just operations that commute with \mathcal{U}_{loc}), and gives the entanglement $E(\mathcal{U}_{\text{loc}}[\rho_{AB}])$ that Alice and Bob can produce between their registers from the state $\mathcal{U}_{\text{loc}}[\rho_{AB}]$ by unconstrained LOCC.

4. Extensions and application to other systems

The general perspective discussed above for investigating entanglement without a shared reference frame can be applied to other situations, although for the most part this issue has not been explored. Condensed-matter systems is one area where these results can be directly applied, because these systems possess a number of practical restrictions on operations. Local particle-number superselection rules often apply in practice; for example, as noted by several authors, the single-electron two-mode Fock state $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$ has ambiguous entanglement properties under this restriction (Wiseman and Vaccaro, 2003; Samuelsson *et al.*, 2005; Beenakker, 2006; Dowling *et al.*, 2006). For this reason, most proposals for creating bipartite entangled states make use of spin or orbital angular momentum degrees of freedom of multiple particles (Beenakker *et al.*, 2003; Samuelsson *et al.*, 2003, 2004). We note, however, that the two-mode single-electron Fock state is an entanglement resource akin to the two-mode single-photon state, which we have shown to be useful through activation or

multicopy entanglement distillation; also, a suitable shared $U(1)$ reference frame could “lift” the restriction of the superselection rule, and the two-mode single-electron Fock state would be unambiguously entangled with such a resource. [Determining a suitable quantum state of such a shared $U(1)$ RF consisting of fermions is an outstanding problem in general. The two-copy entanglement distillation protocol described above applies equally well to fermionic states of the form $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$. The activation protocols described above, however, do not appear to have precise fermionic analogs; specifically, there are several challenges in defining an analog of the optical coherent state for fermions. See Dowling *et al.* (2006).] Moreover, entangled states between angular momentum degrees of freedom of different particles will yield no real advantage over the two-mode single-electron Fock state in situations wherein there is a local $SU(2)$ superselection rule. Such a superselection rule will be in force, for instance, if the parties fail to share a Cartesian frame for spatial orientations. As with quantum optical systems, such considerations emphasize the need to be operational when classifying or quantifying entanglement.

The theory of entanglement for indistinguishable particles is another situation where considerations of entanglement without a shared reference frame are relevant. States of indistinguishable particles can appear entangled due to the necessary symmetrization or antisymmetrization of the wave function. For example, in the position representation of two indistinguishable particles, a wave function of two particles is expressed as

$$\psi_{12}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2) \pm \psi_1(\mathbf{x}_2)\psi_2(\mathbf{x}_1)], \quad (3.34)$$

where the \pm cases correspond to bosons and fermions. The entanglement properties of such a state are the subject of some debate (Paskauskas and You, 2001; Schliemann *et al.*, 2001; Wiseman and Vaccaro, 2003; Dowling *et al.*, 2006). From the perspective of this review, one can view the indistinguishability of particles as a lack of a reference ordering, i.e., lack of a reference frame to uniquely label the particles (Eisert *et al.*, 2000; Bartlett and Wiseman, 2003; von Korff and Kempe, 2004; Jones *et al.*, 2005, 2006). For example, if the particles described in the above state were distinguishable through another degree of freedom, such as their spin, then the entanglement of the above state would be unambiguous. Thus, in many condensed-matter systems it may be worthwhile to consider the possibility of lifting the restriction of indistinguishability, viewed as a lack of a reference ordering, through an appropriate reference frame. (Such a reference frame would necessarily make use of some physical degrees of freedom to uniquely label the particles.)

D. Private shared reference frames as cryptographic key

Two parties, Alice and Bob, are said to possess a private shared RF for some degree of freedom if their reference frames are perfectly correlated with each other, and are completely uncorrelated with any other party.

Such private shared RFs can be used as a novel kind of key for cryptography. To illustrate the general idea, consider the case where Alice and Bob share a private Cartesian frame.¹⁰ They can achieve some private classical communication as follows: Alice transmits to Bob an orientable physical system (e.g., a pencil or a gyroscope) after encoding her message into the relative orientation between this system and her local reference frame (for instance, by turning her bit string into a set of Euler angles). Bob can decrypt the message by measuring the relative orientation between this system and his local reference frame. Because an eavesdropper (Eve) does not have a reference frame correlated with theirs, she cannot infer any information about the message from the transmission.

We consider the quantum version of this example, where Alice sends spin-1/2 particles to Bob via a noiseless channel, as in the communication problem of Sec. III.A.2. Note that whereas in that problem Bob lacked the RF with respect to which the spins were prepared, here Bob shares the RF and it is Eve who lacks it. Thus the superoperator \mathcal{E}_N of Eq. (3.2) now describes the restriction that Eve faces by virtue of lacking the private shared RF. In Sec. III.A.2 we sought to determine how Alice could encode information in such a way that it remained accessible to someone who lacked her RF, whereas here we are interested in the opposite problem: how to encode information in such a way that it is inaccessible to someone who lacks her RF (but accessible to someone who has it). We follow the work of [Bartlett et al. \(2004a\)](#), to which the reader is directed for a more complete analysis.

A few points are worth noting before presenting the results. First, private communication using a private SRF is similar in some ways to private-key cryptography, specifically, the Vernam cipher (one-time pad). For example, the secret key in the Vernam cipher can be used only once to ensure perfect security. Similarly, for our communication schemes, only a single plain text (classical or quantum) can be encoded using a single private SRF. If the same private SRF is used to encode two plain texts, then the relation that holds between two cipher texts carries information about the plain texts, and because it is possible to learn about this relation without making use of the SRF, Eve can obtain this information. This is akin to the fact that in our example of the classical pencil or gyroscope Eve can measure the angular separation of the two pencils.

This analogy prompts us to raise and dismiss the possibility that a private SRF is equivalent, as a resource, to some amount of secret key or entanglement. It is true that a private SRF may, through public communication, yield a secret key. Conversely, as will be seen in Sec. V.J,

a secret key may, through public communication, yield a private SRF. Moreover, if, contrary to what has been assumed here and in Sec. V.J, the parties possess a public SRF, then a private SRF is equivalent to an unbounded amount of secret key (in practice, the size of the key is limited by the bounded size of physical systems that define the SRF or the bounded degree of correlation in the SRF). For instance, the parties can measure the Euler angles relating the private SRF to the public SRF and then express these in binary to obtain secret bits. Nonetheless—and this is the critical point—in the absence of either a public SRF or public communication of unspeakable information, there is no procedure for interconverting secret key and private SRF. Thus the two resources are not equivalent. Similarly, one can show that the resource of a private SRF is distinct from that of entanglement.

a. One qubit

Consider the transmission of a single qubit from Alice to Bob. As they share an RF, Bob represents states of this single qubit in the same way as Alice. On the other hand, Eve, who does not share Alice's RF, describes the state ρ as $\mathcal{E}_1(\rho) = \frac{1}{2}I$, as in Eq. (3.4). She consequently cannot correlate the outcomes of her measurements with Alice's preparations. It follows that using this single qubit and their private shared RF, Alice and Bob can privately communicate one logical qubit, and thus also one logical classical bit.

b. Two qubits: Decoherence-full subspaces

If multiple qubits are transmitted, it is possible for Eve to acquire some information about the preparation even without access to the private shared RF by performing relative measurements on the qubits. For two transmitted qubits in the state ρ , Eve's description is $\mathcal{E}_2(\rho) = p_{j=1}(\frac{1}{3}\Pi_{j=1}) + p_{j=0}\Pi_{j=0}$ as in Eq. (3.11). Despite not sharing the RF, Eve can still discriminate the singlet and triplet subspaces and thus acquire information about the preparation. Nonetheless, Alice can achieve some private quantum communication by encoding the state of a qutrit (a three-dimensional generalization of the qubit) into $\mathcal{H}_{j=1}$, the triplet subspace. Bob, sharing the private RF, can recover this qutrit with perfect fidelity. However, Eve identifies all such qutrit states with $\frac{1}{3}\Pi_{j=1}$, and therefore cannot infer anything about Alice's preparation.

The property of $\mathcal{H}_{j=1}$ that is key for this scheme is that the two-qubit superoperator \mathcal{E}_2 is completely depolarizing on it, as seen explicitly from Eq. (3.10), which we repeat: $\mathcal{E}_2 = (\mathcal{D}_{\mathcal{M}_{j=1}} \circ \mathcal{P}_{j=1}) + \mathcal{P}_{j=0}$. We define subspaces with this property to be decoherence-full subspaces, consistent with the terminology presented in Sec. II.C.

Now consider how many classical bits of information Alice can transmit privately to Bob. An obvious scheme is for her to encode a classical trit as three orthogonal states within the triplet subspace. However, this is not the most efficient scheme. Suppose instead that Alice encodes two classical bits as the four orthogonal states

¹⁰Although it is difficult to imagine how a Cartesian frame defined by fixed stars might be made private, it is clear that if the Cartesian frame is defined by a set of gyroscopes, privacy amounts to no other party having gyroscopes that are correlated with those of Alice and Bob.

$$|i\rangle = \frac{1}{2}|\psi^-\rangle + \frac{\sqrt{3}}{2}|\mathbf{n}_i\rangle|\mathbf{n}_i\rangle, \quad i = 1, \dots, 4, \quad (3.35)$$

where $|\psi^-\rangle$ is the singlet state and the $|\mathbf{n}_i\rangle|\mathbf{n}_i\rangle$ are four states in the triplet subspace with both spins pointed in the same direction, with the four directions forming a tetrahedron on the Bloch sphere; specifically,

$$|\mathbf{n}_1\rangle = |0\rangle, \quad (3.36)$$

$$|\mathbf{n}_2\rangle = \frac{i}{\sqrt{3}}(|0\rangle + \sqrt{2}|1\rangle), \quad (3.37)$$

$$|\mathbf{n}_3\rangle = \frac{-i}{\sqrt{3}}(|0\rangle + e^{2\pi i/3}\sqrt{2}|1\rangle), \quad (3.38)$$

$$|\mathbf{n}_4\rangle = \frac{i}{\sqrt{3}}(|0\rangle + e^{-2\pi i/3}\sqrt{2}|1\rangle), \quad (3.39)$$

as in [Massar and Popescu \(1995\)](#). It is straightforward to verify that

$$\mathcal{E}_2(|i\rangle\langle i|) = \frac{1}{4}I, \quad \forall i, \quad (3.40)$$

i.e., all four states are represented by Eve as the completely mixed state. As these four states are orthogonal, they are completely distinguishable by Bob and so provide an optimal private classical communication scheme.

c. Three qubits: Decoherence-full subsystems

Consider the case where Alice transmits three qubits to Bob. The Hilbert space $\mathcal{H}_{1/2}^{\otimes 3}$ and the superoperator \mathcal{E}_3 decompose into irreps as

$$(\mathcal{H}_{1/2}^{\otimes 3})_8 = \mathcal{H}_{j=3/2}_4 \oplus (\mathcal{M}_{j=1/2}_2 \otimes \mathcal{N}_{j=1/2}_2) \quad (3.41)$$

and

$$\mathcal{E}_3 = \mathcal{D}_{\mathcal{M}_{j=3/2}} \circ \mathcal{P}_{j=3/2} + (\mathcal{D}_{\mathcal{M}_{j=1/2}} \otimes \mathcal{I}_{\mathcal{N}_{j=1/2}}) \circ \mathcal{P}_{j=1/2}, \quad (3.42)$$

as in Eqs. (3.16) and (3.17). Clearly, the four-dimensional subspace $\mathcal{H}_{j=3/2}$ is a decoherence-full subspace. We also see that any state on $\mathcal{H}_{j=1/2}$ that is of the product form $\rho \otimes \sigma$ with respect to the factorization $\mathcal{H}_{j=1/2} = \mathcal{M}_{j=1/2} \otimes \mathcal{N}_{j=1/2}$ is mapped by \mathcal{E}_3 to the state $\frac{1}{2}I_{\mathcal{M}_{j=1/2}} \otimes \sigma$ [see also Eq. (3.18)]. Thus every state of the virtual subsystem $\mathcal{M}_{j=1/2}$ is mapped to the completely mixed state on that subsystem. Such a subsystem is an example of a decoherence-full subsystem.

Alice can therefore achieve private communication of two qubits using the decoherence-full subspace $\mathcal{H}_{j=3/2}$ or a single qubit using the decoherence-full subsystem $\mathcal{M}_{j=1/2}$. Note, however, that for greater numbers of transmitted qubits, the decoherence-full subsystems typically have greater dimensionality than the decoherence-full subspaces, and schemes that encode within them are necessary to achieve optimal efficiency, as discussed below.

For private classical communication, the question of optimal efficiency is much more complex. One scheme would be for Alice to encode two (classical) bits into four orthogonal states within the $j=3/2$ decoherence-full subspace. Alice can also encode two bits into four orthogonal maximally entangled states on the virtual tensor product $\mathcal{M}_{j=1/2} \otimes \mathcal{N}_{j=1/2}$, because the depolarization on $\mathcal{M}_{j=1/2}$ is sufficient to map all of these to $\frac{1}{2}I_{\mathcal{M}_{j=1/2}} \otimes \frac{1}{2}I_{\mathcal{N}_{j=1/2}}$, making them indistinguishable to Eve.

It turns out that the optimally efficient scheme for private classical communication uses both the $j=3/2$ and $1/2$ subspaces. Let $|j=3/2, \mu\rangle$, $\mu=1, \dots, 4$ be four orthogonal states on the $j=3/2$ subspace, and let $|j=1/2, \mu\rangle$, $\mu=1, \dots, 4$ be four maximally entangled states (as described above) on the $j=1/2$ subspace. Define the eight orthogonal states

$$|b, \mu\rangle = \frac{1}{\sqrt{2}}[|j=3/2, \mu\rangle + (-1)^b |j=1/2, \mu\rangle], \quad (3.43)$$

where $b=1, 2$ and $\mu=1, \dots, 4$. Alice can encode three bits into these eight states, which are completely distinguishable by Bob. It is easily shown that the decohering superoperator \mathcal{E}_3 maps all of these states to the completely mixed state on the total Hilbert space; thus these states are completely indistinguishable from Eve's perspective.

d. General results

In general, an optimally efficient private quantum communication scheme for N spin-1/2 systems is given by encoding into the largest decoherence-full subsystem for \mathcal{E}_N of Eq. (3.2). The largest is $\mathcal{M}_{j=N/2}$ and has dimension $N+1$. Thus, given a private Cartesian frame and the transmission of N qubits, Alice and Bob can privately communicate $\log_2(N+1)$ qubits.

The general results for private classical communication are much more complex, and beyond the scope of this review. (Observe the complexity of even the three-qubit example above.) Here we simply state the result, which is that the number of private classical bits that can be communicated using a private shared Cartesian frame and N qubits is $3 \log_2 N$ ([Bartlett et al., 2004a](#)).

These results show that, asymptotically, the private classical capacity ($3 \log_2 N$) is three times the private quantum capacity ($\log_2 N$). By relaxing the requirement of perfect privacy, it is possible to use the properties of random subspaces to nearly triple the private quantum capacity, almost closing the gap between the private classical and quantum capacities ([Bartlett et al., 2005](#)). Finally, we note in passing that bipartite entangled states of $2N$ spins, completely mixed on the total $J=0$ subspace, have been identified as a resource for private quantum and classical communication; it is illustrative to view such states as quantum private shared RFs ([Livine and Terno, 2006](#)).

IV. QUANTUM TREATMENT OF REFERENCE FRAMES

As we have seen in the previous two sections, the lack of a reference frame has the effect of inducing a superselection rule. We have explored examples of how the lack of a phase reference in quantum optics experiments leads to an Abelian SSR and how the lack of a Cartesian frame leads to a non-Abelian SSR.

However, some SSRs are typically viewed as being axiomatic; a canonical example is an SSR for electric charge, which forbids superpositions of eigenstates of different charge (Wick *et al.*, 1952, 1970; Strocchi and Wightman, 1974). In a classic paper, Aharonov and Susskind (1967) challenged the necessity of this SSR, and outlined a gedanken experiment for exhibiting a coherent superposition of charge eigenstates as an example of how this SSR can be obviated in practice. This gedanken experiment highlights the requirement of an appropriate reference frame in order to exhibit superpositions between eigenstates of superselected quantities, and as a result it can be argued that an SSR is simply a practical limitation due to the lack of such a reference frame. This point has been repeated by several authors (Mirman, 1969, 1970; Lubkin, 1970; Giulini, 2000a, 2000b).

In this section, we demonstrate that this result is general: Any SSR associated with a unitary representation of a compact group can be viewed as the lack of an appropriate reference frame, and can be overcome by using an appropriate quantum system to serve as a reference frame.

A. Relational descriptions of phase

1. Quantization of a phase reference

Suppose we have a system that transforms under $U(1)$ and where the associated eigenstates are denoted by $|n\rangle$. For concreteness, we imagine these to be eigenstates of photon number, or of number of bosonic atoms, in some mode. If there is no SSR for $U(1)$, then we can prepare states such as

$$|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, \quad (4.1)$$

$$|\psi_\pi\rangle = (|0\rangle - |1\rangle)/\sqrt{2}, \quad (4.2)$$

which differ only in their phases. We distinguish such a state, which has coherence between $|0\rangle$ and $|1\rangle$, from the incoherent mixture $I/2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ by measuring an ensemble of such systems in the basis $\{|\psi_0\rangle, |\psi_\pi\rangle\}$ and observing whether the outcome is random or not.

Now suppose instead that there is an SSR for $U(1)$ in force for this system. For optical systems, this corresponds as discussed above to the situation where one lacks the phase reference with which these states were prepared. If the states $|n\rangle$ are eigenstates of bosonic atom number (such as are used in describing Bose-Einstein condensates), then such an SSR is often assumed to be an axiomatic restriction [cf. Wick *et al.* (1952); Cirac *et al.* (1996); Leggett (2001); Wiseman and Vaccaro (2003)]. In either case, it becomes impossible to

prepare a coherent superposition of eigenstates of total number. Nonetheless, it is still possible to prepare a pair of systems in such a way that they have a well-defined relative phase (Nemoto and Braunstein, 2003). We consider the pair consisting of our original system, which we denote by S , and a new system, which we denote by R . Defining the states

$$|\chi_{0(\pi)}\rangle_R = (|n-1\rangle_R \pm |n\rangle_R)/\sqrt{2}, \quad (4.3)$$

on R (with $n \geq 1$), we may then define states on the pair with relative phases 0 and π , respectively,

$$|\Psi_0\rangle_{RS} = (|\chi_0\rangle_R |\psi_0\rangle_S - |\chi_\pi\rangle_R |\psi_\pi\rangle_S)/\sqrt{2}, \quad (4.4)$$

$$|\Psi_\pi\rangle_{RS} = (|\chi_0\rangle_R |\psi_\pi\rangle_S - |\chi_\pi\rangle_R |\psi_0\rangle_S)/\sqrt{2}. \quad (4.5)$$

Noting that these states can also be expressed as

$$|\Psi_0\rangle_{RS} = (|n\rangle_R |0\rangle_S + |n-1\rangle_R |1\rangle_S)/\sqrt{2}, \quad (4.6)$$

$$|\Psi_\pi\rangle_{RS} = (|n\rangle_R |0\rangle_S - |n-1\rangle_R |1\rangle_S)/\sqrt{2}, \quad (4.7)$$

it is clear that both of these states are eigenstates of total number with eigenvalue n and are therefore valid preparations under the SSR.

Moreover, within the eigenvalue n eigenspace, one can measure the basis $\{|\Psi_0\rangle, |\Psi_\pi\rangle\}$ in order to statistically distinguish states with a well-defined relative phase from those, like $\frac{1}{2}|n+1\rangle\langle n+1| \otimes |0\rangle\langle 0| + \frac{1}{2}|n\rangle\langle n| \otimes |1\rangle\langle 1|$, which do not have a well-defined relative phase. Clearly, this measurement is also valid within the constraints of the SSR.

In fact, for every preparation, operation, and measurement of the system that is not $U(1)$ invariant, one can find an equivalent preparation, operation, and measurement for the relation between the pair of systems that is $U(1)$ invariant. To do so, we simply use the map

$$\begin{aligned} |0\rangle &\rightarrow |n\rangle_R |0\rangle_S, \\ |1\rangle &\rightarrow |n-1\rangle_R |1\rangle_S, \end{aligned} \quad (4.8)$$

so that, in particular, we have

$$a|0\rangle + b|1\rangle \rightarrow a|n\rangle_R |0\rangle_S + b|n-1\rangle_R |1\rangle_S, \quad (4.9)$$

for $|a|^2 + |b|^2 = 1$.

It is straightforward to generalize the quantization map of Eq. (4.8) to the case of a system which may have more than one photon. If it has at most m_{\max} photons, we simply use the map

$$|m\rangle \rightarrow |n-m\rangle_R |m\rangle_S, \quad (4.10)$$

where we require that $n \geq m_{\max}$. In this case,

$$\sum_{m=0}^{m_{\max}} c_m |m\rangle \rightarrow \sum_{m=0}^{m_{\max}} c_m |n-m\rangle_R |m\rangle_S. \quad (4.11)$$

This extension of the Hilbert space corresponds physically to incorporating the phase reference into the quantum formalism. In other words, it describes the internalization or quantization of the reference frame. To see this, consider the following analogy with classical me-

chanics. Suppose a ball is bounced off of a wall. If we do not treat the wall as a dynamical entity, but rather as an external potential that appears in the equations of motion of the ball, then the solutions to the equations of motion are not translationally invariant. Specifically, if we take a given bouncing trajectory for the ball and translate it in such a way that the bounce no longer coincides with the location of the wall, we do not obtain another solution—the external potential breaks the translation invariance. However, if we internalize the wall, that is, treat its position as a dynamical degree of freedom, then we find that the equations of motion, and the solutions, will be invariant under translations of the entire system (consisting of the ball and the wall).

Similarly, when one writes down a state such as $|\psi_{0(\pi)}\rangle = (|0\rangle \pm |1\rangle) / \sqrt{2}$, the phase of this state is only defined relative to an external phase reference. We can view this external phase reference as a type of external potential, which provides the means for preparing states and performing operations (i.e., giving solutions to the quantum-mechanical equations of motion) that are not invariant under phase shifts. However, if we incorporate the phase reference as an internal system (and we do not compare our internal systems to any other external phase reference), then the only empirically meaningful states and operations are invariant under phase shifts of the entire system (including the internalized phase reference).

Whether one treats the wall in our classical example as an external potential or an internal dynamical system is a choice of the physicist. Similarly, one can treat a reference frame internally or externally; with either choice, one can obtain an empirically adequate description of the experiment (Bartlett, Rudolph, and Spekkens, 2006).

2. Dequantization of a phase reference

It is useful to consider the opposite problem to the one considered above, namely, given a description of an experiment wherein the phase reference is being treated internally, how does one obtain a description wherein it is treated externally? In our classical example of a ball bouncing off a wall, this involves finding the equations of motion for the relative position of the ball to the wall.

We would like to determine the quantum analog of this process. In the context of a quantum reference frame for spatial location, it is relatively straightforward. To externalize the reference frame, one defines a novel tensor product structure of the Hilbert space in terms of the commuting pair of observables $q_R - q_S$ and $p_R + p_S$, where q_R, p_R and q_S, p_S are the position and momentum operators for the reference frame and system, respectively.

The procedure is a bit more subtle in our U(1) example, but also involves identifying a novel tensor product structure of the Hilbert space. The original tensor product structure, corresponding to the reference frame and system division, will be denoted $\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_S$. The product states with respect to this structure $|n\rangle_R |m\rangle_S$ are

simultaneous eigenstates of \hat{N}_R , the number operator for R , and \hat{N}_S , the number operator for S with eigenvalues n and m , respectively. The operators \hat{N}_R and \hat{N}_S form a complete set of commuting operators for the Hilbert space \mathcal{H} .

By choosing a different complete set of commuting operators, we can define an alternate tensor product structure for the Hilbert space. Specifically, we choose \hat{N}_S , the number operator for S , and $\hat{N}_{\text{tot}} = \hat{N}_S + \hat{N}_R$, the total number operator. The state $|m\rangle_R |n\rangle_S$ is also a joint eigenstate of this pair, with eigenvalues m and $m+n$, respectively. Given that \hat{N}_S and \hat{N}_{tot} form a complete set of commuting observables, we may label an element of the basis $\{|n\rangle_R |m\rangle_S\}$ instead by the eigenvalues of \hat{N}_S and \hat{N}_{tot} , that is, $|n\rangle_R |m\rangle_S = |N_{\text{tot}} = m+n, N_S = m\rangle$.

Now, if it were the case that any pair of values, one drawn from the spectra of N_S and the other drawn from the spectra of N_{tot} , could be simultaneous eigenvalues of N_S and N_{tot} , then we could define a new tensor product structure by $|N_{\text{tot}} = l, N_S = m\rangle = |l\rangle \otimes |m\rangle$. However, any pair (l, m) with $m > l$ cannot be simultaneous eigenvalues. This problem can be resolved by restricting our attention to states $|n\rangle_R |m\rangle_S$ where the minimum value of n is larger than the maximum value of m . Recalling the physical significance of these eigenvalues, we see that this corresponds to assuming that the RF has more excitations than the system.

Assuming a system with at most m_{max} excitations, and a reference with a number of excitations that is at least m_{max} , we may focus upon the subspace $\mathcal{H}' = \text{span}\{|n\rangle_R |m\rangle_S, m=0, \dots, m_{\text{max}}, n \geq m_{\text{max}}\}$. It is then straightforward to introduce a tensor product structure on \mathcal{H}' as follows. We define an m_{max} -dimensional Hilbert space \mathcal{H}_{rel} with an orthonormal basis $|m\rangle_{\text{rel}}$ labeled by the eigenvalue m of \hat{N}_S . We call this the relational Hilbert space. We also define a Hilbert space \mathcal{H}_{gl} with an orthonormal basis $|l\rangle_{\text{gl}}$ labeled by the eigenvalue of \hat{N}_{tot} . We call this the global Hilbert space. We then have a vector space isomorphism

$$\mathcal{H}' \cong \mathcal{H}_{\text{gl}} \otimes \mathcal{H}_{\text{rel}}, \quad (4.12)$$

which is made by identifying

$$|l\rangle_{\text{gl}} |m\rangle_{\text{rel}} \equiv |N_{\text{tot}} = l, N_S = m\rangle, \quad (4.13)$$

for all $m \leq m_{\text{max}}$ and $l \geq m_{\text{max}}$.

We can therefore define a linear map from the subspace \mathcal{H}' of $\mathcal{H}_R \otimes \mathcal{H}_S$ to $\mathcal{H}_{\text{gl}} \otimes \mathcal{H}_{\text{rel}}$ in terms of their respective basis states as

$$|n\rangle_R |m\rangle_S \mapsto |m+n\rangle_{\text{gl}} |n\rangle_{\text{rel}}. \quad (4.14)$$

Under this map, we have

$$a|n+1\rangle_R |0\rangle_S + b|n\rangle_R |1\rangle_S \mapsto |n\rangle_{\text{gl}} \otimes (a|0\rangle_{\text{rel}} + b|1\rangle_{\text{rel}}). \quad (4.15)$$

Any U(1)-invariant state on $\mathcal{H}_R \otimes \mathcal{H}_S$ will lead to a state on $\mathcal{H}_{\text{rel}} \otimes \mathcal{H}_{\text{gl}}$ that commutes with \hat{N}_{tot} , i.e., the state will

be diagonal in the number basis of \mathcal{H}_{gl} . By discarding the global degrees of freedom and considering only the reduced density matrix on \mathcal{H}_{rel} , we are essentially moving to a paradigm of description wherein the RF is not treated within the quantum formalism. We call this procedure externalizing or dequantizing the reference frame. For instance, if we follow the map of Eq. (4.15) by a trace over \mathcal{H}_{gl} , we obtain the map

$$a|n+1\rangle_R|0\rangle_S + b|n\rangle_R|1\rangle_S \mapsto a|0\rangle_{\text{rel}} + b|1\rangle_{\text{rel}}, \quad (4.16)$$

which is the inverse of Eq. (4.9), the map describing the internalization or quantization of the phase reference.

3. Optical coherence controversy

This simple analysis of the quantum treatment of reference frames is useful for resolving a controversy concerning whether quantum coherences between photon-number eigenstates are fact or fiction (Molmer, 1997, 1998; Gea-Banacloche, 1998; Rudolph and Sanders, 2001a, 2001b; Nemoto and Braunstein, 2002, 2003, 2004; van Enk and Fuchs, 2002a, 2002b; Fujii, 2003; Sanders *et al.*, 2003; Spekkens and Sipe, 2003; Wiseman, 2003, 2004; Smolin, 2004; Bartlett, Rudolph, and Spekkens, 2006). It is standard practice in quantum optics to model the state of the electromagnetic field generated by a laser to be a coherent state, which is a coherent superposition of photon-number eigenstates. One justification that may be given for such an approach is that if one imagines the source of the radiation to be a classical oscillating dipole (which seems a reasonable assumption) then a simple calculation shows that the field is left in a coherent state. On the other hand, if one quantizes the dipole moments in the gain medium and assumes that these are initially in a thermal state (which must have zero expectation value of the dipole moment operator), and that the coupling between the gain medium and the radiation field conserves photon number (which again seem like reasonable assumptions), then the reduced density operator of the field is found to be in an incoherent mixture of photon-number eigenstates (Molmer, 1997). The fact that distinct states are obtained by the two analyses has led many researchers to conclude that the two descriptions are inconsistent and that one must be wrong.

To gain insight into this controversy, it is useful to consider the gain medium as a phase reference for the radiation field. Rather than considering this case in detail, we return to the example of the previous section, which provides a simplified version of the controversial phenomena. Recall that we also considered two distinct paradigms of description for a system S and a phase reference R . In the first description—the external- R paradigm—only S was treated quantum mechanically, so that the total Hilbert space was \mathcal{H}_S . In the second description—the internal- R paradigm—both S and R were treated quantum mechanically, so that the total Hilbert space was $\mathcal{H}_S \otimes \mathcal{H}_R$. Moreover, the state on \mathcal{H}_S is different in the two cases. For instance, if the state of S in the external- R paradigm is $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ of Eq.

(4.1), after internalizing R , the joint state is $|\Psi_0\rangle$ of Eq. (4.6), and the state on \mathcal{H}_S is $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$.

Thus just as the classical and quantum treatments of the gain medium in the generation of laser light led to distinct state ascriptions for the radiation field, our classical and quantum treatments of the phase reference R lead to distinct state ascriptions for the system S . It is a mistake, however, to conclude that the two descriptions are inconsistent. As we saw in the previous subsection, both descriptions are valid. To resolve the confusion explicitly, we elaborate on the physical interpretation of the states in these Hilbert space.

In the external- R paradigm, we saw that the phase of the quantum state of S (that is, the phase of the ratio of amplitudes of $|0\rangle$ and $|1\rangle$) can only be given meaning relative to the external phase reference R . So it is clear that the state on \mathcal{H}_S describes not just the intrinsic properties of S , but some of its extrinsic properties as well, specifically, its relation to R .

In the internal- R paradigm, any phase of the quantum state of S can also only be given meaning relative to an external phase reference, but R is no longer an external RF, and any phase reference that is still treated externally, say R' , has been assumed not to be correlated with S . Thus we expect S to not have a well-defined phase in this case. The point is that in the internal- R paradigm \mathcal{H}_S also describes extrinsic properties of S , but in this case it is the relation of S to R' , rather than R .

Thus the fact that the quantum states on \mathcal{H}_S are distinct in the two paradigms is not an inconsistency because despite the common notation, they describe different degrees of freedom: one describes the relation of S to R and the other the relation of S to R' .

Moreover, if one wishes to recover the quantum state describing the relation of S to R in the internal- R paradigm, it is clear that one should not look at the quantum state on \mathcal{H}_S because this amounts to tracing over \mathcal{H}_R which corresponds to ignoring R , and one clearly cannot ignore a system when one seeks to find the relation between it and another. But where then is information about the relation between S and R found in $\mathcal{H}_S \otimes \mathcal{H}_R$? The answer is that it is found in a virtual subsystem, specifically in the Hilbert space \mathcal{H}_{rel} . The resolution of the optical coherence controversy is achieved in an analogous manner.

The key insight for resolving these sorts of confusions is that quantum states of systems in an external RF paradigm do not simply describe its intrinsic properties, but also the relation of the system to the external RF; further discussion on this issue can be found in Wiseman (2004) and Bartlett, Rudolph, and Spekkens (2006).

4. Generalization to composite systems

The generalization of the quantization procedure in Sec. IV.A.1 to the case of multiple systems (i.e., modes) is not so straightforward. The problem is that if we wish to describe a pair of systems S_1 and S_2 relative to an RF R , then the reduced density operators on RS_1 and on

RS_2 cannot both be pure entangled states (Coffman *et al.*, 2000). This fact is known as the monogamy of pure entanglement. As a result, the quantum description of the RF and system that was presented above is only adequate if the system in question is the only one that will ever be compared to the RF. However, the most general notion of an RF is something with respect to which the orientation of many systems can be defined. We consider such a generalization presently.

Note that if we demand that there be no limit on the number of systems that can be correlated with the RF, and that the degree of correlation with the RF be equal for all the systems, then the reduced density operator on RS for an arbitrary system S must be unentangled, that is, a separable state. At first glance, this might seem problematic, because it might seem that the entanglement in Eqs. (4.9) and (4.11) is critical for the RF quantization procedure to work. It is true that if we restrict ourselves to a subspace of \mathcal{H}_R of the same dimension as the system of interest, as we did in Eqs. (4.9) and (4.11), then we cannot obtain a faithful representation. However, by allowing ourselves to make use of a larger subspace, we can obtain a good representation, and in the limit of arbitrarily large dimension, we obtain a perfect representation, as before. Defining the unnormalized states

$$|\chi_\phi\rangle_R = \sum_{n=0}^{\infty} e^{i\phi n} |n\rangle_R, \quad (4.17)$$

which have well-defined phase, the quantization map takes the form

$$\rho \rightarrow \int d\phi |\chi_\phi\rangle_R \langle \chi_\phi| \otimes U(\phi) \rho U^\dagger(\phi). \quad (4.18)$$

We must demonstrate that this is a faithful representation of the system that satisfies the $U(1)$ -SSR. Rather than doing so for the phase reference case individually, we proceed directly to present the generalization of this quantization map to an arbitrary group, and prove that the latter has the properties we desire.

B. Quantization of a general reference frame

Consider the quantum description of a system with Hilbert space \mathcal{H}_S , in the case where one possesses an external reference frame for a degree of freedom associated with the group G . The Born rule predicts that for a preparation associated with density operator ρ followed by a transformation associated with operation \mathcal{E} and finally a measurement associated with the POVM $\{E_k\}_k$ the probability of the measurement outcome k is $\text{Tr}[\mathcal{E}(\rho)E_k]$.

Now consider a quantum description for the same system, but where an SSR for G is in force. This SSR implies a restriction on the states, transformations, and measurements. However, as we now demonstrate it is possible to append this system with another quantum system R which serves as a quantum reference frame in

such a way that, although an SSR for G applies to the entire composite system RS , the RF allows us to effectively describe the system as if the SSR did not exist (Kitaev *et al.*, 2004).

Our aim is to map the elements of the old representation (of preparations, operations, measurements) to elements in a new, G -invariant representation on $\mathcal{H}_R \otimes \mathcal{H}_S$. Thus we seek a map

$$\rho \rightarrow \rho^{\text{inv}}, \quad (4.19)$$

$$\{E_k\}_k \rightarrow \{E_k^{\text{inv}}\}_k, \quad (4.20)$$

$$\mathcal{E} \rightarrow \mathcal{E}^{\text{inv}} \quad (4.21)$$

such that $\rho^{\text{inv}} = \mathcal{G}(\rho^{\text{inv}})$ and $E_k^{\text{inv}} = \mathcal{G}(E_k^{\text{inv}})$ are both G -invariant operators on $\mathcal{H}_R \otimes \mathcal{H}_S$, and \mathcal{E}^{inv} is a G -invariant superoperator on $\mathcal{B}(\mathcal{H}_R) \otimes \mathcal{B}(\mathcal{H}_S)$ (this implies that when acting on a G -invariant operator \tilde{A} , \mathcal{E}^{inv} satisfies $(\mathcal{G} \circ \mathcal{E}^{\text{inv}} \circ \mathcal{G})(\tilde{A}) = \mathcal{E}^{\text{inv}}(\tilde{A})$).

In addition, we would like this map to preserve the statistical predictions of the old representation; if all statistics of the Born rule can be reproduced in this new representation, then it is equivalent to the old one. Specifically, we want this map to be such that

$$\text{Tr}_{RS}[\mathcal{E}^{\text{inv}}(\rho^{\text{inv}})E_k^{\text{inv}}] = \text{Tr}_S[\mathcal{E}(\rho)E_k], \quad (4.22)$$

for all states ρ , operations \mathcal{E} , and measurements $\{E_k\}_k$. Such a map does exist (assuming we can allow d_R , the dimension of \mathcal{H}_R , to be arbitrarily large), as we now demonstrate.

The quantum system R that will constitute the RF for G must clearly transform under G in some nontrivial manner. Thus \mathcal{H}_R must carry a representation of G , denoted U_R , which in general will be reducible. In order for R to serve as a complete quantum RF for G , the state $|g\rangle$ on \mathcal{H}_R corresponding to the configuration $g \in G$ must not possess a nontrivial invariant subgroup, i.e., if $U_R(g')|g\rangle \propto |g\rangle$ then g' must be the identity. It follows that the states of R transform as

$$U_R(g')|g\rangle = |g'g\rangle, \quad \forall g, g' \in G. \quad (4.23)$$

For this quantum system to function as a perfect reference frame for G , the different configurations $|g\rangle$ must all be distinguishable. Thus we require that states for different configurations are orthogonal

$$\langle g|g'\rangle = \delta(g^{-1}g'), \quad (4.24)$$

where $\delta(g)$ is the delta function on G defined by $\int dg \delta(g)f(g) = f(e)$ for any continuous function f of G , where e is the identity element in G . The above requirements are the defining properties of the left regular representation of G . In the case of a Lie group, the dimensionality of \mathcal{H}_R must be infinite for such states to exist.

We refer to such an infinite-dimensional quantum RF as *unbounded*.¹¹

We now present the map from operators on \mathcal{H}_S to G -invariant operators on $\mathcal{H}_R \otimes \mathcal{H}_S$:

$$\mathcal{S}: A \mapsto \int_G dg |g\rangle\langle g| \otimes U_S(g) A U_S^\dagger(g), \quad (4.25)$$

where U_S is the representation of G on the system.

Using this map \mathcal{S} , we define the invariant versions of density operators, elements of POVMs, and Kraus operators, respectively, as

$$\rho^{\text{inv}} = \frac{1}{d_R} \mathcal{S}(\rho), \quad (4.26)$$

$$E^{\text{inv}} = \mathcal{S}(E), \quad (4.27)$$

$$K^{\text{inv}} = \mathcal{S}(K), \quad (4.28)$$

where d_R is the dimensionality of the Hilbert space $\mathcal{H}'_R \equiv \text{span}\{|g\rangle, g \in G\}$ spanned by the orbit of the RF states, which may be a subspace of \mathcal{H}_R . (One can easily check that $\text{Tr}_{RS}[\rho^{\text{inv}}] = 1$ if $\text{Tr}_S[\rho] = 1$.)

The following are properties of the \mathcal{S} map: (i) $\mathcal{S}(A)$ is G -invariant; (ii) $\mathcal{S}(A+B) = \mathcal{S}(A) + \mathcal{S}(B)$ and $\mathcal{S}(AB) = \mathcal{S}(A)\mathcal{S}(B)$, so the algebra of operators is reproduced.

The G -invariance of $\mathcal{S}(A)$ follows from

$$\begin{aligned} & [U_R(g') \otimes U_S(g')] \mathcal{S}(A) [U_R^\dagger(g') \otimes U_S^\dagger(g')] \\ &= \int_G dg |g'g\rangle\langle g'g| \otimes U_S(g'g) A U_S^\dagger(g'g) = \mathcal{S}(A), \end{aligned} \quad (4.29)$$

where the final equality follows from the invariance of the Haar measure dg . To prove property (ii), we note that \mathcal{S} is linear by definition, and that

$$\begin{aligned} \mathcal{S}(A)\mathcal{S}(B) &= \int dg dg' |g\rangle\langle g| |g'\rangle\langle g'| \\ &\quad \otimes U_S(g) A U_S^\dagger(g) U_S(g') B U_S^\dagger(g') \\ &= \int dg |g\rangle\langle g| \otimes U_S(g) A B U_S^\dagger(g) = \mathcal{S}(AB), \end{aligned} \quad (4.30)$$

where we have used Eq. (4.24).

From these properties, one can show that if ρ is a density operator, then so is ρ^{inv} , if $\{E_k\}$ is a POVM, then so is $\{E_k^{\text{inv}}\}$, and that if \mathcal{E} is a superoperator with Kraus operators $\{K_\mu\}$ satisfying $\sum_\mu K_\mu^\dagger K_\mu = E$, then the superoperator \mathcal{E}^{inv} having Kraus operators $\{K_\mu^{\text{inv}}\}$ satisfies $\sum_\mu (K_\mu^{\text{inv}})^\dagger K_\mu^{\text{inv}} = E^{\text{inv}}$. Most importantly, one can prove that the new representation satisfies Eq. (4.22) and therefore reproduces the quantum statistics:

$$\begin{aligned} \text{Tr}_{RS}[\rho^{\text{inv}} E_k^{\text{inv}}] &= d_R^{-1} \text{Tr}_{RS}[\mathcal{S}(\rho) \mathcal{S}(E_k)] \\ &= d_R^{-1} \text{Tr}_{RS}[\mathcal{S}(\rho E_k)] \\ &= d_R^{-1} \text{Tr}_{RS} \left[\int_G dg |g\rangle\langle g| \right. \\ &\quad \left. \otimes U_S(g) \rho E_k U_S^\dagger(g) \right] \\ &= d_R^{-1} \text{Tr}_R \left[\int_G dg |g\rangle\langle g| \right] \text{Tr}_S[\rho E_k] \\ &= \text{Tr}_S[\rho E_k]. \end{aligned} \quad (4.31)$$

The case where there is a nontrivial operation \mathcal{E} can be dealt with similarly.

This is a remarkable result. It proves that superselection rules cannot provide any fundamental restrictions on quantum theory. This has particular implications for quantum cryptography as we discuss below. It also proves that all superselection rules associated with unitary representations of compact groups result from a lack of an appropriate reference frame, because, as we have shown, including an unbounded quantum reference frame reproduces a quantum theory that is equivalent to one in which the superselection rule does not apply.

C. Are certain superselection rules fundamental?

We now return to the question, introduced at the beginning of this section, of whether certain superselection rules are more fundamental than others. That is, are certain SSRs axiomatic, as opposed to those which arise in practice when there is not an appropriate RF? This issue bears on several controversies that are the counterparts of the optical coherence controversy in other contexts. It has arisen in the context of coherence between charge eigenstates in superconductivity (Haag, 1962; Kershaw and Woo, 1974; Anderson, 1986) and of coherence between atom number eigenstates in Bose-Einstein condensation (Hoston and You, 1996; Javanainen and Yoo, 1996; Castin and Dalibard, 1997; Yoo *et al.*, 1997; Leggett, 2001). Here, however, the intuition for the coherences being fiction is based on the notion that the superselection rule for charge and for baryon number are axiomatic, so that any quantum state that violates this SSR does not represent reality.

To make the discussion definite, let us compare on the one hand quantities such as charge and baryon number, for which axiomatic SSRs are conventionally assumed to apply, and on the other hand quantities such as linear momentum, angular momentum, and photon number, for which SSRs are generally not assumed to apply. We consider whether our conclusion, that it is possible to effectively lift a superselection rule, should apply to both of these equally.

The example of the phase reference provided in Sec. IV.A applies equally well in the case of atom number (and thus baryon number) as it does to the case of photon number. In both cases, one can certainly create well-

¹¹For finite groups, one need only assume that $\langle g|g'\rangle = \delta_{g,g'}$, where $\delta_{g,g'}$ is the Kronecker delta.

defined relative phases between a pair of systems. Moreover, the reasons for interpreting the larger of the two systems as a reference frame for the other are just as valid in the case of atom number as they are in the case of photon number. Finally, in both cases one can recover a description of the relational degree of freedom, wherein one effectively has lifted the SSR.

Given the generalization to non-Abelian groups, provided in Sec. IV.B, it would appear that all such superselection rules may be lifted in practice. Of course, the technical challenge in doing so is to build a reference frame for the degree of freedom in question. Admittedly, it may be more difficult to construct good reference frames for some degrees of freedom, but there is nothing in principle preventing their construction. For instance, to lift the superselection rule associated with charge, one must simply have a large reference system with respect to which one can coherently exchange charge, as argued by [Aharonov and Susskind \(1967\)](#). As another example, the experimental realization of Bose-Einstein condensation in alkali atoms provided a reference frame for the phase that is conjugate to atom number ([Dowling et al., 2006](#)). We see no obstacle in principle to lifting more general sorts of superselection rules as well.

What sets the two categories apart in practice seems to be the fact that some reference frames, such as those for spatial location or angular position, are ubiquitous, whereas others, such as a frame for the quantity conjugate to charge, tend not to arise through natural causes and are difficult to prepare and maintain. But this may be only a practical and not a fundamental difference.

Another motivation might be given for treating the two categories differently, specifically, that a superselection rule for linear momentum would seem to imply that objects could not be localized in space, and this one might think would be contrary to what is observed. However, all that is ever observed empirically is the localization of systems relative to other systems, and this is consistent with a superselection rule for total linear momentum. If one seeks to describe the entire universe quantum mechanically, as is typically done in quantum cosmology and some approaches to quantum gravity, then it is natural to assume SSRs for all global transformations, so that there is no distinction between charge and linear momentum. One can reach this conclusion by noting that all physical systems that could serve as RFs have been quantized. Alternatively, one can appeal to one of the central lessons of general relativity: that all observable quantities ought to be relational.

D. Superselection rules and quantum cryptography

Information-theoretic security is a form of security that does not rely on assumptions about the computational capabilities of one's adversary. The appeal of quantum cryptography is that it offers protocols achieving this sort of security where classical protocols fail. Quantum key distribution is the primary example of a task for which this is the case. On the other hand, there

exist cryptographic tasks, such as bit commitment, for which it has been shown that even quantum protocols cannot achieve information-theoretic security. The possibility of quantum key distribution arises ultimately from a restriction imposed by the laws of quantum mechanics on would-be eavesdroppers—namely, that quantum information cannot be cloned. By definition, SSRs also impose restrictions on the accessible quantum states and operations. For instance, an SSR for charge forbids the creation of superpositions of eigenstates of differing total charge. It is conceivable therefore, as first suggested by [Popescu \(2002\)](#), that SSRs could place restrictions on would-be cheaters and thereby achieve greater security for some tasks (for instance, unconditional security for bit commitment) ([Mayers, 2002](#); [Verstraete and Cirac, 2003](#); [DiVincenzo et al., 2004](#); [Kitaev et al., 2004](#)).

To motivate the intuition that SSRs might improve the security of quantum protocols, we consider the case of a partially binding and partially concealing bit commitment protocol ([Spekkens and Rudolph, 2001](#)) in the presence of a superselection rule for $SO(3)$. Alice prepares two qubits in either the singlet state $|\psi^-\rangle$, which has total spin 0, or the triplet state $|11\rangle$, which has total spin 1, according to whether she wants to commit a bit $b=0$ or 1, respectively. She sends Bob one of the two qubits as a token of her commitment. Bob cannot distinguish the reduced states $I/2$ and $|1\rangle\langle 1|$ with certainty and so the protocol is partially concealing. At a later stage, she sends him the second qubit, at which point Bob checks her honesty by performing a projective measurement to discriminate $|\psi^-\rangle$ from $|11\rangle$. There is no cheating strategy that allows Alice to unveil an arbitrary bit value, so the protocol is partially binding. Clearly each step in the honest protocol respects the SSR. However, it is quite plausible, at first sight, that an optimal cheating strategy for Alice will not respect the SSR—either because she must prepare a state which is a superposition of two different angular momenta, such as $(|\psi^-\rangle + |11\rangle)/\sqrt{2}$, or because prior to sending the second qubit to Bob she must apply to it some local operation that violates the SSR. If all of Alice's optimal cheating strategies required SSR violation, then the degree of bindingness against Alice and thus the security of the protocol would be greater by virtue of the SSR.

Despite the plausibility of this notion, it turns out that SSRs do not, in general, offer the possibility of cryptographic protocols with greater security ([Kitaev et al., 2004](#)). This result can be proven using the general framework of Sec. IV.B. We begin by demonstrating this for the case of arbitrary two-party cryptographic protocols. Such protocols can be formulated as follows. Alice and Bob each hold a local system in their laboratories, called A and B , respectively, and exchange a message system M back and forth. At the outset, they share a product state $\rho_A \otimes \rho_M \otimes \rho_B$ and in each round of the protocol, one of the parties applies a joint operation on their local systems and the message system and then sends the message system to the other party. At the end, both parties perform a measurement on their local system.

Security in this context is a restriction on the degree to which a cheating Alice can influence the probability distribution over the outcomes of the final measurement of an honest Bob, and a similar restriction with the roles of Alice and Bob reversed. (No restrictions are guaranteed for the case where both parties cheat.) We consider the case of a cheating Alice here.

Because we may include any ancillas used by Alice and Bob in the local systems A and B , we can assume that all operations are unitary. In the honest protocol, the first operation implemented by Alice is V_{A_1} , the first implemented by Bob is V_{B_1} , the second by Alice is V_{A_2} , and so forth. We denote the POVM associated with Bob's final measurement by $\{E_{B,k}\}$ where k labels the possible outcomes. The probability of outcome k is

$$p_B(k) = \text{Tr}[E_{B,k}V(\rho_A \otimes \rho_M \otimes \rho_B)V^\dagger], \quad (4.32)$$

where

$$V = V_{B_n}V_{A_n} \cdots V_{B_2}V_{A_2}V_{B_1}V_{A_1}. \quad (4.33)$$

Suppose that the honest protocol respects the SSR, and that the SSR is associated with a group G . In this case, all states, unitaries, and POVM elements described above are G -invariant operators. We now show that a cheating strategy that violates the SSR can always be simulated by a cheating strategy that respects the SSR, and consequently a cheater that faces an SSR does not suffer any disadvantage in cheating ability compared to one who does not.

Suppose that Alice's optimal SSR-violating cheating strategy is one wherein she replaces each G -invariant operation V_{A_j} with an operation V'_{A_j} that need not be G -invariant. She thereby can cause the probability of outcome k to be $p'_B(k) = \text{Tr}[E_{B,k}V'(\rho_A \otimes \rho_M \otimes \rho_B)V'^\dagger]$ where $V' = V_{B_n}V'_{A_n} \cdots V_{B_2}V'_{A_2}V_{B_1}V'_{A_1}$. We now demonstrate that there is an SSR-respecting cheating strategy that also leads to $p'_B(k)$. The trick is to use the construction of Sec. IV.B. Alice simply extends her local system A to RA , where R is a system that will play the role of a local reference frame. She replaces the G -noninvariant operation V'_{A_j} , which acts nontrivially on AM , with $\$(V'_{A_j})$, where $\$$ is the map defined in Eq. (4.25). $\$(V'_{A_j})$ is a G -invariant unitary operator that acts nontrivially on RAM . Bob's operations must be trivial on R , so that we can write these as $I_R \otimes V_{B_j}$. It is useful to note, however, that because the V_{B_j} are G -invariant, it follows that $I_R \otimes V_{B_j} = \$(V_{B_j})$. Moreover, given that $\$$ preserves the algebra of operators [property (ii) of the $\$$ map], we have $\$(V_{B_n})\$(V'_{A_n}) \cdots \$(V_{B_1})\$(V'_{A_1}) = \$(V_{B_n}V'_{A_n} \cdots V_{B_1}V'_{A_1}) = \(V') . The initial state and Bob's final measurement are also trivial on R and G -invariant. It follows that $d_R^{-1}I_R \otimes \rho_A \otimes \rho_M \otimes \rho_B = d_R^{-1}\$(\rho_A \otimes \rho_M \otimes \rho_B)$ and $I_R \otimes E_{B,k} = \$(E_{B,k})$.

Thus the probability of outcome k in Alice's SSR-respecting cheating strategy is

$$\begin{aligned} & d_R^{-1} \text{Tr}[\$(E_{B,k})\$(V')\$(\rho_A \otimes \rho_M \otimes \rho_B)\$(V'^\dagger)] \\ &= \text{Tr}(E_{B,k}V'\rho_A \otimes \rho_M \otimes \rho_B V'^\dagger) = p'_B(k), \end{aligned} \quad (4.34)$$

where we have used Eq. (4.31). Thus any probability distribution achieved by a cheating strategy that violates the SSR can also be achieved by one that respects it.

It is straightforward to generalize this result to the case of an n -party protocol with k cheating parties. We begin with the case of a pair of cheating parties (Alice and David). If their SSR-violating cheating strategies consist of unitaries V'_{A_j} and V'_{D_j} , then by the same reasoning as applied above they can achieve an equivalent degree of success using SSR-respecting cheating strategies consisting of unitaries $\$(V'_{A_j})$ and $\$(V'_{D_j})$ which are nontrivial on RA and RD , respectively. Because only one of these parties is ever implementing an operation at a given time, they can achieve this strategy by passing the RF R back and forth between them. Moreover, even if Alice and David are prevented from implementing such transmissions during the protocol, there is a resource they may share prior to the protocol, namely, a shared RF, which allows them to do just as well. Suppose their shared RF is constituted of a pair of systems, R_1 and R_2 , in the state

$$\rho_{R_1 R_2} = \int dg |g\rangle_{R_1} \langle g| \otimes |g\rangle_{R_2} \langle g|. \quad (4.35)$$

We show that by using R_2 alone, David can achieve the same operations on MD as could be achieved if Alice had passed him R_1 . We define $\$_1$ and $\$_2$ as the generalizations of $\$$ for $R=R_1$ and $R=R_2$, respectively. If Alice had passed David a copy of R_1 , he could replace V'_{D_j} by the operation $\$_1(V'_{D_j})$. But given that

$$\begin{aligned} & \$_1(V'_{D_j})\rho_{R_1 R_2} \otimes \rho_{MD} \$_1(V'_{D_j}^\dagger) \\ &= \int dg |g\rangle_{R_1} \langle g| \otimes |g\rangle_{R_2} \langle g| \\ & \quad \otimes [U(g)V_{D_j}U^\dagger(g)]\rho_{MD}[U(g)V_{D_j}^\dagger U^\dagger(g)] \\ &= \$_2(V'_{D_j})\rho_{R_1 R_2} \otimes \rho_{MD} \$_2(V'_{D_j}^\dagger), \end{aligned} \quad (4.36)$$

it follows that David achieves the same effect by replacing V'_{D_j} by the operation $\$_2(V'_{D_j})$, which is something that he can achieve locally. The generalization of this argument to an arbitrary number of cheating parties is straightforward.

It should be noted that for Lie groups, the states we have been considering are, strictly speaking, not normalizable. However, one can introduce a sequence of normalizable approximations to these states, parametrized by an integer N , corresponding to RFs of bounded size, such that the results described here are reproduced in the limit $N \rightarrow \infty$, that is, the limit of an unbounded RF. See Kitaev *et al.* (2004) for details.

Superselection rules that are not associated with a compact symmetry group have also been considered, for instance, the superselection rule for univalence which denies the possibility of a coherent superposition of bo-

son and fermion (Doplicher and Roberts, 1990) and superselection rules that can arise in two-dimensional systems that admit non-Abelian anyons. Using different methods, it has been shown that such SSRs also fail to yield any advantages for two-party cryptography (Kitaev *et al.*, 2004), but the question remains open for multi-party protocols.

V. ALIGNING REFERENCE FRAMES

Separated parties often require the use of a shared reference frame. For instance, they may require their clocks to be synchronized or their Cartesian frames to be aligned. Furthermore, although it was shown in Sec. III that lacking a reference frame does not prevent one from achieving information-theoretic tasks such as communication, cryptography, and computation, this restriction can decrease the (nonasymptotic) efficiency with which they can be achieved and often requires more sophisticated encodings. Thus separated parties might opt to initially devote their communication resources to setting up a shared reference frame and thereafter use a standard encoding, rather than perpetually circumventing the lack of such an RF with a relational encoding.

We refer to the process by which observers correlate their local reference frames, that is, by which they refine their knowledge of the relation between them, as reference frame alignment. In order to do so, the parties must exchange systems with the relevant degrees of freedom, which serve as finite samples of the sender's local RF and can be compared to the receiver's local RF to obtain some information about the relative orientation of the two frames. For example, through the exchange of spin-1/2 particles, Alice and Bob can align their local Cartesian frames. Exchanging quantum states of an optical mode allows them to align their phase references.

We discussed in Sec. IV how a quantum system of unbounded size can play the same role as a classical reference frame. By the transmission of such a system, one can achieve perfect alignment of separated classical reference frames. However, one is often restricted to sending systems of bounded size, either to economize on communication resources or because of the impracticality of encodings that require a joint preparation of too many systems. It is therefore of great interest to determine the fundamental quantum limits on the alignment precision that can be achieved for given communication resources. This is the question we address in this section.

It should be noted that if the communication resources are bounded, the end result of an alignment scheme is partial correlation between the local reference frames. The operational consequence of not having complete correlation is that the parties must contend with the decoherence that arises from the weighted G -twirling operation discussed in Sec. II.C. As the imprecision in this alignment approaches zero, the weighted G -twirling operation approaches the identity map. We are concerned here with schemes that minimize this imprecision.

Consider an alignment scheme for some form of reference frame, which makes use of a number N of transmitted quantum systems. The expected error in alignment, measured by the variance, can be theoretically determined as a function of N . The problem has close connections with the field of quantum parameter estimation (Holevo, 1982) and quantum metrology [see Giovannetti, Lloyd, and Maccone (2004)], and a general feature of this behavior is closely related to well-studied results in phase estimation. Specifically, if the N quantum systems are used independently (i.e., entangled signal states are not used, and measurements on individual systems are independent) one can only achieve an error, quantified by the variance, that scales as $1/N$. This result, which is a consequence of the central limit theorem, is commonly known as the standard quantum limit. In contrast, strategies which make use of entanglement between the N systems, as well as joint measurements, can achieve an error (variance) that scales as $1/N^2$. This result, commonly known as the Heisenberg limit, represents the fundamental limit to the scaling of accuracy as allowed by the laws of quantum physics (Giovannetti *et al.*, 2006).

We begin with a discussion of a simple example to provide some intuition about what sorts of states are optimal for the alignment problem. Heuristically, they are states that, when mixed over the action of the group, have significant support on the largest possible dimensionality of Hilbert space, thereby making them as distinguishable as possible. This intuition can be made rigorous in the context of a simple figure of merit: the maximum likelihood of a correct guess. After introducing a more useful figure of merit, the fidelity, we describe in detail strategies for the alignment of phase references, spatial directions, and Cartesian frames, and demonstrate how the Heisenberg limit can be achieved. We also overview results on the alignment problem for a few other sorts of reference frames. For alternate overviews of techniques for aligning directions and Cartesian frames, see Peres and Scudo (2002b) and Bagan and Muñoz-Tapia (2006).

A. Example: Sending a direction with two spins

Suppose Alice and Bob have uncorrelated Cartesian frames and they wish to align their \mathbf{z} axes by Alice transmitting a pair of spin-1/2 particles to Bob. What state of these two spins should Alice prepare, and what measurement should Bob perform in order to optimize their expected success in this task?

A seemingly reasonable strategy would be for Alice to send parallel spins aligned with her \mathbf{z} axis. Assuming Alice's \mathbf{z} axis points in the \mathbf{n} direction relative to Bob's frame, this strategy corresponds to sending Bob the state $|\mathbf{n}\rangle|\mathbf{n}\rangle$, where $(\hat{\mathbf{S}} \cdot \mathbf{n})|\mathbf{n}\rangle = (\hbar/2)|\mathbf{n}\rangle$, relative to his local frame. Bob's task is now one of state estimation—to optimally estimate the pure state $|\mathbf{n}\rangle$ given two copies (Massar and Popescu, 1995). We note that the set of states from which Bob must measure are all on the

three-dimensional symmetric $j=1$ subspace $\mathcal{H}_{j=1}$ of two spins; he can thus restrict his measurement to a POVM on this Hilbert space. We consider the case where Bob performs a covariant measurement, i.e., a continuously parametrized POVM of the form

$$\{E_\Omega = R(\Omega) \otimes^2 E_0 R^\dagger(\Omega) \otimes^2, \quad \Omega \in \text{SU}(2)\}, \quad (5.1)$$

where $E_0 = |e\rangle\langle e|$ is a positive rank-1 operator. (Any POVM of higher rank can be simulated by a rank-1 POVM followed by classical post-processing of the result.) To form a POVM, the vector $|e\rangle$ must satisfy the normalization condition

$$\int d\Omega E_\Omega = I_{j=1}, \quad (5.2)$$

where $I_{j=1}$ is the identity operator on $\mathcal{H}_{j=1}$. It is straightforward to show [for example, by using Eq. (3.10)] that this condition completely constrains the form of the POVM, i.e., it requires that $|e\rangle = \sqrt{3}|00\rangle$ up to an arbitrary choice of single-spin basis $\{|0\rangle, |1\rangle\}$. Let Bob choose $|0\rangle$ to be aligned with his $+z$ direction.

If Alice sends two spins in the state $|\mathbf{n}\rangle|\mathbf{n}\rangle$, and Bob performs the measurement (5.1), then the probability of Bob obtaining the measurement outcome Ω is given by the Born rule

$$p(\Omega|\mathbf{n}) = \text{Tr}[E_\Omega|\mathbf{n}\rangle|\mathbf{n}\rangle\langle\mathbf{n}|\langle\mathbf{n}|] \quad (5.3)$$

$$= 3 \cos^4(\beta/2), \quad (5.4)$$

where $\beta = \cos^{-1}(\mathbf{n} \cdot \Omega\mathbf{z})$ is the angle between \mathbf{n} and $\Omega\mathbf{z}$. If Bob obtains the measurement outcome Ω , then his best guess as to the direction \mathbf{n} is $\mathbf{n}_g = \Omega\mathbf{z}$. A natural way with which to quantify the quality of Bob's guess is to use the fidelity $(1 + \mathbf{n} \cdot \mathbf{n}_g)/2 = \cos^2(\beta/2)$, which gives a value of 1 if he guesses correctly ($\mathbf{n}_g = \mathbf{n}$), a value of 0 if he guesses the opposite direction ($\mathbf{n}_g = -\mathbf{n}$), and which decreases monotonically between these two limits. A random guess would give an average fidelity of $1/2$.

The average fidelity of Bob's guess then is given by averaging over the distribution of transmitted states by Alice (chosen to be uniformly sampled from the sphere) and all possible measurement outcomes by Bob, weighted by the fidelity,

$$F = \int d\mathbf{n} \int d\Omega p(\Omega|\mathbf{n}) \frac{1 + \mathbf{n} \cdot \Omega\mathbf{z}}{2}. \quad (5.5)$$

As the probability $p(\Omega|\mathbf{n})$ and the fidelity depend only on the angle $\beta = \cos^{-1}(\mathbf{n} \cdot \Omega\mathbf{z})$, this expression simplifies to

$$F = \frac{1}{2} \int_0^\pi d\beta \sin \beta [3 \cos^4(\beta/2)] \frac{1 + \cos \beta}{2} = \frac{3}{4}. \quad (5.6)$$

We note that the same average fidelity can be achieved with a finite (four-element) PVM in the basis given by Eq. (3.35) (Massar and Popescu, 1995).

Remarkably, this method where Alice sends two parallel spins is not optimal; a higher average fidelity can be achieved if Alice instead sends two antiparallel spins (Gisin and Popescu, 1999), as we now demonstrate. Let

$|\mathbf{n}\rangle|-\mathbf{n}\rangle$ be the two-qubit state transmitted by Alice; again, Bob must perform a type of state estimation to determine \mathbf{n} . Note that the set of possible states is no longer contained within the $j=1$ symmetric subspace, and thus Bob must now perform a measurement on the entire two-spin Hilbert space. Choosing a covariant POVM of the form (5.1), the new normalization condition now becomes

$$\int d\Omega E_\Omega = I, \quad (5.7)$$

where I is now the identity on the full two-spin Hilbert space. Choosing $E_0 = |e\rangle\langle e|$ to be rank 1, this normalization again completely constrains the POVM (up to an arbitrary choice of single-spin basis by Bob) to be

$$|e\rangle = \sqrt{3}(|\psi^+\rangle + |\psi^-\rangle), \quad (5.8)$$

where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

The probability that Bob obtains the measurement outcome Ω given that Alice prepared $|\mathbf{n}\rangle|-\mathbf{n}\rangle$ is a function only of the angle β between \mathbf{n} and $\Omega\mathbf{z}$, given in this case by

$$p(\Omega|\mathbf{n}) = \frac{(1 + \sqrt{3}\cos \beta)^2}{2}. \quad (5.9)$$

This leads to a fidelity of $F = (1 + \sqrt{3})/(2\sqrt{3}) \approx 0.789$, which is greater than that achieved for the parallel spin case. We note that this fidelity can also be achieved with a finite (four-outcome) PVM of the form

$$|i\rangle = \frac{\sqrt{3}}{2} \frac{|\mathbf{n}_i\rangle|-\mathbf{n}_i\rangle + |-\mathbf{n}_i\rangle|\mathbf{n}_i\rangle}{\sqrt{2}} + \frac{1}{2}|\psi^-\rangle, \quad (5.10)$$

where $|\mathbf{n}_i\rangle$ are along the four directions of the tetrahedron, given by Eqs. (3.36)–(3.39). An experiment demonstrating this protocol has been performed by Jeffrey *et al.* (2006), wherein it was referred to as “quantum orienteering.”

A heuristic explanation of why the antiparallel spins are superior to the parallel spins for this task is obtained by investigating the orbits of transmitted states under the relevant group, in this case, the group of rotations $\text{SU}(2)$. Consider the orbit under the group of a state of parallel spins $|\mathbf{n}\rangle|\mathbf{n}\rangle$,

$$M_{\text{par}} = \{[R(\Omega) \otimes R(\Omega)]|\mathbf{n}\rangle|\mathbf{n}\rangle, \quad \Omega \in \text{SU}(2)\}. \quad (5.11)$$

This orbit has support entirely on the three-dimensional symmetric $j=1$ subspace of the two-spin Hilbert space. In contrast, the orbit under the group of a state of antiparallel spins $|\mathbf{n}\rangle|-\mathbf{n}\rangle$,

$$M_{\text{anti}} = \{[R(\Omega) \otimes R(\Omega)]|\mathbf{n}\rangle|-\mathbf{n}\rangle, \quad \Omega \in \text{SU}(2)\}, \quad (5.12)$$

has support on the full four-dimensional two-spin Hilbert space. The latter orbit spans a larger space, therefore its elements are, loosely speaking, more orthogonal and consequently easier to discriminate. We see in the following that this heuristic idea can be formalized to

provide optimal methods for the alignment of any type of reference frame.

B. General approach to aligning reference frames

Consider the general problem of aligning an RF associated with the group G using the one-way transmission of a quantum system (composed, say, of a number of elementary systems) with Hilbert space \mathcal{H} . For instance, one might be trying to communicate information about a Cartesian frame, associated with the group $SU(2)$, using N spin-1/2 particles and corresponding Hilbert space $\mathcal{H}=(\mathcal{H}_{1/2})^{\otimes N}$. The problem is to devise an optimal protocol for this task, given the allowed communication resources, for some given figure of merit.

The most general statements we can make about optimal RF distribution schemes concern the form of an optimal POVM for a given covariant set of signal states, provided the figure of merit satisfies some very general and natural properties. Optimal states for a general task of this sort can be taken to be pure, given that any mixed state scheme is a convex sum of pure state schemes and therefore can do no better than the best pure state scheme. Thus the signal states form an orbit of pure states

$$|\psi(g)\rangle = U(g)|\psi\rangle, \quad (5.13)$$

where $U(g)$ is the representation of G on \mathcal{H} . We call $|\psi\rangle$ the *fiducial state*.

We now consider a general form for the fiducial state in terms of the decomposition of \mathcal{H} into irreps of G , given by Eqs. (2.22) and (2.23). In terms of the charge sectors \mathcal{H}_q , we express the fiducial state as

$$|\psi\rangle = \sum_q \beta_q |\psi_q\rangle, \quad (5.14)$$

with β_q satisfying $\sum_q |\beta_q|^2 = 1$, and where $|\psi_q\rangle \propto \Pi_q |\psi\rangle$ is the normalized component of the fiducial state on each charge sector \mathcal{H}_q (where Π_q is the projector onto the q th charge sector). Each state $|\psi_q\rangle$ can be viewed as a (generally entangled) state on the tensor product decomposition $\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q$ of Eq. (2.23). Let

$$|\psi_q\rangle = \sum_{m=1}^{d_q} \lambda_m^{(q)} |\phi_m^{(q)}\rangle \otimes |r_m^{(q)}\rangle \quad (5.15)$$

be a Schmidt decomposition of this state on $\mathcal{M}_q \otimes \mathcal{N}_q$, where

$$d_q \equiv \min\{\dim \mathcal{M}_q, \dim \mathcal{N}_q\}. \quad (5.16)$$

We note that if $|\psi_q\rangle$ does not have maximal Schmidt rank (meaning some of the $\lambda_m^{(q)}$ are zero), then the Schmidt vectors are not unique. Let $\tilde{\mathcal{N}}_q \subseteq \mathcal{N}_q$ be the d_q -dimensional space spanned by the Schmidt vectors $\{|r_m^{(q)}\rangle\}$. Let $\{|\phi_m^{(q)}\rangle\}$ be a basis for \mathcal{M}_q , obtained using the Schmidt vectors from Eq. (5.15) and, if necessary, completing this set arbitrarily to a basis.

A general expression for the fiducial state is thus

$$|\psi\rangle = \sum_q \beta_q \sum_{m=1}^{d_q} \lambda_m^{(q)} |\phi_m^{(q)}\rangle \otimes |r_m^{(q)}\rangle, \quad (5.17)$$

which lies on the subspace $\tilde{\mathcal{H}} \subset \mathcal{H}$ given by

$$\tilde{\mathcal{H}} \equiv \sum_q \mathcal{M}_q \otimes \tilde{\mathcal{N}}_q. \quad (5.18)$$

In addition, the support of the orbit of the fiducial state, i.e., the space

$$\mathcal{H}^\psi = \text{span}\{U(g)|\psi\rangle, g \in G\} = \text{supp}[\mathcal{G}(|\psi\rangle\langle\psi|)], \quad (5.19)$$

will also lie within $\tilde{\mathcal{H}}$,

$$\mathcal{H}^\psi \subseteq \tilde{\mathcal{H}}. \quad (5.20)$$

Thus for any choice of a fiducial state $|\psi\rangle$ the measurement may be described by a POVM that is restricted to $\tilde{\mathcal{H}}$. In addition, if the figure of merit we are attempting to optimize satisfies some general conditions (which we discuss below), the optimal POVM can be chosen to be G covariant.¹² Moreover, its elements can be taken to be rank 1.¹³ Thus the optimal POVM must have the form $\{E(g)\}$, given by

$$E(g) = U(g)|e\rangle\langle e|U(g)^\dagger. \quad (5.21)$$

We call $|e\rangle\langle e|$ the fiducial POVM element. Given that these elements form a resolution of identity on $\tilde{\mathcal{H}}$, we have $\int dg E(g) = I_{\tilde{\mathcal{H}}}$, or, equivalently,

$$\mathcal{E}[|e\rangle\langle e|] = I_{\tilde{\mathcal{H}}}. \quad (5.22)$$

Thus an RF alignment scheme of this sort is specified by a fiducial state $|\psi\rangle$ and a fiducial POVM element $|e\rangle\langle e|$. In order to determine an optimal scheme, we first determine the optimal $|e\rangle\langle e|$ for a given $|\psi\rangle$.

The constraint Eq. (5.22) completely fixes the form of $|e\rangle$ to be

$$|e\rangle = \sum_q \sqrt{\dim(\mathcal{M}_q)} \sum_{m=1}^{d_q} |\phi_m^{(q)}\rangle \otimes |r_m^{(q)}\rangle. \quad (5.23)$$

This vector $|e\rangle$ of Eq. (5.23) can be described as follows: it is a coherent superposition across the charge sectors where $|\psi\rangle$ has support, with the amplitude squared in each such charge sector given by the dimensionality of \mathcal{M}_q , and the projections in each such charge sector given by maximally entangled states across $\mathcal{M}_q \otimes \tilde{\mathcal{N}}_q$.

We now demonstrate why $|e\rangle$ must take this form. Equation (5.22) can be expressed as

¹²We note that choosing a covariant POVM is sufficient to obtain an optimal protocol, but not necessary. For practical schemes, it may be valuable to identify finite-element POVMs that also obtain the optimum, as in Sec. V.A.

¹³For a proof of this, see [Chiribella et al. \(2005\)](#). In general, any non-rank-1 POVM can be simulated by a rank-1 POVM followed by classical post-processing of the result; however, this simulation need not be covariant.

$$\sum_q \mathcal{D}_q \otimes \mathcal{I}_q[\Pi_q|e\rangle\langle e|\Pi_q] = \sum_q I_{\mathcal{M}_q} \otimes I_{\tilde{\mathcal{N}}_q}. \quad (5.24)$$

In terms of the charge sectors, we write

$$|e\rangle = \sum_q c_q |e_q\rangle, \quad (5.25)$$

where the c_q are nonzero and $|e_q\rangle \equiv \Pi_q|e\rangle$ is a normalized state in the j th sector. Projecting Eq. (5.24) onto a single charge sector and tracing over \mathcal{M}_q , we find

$$\text{Tr}_{\mathcal{M}_q}[|e_q\rangle\langle e_q|] = I_{\tilde{\mathcal{N}}_q}/d_q, \quad (5.26)$$

which tells us that the reduced density operator on $\tilde{\mathcal{N}}_q$ of $|e_q\rangle$ is the completely mixed state, and consequently that $|e_q\rangle$ is a maximally entangled state across $\mathcal{M}_q \otimes \tilde{\mathcal{N}}_q$. This may be written as

$$|e_q\rangle = \frac{1}{\sqrt{d_q}} \sum_{m=1}^{d_q} |\phi_m^{(q)}\rangle \otimes |r_m^{(q)}\rangle. \quad (5.27)$$

Projecting Eq. (5.24) onto a single charge sector and tracing over both $\tilde{\mathcal{N}}_q$ and \mathcal{M}_q , we conclude that

$$|c_q|^2 = \text{Tr}[I_{\mathcal{M}_q}] \text{Tr}[I_{\tilde{\mathcal{N}}_q}] = \text{dim}(\mathcal{M}_q) d_q. \quad (5.28)$$

We may define $|\phi_m^{(q)}\rangle$ in such a way that the coefficients c_q can be taken to be real and positive. Combining Eqs. (5.25), (5.27), and (5.28) we recover Eq. (5.23).

The covariant POVM then is fixed by the problem, and it remains only to determine the optimal fiducial state $|\psi\rangle$. To do so, one needs to specify a figure of merit.

C. Maximum likelihood estimation

We now consider a particular choice for a figure of merit: the maximum likelihood of a correct guess (Chiribella *et al.*, 2004b). Because our standard example is a continuous group, the likelihood of a correct guess is infinitesimal, and so we must look at the maximum likelihood density—the probability density μ of obtaining the POVM outcome $E(g)$ given that the signal state is $|\psi(g)\rangle$, averaged over the prior distribution over signal states, which we take to be uniform. (A more simple analysis is possible for finite groups.) This density takes the simple form

$$\mu = \int dg \text{Tr}[E(g)|\psi(g)\rangle\langle\psi(g)|] = |\langle e|\psi\rangle|^2. \quad (5.29)$$

As the fiducial POVM element $|e\rangle$ is fixed, optimization is achieved by taking $|\psi\rangle$ to be parallel to $|e\rangle$,

$$|\psi\rangle = \frac{|e\rangle}{\|e\|}, \quad (5.30)$$

where $\|e\| = \sqrt{\langle e|e\rangle}$. It follows from Eq. (5.23) that

$$\|e\| = \sqrt{\sum_q \text{dim}(\mathcal{M}_q) d_q} = \sqrt{\text{dim } \tilde{\mathcal{H}}}. \quad (5.31)$$

The optimal fiducial state, which has the form of Eq. (5.17), must have all Schmidt coefficients $\lambda_m^{(q)}$ equal and nonzero, and the coefficients β_q are completely fixed by the optimal $|e\rangle$; i.e., the optimal fiducial state is

$$|\psi\rangle = \sum_j \sqrt{\frac{\text{dim}(\mathcal{M}_q)}{\text{dim } \tilde{\mathcal{H}}}} \sum_{m=1}^{d_q} |\phi_m^{(q)}\rangle \otimes |r_m^{(q)}\rangle. \quad (5.32)$$

Note that this state satisfies

$$\mathcal{G}[|\psi\rangle\langle\psi|] = I_{\tilde{\mathcal{H}}}/\text{dim } \tilde{\mathcal{H}}. \quad (5.33)$$

We can conclude that the maximum likelihood density of a correct guess takes the general form

$$\begin{aligned} \mu_{\max} &= \|e\|^2 = \text{dim } \tilde{\mathcal{H}} \\ &= \text{rank}(\mathcal{G}[|\psi\rangle\langle\psi|]) \\ &= \sum_q \text{dim } \mathcal{M}_q \min\{\text{dim } \mathcal{M}_q, \text{dim } \mathcal{N}_q\}. \end{aligned} \quad (5.34)$$

Given Eqs. (5.33) and (5.34), we can interpret this result as follows: We maximize the likelihood of a correct guess by choosing the fiducial signal state $|\psi\rangle$ to be such that, under G averaging, the weights of the state $\mathcal{G}[|\psi\rangle\langle\psi|]$ are spread uniformly over the largest possible space. Thus, at least for the case of maximum likelihood estimation, the intuition behind why antiparallel spins do better than parallel spins is found to have a rigorous counterpart, and indeed this intuition is found to generalize to the alignment of any RF whose configurations correspond to the elements of a group. By choosing a fiducial state in this way, the signal states are made as distinguishable as possible.

1. Maximum likelihood estimation of a phase reference

With the general results above, the optimal performance of any particular alignment protocol quantified by maximizing the likelihood can be directly and simply calculated. Suppose, for example, one seeks to align phase references by transmitting at most n_{\max} photons in a single mode. The relevant group in this case is $U(1)$. The charge sectors correspond to total photon number, so we use n rather than q to denote them. Because the irreps of $U(1)$ are one dimensional, we have $\text{dim } \mathcal{M}_n = 1$. In this case, the optimal fiducial POVM element and the optimal fiducial signal state are

$$|e\rangle = \sum_{n=0}^{n_{\max}} |n\rangle, \quad |\psi\rangle = \frac{1}{\sqrt{n_{\max} + 1}} \sum_{n=0}^{n_{\max}} |n\rangle. \quad (5.35)$$

Clearly,

$$\mathcal{G}[|\psi\rangle\langle\psi|] = \sum_{n=0}^{n_{\max}} \frac{1}{n_{\max} + 1} |n\rangle\langle n|, \quad (5.36)$$

so that the maximum likelihood density of a correct guess is

$$\mu_{\max} = \text{rank}(\mathcal{G}[|\psi\rangle\langle\psi|]) = n_{\max} + 1. \quad (5.37)$$

Note that for multiple modes the subspaces \mathcal{H}_n may be multidimensional, and in this case the basis states can be chosen to be any set of eigenstates of \hat{N}_{tot} , i.e., any multimode states that are eigenstates of total photon number.

For comparison, it is useful to consider the maximum likelihood that could be achieved using a coherent state $|\alpha\rangle$ with mean photon number $n_{\max}/2$ (we cut off the amplitude for $n > n_{\max}$ which is negligible for sufficiently large values of n_{\max}),

$$\mu_{\text{CS}} = |\langle\alpha|e\rangle|^2 = \left| \sum_{n=0}^{n_{\max}} \frac{e^{-|\alpha|^2/2} \alpha^n}{\sqrt{n!}} \right|^2, \quad (5.38)$$

which behaves as $|\alpha| = \sqrt{n_{\max}/2}$ for large values of n_{\max} .

Thus the phase eigenstate offers a quadratic improvement in n_{\max} over the coherent state. Heuristically, this is due to the fact that for the Poissonian number distribution of a coherent state most of the support lies within $\pm\sqrt{\bar{n}}$ of the mean photon number $\bar{n} = n_{\max}/2$. Thus the majority of the support of the U(1) orbit of a coherent state is carried by a subspace of the Hilbert space with dimension that scales as $\sqrt{\bar{n}}$. For the optimal state $|\psi\rangle = \frac{1}{\sqrt{2\bar{n}+1}} \sum_{n=0}^{2\bar{n}} |n\rangle$, in contrast, the dimensionality of this subspace scales as \bar{n} .

The quadratic improvement achievable in such cases is typically explained by noting that there is an uncertainty relation between phase and photon number, and to achieve the smallest possible variance in phase, one requires the largest possible variance in photon number. This is certainly a useful tool for understanding the successes of different schemes. Nonetheless, as we shall show presently, whereas the optimal strategies for aligning RFs associated with non-Abelian groups can also be understood in terms of the support of the group orbit of the signal state, it is at present unclear whether an argument in terms of an uncertainty principle can be provided in such cases.

2. Maximum likelihood estimation of a Cartesian frame

We now consider the task of optimally aligning a full spatial (Cartesian) frame through the exchange of spin-1/2 particles, based on maximizing the likelihood of a correct estimation. For this example we will be required to use the multiplicity of irreducible representations of SO(3) that occur in $(\mathcal{H}_{1/2})^{\otimes N}$ to obtain the optimal scheme.

We restrict ourselves to the case of N an even number for simplicity. N spin-1/2 particles carry a tensor representation $R^{\otimes N}$ of SO(3); this representation is reducible, and the irreducible representations (labeled by j) appear

with nontrivial multiplicities. As analyzed in Sec. III.A.2, the total Hilbert space of N spin-1/2 particles can be decomposed as

$$(\mathcal{H}_{1/2})^{\otimes N} = \bigoplus_{j=0}^{N/2} \mathcal{M}_j \otimes \mathcal{N}_j, \quad (5.39)$$

where \mathcal{M}_j carry irreducible representations R_j of SO(3) and have dimensionality $2j+1$, and \mathcal{N}_j carry the trivial representation of SO(3) and have dimensionality given by Eq. (3.21). For this example, the dimension of each decoherence-free subsystem \mathcal{N}_j is greater than or equal to the dimension of the corresponding decoherence-full subsystem \mathcal{M}_j for all j except $j=N/2$ (where $\dim \mathcal{N}_j=1$). Thus the maximum dimension of \mathcal{H}^ψ is, from Eqs. (5.20) and (5.34),

$$\dim \mathcal{H}^\psi = (N+1) + \sum_{j=0}^{N/2-1} (2j+1)^2 = \frac{1}{6}N^3 + \frac{5}{6}N + 1. \quad (5.40)$$

For each $j < N/2$ we choose a $(2j+1)$ -dimensional subspace $\mathcal{N}_j^\psi \subset \mathcal{N}_j$ with basis $|j, \alpha(m)\rangle$. The optimal signal state thus has the form

$$|\psi^{(N)}\rangle = \frac{1}{\dim \mathcal{H}^\psi} \left(\sqrt{N+1} |N/2, N/2\rangle + \sum_{j=0}^{N/2-1} \sqrt{2j+1} \sum_{m=-j}^j |j, m\rangle \otimes |j, \alpha(m)\rangle \right), \quad (5.41)$$

and the maximum likelihood density of a correct guess is

$$\mu_{\max} = \frac{1}{6}N^3 + \frac{5}{6}N + 1, \quad (5.42)$$

which scales as $N^3/6$ for large N .

D. General figures of merit

As discussed above, maximizing the likelihood of the correct guess led us directly to a general principle for choosing the fiducial signal state. However, as a figure of merit, the maximum likelihood density is not a very practical choice—it rewards only a perfectly correct guess. In many situations, one would desire a figure of merit that would quantify the performance of a scheme by the amount of Shannon information gained by the recipient about the sender's reference frame. However, such figures of merit usually lead to intractable optimization problems.

A more common and tractable approach is to introduce a payoff function $f(g', g)$ which specifies the payoff for guessing group element g' when the actual group element is g (Chiribella *et al.*, 2005). Assuming a uniform prior for the signal states, the figure of merit for the alignment scheme can then be the average payoff

$$\bar{f} = \int dg dg' p(g'|g) f(g', g), \quad (5.43)$$

where $p(g'|g)$ is the probability of guessing g' when the signal state is g for the scheme in question. In particular, the commonly used fidelity, which quantifies the variance of the average guess and which leads to direct comparisons with the standard quantum limit, is one choice of payoff function; we will determine this fidelity and explore protocols that optimize its average in the examples that follow.

The task of reference frame alignment imposes some natural constraints on the form of payoff functions. We note that the group elements g and g' denote an orientation relative to some background RF, i.e., the identity group element corresponds to being aligned with this background RF. However, it is desirable to construct protocols that are independent of any background RF; for example, if the background RF was transformed by a group element $h \in G$, and g and g' were now defined with respect to this transformed background RF, the payoff function should be the same. Such protocols are associated with a payoff function $f(g', g)$ that is right invariant, i.e., which satisfies

$$f(g'h^{-1}, gh^{-1}) = f(g', g), \quad \forall h \in G. \quad (5.44)$$

In addition, the payoff function should be a function only of the relative transformation relating the transmitted state (determined by g) and the measurement outcome (determined by g'). This requirement of a protocol demands that the payoff function be left invariant,

$$f(hg', hg) = f(g', g), \quad \forall h \in G. \quad (5.45)$$

Payoff functions that are left invariant are also referred to as covariant. It is always possible to find a covariant POVM that is optimal for any estimation problem with a covariant (left-invariant) payoff function (Holevo, 1982), and it is for this reason that we focused our attention on covariant POVMs early in this section.

Any function $f(g', g)$ that is left invariant can be written as a function $f(g', g) = \tilde{f}(g'^{-1}g)$; if this function is also right invariant, then it satisfies $\tilde{f}(hg'^{-1}gh^{-1}) = \tilde{f}(g'^{-1}g)$, and thus is a *class function*, i.e., a function on the conjugacy classes of G . (Recall two group elements, g_1 and g_2 , are in the same conjugacy class if there exists another group element h such that $g_1 = hg_2h^{-1}$.) Any class function \tilde{f} can be expanded as a sum of the characters¹⁴ $\chi_q(g)$ of G as

$$\tilde{f}(g'^{-1}g) = \sum_q a_q \chi_q(g'^{-1}g), \quad (5.46)$$

where a_q are arbitrary coefficients. We restrict our attention to real, positive-valued payoff functions, which will allow us to perform a simple maximization.

We note that the maximum likelihood estimation task described above corresponds to choosing a payoff function $f(g', g) = \delta(g'^{-1}g)$, a delta function. This payoff function is both left and right invariant, and its expansion in terms of characters as in Eq. (5.46) corresponds to choosing all a_q positive and equal to the dimension of the irrep.

As a consequence of the covariance of both the set of signal states and the POVM, the probability $p(g'|g)$ is also a function of $g'^{-1}g$, i.e.,

$$p(g'|g) = |\langle e|U(g'^{-1}g)|\psi\rangle|^2 \equiv \tilde{p}(g'^{-1}g). \quad (5.47)$$

The average payoff of Eq. (5.43) then simplifies as

$$\bar{f} = \int dg dg' \tilde{p}(g'^{-1}g) \tilde{f}(g'^{-1}g) = \int dg \tilde{p}(g) \tilde{f}(g), \quad (5.48)$$

which follows from the invariance of the measure dg . Using the explicit form of Eq. (5.47), we have

$$\bar{f} = \int dg \langle \psi|U^\dagger(g)|e\rangle \langle e|U(g)|\psi\rangle \tilde{f}(g). \quad (5.49)$$

Defining

$$M \equiv \int dg U^\dagger(g)|e\rangle \langle e|U(g) \tilde{f}(g), \quad (5.50)$$

we may rewrite \bar{f} as

$$\bar{f} = \langle \psi|M|\psi\rangle. \quad (5.51)$$

Equation (5.51) is the generalization of Eq. (5.29) to an arbitrary covariant payoff function. As the fiducial POVM element is completely constrained to be of the form of Eq. (5.23), the operator M is therefore determined by the figure of merit. In order to maximize the average payoff \bar{f} then, one must find a fiducial state $|\psi\rangle$ of the form (5.17) that lies in the eigenspace of M with the largest eigenvalue. Specifically, we solve the eigenvalue equation

$$M|\psi\rangle = \lambda^{\max}|\psi\rangle, \quad (5.52)$$

and the use of this state yields a maximal average payoff of

$$\bar{f}^{\max} = \lambda^{\max}. \quad (5.53)$$

For the problem of optimally aligning reference frames using a left- and right-invariant payoff function, we use the following result of Chiribella *et al.* (2005) without proof:¹⁵ the optimal fiducial signal state can be chosen to have the form

¹⁴The characters $\chi_q(g)$ of a group G form a basis of class functions; they are given by the trace of the irreducible representations T_q of G , i.e., $\chi_q(g) = \text{Tr}[T_q(g)]$.

¹⁵The proof relies on determining an upper bound on the average payoff, and then demonstrating that states of the form (5.54) saturate this bound.

$$|\psi\rangle = \sum_q \beta_q \sum_{m=1}^{d_q} |\phi_m^{(q)}\rangle \otimes |r_m^{(q)}\rangle, \quad (5.54)$$

for coefficients β_q satisfying $\sum_q |\beta_q|^2 = 1$. These coefficients are determined by the specific choice of payoff function. We note, however, that this result greatly simplifies the optimization problem: the number of coefficients is now given by the number of irreps appearing in the decomposition of U , rather than by the dimension of the Hilbert space.

1. Fidelity of aligning a phase reference

We now reconsider the problem of aligning a phase reference, as in Sec. V.C.1, but with an alternate (and commonly used) payoff function: the function $f(\theta', \theta) = \cos^2[(\theta' - \theta)/2]$, which takes the value 1 for the correct guess ($\theta' = \theta$) and 0 for $\theta' = \theta + \pi$. Note that this payoff function is left- and right-invariant, and can be written as $\tilde{f}(\theta) = \cos^2(\theta/2)$, where θ now denotes the relative angle between the signal and guess. This figure of merit is commonly referred to as the fidelity.

The Hilbert space for this task will be restricted as follows: we allow arbitrarily few or many modes, but the maximum total photon number is restricted to n_{\max} . (An alternate approach would be to bound the mean photon number; however, this adds considerable complexity to the problem.)

As mentioned above, for any alignment scheme based on independent uses of N modes with at most a single photon in each, that is, N single-rail qubits, the average fidelity will approach $\bar{f} = 1$ as $1/N$, from the central limit theorem. This scaling is referred to as the standard quantum limit; the optimal scheme outperforms this scaling, as we now demonstrate. We now optimize over choices of signal state $|\psi^{(N)}\rangle$ in order to maximize the expected payoff, quantified by the average fidelity \bar{f} of Eq. (5.51).

Let $\{|n\rangle, n=0, 1, \dots, n_{\max}\}$ be an arbitrary set of eigenstates of the total number operator \hat{N}^{tot} ; the details of these states, including their mode structure, is irrelevant to the task. The fiducial POVM element is of the form $|e\rangle = \sum_{n=0}^{n_{\max}} |n\rangle$.

The operator M of Eq. (5.50) is given in this instance by the matrix

$$\begin{aligned} M_{nn'} &= \langle n | M | n' \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(n-n')\theta} \cos^2(\theta/2) \\ &= \frac{1}{2} \delta_{n,n'} + \frac{1}{4} \delta_{n,n'+1} + \frac{1}{4} \delta_{n+1,n'}. \end{aligned} \quad (5.55)$$

Note that

$$M = \frac{1}{4} \tilde{M} + \frac{1}{2} I, \quad (5.56)$$

where $\tilde{M}_{nn'} = \delta_{n,n'+1} + \delta_{n+1,n'}$. As any eigenvector of \tilde{M} is an eigenvector of M , it suffices to find the eigenvalues

and eigenvectors of \tilde{M} . The maximum average fidelity is then

$$\bar{f}^{\max} = \frac{1}{2} + \frac{1}{4} \lambda^{\max}(\tilde{M}), \quad (5.57)$$

and is achieved when $|\psi\rangle$ is the eigenvector of \tilde{M} associated with the maximum eigenvalue $\lambda^{\max}(\tilde{M})$.

The characteristic equation we must solve is

$$\det(\tilde{M} - \lambda I) = 0, \quad (5.58)$$

where I is the identity. Defining $G_k \equiv \tilde{M} - \lambda I$ where k is the dimension of the vector space on which \tilde{M} acts, one finds that

$$\det G_k = -\lambda \det G_{k-1} - \det G_{k-2}, \quad (5.59)$$

for which the solution is

$$\det G_k = U_k(-\lambda/2), \quad (5.60)$$

where U_k are the Chebyshev polynomials of the second kind, given by

$$U_k(\cos \theta) = \frac{\sin[(k+1)\theta]}{\sin \theta}. \quad (5.61)$$

Given that $U_k(x) = \pm U_k(-x)$, it follows that the characteristic equation is $U_{N+1}(\lambda/2) = 0$, and thus the largest eigenvalue is $\lambda^{\max} = 2 \cos[\pi/(N+2)]$. The maximum average fidelity for the distribution of a phase reference is thus

$$\bar{f}^{\max} = \frac{1}{2} \{1 + \cos[\pi/(N+2)]\}. \quad (5.62)$$

To find the eigenvector $|\psi\rangle$ associated with the largest eigenvalue, we must solve

$$\tilde{M}|\psi\rangle = \lambda^{\max}|\psi\rangle. \quad (5.63)$$

Let β_n be the coefficients of $|\psi\rangle = \sum_{n=0}^{n_{\max}} \beta_n |n\rangle$. The definition of \tilde{M} leads to

$$\beta_{n+1} + \beta_{n-1} = \lambda^{\max} \beta_n, \quad (5.64)$$

for $1 \leq j \leq N-1$. At $n=0$, we have $\beta_1 = \lambda^{\max} \beta_0$, and at $n=N$, we have $\beta_{N-1} = \lambda^{\max} \beta_N$. The solution is $\beta_n = U_n(\lambda^{\max}/2)$, and the coefficients β_n fall to zero at $N+1$ as required. The optimal state thus has the form

$$|\psi^{(N)}\rangle = \mathcal{N} \sum_{n=0}^N \sin\left[\frac{(n+1)\pi}{N+2}\right] |n\rangle, \quad (5.65)$$

where the normalization \mathcal{N} is approximately $\mathcal{N} \approx (N/2 + 1)^{-1/2}$ in the large- N limit (Berry and Wiseman, 2000).

In the limit of large N , the average fidelity behaves as

$$\bar{f}^{\max} \approx 1 - \frac{\pi^2}{4N^2} \quad \text{for } N \gg 1. \quad (5.66)$$

Thus this optimal protocol for the alignment of a phase reference has an error (variance) which decreases as $1/N^2$, i.e., at the Heisenberg limit.

2. Fidelity of aligning a Cartesian frame

We now consider the task of aligning a Cartesian frame through the exchange of spin-1/2 particles, using the fidelity as the figure of merit (Bagan *et al.*, 2004b; Chiribella *et al.*, 2004a, 2005).

We first develop the payoff function, which we require to be both left and right invariant. One such possibility is to use the mean deviation between Alice's coordinate axes and Bob's, i.e.,

$$f(\Omega'|\Omega) = 1 - \frac{1}{8} \sum_{i=x,y,z} |\Omega \mathbf{n}_{iA} - \Omega' \mathbf{n}_{iB}|^2, \quad (5.67)$$

where $\Omega \mathbf{n}$ denotes the vector obtained by rotating the vector \mathbf{n} by $\Omega \in \text{SO}(3)$. This function can be expressed in terms of the characters $\chi_j(\Omega)$ for $\text{SO}(3)$; because these characters will be useful in the following, we briefly review them here. The characters of $\text{SO}(3)$ are given by the trace of the irreps R_j as

$$\chi_j(\Omega) = \text{Tr}[R_j(\Omega)]. \quad (5.68)$$

Recall that any rotation Ω in $\text{SO}(3)$ can be expressed as a rotation by ω , in the range $0 \leq \omega < 2\pi$, about some axis. Conjugation by another rotation in $\text{SO}(3)$ simply changes the axis, not the value of ω . Thus conjugacy classes are labeled by an angle ω (a rotation by ω about some axis), and characters being class functions are functions only of ω . Explicitly, they are given by

$$\chi_j(\Omega) = \chi_j(\omega) = \frac{\sin[(2j+1)\omega/2]}{\sin(\omega/2)}. \quad (5.69)$$

The payoff function can be expressed in terms of the character $\chi_{j=1}$ of $R_{j=1}$ [the representation of $\text{SO}(3)$ that acts on spatial vectors] as

$$f(\Omega'|\Omega) = \frac{1}{4} + \frac{1}{4}\chi_1(\Omega'^{-1}\Omega). \quad (5.70)$$

As it is a covariant function only on the conjugacy class, we can express it as

$$\tilde{f}(\omega) = \frac{1}{4} + \frac{1}{4}\chi_1(\omega). \quad (5.71)$$

The fiducial POVM element can be written as

$$|e^{(N)}\rangle = \sqrt{N+1} \left| \frac{N}{2}, \frac{N}{2} \right\rangle + \sum_{j=0}^{N/2-1} (2j+1) |e_j\rangle, \quad (5.72)$$

where

$$|e_j\rangle \equiv \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j |j, m\rangle \otimes |j, \alpha(m)\rangle, \quad (5.73)$$

are maximally entangled.

From Eq. (5.54), the optimal fiducial signal state has the form

$$|\psi^{(N)}\rangle = \beta_{N/2} \left| \frac{N}{2}, \frac{N}{2} \right\rangle + \sum_{j=0}^{N/2-1} \beta_j |e_j\rangle, \quad (5.74)$$

where the coefficients β_j are to be determined. For simplicity and brevity, we only solve this eigenvalue problem in the limit of large N . In this limit, the $\beta_{N/2}$ term

(the only exceptional term) can be ignored.

As with the phase distribution problem, the goal is to find the state $|\psi\rangle$ that maximizes

$$\tilde{f} = \langle \psi | M | \psi \rangle = \sum_{j,j'} \beta_j^* \beta_{j'} M_{jj'}, \quad (5.75)$$

where $M_{jj'}$ is the matrix

$$M_{jj'} = \int d\Omega \langle e_j | R_j(\Omega) | e_j \rangle \langle e_{j'} | R_{j'}(\Omega)^\dagger | e_{j'} \rangle \tilde{f}(\Omega). \quad (5.76)$$

We note that

$$\begin{aligned} \langle e_j | R_j(\Omega) | e_j \rangle &= \frac{1}{2j+1} \sum_{m,m'=-j}^j \langle j, m | R_j(\Omega) | j, m' \rangle \\ &\quad \times \langle \alpha(m) | \alpha(m') \rangle \\ &= \frac{1}{2j+1} \sum_{m=-j}^j \langle j, m | R_j(\Omega) | j, m \rangle \\ &= \frac{1}{2j+1} \chi_j(\Omega). \end{aligned} \quad (5.77)$$

Thus

$$M_{jj'} \propto \int d\omega \chi_j(\omega) \chi_{j'}^*(\omega) \left[\frac{1}{4} + \frac{1}{4}\chi_1(\omega) \right]. \quad (5.78)$$

To evaluate this integral, one can make use of the orthogonality properties of group characters; see Chiribella *et al.* (2004a) for details. We find that

$$M_{jj'} \propto \frac{1}{4} \delta_{j,j'} + \frac{1}{4} (\delta_{j,j'-1} + \delta_{j-1,j'}). \quad (5.79)$$

The eigenvalue problem then is essentially identical to that solved for the distribution of a phase reference in the previous section. In this limit this maximum average fidelity scales as

$$\tilde{f}^{\max} \simeq 1 - \frac{\pi^2}{N^2} \quad \text{for } N \gg 1. \quad (5.80)$$

Thus this scheme also scales at the Heisenberg limit.

We note that this particular task has given rise to some controversy and errors in the literature. In particular, a mistaken claim of optimality for this task in Bagan, Baig, and Munoz-Tapia (2001), which resulted from a failure to include the multiplicity of irreducible representations, led to some confusion over the use of covariant measurements in this task (Peres and Scudo, 2002a).

E. Reference frames associated with coset spaces

A directional RF, for the \mathbf{z} axis say, can be obtained from a full Cartesian RF by throwing away the information about the azimuthal angle. To specify a direction therefore it is sufficient to specify an equivalence class of Cartesian frames, those related by an $\text{SO}(2)$ transformation about this axis. Hence a directional RF is associated with an element of the coset space $\text{SO}(3)/\text{SO}(2)$. This

coset space is equivalent to S_2 , the space of points on a three-dimensional sphere, which corresponds to the possible directions in space.

Thus certain reference frames have distinct configurations which do not correspond to the elements of a group, but rather those of a coset space of a group. If we consider a reference frame for a group G but are unconcerned about the difference between those related by a subgroup G_0 of transformations, then we can speak of a reference frame for the coset space G/G_0 . We may incorporate such cases into the framework specified above by choosing our figure of merit to reflect the unimportance of the subgroup in the estimation task; i.e., choose a payoff function $\tilde{f}(g'^{-1}g)$ that satisfies

$$\tilde{f}(g'^{-1}gg_0) = \tilde{f}(g'^{-1}g), \quad \forall g_0 \in G_0. \quad (5.81)$$

In other words, we imagine choosing signal states and POVMs that are covariant for a group G that is a covering group for the coset space in question. Let z be a set of coset representatives, i.e., $z \in G/G_0$, and let dz be a left-invariant measure on G/G_0 . Then, using Eq. (5.48),

$$\begin{aligned} \tilde{f} &= \int dg \tilde{p}(g) \tilde{f}(g) = \int_{G/G_0} dz \left(\int_{G_0} dg_0 \tilde{p}(zg_0) \right) \tilde{f}(z) \\ &= \int_{G/G_0} dz \tilde{p}_{\text{inv}}(z) \tilde{f}(z). \end{aligned} \quad (5.82)$$

Here we have defined

$$\begin{aligned} \tilde{p}_{\text{inv}}(z) &\equiv \int_{G_0} dg_0 \tilde{p}(zg_0) \\ &= \text{Tr} \left[\left(\int_{G_0} dg_0 U(zg_0) |e\rangle\langle e| U^\dagger(zg_0) \right) |\psi\rangle\langle\psi| \right] \\ &= \text{Tr} [U(z) E_{\text{inv}} U^\dagger(z) |\psi\rangle\langle\psi|], \end{aligned} \quad (5.83)$$

where

$$E_{\text{inv}} = \int_{G_0} dg_0 U(g_0) |e\rangle\langle e| U^\dagger(g_0), \quad (5.84)$$

is G_0 -invariant. Thus for any covariant measurement with fiducial POVM element $|e\rangle$ that achieves the optimum figure of merit, there exists a G_0 -invariant covariant measurement with fiducial POVM element E_{inv} that achieves the same optimum. For this reason, we may as well restrict the fiducial signal state and fiducial POVM element to be G_0 -invariant.

We note that, if the group G_0 is non-Abelian, it may not be possible to find a pure state that is invariant under the subgroup. In such a situation, if one wishes to work with G_0 -invariant states and measurements, then one will have to use mixed fiducial states and POVM elements (Chiribella and D'Ariano, 2004). We now consider an example with an Abelian group G_0 , for which these complications do not arise.

1. Aligning a direction

Consider the task of optimally aligning a direction in space through the exchange of spin-1/2 particles. This was first considered for just two particles by Gisin and Popescu (1999) and Massar (2000), as discussed in Sec. V.A. The problem was subsequently considered for an arbitrary number of particles by Bagan, Baig, Brey, *et al.* (2000, 2001) and Peres and Scudo (2001a). [For a related investigation, wherein it is addressed how to perform this task using product states, see Bagan, Baig, and Munoz-Tapia (2001).]

Let $|\psi^{(N)}\rangle$ be the fiducial signal state. We restrict our attention to the case of N even. Because we are concerned only with aligning a direction and not a full Cartesian frame, we can choose $|\psi^{(N)}\rangle$ to be invariant under rotations about the z axis without loss of generality. Any such pure invariant state is an eigenstate of J_z ; thus choose $|\psi_m^{(N)}\rangle$ to be an eigenstate of J_z with eigenvalue $\hbar m$. Clearly, m must be in the range $-N/2, \dots, N/2$.

First, some notation. It is standard to express a rotation in $\text{SO}(3)$ in terms of its Euler angles (α, β, γ) . Specifically, a unitary rotation operator can be expressed as

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma), \quad (5.85)$$

where R_y and R_z are $\text{SO}(2)$ rotations about the y and z axes, respectively. For any element in $\text{SO}(3)$, a set of Euler angles can be found in the range $0 \leq \alpha, \gamma < 2\pi$, and $0 \leq \beta < \pi$. The invariant subgroup is $G_0 = \text{SO}(2)$ in this problem, rotations about the z axis; thus the parameters (α, β) provide coordinates for the coset space $\text{SO}(3)/\text{SO}(2)$.

a. Maximum likelihood

We now maximize the likelihood of a correct guess. Restricting the fiducial POVM element to be $\text{SO}(2)$ invariant, it takes the form

$$|e_m^{(N)}\rangle = \sum_{j=m}^{N/2} \sqrt{2j+1} |j, m\rangle. \quad (5.86)$$

As we wish to include all possible irreps j , following the general construction of Sec. V.B, we should choose $m=0$, i.e., a fiducial POVM element

$$|e^{(N)}\rangle = \sum_{j=0}^{N/2} \sqrt{2j+1} |j, 0\rangle. \quad (5.87)$$

The signal state should then be parallel to this vector of the form

$$|\psi^{(N)}\rangle = \frac{1}{(N/2+1)^2} \sum_{j=0}^{N/2} \sqrt{2j+1} |j, 0\rangle, \quad (5.88)$$

and the maximum likelihood density of a correct guess is

$$\mu_{\text{max}} = (N/2+1)^2. \quad (5.89)$$

b. Fidelity

A natural payoff function for this problem is the inner product between Bob's guess direction \mathbf{n}_g and Alice's transmitted \mathbf{n} , given by $\tilde{f}(\theta) = (1 + \mathbf{n}_g \cdot \mathbf{n}) = \cos^2(\theta/2)$, where θ is the angle between their directions. This payoff function is also known as the fidelity. We provide the details for this optimization as well. Note that this is the generalization of the example provided in Sec. V.A from two to an arbitrary number of spin-1/2 systems.

The fiducial POVM element is essentially¹⁶ constrained to be that of Eq. (5.87), and now the signal state takes the general form

$$|\psi^{(N)}\rangle = \sum_{j=0}^{N/2} b_j |j, 0\rangle, \quad (5.90)$$

where the coefficients b_j are to be determined. The Born rule yields

$$\tilde{p}(\alpha, \beta, 0) = |\langle \psi^{(N)} | R^{\otimes N}(\alpha, \beta, 0) | e^{(N)} \rangle|^2. \quad (5.91)$$

This quantity is independent of α , and thus the relevant conditional probability is

$$\tilde{p}(\beta) = |\langle \psi^{(N)} | R^{\otimes N}(0, \beta, 0) | e^{(N)} \rangle|^2. \quad (5.92)$$

Note that $R^{\otimes N}(0, \beta, 0) = R_y^{\otimes N}(\beta)$ (a rotation about the y axis) and the reduced Wigner matrix $d_{00}^j(\beta)$ is given by

$$d_{00}^j(\beta) \equiv \langle j, 0 | R_y(\beta) | j, 0 \rangle = P_j(\cos \beta), \quad (5.93)$$

where $P_j(x)$ is a Legendre polynomial.

The operator M of Eq. (5.50) is given by the matrix

$$\begin{aligned} M_{jj'} &= \langle j, 0 | M | j', 0 \rangle \\ &= \frac{1}{2} \int_0^\pi \sin \beta d\beta P_j(\cos \beta) P_{j'}(\cos \beta) \cos^2(\beta/2) \\ &= \frac{1}{4} \int_{-1}^1 dx P_j(x) P_{j'}(x) [P_0(x) + P_1(x)] \\ &= \frac{1}{4} \left(\frac{2}{2j+1} \delta_{j,j'} + \frac{2j}{(2j+1)(2j'+1)} \delta_{j,j'+1} \right. \\ &\quad \left. + \frac{2j'}{(2j+1)(2j'+1)} \delta_{j,j'-1} \right), \end{aligned} \quad (5.94)$$

where we have expanded the payoff function in terms of Legendre polynomials.

This eigenvalue problem is essentially the same as those solved in the previous section. The maximum average fidelity is given by

¹⁶The choice of $m=0$, necessary to optimize the maximum likelihood problem, is not *a priori* optimal for maximizing the average fidelity. However, m can be left free and then optimized at the end, with the result that $m=0$ is indeed optimal for this task (Peres and Scudo, 2001a).

$$\bar{f}^{\max} = \frac{1 + x_{N/2+1}}{2}, \quad (5.95)$$

where $x_{N/2+1}$ is the largest zero of the Legendre polynomial $P_{N/2+1}(x)$. In the limit of large N , this maximum average fidelity scales as

$$\bar{f}^{\max} \simeq 1 - \frac{\zeta^2}{N^2} \quad \text{for } N \gg 1, \quad (5.96)$$

where $\zeta \simeq 2.4$. Thus this optimal scheme also scales at the Heisenberg limit (Bagan, Baig, Brey, *et al.*, 2001; Peres and Scudo, 2001a).

F. Relation to phase/parameter estimation

We note that the task of aligning a phase reference is essentially equivalent to the task of estimating an unknown phase. Specifically, instead of viewing the problem of noiseless transmission of a quantum system between parties who do not share a phase reference, the problem could instead be viewed as one of transmission of the same quantum system between parties who do share a phase reference, but where the transmitting channel induces an unknown phase shift on the system.

In this light, we note that the protocol presented in Sec. V.D.1 is equivalent to the optimal solution for phase estimation using the same figure of merit (Berry and Wiseman, 2000). Techniques for quantum-limited phase estimation have been well studied, and there exist a wide variety of alternate methods that could each be applied, in some form, to the task of aligning a phase reference. For an overview of phase estimation techniques from different viewpoints, we refer the reader to the review article on quantum metrology by Giovannetti, Lloyd, and Maccone (2004), or the text of Nielsen and Chuang (2000) which discusses phase estimation techniques from a quantum algorithm perspective. Also, see Giovannetti *et al.* (2006) for a unified framework of these techniques.

Similarly, the task of aligning a reference frame for G through the transmission of a quantum system is essentially equivalent to estimating an unknown element $g \in G$ given a quantum channel that acts on the same quantum system with the unitary $U(g)$. This latter task is generally referred to as *parameter estimation*.

We note then that the scheme for aligning a Cartesian frame presented in Sec. V.D.2 is closely related to a method for estimating an unknown $SU(2)$ [or more generally $SU(d)$] transformation (Acin *et al.*, 2001). We briefly review this latter scheme, because of its close relation to the topic at hand. Let $R(\Omega)$ be the unitary representation of an unknown rotation $\Omega \in SU(2)$, which acts on states of a Hilbert space \mathcal{H} ; the task is to estimate Ω through one application of $R(\Omega)$ to some quantum state. For this problem, we allow the use of an ancillary system, with Hilbert space \mathcal{K} of arbitrary dimension; this ancilla is assumed to transform trivially under $SU(2)$. [That is, $SU(2)$ acts as $R(\Omega) \otimes I$ on $\mathcal{H} \otimes \mathcal{K}$.] Without loss of generality, one can choose $\dim \mathcal{K}$

$= \dim \mathcal{H}$, and choose a basis for \mathcal{K} with the same labels as \mathcal{H} . We choose the standard $SU(2)$ angular momentum basis $|j, m, \alpha\rangle$, where α labels the multiplicity.

An optimal state $|\psi\rangle$ on $\mathcal{H} \otimes \mathcal{K}$ for maximizing the likelihood of a correct guess, up to a normalization constant, is

$$|\psi\rangle \propto \sum_{j=0}^{N/2-1} \frac{1}{\sqrt{(2j+1)c_j}} \sum_{m=-j}^j \sum_{\alpha=1}^{c_j} |j, m, \alpha\rangle_{\mathcal{H}} |j, m, \alpha\rangle_{\mathcal{K}}, \quad (5.97)$$

where c_j is the multiplicity of the j th representation. The fiducial POVM element will be parallel to this vector. We note that this state is a superposition over irreps j of a maximally entangled state between an irrep j on \mathcal{H} and an equally sized space on \mathcal{K} . The optimality of this state for alignment follows from the general arguments presented in Sec. V.C: the optimal fiducial state is one that maximizes the dimension of the group orbit. Without the help of an ancilla, this is achieved within a given irrep j by entangling the gauge space \mathcal{M}_j (on which the group acts nontrivially) with the multiplicity space \mathcal{N}_j (on which it acts trivially). In the present context, it is achieved by entangling the system with the ancilla.

As such methods for parameter estimation have importance for quantum computing in terms of the characterization of quantum gates, it is interesting to consider how the methods of reference frame alignment may be applied to such characterization problems as well.

Finally, work on magnetometry—the use of magnets as direction indicators to determine the strength and direction of a magnetic field—is also a problem of parameter estimation, closely related to the problem of reference frame distribution. Practical proposals for quantum-limited magnetometry make use of spin-squeezed clouds of cold atoms to measure the three components of an unknown magnetic field through a form of phase estimation (Petersen *et al.*, 2005). It would be interesting to investigate whether spin-squeezed states of indistinguishable particles (e.g., atoms) can be used for the distribution of a direction or frame as efficiently as the optimal protocols derived above, which make use of (distinguishable) qubits in highly entangled states and corresponding entangling measurements.

G. Communication complexity of alignment

We have thus far only considered protocols for RF alignment wherein there is a single round of communication from Alice to Bob. We now consider multiround protocols (Rudolph and Grover, 2003; de Burgh and Bartlett, 2005; Giovannetti *et al.*, 2006). Whereas the single-round protocols generally require entanglement between the transmitted systems to achieve the Heisenberg limit, multiround protocols have the advantage that they can achieve this limit despite using no entanglement.

With multiround protocols, it is natural to frame the problem as one of communication complexity, wherein

one investigates the resources of rounds of communication along with the standard resources of number of transmitted quantum or classical bits. To conform with standard notions of communication complexity, it is useful to consider the alignment problem with two departures from the approach adopted in analyzing the previous alignment protocols. First, we consider the worst case scenarios (rather than the average case considered above); second, we avoid the use of payoff functions, such as the fidelity, with a view to obtaining a more precise estimate of how well any given instance of the protocol has performed. As such, we consider strategies for aligning spatial reference frames that allow Bob to directly determine the angle which relates his and Alice's RFs to some specified accuracy with a bounded probability of error in the worst case scenario. More precisely, if θ is an angle relating Alice and Bob's RFs, and θ' is the estimation of θ inferred by Bob, then we will be interested in the amount (and type) of communication required for protocols that achieve $P_{\text{error}} = \Pr[|\theta - \theta'| \geq \delta] \leq \epsilon$, for some fixed $\epsilon, \delta > 0$. By setting $\delta = 1/2^{k+1}$ we say that with probability $1 - \epsilon$ Bob has a k -bit approximation to θ .

We now describe such a protocol for the case of sharing a phase reference through the exchange of qubits, i.e., the same task as investigated in Sec. V.D.1. This protocol can also be applied to the task of aligning a Cartesian frame (Rudolph and Grover, 2003). The effects of decoherence on these protocols has been shown to be equivalent to that of decoherence on the “standard” protocol of Sec. V.D.1 (Boixo *et al.*, 2006).

Let θ_{BA} be the unknown angle (misalignment) that relates Bob's phase reference to Alice's. In this protocol, Alice and Bob use an algorithm that estimates each bit of the phase angle θ_{BA} independently. We define the phase angle $\theta_{BA} = \pi T$, where T has the binary expansion $T = 0.t_1 t_2 t_3 \dots$. Alice and Bob will attempt to determine T to k bits of precision, and accept a total error probability $P_{\text{error}} \leq \epsilon$. If the total error probability is to be bounded by ϵ , then each $t_i, i = 1, \dots, k$, must be estimated with an error probability of ϵ/k . [An error in any one bit causes the protocol to fail, so the total error probability in estimating all k bits is $P_{\text{error}} = 1 - (1 - \epsilon/k)^k \leq \epsilon$.]

To estimate the first bit t_1 , Alice prepares a single qubit in the state $(|0\rangle_A + |1\rangle_A)/\sqrt{2}$ (relative to her phase reference) and sends the qubit to Bob. Bob then performs his operation X_B and sends the qubit back to Alice, where X_B is the Pauli bit-flip operator according to Bob. She then performs her operation X_A . The resulting combined transformation $X_A X_B$ is described in Alice's frame as

$$X_A X_B = X_A (e^{-i\theta_{BA}Z/2} X_A e^{+i\theta_{BA}Z/2}) = e^{+i\theta_{BA}Z}, \quad (5.98)$$

where Z is the Pauli phase-flip operator. Finally, Alice performs a Hadamard transformation H_A (in her frame) and measures the observable $O_A = -Z$. The expected value of this observable is $\langle O_A \rangle = \cos(2\theta_{BA})$, with an uncertainty of

$$\Delta O_A = \sqrt{\langle O_A^2 \rangle - \langle O_A \rangle^2} = \sin(2\theta_{BA}). \quad (5.99)$$

Expressing $\langle O_A \rangle$ in terms of T , we have

$$\langle O_A \rangle = \cos(2\theta_{BA}) = \cos(2\pi \cdot t_1 t_2 \cdots). \quad (5.100)$$

By repeating this procedure n_1 times, i.e., sending n_1 independent qubits and averaging the results, Alice obtains $\overline{\langle O_A \rangle}$, the estimate of $\langle O_A \rangle$. If n_1 is chosen such that $|\overline{\langle O_A \rangle} - \langle O_A \rangle| \leq 1/2$ with some error probability, then $|\bar{T} - T| \leq 1/4$, thus determining the first bit t_1 with this same probability. The required number of iterations n_1 to achieve the desired error is given by the Chernoff bound, with $\delta = 1/4$. That is, the probability that the first bit of Alice's estimate $\overline{\langle O_A \rangle}$ differs from the first bit of the actual value $\langle O_A \rangle$ decreases exponentially in the number of repetitions n_1 , and is bounded explicitly by

$$\Pr[|\overline{\langle O_A \rangle} - \langle O_A \rangle| \geq 1/2] \leq \epsilon/k \leq 2e^{-n_1/32}. \quad (5.101)$$

Thus, allowing a probability of error ϵ/k in this bit, we require $n_1 \geq 32 \ln(2k/\epsilon)$ iterations.

Now we define a similar procedure for estimating an arbitrary bit t_{j+1} . Alice prepares the energy eigenstate $|0\rangle$, and performs her H_A operation. Alice and Bob then pass the qubit back and forth to each other 2^j times, each time Bob performs his X_B operation and Alice performs her X_A operation. That is, they jointly implement the operation $(X_A X_B)^{2^j}$. Finally Alice performs her H_A operation. Expressing these operations in Alice's frame, the protocol to estimate t_{j+1} produces the state

$$\begin{aligned} |\psi_j\rangle_A &= H_A (X_A X_B)^{2^j} H_A |0\rangle = H_A (e^{+i\omega t_{AB} Z})^{2^j} H_A |0\rangle \\ &= H_A e^{+i2^j \omega t_{BA} Z} H_A |0\rangle \\ &= [i \sin(2^j \omega t_{BA}) |0\rangle + \cos(2^j \omega t_{BA}) |1\rangle]_A. \end{aligned} \quad (5.102)$$

Alice then measures the observable $O_A = -Z$. The expected value of this observable is

$$\begin{aligned} \langle O_A \rangle &= \cos(2^{j+1} \omega t_{BA}) = \cos(2^j [2\pi \cdot t_1 t_2 \cdots]) \\ &= \cos(2\pi t_1 t_2 \cdots t_j \cdot t_{j+1} t_{j+2} \cdots) \\ &= \cos(2\pi \cdot t_{j+1} t_{j+2} \cdots). \end{aligned} \quad (5.103)$$

Equation (5.103) has the same form as one iteration of the scheme to estimate the first bit t_1 ; Alice and Bob simply require more exchanges to implement $(X_A X_B)^{2^j}$. To get a probability estimate for each bit t_{j+1} , this more complicated procedure is repeated n_{j+1} times. Because we require equal probabilities for correctly estimating each bit, we can set all n_{j+1} equal to the same value, $n \sim 32 \ln(2k/\epsilon)$. Thus the total amount of qubit communication N_c required to obtain bits t_1 through t_k by this procedure is

$$N_c = n \times \sum_{j=1}^k 2^{j-1} = n(2^k - 1) = O(2^k \ln(2k/\epsilon)). \quad (5.104)$$

To facilitate comparison with the previous sections, wherein the focus was on maximizing the average fidelity, we imagine that Alice and Bob use the above proto-

col to obtain, with probability $(1-\epsilon)$, an angle θ' which is a k -bit estimator of the true angle θ , i.e., they obtain $|\theta - \theta'| \leq 2\pi/2^{k+1}$ with probability $(1-\epsilon)$. The fidelity of this estimate is $f = \cos^2[(\theta - \theta')/2]$. Since $\cos x \geq 1 - x^2$ we have that $\bar{f} \geq 1 - (2\pi/2^k)^2$. To compare with the average case fidelity computed previously, we assume that when the protocol fails (which happens with probability ϵ) the fidelity obtained is 0, i.e., worse than a random guess. We then have that the expected fidelity from this protocol satisfies

$$\bar{f} \geq (1-\epsilon) \left[1 - \left(\frac{2\pi}{2^k} \right)^2 \right]. \quad (5.105)$$

If we take $\epsilon = 1/2^{2k}$, then the total number of qubit communications scales as $k2^k$ for large k , while the expected fidelity \bar{f} scales as $2^{-2k} \approx 1 - (\log_2 N_c)/N_c^2$. Thus, remarkably, this protocol beats the standard quantum limit of $1/N_c$, yet does not require entangled states or collective measurements. This is achieved at the cost of an increased complexity in coherent qubit communications.

H. Clock synchronization

If Alice and Bob share a common frequency standard, then they can use techniques for phase reference alignment that were outlined above to perform clock synchronization, i.e., aligning a temporal reference (Jozsa *et al.*, 2000). To see the relation between phase alignment and clock synchronization, it is easiest to work in a rotating frame (interaction picture) in which states are described as $|\psi\rangle_I = e^{iH_0 t/\hbar} |\psi\rangle$, and observables and transformations as $A_I = e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}$, where H_0 is the free Hamiltonian. In this rotating frame, states are stationary under free evolution, and the problem of clock synchronization is reduced to one of aligning a phase reference.

In one class of clock synchronization protocols based on phase estimation, the systems exchanged are ticking qubits: nondegenerate two-level quantum systems that undergo time evolution (Chuang, 2000). Much like phase estimation, the use of entangling operations and/or measurements can lead to different scalings in the synchronization accuracy. For example, a protocol that uses only separable (unentangled) ticking qubits and single-qubit measurements requires $O(2^{2k})$ ticking qubit communications (a coherent transfer of a single qubit from Alice to Bob) to achieve an accuracy in the time offset of Alice's and Bob's clocks of k bits. That is, the synchronization accuracy scales as the standard quantum limit. In comparison, a protocol by Chuang (2000) makes use of the quantum Fourier transform and an exponentially large range of qubit ticking frequencies. This protocol requires only $O(k)$ quantum messages to achieve k bits of precision, an exponential advantage over the standard quantum limit. Although this protocol gives insight into the ways that quantum resources may allow an advantage in clock synchronization, it is unsatisfactory for two reasons: (i) its use of exponentially demanding physical

resources is arguably the origin of the enhanced efficiency (Chuang, 2000; Giovannetti *et al.*, 2001), and (ii) Alice and Bob need to *a priori* share a synchronized clock in order to implement the required operations as defined in Chuang (2000). These problems are not present in subsequent protocols based on ticking qubits, which used the techniques for phase estimation presented in Sec. V.G to design a clock synchronization protocol that operates near the Heisenberg limit (de Burgh and Bartlett, 2005). The Heisenberg limit for clock synchronization can be achieved by making use of the phase estimation protocol of Sec. V.D.1.

Distinct from approaches based on phase estimation, there has been considerable interest in another class of clock synchronization protocols which make use of entanglement (Jozsa *et al.*, 2000); see also Preskill (2000); Burt *et al.* (2001); Jozsa *et al.* (2001); Yurtsever and Dowling (2002). In a variant of this approach, a third party Charlie distributes a large number of boxes to Alice and Bob, where each box contains one spin of a spin singlet. Each box also contains a classical magnetic field aligned in the z direction, such that the free Hamiltonian for each spin is $H_0 = \chi \sigma_z$ for some constant energy χ . (We note that this establishes a shared RF between Alice and Bob for this particular direction.) Clock synchronization can be achieved by Alice performing measurements on her spins in her x direction at time $t=0$. By informing Bob (via any classical channel) as to the subensemble of the singlets for which she obtained the $+x$ outcome, Bob can identify the subset of his particles which are all precessing (via the free Hamiltonian) around the common z direction in phase with Alice's clock. However, because Bob does not necessarily share a common x direction with Alice, he cannot actually read out this phase information. This complication can be neatly circumvented with a slight modification—on half of the particles, Alice and Bob use a different magnetic field strength in the z direction to establish two different precession frequencies. Bob can now choose any x direction to measure each subensemble, because both ensembles of precessing spins are offset by the same unknown phase shift with respect to Alice's spins, and he can achieve synchronization by merely observing the beats between the oscillations.

In the language of this review, we can understand this protocol as one in which standard *refbits* (see Sec. VI.D for more details) are being distilled from the initial singlets and put to use as a bounded shared RF. Note that if such synchronization was Charlie's intention all along, then such a protocol would not be a particularly efficient use of resources—he could just as simply have distributed to Alice and Bob a shared RF state [such as those given in Eq. (3.28), for example] which require no entanglement whatsoever (Preskill, 2000).¹⁷ This fact, together with the result that this entanglement cannot be

purified (an issue we return to in Sec. VI.E), suggests that shared entanglement between two parties does not provide an advantage for clock synchronization (or other forms of RF alignment).

A third class of protocols for clock synchronization makes use of precise timing of light signals exchanged between parties, and for which the quantum limits have recently been investigated. Instead of classical coherent state light pulses for the signals, one can use highly entangled states of many photons and beat the standard quantum limit (Giovannetti *et al.*, 2001). Essentially, the advantage is due to entanglement-induced bunching in arrival time of individual photons, enabling more accurate timing measurements. The key disadvantage of this technique is that the loss of a single photon destroys the entanglement and renders the measurement useless (Giovannetti, Lloyd, and Maccone, 2001, 2004), although techniques have been developed to trade off the quantum advantage in return for robustness against loss (Giovannetti *et al.*, 2002). Furthermore, the effect of dispersion is known to be an important issue with such protocols, with the use of entanglement possibly offering an advantage here as well (Fitch and Franson, 2002; Giovannetti, Lloyd, Maccone, *et al.*, 2004). We note that such protocols differ from those based on phase estimation in that they make use of relativistic principles (specifically, the constancy of the speed of light).

I. Other instances of alignment

We considered above the alignment of Cartesian frames using N spin-1/2 systems. A different approach to this alignment problem is to use a single hydrogen atom (Peres and Scudo, 2001b). The analysis of this task is similar to that presented above, with the notable difference that multiplicities of representations of $SO(3)$ are not available with the hydrogen atom. Also, the use of elliptic Rydberg states of the hydrogen atom have been considered for this problem, with resulting fidelity comparable with that of the optimal scheme for a hydrogen atom (Lindner *et al.*, 2003).

Although a phase reference (clock) and spatial direction (or Cartesian frame) may be the most ubiquitous types of reference systems, it is possible to distribute more general and exotic types of reference frames through the exchange of appropriate quantum systems.

For example, consider the distribution of a reference ordering, i.e., a labeling of N objects (von Korff and Kempe, 2004). Through the use of techniques similar to those described in Sec. V.D.2 for the distribution of a Cartesian frame, one can construct an optimal protocol that distributes an ordering of N particles using N systems with dimensionality N/e . (In contrast, the classical problem requires N systems each with N distinguishable states.)

Another problem is the distribution of a reference frame for chirality (Diosi, 2000; Gisin, 2004; Collins *et al.*, 2005). Such quantum systems have been given the moniker “quantum gloves.” Clearly, a full Cartesian frame includes a reference for chirality; however, distrib-

¹⁷An analog of this scheme for Cartesian frame alignment has been proposed (Rudolph, 1999a, 1999b), and similar reservations apply.

uting a full Cartesian frame purely for the purposes of distributing a reference chirality is not very economical, and more efficient methods are possible. Also, a chiral reference can be distributed perfectly, i.e., with no error, with only finite quantum resources. Methods for the distribution of a chiral reference using only two kinds of particle (i.e., a proton and an electron) and only four spinless particles, along with other interesting combinations, can be found in [Collins *et al.* \(2005\)](#).

One can also consider the problem of secret sharing of unspeakable information ([Bagan, Calsamiglia, *et al.*, 2006](#)). In such a protocol, a quantum state is shared between several parties with the aim that a reference frame can only be determined if all of the parties come together (to perform collective operations and measurements). Parties working alone, or together using only LOCC, cannot determine the reference frame with the same precision.

J. Private communication of unspeakable information

With the development of optimal schemes for the distribution of a reference frame, it is natural to consider how two parties, Alice and Bob, can perform such a distribution privately by using some number of shared secret bits—a classical key—to randomize the signal state ([Chiribella *et al.*, 2006](#)). In other words, we consider the problem of how well two parties can convert a private classical key into a private shared RF (of bounded size) using a public channel and given that they do not previously possess a shared RF (private or public). For concreteness, we investigate the private communication of a Cartesian frame.

Consider the optimal scheme for the distribution of a Cartesian frame using fidelity as the figure of merit (which achieves the Heisenberg limit). The fiducial signal state for this scheme is given by Eq. (5.74). (The scheme will be essentially identical for any figure of merit.) As with any private quantum communication, Alice and Bob can choose unitary operators from a set of unitaries, based on their classical key, to randomize any quantum state as viewed by an eavesdropper Eve who does not share this key, using the techniques of [Ambainis *et al.* \(2000\)](#). In general, to completely randomize a state on a Hilbert space of dimension $d=2^N$, Alice and Bob require a key consisting of $2N$ secret bits. However, we note that Alice and Bob do not share a reference frame to begin with and thus they can only perform correlated operations on the multiplicity spaces $\tilde{\mathcal{N}}_j$. Despite this restriction, a complete randomization can be achieved for the fiducial state of Eq. (5.74). To see this, recall that this state can be expressed as a coherent superposition, over all representations (charge sectors) of $\text{SO}(3)$, of maximally entangled states across the representation space \mathcal{M}_j and an equally sized subspace $\tilde{\mathcal{N}}_j$ of the multiplicity space. Because of this particular structure of the fiducial state, a complete randomization over just the subsystems $\tilde{\mathcal{N}}_j$ will take every signal state to the same mixed state (and thus achieves a

complete randomization on the Hilbert space that is the span of the supports of the signal states). The dimension of each subsystem $\tilde{\mathcal{N}}_j$ is equal to that of \mathcal{M}_j , namely, $d_j = 2j+1$, thus requiring $2 \log_2(2j+1)$ secret bits to completely randomize. The total number of secret bits required to completely randomize all signal states is

$$\log_2 \left(\sum_{j=0}^{N/2} (2j+1)^2 \right) \approx 3 \log_2 N. \quad (5.106)$$

Thus through the transmission of N qubits on a public channel and using $3 \log_2 N$ classical bits of private key, one can achieve the distribution of a private Cartesian frame at the Heisenberg limit, which is to say with an error that scales as $1/N^2$. We note that this number $3 \log_2 N$ is identical to the number of classical bits that can be transmitted privately given a private shared Cartesian frame as key, as discussed in Sec. III.D.

K. Dense coding of unspeakable information

Consider the following problem. Alice wants to send Bob classical information, but at the time that Alice learns which message she would like to send Bob, the cost of using the quantum channel is very high, whereas earlier, before Alice learns the message, the cost of using the channel is low. Dense coding allows Alice to make use of the channel at the early time, prior to learning the message she wishes to send, in order to increase the amount of information she succeeds in transmitting to Bob at the later time.

Suppose that Alice wishes to send to Bob a direction in space rather than a classical message. Suppose, moreover, that at the time where Alice learns the direction she would like to send Bob, the cost of using the quantum channel is very high, whereas earlier, before Alice learns the direction, the cost of using the channel is low. One would have a natural analog of dense coding to unspeakable information (directional information in this case) if use of the channel at the early time allowed Bob to estimate Alice's direction with greater accuracy at the later time.

The following is such an analog. Alice prepares a pair of spin-1/2 systems in a singlet state, and in the first use of the channel sends one of these to Bob. Later, when she has a sample of the classical direction $\hat{\mathbf{n}}$ that she would like to send to Bob, she implements a unitary rotation of π degrees about $\hat{\mathbf{n}}$ on her spin-1/2 system and sends it to Bob. Through Alice's operation, the singlet is transformed into $(|+\hat{\mathbf{n}}\rangle|-\hat{\mathbf{n}}\rangle + |-\hat{\mathbf{n}}\rangle|+\hat{\mathbf{n}}\rangle)/\sqrt{2}$, which is a two-spin state that can be used to indicate the direction $\hat{\mathbf{n}}$; in fact, given that the image of such a state under $\text{SU}(2)$ averaging covers the entire symmetric subspace, this state is as good a direction indicator as the parallel spin state of Sec. V.A. In her second use of the quantum channel, she sends her spin to Bob, and Bob estimates the direction. The optimal average fidelity that can be achieved for such a state and measurement was shown in Sec. V.A to be $3/4$, which is greater than the fidelity of

2/3 that could be achieved using a single spin-1/2 system.¹⁸ Thus this scheme provides an analog of dense coding for unspeakable quantum information. The optimization of this sort of dense coding scheme has not been investigated to date.

A slightly different analog of dense coding of unspeakable information was considered by Bagan *et al.* (2004a), building on the results of Acin *et al.* (2001). This protocol involves Alice initially sending Bob half of an entangled state over multiple spin systems. It is assumed that subsequently the entire laboratory of the sender is subject to the same SU(2) transformation that her half of the entangled pairs are subject to. Under this assumption the three parameters describing the relation of her spin-1/2 system to that of the receiver also describe the relation of her local Cartesian frame to that of the receiver. In this scenario, the sender is essentially passive: both the spin and the local Cartesian frame must be acted upon by some external agency. Unfortunately, it is not clear whether this is of practical significance in the most common case where the SU(2) transformation acting on the local Cartesian frame is a rotation in space. For instance, if a rotation of the entire laboratory is realized by an external torque, it is not clear that the state of a spin-1/2 system stored in this laboratory (i.e., in some trapping potential) will necessarily undergo the same rotation.¹⁹ Nonetheless, the optimal solution for this sort of scheme has been provided for an arbitrary number of spin-1/2 systems (Bagan *et al.*, 2004a). The optimal state bears a strong similarity to the optimal state for aligning Cartesian RFs, presented in Sec. V.D.2.

L. Error correction of unspeakable information?

We end this section with a cautionary note on the potential use of quantum methods for aligning reference frames, made by Preskill (2000) for the specific task of clock synchronization: that the standard techniques of quantum error correction cannot be directly applied to unspeakable information.

Consider a situation wherein Alice and Bob wish to align their respective frames by exchanging quantum systems via some noisy quantum channel. Let \mathcal{F} be the decohering superoperator describing the channel. The form of this noise is critical to their ability to complete this task; here we consider only two extreme cases. One is if the noise is of the form $\mathcal{F} = \oplus_q I_{\mathcal{M}_q} \otimes \mathcal{D}_{\mathcal{N}_q}$ in terms of the decomposition of Eq. (2.24), where $\mathcal{D}_{\mathcal{N}_q}$ is the completely depolarizing superoperator on \mathcal{N} . This noise affects only the multiplicity subsystems; in other words, it

acts only on the relational degrees of freedom of the transmitted systems. In such a case, RF alignment is still possible (although possibly at a decreased efficiency, as the optimal protocols took advantage of these multiplicity subsystems). Alice and Bob can choose to transmit states that are encoded entirely within the gauge subsystems \mathcal{M}_q , as these subsystems are decoherence-free in terms of the noise.

On the other hand, if the noise is of the form $\mathcal{F} = \oplus_q \mathcal{D}_{\mathcal{M}_q} \otimes I_{\mathcal{N}_q}$ in terms of the decomposition of Eq. (2.24), then the gauge subsystems \mathcal{M}_q will experience complete decoherence. However, Alice and Bob cannot choose to execute their alignment protocol entirely within the decoherence-free multiplicity subsystems \mathcal{N}_q , because these subsystems cannot carry unspeakable information (at least, not of this type). Whereas speakable information can be encoded into any desired subsystem, unspeakable information must be encoded into subsystems carrying the appropriate degree of freedom.

This latter case can be worded as a simple physical example. Consider the alignment of a phase reference, using a noisy channel that simply adds a constant but unknown phase shift. If Alice and Bob use one of the techniques of this section to attempt to align their phase references using this channel, Bob will acquire an estimate of the phase difference between his and Alice's RFs. However, because Bob knows this estimate may differ from the actual difference by some unknown shift, caused by the channel, he in fact has learned nothing about the relation between his phase reference and Alice's. There is no protocol that they can perform that will distinguish the unknown phase shift relating their RFs and the unknown phase shift applied by the channel, and thus alignment cannot be performed using this channel (Preskill, 2000; Yurtsever and Dowling, 2002).

VI. QUANTUM INFORMATION WITH BOUNDED REFERENCE FRAMES

In the reference frame alignment schemes of Sec. V, we determined which quantum states of a given bounded size were optimal in serving as a sample of the sender's classical reference frame. The systems were ultimately measured relative to the receiver's classical reference frame, so that the unspeakable information that they contained was essentially amplified to the macroscopic scale with some associated uncertainty. However, there will be situations for which this amplification process is not ideal, and instead one should make direct use of the quantum RF itself.²⁰ Whatever purpose the recipient had in mind in trying to align his classical RF with that of the sender's, one can ask to what extent he could achieve this same purpose by storing his quantum sample of the sender's reference frame in his laboratory and thereafter using it in place of his classical RF.

Furthermore, many quantum experiments involve me-

¹⁸The optimal average fidelity that can be achieved for a symmetric product state consisting of N spin-1/2 systems (a parallel state) is $(N+1)/(N+2)$, and this result generalizes to any symmetric pure state (Massar and Popescu, 1995).

¹⁹That is, unless the spin degrees of freedom are coupled to other fields in the laboratory. This coupling itself would negate the protocol, however, as it implies some ongoing active transformations on the stored spins.

²⁰For example, Janzing and Beth (2003) consider the constraints on amplifying and copying quantum RFs for phase.

soscopic or even microscopic systems that can be understood as playing the role of a reference frame. For instance, a Bose-Einstein condensate may act as a reference frame for the phase conjugate to atom number, even though it may contain a relatively small number of atoms. We are therefore led to consider the question of how well a bounded-size quantum system may stand in for a classical reference frame.

In Sec. IV we have already considered the problem of treating reference frames within the quantum formalism, but the system instantiating the reference frame was assumed to be of unbounded size. Here we are interested in bounded-size quantum reference frames. We focus, in particular, on the implications of such RFs on one's ability to perform quantum-information processing tasks, specifically: the fundamental primitive of quantum state estimation, operations and measurements in quantum computation, and the quantum cryptographic protocols of data hiding and bit commitment. Furthermore, we demonstrate that for bounded shared RFs, like entanglement, it is possible to develop a general theory of the manner in which this resource is distributed, transformed from one form to another, distilled, degraded with use, quantified, etc.

A. Measurements and state estimation with bounded reference frames

State estimation is a fundamental primitive of quantum-information processing. In this section, we discuss the role of reference frames in performing measurements required for state estimation, and the effect of bounding the size of this RF.

1. A directional example

Consider the task of estimating whether the state of a spin-1/2 system is aligned or antialigned with some perfect (unbounded) directional RF, given the promise that it is one of the two. If one is able to compare the system with this RF, then this task can be easily achieved, as it corresponds simply to discriminating a pair of orthogonal states, $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$, where we take \mathbf{z} to be the axis defined by the directional RF. Specifically, a measurement of $\mathbf{S}\cdot\mathbf{z}$, the spin along \mathbf{z} , determines the answer with certainty. In contrast, if one is not able to make use of this RF, then a superselection rule is in force, the measurement of $\mathbf{S}\cdot\mathbf{z}$ becomes impossible, and the states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ become completely indistinguishable.

There is an intermediate scenario between these two extreme cases, however, wherein one only has access to a sample of the RF—one that is of bounded size. In this case, $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ become partially distinguishable, as we now demonstrate with an example. We consider the case wherein the directional RF is a spin- j system, for some arbitrary but finite j , prepared in an SU(2) coherent state $|j\mathbf{z}\rangle$ (the eigenstate of $\mathbf{J}\cdot\mathbf{z}$ associated with the maximum eigenvalue).

Because the task is to estimate the relations between the bounded RF and the system, it is possible to restrict

the measurement to one that is invariant under collective rotations (i.e., rotations of both the bounded RF and the spin-1/2 system by the same amount). In other words, one can consider a global superselection rule associated with the group SU(2) to apply, because the system serving as an RF for direction is treated internally. As a result, the form of the measurement is highly constrained. Note that the joint Hilbert space $\mathcal{H}_j\otimes\mathcal{H}_{1/2}$ of the bounded RF and system decomposes into a sum $\mathcal{H}_{j+1/2}\oplus\mathcal{H}_{j-1/2}$ of a $J=j+1/2$ and a $J=j-1/2$ irreducible representation of SU(2), the group of collective rotations. By Schur's lemmas (see the proof in Sec. II) a positive operator on this space that is SU(2)-invariant must have the form $p_+\Pi_{j+1/2}+p_-\Pi_{j-1/2}$, where $\Pi_{j\pm 1/2}$ is the projector onto $\mathcal{H}_{j\pm 1/2}$. Thus a rotationally invariant measurement is represented by a POVM with elements of this form. However, any such POVM may be obtained by classical post-processing of the outcome of the two-element projective measurement $\{\Pi_{j+1/2}, \Pi_{j-1/2}\}$, so that the latter is the most informative rotationally invariant POVM. The POVM elements $\Pi_{j+1/2}$ and $\Pi_{j-1/2}$ are associated with the measurement outcomes "aligned" and "antialigned," respectively.

Denote the probability that the state $|\pm\mathbf{z}\rangle$ is found to be aligned with the bounded RF by $p(+|\pm)$ and the probability that it is found to be antialigned by $p(-|\pm)$. The Born rule

$$p(\pm|\pm) = \langle j\mathbf{z}|\langle\pm\mathbf{z}|\Pi_{j\pm 1/2}|j\mathbf{z}\rangle|\pm\mathbf{z}\rangle \quad (6.1)$$

yields

$$p(+|+) = \frac{2j+1}{2j+2}, \quad p(-|+) = \frac{1}{2j+2},$$

$$p(+|-) = 0, \quad p(-|-) = 1. \quad (6.2)$$

Assuming equal prior probabilities for $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$, the average probability of successful discrimination is

$$p_{\text{success}} = \frac{1}{2}p(+|+) + \frac{1}{2}p(-|-) = 1 - \frac{1}{4(j+1)}. \quad (6.3)$$

The smallest possible RF corresponds to taking $j=1/2$, in which case $p_{\text{success}}=5/6$; see [Pryde et al. \(2005\)](#) for an experiment based on this example. For large j , we have a probability of success that approaches 1 linearly in $1/j$, and we recover perfect distinguishability as $j\rightarrow\infty$, corresponding to the case of an unbounded RF.

This example can be extended to the problem of estimating the relative angle between a spin j_1 and a spin j_2 ; the optimal measurement is the projective measurement $\{\Pi_J, J=|j_1-j_2|, \dots, j_1+j_2\}$ onto the subspaces of total angular momentum J , for the same reasons as above ([Bartlett et al., 2004b](#)). The optimization problem becomes nontrivial when we allow for states of the bounded RF and/or the system that span multiple irreps, i.e., states that are not eigenstates of \hat{J}^2 . Measuring a spin- j system relative to a bounded directional RF con-

sisting of a pair of spins has been considered by [Bagan, Iblisidor, and Munoz-Tapia \(2006\)](#) and [Lindner *et al.* \(2006\)](#).

These sorts of results serve to illustrate how, for Lie groups at least, a measurement relative to a bounded RF cannot perfectly simulate one relative to an unbounded RF. Note that we have only considered the inferential but not the transformative aspect of the measurement, that is, we have not considered how the quantum state of the system is updated as a result of the measurement. The work of [Wigner \(1952\)](#) and of [Araki and Yanase \(1960\)](#) demonstrates, however, that for rank-1 projective measurements one cannot perfectly simulate a von Neumann update rule when an unbounded RF is replaced by a bounded one.

2. Measuring relational degrees of freedom

We note that measurements relative to a bounded quantum RF are an example of measurements of relational degrees of freedom. While a complete discussion of relational formulations of quantum theory is beyond the scope of this review, we briefly make some connections between problems involving reference frames and relational ones.

The estimation of relative parameters for various degrees of freedom encompasses such natural tasks as estimating the distance between two massive particles, the phase between two modes of an electromagnetic field (the essential aim of a homodyne measurement), or the angle between a pair of spins as described above, all of which are clearly related to issues of bounded reference frames. Such measurements have been discussed in connection with their ability to induce a relation between quantum systems that had no relation prior to the measurement, e.g., inducing a relative phase between two Fock states ([Javanainen and Yoo, 1996](#); [Molmer, 1997](#); [Sanders *et al.*, 2003](#)) or a relative position between two momentum eigenstates ([Rau *et al.*, 2003](#); [Cable *et al.*, 2005](#)).

Measurements of relative parameters are critical for achieving programmable quantum measurements ([Dušek and Bužek, 2002](#); [Fiurášek *et al.*, 2002](#)). Such measurements use the state of a quantum system (the reference system) to determine the form of a measurement performed on another system. For example, a reference system in an $SU(2)$ coherent state as above can be used to “program” the choice of measurement basis of a spin-1/2 system. The problem of optimizing programmable quantum measurements, including determining the form of the joint measurement and the optimal state of the reference system, is directly related to the problem of optimizing measurements of relative parameters between a system and a bounded reference frame.

Note that problems of relative parameter estimation are complementary to those of determining the optimal measurement schemes for estimating collective parameters for a rotational degree of freedom, the subject of Sec. V.

B. Quantum computation with bounded reference frames

1. Precision of quantum gates

In the majority of architectures proposed for quantum computation, external classical fields are utilized to implement single qubit logical operations. As an example, we focus on the use of coherent states of the electromagnetic field, which are particularly ubiquitous for quantum computing architectures—in the form of either lasers or radio frequency fields generated by an oscillating current. From the perspective of this review, we interpret such fields as defining a reference frame—in this case, a clock—with respect to which coherent superpositions of the computational basis (energy eigenstate) states are necessarily defined. Note that the fact that the RF is interacting directly with the qubits does not weaken such a viewpoint—at some stage in the quantum-information processing a clock must physically interact with the quantum computer (perhaps via intermediary systems); if it did not, then there would be no operational difference if we enforced a superselection rule for energy.²¹

If the reference frame is bounded—quantified in this example as a finite mean photon number of the laser field—the operations performed with respect to this bounded RF may have imperfect precision, and generally the system and the field become entangled. A simple and standard model of a single two-level atom resonantly interacting with a single mode (cavity) field via a Jaynes-Cummings interaction serves to illustrate the basic idea ([van Enk and Kimble, 2001](#)). The interaction Hamiltonian takes the form

$$H = i\hbar g(\hat{S}^+ \hat{a} - \hat{a}^\dagger \hat{S}^-), \quad (6.4)$$

where \hat{a} is the field mode annihilation operator, and \hat{S}^\pm are the atomic raising and lowering operators. If the field mode is initially prepared as a coherent state $|\alpha\rangle = e^{-(1/2)|\alpha|^2} \sum_n (\alpha^n / \sqrt{n!}) |n\rangle$ with very large amplitude $|\alpha|^2 \rightarrow \infty$, it is common to replace the field operators \hat{a}, \hat{a}^\dagger with classical c -numbers α and α^* . In the language of this review, this is the process of externalizing the reference frame. For an atom initially in the excited state, evolution under this classical field then yields the well-known Rabi oscillations between the ground $|0\rangle$ and excited states $|1\rangle$. Specifically, the state at time t is

$$|\psi_C(t)\rangle = \sin(g|\alpha|t)|0\rangle + \cos(g|\alpha|t)|1\rangle. \quad (6.5)$$

Consider now what occurs if we choose not to externalize the driving field, and in particular describe it via a finite amplitude coherent state. The evolution of the

²¹Previously, we have been concerned primarily with SSRs for photon number when dealing with optical examples. However, in this section, as we are discussing the coupling of atoms to the optical fields under (additively) energy conserving Hamiltonians, we extend the type of SSR under consideration to one for total energy.

atom and field under the Hamiltonian of Eq. (6.4) can be solved exactly, yielding

$$|\psi_Q(t)\rangle = \sum_{n=1}^{\infty} A_n(t)|0\rangle|n\rangle + B_{n-1}(t)|1\rangle|n-1\rangle, \quad (6.6)$$

where

$$A_n(t) = \frac{e^{-1/2|\alpha|^2} \alpha^{n-1}}{\sqrt{(n-1)!}} \sin(g\sqrt{nt}), \quad (6.7)$$

$$B_{n-1}(t) = \frac{e^{-1/2|\alpha|^2} \alpha^{n-1}}{\sqrt{(n-1)!}} \cos(g\sqrt{nt}). \quad (6.8)$$

To compare this evolution under a bounded reference frame to the ideal (unbounded) case we should dequantize the reference frame²² using the techniques of Sec. IV.A.2. Following the procedure outlined therein, we now move into the tensor product structure induced by energy difference (relational) versus total energy (global). In terms of this tensor product structure we have

$$|\psi_Q(t)\rangle = \sum_{n=0}^{\infty} |\phi_n(t)\rangle_{\text{rel}} |n-1\rangle_{\text{gl}}, \quad (6.9)$$

where $|\phi_n(t)\rangle_{\text{rel}} = A_n(t)|0\rangle_{\text{rel}} + B_{n-1}(t)|1\rangle_{\text{rel}}$ is unnormalized.

The reduced density matrix of the relational system (i.e., we trace out the global degree of freedom) is therefore a mixed state,

$$\rho_Q(t) = \sum_{n=0}^{\infty} |\phi_n(t)\rangle\langle\phi_n(t)|, \quad (6.10)$$

and it is this mixed state we wish to compare with the pure state (6.5) expected in the unbounded, externalized description.

We perform the comparison by computing the fidelity $F(t) = \langle\psi_C(t)|\rho_Q(t)|\psi_C(t)\rangle$ between the mixed state obtained through the full quantum treatment and the pure state of Eq. (6.5) obtained via the above approximation. If we consider the specific choice of evolution time $t = \pi/(2|\alpha|g)$, which corresponds to performing a σ_x gate, then this fidelity is very well approximated (even for small values of $|\alpha|$) by (Haroche, 1984)

$$\begin{aligned} F\left(t = \frac{\pi}{2|\alpha|g}\right) &\cong \frac{1}{2} \left(1 - \cos\left(\frac{\pi\sqrt{1+|\alpha|^2}}{|\alpha|}\right) e^{-\pi^2/8(|\alpha|^2+1)}\right) \\ &= 1 - \frac{\pi^2}{16|\alpha|^2} + \mathcal{O}\left(\frac{1}{|\alpha|^4}\right). \end{aligned} \quad (6.11)$$

We see that the gate operation results in a state that is in error (as quantified by the fidelity) by an amount that is inversely proportional to the mean number of photons in the driving field.

²²This dequantization was not carried out by van Enk and Kimble (2001), although for this example it does not make a quantitative difference to the overall conclusions about gate fidelity.

The extent to which such a model captures the essential features of currently proposed quantum computing architectures has been the subject of considerable debate, cf. van Enk and Kimble (2001); Gea-Banacloche (2002a, 2002b); Itano (2003); Silberfarb and Deutsch (2003); Gea-Banacloche and Ozawa (2005); Nha and Carmichael (2005). What is clear is that such effects are generally about two orders of magnitude smaller than the typical spontaneous emission rates in these systems. However, in situations wherein the reference frame is small (for example, if quantum computers together with the control fields were to be built on chips in an integrated manner) or in systems which have negligible spontaneous emission, then it is not unreasonable that such considerations will have to be incorporated into analyses of fault tolerance.

2. Degradation of a quantum reference frame

As we have demonstrated, a bounded reference frame can result in nontrivial limitations on one's ability to perform operations and measurements on quantum systems, and thus limitations on quantum-information processing tasks such as quantum computing. However, this imprecision is not the only limitation enforced by quantum mechanics. In addition, any measurement that acquires information about the relations between the system and RF must necessarily disturb them uncontrollably. The resulting disturbance to the RF can be understood as a measurement *back action*. The effect of this back action has been studied for reference frames for spatial position (Aharonov and Kaufherr, 1984), for directional reference frames (Aharonov *et al.*, 1998), and for clocks (Casher and Reznik, 2000). Here we investigate how measurement back action on a bounded RF can lead to its degradation, i.e., a reduction of its suitability to perform future measurements.

The conventional approach wherein reference frames suffer no back action may yield a poor approximation to the full quantum treatment, as suggested above. This issue may be particularly important for quantum computation, where a large number of high-precision measurements must be performed. In some implementations, such measurements are performed relative to a reference frame that is usually described by a finite quantum system; for example, the proposed single-spin measurement technique using magnetic resonance force microscopy (Rugar *et al.*, 2004), or the single-electron transistors used for measurement of superconducting qubits (Makhlin *et al.*, 2001). We now demonstrate that the number of measurements for which a quantum reference frame can be used scales quadratically rather than linearly in the size of the reference frame, which is a promising result for the prospect of using microscopic or mesoscopic reference frames in performing repeated high-precision measurements.

In the following example, we investigate the degradation of a quantum reference direction as it is used for repeated measurements. We use a spin- j system for our quantum reference direction (the RF), with Hilbert

space \mathcal{H}_j . We choose the initial quantum state of the spin- j system to be $\rho^{(0)} = |j, j\rangle\langle j, j|$; this choice of initial state simplifies the analysis, and in addition it has been determined to be the initial state that maximizes the initial success probability (Bartlett, Rudolph, Spekkens, and Turner, 2006). This quantum RF is aligned in the $+z$ direction relative to a background frame.

The systems to be measured will be spin-1/2 systems, each with a Hilbert space $\mathcal{H}_{1/2}$. We choose the initial state of each such system to be the completely mixed state $I/2$, and our quantum RF will be used to measure many such independent spin-1/2 systems sequentially. We assume trivial dynamics between measurements, and thus our time index will simply be an integer specifying the number of measurements that have taken place. The state of the RF following the n th measurement is $\rho^{(n)}$, with $\rho^{(0)}$ denoting the initial state of the RF prior to any measurement. We consider the state of the RF from the perspective of someone who has not kept a record of the outcomes of previous measurements. Thus, at every measurement, we average over the possible outcomes with their respective weights to obtain the final density operator.

The measurement which optimally determines whether a spin-1/2 particle is aligned or antialigned to a spin- j system was determined in Sec. VI.A.1 to be the two-outcome projective measurement $\{\Pi_+ \equiv \Pi_{j+1/2}, \Pi_- \equiv \Pi_{j-1/2}\}$ on $\mathcal{H}_j \otimes \mathcal{H}_{1/2}$. We use this measurement here. It can be shown that of the many ways of implementing this measurement, the update rule that degrades the reference frame the least is the standard Lüders update rule (Bartlett, Rudolph, Spekkens, and Turner, 2006). Thus the resulting evolution of the quantum RF as a result of the n th measurement is

$$\rho^{(n+1)} = \mathcal{E}(\rho^{(n)}), \quad (6.12)$$

where the superoperator \mathcal{E} is given by

$$\mathcal{E}(\rho) = \text{Tr}_S \left(\sum_{c \in \{+, -\}} \Pi_c(\rho \otimes I/2) \Pi_c \right), \quad (6.13)$$

with Tr_S the partial trace over $\mathcal{H}_{1/2}$.

The map \mathcal{E} can be written using the operator-sum representation as

$$\mathcal{E}(\rho) = \frac{1}{2} \sum_{c \in \{+, -\}} \sum_{a, b \in \{0, 1\}} E_{ab}^c \rho E_{ab}^{c\dagger}, \quad (6.14)$$

where $E_{ab}^c \equiv \langle a | \Pi_c | b \rangle$ is a Kraus operator on \mathcal{H}_j and $\{|0\rangle, |1\rangle\}$ is a basis for $\mathcal{H}_{1/2}$. These operators can be straightforwardly determined in terms of Clebsch-Gordan coefficients.

We quantify the quality of a quantum RF as the average probability of a successful estimation of the orientation of a fictional test spin-1/2 system which is, with equal probability, either aligned or antialigned with the background $+z$ axis. Denote the pure state of the test spin-1/2 system that is aligned (antialigned) with the initial RF by $|0\rangle$ ($|1\rangle$). For a spin-1/2 system prepared in the state $|0\rangle$ or $|1\rangle$ with equal probability, the average probability of success using a quantum RF state ρ is

$$\bar{P}_s = \frac{1}{2} \text{Tr}_R[\rho(E_{00}^+ + E_{11}^-)]. \quad (6.15)$$

The solution for $\rho^{(n)}$, given the initial state $\rho^{(0)} = |j, j\rangle\langle j, j|$, yields an average probability of success $\bar{P}_s(n)$ that decreases with n as

$$\bar{P}_s(n) = \frac{1}{2} + \frac{j}{2j+1} \left(1 - \frac{2}{(2j+1)^2} \right)^n. \quad (6.16)$$

The initial slope R of this function bounds the rate of degradation. It is

$$R \equiv \bar{P}_s(1) - \bar{P}_s(0) = -2j/(2j+1)^3. \quad (6.17)$$

Thus, in the large j limit, we have the rate of degradation with n satisfying $R \geq -1/(4j^2)$.

Let $\epsilon < 1$ be a fixed allowed error probability for the spin-1/2 direction estimation problem. After n measurements, the probability of successful estimation is lower bounded by $1+nR$, so the number of measurements required to ensure that this bound be greater than $1-\epsilon$ is $-\epsilon/R$. Consequently, the number of measurements that can be implemented relative to the spin- j RF with probability of error less than ϵ is

$$n_{\max} \simeq \epsilon j^2. \quad (6.18)$$

This result implies that the number of measurements for which an RF is useful, that is, the longevity of an RF, increases quadratically rather than linearly with the size of the RF. Thus in order to maximize the number of measurements that can be achieved with a given error threshold, one should combine all of one's RF resources into a single large RF and perform all measurements relative to it, rather than use a number of smaller RFs individually. We note that this degradation, as quantified by the decreasing average probability of success $\bar{P}_s(n)$, can be modeled precisely as the distribution of a classical reference direction undergoing a random walk (Bartlett, Rudolph, Sanders, and Turner, 2006).

Using similar methods, it has also been demonstrated that a bounded quantum phase reference, realized as a single-mode quantum state of the electromagnetic field with bounded photon number, also leads to a longevity that scales quadratically in this size (mean photon number) (Bartlett, Rudolph, Spekkens, and Turner, 2006). It is an open problem to determine if this quadratic scaling is a general result.

We have discussed the degradation associated with a loss of purity of the reference frame state. Another mechanism of degradation is for the reference frame to become misaligned with the background reference frame of which it is a token. Poulin and Yard (2006) have shown how a reference frame can suffer this sort of misalignment when the systems being measured have a non-zero polarization (that is, are not described by the completely mixed state), and that in the presence of such drift, the longevity scales linearly, rather than quadratically, in the size of the reference system.

C. Quantum cryptography with bounded reference frames

In Sec. IV.D, we demonstrated that SSRs cannot provide any fundamental limitations on quantum cryptographic protocols, essentially because quantum reference systems which obey the SSR can enable it to be effectively lifted. However, this result does not mean that SSRs are uninteresting for cryptography. In quantum cryptography, it is typical to focus on unconditional security—security not premised upon assumptions about the resources of one’s adversaries, but only upon the validity of the laws of quantum mechanics. In classical cryptography, in contrast, security is typically conditional—it is generally premised upon assumptions about the computational capabilities of one’s opponents. Other types of conditional security can be premised upon assumptions about other noncomputational capabilities or resources available to the adversarial parties. In this section, we consider the specific case where the physical resource about which assumptions are being made is some kind of RF, the lack of which in turn induces an effective SSR. This is effectively an assumption of bounded resources, because given unlimited resources the SSR can be lifted as in Sec. IV.B. We use the specific examples of data hiding and bit commitment to illustrate protocols that achieve this sort of security.²³

1. Data hiding with a superselection rule

In a quantum data hiding protocol, one party (Charlie) wants to share a single datum by distributing systems among two other parties (Alice and Bob) in such a way that the bit can only be recovered if the parties have some mechanism for performing joint measurements on the distributed systems. Such measurements could be performed by the parties coming together, by using a quantum channel, or by performing teleportation (using prior entanglement) with a classical channel. These possibilities are generally considered equivalent. Thus Charlie must assume that the two parties have no access to the specific physical resource of a quantum channel. It has been proven that perfect quantum data hiding is not possible even with this assumption (Terhal *et al.*, 2001).

If, in addition, Charlie can assume that the two parties do not share a phase reference (that is, they are subject to a local Abelian SSR) then perfect data hiding can be achieved (Verstraete and Cirac, 2003). For example, without a shared phase reference for their optical modes as in Sec. III.C.1, Alice and Bob cannot distinguish the pair of orthogonal pure states $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle) / \sqrt{2}$ using LOCC because $\mathcal{U}_A \otimes \mathcal{U}_B(|\psi^\pm\rangle) = \mathcal{U}_A \otimes \mathcal{U}_B(|\psi^\mp\rangle)$, and so this pair of states could be used to encode the classical bit.

As discussed in Sec. IV.B, it is also clear that shared reference systems could be used by Alice and Bob to

break such data hiding protocol. Such reference systems need not be entangled, which shows that breaking of data hiding in this case is quite different to the case of using entanglement to implement a quantum channel. However, if Charlie has reason to believe that the reference systems shared by Alice and Bob are bounded in size, then it is still possible for him to achieve data hiding. He does this by using such a large number of systems to encode the bit that any bounded shared reference does not suffice to extract all required data.

2. Ancilla-free bit commitment

A particularly simple class of bit commitment protocols (Spekkens and Rudolph, 2001) involve Alice preparing one of two orthogonal states $|\chi_{0,1}\rangle$, according to whether she wishes to commit a bit $b=0,1$. Here $|\chi_{0,1}\rangle$ are (generally entangled) states over a proof system and a token system. She sends the token system to Bob as her commitment. To unveil the bit, she sends Bob the proof system, and he verifies her commitment by projecting onto $|\chi_{0,1}\rangle$. A simple version of such a protocol was discussed in Sec. IV.D above.

Consider the situation wherein Alice and Bob are constrained such that they cannot make use of a reference frame, either shared or local. This constraint enforces the protocol to be ancilla-free—for instance, we do not allow either party to prepare ancillary systems which could then act as an effective RF in the manner described in Sec. IV.B. It turns out that under such a constraint, arbitrarily secure bit commitment is possible (DiVincenzo *et al.*, 2004).

It is illustrative to first consider a case where such a restriction does not help. Consider ancilla-free bit commitment in the case that Alice and Bob lack a phase reference. As Alice must prepare the initial states $|\chi_{0,1}\rangle$, they must each lie completely in a single superselection sector, i.e., eigenstates of total photon number, and take the form

$$|\chi_b\rangle = \sum_n c_n^b |N-n\rangle_P |n\rangle_T. \quad (6.19)$$

Because the reduced density matrices of the token system $\rho_b = \sum_n |c_n^b|^2 |n\rangle_T \langle n|$ are diagonal in the number basis, the fact that Bob can only perform measurements diagonal in this basis actually is no restriction on him at all—he can cheat (by trying to distinguish these states) just as well as he could in an unconstrained protocol.

Consider now if Alice is cheating, i.e., she attempts to commit her bit only after the commitment stage. An optimal cheating strategy for Alice is to prepare a state $|\tilde{\chi}\rangle \propto |\chi_0\rangle + U_P \otimes I_T |\chi_1\rangle$, where the unitary matrix U_P on the proof system is one which maximizes the overlap $\langle \chi_0 | U_P \otimes I_T | \chi_1 \rangle$ (Spekkens and Rudolph, 2001). If, after the commitment stage, she decides to commit $b=0$, then she simply sends the proof system as is. If instead she decides to commit $b=1$, then she applies U_P^\dagger to the proof system before sending it to Bob. The question is whether Alice can perform this optimal cheating strategy despite the SSR. Consider first the unitary U_P which

²³We note in passing that a different type of assumption, namely, that Alice and Bob share partially misaligned reference frames, can be used as a kind of guaranteed noisy channel, and, as in classical cryptography, such channels can be used for secure two party protocols (Harrow *et al.*, 2006).

maximizes $\langle \chi_0 | U_P \otimes I_T | \chi_1 \rangle$. If $U_P |N-n\rangle \equiv |A_{N-n}\rangle$, where $|A_i\rangle$ is a state not necessarily respecting the SSR, then $\langle \chi_0 | U_P \otimes I_T | \chi_1 \rangle = \sum_n |c_n|^2 \langle A_{N-n} | N-n \rangle$. Clearly the maximization of this expression will be achieved by choosing $|A_{N-n}\rangle = e^{i\phi_{N-n}} |N-n\rangle$, i.e., for a unitary U_P which is diagonal in the number state basis. Furthermore, the state $|\tilde{\chi}\rangle$ then takes the generic form $\sum_n c_n |N-n\rangle_P |n\rangle_T$ which respects the SSR. Under the assumptions of an ancilla-free SSR protocol, any state prepared by Alice or any unitary operator she applies is constrained to be diagonal in the number basis—as we have seen, in this case she can still achieve the optimal cheating strategy despite such a constraint.

The Abelian SSR induced by lack of a phase reference therefore does not help devise a more secure ancilla-free bit commitment. It can be shown, however, that a different type of Abelian SSR does lead to arbitrarily secure ancilla-free bit commitment. We follow the work of DiVincenzo *et al.* (2004). Consider a number of spin systems, with a local Abelian SSR given as follows: all local operations must commute with the total local angular momentum operator J^2 . Thus all states and operations must be diagonal in total spin quantum number j . However, they can have coherence between the different m eigenvalues of J_z . This superselection rule is distinct from any that we have considered in this review, and does not appear to be related to the lack of an appropriate reference frame. The key property of this Abelian SSR that will be useful for ancilla-free bit commitment, and is distinct from the other Abelian SSRs we consider, is that the quantum number j labeling the local superselection sectors is nonadditive.

Using the standard $|j, m\rangle$ notation for the uncoupled basis, consider the bit commitment protocol which is defined by the following two states $|\chi_b\rangle$ of total spin $j=1$:

$$\begin{aligned} |\chi_b\rangle = & \frac{1}{\sqrt{2}} |1, 1\rangle_P |\phi_0^b\rangle_T + \frac{1}{\sqrt{3}} |1, 0\rangle_P |\phi_1^b\rangle_T \\ & + \frac{1}{\sqrt{6}} |1, -1\rangle_P |\phi_2^b\rangle_T, \end{aligned} \quad (6.20)$$

where

$$|\phi_0^b\rangle_T = \frac{2}{3} |0, 0\rangle_T + (-1)^b \frac{1}{2} |1, 0\rangle_T + \frac{\sqrt{2}}{6} |2, 0\rangle_T, \quad (6.21)$$

$$|\phi_1^b\rangle_T = (-1)^b \frac{\sqrt{3}}{2} |1, 1\rangle_T - \frac{1}{2} |2, 1\rangle_T, \quad (6.22)$$

$$|\phi_2^b\rangle_T = |2, 2\rangle_T. \quad (6.23)$$

We note that although the proof system is also an eigenstate of J_T^2 with eigenvalue $j=1$, the token system is not; this is a result of the nonadditive nature of this SSR. If we look at the reduced density matrices $\rho_{0,1}$ on the token system in the uncoupled basis, we see that they are block diagonal (incoherent mixtures) in the eigenspaces of J_z , with eigenvalues $m=0, 1, 2$. Within each block, the diagonal elements of $|\phi_m^b\rangle\langle\phi_m^b|$ are the same—that is, they are

indistinguishable by their total spin. Under the SSR, Bob is restricted to performing measurements which are diagonal in total spin, and so these two states are completely indistinguishable.

In a general bit commitment scenario, indistinguishability of the token systems by Bob would imply that Alice has complete control—that she should be able to perfectly change her commitment after the commitment stage. However, this is not the case for this example—the two states $\rho_{0,1}$ have a nonunit fidelity $F(\rho_0, \rho_1) < 1$. Because the fidelity sets a bound (for these type of protocols) on how well Alice can control the outcome (regardless of any restrictions on her), we see that some security against Alice will be possible.

Generalizations of the above pair of states $|\chi_b\rangle$ can be defined for which, as j becomes large, $F(\rho_0, \rho_1) \rightarrow 0$, implying perfect security against Alice (DiVincenzo *et al.*, 2004). We believe that a fruitful avenue for future research would be to determine if such constraints on ancilla-free bit commitment can be achieved using a non-Abelian superselection rule of the form discussed in this review.

D. Quantifying bounded shared reference frames

Much of quantum-information theory is concerned with tradeoffs in the utilization of various types of fundamental resources. The canonical example is quantum teleportation, which demonstrates that one *ebit* (Bell pair) of shared entanglement plus two communicated classical bits are equivalent to the communication of a single qubit. In this section, we demonstrate that a shared reference frame is also a quantifiable resource, akin to entanglement, which allows parties to perform tasks that they were unable to perform without it, or to perform tasks more efficiently.

Consider the activation example of Sec. III.C.2, which involved two parties (Alice and Bob) who do not share the phase reference of a third party (Charlie) or each other. In this context, the two-mode single-photon state $(|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B) / \sqrt{2}$ could not be used to perform quantum teleportation or to violate a Bell inequality. However, if Alice and Bob were also provided with the bipartite state $|+\rangle_A |+\rangle_B$, where $|+\rangle = (|0\rangle + |1\rangle) / \sqrt{2}$ described relative to Charlie's phase reference, they could activate the entanglement in the former state through LOCC. Although Alice and Bob do not share Charlie's phase reference, clearly the state $|+\rangle_A |+\rangle_B$ provides a bounded version of it. This bounded shared phase reference can be used to activate the entanglement of the two-mode single-photon state, as can an unbounded classical shared phase reference. However, unlike the latter, the bounded shared phase reference $|+\rangle_A |+\rangle_B$ can only activate the entanglement with probability $1/2$, and in addition is consumed in the process; it is a shared reference frame that can be depleted, in this case, through a single use. The state $|+\rangle_A |+\rangle_B$ can be considered an elementary unit of Charlie's phase reference, much like an *ebit* (a Bell pair) is considered an elemen-

tary unit of entanglement. As a result of this analogy, the state $|+\rangle_A|+\rangle_B$ has been denoted a *refbit*.

Continuing this example, we note that the two-mode single-photon state $(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$ can also be viewed as a resource for activating another copy of this same state. (This process can alternatively be viewed as two-copy entanglement distillation, as in Sec. III.C.2.) This state is invariant under global phase changes, and thus is completely uncorrelated with Charlie's (or any other party's) phase reference, but it nevertheless provides a bounded version of a shared phase reference for Alice and Bob. It is useful to view this state as the elementary unit of a shared phase reference between Alice and Bob, uncorrelated with any other.

Because of the dual purpose of this state—either as an elementary unit of a shared phase reference or as a state from which entanglement can be activated with the use of a shared phase reference—it has been named and categorized in many different ways depending on its intended use. So which way should it be viewed? The answer is that this state can serve as a resource for both entanglement and a shared reference frame, and that one must trade off its usefulness for one purpose against the other. In fact, a wide variety of tradeoffs between refbits, ebits, cbits, and other resources can be derived (van Enk, 2005a, 2006), which emphasizes the utility of thinking of reference frames as yet another form of resource.

Now that we have identified standard elementary unit(s) of a shared reference frame, we can quantify how well a given quantum state serves as a shared reference frame by the state's asymptotic interconvertibility to this standard form, using local operations and classical communication.²⁴ A remarkable property of entanglement of pure bipartite states is that, by observing the properties of asymptotically reversible transformations using LOCC, entanglement can be quantified by a single additive measure: the reversible conversion efficiency to a standard form of entanglement, the ebit. For an Abelian superselection rule, the resource of a quantum shared reference frame can be quantified by a single additive measure in a similar fashion. Thus the nonlocal properties of pure bipartite quantum states in the presence of an Abelian superselection rule are completely characterized by two additive measures: the entanglement (for which we can use an operational measure such as E_{SSR} , the entanglement in the presence of an SSR, discussed in Sec. III.C.3) and another measure quantifying the state's ability to serve as a shared RF. In the following, we investigate one such measure for the latter, the superselection induced variance (SIV) (Schuch *et al.*, 2004a, 2004b). We note that measures for quantifying quantum shared RFs in the presence of non-Abelian superselection rules have not been explored to date.

²⁴For an alternate measure of how well a quantum state can serve as a shared reference frame, based on entropic properties, see Vaccaro *et al.* (2005).

Let Alice and Bob each have in their possession a number of optical modes, and consider a situation as in Sec. III.C.1 where they do not share a phase reference. Thus Alice and Bob are restricted by an Abelian local superselection rule for photon number. Let \mathcal{H}_A (\mathcal{H}_B) be the local Hilbert space for Alice's (Bob's) modes, and \hat{N}_A (\hat{N}_B) be the total local photon-number operator for these modes. Consider a bipartite quantum state $|\phi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ that is an eigenstate of total photon number $\hat{N}_A + \hat{N}_B$. (This condition ensures that the state is not correlated with another party's phase reference.) The superselection induced variance $V(\phi)$ of this state is defined to be the variance in the local photon number,

$$V(\phi) \equiv 4(\langle\phi|\hat{N}_A^2 \otimes I_B|\phi\rangle - \langle\phi|\hat{N}_A \otimes I_B|\phi\rangle^2). \quad (6.24)$$

This SIV satisfies the following properties: (i) it is additive, meaning $V(\phi \otimes \phi') = V(\phi) + V(\phi')$ for any $|\phi\rangle, |\phi'\rangle$; (ii) it is symmetric under exchange of A and B ; and (iii) it is a bipartite monotone, in that it is nonincreasing under LOCC operations that can be performed by Alice and Bob without a shared phase reference. The state $(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$, which we identified above as an elementary unit of shared phase reference, has an SIV of 1.

Two measures—the entanglement and the SIV—completely quantify the nonlocal resources of a bipartite state. To prove this result, the general idea is to show that Alice and Bob can, through LOCC restricted by the superselection rule, reversibly convert an asymptotic number of copies of the bipartite state into a number of states with only the first type of resource (entanglement) and none of the second (SIV), and a number of states with only the second and none of the first. Let Alice and Bob share N copies of a bipartite state $|\phi\rangle$, which has entanglement $E_{\text{SSR}}(\phi)$ and SIV $V(\phi)$. In addition, let Alice and Bob each have in their possession an arbitrary number of quantum registers—quantum systems that are not restricted by any superselection rule, such as were discussed in Sec. III.C.3; these registers are initiated in an arbitrary unentangled state $|\tilde{0}\rangle|\tilde{0}\rangle$. Then the transformation

$$|\phi\rangle^{\otimes N} \otimes (|\tilde{0}\rangle|\tilde{0}\rangle)^{\otimes E_{\text{SSR}}(\phi)N} \rightarrow \left(\sum_n c_n |n\rangle|N-n\rangle \right) \otimes |\tilde{\psi}^-\rangle^{\otimes E_{\text{SSR}}(\phi)N} \quad (6.25)$$

is asymptotically reversible, and Alice and Bob can perform this transformation with LOCC restricted by the SSR, where the coefficients c_n are Gaussian distributed with variance $NV(\phi)/4$, and $|\tilde{\psi}^-\rangle$ is a maximally entangled Bell state of a pair of qubits of Alice's and Bob's quantum registers.

We now analyze the two states on the right side of Eq. (6.25). The first state $\sum_n c_n |n\rangle|N-n\rangle$ has SIV of $NV(\phi)$. Such a state serves as a good standard shared RF for large N (Vaccaro *et al.*, 2003). Although it is a nonseparable pure state, the entanglement in the presence of an SSR, E_{SSR} , of this state is zero. In contrast, the state of

the unrestricted registers $|\tilde{\psi}\rangle^{\otimes E_{\text{SSR}}(\phi)N}$ clearly contains an amount of entanglement equal to $E_{\text{SSR}}(\phi)N$ standard ebits. As this system is a quantum register and not a system with a phase degree of freedom, it clearly has no function as a shared phase reference; thus the SIV of this state is zero. The two states on the right side of Eq. (6.25) then represent standard forms for each type of nonlocal resource—superselection induced variance, and entanglement in the presence of an SSR—and contain none of the other type.

A proof that the transformation (6.25) is asymptotically reversible with LOCC restricted by the SSR can be found in Schuch *et al.* (2004a). In their proof, they used the entropy of entanglement E rather than E_{SSR} ; we note that in the asymptotic limit for an Abelian SSR, $E_{\text{SSR}}(\phi^{\otimes N}) \rightarrow E(\phi^{\otimes N})$ for any pure state $|\phi\rangle$ (Wiseman and Vaccaro, 2003). Thus their proof applies directly to the above statement. This result can be extended to apply to mixed states (Schuch *et al.*, 2004b). Finally, we note that explicit protocols for activation—creating states with $E_{\text{SSR}} \neq 0$ using states with $E_{\text{SSR}} = 0$ using a quantum shared reference frame state—have been developed (Vaccaro *et al.*, 2003; Bartlett, Doherty, *et al.*, 2006).

E. Purification of bounded shared reference frames?

As noted in the previous section, the state $(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$ can be viewed as the elementary unit of a shared phase reference between Alice and Bob, uncorrelated with any other. This state has the appearance of a maximally entangled Bell state (see Sec. III.C), and so a natural question is to ask whether a number of imperfect (noisy) states can be purified to a smaller number of superior states of this form. Of course, for such a process to be of any use it would need to be implementable without the use of some other, unbounded shared RF. An affirmative answer would mean that shared RFs are a resource that can be purified, just like entanglement. Unfortunately, however, such a task does not appear to be possible, as we now demonstrate, following Preskill (2000).

Consider a noisy shared RF state that is a mixture of the state $(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$ with probability $p > 1/2$ and the state $(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)/\sqrt{2}$ with probability $1 - p$. Let Alice and Bob share two copies of this mixed state. With these states, they attempt to perform the following simple entanglement purification protocol (Bennett *et al.*, 1996): they each apply a controlled-NOT (CNOT) on the two qubits in their possession, and then perform an X measurement on the target qubit. Each party obtains a measurement outcome ± 1 , which they communicate with each other classically, and compare whether the results are the same or different. Effectively, though this process they have measured the joint nonlocal operator

$$(X_A X_B)_1 \cdot (X_A X_B)_2. \quad (6.26)$$

In the standard entanglement purification protocol, if Alice and Bob keep only those states where they obtain the same measurement results, the resulting states will have higher fidelity with the state $(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$.

Note, however, that this protocol requires operations which are not $U(1)$ invariant. For example, a measurement of X must be performed relative to that party's local phase reference. Let Alice and Bob make use of unbounded local phase references in this protocol. Note that the operator X_A is defined with respect to Alice's phase reference, and X_B with respect to Bob's. If their phase references differ by a phase shift θ_{BA} , then these two operations are related by $X_B = e^{-i\theta_{BA}Z/2} X_A e^{+i\theta_{BA}Z/2}$; see Sec. V.G. Thus the state to which they are purifying in this instance is

$$(|1\rangle_A|0\rangle_B + e^{-i\theta_{BA}}|0\rangle_A|1\rangle_B)/\sqrt{2}. \quad (6.27)$$

If Alice's and Bob's local phase references are uncorrelated, as we assumed, then θ_{BA} is completely unknown, and the protocol does not yield a state with higher fidelity with $(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$.

F. Treating bounded reference frames as decoherence

We conclude this section with a discussion of an approach to describing the effect of using bounded RFs. As demonstrated above, bounded RFs limit one's ability to prepare states and to perform quantum operations and measurements on a system, and the nature of these limitations is similar in many ways to that of decoherence. One is then led to ask whether it is possible to treat bounded RFs externally rather than internally (in the sense of Sec. IV.A.2) by positing an unavoidable decoherence. In other words, if such a description existed, then the bounded size of the RF could be said to effectively reduce the purity and/or coherence of systems described with respect to it.

While no completely general description of treating bounded RFs in this manner has yet been developed, specific examples of such decoherence mechanisms and their consequences have been discussed in various relational approaches to quantum theory (Page and Wootters, 1983; Gambini *et al.*, 2004a, 2004b; Milburn and Poulin, 2006; Poulin, 2006). These discussions have primarily focused on the tricky issue of internalizing time in quantum theory. Unsurprisingly, given the interpretation of certain types of phase references as clocks, these relational formulations generally follow along the lines of the procedures we have already reviewed. One begins by treating all systems which can serve as a clock as internal, constructs (pure or mixed) states that are invariant under global time shifts, identifies relational spaces in the decoherence-free subsystems, refactorizes the Hilbert space in terms of the induced tensor product, and finally interprets the new formulation as the true dynamical description. As expected, a form of decoherence in this new description is found whenever the size

of the internalized reference system(s) is bounded. It is an interesting open problem to identify the appropriate decoherence maps (if they exist) that describe the dynamics of a system relative to a bounded (particularly non-Abelian) RF. While one can debate the appropriateness of these approaches from a foundational perspective, such an approach would certainly be useful for addressing questions in the field of quantum information.

VII. OUTLOOK

The study of reference frames and superselection rules in the context of quantum-information theory is an unfinished task. In this section, we provide an overview of the topics discussed together with some open problems and research directions, while outlining the practical and foundational significance of this sort of investigation.

It is useful to divide the practical applications into two broad categories corresponding to whether their purpose is the manipulation of speakable information or of unspeakable information, that is, corresponding to the nature of their inputs and outputs.

The first category contains the standard problems of interest in quantum-information theory, both those that use quantum systems to manipulate classical information and those whose inputs and outputs are themselves quantum information. Even though these ultimately process speakable information (whether classical or quantum), protocols for such tasks must always encode this information using some degree of freedom, which requires some form of RF. Thus we are led to ask how much the absence of a particular RF or of a shared RF among separated parties decreases the efficiency of various information-processing tasks, or increases the practical difficulty of implementing them. What is the answer to such questions in the case where one has an imprecise RF or two parties share RFs that are only partially correlated? Such questions have been considered here for a variety of tasks, such as quantum and classical communication (Sec. III.A), quantum key distribution (Sec. III.B), and implementing quantum gates (Sec. VI.B.1). There are many more tasks that could be considered. Most of the communication and cryptographic problems considered to date have determined the efficiency only in the case where one demands perfect fidelity encoding and decoding and perfect security. Furthermore, these sorts of questions have been scarcely addressed for the case of shared RFs that are partially correlated. Finally, although there have been a few experiments demonstrating the viability of some of these schemes, such as relational encodings (Sec. III.A.3), the development of realistic physical implementations remains as much a source of experimental challenges as any other quantum technology.

The second category of applications consists of tasks that explicitly involve the manipulation of unspeakable information, such as clock synchronization or the alignment of Cartesian frames. Quantum considerations be-

come important to achieve the optimal precision and it is the tools of quantum-information theory that are best suited to a treatment of the problem. We may describe the alignment of remote reference frames as the communication of unspeakable information, and as soon as one starts describing and thinking about such tasks in the language of information theory, many new tasks suggest themselves. Examples mentioned in this review are dense coding of unspeakable information (Sec. V.K), using private shared RFs as a cryptographic key (Sec. III.D), the private communication of unspeakable information (Sec. V.J), and secret sharing of unspeakable information. Many more analogies of this sort could be considered. Indeed, for almost any information-theoretic task of interest today, it is interesting to muse about possible analogs of it for unspeakable information. (A particularly intriguing question to consider is whether there is such an analog for computation.) On the experimental side, the implementation of quantum protocols for even the best studied of these sorts of tasks, the alignment of reference frames, has, with the exception of phase estimation, only just begun to be investigated.

As emphasized in the Introduction, imposing a restriction on operations generically leads to the identification of a novel resource to overcome this restriction, and we are then compelled to develop a theory for how that resource may be manipulated. For instance, under the restriction of LOCC, entanglement becomes a resource, and the theory of how this resource can be manipulated—the theory of entanglement—has been the subject of a significant amount of work in recent years. Others have considered the theory of communication under natural restrictions such as local operations and public communication (Collins and Popescu, 2002) or restrictions to only Gaussian quantum-optical states and operations (Eisert and Plenio, 2003). A superselection rule (either local and global) is another sort of natural restriction, and under this restriction any quantum state that acts as an RF becomes a resource. The theory of such resources might aptly be called the theory of quantum reference frames or the theory of unspeakable quantum information. It endeavors to answer questions such as how this resource is depleted with use, transformed from one form to another, shared among several parties, etc. Such a theory has only begun to be developed. The limited results on bounded quantum reference frames, described in Sec. VI (see also Sec. III.C) are evidence of this. Moreover, in a sense there is a family of theories to be developed, because we obtain different results depending on the group G with which the superselection rule is associated. Many investigations to date have applied only to the cases of the group $U(1)$ and/or the group $SU(2)$. Ultimately, one would like to have a generic theory of unspeakable information that applies to any group; in particular, noncompact groups (such as the Lorentz or Poincaré groups) may require more general mathematical tools than those discussed here.

It is worth noting that this research is not necessarily

driven by applications. In this sense, it is similar to the study of entanglement in quantum-information theory, which although initially motivated by its practical applications, has increasingly become an interesting subject in its own right. Of course, just as the development of the theory of entanglement has led to many unforeseen practical dividends, we may expect this of a general theory of unspeakable information as well.

Finally, applying the tools of quantum-information theory to the study of RFs and SSRs can shed light on foundational issues in quantum theory. Examples from this review include whether there exist axiomatic superselection rules and the controversy over the nonlocality of a single photon. Another issue which is likely to be clarified by an analysis in terms of quantum reference frames is that of spontaneous symmetry breaking, which is significant in both condensed matter physics and quantum field theory and the foundational status of which is notoriously murky. Foundational issues related to particle statistics may also benefit from such an analysis. For instance, an interesting open question is whether the univalence superselection rule, which forbids coherent superpositions of bosons and fermions (Giulini, 1996), may be lifted by an appropriate reference frame (Dowling *et al.*, 2006). Because this SSR is not associated with a compact group, answering this question requires a formalism more general than the one presented here.

ACKNOWLEDGMENTS

We thank Gilad Gour, Pieter Kok, and Steven van Enk for helpful comments on the manuscript. S.D.B. acknowledges support from the Australian Research Council. T.R. acknowledges support from the Engineering and Physical Sciences Research Council of the United Kingdom. R.W.S. acknowledges support from the Royal Society.

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