Supersymmetric solitons

M. Shifman

William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA

A. Yung

William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA; Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia; and Institute of Theoretical and Experimental Physics, Moscow 117259, Russia

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In the past ten years it has become clear that methods and techniques based on supersymmetry provide deep insights into quantum chromodynamics and other nonsupersymmetric gauge theories at strong coupling. This review summarizes major advances in critical (Bogomol'nyi-Prasad-Sommerfield–saturated) solitons in supersymmetric theories and their implications for understanding basic dynamical regularities of nonsupersymmetric theories. After a brief introduction to the theory of critical solitons (including a historical introduction), three topics are discussed: (i) non-Abelian strings in $\mathcal{N}=2$ and confined monopoles, (ii) reducing the level of supersymmetry, and (iii) domain walls as D-brane prototypes.

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CONTENTS

I. INTRODUCTION

To begin with, we ask why supersymmetric solitons have attracted so much attention in the past 15 years, mainly in connection with non-Abelian gauge theories, and why have advances in this area been so profound. To answer these questions, one must return to the 1970s, when quantum chromodynamics (QCD) was established as the theory of strong interactions. What is the most remarkable feature of quantum chromodynamics and QCD-like theories? The fact that at the Lagrangian level one deals with quarks and gluons while experimentalists detect pions, protons, glueballs, and other color singlet states—never quarks and gluons—is the single most salient feature of non-Abelian gauge theories at strong coupling. Color confinement makes colored degrees of freedom inseparable. In a bid to understand this phenomenon, Nambu, 't Hooft, and Mandelstam suggested in the mid-1970s (independently and practically simultaneously) a non-Abelian dual Meissner effect. At that time, their suggestion was more of a dream than a physical scenario. According to their vision, non-Abelian monopoles condense in the vacuum, resulting in the formation of non-Abelian chromoelectric flux tubes between color charges, e.g., between a probe heavy quark and antiquark. Attempts to separate these probe quarks would lead to stretching of the flux tubes, so that the energy of the system grows linearly with separation. That is how linear confinement was visualized. However, at that time the notions of non-Abelian flux tubes and non-Abelian monopoles (let alone condensed monopoles in non-Abelian gauge theories) were nonexistent. Nambu, 't Hooft, and Mandelstam operated with nonexistent objects.

One may ask, from where did these theorists get their inspiration? There was one physical phenomenon known from long ago and well understood theoretically that yielded a rather analogous picture.

In 1933, Meissner discovered that magnetic fields could not penetrate inside superconducting media. The expulsion of magnetic fields by superconductors is referred to as the Meissner effect. Twenty years later, Abrikosov posed the following question: What happens if one immerses a magnetic charge and an anticharge in type-II superconductors (which in fact he discovered)? One can visualize a magnetic charge as an end point of a very long and very thin solenoid. We refer to the N end point of such a solenoid as a positive magnetic charge and the S end point as a negative magnetic charge.

In empty space, the magnetic field will spread in the bulk, while the energy of the magnetic charge-anticharge configuration will obey the Coulomb $1/r$ law. The force between them will die off as $1/r^2$.

What changes if the magnetic charges are placed inside a large type-II superconductor?

Inside the superconductor, Cooper pairs condense, all electric charges are screened, while the photon acquires a mass. According to modern terminology, the electromagnetic U(1) gauge symmetry is Higgsed. The magnetic field cannot be screened in this way; in fact, the magnetic flux is conserved. At the same time, the superconducting medium does not tolerate the magnetic field.

The clash of contradictory requirements is solved through a compromise. A thin tube is formed between the magnetic charge and anticharge immersed in the superconducting medium. Inside this tube, superconductivity is ruined—which allows the magnetic field to spread from the charge to the anticharge through this tube. The tube transverse size is proportional to the inverse photon mass while its tension is proportional to the Cooper pair condensate. These tubes are referred to as Abrikosov vortices. In fact, for arbitrary magnetic fields he predicted lattices of such flux tubes. A dramatic (and sometimes tragic) history of this discovery is nicely described in Abrikosov's Nobel Lecture.

Returning to the magnetic charges immersed in the type-II superconductor under consideration, one can see that increasing the distance between these charges (as long as they are inside the superconductor) does not lead to their decoupling—the magnetic flux tubes become longer, leading to a linear growth of the system's energy.

The Abrikosov vortex lattices were experimentally observed in the 1960s. This physical phenomenon inspired the ideas of Nambu, 't Hooft, and Mandelstam on non-Abelian confinement. Many have tried to quantify these ideas in the context of non-Abelian gauge theories. It took three decades to pave the way for the current understanding, and the road was not straight. It turns out that it goes through the realm of supersymmetry, and special solitons occurring in supersymmetric gauge theories have played a crucial role in most of the achievements made in the understanding of nonperturbative gauge dynamics. These supersymmetric solitons are referred to as Bogomol'nyi-Prasad-Sommerfield (BPS) –saturated. BPS-saturated flux tubes, monopoles, and domain walls are the main subjects of the present review.

It has long been known that supersymmetric theories may have BPS sectors in which some data can be computed at strong coupling even when the full theory is not solvable. Historically, this is how the first exact results on particle spectra were obtained (Witten and Olive, 1978). Seiberg-Witten's breakthrough results (Seiberg and Witten, 1994a, 1994b) in the mid-1990s provided additional motivation for studies of the BPS sectors.

BPS solitons can emerge in those supersymmetric theories in which superalgebras are centrally extended. In many instances the corresponding central charges are seen at the classical level. In some interesting models, central charges appear as quantum anomalies.

The first studies of BPS solitons (sometimes referred to as critical solitons) in supersymmetric theories at weak coupling date back to the 1970s. de Vega and Schaposnik (1976) were the first to point out that a model in which classical equations of motion can be reduced to first-order BPS equations (Prasad and Sommerfield, 1975; Bogomol'nyi, 1976) is, in fact, a bosonic reduction of a supersymmetric theory. Already in 1977, critical soliton solutions were obtained in the superfield form in some two-dimensional models (Di Vecchia and Ferrara, 1977). In the same year, miraculous cancellations occurring in calculations of quantum corrections to soliton masses were noted by D'Adda *et al.* (1978) [see also Hruby (1980)]. It was observed that for BPS solitons, the boson and fermion modes are degenerate and their number is balanced. It was believed that the soliton masses receive no quantum corrections. The modern correct version of this statement is as follows: If a soliton is BPS saturated at the classical level and belongs to a shortened supermultiplet, it stays BPS-saturated after quantum corrections, and its mass coincides exactly with the central charge it saturates. The latter may or may not be renormalized. Often—but not always—central charges that do not vanish at the classical level and have quantum anomalies are renormalized. Those that emerge as anomalies and have no classical part typically receive no renormalizations. In many instances, holomorphy protects central charges against renormalizations.

Critical solitons play a special role in gauge field theories. Numerous parallels between such solitonic objects and basic elements of string theory have been revealed in recent years. At first, the relation between string theory and supersymmetric gauge theories was mostly a "one-way street"—from strings to field theory. Now it is becoming exceedingly more evident that field-theoretic methods and results, in their turn, provide insights into string theory.

String theory, which emerged from dual hadronic models in the late 1960s and 1970s, and became a canidate for the "theory of everything" in the 1980s and 1990s, when it experienced an unprecedented expansion, has seemingly entered a "return-to-roots" stage. The task of finding solutions to "down-to-earth" problems of QCD and other gauge theories by using results and techniques of string/D-brane theory is currently recognized as one of the most important and exciting goals of the community. In this area, the internal logic of development of string theory is fertilized by insights and hints obtained from field theory. In fact, this is a very healthy process of cross-fertilization.

If supersymmetric gauge theories are, in a sense, dual to string/D-brane theory—as is generally believed to be the case—they must support domain walls (of the D-brane type) (Polchinski, 1995), and we know that they do (Dvali and Shifman, 1997; Witten, 1997). A D-brane is defined as a hypersurface on which a string may end. In field theory, both the brane and the string arise as BPS solitons, the brane as a domain wall and the string as a flux tube. If their properties reflect those inherent to string theory, at least to an extent, the flux tube must end on the wall. Moreover, the wall must house gauge fields living on its worldvolume. The end of the string plays the role of a source for these gauge fields.

The purpose of this review is to summarize developments in critical solitons in two, three and four dimensions, with emphasis on four dimensions and on most recent results. A large variety of BPS-saturated solitons exist in four-dimensional field theories: domain walls, flux tubes (strings), monopoles and dyons, and various junctions of the above objects. A list of recent discoveries includes localization of gauge fields on domain walls, non-Abelian strings that can end on domain walls, developed boojums, confined monopoles attached to strings, and other remarkable findings. The BPS nature of these objects allows one to obtain a number of exact results. In many instances, nontrivial dynamics of the bulk theories we consider leads to effective low-energy theories in the world volumes of domain walls and strings (they are related to zero modes) exhibiting novel dynamical features that are interesting by themselves.

We do not try to review the vast literature that has accumulated since the mid-1990s in its entirety. That would be analogous to charting vast unmapped territory which has not been fully explored. Instead, we suggest "travel diaries" of the participants of the exploratory expedition. Recent publications (Harvey, 1997; Tong, 2005; Eto *et al.*, 2006b; Schaposnik, 2006) facilitate our task, since they present the current developments in this field from a complementary point of view.

The paper is organized in two parts. The first part (Secs. II and III) is a bird's-eye view of the territory. It gives a brief and largely nontechnical introduction to basic ideas behind supersymmetric solitons and particular applications. It is designed to present a general perspective that would be understandable to anyone with an elementary knowledge of classical and quantum fields, and supersymmetry.

Here we present some historical remarks, catalog relevant centrally extended superalgebras, and review basic building blocks with which we consistently deal—domain walls, flux tubes, and monopoles—in their classic form. The word "classic" is used here not to mean "before quantization" but, rather, to mean "recognized and cherished in the community for years."

The second part (Secs. IV–VII) is built upon other principles. It is intended for those who would like to delve into this subject thoroughly, with its specific methods and technical devices. We put special emphasis on recent developments having direct relevance to QCD and gauge theories at large, such as non-Abelian flux tubes (strings), non-Abelian monopoles confined on these strings, gauge field localization on domain walls, etc. We start by presenting our benchmark model, which has extended $\mathcal{N}=2$ supersymmetry. Here we go well beyond conceptual foundations, investing effort in detailed discussions of particular problems and aspects of our choosing. Naturally, we choose those problems and aspects that are instrumental in the phenomena mentioned above.

Our subsequent logic is from $\mathcal{N}=2$ to 1 and further.

TABLE I. The minimal number of supercharges, the complex dimension of the spinorial representation, and the number of additional conditions (i.e., the Majorana and/or Weyl conditions).

D	2	3	4	5	6	7	8	9	10
ν_Q	(1^*) 2	2	4	8	8	8	16	16	16
Dim $(\psi)_C$	2	2	4	4	8	8	16	16	32
No. cond.	2	1	1	0	1	1	1	1	2

Indeed, in certain instances we are able to descend to nonsupersymmetric gauge theories that are very close relatives of QCD. In particular, we present a fully controllable weakly coupled model of the Meissner effect that exhibits quite nontrivial (strongly coupled) dynamics on the string worldsheet. One can draw direct parallels between this construction and the issue of k-strings in QCD.

II. CENTRAL CHARGES IN SUPERALGEBRAS

In this section we review general issues related to central charges (CC) in superalgebras.

A. History

The first superalgebra in four-dimensional field theory was derived by Golfand and Likhtman (1971) in the form

$$\{\bar{Q}_{\dot{\alpha}}Q_{\beta}\} = 2P_{\mu}(\sigma^{\mu})_{\alpha\beta}, \quad \{\bar{Q}_{\alpha}\bar{Q}_{\beta}\} = \{Q_{\alpha}Q_{\beta}\} = 0, \qquad (2.1)$$

i.e., with no central charges. The possible occurrence of CC (elements of superalgebra commuting with all other operators) was first mentioned by Lopuszanski and Sohnius (1974), where the last two anticommutators were modified as

$$\{Q^I_{\alpha}Q^G_{\beta}\} = Z^{IG}_{\alpha\beta}. \qquad (2.2)$$

The superscripts I, G mark extended supersymmetry. A more complete description of superalgebras with CC in quantum field theory was worked out by Haag *et al.* (1975). The only central charges analyzed in this paper were Lorentz scalars (in four dimensions), $Z_{\alpha\beta} \sim \varepsilon_{\alpha\beta}$. Thus, by construction, they could be relevant only to extended supersymmetries.

A few years later, Witten and Olive (1978) showed that in supersymmetric theories with solitons, central extension of superalgebras is typical; topological quantum numbers play the role of central charges.

It was generally understood that superalgebras with (Lorentz-scalar) central charges can be obtained from superalgebras without central charges in higher-dimensional space-time by interpreting some of the extra components of the momentum as CC's [see, e.g., Gates *et al.* (1983)]. When one compactifies extra dimensions, one obtains an extended supersymmetry; the extra components of the momentum act as scalar central charges.

Algebraic analysis extending that of Haag *et al.* (1975) carried out in the early 1980s [see, e.g., van Holten and Van Proeyen (1982)] indicated that the super-Poincaré algebra admits CC's of a more general form, but the dynamical role of additional tensorial charges was not recognized until much later. Now it is common knowledge that central charges that originate from operators other than the energy-momentum operator in higher dimensions can play a crucial role. These tensorial central charges take nonvanishing values on extended objects such as strings and membranes.

Central charges that are antisymmetric tensors in various dimensions were introduced (in the supergravity context, in the presence of p-branes) in de Azcarraga *et al.* (1989) [see also Abraham and Townsend (1991) and Townsend (1999)]. These CC's are relevant to extended objects of the domain-wall type (membranes). Their occurrence in four-dimensional super-Yang-Mills theory (as a quantum anomaly) was first observed by Dvali and Shifman (1997). A general theory of central extensions of superalgebras in three and four dimensions was discussed by Ferrara and Porrati (1998). It is worth noting that those central charges that have the Lorentz structure of Lorentz vectors were not considered by Ferrara and Porrati (1998). The gap was closed by Gorsky and Shifman (2000).

B. Minimal SUSY

The minimal number of supercharges ν_Q in various dimensions is given in Table I. Two-dimensional theories with a single supercharge, although algebraically possible, are quite exotic. In conventional models in $D=2$ with local interactions, the minimal number of supercharges is two.

The minimal number of supercharges in Table I is given for a real representation. Therefore, it is clear that the maximal possible number of CC's is determined by the dimension of the symmetric matrix $\{Q_iQ_j\}$ of the size $\nu_Q \times \nu_Q$, namely,

$$\nu_{\mathrm{CC}} = \frac{\nu_Q(\nu_Q+1)}{2}. \qquad (2.3)$$

In fact, D anticommutators have the Lorentz structure of the energy-momentum operator P_{μ}. Therefore, in general up to D central charges could be absorbed in P_{μ}. In certain situations this number can be smaller, since although algebraically the corresponding CC's have the same structure as P_{μ}, they are dynamically distinguishable. The point is that P_{μ} is uniquely defined through the conserved and symmetric energy-momentum tensor of the theory.

Additional dynamical and symmetry constraints can further diminish the number of independent central charges; see, e.g., Sec. II.B.1.

The total set of CC's can be arranged by classifying CC's with respect to their Lorentz structure. Below we present this classification for $D=2$, 3, and 4, with special emphasis on the four-dimensional case. In Sec. II.C, we deal with $\mathcal{N}=2$ superalgebras.

1. $D=2$

Consider two-dimensional theories with two supercharges. From the discussion above, on purely algebraic grounds, three CC's are possible: one Lorentz-scalar and a two-component vector,

$$\{Q_\alpha,Q_\beta\}=2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu+Z_\mu)+i(\gamma^5\gamma_0)_{\alpha\beta}Z. \qquad (2.4)$$

$Z^\mu\neq 0$ would require the existence of a vector order parameter taking distinct values in different vacua. Indeed, if this central charge existed, its current would have the form

$$\zeta^\mu_\nu=\varepsilon_{\nu\rho}\partial^\rho A^\mu, \quad Z^\mu=\int \zeta^\mu_0 dz,$$

where A^μ is the above-mentioned order parameter. However, $\langle A^\mu\rangle\neq 0$ will break Lorentz invariance and supersymmetry of the vacuum state. This option will not be considered. Limiting ourselves to supersymmetric vacua, we conclude that a single (real) Lorentz-scalar central charge Z is possible in $\mathcal{N}=1$ theories. This central charge is saturated by kinks.

2. $D=3$

The central charge allowed in this case is a Lorentz vector Z_μ, i.e.,

$$\{Q_\alpha,Q_\beta\}=2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu+Z_\mu). \qquad (2.5)$$

One should arrange Z_μ to be orthogonal to P_μ. In fact, this is the scalar central charge of Sec. II.B.1 elevated by one dimension. Its topological current can be written as

$$\zeta_{\mu\nu}=\varepsilon_{\mu\nu\rho}\partial^\rho A, \quad Z_\mu=\int d^2x\,\zeta_{\mu 0}. \qquad (2.6)$$

By an appropriate choice of the reference frame, Z_μ can always be reduced to a real number times (0,0,1). This central charge is associated with a domain line oriented along the second axis.

Although from the general relation (2.5) it is pretty clear why BPS vortices cannot appear in theories with two supercharges, it is instructive to discuss this question from a slightly different standpoint. Vortices in three-dimensional theories are localized objects, namely, particles [BPS vortices in 2+1 dimensions were previously considered by Hloušek and Spector (1992) and Davis *et al.* (1997); see also references therein]. The number of broken translational generators is d, where d is the soliton's co-dimension, $d=2$ in this case. Then at least d supercharges are broken. Since we have only two supercharges in the problem at hand, both must be broken. This simple argument tells us that for a vortex, the matching between bosonic and fermionic zero modes in the (super)translational sector is one-to-one.

Consider now a putative 1/2-BPS vortex in a theory with minimal $\mathcal{N}=1$ SUSY in (2+1)D. Such a configuration would require a world volume description with two bosonic zero modes, but only one fermionic mode. This is not permitted by the argument above, and indeed no configurations of this type are known. Vortices always exhibit at least two fermionic zero modes and can be BPS saturated only in $\mathcal{N}=2$ theories.

3. $D=4$

Maximally one can have 10 CC's that are decomposed into Lorentz representations as $(0,1)+(1,0)+(1/2,1/2)$,

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\}=2(\gamma^\mu)_{\alpha\dot\alpha}(P_\mu+Z_\mu), \qquad (2.7)$$

$$\{Q_\alpha,Q_\beta\}=(\Sigma^{\mu\nu})_{\alpha\beta}Z_{[\mu\nu]}, \qquad (2.8)$$

$$\{\bar{Q}_{\dot\alpha},\bar{Q}_{\dot\beta}\}=(\bar{\Sigma}^{\mu\nu})_{\dot\alpha\dot\beta}\bar{Z}_{[\mu\nu]}, \qquad (2.9)$$

where $(\Sigma^{\mu\nu})_{\alpha\beta}=(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha}_\beta$ is a chiral version of $\sigma^{\mu\nu}$ [see, e.g., Shifman and Vainshtein (1999)]. The antisymmetric tensors $Z_{[\mu\nu]}$ and $\bar{Z}_{[\mu\nu]}$ are associated with domain walls, and reduce to a complex number and a spatial vector orthogonal to the domain wall. The (1/2,1/2) CC Z_μ is a Lorentz vector orthogonal to P_μ. It is associated with strings (flux tubes) and reduces to one real number and a three-dimensional unit spatial vector parallel to the string.

C. Extended SUSY

In four dimensions, one can extend superalgebra up to $\mathcal{N}=4$, which corresponds to 16 supercharges. Reducing this to lower dimensions, we get a rich variety of extended superalgebras in $D=3$ and 2. In fact, in two dimensions the Lorentz invariance provides a much weaker constraint than in higher dimensions, and one can consider a wider set of (p,q) superalgebras comprising $p+q=2$, 4, 8, or 16 supercharges. We will not pursue a general solution; instead, we limit our task to (i) analysis of central charges in $\mathcal{N}=2$ in four dimensions; (ii) reduction of the minimal SUSY algebra in $D=4$ to $D=2$ and 3, namely, the $\mathcal{N}=2$ SUSY algebra in those dimensions. Thus, in two dimensions we consider only the nonchiral $\mathcal{N}=(2,2)$ case. As should be clear from the discussion above, in the dimensional reduction the maximal number of CC's stays intact. What changes is the decomposition in Lorentz and R-symmetry irreducible representations.

1. $\mathcal{N}$=2 in D=2

We focus on the nonchiral $\mathcal{N}=(2,2)$ case corresponding to dimensional reduction of the $\mathcal{N}=1$, $D=4$ algebra. The tensorial decomposition is as follows:

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}[(P_\mu + Z_\mu)\delta^{IJ} + Z^{(IJ)}_\mu] + 2i(\gamma^5\gamma^0)_{\alpha\beta}Z^{\{IJ\}} + 2i\gamma^0_{\alpha\beta}Z^{[IJ]}. \quad (2.10)$$

Here $Z^{[IJ]}$ is antisymmetric in I,J (and $I,J=1,2$); $Z^{\{IJ\}}$ is symmetric while $Z^{(IJ)}$ is symmetric and traceless. All these CC's are real. We can discard all vectorial central charges Z^{IJ}_μ for the same reasons as in Sec. II.B.1. Then we are left with two Lorentz singlets $Z^{(IJ)}$, which represent the reduction of the domain wall charges in $D=4$ and two Lorentz singlets Tr $Z^{\{IJ\}}$ and $Z^{[IJ]}$, arising from P_2 and the vortex charge in $D=3$ (see Sec. II.C.2). These central charges are saturated by kinks.

Summarizing, the (2,2) superalgebra in $D=2$ is

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu\delta^{IJ} + 2i(\gamma^5\gamma^0)_{\alpha\beta}Z^{\{IJ\}} + 2i\gamma^0_{\alpha\beta}Z^{[IJ]}. \quad (2.11)$$

It is instructive to rewrite Eq. (2.11) in terms of complex supercharges Q_α and $Q^\dagger_\beta$ corresponding to four-dimensional $Q_\alpha, \bar{Q}_{\dot\alpha}$, see Sec. II.B.3. Then

$$\{Q_\alpha, Q^\dagger_\beta\}(\gamma^0)_{\beta\gamma} = 2\left[P_\mu\gamma^\mu + Z\frac{1-\gamma_5}{2} + Z^\dagger\frac{1+\gamma_5}{2}\right]_{\alpha\gamma},$$

$$\{Q_\alpha, Q_\beta\}(\gamma^0)_{\beta\gamma} = -2Z'(\gamma_5)_{\alpha\gamma},$$

$$\{Q^\dagger_\alpha, Q^\dagger_\beta\}(\gamma^0)_{\beta\gamma} = 2Z'^\dagger(\gamma_5)_{\alpha\gamma}. \quad (2.12)$$

The algebra contains two complex central charges, Z and Z'. In terms of components $Q_\alpha=(Q_R, Q_L)$, the nonvanishing anticommutators are

$$\{Q_L, Q^\dagger_L\} = 2(H+P), \quad \{Q_R, Q^\dagger_R\} = 2(H-P),$$

$$\{Q_L, Q^\dagger_R\} = 2iZ, \quad \{Q_R, Q^\dagger_L\} = -2iZ^\dagger,$$

$$\{Q_L, Q_R\} = 2iZ', \quad \{Q^\dagger_R, Q^\dagger_L\} = -2iZ'^\dagger. \quad (2.13)$$

It exhibits the automorphism $Q_R \leftrightarrow Q^\dagger_R, Z \leftrightarrow Z'$ associated (Dorey, 1998) with the transition to a mirror representation (Hanany and Hori, 1998) The complex central charges Z and Z' can be expressed in terms of real $Z^{\{IJ\}}$ and $Z^{[IJ]}$,

$$Z = Z^{[12]} + \frac{i}{2}(Z^{\{11\}} + Z^{\{22\}}),$$

$$Z' = \frac{Z^{\{12\}} + Z^{\{21\}}}{2} - i\frac{Z^{\{11\}} - Z^{\{22\}}}{2}. \quad (2.14)$$

Typically, in a given model either Z or Z' vanish. [Both $Z \neq 0$ and $Z' \neq 0$ simultaneously in a contrived model (Losev and Shifman, 2003) in which the Lorentz symmetry and a part of supersymmetry are spontaneously broken.]

2. $\mathcal{N}$=2 in D=3

The superalgebra can be decomposed into Lorentz and R-symmetry tensorial structures as follows:

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}[(P_\mu + Z_\mu)\delta^{IJ} + Z^{(IJ)}_\mu] + 2i\gamma^0_{\alpha\beta}Z^{[IJ]}, \quad (2.15)$$

where all central charges above are real. The maximal set of 10 CC's enter as a triplet of spacetime vectors Z^{IJ}_μ and a singlet $Z^{[IJ]}$. The singlet CC is associated with vortices (or lumps), and corresponds to the reduction of the (1/2,1/2) charge or the fourth component of the momentum vector in $D=4$. The triplet Z^{IJ}_μ is decomposed into an R-symmetry singlet Z_μ, algebraically indistinguishable from the momentum, and a traceless symmetric combination $Z^{(IJ)}_\mu$. The former is equivalent to the vectorial charge in the $\mathcal{N}=1$ algebra, while $Z^{(IJ)}_\mu$ can be reduced to a complex number and vectors specifying the orientation. We see that these are the direct reduction of the (0,1) and (1,0) wall charges in $D=4$. They are saturated by domain lines.

3. On extended supersymmetry (eight supercharges) in D=4

Complete algebraic analysis of all tensorial central charges in this problem is analogous to the previous cases and is rather straightforward. With eight supercharges, the maximal number of CC's is 36. The dynamical aspect is less developed—only a modest fraction of the above 36 CC's are known to be nontrivially realized in models studied. We limit ourselves to a few remarks regarding the well-established CC's. We use a complex (holomorphic) representation of the supercharges. Then the supercharges are labeled as follows:

$$Q^F_\alpha, \bar{Q}_{\dot\alpha G}, \quad \alpha,\dot\alpha = 1,2, \quad F,G = 1,2. \quad (2.16)$$

On general grounds one can write

$$\{Q^F_\alpha, \bar{Q}_{\dot\alpha G}\} = 2\delta^F_G P_{\alpha\dot\alpha} + 2(Z^F_G)_{\alpha\dot\alpha},$$

$$\{Q^F_\alpha, Q^G_\beta\} = Z^{\{FG\}}_{\{\alpha\beta\}} + \varepsilon_{\alpha\beta}\varepsilon^{FG}Z,$$

$$\{\bar{Q}_{\dot\alpha F}, \bar{Q}_{\dot\beta G}\} = (\bar{Z}_{\{FG\}})_{\{\dot\alpha\dot\beta\}} + \varepsilon_{\alpha\beta}\varepsilon^{FG}\bar{Z}. \quad (2.17)$$

Here $(Z^F_G)_{\alpha\dot\alpha}$ are four vectorial central charges (1/2, 1/2) (16 components altogether) while $Z^{\{FG\}}_{\{\alpha\beta\}}$ and the complex conjugate are (1,0) and (0,1) central charges. Since the matrix $Z^{\{FG\}}_{\{\alpha\beta\}}$ is symmetric with respect to F,G, there are three flavor components, while the total number of components residing in (1,0) and (0,1) central charges is 18. Finally, there are two scalar central charges, Z and $\bar{Z}$.

Dynamically, the above central charges can be described as follows. The scalar CC's Z and $\bar{Z}$ are saturated by monopoles and/or dyons. One vectorial central charge Z_μ (with the additional condition $P^\mu Z_\mu=0$) is saturated (Vainshtein and Yung, 2001) by an Abrikosov-Nielsen-Olesen (ANO) string (Abrikosov, 1957; Nielsen and Olesen, 1973). A (1,0) central charge with $F=G$ is saturated by domain walls (Shiftman and Yung, 2004a).

We now briefly discuss the Lorentz-scalar central charges in Eq. (2.17) that are saturated by monopoles and/or dyons. They will be referred to as monopole central charges. Historically they were the first to be introduced within the framework of an extended 4D superalgebra (Lopuszenski and Sohnius, 1974; Haag *et al.*, 1975). On the dynamical side, they appeared as the first example of the topological charge ↔ central charge relation revealed by Witten and Olive (1978). Twenty years later, the $\mathcal{N}=2$ model in which these central charges first appeared was solved by Seiberg and Witten (1994a, 1994b) and the exact masses of the BPS-saturated monopoles and/or dyons found. No direct comparison with the operator expression for the central charges was carried out, however. Rebhan *et al.* (2004a) noted that for the Seiberg-Witten formula to be valid, a boson-term anomaly should exist in the monopole central charges. Even before Rebhan *et al.* (2004a) a fermion-term anomaly was identified (Shiftman and Yung, 2004a) which plays a crucial role (Shiftman and Yung, 2004b) for the monopoles in the Higgs regime (confined monopoles).

III. THE MAIN BUILDING BLOCKS

A. Domain walls

1. Preliminaries

In four dimensions, domain walls are two-dimensional extended objects. In three dimensions they become domain lines, while in two dimensions they reduce to kinks that can be considered as particles since they are localized. Embeddings of bosonic models supporting kinks in $\mathcal{N}=1$ supersymmetric models in two dimensions were first discussed by Di Vecchia and Ferrara (1977) and Witten and Olive (1978). Occasional remarks on kinks in models with four supercharges of the type of the Wess-Zumino models (Wess and Bagger, 1992) can be found in the literature of the 1980s, but they went unnoticed. The only issue that caused much interest and debate in the 1980s was the issue of quantum corrections to the BPS kink mass in 2D models with $\mathcal{N}=1$ supersymmetry.

The mass of the BPS saturated kinks in two dimensions must be equal to the central charge Z in Eq. (2.4). The simplest two-dimensional model with the minimal superalgebra, admitting solitons, was considered by D'Adda and Di Vecchia (1978). In components, the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + \bar{\psi} i \partial\!\!\!/ \psi + F^2) + \mathcal{W}'(\phi)F - \frac{1}{2}\mathcal{W}''(\phi)\bar{\psi}\psi, \tag{3.1}$$

where $\mathcal{W}(\phi)$ is a real superpotential, which in the simplest case takes the form

$$\mathcal{W}(\Phi) = \frac{m^2}{4\lambda}\Phi - \frac{\lambda}{3}\Phi^3. \tag{3.2}$$

Moreover, the auxiliary field F can be eliminated by virtue of the classical equation of motion $F=-\mathcal{W}'$. This is a real reduction (two supercharges) of the Wess-Zumino model, which has one real scalar field ϕ and one two-component real spinor ψ.

The story of kinks in this model is long and dramatic. In the very beginning, it was argued (D'Adda and Di Viecchia, 1978) that, due to a residual supersymmetry, the mass of the soliton calculated at the classical level remains intact at the one-loop level. A few years later, it was noted (Kaul and Rajaraman, 1983) that the nonrenormalization theorem (D'Adda and Di Viecchia, 1978) cannot possibly be correct, since the classical soliton mass is proportional to m^3/λ^2 (where m and λ are the bare mass parameter and coupling constant, respectively), and the physical mass of the scalar field gets a logarithmically infinite renormalization. Since the soliton mass is an observable physical parameter, it must stay finite in the limit $M_{uv}\to\infty$, where M_{uv} is the ultraviolet cutoff. This implies, in turn, that the quantum corrections cannot vanish—they "dress" m in the classical expression, converting the bare mass parameter into the renormalized one. The one-loop renormalization of the soliton mass was first calculated by Kaul and Rajaraman (1983). Technically the emergence of the one-loop correction was attributed to a "difference in the density of states in continuum in the boson and fermion operators in the soliton background field." Subsequent work (Imbimbo and Mukhi, 1984) dealt with the renormalization of the central charge, with the conclusion that the central charge is renormalized in just the same way as the kink mass, so that the saturation condition is not violated.

Then many repeated one-loop calculations for the kink mass and/or central charge.[1] The results reported and the conclusion of saturation or nonsaturation oscillated with time, with little sign of convergence. Needless to say, all agreed that the logarithmically divergent term in Z matched the renormalization of m. However, the finite (nonlogarithmic) term varied from work to work, sometimes even in the successive works of the same authors, e.g., Rebhan and van Nieuwenhuizen (1997) and Nastase *et al.* (1999), or Uchiyama (1984, 1986a) and Uchiyama (1986b). According to Nastase *et al.* (1999) the BPS saturation is violated at one loop. This assertion reversed the earlier trend (Kaul and Rajaraman, 1983; Yamagishi, 1984; Chatterjee ans Majumdar, 1985) according to which the kink mass and the corresponding central charge are renormalized in a concerted way.

[1]See D'Adda *et al.* (1978); Horsley (1979); Schönfeld (1979); Rouhani (1981); Chatterjee and Majumdar (1984, 1985); Uchiyama (1984, 1986a, 1986b); Yamagishi (1984); Rebhan and van Nieuwenhuizen (1997); Graham and Jaffe (1999); and Nastase *et al.* (1999).

The story culminated in 1998 with the discovery of a quantum anomaly in the central charge (Shiftman *et al.*, 1999). Classically, the kink central charge Z is equal to the difference between the values of the superpotential $\mathcal{W}$ at spatial infinities,

$$Z = \mathcal{W}[\phi(z=\infty)] - \mathcal{W}[\phi(z=-\infty)]. \tag{3.3}$$

This is known from the pioneering paper of Witten and Olive (1978). Due to the anomaly, the central charge gets modified in the following way:

$$\mathcal{W} \to \mathcal{W} + \frac{\mathcal{W}''}{4\pi}, \tag{3.4}$$

where the term proportional to $\mathcal{W}''$ is anomalous (Shiftman *et al.*, 1999). The right-hand side of Eq. (3.4) must be substituted in the expression for the central charge (3.3) instead of $\mathcal{W}$. Inclusion of the additional anomalous term restores the equality between the kink mass and its central charge. The BPS nature is preserved, which is correlated with the fact that the kink supermultiplet is short in the case at hand (Losev *et al.*, 2001, 2002). All subsequent investigations confirmed this conclusion [see, e.g., the review paper by Goldhaber *et al.* (2004)].

Critical domain walls in theories with four supercharges started attracting attention in the 1990s. The most popular model of this time supporting such domain walls is the generalized Wess-Zumino model with the Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\bar{\Phi},\Phi) + \left(\int d^2\theta \mathcal{W}(\Phi) + \text{H.c.}\right), \tag{3.5}$$

where K is the Kähler potential and Φ stands for a set of the chiral superfields. This model can be considered in two and four dimensions. A popular choice was a trivial Kähler potential,

$$K = \bar{\Phi}\Phi.$$

BPS walls in this system satisfy the first-order differential equations (Fendly *et al.*, 1990; Abraham and Townsend, 1991; Cecotti and Vafa, 1993; Chibisov and Shiftman, 1997)

$$g_{\bar{a}b}\partial_z \Phi^b = e^{i\gamma}\partial_{\bar{a}}\bar{\mathcal{W}}, \tag{3.6}$$

where the Kähler metric is given by

$$g_{\bar{a}b} = \frac{\partial^2 K}{\partial \bar{\Phi}^{\bar{a}}\partial \Phi^b} \equiv \partial_{\bar{a}}\partial_b K, \tag{3.7}$$

and γ is the phase of the (1,0) central charge Z as defined in Eq. (2.8). The phase γ depends on the choice of the vacua between which the given domain wall interpolates,

$$Z = 2(\mathcal{W}_{\text{vac}_f} - \mathcal{W}_{\text{vac}_i}). \tag{3.8}$$

A useful consequence of the BPS equations is that

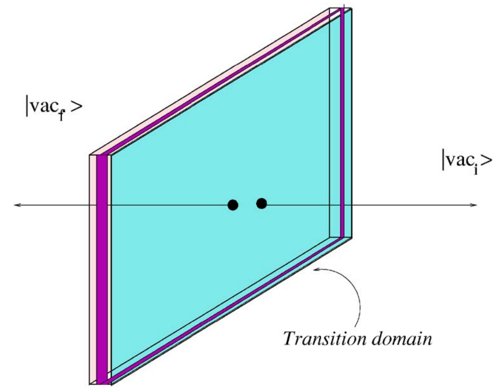

FIG. 1. (Color online) A field configuration interpolating between two distinct degenerate vacua.

$$\partial_z \mathcal{W} = e^{i\gamma}\|\partial_a \mathcal{W}\|^2, \tag{3.9}$$

and thus the domain wall describes a straight line in the $\mathcal{W}$ plane connecting the two vacua.

Construction and analysis of BPS saturated domain walls in four dimensions crucially depends on the realization of the fact that the central charges relevant to critical domain walls are not Lorentz scalars; rather they transform as $(1,0)+(0,1)$ under the Lorenz transformations. It was a textbook statement ascending to the pioneering paper of Haag *et al.* (1975) that $\mathcal{N}=1$ superalgebras in four dimensions leave no place for central charges. This statement is correct only with respect to Lorenz-scalar central charges. Townsend was the first to note (Townsend, 1988) that supersymmetric branes, being BPS-saturated, require the existence of tensorial central charges antisymmetric in the Lorenz indices. That the anticommutator $\{Q_\alpha, Q_\beta\}$ in the four-dimensional Wess-Zumino model contains the (1,0) central charge is obvious. This anticommutator vanishes, however, in super-Yang-Mills theory at the classical level.

2. D-branes in gauge field theory

In 1996, Dvali and Shifman found in supersymmetric gluodynamics (Davil and Shiftman, 1997) an anomalous (1,0) central charge in superalgebra, not seen at the classical level. They argued that this central charge is saturated by domain walls interpolating between vacua with distinct values of the order parameter, the gluino condensate $\langle\lambda\lambda\rangle$, labeling N distinct vacua of super-Yang-Mills theory with the gauge group SU(N).

What is the domain wall? It is a field configuration interpolating between vacuum i and vacuum f with some transition domain in the middle. Say, to the left you have vacuum i, to the right you have vacuum f, and in the middle you have a transition domain that is referred to as the wall (Fig. 1).

There is a large variety of walls in supersymmetric gluodynamics. Minimal, or elementary, walls interpolate between vacua n and $n+1$, while k walls interpolate between n and $n+k$, see Fig. 2. Dvali and Shiftman (1997) suggested a mechanism for localizing gauge fields on the

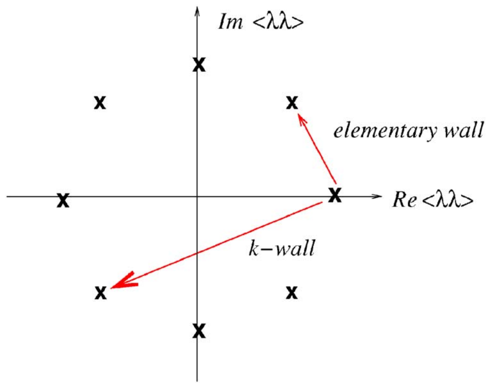

FIG. 2. (Color online) N vacua for SU(N). The vacua are labeled by the vacuum expectation value $\langle\lambda\lambda\rangle = -6N\Lambda^3 \exp(2\pi i k/N)$, where $k=0,1,\ldots,N-1$. Elementary walls interpolate between two neighboring vacua.

wall through bulk confinement. Later, this mechanism was implemented in models at weak coupling, as we will see below.

Shortly after, Witten interpreted the BPS walls in supersymmetric gluodynamics as analogs of D-branes (Witten, 1997). This is because their tension scales as $N\sim 1/g_s$ rather than $1/g_s^2$ typical of solitonic objects (here g_s is the string constant). Many promising consequences ensued. One of them was the Acharya-Vafa derivation of the wall world-volume theory (Acharya and Vafa, 2001). Using a wrapped D-brane picture and certain dualities, they identified the k-wall world-volume theory as $(1+2)$ dimensional U(k) gauge theory with the field content of $\mathcal{N}=2$ and the Chern-Simons term at level N breaking $\mathcal{N}=2$ down to $\mathcal{N}=1$.

In $\mathcal{N}=1$ gauge theories with arbitrary matter content and superpotential, the general relation (2.7) takes the form

$$\{Q_\alpha, Q_\beta\} = -4\Sigma_{\alpha\beta}\bar{Z}, \tag{3.10}$$

where

$$\Sigma_{\alpha\beta} = -\frac{1}{2}\int dx_{[\mu}\, dx_{\nu]}(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha}_\beta \tag{3.11}$$

is the wall area tensor, and

$$Z = \frac{2}{3}\Delta\left\{\left[3\mathcal{W} - \sum_f Q_f\frac{\partial\mathcal{W}}{\partial Q_f}\right] - \left[\frac{3N - \sum_f T(R_f)}{16\pi^2}\mathrm{Tr}\, W^2 + \frac{1}{8}\sum_f \gamma_f \bar{D}^2(\bar{Q}_f e^V Q_f)\right]\right\}_{\theta=0}. \tag{3.12}$$

In this expression, Δ implies taking the difference at two spatial infinities in the direction perpendicular to the surface of the wall. The second term in the first line presents the gauge anomaly in the central charge. The second line is a total superderivative. Therefore, it vanishes after averaging over any supersymmetric vacuum state. Hence, it can be safely omitted. The first term in the first line presents the classical result, cf. Eq. (3.8). At the classical level, $Q_f(\partial\mathcal{W}/\partial Q_f)$ is a total superderivative also, which can be seen from the Konishi anomaly (Clark *et al.*, 1979; Konishi, 1984; Konishi and Shizuya, 1985),

$$\bar{D}^2(\bar{Q}_f e^V Q_f) = 4Q_f\frac{\partial\mathcal{W}}{\partial Q_f} + \frac{T(R_f)}{2\pi^2}\mathrm{Tr}\, W^2. \tag{3.13}$$

If we discard this total superderivative (ignoring quantum effects), we return to $Z=2\Delta(\mathcal{W})$, the formula obtained in the Wess-Zumino model. At the quantum level, $Q_f(\partial\mathcal{W}/\partial Q_f)$ ceases to be a total superderivative because of the Konishi anomaly. It is still convenient to eliminate $Q_f(\partial\mathcal{W}/\partial Q_f)$ in favor of Tr W^2 by virtue of the Konishi relation (3.13). In this way, one arrives at

$$Z = 2\Delta\left\{\mathcal{W} - \frac{N - \sum_f T(R_f)}{16\pi^2}\mathrm{Tr}\, W^2\right\}_{\theta=0}. \tag{3.14}$$

We see that the superpotential $\mathcal{W}$ is amended by the anomaly; in the operator form,

$$\mathcal{W} \to \mathcal{W} - \frac{N - \sum_f T(R_f)}{16\pi^2}\mathrm{Tr}\, W^2. \tag{3.15}$$

Of course, in pure Yang-Mills theory only the anomaly term survives.

We developed in 2002 a benchmark $\mathcal{N}=2$ model, weakly coupled in the bulk (and, thus, fully controllable), which supports both BPS walls and BPS flux tubes. We demonstrated that a gauge field is indeed localized on the wall; for the minimal wall this is a U(1) field while for nonminimal walls the localized gauge field is non-Abelian. We also found a BPS wall-string junction related to the gauge field localization, see Sec. VI. The field-theory string does end on the BPS wall after all. The end point of the string on the wall, after Polyakov's dualization, becomes a source of the electric field localized on the wall. In 2005, Sakai and Tong analyzed generic wall-string configurations. Following condensed-matter physicists, they called them *boojums*.[2]

Equation (3.12) implies that in pure gluodynamics (super-Yang-Mills theory without matter), the domain-wall tension is

$$T = \frac{N}{8\pi^2}\left|\langle\mathrm{Tr}\,\lambda^2\rangle_{\mathrm{vac}_f} - \langle\mathrm{Tr}\,\lambda^2\rangle_{\mathrm{vac}_i}\right|, \tag{3.16}$$

where $\mathrm{vac}_{i,f}$ stands for the initial (final) vacuum between which the given wall interpolates. Furthermore, the gluino condensate $\langle\mathrm{Tr}\,\lambda^2\rangle_{\mathrm{vac}}$ was calculated long ago (Shifman and Vainshtein, 1988), using the same methods that were later advanced and perfected by Seiberg and Seiberg and Witten in their quest for dualities in $\mathcal{N}=1$

[2]"Boojum" comes from Lewis Carroll's children's book *Hunting of the Snark*. Condensed-matter physicists adopted the name to describe solitonic objects of the wall-string junction type in helium-3.

super-Yang-Mills theories (Seiberg, 1995; Intriligator and Seiberg, 1996a, 1996b) and the dual Meissner effect in $\mathcal{N}=2$ [see Seiberg and Witten (1994a, 1994b)]. Namely,

$$2\langle \mathrm{Tr}\,\lambda^2\rangle = \langle \lambda^a_\alpha \lambda^{a,\alpha}\rangle = -6N\Lambda^3 \exp\left(\frac{2\pi i k}{N}\right). \qquad (3.17)$$

Here $k=0,1,\ldots,N-1$ labels the N distinct vacua of the theory, see Fig. 2, and Λ is a dynamical scale, defined in the standard manner [i.e., in accordance with Hinchliffe (2006)] in terms of the ultraviolet parameters M_{uv} (the ultraviolet regulator mass) and g_0^2 (the bare coupling constant),

$$\Lambda^3 = \frac{2}{3}M_{\mathrm{uv}}^3\left(\frac{8\pi^2}{Ng_0^2}\right)\exp\left(-\frac{8\pi^2}{Ng_0^2}\right). \qquad (3.18)$$

In each given vacuum, the gluino condensate scales with the number of colors as N. However, the difference of the values of the gluino condensates in two vacua that lie not too far away from each other scales as N^0. Taking into account Eq. (3.16), we conclude that the wall tension in supersymmetric gluodynamics

$$T \sim N.$$

(This statement just rephrases Witten's argument why the above walls should be considered as analogs of D-branes.)

The volume energy density in both vacua, to the left and to the right of the wall, vanishes due to supersymmetry. Inside the transition domain, where the order parameter changes its value gradually, the volume energy density is expected to be proportional to N^2, because there are N^2 excited degrees of freedom. Therefore, $T \sim N$ implies that the wall thickness in supersymmetric gluodynamics must scale as N^{-1}. This is very unusual, because normally we would say that the glueball mass is $O(N^0)$, hence everything built of regular glueballs should have thickness of order $O(N^0)$.

If the wall thickness is $O(N^{-1})$, the question of what consequences ensue immediately comes to mind. This issue is far from being completely understood; for relevant discussions, see Dvali and Kakushadze (1999), Gabadadze and Shifman (2000), and Armoni and Shifman (2003).

As was mentioned, there is a large variety of walls in supersymmetric gluodynamics as they can interpolate between vacua with arbitrary values of k. Even if $k_f = k_i+1$, i.e., the wall is elementary, we are dealing with several walls, all having one and the same tension—we call them degenerate walls. The first indication of wall degeneracy was obtained by Kovner *et al.* (1997), where two degenerate walls were observed in SU(2) theory. Later, Acharya and Vafa calculated the k-wall multiplicity (Acharya and Vafa, 2001) within the framework of D-brane/string formalism,

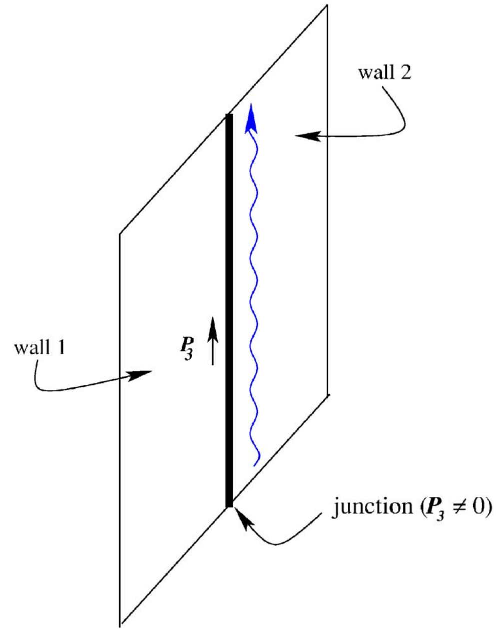

FIG. 3. (Color online) Two distinct degenerate domain walls separated by the wall junction.

$$\nu_k = C_N^k = \frac{N!}{k!(N-k)!}. \qquad (3.19)$$

For $N=2$, only elementary walls exist, and $\nu=2$. In the field-theoretic setting, Eq. (3.19) was derived by Ritz *et al.* (2002). The derivation is based on the fact that the index ν is topologically stable—continuous deformations of the theory do not change ν. Thus, one can add an appropriate set of matter fields sufficient for complete Higgsing of supersymmetric gluodynamics. The domain-wall multiplicity in the effective low-energy theory obtained in this way is the same as in supersymmetric gluodynamics, albeit the effective low-energy theory, a Wess-Zumino-type model, is much simpler.

3. Domain wall junctions

Two degenerate domain walls can coexist in one plane—a new phenomenon that was first discussed by Ritz *et al.* (2004). It is illustrated in Fig. 3. Two distinct degenerate domain walls lie on the plane; the transition domain between wall 1 and wall 2 is the domain-wall junction (domain line).

Each individual domain wall is 1/2 BPS saturated. The wall configuration with the junction line (Fig. 3) is 1/4 BPS saturated. We start from $\mathcal{N}=1$ four-dimensional bulk theory (four supercharges). Naively, the effective theory on the plane must preserve two supercharges, while the domain line must preserve one supercharge. In fact, they have four and two conserved supercharges, respectively. This is another new phenomenon—supersymmetry enhancement—discovered by Ritz *et al.* (2004). One can excite the junction line endowing it with momentum in the direction of the line, without altering its BPS status. A domain line with a plane wave propa-

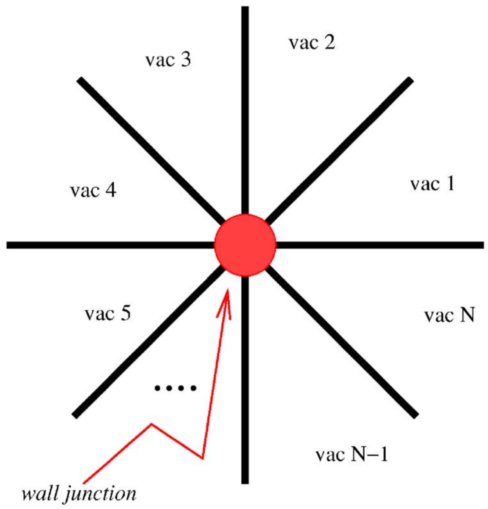

FIG. 4. (Color online) The cross section of the wall junction.

gating on it (Fig. 3) preserves the property of the BPS saturation; see Ritz *et al.* (2004).

We pass now to more conventional wall junctions. Assume that the theory under consideration has a spontaneously broken Z_N symmetry, with $N \geqslant 3$, and, correspondingly, N vacua. Then one can have N distinct walls connected in the asterisk-like pattern; see Fig. 4. This field configuration possesses an obvious axial symmetry: the vacua are located cyclically.

This configuration is absolutely topologically stable, as stable as the wall itself. Moreover, it can be BPS saturated for any value of N. It was noted (Abraham and Townsend, 1991) that theories with either a U(1) or Z_N global symmetry may contain 1/4-BPS objects with axial geometry. The corresponding Bogomol'nyi equations were derived by Chibisov and Shifman (1997) and shortly after rediscovered by Gibbons and Townsend (1999). Further advances in the issue of the domain-wall junctions of the hub-and-spoke type were presented by Oda *et al.* (1999), Carroll *et al.* (2000), Binosi and ter Veldhusi (2000), and Shifman and ter Veldhuis (2000); see also later works of Gauntlett *et al.* (2001a), Kakimoto and Sakai (2003), Eto *et al.* (2005b, 2006a), and Eto, Fujimori, *et al.* (2007). We single out the work of Oda *et al.* (1999) where the first analytic solution for a BPS wall junction was found in a specific generalized Wess-Zumino model. Among stimulating findings in this work is the fact that the junction tension turned out to be negative in this model. The model has Z_3 symmetry. It is derived from a SU(2) Yang-Mills theory with extended supersymmetry ($\mathcal{N}=2$) and one matter flavor perturbed by an adjoint scalar mass. The original model contains three pairs of chiral superfields and, in addition, one extra chiral superfield. In fact, Oda *et al.* (1999) model can be simplified and adjusted to cover the case of arbitrary N, which was done by Shifman and ter Veldhuis (2000). The latter work demonstrates that the tension of the wall junctions is as a rule negative, although exceptional models with the positive tension are possible too. Note that the negative sign of the wall junction tension does not lead to instability since wall junctions do not exist in isolation. They are always attached to walls that stabilize this field configuration.

Returning to SU(N) supersymmetric gluodynamics ($N \geqslant 3$), one expects to get in this theory the 1/4-BPS junctions of the type depicted in Fig. 4. Of course, this theory is strongly coupled; therefore, the classical Bogomol'nyi equations are irrelevant. However, assuming that such wall junctions do exist, one can find their tension at large N even without solving the theory. To this end, one uses (Gabadadze and Shifman, 2000; Shifman and ter Veldhuis, 2000) the expression for the (1/2,1/2) central charge[3] in terms of the contour integral over the axial current (Gorsky and Shifman, 2000). At large N, the latter integral is determined by the absolute value of the gluino condensate and the overall change of the phase of the condensate when one makes the 2π rotation around the hub. In this way, one arrives at the prediction

$$T_{\text{wall junction}} \sim N^2. \tag{3.20}$$

The coefficient in front of the N^2 factor is model dependent.

Can one interpret this N^2 dependence of the hub of the junction? Assume that each wall has thickness $1/N$ and there are N of them. Then it is natural to expect the radius of the intermediate domain where all walls join together to be of the order $(1/N)\times N \sim N^0$. This implies, in turn, that the area of the hub is $O(N^0)$. If the volume energy density inside the junction is N^2 (i.e., the same as inside the walls), one immediately gets Eq. (3.20).

B. Vortices in $D=3$ and flux tubes in $D=4$

Vortices were among the first examples of topological defects treated in the Bogomol'nyi limit (Prasad and Sommerfield, 1975; de Vega and Schaposnik, 1976; Witten and Olive, 1978) [see also Taubes (1980)]. Explicit embedding of the bosonic sector in supersymmetric models dates back to the 1980s. Bezerra de Mello (1990) considered a three-dimensional Abelian Higgs model. That model had $\mathcal{N}=1$ supersymmetry (two supercharges) and thus, according to Sec. II.B.2, contained no central charge that could be saturated by vortices. Hence, the vortices discussed by Bezerra de Mello (1990) were not critical. BPS-saturated vortices can and do occur in $\mathcal{N}=2$ three-dimensional models (four supercharges) with a nonvanishing Fayet-Iliopoulos term (Shmidt, 1992; Edelstein *et al.*, 1994). Such models can

[3]There is a subtle point here that must be noted. For a wall of the hub-and-spokes type, the overall tension is the sum of two tensions: the tension of the walls and the tension of the hub. The first is determined by the (1,0) central charge, the second by (1/2,1/2). Each separately is somewhat ambiguous in the case at hand. The ambiguity cancels in the sum (Gorsky and Shifman, 2000).

be obtained by dimensional reduction from four-dimensional $\mathcal{N}=1$ models. We start with a brief excursion in SQED.

1. SQED in 3D

The starting point is SQED with the Fayet-Illiopoulos term ξ in four dimensions, The SQED Lagrangian is

$$\mathcal{L} = \left\{ \frac{1}{4e^2} \int d^2\theta W^2 + \text{H.c.} \right\} + \sum_f \int d^4\theta \bar{Q}^f e^{n_f V} Q_f - \xi \int d^2\theta d^2\bar{\theta} V(x,\theta,\bar{\theta}), \tag{3.21}$$

where e is the electric coupling constant, Q_f is the chiral matter superfield (with charge $n_f = \pm 1$), and W_α is the supergeneralization of the photon field strength tensor,

$$W_\alpha = \tfrac{1}{8}\bar{D}^2 D_\alpha V = i(\lambda_\alpha + i\theta_\alpha D - \theta^\beta F_{\alpha\beta} - i\theta^2 \partial_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}). \tag{3.22}$$

In four dimensions, the absence of the chiral anomaly in SQED requires the matter superfields to enter in pairs of the opposite charge. Otherwise the theory is anomalous; the chiral anomaly renders it noninvariant under gauge transformations. Thus, the minimal matter sector includes two chiral superfields Q and $\tilde{Q}$, with $n=1$ and $\tilde{n}=-1$, respectively. In three dimensions there is no chirality. Therefore, one can consider 3D SQED with a single matter superfield Q, with $n=1$. Classically it is perfectly fine just to discard the superfield $\tilde{Q}$ from the Lagrangian (3.21). However, such crudely truncated theory may be inconsistent at the quantum level (Alvarez-Gaumé and Witten, 1984; Redlich, 1984a, 1984b; Aharony *et al.*, 1997). Gauge invariance in loops requires, as we will see shortly, simultaneous introduction of a Chern-Simons term in the one matter superfield model (Alvarez-Gaumé and Witten, 1984; Redlich, 1984a, 1984b; Aharony *et al.*, 1997). The Chern-Simons term breaks parity. That is the reason this phenomenon is sometimes referred to as *parity anomaly*.

A perfectly safe way to get rid of $\tilde{Q}$ is as follows. Start from the two-superfield model (3.21), which is certainly self-consistent at both the classical and quantum levels. The one-superfield model can be obtained from two superfields by making $\tilde{Q}$ heavy and integrating it out. If one manages to introduce a mass $\tilde{m}$ for $\tilde{Q}$ without breaking $\mathcal{N}=2$ supersymmetry, the large-$\tilde{m}$ limit can be viewed as an excellent regularization procedure.

Such mass terms are well known; for a review, see Nishino and Gates (1993), Aharony *et al.* (1997), and de Boer *et al.* (1997). They go under the name of real masses, are specific to theories with U(1) symmetries dimensionally reduced from $D=4$ to 3, and present a direct generalization of *twisted masses* in two dimensions (Alverez-Gaumé and Freedman, 1983; Gates, 1984; Gates *et al.*, 1984). To introduce a real mass, one couples matter fields to a background vector field with a nonvanishing component along the reduced direction. For instance, in the case at hand we introduce a background field V_b as

$$\Delta\mathcal{L}_m = \int d^4\theta \bar{\tilde{Q}} e^{V_b} \tilde{Q}, \quad V_b = \tilde{m}(2i)(\theta^1\bar{\theta}^{\dot{2}} - \theta^2\bar{\theta}^{\dot{1}}). \tag{3.23}$$

The reduced spatial direction is along the y axis. We couple V_b to the U(1) current of $\tilde{Q}$ ascribing to $\tilde{Q}$ charge 1 with respect to the background field. At the same time, Q is assumed to have V_b charge zero and, thus, has no coupling to V_b. Then, the background field generates a mass term only for $\tilde{Q}$, without breaking $\mathcal{N}=2$.

After reduction to three dimensions and passing to components (in the Wess-Zumino gauge), we arrive at the action in the following form (in the three-dimensional notation):

$$S = \int d^3x \Big\{ -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2e^2}(\partial_\mu a)^2 + \frac{1}{e^2}\bar{\lambda} i\partial\!\!\!/ \lambda + \frac{1}{2e^2}D^2 - \xi D + \sum_f n_{ef}\bar{q}^f q_f D + \sum_f [\mathcal{D}^\mu \bar{q}^f \mathcal{D}_\mu q_f + \bar{\psi}^f i\mathcal{D}\!\!\!\!/\, \psi_f] - a^2\bar{q}q - (\tilde{m}+a)^2\bar{\tilde{q}}\tilde{q} + a\bar{\psi}\psi - (\tilde{m}+a)\bar{\tilde{\psi}}\tilde{\psi} + \sum_f n_{ef}[\sqrt{2}(\bar{\lambda}\psi_f)\bar{q}^f + \text{H.c.}] \Big\}. \tag{3.24}$$

Here a is a real scalar field,

$$a = -A_2, \quad i\mathcal{D}_\mu = i\partial_\mu + n_e A_\mu,$$

λ is the photino field, and q_f and ψ_f are matter fields belonging to Q and $\tilde{Q}$ at $f=1,2$, respectively. Finally, D is an auxiliary field, the last component of the superfield V. Eliminating D via the equation of motion, we get the scalar potential

$$V = \frac{e^2}{2}\Big(\xi - \sum_f n_{ef}\bar{q}^f q_f\Big)^2 + a^2\bar{q}q + (\tilde{m}+a)^2\bar{\tilde{q}}\tilde{q}, \tag{3.25}$$

which implies a potentially rather rich vacuum structure. For our purposes—the BPS-saturated vortices—only the Higgs phase is of importance. We assume that

$$\xi > 0, \quad \tilde{m} \geqslant 0. \tag{3.26}$$

If $\tilde{\psi}$ and $\tilde{q}$ are viewed as regulators (i.e., $\tilde{m}\to\infty$), they can be integrated out leaving the one matter superfield model. It is obvious that by integrating them out, we get a Chern-Simons term at one loop,[4] with a well-defined coefficient that does not vanish in the limit $\tilde{m}=\infty$. We prefer to keep $\tilde{m}$ as a free parameter, assuming that $\tilde{m} \neq 0$.

[4]In passing from two to one matter superfield, in order to justify integrating out $\tilde{Q}$, one must consider $\tilde{m} \gg e\sqrt{\xi}$. Given that $e^2/\xi \ll 1$, the condition $\tilde{m} \gg e\sqrt{\xi}$ does not necessarily imply that $\tilde{m} \gg \xi$.

From the standpoint of vortex studies, the model (3.21) *per se* is not quite satisfactory due to the existence of the flat direction [correspondingly, there is a gapless mode that renders the theory ill-defined in the infrared; see Shifman and Yung (2005)]. The flat direction is eliminated at $\tilde{m}\neq 0$. Thus, there are free relevant parameters of dimension of mass, e^2, ξ, and $\tilde{m}$. The weak-coupling regime implies that $e^2/\xi \ll 1$.

If $\tilde{m}\neq 0$, the field $\tilde{q}$ can (and must) be set to zero, and $\tilde{q},\tilde{\psi}$ play a role only at the level of quantum corrections, providing a well-defined regularization in loops. If $\tilde{q}=0$, the vanishing of the D term in the vacuum requires $\bar{q}q_{\rm vac}=\xi$. Then the term $a^2\bar{q}q$ in Eq. (3.25) implies that $a=0$ in the vacuum. Up to gauge transformations, the vacuum is unique. The Higgs phase is enforced by our choice $\tilde{m}\neq 0$ and $\xi\neq 0$.

a. Central charge

The general form of the centrally extended $\mathcal{N}=2$ superalgebra in $D=3$ was discussed in Sec. II.C.2. The central charge relevant to the problem at hand—vortices—is presented by the last term in Eq. (2.15). It can be derived using the complex representation for supercharges and reducing from $D=4$ to 3. In four dimensions (Gorsky and Shifman, 2000),

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha} + 2Z_{\alpha\dot\alpha} \equiv 2(P_\mu + Z_\mu)(\sigma^\mu)_{\alpha\dot\alpha}, \qquad (3.27)$$

where P_μ is the momentum operator and

$$Z_\mu = \xi \int d^3x\,\epsilon_{0\mu\nu\rho}(\partial^\nu A^\rho) + \cdots\ . \qquad (3.28)$$

Here the ellipses denote full spatial derivatives of currents[5] that fall off exponentially fast at infinity. Such terms are inessential.

In three dimensions, the central charge of interest reduces to P_2+Z_2. Thus, in terms of complex supercharges, the appropriate centrally extended algebra takes the form

$$\{Q,(Q^\dagger)\gamma^0\} = 2(P_0\gamma^0 + P_1\gamma^x + P_3\gamma^z) + 2\left\{\frac{1}{e^2}\int d^2x\,\vec{\nabla}(\vec{E}a) + \frac{\tilde{m}}{2}q - \xi\int d^2x B\right\}, \qquad (3.29)$$

where $\vec{E}$ is the electric field, B is the magnetic field,

$$B = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}, \qquad (3.30)$$

and q is a conserved Noether charge,

[5]Moreover, these currents are not unambiguously defined; see Gorsky and Shifman (2000).

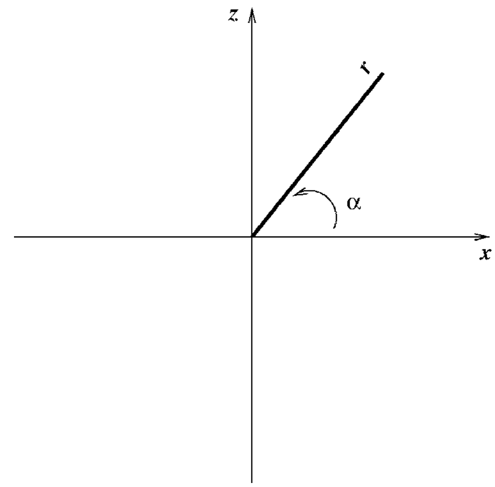

FIG. 5. Polar coordinates on the x,z plane.

$$q = \int d^2x j^0, \quad j^\mu \equiv \bar{\tilde{\psi}}^f\gamma^\mu\tilde{\psi}_f + \bar{\tilde{q}}^f i\overleftrightarrow{\mathcal{D}}_\mu\tilde{q}_f. \qquad (3.31)$$

The second line in Eq. (3.29) presents the vortex-related central charge.[6] The term proportional to a gives a vanishing contribution to the central charge. However, the q term (sometimes omitted in the literature) plays an important role. It combines with the ξ term in the expression for the vortex mass converting the bare value of ξ into the renormalized one. In the problem at hand, the vortex mass gets renormalized at one loop, and so does the Fayet-Iliopoulos parameter.

b. BPS equation for the vortex

At the classical level, the fields a and $\tilde{\phi}$ play no role. They will be set as

$$\tilde{q}=0, \quad a=0. \qquad (3.32)$$

The first-order equations describing the ANO vortex in the Bogomol'nyi limit (Prasad and Sommerfield, 1975; de Vega and Schaposnik, 1976; Witten and Olive, 1978) take the form

$$B - e^2(|q|^2 - \xi) = 0,$$
$$(\mathcal{D}_x + i\mathcal{D}_z)q = 0 \qquad (3.33)$$

with the boundary conditions

$$q \to \sqrt{\xi}e^{ik\alpha} \quad \text{at } r\to\infty,$$
$$q \to 0 \quad \text{at } r\to 0, \qquad (3.34)$$

where α is the polar angle on the x,z plane, while r is the distance from the origin in the same plane (Fig. 5).

[6]The emergence of the U(1) Noether charge $\tilde{m}q/2$ in the central charge is in one-to-one correspondence with a similar phenomenon taking place in the 2D CP($N-1$) models with the twisted mass (Shifman *et al.*, 2006).

Moreover k is an integer counting the number of windings.

If Eqs. (3.33) are satisfied, the flux of the magnetic field is $2\pi k$ (the winding number k determines the quantized magnetic flux), and the vortex mass (string tension) is

$$M = 2\pi\xi k. \tag{3.35}$$

The linear dependence of the k-vortex mass on k implies the absence of their potential interaction.

For the elementary $k=1$ vortex, it is convenient to introduce two profile functions $s(r)$ and $f(r)$ as follows:

$$q(x) = \phi(r)e^{i\alpha}, \quad A_n(x) = -\frac{1}{n_e}\varepsilon_{nm}\frac{x_m}{r^2}[1-f(r)]. \tag{3.36}$$

The ansatz (3.36) goes through the set of equations (3.33), and we get the following two equations on the profile functions:

$$-\frac{1}{r}\frac{df}{dr} + n_e^2e^2(\phi^2-\xi) = 0, \quad r\frac{d\phi}{dr} - f\phi = 0. \tag{3.37}$$

The boundary conditions for the profile functions are rather obvious from the form of the ansatz (3.36) and from our previous discussion. At large distances,

$$\phi(\infty) = \sqrt{\xi}, \quad f(\infty) = 0. \tag{3.38}$$

At the same time, at the origin the smoothness of the field configuration at hand (the absence of singularities) requires

$$\phi(0) = 0, \quad f(0) = 1. \tag{3.39}$$

These boundary conditions are such that the scalar field reaches its vacuum value at infinity. Equations (3.37) with the above boundary conditions lead to a unique solution for the profile functions, although its analytic form is not known. The vortex size is $\sim e^{-1}\xi^{-1/2}$.

c. Fermion zero modes

Quantization of vortices requires knowledge of the fermion zero modes for the given classical solution. More precisely, since the solution under consideration is static, we are interested in the zero-eigenvalue solutions of the static fermion equations, which, therefore, effectively become two rather than three dimensional,

$$i(\gamma^x\mathcal{D}_x + \gamma^z\mathcal{D}_z)\psi + \sqrt{2}\lambda q = 0. \tag{3.40}$$

This equation is obtained from Eq. (3.24), where we dropped the terms with tildes (since $\tilde{q}=0$) and, correspondingly, set $n_e \to 1$. The fermion operator is Hermitean, implying that every solution for $\{\psi,\lambda\}$ is accompanied by that for $\{\bar{\psi},\bar{\lambda}\}$.

Since the solution to Eq. (3.33) discussed above is 1/2 BPS, two of the four supercharges annihilate it while the other two generate the fermion zero modes—superpartners of translational modes. One can show (Rebhan *et al.*, 2004b) that these are the only normalizable fermion zero modes in the problem.

d. Short versus long representations

The (1+2)-dimensional model under consideration has four supercharges. The corresponding regular superrepresentation is four dimensional (i.e., it contains two bosonic and two fermionic states).

The vortex we discuss has two fermion zero modes. Viewed as a particle in 1+2 dimensions, it forms a superdoublet (one bosonic state plus one fermionic). Hence, this is a short multiplet. This implies, of course, that the BPS bound must remain saturated when quantum corrections are switched on. Both the central charge and the vortex mass get corrections (Rebhan *et al.*, 2004b), but they remain equal to each other.

2. ANO string in four dimensions

The vacuum manifold is a "hyperboloid,"

$$\bar{q}q - \bar{\tilde{q}}\tilde{q} = \xi, \quad a = 0. \tag{3.41}$$

Thus, we deal with the Higgs branch of real dimension 2. In fact, the vacuum manifold can be parametrized by a complex modulus $\tilde{q}q$. On this Higgs branch, the photon field and a, plus their superpartners, form a massive supermultiplet, while $\tilde{q}q$ and superpartners form a massless one.

As was shown by Penin *et al.* (1996) no finite-size vortices exist at a generic point on the vacuum manifold, due to the absence of the mass gap (presence of the massless Higgs excitations). The moduli fields get involved in the solution at the classical level generating a logarithmically convergent tail. Still certain infrared regularizations remove this logarithmic divergence and vortices become well defined but they are not BPS; see Yung (1999).

At the base of the Higgs branch, at $\tilde{q}=0$ the classical solution of the BPS equations *per se* is well defined. The fact that there is a flat direction and, hence, massless particles in the bulk theory does not disappear, of course. Even though at $\tilde{q}=0$ the classical string solution is well defined, infrared problems arise at the loop level. One can avoid massless particles in the spectrum if one embeds the theory (3.24) in SQED with eight supercharges; see Shifman and Yung (2005). Then the Higgs branch is eliminated, and one is left with isolated vacua. After the embedding is done, one can break $\mathcal{N}=2$ down to $\mathcal{N}=1$, if one so desires.

C. Monopoles

In this section, we discuss magnetic monopoles—interesting objects that carry magnetic charges. They emerge as free magnetically charged particles in non-Abelian gauge theories in which the gauge symmetry is spontaneously broken down to an Abelian subgroup.[7]

[7]In the confining regime, monopoles can be obtained in some theories with no adjoint fields, in which the gauge symmetry is broken completely (Gorsky *et al.*, 2007). This is a recent development.

The simplest example was found by 't Hooft (1974) and Polyakov (1974). The model they considered had been invented by Georgi and Glashow (1972) for different purposes. As it often happens, the Georgi-Glashow model turned out to be more valuable than the original purpose, which is long forgotten, while the model itself is alive and well and is being constantly used by theorists.

1. Georgi-Glashow model: Vacuum and elementary excitations

We begin with a brief description of the Georgi-Glashow model. The gauge group is SU(2) and the matter sector consists of one real scalar field ϕ^a in the adjoint representation [i.e., SU(2) triplet]. The Lagrangian of the model is

$$L = -\frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) - \frac{1}{8}\lambda(\phi^a\phi^a - v^2)^2, \tag{3.42}$$

where the covariant derivative in the adjoint acts as

$$D_\mu\phi^a = \partial_\mu\phi^a + \varepsilon^{abc}A^b_\mu\phi^c. \tag{3.43}$$

Below we focus on the limit of BPS monopoles. This limit corresponds to a vanishing scalar coupling, $\lambda \to 0$. The only role of the last term in Eq. (3.42) is to provide a boundary condition for the scalar field. As seen from Sec. II, the monopole central charge exists only in $\mathcal{N}=2$ and 4 superalgebras. Therefore, one should understand the theory (3.42) (at $\lambda=0$) as embedded in super-Yang-Mills theories with extended superalgebra. In Secs. IV–VII we extensively discuss such embeddings in the context of $\mathcal{N}=2$.

The classical definition of magnetic charges refers to theories that support long-range (Coulomb) magnetic field. Therefore, in consideration of the isolated monopole, the pattern of the symmetry breaking should be such that some of the gauge bosons remain massless. In the Georgi-Glashow model (3.42), the pattern is as follows:

$$\mathrm{SU}(2) \to \mathrm{U}(1). \tag{3.44}$$

To see that this is indeed the case, we note that the ϕ^a self-interaction term [the last term in Eq. (3.42)] forces ϕ^a to develop a vacuum expectation value,

$$\langle\phi^a\rangle = v\,\delta^{3a}. \tag{3.45}$$

The direction of the vector ϕ^a in the SU(2) space (to be referred to as "color space" or "isospace") can be chosen arbitrarily. One can always reduce it to the form (3.45) by a global color rotation. Thus, Eq. (3.45) can be viewed as a (unitary) gauge condition on the field ϕ.

This gauge is convenient for discussing the particle content of the theory, namely elementary excitations. Since the color rotation around the third axis does not change the vacuum expectation value of ϕ^a,

$$\exp\left\{i\alpha\frac{\tau_3}{2}\right\}\phi_{\rm vac}\exp\left\{-i\alpha\frac{\tau_3}{2}\right\} = \phi_{\rm vac}, \quad \phi_{\rm vac} = v\frac{\tau_3}{2}, \tag{3.46}$$

the third component of the gauge field remains massless—we will call it photon,

$$A^3_\mu \equiv A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.47}$$

The first and second components form massive vector bosons,

$$W^\pm_\mu = \frac{1}{\sqrt{2}g}(A^1_\mu \pm A^2_\mu). \tag{3.48}$$

As usual in the Higgs mechanism, the massive vector bosons eat up the first and second components of the scalar field ϕ^a. The third component, the physical Higgs field, can be parametrized as

$$\phi^3 = v + \varphi, \tag{3.49}$$

where φ is the physical Higgs field. In terms of these fields, the Lagrangian (3.42) can be rewritten as

$$\begin{aligned} L = &-\frac{1}{4g^2}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi)^2 - (D_\alpha W^+_\mu)(D_\alpha W^-_\mu) \\ &+ (D_\mu W^+_\mu)(D_\nu W^-_\nu) + g^2(v+\phi)^2 W^+_\mu W^-_\mu \\ &- 2W^+_\mu F_{\mu\nu}W^-_\nu + \frac{g^2}{4}(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)^2, \end{aligned} \tag{3.50}$$

where the covariant derivative now includes only the photon field,

$$D_\alpha W^\pm = (\partial_\alpha \pm iA_\alpha)W^\pm. \tag{3.51}$$

The last line presents the magnetic moment of the charged (massive) vector bosons and their self-interaction. In the limit $\lambda \to 0$, the physical Higgs field is massless. The mass of the $W^\pm$ bosons is

$$M_W = gv. \tag{3.52}$$

2. Monopoles—Topological argument

We explain why this model has a topologically stable soliton [see, e.g., Coleman (1983)].

Assume that the monopole's center is at the origin and consider a large sphere $\mathcal{S}_R$ of radius R with the center at the origin. Since the mass of the monopole is finite, by definition, $\phi^a\phi^a=v^2$ on this sphere. ϕ^a is a three-component vector in the isospace subject to the constraint $\phi^a\phi^a=v^2$, which gives us a two-dimensional sphere $\mathcal{S}_G$. Thus, we deal here with mappings of $\mathcal{S}_R$ into $\mathcal{S}_G$. Such mappings split in distinct classes labeled by an integer n, counting how many times the sphere $\mathcal{S}_G$ is swept when we sweep once the sphere $\mathcal{S}_R$, since

$$\pi_2[\mathrm{SU}(2)/\mathrm{U}(1)] = Z. \tag{3.53}$$

$\mathcal{S}_G=\mathrm{SU}(2)/\mathrm{U}(1)$ because for each given vector ϕ^a there is a U(1) subgroup that does not rotate it. The SU(2) group space is a three-dimensional sphere while that of SU(2)/U(1) is a two-dimensional sphere.

An isolated monopole field configuration (the 't Hooft–Polyakov monopole) corresponds to a mapping with $n=1$. Since it is impossible to continuously deform it to the topologically trivial mapping, the monopoles are topologically stable.

3. Mass and magnetic charge

Classically the monopole mass is given by the energy functional

$$E = \int d^3x \left\{ \frac{1}{2g^2} B_i^a B_i^a + \frac{1}{2}(D_i\phi^a)(D_i\phi^a) \right\}, \tag{3.54}$$

where

$$B_i^a = -\tfrac{1}{2}\varepsilon_{ijk}F_{jk}^a. \tag{3.55}$$

The fields are assumed to be time independent, $B_i^a = B_i^a(\vec{x})$, $\phi^a = \phi^a(\vec{x})$. For static fields it is natural to assume that $A_0^a=0$. This assumption will be verified *a posteriori*, after we find the field configuration minimizing the functional (3.54). Equation (3.54) assumes the limit $\lambda \to 0$. However, in performing minimization we should keep in mind the boundary condition $\phi^a(\vec{x})\phi^a(\vec{x}) \to v^2$ at $|\vec{x}| \to \infty$.

Equation (3.54) can be rewritten as follows:

$$E = \int d^3x \left\{ \frac{1}{2}\left(\frac{1}{g}B_i^a - D_i\phi^a\right)\left(\frac{1}{g}B_i^a - D_i\phi^a\right) + \frac{1}{g}B_i^a D_i\phi^a \right\}. \tag{3.56}$$

The last term on the right-hand side is a full derivative. Indeed, after integrating by parts and using the equation of motion $D_iB_i^a=0$, we get

$$\int d^3x \left\{ \frac{1}{g} B_i^a D_i \phi^a \right\} = \frac{1}{g}\int d^3x\, \partial_i (B_i^a\phi^a) = \frac{1}{g}\int_{\mathcal{S}_R} d^2S_i (B_i^a\phi^a). \tag{3.57}$$

In the last line, we made use of Gauss's theorem and passed from the volume integration to that over the surface of the large sphere. Thus, the last term in Eq. (3.56) is topological.

The combination $B_i^a\phi^a$ can be viewed as a gauge-invariant definition of the magnetic field $\vec{\mathcal{B}}$. More exactly,

$$\mathcal{B}_i = \frac{1}{v}B_i^a\phi^a. \tag{3.58}$$

Indeed, far away from the monopole core one can always assume ϕ^a to be aligned in the same way as in the vacuum (in an appropriate gauge), $\phi^a = v\delta^{3a}$. Then $\mathcal{B}_i = B_i^3$. The advantage of the definition (3.58) is that it is gauge independent.

Furthermore, the magnetic charge Q_M inside a sphere $\mathcal{S}_R$ can be defined through the flux of the magnetic field through the surface of the sphere,[8]

$$Q_M = \int_{\mathcal{S}_R} d^2S_i \frac{1}{g}\mathcal{B}_i. \tag{3.59}$$

From Eq. (3.71) (see below), we see that

$$\mathcal{B}_i \equiv \frac{1}{v}B_i^a\phi^a \to n^i\frac{1}{r^2} \text{ at } r\to\infty, \tag{3.60}$$

and, hence,

$$Q_M = \frac{4\pi}{g}. \tag{3.61}$$

Combining Eqs. (3.59), (3.58), and (3.57), we conclude that

$$E = vQ_M + \int d^3x \left\{ \frac{1}{2}\left(\frac{1}{g}B_i^a - D_i\phi^a\right)\left(\frac{1}{g}B_i^a - D_i\phi^a\right) \right\}. \tag{3.62}$$

The minimum of the energy functional is attained at

$$\frac{1}{g}B_i^a - D_i\phi^a = 0. \tag{3.63}$$

The mass of the field configuration realizing this minimum—the monopole mass—is equal to

$$M_M = \frac{4\pi v}{g}. \tag{3.64}$$

Thus, the mass of the critical monopole is in one-to-one relation with its magnetic charge. Equation (3.63) is the Bogomol'nyi equation in the monopole problem. If it is satisfied, the second-order differential equations of motion are satisfied too.

4. Solution of the Bogomol'nyi equation for monopoles

To solve the Bogomol'nyi equations, we need to find an appropriate ansatz for ϕ^a. As one sweeps S_R, the vector ϕ^a must sweep the group space sphere. The simplest choice is to identify these two spheres point by point,

$$\phi^a = v\frac{x^a}{r} = vn^a, \quad r\to\infty, \tag{3.65}$$

where $n^i \equiv x^i/r$. This field configuration belongs to the class with $n=1$. The SU(2) group index a got entangled with the coordinate $\vec{x}$. Polyakov proposed to refer to such fields as "hedgehogs."

[8]Different conventions for the charge normalization may vary. Dirac (1931) used the convention $e^2=\alpha$ and the electromagnetic Hamiltonian $\mathcal{H}=(8\pi)^{-1}(\vec{E}^2+\vec{B}^2)$. Then, the electric charge is defined through the flux of the electric field as $e=(4\pi)^{-1}\int_{\mathcal{S}_R} d^2S_iE_i$, and analogously for the magnetic charge. We use the convention according to which $e^2=4\pi\alpha$, and the electromagnetic Hamiltonian $\mathcal{H}=(2g^2)^{-1}(\vec{E}^2+\vec{B}^2)$. Then $e=g^{-1}\int_{\mathcal{S}_R} d^2S_iE_i$ while $Q_M=g^{-1}\int_{\mathcal{S}_R} d^2S_iB_i$.

Next, observe that finiteness of the monopole energy requires the covariant derivative $D_i\phi^a$ to fall off faster than $r^{-3/2}$ at large r, cf. Eq. (3.54). Since

$$\partial_i\phi^a = v\frac{1}{r}\{\delta^{ai} - n^a n^i\} \sim \frac{1}{r}, \tag{3.66}$$

one must choose A_i^b in such a way as to cancel Eq. (3.66). It is not difficult to see that

$$A_i^a = \varepsilon^{aij}\frac{1}{r}n^j, \quad r\to\infty. \tag{3.67}$$

Then the term $1/r$ is canceled in $D_i\phi^a$.

Equations (3.65) and (3.67) determine the index structure of the field configuration which we use. The appropriate ansatz is now

$$\phi^a = vn^a H(r), \quad A_i^a = \varepsilon^{aij}\frac{1}{r}n^j F(r), \tag{3.68}$$

where H and F are functions of r with the boundary conditions

$$H(r)\to 1, \quad F(r)\to 1 \quad \text{at } r\to\infty \tag{3.69}$$

and

$$H(r)\to 0, \quad F(r)\to 0 \quad \text{at } r\to 0. \tag{3.70}$$

The boundary condition (3.69) is equivalent to Eqs. (3.65) and (3.67), while the boundary condition (3.70) guarantees that our solution is nonsingular at $r\to 0$.

After some straightforward algebra, we get

$$B_i^a = (\delta^{ai} - n^a n^i)\frac{1}{r}F' + n^a n^i\frac{1}{r^2}(2F - F^2),$$

$$D_i\phi^a = v\left\{(\delta^{ai} - n^a n^i)\frac{1}{r}H(1-F) + n^a n^i H'\right\}, \tag{3.71}$$

where the prime denotes differentiation with respect to r.

We return now to the Bogomol'nyi equations (3.63). This is a set of nine first-order differential equations. Our ansatz has only two unknown functions. The fact that the ansatz goes through and we get two scalar equations on two unknown functions from the Bogomol'nyi equations is a highly nontrivial check. Comparing Eqs. (3.63) and (3.71), we get

$$\frac{1}{g}F' = vH(1-F),$$

$$H' = \frac{1}{gv}\frac{1}{r^2}(2F - F^2). \tag{3.72}$$

The functions H and F are dimensionless. It is convenient to make the radius r dimensionless too. A natural unit of length in the problem at hand is $(gv)^{-1}$. From now on we measure r in these units,

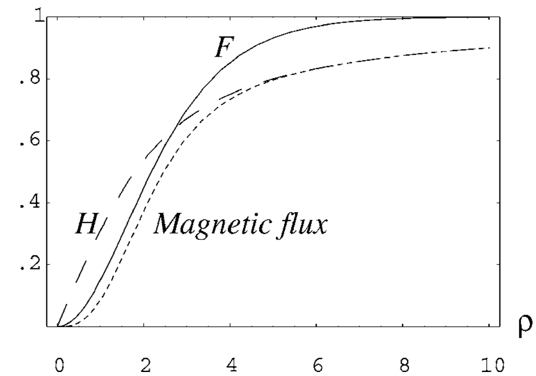

FIG. 6. The functions F (solid line) and H (long dashes) in the critical monopole solution vs ρ. The short-dashed line shows the flux of the magnetic field $\mathcal{B}_i$ (in the units 4π) through the sphere of radius ρ.

$$\rho = r(gv). \tag{3.73}$$

The functions H and F are considered as functions of ρ, while the prime denotes differentiation over ρ. Then the system (3.72) takes the form

$$F' = H(1-F),$$

$$H' = \frac{1}{\rho^2}(2F - F^2). \tag{3.74}$$

These equations have known analytical solutions,

$$F = 1 - \frac{\rho}{\sinh\rho},$$

$$H = \frac{\cosh\rho}{\sinh\rho} - \frac{1}{\rho}. \tag{3.75}$$

At large ρ, the functions H and F tend to unity [cf. Eq. (3.69)] while at $\rho\to 0$,

$$F = 0(\rho^2), \quad H = 0(\rho).$$

They are plotted in Fig. 6. Calculating the flux of the magnetic field through the large sphere, we verify that for the solution (3.75) $Q_M = 4\pi/g$.

5. Collective coordinates (moduli)

The monopole solution presented in the previous section breaks a number of valid symmetries of the theory, for instance, translational invariance. As usual, the symmetries are restored after the introduction of the collective coordinates (moduli), which convert a given solution into a family of solutions.

Our first task is to count the number of moduli in the monopole problem. A straightforward way to count this number is counting linearly independent zero modes. To this end, one represents the fields A_μ and ϕ as a sum of the monopole background plus small deviations,

$$A^a_\mu = A^{a(0)}_\mu + a^a_\mu, \quad \phi^a = \phi^{a(0)} + (\delta\phi)^a, \tag{3.76}$$

where the superscript (0) marks the monopole solution. At this point, it is necessary to impose a gauge-fixing condition. A convenient condition is

$$\frac{1}{g} D_i a^a_i - \varepsilon^{abc}\phi^b(\delta\phi)^c = 0, \tag{3.77}$$

where the covariant derivative in the first term contains only the background field.

Substituting the decomposition (3.76) in the Lagrangian, one finds the quadratic form for $\{a,(\delta\phi)\}$, and determines the zero modes of this form [subject to the condition (3.77)].

We will not track this procedure in detail, referring the reader to the original literature (Mottola, 1978). Instead, we suggest a simple heuristic consideration.

We ask ourselves, what are the valid symmetries of the model at hand? They are as follows: (i) three translations, (ii) three spatial rotations, and (iii) three rotations in the SU(2) group. Not all these symmetries are independent. It is not difficult to check that the spatial rotations are equivalent to the SU(2) group rotations for the monopole solution. Thus, we should not count them independently. This leaves us with six symmetry transformations.

One should not forget, however, that two of those six act nontrivially in the trivial vacuum. Indeed, the latter is characterized by the condensate (3.45). While rotations around the third axis in the isospace leave the condensate intact [see Eq. (3.46)], the rotations around the first and second axes do not. Thus, the number of moduli in the monopole problem is $6-2=4$. These four collective coordinates have a very transparent physical interpretation. Three of them correspond to translations. They are introduced in the solution through the substitution

$$\vec{x} \to \vec{x} - \vec{x}_0. \tag{3.78}$$

The vector $\vec{x}_0$ now plays the role of the monopole center. The unit vector $\vec{n}$ is now defined as $\vec{n}=(\vec{x}-\vec{x}_0)/|\vec{x}-\vec{x}_0|$.

The fourth collective coordinate is related to the unbroken U(1) symmetry of the model. This is the rotation around the direction of alignment of the field ϕ. In the trivial vacuum, ϕ^a is aligned along the third axis. The monopole generalization of Eq. (3.46) is

$$A^{(0)} \to U^{-1}A^{(0)}U - iU^{-1}\partial U,$$

$$\phi^{(0)} \to U^{-1}\phi^{(0)}U = \phi^{(0)},$$

$$U = \exp\{i\alpha\phi^{(0)}/v\}, \tag{3.79}$$

where the fields $A^{(0)}$ and $\phi^{(0)}$ are understood here in the matrix form,

$$A^{(0)} = A^{a(0)}(\tau^a/2), \quad \phi^{(0)} = \phi^{a(0)}(\tau^a/2).$$

Unlike the vacuum field, which is not changed under Eq. (3.46), the monopole solution for the vector field changes its form. The change looks like a gauge transformation. Note, however, that the gauge matrix U does not tend to unity at $r\to\infty$. Thus, this transformation is in fact a global U(1) rotation. The physical meaning of the collective coordinate α will become clear shortly. Now we note that (i) for small α, Eq. (3.79) reduces to

$$\delta A^a_i = \alpha\frac{1}{v}(D_i\phi^{(0)})^a, \quad \delta\phi = 0, \tag{3.80}$$

and this is compatible with the gauge condition (3.77); (ii) the variable α is compact, since the points α and $\alpha+2\pi$ can be identified (the transformation of $A^{(0)}$ is identically the same for α and $\alpha+2\pi$). In other words, α is an angle variable.

Having identified all four moduli relevant to the problem, we can proceed to the quasiclassical quantization. The task is to obtain quantum mechanics of the moduli. We start from the monopole center coordinate $\vec{x}_0$. To this end, we assume that $\vec{x}_0$ depends weakly on time t, so that the only time dependence of the solution enters through $\vec{x}_0(t)$. The time dependence is important only in time derivatives, so that the quantum-mechanical Lagrangian of moduli can be obtained from the following expression:

$$\mathcal{L}_{\rm QM} = -M_M + \frac{1}{2}(\dot{x}_0)_k(\dot{x}_0)_j \int d^3x \left\{ \left[\frac{1}{g}F^{a(0)}_{ik}\right] \times \left[\frac{1}{g}F^{a(0)}_{ij}\right] + [D_k\phi^{a(0)}][D_j\phi^{a(0)}]\right\}, \tag{3.81}$$

where $\partial_k A$ and $\partial_k\phi$ were supplemented by appropriate gauge transformations to satisfy the gauge condition (3.77).

Averaging over the angular orientations of $\vec{x}$ yields

$$\mathcal{L}_{\rm QM} = -M_M + \frac{1}{2}(\dot{\vec{x}}_0)^2 \int d^3x \left\{\frac{2}{3}\frac{1}{g^2}B^{a(0)}_i B^{a(0)}_i + \frac{1}{3}D_i\phi^{a(0)}D_i\phi^{a(0)}\right\} = -M_M + \frac{M_M}{2}(\dot{\vec{x}}_0)^2. \tag{3.82}$$

This last result follows if one combines Eqs. (3.54) and (3.63). Of course, this final answer could have been guessed from the very beginning since this is merely but the Lagrangian describing free nonrelativistic motion of a particle of mass M_M endowed with the coordinate $\vec{x}_0$.

Now, having tested the method in the case in which the answer was obvious, we apply it to the fourth collective coordinate α. Using Eq. (3.80), we get

$$\mathcal{L}_{\alpha\rm QM} = \frac{1}{2}\frac{M_M}{M_W^2}\dot{\alpha}^2, \tag{3.83}$$

or, equivalently,

$$\mathcal{H}_\alpha = \frac{1}{2}\frac{M_W^2}{M_M}p_\alpha^2, \quad p_\alpha \equiv -i\frac{d}{d\alpha}, \tag{3.84}$$

where $\mathcal{H}_\alpha$ is the part of the Hamiltonian relevant to α. The full quantum-mechanical Hamiltonian describing the moduli dynamics is, thus,

$$\mathcal{H} = M_M + \frac{p^2}{2M_M} + \frac{1}{2}\frac{M_W^2}{M_M}p_\alpha^2, \quad p \equiv -i\frac{d}{dx_0}. \tag{3.85}$$

It describes free motion of a spinless particle endowed with an internal (compact) variable α. While the spatial part of $\mathcal{H}$ does not raise any questions, the α dynamics deserves an additional discussion.

The α motion is free, but one should not forget that α is an angle. Because of the 2π periodicity, the corresponding wave functions must have the form

$$\Psi(\alpha) = e^{ik\alpha}, \tag{3.86}$$

where k is an integer, $k=0,\pm 1,\pm 2,\ldots$. Strictly speaking, only the ground state $k=0$ describes the monopole—a particle with the magnetic charge $4\pi/g$ and vanishing electric charge. Excitations with $k\neq 0$ correspond to a particle with the magnetic charge $4\pi/g$ and the electric charge kg, the dyon.

To see that this is indeed the case, we note that for $k\neq 0$ the expectation value of p_α is k and, hence, the expectation value of $\dot\alpha=(M_W^2/M_M)p_\alpha$ is $M_W^2 k/M_M$. Moreover, we define a gauge-invariant electric field $\mathcal{E}_i$ [analogous to $\mathcal{B}_i$ of Eq. (3.58)] as

$$\mathcal{E}_i \equiv \frac{1}{v}E_i^a\phi^a = \frac{1}{v}\phi^{a(0)}\dot{A}_i^{a(0)} = \frac{1}{v^2}\dot\alpha\phi^{a(0)}(D_i\phi^{a(0)}). \tag{3.87}$$

Since for the critical monopole $D_i\phi^{a(0)}=(1/g)B_i^{a(0)}$, we see that

$$\mathcal{E}_i = \dot\alpha\frac{1}{M_W}\mathcal{B}_i, \tag{3.88}$$

and the flux of the gauge-invariant electric field over the large sphere is

$$\frac{1}{g}\int_{\mathcal{S}_R} d^2S_i\mathcal{E}_i = \frac{M_W^2 k}{M_M}\frac{1}{M_W}\frac{1}{g}\int_{\mathcal{S}_R} d^2S_i\mathcal{B}_i, \tag{3.89}$$

where we replaced $\dot\alpha$ by its expectation value. Thus, the flux of the electric field reduces to

$$\frac{1}{g}\int_{\mathcal{S}_R} d^2S_i\mathcal{E}_i = kg, \tag{3.90}$$

which proves the above assertion of the electric charge kg.

It is interesting to note that the mass of the dyon can be written as

$$M_D = M_M + \frac{1}{2}\frac{M_W^2}{M_M}k^2 \approx \sqrt{M_M^2 + M_W^2k^2} = v\sqrt{Q_M^2+Q_E^2}. \tag{3.91}$$

Although from our derivation it might seem that the square root result is approximate, in fact, the prediction for the dyon mass $M_D=v(Q_M^2+Q_E^2)^{1/2}$ is exact; it follows from the BPS saturation and the central charges in the $\mathcal{N}=2$ model (see Sec. II).

Magnetic monopoles were introduced by Dirac (1931). He considered macroscopic electrodynamics and derived a self-consistency condition for the product of the magnetic charge of the monopole Q_M and the elementary electric charge e,[9]

$$Q_M e = 2\pi. \tag{3.92}$$

This is known as the Dirac quantization condition. For the 't Hooft–Polyakov monopole, we have just derived that $Q_M g=4\pi$, twice as large as in the Dirac quantization condition. Note, however, that g is the electric charge of the W bosons. It is not the minimal possible electric charge that can be present in the theory at hand. If quarks in the fundamental (doublet) representation of SU(2) were introduced in the Georgi-Glashow model, their U(1) charge would be $e=g/2$, and the Dirac quantization condition would be satisfied for such elementary charges.

6. The θ term induces a fractional electric charge for the monopole (the Witten effect)

There is a P- and T-odd term, the θ term, which can be added to the Lagrangian for the Yang-Mills theory without spoiling renormalizability. It is given by

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2}F_{\mu\nu}^a\tilde{F}^{a\mu\nu} = -\frac{\theta}{8\pi^2}\vec{E}^a\cdot\vec{B}^a. \tag{3.93}$$

This interaction violates P and CP but not C. As is well known, this term is a surface term and does not affect the classical equations of motion. There is, however, θ dependence in instanton effects that involve nontrivial long-range behavior of the gauge fields. As was realized by Witten (1979a), in the presence of magnetic monopoles θ also has a nontrivial effect, namely it shifts the allowed values of electric charge in the monopole sector of the theory.

Since the equations of motions do not change, the monopole solution obtained above stays intact. What changes is the effective quantum-mechanical Lagrangian. As usual, we assume an adiabatic time dependence of moduli. In the case at hand, we must replace the constant phase modulus α by $\alpha(t)$. This generates the electric field

[9]In Dirac's original convention, the charge quantization condition is, in fact, $Q_M e=1/2$.

$$E_i^a = \dot{\alpha}(\delta A_i^a/\delta\alpha) = \frac{\dot{\alpha}}{v}(D_i\phi^{(0)})^a,$$

where Eq. (3.80) is used. The magnetic field does not change, and can be expressed through $(D_i\phi^{(0)})^a$ using Eq. (3.63). As a result, the quantum-mechanical Lagrangian for α acquires a full derivative term,

$$\mathcal{L}_{\alpha\mathrm{QM}} = \frac{1}{2\mu}\dot{\alpha}^2 - \frac{\theta}{2\pi}\dot{\alpha}, \quad \mu = \frac{M_W^2}{M_M}. \tag{3.94}$$

This changes the expression for the canonic momentum conjugated to α. If previously p_α was $\dot{\alpha}/\mu$, now

$$p_\alpha = \frac{\dot{\alpha}}{\mu} - \frac{\theta}{2\pi}. \tag{3.95}$$

Correspondingly,

$$\dot{\alpha} = \mu\left(p_\alpha + \frac{\theta}{2\pi}\right). \tag{3.96}$$

From Sec. III.C.5, we know that the electric charge of the field configuration under consideration is [see Eq. (3.90)]

$$Q_E = \frac{1}{M_W g}\langle\dot{\alpha}\rangle\int_{S_R} d^2S_i\mathcal{B}_i. \tag{3.97}$$

Substituting Eq. (3.96) and $\langle p_\alpha\rangle = k$, we arrive at

$$Q_E = \left(k + \frac{\theta}{2\pi}\right)g. \tag{3.98}$$

We see that at $\theta\neq 0$ the electric charge of the dyon is noninteger. As θ changes from zero to the physically equivalent point $\theta=2\pi$, the dyon charges shift by one unit. The dyon spectrum as a whole remains intact.

D. Monopoles and fermions

The critical 't Hooft–Polyakov monopoles just discussed can be embedded in $\mathcal{N}=2$ super-Yang-Mills. There are no $\mathcal{N}=1$ models with the 't Hooft–Polyakov monopoles [albeit $\mathcal{N}=1$ theories supporting confined monopoles are found (Gorsky *et al.*, 2007)]. The minimal model with the BPS-saturated 't Hooft–Polyakov monopole is the $\mathcal{N}=2$ generalization of supersymmetric gluodynamics, with the gauge group SU(2). In terms of $\mathcal{N}=1$ superfields, it contains one vector superfield in the adjoint describing gluon and gluino, plus one chiral superfield in the adjoint describing a scalar $\mathcal{N}=2$ superpartner for gluon and a Weyl spinor, an $\mathcal{N}=2$ superpartner for gluino.

The couplings of fermion fields to boson fields are of a special form; they are fixed by $\mathcal{N}=2$ supersymmetry. In this section, we first present the Lagrangian of $\mathcal{N}=2$ supersymmetric gluodynamics, including the part with the adjoint fermions, and then consider effects due to the adjoint fermions. We conclude Sec. III.D with a comment on fermions in the fundamental representation in the monopole background.

1. $\mathcal{N}=2$ super-Yang-Mills (without matter)

Two $\mathcal{N}=1$ superfields are used to build the model,

$$W_\alpha = i(\lambda_\alpha + i\theta_\alpha D - \theta^\beta F_{\alpha\beta} - i\theta^2 D_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}) \tag{3.99}$$

and

$$\mathcal{A} = a + \sqrt{2}\psi\theta + \theta^2 F. \tag{3.100}$$

Here the notation is spinorial, and all fields are in the adjoint representation of SU(2). The corresponding generators are

$$(T^a)_{bd} = i\varepsilon_{bad}. \tag{3.101}$$

The Lagrangian contains kinetic terms and their supergeneralizations. In components

$$\mathcal{L} = \frac{1}{g^2}\Big\{ -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \lambda^{\alpha,a}iD_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha},a} + \frac{1}{2}D^aD^a + \psi^{\alpha,a}i\mathcal{D}_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha},a} + (D^\mu\bar{a})(D_\mu a) - \sqrt{2}\varepsilon_{abc}(\bar{a}^a\lambda^{\alpha,b}\psi^c_\alpha + a^a\bar{\lambda}^b_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha},c}) - \frac{i}{2}\varepsilon_{abc}D^a\bar{a}^b a^c\Big\}. \tag{3.102}$$

As usual, the D field is auxiliary and can be eliminated via an equation of motion,

$$D^a = \frac{i}{2}\varepsilon_{abc}\bar{a}^b a^c. \tag{3.103}$$

There is a flat direction: if the field a is real, all D terms vanish. If a is chosen to be purely real or purely imaginary and the fermion fields ignored, we obviously return to the Georgi-Glashow model discussed above. In the general case, combining λ^α and $\bar{\psi}_{\dot{\alpha}}$ into one Dirac spinor

$$\Psi = \begin{pmatrix}\lambda^\alpha \\ \bar{\psi}_{\dot{\alpha}}\end{pmatrix}, \tag{3.104}$$

we get

$$\mathcal{L} = \frac{1}{g^2}\Big\{ -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + (D^\mu\bar{a})(D_\mu a) + \bar{\Psi}i\mathcal{D}\Psi + \frac{1}{8}(\varepsilon_{abc}\bar{a}^b a^c)^2 - \frac{\sqrt{2}}{2}\varepsilon_{abc}[\bar{a}^a\bar{\Psi}^b(1+\gamma^5)\Psi^a - a^a\bar{\Psi}^b(1-\gamma^5)\Psi^a]\Big\}, \tag{3.105}$$

where the Dirac matrices are taken in the spinor representation. For purely imaginary $a\equiv i\phi/\sqrt{2}$ (where ϕ is real), the fermion interaction takes the form

$$\mathcal{L}_\Psi = -\frac{1}{g^2}\bar{\Psi}\phi\Psi. \tag{3.106}$$

2. Zero modes for adjoint fermions

Equations for the fermion zero modes can be derived from the Lagrangian (3.102),

$$iD_{\alpha\dot{\alpha}}\lambda^{\alpha,c} - \sqrt{2}\varepsilon_{abc}a^a\bar{\psi}^b_{\dot{\alpha}} = 0$$

$$iD_{\alpha\dot{\alpha}}\psi^{\alpha,c} + \sqrt{2}\varepsilon_{abc}a^a\bar{\lambda}^b_{\dot{\alpha}} = 0, \tag{3.107}$$

plus Hermitean conjugate. After a brief reflection, we can get two complex (four real) zero modes.[10] Two of them are obtained if we substitute

$$\lambda^\alpha = F^{\alpha\beta}, \quad \bar{\psi}_{\dot{\alpha}} = \sqrt{2}D_{\alpha\dot{\alpha}}\bar{a}. \tag{3.108}$$

The other two solutions correspond to the following substitution:

$$\psi^\alpha = F^{\alpha\beta}, \quad \bar{\lambda}_{\dot{\alpha}} = \sqrt{2}D_{\alpha\dot{\alpha}}\bar{a}. \tag{3.109}$$

This result is easy to understand. Our starting theory has eight supercharges. The classical monopole solution is BPS saturated, implying that four of these eight supercharges annihilate the solution (these are the Bogomol'nyi equations), while the action of the other four supercharges produces the fermion zero modes.

With four real fermion collective coordinates, the monopole supermultiplet is four dimensional: it includes two bosonic states and two fermionic. (The above counting refers just to the monopole, without its antimonopole partner. The antimonopole supermultiplet also includes two bosonic and two fermionic states.) From the standpoint of $\mathcal{N}=2$ supersymmetry in four dimensions, this is a short multiplet. Hence, the monopole states remain BPS saturated to all orders in perturbation theory [in fact, the criticality of the monopole supermultiplet is valid beyond perturbation theory (Seiberg and Witten, 1994a, 1994b)].

3. Zero modes for fermions in the fundamental representation

This topic, related to charge fractionalization, is a marginal topic for this review and, therefore, we limit ourselves to a brief comment. The interested reader is referred to Harvey (1997) and Rubakov (2002) for further details. The fermion part of the Lagrangian can be obtained from Eqs. (3.105) and (3.106) with replacement of the adjoint Dirac fermion by the fundamental one, which we denote by χ,

$$\mathcal{L} = \frac{1}{g^2}\left\{-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D^\mu\phi)(D_\mu\phi) + \bar{\chi}i\mathcal{D}\chi - \bar{\chi}\phi\chi\right\}. \tag{3.110}$$

The Dirac equation then takes the form

$$(i\gamma^\mu D_\mu - \phi)\chi = 0. \tag{3.111}$$

Needless to say, the gamma matrices can now be chosen in any representation. The one that is convenient here is

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}. \tag{3.112}$$

For the static 't Hooft–Polyakov monopole configuration (with $A_0=0$), the zero mode equations reduce to two decoupled equations,

$$\mathcal{D}\chi^- \equiv (i\sigma^i D_i + \phi)\chi^- = 0,$$

$$\mathcal{D}^\dagger\chi^+ \equiv (i\sigma^i D_i - \phi)\chi^+ = 0, \quad i = 1,2,3, \tag{3.113}$$

provided we parametrize $\chi(\vec{x})$ in terms of the following two-component spinors:

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}. \tag{3.114}$$

Now we can use the Callias theorem, which says

$$\dim\ker\mathcal{D} - \dim\ker\mathcal{D}^\dagger = n_m, \tag{3.115}$$

where n_m is the topological number, $n_m=1$ for the monopole, and $n_m=-1$ for the antimonopole. This implies, in turn, that Eq. (3.113) has one complex zero mode, i.e., in this case we characterize the monopole by one complex fermion collective coordinate (and its conjugate, of course). This fact leads to a drastic consequence: the monopole acquires a half-integer electric charge. It becomes a dyon with charge $1/2$ even in the absence of the θ term. This phenomenon—the charge fractionalization in the cases with a single complex fermion collective coordinate—is well known (Harvey, 1997; Rubakov, 2002; Shifman *et al.*, 2006) and dates back to Jackiw and Rebbi (1976).

IV. NON-ABELIAN STRINGS

Ever since Mandelstam (1976) and 't Hooft (1981) put forward the hypothesis of the dual Meissner effect to explain color confinement in non-Abelian gauge theories, many have tried to find a controllable approximation in which one could reliably demonstrate the occurrence of the dual Meissner effect in these theories. A breakthrough achievement was the Seiberg-Witten solution (Seiberg and Witten, 1994a) of $\mathcal{N}=2$ supersymmetric Yang-Mills theory. They found massless monopoles and, adding a small ($\mathcal{N}=2$)-breaking deformation, proved that they condense creating strings carrying a chromoelectric flux. It was a great success in the qualitative understanding of color confinement.

A more careful examination shows, however, that details of the Seiberg-Witten confinement are quite different from those expected in QCD-like theories. Indeed, a crucial aspect of Seiberg and Witten (1994a) is that the SU(N) gauge symmetry is first broken, at a high scale, down to $U(1)^{N-1}$, which is then completely broken, at a much lower scale where monopoles condense. Correspondingly, the strings in the Seiberg-Witten solution are, in fact, Abelian strings (Abrikosov, 1957; Nielsen and Olesen, 1973) of the Abrikosov-Nielsen-Olesen (ANO) type, which results, in turn, in confinement whose structure does not resemble at all that of QCD. In

[10]This means that the monopole is described by two complex fermion collective coordinates, or four real.

particular, the hadronic spectrum is much richer than that in QCD (Douglas and Shenker, 1995; Hanany *et al.*, 1998; Strassler, 1998; Yung, 2000; Vainshtein and Yung, 2001). To see this note that for the low-energy gauge group $U(1)^{N-1}$, one has $N-1$ Abelian strings associated with each of $N-1$ Abelian factors. Since

$$\pi_1[U(1)^{N-1}] = Z^{N-1}, \quad (4.1)$$

Abelian strings and therefore the meson spectrum come in $N-1$ infinite towers. This feature is not expected in real world QCD. Moreover, there is no experimental indication of dynamical Abelization in QCD.

In this section, we review a discovery of non-Abelian strings (Auzzi *et al.*, 2003; Hanany and Tong, 2003, 2004; Shifman and Yung, 2004b) that appear in certain regimes in $\mathcal{N}=2$ supersymmetric gauge theories. The most important feature of these strings is that they acquire orientational zero modes associated with rotation of their color flux inside the non-Abelian subgroup SU(N) of the gauge group. This makes these strings genuinely non-Abelian.

Actually the flux tubes in non-Abelian theories at weak coupling were studied previously (de Vega and Schaposnik, 1986a, 1986b; Heo and Vachaspati, 1998; Schaposnik and Suranyi, 2000; Suranyi, 2000; Kneipp and Brockill, 2001; Marshakov and Yung, 2002; Konishi and Spanu, 2003). These strings are called Z_N strings because they are related to the center of gauge group SU(N). Consider, say, SU(N) gauge theory with several scalar fields in adjoint representation. Suppose adjoint scalars condense breaking the gauge group down to its center Z_N. Then string solutions are classified according to

$$\pi_1\left(\frac{SU(N)}{Z_N}\right) = Z_N. \quad (4.2)$$

In these previous constructions of Z_N strings (de Vega and Schaposnik, 1986a, 1986b; Heo and Vachaspati, 1998; Schaposnik and Suranyi, 2000; Suranyi, 2000; Kneipp and Brockill, 2001; Marshakov and Yung, 2002; Konishi and Spanu, 2003), the flux was always directed in a fixed group direction (corresponding to a Cartan subalgebra), and no moduli that would freely govern its orientation in the group space were ever obtained. Therefore, we prefer to call these Z_N strings Abelian in contrast to non-Abelian strings, which have orientational moduli.

In this section, we discuss a particular class of $\mathcal{N}=2$ supersymmetric gauge theories in which non-Abelian strings were found. One can address the following question: What is so special about these models that makes an Abelian Z_N string become a non-Abelian? Models we are going to consider have both gauge and flavor symmetries broken by the condensation of scalar fields. The common feature of these models is that some global diagonal combination of color and flavor groups survives the breaking. We consider a case in which this diagonal group is $SU(N)_{C+F}$, where the subscript $C+F$ means a combination of global color and flavor groups. The presence of this unbroken subgroup is responsible for the appearance of orientational zero modes of the string, which ensure its non-Abelian nature.

Clearly the presence of supersymmetry is not important for the construction of non-Abelian strings. In particular, in this section we consider BPS non-Abelian strings in $\mathcal{N}=2$ supersymmetric gauge theories while in the next section we review non-Abelian strings in $\mathcal{N}=1$ supersymmetric and nonsupersymmetric theories.

A. Basic model: $\mathcal{N}=2$ SQCD

The model we deal with derives from $\mathcal{N}=2$ SQCD with the gauge group SU($N+1$) and $N_f=N$ flavors of the fundamental matter hypermultiplets, which we call quarks (Seiberg and Witten, 1994b). At a generic point on the Coulomb branch of this theory, the gauge group is broken down to $U(1)^N$. We will be interested, however, in a particular subspace of the Coulomb branch, on which the gauge group is broken down to $SU(N)\times U(1)$. We enforce this regime by a special choice of the quark mass terms.

The breaking $SU(N+1)\to SU(N)\times U(1)$ occurs at the scale m, which is supposed to lie very high, $m \gg \Lambda_{SU(N+1)}$, where $\Lambda_{SU(N+1)}$ is the scale of SU($N+1$) theory. Correspondingly, the masses of the gauge bosons from $SU(N+1)/SU(N)\times U(1)$ sector and their superpartners are very large, proportional to m, and so are the masses of the ($N+1$)th color component of the quark fields in the fundamental representation. We are interested in phenomena at the scales $\ll m$. Therefore, our starting point is in fact the $SU(N)\times U(1)$ model with $N_f=N$ matter fields in the fundamental representation of SU(N), as it emerges after the $SU(N+1)\to SU(N)\times U(1)$ breaking. These matter fields are also coupled to the U(1) gauge field.

The field content of $SU(N)\times U(1)$ $\mathcal{N}=2$ SQCD with N flavors is as follows. The $\mathcal{N}=2$ vector multiplet consists of the U(1) gauge fields A_μ and SU(N) gauge field A^a_μ (here $a=1,\ldots,N^2-1$), their Weyl fermion superpartners $(\lambda^1_\alpha,\lambda^2_\alpha)$ and $(\lambda^{1a}_\alpha,\lambda^{2a}_\alpha)$, and complex scalar fields a and a^a, the latter in the adjoint of SU(N). The spinorial index of λ's runs over $\alpha=1,2$. In this sector, the global $SU(2)_R$ symmetry inherent to the $\mathcal{N}=2$ model manifests itself through rotations $\lambda^1\leftrightarrow\lambda^2$.

The quark multiplets of $SU(N)\times U(1)$ theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and the Weyl fermions ψ^{kA} and $\tilde{\psi}_{Ak}$, all in the fundamental representation of the SU(N) gauge group. Here $k=1,\ldots,N$ is the color index while A is the flavor index, $A=1,\ldots,N$. Note that the scalars q^{kA} and $\bar{\tilde{q}}^{kA}$ form a doublet under the action of the global $SU(2)_R$ group.

The original SU($N+1$) theory was perturbed by adding a small mass term for the adjoint matter, via the superpotential $\mathcal{W}=\mu\,\mathrm{Tr}\,\Phi^2$. Generally speaking, this superpotential breaks $\mathcal{N}=2$ down to $\mathcal{N}=1$. The Coulomb branch shrinks to a number of isolated $\mathcal{N}=1$ vacua (Seiberg and Witten, 1994a, 1994b; Douglas and Shenker, 1995; Argyres *et al.*, 1996; Carlino *et al.*, 2000). In the limit of $\mu \to 0$, these vacua correspond to special singular points on the Coulomb branch in which N monopoles and/or dyons or quarks become massless. The first ($N+1$) of these points (often referred to as the Seiberg-Witten vacua) are always at strong coupling. They correspond to $\mathcal{N}=1$ vacua of pure SU($N+1$) gauge theory.

The massless quark points—they present vacua of a distinct type, to be referred to as quark vacua—may or may not be at weak coupling, depending on the values of the quark mass parameters m_A. If $m_A \gg \Lambda_{SU(N+1)}$, the quark vacua do lie at weak coupling. Below we are interested only in the quark vacua assuming that the condition $m_A \gg \Lambda_{SU(3)}$ is met.

In the low-energy SU(N)×U(1) theory, which is our starting point, the perturbation $\mathcal{W}=\mu \mathrm{Tr}\,\Phi^2$ can be truncated, leading to a crucial simplification. Indeed, since the $\mathcal{A}$ chiral superfield, the $\mathcal{N}=2$ superpartner of the U(1) gauge field,[11]

$$\mathcal{A} \equiv a + \sqrt{2}\lambda^2\theta + F_a\theta^2, \tag{4.3}$$

is not charged under the gauge group SU(N)×U(1), one can introduce the superpotential linear in $\mathcal{A}$,

$$\mathcal{W}_{\mathcal{A}} = -\frac{N}{2\sqrt{2}}\xi\mathcal{A}. \tag{4.4}$$

Here we expand $\mathrm{Tr}\,\Phi^2$ around its vacuum expectation value (VEV) (see below) and truncate the series keeping only the linear term in $\mathcal{A}$. The truncated superpotential is a Fayet-Iliopoulos (FI) F term.

Now, we explain this in more detail. In $N=1$ supersymmetric theory with a gauge group SU(N)×U(1), one can add the FI term to the action (Fayet and Iliopoulos, 1974) (we call it the FI D term here)

$$\xi_3 D, \tag{4.5}$$

where D is the D component of the U(1) gauge field. In $N=2$ SUSY theory, field D belongs to the SU(2)$_R$ triplet together with F components of the chiral field $\mathcal{A}$, F, and $\bar{F}$. Namely, we introduce the triplet F_p ($p=1,2,3$) using the relations[12]

$$D = F_3,$$

$$F = \frac{1}{\sqrt{2}}(F_1 + iF_2),$$

$$\bar{F} = \frac{1}{\sqrt{2}}(F_1 - iF_2). \tag{4.6}$$

Now the generalized FI term can be written as

$$S_{\mathrm{FI}} = -\frac{N}{2}\int d^4x\,\xi_p F_p. \tag{4.7}$$

Comparing this with Eq. (4.4), we identify

$$\xi = (\xi_1 - i\xi_2),$$

$$\bar{\xi} = (\xi_1 + i\xi_2). \tag{4.8}$$

This is the reason we call superpotential (4.4) the FI F term.

A remarkable feature of the FI term is that it does not break $\mathcal{N}=2$ supersymmetry (Hanany *et al.*, 1998; Vainshtein and Yung, 2001). Keeping higher-order terms of the expansion of $\mu\,\mathrm{Tr}\,\Phi^2$ in powers of $\mathcal{A}$ would inevitably explicitly break $\mathcal{N}=2$. For our purposes, it is crucial that our model is exactly $\mathcal{N}=2$ supersymmetric. This ensures that flux tube solutions of the model are BPS saturated. If higher-order terms in $\mathcal{A}$ are taken into account, $\mathcal{N}=2$ supersymmetry is broken down to $\mathcal{N}=1$ and strings are no longer BPS. The superconductivity in the model becomes of the type I (Vainshtein and Yung, 2001).

1. SU(N)×U(1) $\mathcal{N}=2$ QCD

The bosonic part of our SU(N)×U(1) theory has the form[13] (Auzzi *et al.*, 2003)

$$S = \int d^4x\left[\frac{1}{4g_2^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{g_2^2}|D_\mu a^a|^2 + \frac{1}{g_1^2}|\partial_\mu a|^2 + |\nabla_\mu q^A|^2 + |\nabla_\mu \bar{\tilde{q}}^A|^2 + V(q^A,\tilde{q}_A,a^a,a)\right]. \tag{4.9}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(N), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2}A_\mu - iA^a_\mu T^a, \tag{4.10}$$

where we suppress the color SU(N) indices, and T^a are the SU(N) generators normalized as $\mathrm{Tr}(T^aT^b)=\frac{1}{2}\delta^{ab}$. The coupling constants g_1 and g_2 correspond to the U(1) and SU(2) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$.

The potential $V(q^A,\tilde{q}_A,a^a,a)$ in the action (4.9) is a sum of D and F terms,

[11]The superscript 2 in Eq. (4.3) is the global SU(2)$_R$ index of λ rather than λ squared.

[12]The index p is a SU(2)$_R$ index rather than the color index.

[13]Here and in the remainder of this review we use formally Euclidean notations. This is appropriate since we study static (time-independent) field configurations, and $A_0=0$. Then the Euclidean action is merely the energy functional.

$$V(q^A,\tilde{q}_A,a^a,a) = \frac{g_2^2}{2}\left(\frac{i}{g_2^2}f^{abc}\bar{a}^b a^c + \bar{q}_A T^a q^A - \tilde{q}_A T^a \bar{\tilde{q}}^A\right)^2 + \frac{g_1^2}{8}(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi_3)^2 + 2g^2|\tilde{q}_A T^a q^A|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A - \frac{N}{2}\xi\right|^2 + \frac{1}{2}\sum_{A=1}^{N}\{|(a+\sqrt{2}m_A + 2T^a a^a)q^A|^2 + |(a+\sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A|^2\}, \tag{4.11}$$

where the sum over repeated flavor indices A is implied and f^{abc} are structure constants of the SU(N) group. The first and second lines represent D terms, the third line the F_A terms, while the fourth and fifth lines represent the squark F terms. Using $\mathrm{SU}(2)_R$ rotations, we can always direct FI parameter vector ξ_p in a given direction. Below we consider the case of the FI F term with real ξ, in other words, we use $\mathrm{SU}(2)_R$ rotations to get

$$\xi_3 = 0, \quad \xi_2 = 0, \quad \xi = \xi_1. \tag{4.12}$$

2. Vacuum structure and excitation spectrum

Now we briefly outline the vacuum structure and the excitation mass spectrum of our basic $\mathrm{SU}(N)\times\mathrm{U}(1)$ model. The underlying $\mathcal{N}=2$ SU($N+1$) QCD has a variety of vacua (Argyres *et al.*, 1996; Carlino *et al.*, 2000; Marshakov and Yung, 2000). Besides N strong-coupling vacua that exist in pure gauge theory, there are a number of so-called r quark vacua, where r is the number of quark flavors that develop VEV's in a given vacuum. Here we focus on a particular isolated vacuum with a maximal possible value of r, $r=N$.[14] The theory (4.9) is merely the low-energy truncation of the full SU($N+1$) QCD, which describes physics around this vacuum.

The vacua of the theory (4.9) are determined by the zeros of the potential (4.11). The adjoint fields develop the following VEV's:

$$\langle\Phi\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & m_N \end{pmatrix}, \tag{4.13}$$

where we defined the scalar adjoint matrix as

$$\Phi = \tfrac{1}{2}a + T^a a^a. \tag{4.14}$$

For generic values of quark masses, the SU(N) subgroup of the gauge group is broken down to $\mathrm{U}(1)^{N-1}$. However, for a special choice

$$m_1 = m_2 = \cdots = m_N, \tag{4.15}$$

in which we are mostly interested in this section, the $\mathrm{SU}(N)\times\mathrm{U}(1)$ gauge group remains classically unbroken. In fact, the common value m of quark masses determines the scale of breaking of SU($N+1$) gauge symmetry of the underlying theory down to $\mathrm{SU}(N)\times\mathrm{U}(1)$ gauge symmetry of our low-energy theory (4.9).

If the value of the FI parameter is taken real, we can exploit gauge rotations to make the quark VEV's real too. Then in this case they take the color-flavor locked form

$$\langle q^{kA}\rangle = \langle\bar{\tilde{q}}^{kA}\rangle = \sqrt{\frac{\xi}{2}}\begin{pmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix},$$

$$k = 1,\ldots,N \quad A = 1,\ldots,N, \tag{4.16}$$

where we write the quark fields as an $N\times N$ matrix in color and flavor indices. This particular form of the squark condensates is dictated by the third line in Eq. (4.11). Note that the squark fields stabilize at nonvanishing values entirely due to the U(1) factor—the second term in the third line.

The vacuum field (4.16) results in the spontaneous breaking of both gauge and flavor SU(N)'s. A diagonal global SU(N) survives, however, namely,

$$\mathrm{U}(N)_{\mathrm{gauge}} \times \mathrm{SU}(N)_{\mathrm{flavor}} \to \mathrm{SU}(N)_{C+F}. \tag{4.17}$$

Thus, color-flavor locking takes place in the vacuum. A version of this scheme of symmetry breaking has already been suggested (Bardakci and Halpern, 1972).

We move on to the issue of the excitation spectrum in this vacuum (Vainshtein and Yung, 2001; Auzzi *et al.*, 2003). The mass matrix for the gauge fields (A^a_μ, A_μ) can be read off from the quark kinetic terms in Eq. (4.9). It shows that all SU(N) gauge bosons become massive, with one and the same mass

$$M_{\mathrm{SU}(N)} = g_2\sqrt{\xi}. \tag{4.18}$$

The equality of the masses is no accident. It is a consequence of the unbroken $\mathrm{SU}(N)_{C+F}$ symmetry (4.17).

The mass of the U(1) gauge boson is

$$M_{\mathrm{U}(1)} = g_1\sqrt{\frac{N}{2}\xi}. \tag{4.19}$$

The mass spectrum of the adjoint scalar excitations is the same as for the gauge bosons. This is enforced by $\mathcal{N}=2$.

What is the mass spectrum of the quark excitations? It can be read off the potential (4.11). From $4N^2$ real degrees of freedom of quark scalars q and $\tilde{q}$, N^2 are eaten up by the Higgs mechanism making gauge bosons massive. The remaining $3N^2$ states split in three and $3(N^2-1)$ states with masses (4.19) and (4.18), respectively. Combining these states with massive gauge boson states and adjoint scalar states, we get one long $\mathcal{N}=2$ BPS multiplet (eight real bosonic plus eight fermionic degrees of

[14] There are singular points on the Coulomb branch of the underlying SU($N+1$) theory where more than N quark flavors become massless. These singularities are roots of Higgs branches (Argyres *et al.*, 1996; Carlino *et al.*, 2000; Marshakov and Yung, 2000).

freedom) with mass (4.19) and N^2-1 long $\mathcal{N}=2$ BPS multiplets with mass (4.18) (Vainshtein and Yung, 2001; Auzzi *et al.*, 2003). Note that these supermultiplets come in representations of the unbroken $\mathrm{SU}(N)_{C+F}$ group, namely, scalar and adjoint ones.

To conclude this section, we discuss quantum effects in the theory (4.9). At high scale m, the $\mathrm{SU}(N+1)$ gauge group is broken down to $\mathrm{SU}(N)\times\mathrm{U}(1)$ by condensation of adjoint fields provided the condition (4.15) is met. The $\mathrm{SU}(N)$ sector is asymptotically free, and if uninterrupted it would run into strong coupling. This would invalidate our quasiclassical analysis. Moreover, strong-coupling effects on the Coulomb branch would break the $\mathrm{SU}(N)$ gauge subgroup [and the $\mathrm{SU}(N)_{C+F}$ group] down to $\mathrm{U}(1)^{N-1}$ by the Seiberg-Witten mechanism (Seiberg and Witten, 1994a, 1994b) and no non-Abelian strings would emerge.

One way out has been proposed by Argyres *et al.* (1996) and Carlino *et al.* (2000). One can add more flavors to the theory making $N_f > 2N$. Then the $\mathrm{SU}(N)$ sector is not asymptotically free and does not run into strong coupling. However, ANO strings in a theory with many flavors (on the Higgs branches) become semilocal strings (Achucarro and Vachaspati, 2000) (see also Sec. IV.G) and confinement is lost. We consider a different route here taking the FI parameter ξ large,

$$\xi \gg \Lambda_{\mathrm{SU}(N)}. \tag{4.20}$$

This condition ensures weak coupling in the $\mathrm{SU}(N)$ sector because $\mathrm{SU}(N)$ gauge coupling does not run below the scale of quark VEV's, which is determined by ξ. Explicitly,

$$\frac{8\pi^2}{g_2^2}(\xi) = N \ln \frac{\sqrt{\xi}}{\Lambda_{\mathrm{SU}(N)}} \gg 1. \tag{4.21}$$

Alternatively, one has

$$\Lambda_{\mathrm{SU}(N)}^N = \xi^{N/2} \exp\left(-\frac{8\pi^2}{g_2^2(\xi)}\right). \tag{4.22}$$

B. Z_N Abelian strings

Strictly speaking, $\mathcal{N}=2$ QCD with the $\mathrm{SU}(N+1)$ gauge group does not have stable flux tubes. They are unstable due to monopole-antimonopole pair creation in the $\mathrm{SU}(N+1)/\mathrm{SU}(N)\times\mathrm{U}(1)$ sector. However, at large m these monopoles become heavy. In fact, there are no monopoles in the low-energy theory (4.9) (they can be considered as infinitely heavy); therefore, the theory (4.9) has stable string solutions. Moreover, when the perturbation $\mu\,\mathrm{Tr}\,\Phi^2$ is truncated to the FI term (4.4), the theory enjoys $\mathcal{N}=2$ supersymmetry and has BPS string solutions (Fuertes and Guilarte, 1998; Hanany *et al.*, 1998; Edelstein *et al.*, 2000; Vainshtein and Yung, 2001; Marshakov and Yung, 2002; Auzzi *et al.*, 2003). Note that here we discuss magnetic flux tubes. They are formed in the Higgs phase of the theory upon condensation of quarks and lead to the confinement of monopoles.

Now, we briefly review BPS string solutions in the model (4.9). (Marshakov and Yung, 2002; Auzzi *et al.*, 2003; Hanany and Tong 2003). Here we consider the case of equal quark masses (4.15) when the global $\mathrm{SU}(N)_{C+F}$ group is unbroken. First we review Abelian solutions for Z_N strings, and in the next subsection we show that in this limit they acquire orientational moduli. In fact, the Z_N Abelian strings considered are partial solutions of vortex equations [see Eq. (4.32)], which are exchanged by the action of the discrete Z_N subgroup. In the equal quark masses limit (4.15), the global $\mathrm{SU}(N)_{C+F}$ group is restored and a general solution for an elementary non-Abelian string has a continuous moduli space isomorphic to $CP(N-1)$, with Z_N strings having N discrete points on it.

In the case of generic unequal quark masses, the $\mathrm{SU}(N)_{C+F}$ group is broken and the continuous moduli space of string solutions is lifted. Only Z_N Abelian strings survive this breaking. The case of generic quark masses is considered in Sec. IV.D.4.

It turns out that string solutions do not involve adjoint fields a and a^a; strings are built from gauge and quark fields only. Therefore, in order to find the classical solution, we can put adjoint fields equal to their VEV's (4.13) in the action (4.9).[15] Moreover, we use the ansatz

$$q^{kA} = \bar{\tilde{q}}^{kA} = \frac{1}{\sqrt{2}}\varphi^{kA} \tag{4.23}$$

reducing the number of squark degrees of freedom to one complex field for each color and flavor. With these simplifications, the action of model (4.9) reads

$$S = \int d^4x \left\{ \frac{1}{4g_2^2}(F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + |\nabla_\mu \varphi^A|^2 + \frac{g_2^2}{2}(\bar{\varphi}_A T^a \varphi^A)^2 + \frac{g_1^2}{8}(|\varphi^A|^2 - N\xi)^2 \right\}, \tag{4.24}$$

while the VEV's of quark fields (4.16) become

$$\langle\varphi\rangle = \sqrt{\xi}\,\mathrm{diag}\{1,1,\ldots,1\}. \tag{4.25}$$

Since it includes a spontaneously broken gauge U(1), the model supports conventional ANO strings (Abrikosov, 1957; Nielsen and Olesen, 1973) in which one can discard the $\mathrm{SU}(N)_{\mathrm{gauge}}$ part of the action. The topological stability of the ANO string is due to the fact that $\pi_1[\mathrm{U}(1)] = Z$. These are not the strings in which we are interested. At first glance, the triviality of the homotopy group, $\pi_1[\mathrm{SU}(N)] = 0$, implies that there are no other topologically stable strings. This impression is false. One can combine the Z_N center of $\mathrm{SU}(N)$ with the elements $\exp(2\pi i k/N) \in \mathrm{U}(1)$ to get a topologically stable string solution possessing both windings, in $\mathrm{SU}(N)$ and U(1). In other words,

[15] This is consistent with equations of motion. Of course this procedure is not correct on the quantum level.

$$\pi_1[\mathrm{SU}(N)\times \mathrm{U}(1)/Z_N]\neq 0. \tag{4.26}$$

It is easy to see that this nontrivial topology amounts to winding one element of φ, say φ^{11} or φ^{22}, for instance,[16]

$$\varphi_{\text{string}}=\sqrt{\xi}\,\mathrm{diag}(1,1,\ldots,e^{i\alpha(x)}),\quad x\to\infty. \tag{4.27}$$

Such strings can be called elementary; their tension is $(1/N)$th that of the ANO string. The ANO string can be viewed as a bound state of N elementary strings.

More concretely, the Z_N string solution (a progenitor of the non-Abelian string) can be written as follows (Auzzi *et al.*, 2003):

$$\varphi=\begin{pmatrix}\phi_2(r) & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \phi_2(r) & 0\\ 0 & 0 & \cdots & e^{i\alpha}\phi_1(r)\end{pmatrix},$$

$$A_i^{\mathrm{SU}(N)}=\frac{1}{N}\begin{pmatrix}1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & -(N-1)\end{pmatrix}\times(\partial_i\alpha)[-1+f_{NA}(r)],$$

$$A_i^{\mathrm{U}(1)}=\frac{I}{2}A_i=\frac{I}{N}(\partial_i\alpha)[1-f(r)],$$

$$A_0^{\mathrm{U}(1)}=A_0^{\mathrm{SU}(N)}=0, \tag{4.28}$$

where $i=1,2$ labels coordinates in the plane orthogonal to the string axis, r and α are the polar coordinates in this plane, and I is the unit $N\times N$ matrix. The profile functions $\phi_1(r)$ and $\phi_2(r)$ determine the profiles of the scalar fields, while $f_{NA}(r)$ and $f(r)$ determine the SU(N) and U(1) fields of the string solutions, respectively. These functions satisfy the following boundary conditions:

$$\phi_1(0)=0,$$

$$f_{NA}(0)=1, f(0)=1 \tag{4.29}$$

at $r=0$, and

$$\phi_1(\infty)=\sqrt{\xi},\quad \phi_2(\infty)=\sqrt{\xi},$$

$$f_{NA}(\infty)=0,\quad f(\infty)=0 \tag{4.30}$$

at $r=\infty$.

Now we derive first-order equations that determine the profile functions making use of the Bogomol'nyi representation (Bogomol'nyi, 1976) for model (4.24). We have

[16] As explained, α is the angle of the coordinate $\vec{x}_\perp$ in the perpendicular plane.

$$T=\int d^2x\left\{\left[\frac{1}{\sqrt{2}g_2}F_3^{*a}+\frac{g_2}{\sqrt{2}}(\bar{\varphi}_A T^a\varphi^A)\right]^2+\left[\frac{1}{\sqrt{2}g_1}F_3^{*}+\frac{g_1}{2\sqrt{2}}(|\varphi^A|^2-N\xi)\right]^2+|\nabla_1\varphi^A+i\nabla_2\varphi^A|^2+\frac{N}{2}\xi F_3^*\right\}, \tag{4.31}$$

where $F_3^*=\frac{1}{2}F_{12}$ and $F_3^{*a}=\frac{1}{2}F_{12}^a$ and we assume that fields depend only on coordinates x_i, $i=1,2$.

The Bogomol'nyi representation (4.31) leads to the following first-order equations:

$$F_3^*+\frac{g_1^2}{2}(|\varphi^A|^2-N\xi)=0,$$

$$F_3^{*a}+g_2^2(\bar{\varphi}_A T^a\varphi^A)=0,$$

$$(\nabla_1+i\nabla_2)\varphi^A=0. \tag{4.32}$$

Once these equations are satisfied, the energy of the BPS object is given by the last surface term in Eq. (4.31). Note that representation (4.31) can also be written with a different sign in front of terms proportional to gauge fluxes. This would give first-order equations for the antistring with negative values of gauge fluxes.

For the elementary string, we substitute ansatz (4.28) into Eqs. (4.32) to get first-order equations for profile functions of Z_N string. We have (Marshakov and Yung, 2002; Auzzi *et al.*, 2003)

$$r\frac{d}{dr}\phi_1(r)-\frac{1}{N}[f(r)+(N-1)f_{NA}(r)]\phi_1(r)=0,$$

$$r\frac{d}{dr}\phi_2(r)-\frac{1}{N}[f(r)-f_{NA}(r)]\phi_2(r)=0,$$

$$-\frac{1}{r}\frac{d}{dr}f(r)+\frac{g_1^2N}{4}[(N-1)\phi_2(r)^2+\phi_1(r)^2-N\xi]=0,$$

$$-\frac{1}{r}\frac{d}{dr}f_{NA}(r)+\frac{g_2^2}{2}[\phi_1(r)^2-\phi_2(r)^2]=0. \tag{4.33}$$

These equations are a generalization of Bogomol'nyi equations for ANO string (Bogomol'nyi, 1976) see also Eq. (3.37) to the case of the Z_N string. They were solved numerically for the U(2) case ($N=2$) in Auzzi *et al.* (2003). Clearly, the solutions to the first-order equations automatically satisfy the second-order equations of motion.

The tension of this elementary string is

$$T_1=2\pi\xi. \tag{4.34}$$

Since our string is a BPS object, this result is exact and has neither perturbative nor nonperturbative corrections. Note that the tension of the ANO string is N times larger,

$$T_{\rm ANO} = 2\pi N \xi, \tag{4.35}$$

in our normalization.

We have many string solutions of type (4.28). They can be obtained by changing the position of the quark field, which winds at infinity in Eq. (4.28). Altogether we have N elementary Z_N strings.

Of course first-order equations (4.33) can also be obtained using supersymmetry. We start from the supersymmetry transformations for the fermion fields in the theory (4.9),

$$\delta\lambda^{f\alpha} = \tfrac{1}{2}(\sigma_\mu \bar\sigma_\nu \epsilon^f)^\alpha F_{\mu\nu} + \epsilon^{\alpha p} D^m (\tau^m)^f_p + \cdots,$$

$$\delta\lambda^{af\alpha} = \tfrac{1}{2}(\sigma_\mu \bar\sigma_\nu \epsilon^f)^\alpha F^a_{\mu\nu} + \epsilon^{\alpha p} D^{am} (\tau^m)^f_p + \cdots,$$

$$\delta\bar{\tilde\psi}^{kA}_{\dot\alpha} = i\sqrt{2}\,\bar{\nabla\!\!\!/}_{\dot\alpha\alpha} q^{kA}_f \epsilon^{\alpha f} + \cdots,$$

$$\delta\bar\psi_{\dot\alpha Ak} = i\sqrt{2}\,\bar{\nabla\!\!\!/}_{\dot\alpha\alpha} \bar{q}_{fAk} \epsilon^{\alpha f} + \cdots. \tag{4.36}$$

Here $f=1,2$ is the $\mathrm{SU}(2)_R$ index so $\lambda^{f\alpha}$ and $\lambda^{af\alpha}$ are fermions from the $\mathcal{N}=2$ vector supermultiplets of the U(1) and SU(2) factors, respectively, while q^{kAf} denotes the $\mathrm{SU}(2)_R$ doublet of squark fields q^{kA} and $\bar{\tilde q}^{Ak}$ in the quark hypermultiplets. The parameters of supersymmetry (SUSY), transformations in the microscopic theory are denoted as $\epsilon^{\alpha f}$. Furthermore, the D terms in Eq. (4.36) are

$$D^1 + iD^2 = i\frac{g_1^2}{2}(\mathrm{Tr}|\varphi|^2 - N\xi), \quad D^3 = 0 \tag{4.37}$$

for the U(1) field, and

$$D^{a1} + iD^{a2} = ig_2^2 \mathrm{Tr}(\bar\varphi T^a \varphi), \quad D^{a3} = 0 \tag{4.38}$$

for the SU(N) field. The ellipses in Eqs. (4.36) represent terms involving the adjoint scalar fields that vanish on the string solution (for equal quark masses) because the adjoint fields are given by their vacuum expectation values (4.13).

Vainshtein and Yung (2001) have shown that the four supercharges selected by the conditions

$$\epsilon^{12} = -\epsilon^{11}, \quad \epsilon^{21} = \epsilon^{22} \tag{4.39}$$

act trivially on the BPS string. Namely, imposing conditions (4.39) and requiring that the left-hand sides of Eqs. (4.36) are zero, we get first-order equations (4.33) upon substitution of ansatz (4.28).[17]

C. Elementary non-Abelian strings

The elementary Z_N strings in the model (4.9) give rise to *bona fide* non-Abelian provided the condition (4.15) is satisfied (Auzzi *et al.*, 2003; Hanany and Tong, 2003; Hanany and Tong, 2004; Shifman and Yung, 2004a). This means that, besides trivial translational moduli, they have moduli corresponding to spontaneous breaking of a non-Abelian symmetry. Indeed, while the flat vacuum (4.16) is $\mathrm{SU}(N)_{C+F}$ symmetric, the solution (4.28) breaks this symmetry down[18] to $\mathrm{U}(1)\times\mathrm{SU}(N-1)$ (at $N>2$). This ensures the presence of $2(N-1)$ orientational moduli.

To obtain the non-Abelian string solution from the Z_N string (4.28), we apply the diagonal color-flavor rotation preserving the vacuum (4.16). To this end, it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity, while the string flux comes from the vicinity of the origin. In this gauge, we have

$$\varphi = U \begin{pmatrix} \phi_2(r) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \phi_2(r) & 0 \\ 0 & 0 & \cdots & \phi_1(r) \end{pmatrix} U^{-1},$$

$$A_i^{\mathrm{SU}(N)} = \frac{1}{N} U \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -(N-1) \end{pmatrix} U^{-1} \times (\partial_i \alpha) f_{NA}(r),$$

$$A_i^{\mathrm{U}(1)} = -\frac{1}{N}(\partial_i\alpha) f(r),$$

$$A_0^{\mathrm{U}(1)} = A_0^{\mathrm{SU}(N)} = 0, \tag{4.40}$$

where U is a matrix $\in \mathrm{SU}(N)_{C+F}$. This matrix parametrizes orientational zero modes of the string associated with flux rotation in SU(N). The presence of these modes makes the string genuinely non-Abelian. Since the diagonal color-flavor symmetry is not broken by the VEV's of the scalar fields in the bulk (color-flavor locking), it is physical and has nothing to do with the gauge rotations eaten by the Higgs mechanism. The orientational moduli encoded in the matrix U are not gauge artifacts.

The orientational zero modes of a non-Abelian string were first observed by Auzzi *et al.* (2003) and Hanany and Tong (2003). Hanany and Tong (2003) provided a general index theorem that shows that the dimension of elementary string moduli space is $2N=2(N-1)+2$, where 2 stands for translational moduli while $2(N-1)$ is the dimension of the internal moduli space.[19] In Auzzi *et al.* (2003), an explicit solution for non-Abelian string, which we reviewed here, was worked out.

In fact, nontranslational zero modes of strings were discussed earlier in a $\mathrm{U}(1)\times\mathrm{U}(1)$ model (Witten, 1985;

[17]If instead of Eq. (4.39) we impose different combinations of SUSY transformation parameters to vanish [change signs in Eq. (4.39)], we get the equations for antistring with opposite directions of gauge fluxes.

[18]At $N=2$, the string solution breaks SU(2) down to U(1).

[19]The index theorem by Hanany and Tong (2003) deals with more general multiple strings. It was shown that the dimension of the moduli space of k-string is $2kN$.

Hindmarsh, 1989), and later, in more contrived models, by Alford *et al.* (1991). It is worth emphasizing that, along with some apparent similarities, there are drastic distinctions between the non-Abelian strings reviewed here and the strings discussed in the 1980s. In particular, in the example treated by Alford *et al.* (1991), the gauge group is not completely broken in the vacuum, and, therefore, there are massless gauge fields in the bulk. If the unbroken generator acts nontrivially on the string flux (which is proportional to a broken generator), then it can and does create zero modes. Some divergence problems ensue.

In the case considered here, the gauge group is completely broken (up to a discrete subgroup Z_N). The theory in the bulk is fully Higgsed. The unbroken group $SU(N)_{C+F}$, a combination of the gauge and flavor groups, is global. There are no massless fields in the bulk. It is possible to model the example considered by Alford *et al.* (1991) if we gauge the unbroken global symmetry $SU(N)_{C+F}$ of model (4.9) with respect to yet another gauge field B_μ.

We also note that the generalization of the solutions for non-Abelian strings for six-dimensional gauge theory with eight supercharges has been done by Eto *et al.* (2004) while non-Abelian strings in strongly coupled vacua have been considered by Bolognesi (2005).

D. Worldsheet effective theory

The non-Abelian string solution (4.40) is characterized by two translational moduli [the position of the string in (1,2) plane] and $2(N-1)$ orientational moduli. Below we review the effective two-dimensional low-energy theory on the string worldsheet. As usual, the translational moduli decouple and we focus on the internal dynamics of the orientational moduli. Our string is a 1/2-BPS state in $\mathcal{N}=2$ supersymmetric gauge theory with eight supercharges. Thus it has four supercharges acting in the worldsheet theory. This means that we have extended $\mathcal{N}=2$ supersymmetric effective theory on the string worldsheet. This theory turns out to be a two-dimensional $CP(N-1)$ model (Auzzi *et al.*, 2003; Hanany and Tong, 2003, 2004; Shifman and Yung, 2004a). In this section, we first present a derivation of this theory and then discuss underlying physics.

1. Derivation of the $CP(N-1)$ model

Now following Auzzi *et al.* (2003), Shifman and Yung (2004a), and Gorsky *et al.* (2005), we derive the effective low-energy theory for the moduli residing in the matrix U. As is clear from the string solution (4.40), not each element of the matrix U will give rise to a modulus. The $SU(N-1)\times U(1)$ subgroup remains unbroken by the string solution under consideration; therefore, the moduli space is

$$\frac{SU(N)}{SU(N-1)\times U(1)} \sim CP(N-1). \tag{4.41}$$

Keeping this in mind, we parametrize the matrices entering Eq. (4.40) as follows:

$$\frac{1}{N}\left\{U\begin{pmatrix}1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & -(N-1)\end{pmatrix}U^{-1}\right\}_p^l = -n^l n_p^* + \frac{1}{N}\delta_p^l, \tag{4.42}$$

where n^l is a complex vector in the fundamental representation of $SU(N)$ and

$$n_l^* n^l = 1 \tag{4.43}$$

($l,p=1,\ldots,N$ are color indices). As we show below, one $U(1)$ phase will be gauged in the effective sigma model. This gives the correct number of degrees of freedom, namely $2(N-1)$.

With this parametrization, the string solution (4.40) can be rewritten as

$$\varphi = \frac{1}{N}[(N-1)\phi_2 + \phi_1] + (\phi_1 - \phi_2)\left(n\cdot n^* - \frac{1}{N}\right),$$

$$A_i^{SU(N)} = \left(n\cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\frac{x_i}{r^2}f_{NA}(r),$$

$$A_i^{U(1)} = \frac{1}{N}\varepsilon_{ij}\frac{x_i}{r^2}f(r), \tag{4.44}$$

where for brevity we suppress all $SU(N)$ indices. The notation is self-evident.

Assume that the orientational moduli are slowly varying functions of the string worldsheet coordinates x_k, $k=0,3$. Then the moduli n^l become fields of a (1+1)-dimensional sigma model on the worldsheet. Since n^l parametrize the string zero modes, there is no potential term in this sigma model.

To obtain the kinetic term, we substitute our solution (4.44), which depends on the moduli n^l, in the action (4.24), assuming that the fields acquire a dependence on the coordinates x_k via $n^l(x_k)$. In doing so, we observe that we have to modify the solution, including in it the $k=0,3$ components of the gauge potential that are no longer vanishing. In the $CP(1)$ case, as was shown by Shifman and Yung (2004a), the potential A_k must be orthogonal [in the $SU(N)$ space] to the matrix (4.42) as well as to its derivatives with respect to x_k. Generalization of these conditions to the $CP(N-1)$ case leads to the following ansatz:

$$A_k^{SU(N)} = -i[\partial_k n\cdot n^* - n\cdot\partial_k n^* - 2n\cdot n^*(n^*\partial_\alpha n)] \times \rho(r), \quad \alpha=0,3, \tag{4.45}$$

where we assume the contraction of the color indices inside the parentheses,

$$(n^* \partial_k n) \equiv n_l^* \partial_k n^l,$$

and introduce a new profile function $\rho(r)$.

The function $\rho(r)$ in Eq. (4.45) is determined through a minimization procedure (Auzzi *et al.*, 2003; Shifman and Yung, 2004a; Gorsky *et al.*, 2005), which generates ρ's own equation of motion. Now we review its derivation. But first we note that $\rho(r)$ vanishes at infinity,

$$\rho(\infty) = 0. \tag{4.46}$$

The boundary condition at $r=0$ will be determined shortly.

The kinetic term for n^l comes from the gauge and quark kinetic terms in Eq. (4.24). Using Eqs. (4.44) and (4.45) to calculate the SU(N) gauge field strength, we find

$$F_{ki}^{\mathrm{SU}(N)} = (\partial_k n \cdot n^* + n \cdot \partial_k n^*)\varepsilon_{ij}\frac{x_j}{r^2}f_{NA}[1-\rho(r)] + i[\partial_k n \cdot n^* - n \cdot \partial_k n^* - 2n \cdot n^*(n^* \partial_k n)]\frac{x_i}{r}\frac{d\rho(r)}{dr}. \tag{4.47}$$

In order to have a finite contribution from the term $\mathrm{Tr}\, F_{ki}^2$ in the action, we have to impose the constraint

$$\rho(0) = 1. \tag{4.48}$$

Substituting the field strength (4.47) into the action (4.24) and including, in addition, the kinetic term of the quarks, after tedious algebra we arrive at

$$S^{(1+1)} = 2\beta \int dt dz \{(\partial_k n^* \partial_k n) + (n^* \partial_k n)^2\}, \tag{4.49}$$

where the coupling constant β is given by

$$\beta = \frac{2\pi}{g_2^2} I, \tag{4.50}$$

and I is a basic normalizing integral,

$$I = \int_0^\infty r dr \left\{ \left(\frac{d}{dr}\rho(r)\right)^2 + \frac{1}{r^2} f_{NA}^2 (1-\rho)^2 + g_2^2 \left[\frac{\rho^2}{2}(\phi_1^2 + \phi_2^2) + (1-\rho)(\phi_2 - \phi_1)^2 \right] \right\}. \tag{4.51}$$

The theory in Eq. (4.49) is in fact the two-dimensional $CP(N-1)$ model. To see that this is indeed the case, we can eliminate the second term in Eq. (4.49) by virtue of introduction of a nonpropagating U(1) gauge field. We review this in Sec. IV.D.3, and then discuss the underlying physics of the model. Thus, we obtain the $CP(N-1)$ model as an effective low-energy theory on the worldsheet of the non-Abelian string. Its coupling β is related to the four-dimensional coupling g_2^2 via the basic normalizing integral (4.51). This integral can be viewed as an action for the profile function ρ.

Varying Eq. (4.51) with respect to ρ, one obtains the second-order equation that the function ρ must satisfy, namely,

$$-\frac{d^2}{dr^2}\rho - \frac{1}{r}\frac{d}{dr}\rho - \frac{1}{r^2}f_{NA}^2(1-\rho) + \frac{g_2^2}{2}(\phi_1^2 + \phi_2^2)\rho - \frac{g_2^2}{2}(\phi_1 - \phi_2)^2 = 0. \tag{4.52}$$

After some algebra and extensive use of the first-order equations (4.33), one can show that the solution of Eq. (4.52) is given by

$$\rho = 1 - \frac{\phi_1}{\phi_2}. \tag{4.53}$$

This solution satisfies the boundary conditions (4.46) and (4.48).

Substituting this solution back into the expression for the normalizing integral (4.51), one can check that this integral reduces to a total derivative and is given by the flux of the string determined by $f_{NA}(0)=1$. Therefore, we arrive at

$$I = 1. \tag{4.54}$$

This result can be traced back to the fact that our theory (4.24) is $\mathcal{N}=2$ supersymmetric theory, and the string is BPS saturated. We will see in Sec. IV.E that this fact is very important for the interpretation of confined monopoles as sigma model kinks. Generally speaking, for non-BPS strings I could be a certain function of N [see Markov *et al.* (2005) for a particular example].

From Eq. (4.51), we get

$$\beta = \frac{2\pi}{g_2^2}. \tag{4.55}$$

The two-dimensional coupling is determined by four-dimensional non-Abelian coupling. This relation is obtained at the classical level. In quantum theory, both couplings run. So we have to specify a scale at which the relation (4.55) takes place. The two-dimensional $CP(N-1)$ model (4.49) is an effective low-energy theory good for the description of internal string dynamics at small energies, much less than the inverse thickness of the string, which is given by masses of the gauge/quark multiplets (4.18) and (4.19) in our bulk SU(N)×U(1) theory. Thus, $g\sqrt{\xi}$ plays the role of a physical ultraviolet (uv) cutoff in Eq. (4.49). This is the scale at which Eq. (4.55) holds. Below this scale, the coupling β runs according to its two-dimensional renormalization-group flow; see Sec. IV.D.3.

Thus the model (4.49) describes the low-energy limit. In principle, the zero-mode interaction has higher derivative corrections that run in powers of

$$(g_2\sqrt{\xi})^{-1}\partial_\alpha, \tag{4.56}$$

where $g_2\sqrt{\xi}$ gives the order of magnitude of masses in the bulk theory. The sigma model (4.49) is adequate at scales below $g_2\sqrt{\xi}$, where higher-derivative corrections are negligibly small.

To conclude this subsection, we present the model (4.49) for $N=2$. In this case, the CP^1 model is equivalent to the O(3) sigma model and the action (4.49) can be rewritten as

$$S^{(1+1)} = \frac{\beta}{2}\int dtdz(\partial_k S^a)^2, \tag{4.57}$$

where S^a is a real unit vector on a sphere S_2, $(S^a)^2=1$, $a=1,2,3$, defined as

$$S^a = -n^* \tau^a n. \tag{4.58}$$

The model (4.57) as an effective theory on the worldsheet of the non-Abelian string in SU(2)×U(1) $\mathcal{N}=2$ QCD was first derived by Auzzi *et al.* (2003) in the field theory framework. This derivation was generalized for arbitrary N in Gorsky *et al.* (2005), while the brane construction of Eq. (4.49) was presented by Hanany and Tong (2003).

2. Fermion zero modes

In the last subsection, we derived the bosonic part of the effective $\mathcal{N}=2$ supersymmetric $CP(N-1)$ model. Now we find fermion zero modes of a non-Abelian string and show that the internal worldsheet dynamics is given by $\mathcal{N}=2$ supersymmetric $CP(N-1)$ model as expected. This program was fulfilled for the $N=2$ case in Shifman and Yung (2004a). Here we review this construction.

The string solution (4.44) for SU(2)×U(1) theory reduces to

$$\varphi = U\begin{pmatrix}\phi_2(r) & 0\\ 0 & \phi_1(r)\end{pmatrix}U^{-1},$$

$$A_i^a(x) = -S^a\varepsilon_{ij}\frac{x_j}{r^2}f_{NA}(r),$$

$$A_i(x) = \varepsilon_{ij}\frac{x_j}{r^2}f(r), \tag{4.59}$$

while the parametrization (4.42) reduces to

$$S^a\tau^a = U\tau^3U^{-1}, \quad a=1,2,3, \tag{4.60}$$

with the help of Eq. (4.58).

Our string solution is 1/2 BPS saturated. This means that four supercharges, out of eight of the four-dimensional theory (Sec. IV.A), act trivially on the string solution (4.59). The remaining four supercharges generate four fermion zero modes, which we call supertranslational modes because they are superpartners to two translational zero modes. The corresponding four fermionic moduli are superpartners to the coordinates x_0 and y_0 of the string center. The supertranslational fermion zero modes were found by Vainshtein and Yung (2001) for the U(1) ANO string in $\mathcal{N}=2$ theory, but the transition to the model considered here is absolutely straightforward.

We focus on four additional fermion zero modes that arise only for the non-Abelian string. They are superpartners of the bosonic orientational moduli S^a; therefore, we refer to these modes as superorientational. Now we work out these four zero modes explicitly and study the impact of their presence in the $CP(1)$ model on the string worldsheet.

The fermionic part of the action of the model (4.9) for the case $N=2$ reads

$$\begin{aligned}S_{\text{ferm}} = \int d^4x\Big\{ &\frac{i}{g_2^2}\bar{\lambda}_f^a\bar{\not{D}}\lambda^{af} + \frac{i}{g_1^2}\bar{\lambda}_f\bar{\not{\partial}}\lambda^f + \mathrm{Tr}[\bar{\psi}i\bar{\not{\nabla}}\psi]\\ &+ \mathrm{Tr}[\tilde{\psi}i\not{\nabla}\bar{\tilde{\psi}}] + \frac{1}{\sqrt{2}}\varepsilon^{abc}\bar{a}^a(\lambda_f^b\lambda^{cf})\\ &+ \frac{1}{\sqrt{2}}\varepsilon^{abc}(\bar{\lambda}^{bf}\bar{\lambda}_f^c)a^c + \frac{i}{\sqrt{2}}\mathrm{Tr}[\bar{q}_f(\lambda^f\psi) + (\tilde{\psi}\lambda_f)q^f\\ &+ (\bar{\psi}\bar{\lambda}_f)q^f + \bar{q}^f(\bar{\lambda}_f\bar{\tilde{\psi}})] + \frac{i}{\sqrt{2}}\mathrm{Tr}[\bar{q}_f\tau^a(\lambda^{af}\psi)\\ &+ (\tilde{\psi}\lambda_f^a)\tau^a q^f + (\bar{\psi}\bar{\lambda}_f^a)\tau^a q^f + \bar{q}^f\tau^a(\bar{\lambda}_f^a\bar{\tilde{\psi}})]\\ &+ \frac{i}{\sqrt{2}}\mathrm{Tr}[\tilde{\psi}(a+a^a\tau^a)\psi] + \frac{i}{\sqrt{2}}\mathrm{Tr}[\bar{\psi}(a+a^a\tau^a)\bar{\tilde{\psi}}]\Big\},\end{aligned} \tag{4.61}$$

where matrix color-flavor notations are used for matter fermions $(\psi^\alpha)^{kA}$ and $(\tilde{\psi}^\alpha)_{Ak}$ and traces are performed over color-flavor indices. Contraction of spinor indices is assumed inside parentheses, say $(\lambda\psi)\equiv\lambda_\alpha\psi^\alpha$.

As mentioned in Sec. IV.B, the four supercharges selected by the conditions (4.39) act trivially on the BPS string in the theory with the FI F term. To generate the superorientational fermion zero modes, the following method was used by Shifman and Yung (2004a). Assume that the orientational moduli S^a in the string solution (4.59) have a slow dependence on the worldsheet coordinates x_0 and x_3 (or t and z). Then the four (real) supercharges selected by the conditions (4.39) no longer act trivially. Instead, their action now gives fermion fields proportional to x_0 and x_3 derivatives of S^a. This is exactly what one expects from the residual $\mathcal{N}=2$ supersymmetry in the worldsheet theory. The above four supercharges generate the worldsheet supersymmetry in the $\mathcal{N}=2$ two-dimensional CP^1 model.

$$\delta\chi_1^a = i\sqrt{2}[(\partial_0 + i\partial_3)n^a\varepsilon_2 + \varepsilon^{abc}n^b(\partial_0 + i\partial_3)n^c\eta_2],$$

$$\delta\chi_2^a = i\sqrt{2}[(\partial_0 - i\partial_3)n^a\varepsilon_1 + \varepsilon^{abc}n^b(\partial_0 - i\partial_3)n^c\eta_1], \tag{4.62}$$

where χ_α^a ($\alpha=1,2$ is the spinor index) are real two-dimensional fermions of the $CP(1)$ model. They are superpartners of S^a and subject to the orthogonality condition $S^a\chi_\alpha^a=0$. Real parameters of the $\mathcal{N}=2$ two-dimensional SUSY transformation ε_α and η_α are identified with parameters of the four-dimensional SUSY transformations [with the constraint Eq. (4.39)] as

$$\varepsilon_1 - i\eta_1 = \frac{1}{\sqrt{2}}(\epsilon^{21} + \epsilon^{22}) = \sqrt{2}\epsilon^{22},$$

$$\varepsilon_2 + i\eta_2 = \frac{1}{\sqrt{2}}(\epsilon^{11} - \epsilon^{12}) = \sqrt{2}\epsilon^{11}. \tag{4.63}$$

The worldsheet supersymmetry was used to re-express the fermion fields obtained upon the action of these four supercharges in terms of the (1+1)-dimensional fermions. This procedure give us the superorientational fermion zero modes (Shifman and Yung, 2004a),

$$\bar{\psi}_{Ak\dot{2}} = \left(\frac{\tau^a}{2}\right)_{Ak} \frac{1}{2\phi_2}(\phi_1^2 - \phi_2^2)[\chi_2^a + i\varepsilon^{abc}n^b\chi_2^c],$$

$$\bar{\tilde{\psi}}_{\dot{1}}^{kA} = \left(\frac{\tau^a}{2}\right)^{kA} \frac{1}{2\phi_2}(\phi_1^2 - \phi_2^2)[\chi_1^a - i\varepsilon^{abc}n^b\chi_1^c],$$

$$\bar{\psi}_{Ak\dot{1}} = 0, \quad \bar{\tilde{\psi}}_{\dot{2}}^{kA} = 0,$$

$$\lambda^{a22} = \frac{i}{2}\frac{x_1 + ix_2}{r^2} f_{NA} \frac{\phi_1}{\phi_2}[\chi_1^a - i\varepsilon^{abc}n^b\chi_1^c],$$

$$\lambda^{a11} = \frac{i}{2}\frac{x_1 - ix_2}{r^2} f_{NA} \frac{\phi_1}{\phi_2}[\chi_2^a + i\varepsilon^{abc}n^b\chi_2^c],$$

$$\lambda^{a12} = \lambda^{a11}, \quad \lambda^{a21} = \lambda^{a22}, \tag{4.64}$$

where the dependence on x_i is encoded in the profile functions of the string, see Eq. (4.59).

Now we check directly that zero modes (4.64) satisfy Dirac equations of motion. From the fermion action of the model (4.61) we get relevant Dirac equations for λ^a,

$$\frac{i}{g_2^2}\bar{\not{D}}\lambda^{af} + \frac{i}{\sqrt{2}}\mathrm{Tr}(\bar{\psi}\tau^a q^f + \bar{q}^f\tau^a\bar{\tilde{\psi}}) = 0 \tag{4.65}$$

and for matter fermions,

$$i\not{\nabla}\bar{\psi} + \frac{i}{\sqrt{2}}[\bar{q}_f\lambda^f - (\tau^a\bar{q}_f)\lambda^{af} + (a - a^a\tau^a)\tilde{\psi}] = 0,$$

$$i\not{\nabla}\bar{\tilde{\psi}} + \frac{i}{\sqrt{2}}[\lambda_f q^f + \lambda_f^a(\tau^a q^f) + (a + a^a\tau^a)\psi] = 0. \tag{4.66}$$

Next we substitute orientational fermion zero modes (4.64) into these equations. After some algebra, one can check that Eqs. (4.64) do satisfy Dirac equations (4.65) and (4.66) provided first-order equations for string profile functions (4.33) are fulfilled.

It is instructive to check that the zero modes (4.64) do produce the fermion part of the $\mathcal{N}$=2 two-dimensional CP^1 model. To this end we return to the usual assumption that the fermion collective coordinates χ_α^a in Eq. (4.64) have an adiabatic dependence on the worldsheet coordinates x_k (k=0,3). This is quite similar to the procedure of the preceding section for bosonic moduli. Substituting Eq. (4.64) into the kinetic terms of fermions in the bulk theory (4.61), and taking into account the derivatives of χ_α^a with respect to the worldsheet coordinates, we arrive at

$$\beta\int dtdz\{\tfrac{1}{2}\chi_1^a(\partial_0 - i\partial_3)\chi_1^a + \tfrac{1}{2}\chi_2^a(\partial_0 + i\partial_3)\chi_2^a\}, \tag{4.67}$$

where β is given by the same integral (4.55) as for the bosonic kinetic term, see Eq. (4.57).

Now we can use the worldsheet $\mathcal{N}$=2 supersymmetry to reconstruct the four fermion interactions. The SUSY transformations in the $CP(1)$ model look like [see Novikov *et al.* (1984) for a review]

$$\delta\chi_1^a = i\sqrt{2}(\partial_1 + i\partial_3)n^a\varepsilon_2 - \sqrt{2}\varepsilon_1 n^a(\chi_1^a\chi_2^a),$$

$$\delta\chi_2^a = i\sqrt{2}(\partial_1 - i\partial_3)n^a\varepsilon_1 + \sqrt{2}\varepsilon_2 n^a(\chi_1^a\chi_2^a),$$

$$\delta n^a = \sqrt{2}(\varepsilon_1\chi_2^a + \varepsilon_2\chi_1^a), \tag{4.68}$$

where we set η_α=0 for simplicity. Imposing this supersymmetry leads to the following effective theory on the string worldsheet:

$$S_{CP(1)} = \beta\int dtdz\{\tfrac{1}{2}(\partial_k S^a)^2 + \tfrac{1}{2}\chi_1^a i(\partial_0 - i\partial_3)\chi_1^a + \tfrac{1}{2}\chi_2^a i(\partial_0 + i\partial_3)\chi_2^a - \tfrac{1}{2}(\chi_1^a\chi_2^a)^2\}. \tag{4.69}$$

This is the action of $\mathcal{N}$=2 $CP(1)$ sigma model (Novikov *et al.* 1984).

3. Physics of the $\mathcal{N}$=2 $CP(N-1)$ model

As usual in two dimensions, the Lagrangian of our effective theory on the string worldsheet can be cast in many different (but equivalent) forms. In particular, the $\mathcal{N}$=2 supersymmetric $CP(N-1)$ model (4.49) can be understood as a strong-coupling limit U(1) gauge theory (Witten, 1993). The bosonic part of the action is

$$S = \int d^2x\left\{2\beta|\nabla_k n^\ell|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k\sigma|^2 + 4\beta|\sigma|^2|n^\ell|^2 + 2e^2\beta^2(|n^\ell|^2 - 1)^2\right\}, \tag{4.70}$$

where $\nabla_k = \partial_k - iA_k$ and σ is a complex scalar field. The condition (4.43) is implemented in the limit $e^2 \to \infty$. Moreover, in this limit the gauge field A_k and its $\mathcal{N}$=2 bosonic superpartner σ become auxiliary and can be eliminated by virtue of the equations of motion,

$$A_k = -\frac{i}{2}n_\ell^*\overleftrightarrow{\partial}_k n^\ell, \quad \sigma = 0. \tag{4.71}$$

Substituting Eq. (4.71) into the Lagrangian, we can rewrite the action in the form (4.49).

The coupling constant β is asymptotically free (Polykov, 1975). As a function of energy E, it is given by

$$4\pi\beta = N \ln \frac{E}{\Lambda_\sigma}, \tag{4.72}$$

where Λ_σ is the dynamical scale of the sigma model. As mentioned previously, the ultraviolet cutoff of the sigma model at the string worldsheet is determined by $g_2\sqrt{\xi}$. At this uv cutoff scale, Eq. (4.55) holds. Hence,

$$\Lambda_\sigma^N = \xi^{N/2} e^{-8\pi^2/g_2^2} = \Lambda_{\mathrm{SU}(N)}^N, \tag{4.73}$$

where we take into account Eq. (4.22) for the dynamical scale $\Lambda_{\mathrm{SU}(N)}$ of the SU(N) factor of the bulk theory. Note that in the bulk theory *per se*, because of the VEV's of the squark fields, the coupling constant is frozen at $g_2\sqrt{\xi}$; there are no logarithms below this scale. The logarithms of the theory on the string worldsheet take over. Moreover, the dynamical scales of the bulk and worldsheet theories turn out to be the same. We explain the reason why the dynamical scale of the (1+1)-dimensional effective theory on the string worldsheet equals that of the SU(N) factor of the (3+1)-dimensional gauge theory, in Sec. IV.F.

The $CP(N-1)$ model was solved by Witten in the large-N limit (Witten, 1979b). Here we briefly summarize Witten's results and translate them in terms of strings in four dimensions (Shifman and Yung, 2004a).

Classically, the field n^ℓ can have arbitrary direction; therefore, one might naively expect spontaneous breaking of SU(N) and the occurrence of massless Goldstone modes. However, this cannot happen in two dimensions. Quantum effects restore the symmetry. Moreover, the condition (4.43) becomes in effect relaxed. Due to strong coupling we have more degrees of freedom than in the original Lagrangian, namely all N fields n become dynamical and acquire masses Λ_σ.

As shown by Witten (1979b), the model at large N has N vacua. These N vacua differ from each other by the expectation value of the chiral bifermion operator, see, e.g., Novikov *et al.* (1984). At strong coupling, the chiral condensate is the order parameter. The U(1) chiral symmetry of the $CP(1)$ model is broken down to a discrete Z_{2N} symmetry by chiral anomaly. The fermion condensate breaks it down to Z_N, hence the N-fold degeneracy.

The physics of the model becomes even more transparent in the mirror representation (Hori and Vafa, 2000), which can be written for arbitrary N. In this representation one describes the $CP(N-1)$ model in terms of the Coulomb gas of instantons [see Fateev *et al.* (1979a, 1979b) and Polyakov (1987), where it was done for the nonsupersymmetric $CP(1)$ model] to prove its equivalence to an affine Toda theory. The $CP(N-1)$ model (4.70) is dual to the following $\mathcal{N}=2$ affine Toda model (Fendley and Intriligator, 1992a; Cecotti and Vafa, 1993; Eguchi *et al.* 1997; Hori and Vafa, 2000):

$$\begin{aligned} S_T = \int d^2x d^2\theta d^2\bar{\theta}\beta^{-1} \sum_{i=1}^{N-1} \overline{Y}_i Y_i \\ + \left\{ \Lambda_\sigma \int d^2x d^2\theta \left(\sum_{i=1}^{N-1} \exp(Y_i) \right.\right. \\ \left.\left. + \prod_{i=1}^{N-1} \exp(-Y_i) \right) + \text{H.c.} \right\}. \end{aligned} \tag{4.74}$$

Here the last term is a dual instanton-induced superpotential. In fact, the exact form of the kinetic term is not known because it is not protected from quantum correction in β. However, the superpotential in Eq. (4.74) is exact. As soon as the vacuum structure is determined entirely by a superpotential, it can be read off Eq. (4.74).

The scalar potential of this affine Toda theory has N minima. For example, for $N=2$ this theory becomes $\mathcal{N}=2$ supersymmetric sine-Gordon theory with scalar potential

$$V_{\mathrm{SG}} = \frac{\beta}{4\pi^2} \Lambda_{CP(1)}^2 |\sinh y|^2, \tag{4.75}$$

which has two minima, at $y=0$ and $\pm i\pi$ (the points $y=i\pi$ and $-i\pi$ must be identified; they present one and the same vacuum).

This mirror model explicitly exhibits a mass gap of the order of Λ_σ. It shows that there are no Goldstone bosons [corresponding to the absence of the spontaneous breaking of the SU$(N)_{C+F}$ symmetry]. In terms of strings in four-dimensional bulk theory, this means in turn that the string orientation vector n^ℓ has no particular direction; it is smeared all over. The strings with which we deal here are genuinely non-Abelian. N vacua of the worldsheet theory (4.70) correspond to N elementary non-Abelian strings of the bulk theory. Note that these strings are in a highly quantum regime. They are not the Z_N strings of the quasiclassical $U(1)^{N-1}$ theory since the vector n^ℓ has no particular direction.

4. Unequal quark masses

The fact that we have N distinct vacua in the worldsheet theory—N distinct elementary strings—is not intuitive in the above consideration. This is understandable. At the classical level, the $\mathcal{N}=2$ two-dimensional CP^N sigma model has a continuous vacuum manifold. This is in one-to-one correspondence with continuously many strings parametrized by n^ℓ. The continuous degeneracy is lifted only upon inclusion of quantum effects that occur (in the sigma model) at strong coupling. Gone with this lifting is the moduli nature of the fields n^ℓ. They become massive which is difficult to understand.

To facilitate contact between the bulk and worldsheet theories, it is instructive to start from a deformed bulk theory so that the string moduli are lifted already at the classical level. Then the origin of the N-fold degeneracy of the non-Abelian strings become transparent. This will help us to understand other features listed above. After this understanding is achieved, nothing prevents us from returning to our case of strings with non-Abelian moduli at the classical level, by smoothly suppressing the moduli-breaking deformation. The N-fold degeneracy will remain intact as it follows from the Witten index (Witten, 1982).

Thus, we drop the assumption (4.15) of equal masses of quark flavors and introduce small mass differences. At nonequal quark masses the $U(N)$ gauge group is broken by the condensation of adjoint scalars down to $U(1)^N$; see Eq. (4.13). Off-diagonal gauge bosons as well as off-diagonal fields of quark matrix q^{kA} (together with their fermion superpartners) acquire masses proportional to various mass differences $(m_A - m_B)$. The effective low energy theory now contains only diagonal gauge and quark fields. The reduced action suitable for the search of a string solution takes the form

$$S = \int d^4x \left\{ \frac{1}{4g_2^2}(F^h_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + |\nabla_\mu \varphi^A|^2 + \frac{g_2^2}{2}(\bar{\varphi}_A T^h \varphi^A)^2 + \frac{g_1^2}{8}(|\varphi^A|^2 - N\xi)^2 \right\}, \tag{4.76}$$

where the index $h = 1, \ldots, (N-1)$ runs over Cartan generators of the gauge group SU(N) while the matrix φ^{kA} is reduced to its diagonal components.

The same steps that lead us to Eqs. (4.32) now give first-order equations for strings in the Abelian model (4.76),

$$F_3^* + \frac{g_1^2}{2}(|\varphi^A|^2 - N\xi) = 0,$$

$$F_3^{*h} + g_2^2(\bar{\varphi}_A T^h \varphi^A) = 0,$$

$$(\nabla_1 + i\nabla_2)\varphi^A = 0. \tag{4.77}$$

As soon as Z_N string solutions (4.28) have diagonal form, they automatically satisfy the above first-order equations.

However, Abelian Z_N strings (4.28) are now the only discrete solutions of these equations. The global $\mathrm{SU}(N)_{C+F}$ group is broken down to $U(1)^{N-1}$, and the continuous $CP(N-1)$ moduli space of non-Abelian string is lifted. In fact, the vector n^ℓ gets fixed in N possible positions

$$n^\ell = \delta^{\ell \ell_0}, \quad \ell_0 = 1, \ldots, N, \tag{4.78}$$

which correspond to Abelian Z_N strings; see Eqs. (4.28) and (4.44). If mass differences are much less than $\sqrt{\xi}$, the set of parameters n^ℓ becomes quasimoduli.

Now, our aim is to derive the effective two-dimensional theory on the string worldsheet for the case of unequal quark masses. At small mass differences, we can still introduce the orientational quasimoduli n^ℓ. In terms of the effective two-dimensional theory on the string worldsheet, nonequal masses lead to a shallow potential for the quasimoduli n^ℓ. We now derive this potential. Below for simplicity we review the derivation done in Shifman and Yung (2004a) for the SU(2) $\times$U(1) case. The case of general N is considered in Hanany and Tong (2004). In the $N=2$ case, the two minima of the potential at $S = \{0, 0, \pm 1\}$ correspond to two *bona fide* Z_2 strings.

We start from the expression for the non-Abelian string in the singular gauge (4.59) parametrized by moduli S^a and substitute it into the action (4.9). The only modification that we have to make is to supplement our ansatz (4.59) by that for the adjoint scalar field a^a; the neutral scalar field a will stay fixed at its vacuum expectation value $a = -\sqrt{2}m$.

At large r, the field a^a tends to its VEV directed along the third axis in the color space given by

$$\langle a^3 \rangle = -\frac{\Delta m}{\sqrt{2}}, \quad \Delta m = m_1 - m_2; \tag{4.79}$$

see Eq. (4.13). At the same time, at $r=0$ it must be directed along the vector S^a. The reason for this behavior is easy to understand. The kinetic term for a^a in Eq. (4.9) contains the commutator term of the adjoint scalar and the gauge potential. The gauge potential is singular at the origin, as seen from Eq. (4.59). This implies that a^a must be directed along S^a at $r=0$. Otherwise, the string tension would become divergent. The following ansatz for a^a ensures this behavior:

$$a^a = -\frac{\Delta m}{\sqrt{2}}[\delta^{a3} b + S^a S^3(1-b)]. \tag{4.80}$$

Here we introduced a new profile function $b(r)$ that will be determined from a minimization procedure. Note that at $S^a = (0, 0, \pm 1)$ the field a^a is given by its VEV, as expected. The boundary conditions for the function $b(r)$ are

$$b(\infty) = 1, \quad b(0) = 0. \tag{4.81}$$

Substituting Eq. (4.80) in conjunction with Eq. (4.59) into the action (4.9), we get the potential

$$V_{CP(1)} = \gamma \int d^2x \frac{\Delta m^2}{2}(1 - S_3^2), \tag{4.82}$$

where γ is given by the integral

$$\gamma = \frac{2\pi}{g_2^2}\int_0^\infty r dr \left\{ \left(\frac{d}{dr}b(r)\right)^2 + \frac{1}{r^2} f_{NA}^2 b^2 + g_2^2\left[\frac{1}{2}(1-b)^2(\phi_1^2 + \phi_2^2) + b(\phi_1 - \phi_2)^2\right]\right\}. \tag{4.83}$$

Here the first two terms in the integrand come from the kinetic term of the adjoint scalar field a^a while the term in the square brackets comes from the potential in the action (4.9).

Minimization with respect to $b(r)$, with the constraint (4.81), yields

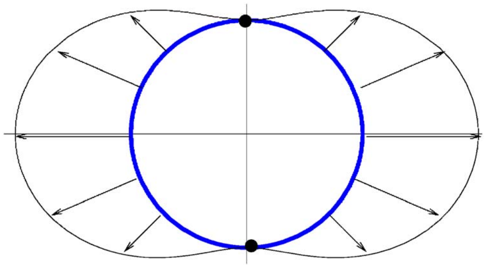

FIG. 7. (Color online) Meridian slice of the target space sphere (thick solid line). Arrows present the scalar potential in Eq. (4.85), their length representing the strength of the potential. Two vacua of the model are denoted by closed circles.

$$b(r) = 1 - \rho(r) = \frac{\phi_1}{\phi_2}(r); \tag{4.84}$$

cf. Eqs. (4.51) and (4.53). Thus, $\gamma = I2\pi/g_2^2 = 2\pi/g_2^2$. We see that the normalization integrals are the same for both the kinetic and potential terms in the worldsheet sigma model, $\gamma = \beta$. As a result, we arrive at the following effective theory on the string worldsheet:

$$S_{CP(1)} = \beta \int d^2x \left\{ \frac{1}{2}(\partial_k S^a)^2 + \frac{|\Delta m|^2}{2}(1 - S_3^2) \right\}. \tag{4.85}$$

This is the only functional form that allows $\mathcal{N}=2$ completion.[20]

The fact that we obtain this form shows that our ansatz is fully adequate. The informative aspect of the procedure is (i) the confirmation of the ansatz (4.80) and (ii) constructive calculation of the constant in front of $1 - S_3^2$ in terms of the bulk parameters. The mass-splitting parameter Δm of the bulk theory coincides exactly with the twisted mass of the worldsheet model.

The $CP(1)$ model (4.85) has two vacua located at $S^a = (0,0,\pm 1)$; see Fig. 7. Clearly, these two vacua correspond to two elementary Z_2 strings.

For the case of general N, the potential in the $CP(N-1)$ model has been worked out by Hanany and Tong (2004). It has the form

$$V_{CP(N-1)} = 2\beta \left\{ \sum_\ell |\tilde{m}_\ell|^2 |n^\ell|^2 - \left| \sum_\ell \tilde{m}_\ell |n^\ell|^2 \right|^2 \right\}, \tag{4.86}$$

where

$$\tilde{m}_\ell = m_\ell - m, \quad m \equiv \frac{1}{N}\sum_\ell m_\ell. \tag{4.87}$$

This potential has N vacua (4.78), which correspond to N Z_N strings in the bulk theory.

The $CP(N-1)$ model with the potential (4.86) is the bosonic part of an $\mathcal{N}=2$ two-dimensional sigma model, which is usually referred to as the $CP(N-1)$ model with the twisted mass. This is a generalization of the massless $CP(N-1)$ model, which preserves four supercharges. Twisted chiral superfields in two dimensions were introduced by Alvarez-Gaumé and Freedman (1983); Gates (1984); and Gates *et al.* (1984), while twisted mass as an expectation value of twisted chiral multiplet was suggested by Hanany and Hori (1998). $CP(N-1)$ models with twisted mass were studied by Dorey (1998) and, in particular, BPS spectra in these theories were determined exactly.

From the bulk theory point of view, the two-dimensional $CP(N-1)$ model is an effective worldsheet theory for the non-Abelian string, and emergence of $\mathcal{N}=2$ supersymmetry should be expected. As we know, the BPS nature of the strings under consideration does require the worldsheet theory to have four supercharges.

The $CP(N-1)$ model with twisted mass can be *rewritten* as a strong-coupling limit of U(1) gauge theory (Dorey, 1998). With twisted masses of n^ℓ fields taken into account, the bosonic part of the action (4.70) becomes

$$S = \int d^2x \left\{ 2\beta |\nabla_k n^\ell|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k \sigma|^2 + 4\beta \left| \sigma - \frac{\tilde{m}_\ell}{\sqrt{2}} \right|^2 |n^\ell|^2 + 2e^2\beta^2(|n^\ell|^2 - 1)^2 \right\}. \tag{4.88}$$

In the limit $e^2 \to \infty$, the σ field can be excluded via an algebraic equation of motion, which leads to the potential (4.86).

As already mentioned, this sigma model gives an effective description of our string at low energies, i.e., energies much lower than the inverse string thickness. Typical momenta in the theory (4.88) are of the order of $\tilde{m}$. Therefore, for the action (4.88) to be applicable, we must impose the condition

$$|\tilde{m}_\ell| \ll g_2\sqrt{\xi}. \tag{4.89}$$

The description in terms of the $CP(N-1)$ model with twisted mass gives us a better understanding of the dynamics of non-Abelian strings. If masses $\tilde{m}_\ell$ are much larger than the scale of the $CP(N-1)$ model Λ_σ, the coupling constant β is frozen at large scale (of order of masses $\tilde{m}_\ell$) and the theory is in the weak coupling. Semiclassical analysis is applicable. The theory (4.88) has N vacua located at

$$n^\ell = \delta^{\ell\ell_0}, \quad \sigma = \frac{\tilde{m}_{\ell_0}}{\sqrt{2}}, \quad \ell_0 = 1, \ldots, N. \tag{4.90}$$

They correspond to Abelian Z_N strings of the bulk theory; see Eq. (4.44). As we reduce mass differences $\tilde{m}_\ell$ and approach the value Λ_σ, the $CP(N-1)$ model enters the strong-coupling regime. At $\tilde{m}_\ell = 0$, the global $SU(N)_{C+F}$ symmetry of the bulk theory is restored. Now n^ℓ has no particular direction. The condition (4.43) is relaxed. Still we have N vacua in the worldsheet theory (Witten index). They are seen in the mirror description (see the preceding subsection). These vacua correspond to N elementary non-Abelian strings in the strong-coupling quantum regime. We see that for BPS strings, the transition from Abelian to non-Abelian regimes

[20]Note that although the global $SU(2)_{C+F}$ is broken by Δm, the extended $\mathcal{N}=2$ supersymmetry is not.

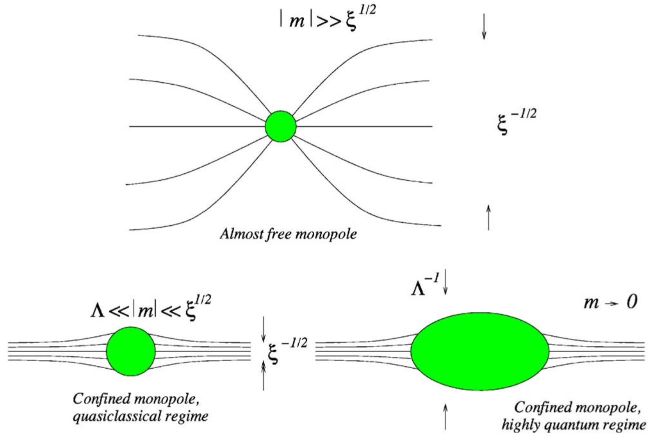

FIG. 8. (Color online) Evolution of the confined monopoles.

goes smoothly. As we discuss in Sec. V, this is not the case for non-BPS strings. In the latter case, two regimes are separated by the phase transition (Gorsky *et al.*, 2005, 2006).

E. Confined monopoles as kinks of the $CP(N-1)$ model

Our bulk theory (4.9) is in the Higgs phase so monopoles present in this theory should be in the confinement phase. If we start from the theory with SU$(N+1)$ gauge group, which is broken to the SU$(N)\times$U(1) gauge group of the theory (4.9) by condensation of the adjoint scalar a, the monopoles of the SU$(N+1)/$SU$(N)\times$U(1) sector can be attached to the ends of the Z_N strings considered here. However, in the bulk theory (4.9) these monopoles are considered as being infinitely heavy and the Z_N strings are stable. However, monopoles from the SU(N) gauge group are still present in the theory (4.9). As we switch on the FI parameter, ξ quarks condense triggering confinement of these monopoles. As shown in this section, these monopoles become string junctions of non-Abelian strings and are seen as kinks in the worldsheet theory interpolating between different vacua of the $CP(N-1)$ model (Hanany and Tong, 2004; Shifman and Yung, 2004a; Tong, 2004).

Our task is to trace the evolution of the confined monopoles starting from the quasiclassical regime, deep into the quantum regime. For illustrative purposes we start from the limit of weakly confined monopoles, when they present just slightly distorted 't Hooft–Polyakov monopoles (Fig. 8). We start from the limit $|\tilde{m}_A| \gg \sqrt{\xi}$ and take all masses of the same order. In this limit, the scalar quark expectation values can be neglected, and the vacuum structure is determined by VEV's of the adjoint a^a field; see Eq. (4.13). In the nondegenerate case, the gauge symmetry SU(N) of our bulk model is broken down to U$(1)^{N-1}$ modulo possible discrete subgroups. This is the standard situation for the occurrence of the SU(N) 't Hooft–Polyakov monopoles. The monopole core size is of the order of $|\tilde{m}|^{-1}$. The 't Hooft–Polyakov solution remains valid up to much larger distances of the order of $\xi^{-1/2}$. At distances larger than $\sim\xi^{-1/2}$, the quark VEV's become important. As usual, the U(1) charge condensation leads to the formation of the U(1) magnetic flux tubes, with the transverse size of the order of $\xi^{-1/2}$ (see the upper picture in Fig. 8). The flux is quantized; the flux tube tension is small in the scale of the square of the monopole mass. Therefore, what we deal with in this limit is basically a very weakly confined 't Hooft–Polyakov monopole.

We now verify that the confined monopole is a junction of two strings. Consider the junction of two Z_N strings corresponding to two neighboring vacua of the $CP(N-1)$ model. For the ℓ_0th vacuum, n^ℓ is given by Eq. (4.90) while for the (ℓ_0+1)th vacuum it is given by the same equations with $\ell_0 \to \ell_0+1$. The flux of this junction is given by the difference of the fluxes of these two strings. Using Eq. (4.44), we get that the flux of the junction is

$$4\pi \times \mathrm{diag}\tfrac{1}{2}\{\ldots,0,1,-1,0,\ldots\} \tag{4.91}$$

with the nonvanishing entries located at positions ℓ_0 and ℓ_0+1. These are exactly the fluxes of $N-1$ distinct 't Hooft–Polyakov monopoles occurring in the SU(N) gauge theory provided that SU(N) is spontaneously broken down to U$(1)^{N-1}$. We see that in the quasiclassical limit of large $|\tilde{m}_A|$, the Abelian monopoles play the role of the Abelian Z_N string junctions. Note that in various models, the fluxes of monopoles and strings were shown (Bais, 1981; Hindmarsh and Kibble, 1985; Everett and Aryal, 1986; Preskill and Vilenkin, 1993; Morshakov and Yung, 2002; Kneipp, 2003, 2004; Auzzi *et al.*, 2004a, 2004b; Auzzi, Bolognesi, *et al.*, 2005; Eto *et al.*, 2006) to match each other so that the monopoles can be confined by strings in the Higgs phase.

Now, if we reduce $|\tilde{m}_A|$, placing it the range

$$\Lambda \ll |\tilde{m}_A| \ll \sqrt{\xi}, \tag{4.92}$$

the size of the monopole $(\sim|\tilde{m}|^{-1})$ becomes larger than the transverse size of the attached strings. The monopole gets squeezed in earnest by the strings—it becomes a *bona fide* confined monopole (the lower left corner of Fig. 8). A natural question is: How is this confined monopole seen in the effective two dimensional CP^{N-1} model (4.88) on the string worldsheet? As soon as Z_N strings of the bulk theory correspond to N vacua of the CP^{N-1} model, the string junction (confined monopole) is a domain wall (kink) interpolating between these vacua; see Fig. 7.

Below we demonstrate that in the semiclassical regime (4.92), the solution for the string junction of the bulk theory corresponds to the kink in the worldsheet theory. Then we show that masses of monopole and kink match. This has been done by Shifman and Yung (2004a) for the $N=2$ case. Below we review this derivation.

1. First-order equations for the string junction

In this section, we derive the first-order equations for the 1/4-BPS junction of the Z_N strings of SU(N) $\times$U(1) theory in the quasiclassical limit (4.92). In this limit, $\tilde{m}_A$ is small enough so we can use our effective low-energy description in terms of the $CP(N-1)$ model with the twisted mass (4.88). On the other hand, $\tilde{m}_A$ is much larger than the scale of $CP(N-1)$ model, so the latter is in the weak-coupling regime, which allows one to apply the quasiclassical treatment.

The geometry of our junction is shown in the left corner of Fig. 8. Both strings are stretched along the z axis. We assume that the monopole sits near the origin, the $n^\ell=\delta^{\ell\ell_0}$ string is at negative z, while the $n^\ell=\delta^{\ell\ell_0+1}$ string is at positive z. The perpendicular plane is parametrized by x_1 and x_2. What is sought is a static solution of the BPS equations, with all relevant fields depending only on x_1, x_2, and z.

Ignoring the time variable, we can represent the energy functional of our theory (4.24) as follows [Bogomol'nyi representation (Bogomol'nyi, 1976)]:

$$E=\int d^3x\left\{\left[\frac{1}{\sqrt{2}g_2}F_3^{*a}+\frac{g_2}{2\sqrt{2}}(\bar{\varphi}_A\tau^a\varphi^A)+\frac{1}{g_2}D_3a^a\right]^2+\left[\frac{1}{\sqrt{2}g_1}F_3^{*}+\frac{g_1}{2\sqrt{2}}(|\varphi^A|^2-2\xi)+\frac{1}{g_1}\partial_3 a\right]^2+\frac{1}{g_2^2}\left|\frac{1}{\sqrt{2}}(F_1^{*a}+iF_2^{*a})+(D_1+iD_2)a^a\right|^2+\frac{1}{g_1^2}\left|\frac{1}{\sqrt{2}}(F_1^{*}+iF_2^{*})+(\partial_1+i\partial_2)a\right|^2+|\nabla_1\varphi^A+i\nabla_2\varphi^A|^2+\left|\nabla_3\varphi^A+\frac{1}{\sqrt{2}}(a^a\tau^a+a+\sqrt{2}m_A)\varphi^A\right|^2\right\} \tag{4.93}$$

plus surface terms. As compared with the Bogomol'nyi representation (4.31) for strings, we keep here also terms involving adjoint fields. Following our conventions, we assume the quark masses to be real, implying that the vacuum expectation values of the adjoint scalar fields are real too. The surface terms mentioned above are

$$E_{\text{surface}}=\xi\int d^3xF_3^*+\sqrt{2}\xi\int d^2x\langle a\rangle\Big|_{z=-\infty}^{z=\infty}-\sqrt{2}\frac{\langle a^a\rangle}{g_2^2}\int dS_nF_n^{*a}, \tag{4.94}$$

where the integral in the last term runs over a large two-dimensional sphere at $\vec{x}^2\to\infty$. The first term on the right-hand side is related to strings, the second to domain walls, while the third to monopoles (string junctions).

The Bogomol'nyi representation (4.93) leads us to the following first-order equations:

$$F_1^*+iF_2^*+\sqrt{2}(\partial_1+i\partial_2)a=0,$$

$$F_1^{*a}+iF_2^{*a}+\sqrt{2}(D_1+iD_2)a^a=0,$$

$$F_3^*+\frac{g_1^2}{2}(|\varphi^A|^2-2\xi)+\sqrt{2}\partial_3 a=0,$$

$$F_3^{*a}+\frac{g_2^2}{2}(\bar{\varphi}_A\tau^a\varphi^A)+\sqrt{2}D_3a^a=0,$$

$$\nabla_3\varphi^A=-\frac{1}{\sqrt{2}}(a^a\tau^a+a+\sqrt{2}m_A)\varphi^A,$$

$$(\nabla_1+i\nabla_2)\varphi^A=0. \tag{4.95}$$

These are our master equations. Once these equations are satisfied, the energy of the BPS object is given by Eq. (4.94).

We now discuss the central charges (the surface terms) of the string, domain wall, and monopole in more detail. In the string case, the three-dimensional integral in the first term in Eq. (4.94) gives the length of the string times its flux. In the wall case, the two-dimensional integral in the second term in Eq. (4.94) gives the area of the wall times its tension. Finally, in the monopole case, the integral in the last term in Eq. (4.94) gives the magnetic-field flux. This means that the first-order master equations (4.95) can be used to study strings, domain walls, monopoles, and all their possible junctions.

It is instructive to check that the wall, string, and monopole solutions, separately, satisfy these equations. For the domain wall, this check was done by Shifman and Yung (2004a), where these equations were used to study the string-wall junctions (reviewed in Sec. VII). Now consider the string solution. The scalar fields a and a^a are given by their VEV's. The gauge flux is directed along the z axis, so that $F_1^*=F_2^*=F_1^{*a}=F_2^{*a}=0$. All fields depend only on the perpendicular coordinates x_1 and x_2. As a result, the first two equations and the fifth one in Eqs. (4.95) are trivially satisfied. The third and fourth equations reduce to the first two equations in Eq. (4.32). The last equation in Eqs. (4.95) reduces to the last equation in Eqs. (4.32).

Now we turn to the monopole solution. The 't Hooft–Polyakov monopole equations ('t Hooft, 1974; Polyakov, 1974) arise from those in Eq. (4.95) in the limit $\xi=0$. Then all quark fields vanish, and Eq. (4.95) reduces to the standard first-order equations for the BPS 't Hooft–Polyakov monopole,

$$F_k^{*a}+\sqrt{2}D_ka^a=0. \tag{4.96}$$

The U(1) scalar field a is given by its VEV while the U(1) gauge field vanishes.

Equation (4.94) shows that the central charge of the SU(2) monopole is determined by $\langle a^a\rangle$, which is proportional to the quark mass difference; see Eq. (4.13). Thus, for the monopole on the Coulomb branch (i.e., with ξ vanishing), Eq. (4.94) yields

$$M_m=\frac{4\pi(m_{\ell_0+1}-m_{\ell_0})}{g_2^2}. \tag{4.97}$$

This coincides, of course, with the Seiberg-Witten result (Seiberg and Witten, 1994a) in the weak-coupling limit. As will be shown shortly, the same expression continues to hold even if $\tilde{m}_A\ll\sqrt{\xi}$ (provided that $\tilde{m}_A$ is still much larger than $\Lambda_{SU(N)}$). An explanation is given in Sec. IV.F.

The Abelian version of the first-order equations (4.95) was derived by Shifman and Yung (2003), where they were used to find the 1/4-BPS-saturated solution for the wall-string junction. Non-Abelian equations (4.95) in the $SU(2)\times U(1)$ theory were derived by Tong (2004), where a confined monopole as a string junction was considered at $\Delta m \neq 0$. Then non-Abelian equations (4.95) were used in the analysis (Shifman and Yung, 2004a) of the wall-string junctions for non-Abelian strings ending on a stack of domain walls, see Sec. VII. Next, Eqs. (4.95) were solved for the confined monopole as a string junction by Shifman and Yung (2004a) for $SU(2)\times U(1)$ theory. Below we review this solution. Later all 1/4-BPS solutions for junctions (in particular, string junctions of semilocal strings) were found by Isozumi *et al.* (2005).

2. String junction solution in the quasiclassical regime

We now apply our master equations for the $N=2$ case in order to find the junction of the $S^a=(0,0,1)$ string and the $S^a=(0,0,-1)$ string via the SU(2) monopole in the quasiclassical limit. We assume that the $S^a=(0,0,1)$ string is at negative z, while the $S^a=(0,0,-1)$ string is at positive z. We show that the solution of the BPS equations (4.95) of the four-dimensional bulk theory is determined by the kink solution in the two-dimensional sigma model (4.85).

To this end, we look for the solution of Eqs. (4.95) in the following ansatz. Assume that the solution for the string junction is given, to leading order in $\Delta m/\sqrt{\xi}$, by the same string configuration (4.59), (4.45), and (4.80) that we dealt with previously (in the case $\Delta m \neq 0$) with S^a slowly varying functions of z, to be determined below, replacing the constant moduli vector S^a.

Now the function $S^a(z)$ satisfies the boundary condition

$$S^a(-\infty) = (0,0,1), \tag{4.98}$$

while

$$S^a(\infty) = (0,0,-1). \tag{4.99}$$

This ansatz corresponds to the non-Abelian string in which the vector S^a slowly rotates from Eq. (4.98) at $z \to -\infty$ to Eq. (4.99) at $z \to \infty$. Now we show that the representation (4.59), (4.45), and (4.80) solves the master equations (4.95) provided the functions $S^a(z)$ are chosen in a special way.

Note that the first equation in Eqs. (4.95) is trivially satisfied because the field a is constant and $F_1^*=F_2^*=0$. The last equation reduces to the first two equations in Eqs. (4.33) because it does not contain derivatives with respect to z and, therefore, is satisfied for arbitrary functions $n^a(z)$. The same remark applies also to the third equation in Eqs. (4.95), which reduces to the third equation in Eqs. (4.33).

Now consider the fifth equation in Eqs. (4.95). Substituting our ansatz in this equation and using Eq. (4.53) for ρ, we find that this equation is satisfied provided $S^a(z)$ are chosen to be the solutions of

$$\partial_3 S^a = \Delta m(\delta^{a3} - S^a S^3). \tag{4.100}$$

Below we show that these equations are first-order equations for a kink in the massive $CP(1)$ model.

By the same token, we consider the second equation in Eqs. (4.95). Upon substitution our ansatz, this reduces to Eq. (4.100) too. Finally, consider the fourth equation in Eqs. (4.95). One can see that in fact it contains an expansion in the parameter $\Delta m^2/\xi$. This means that the solution we have just built is not exact; it has $O(\Delta m^2/\xi)$ corrections. To the leading order in this parameter, the fourth equation in Eqs. (4.95) reduces to the last equation in Eqs. (4.33). In principle, one could go beyond the leading order. Solving the fourth equation in Eqs. (4.95) in the next-to-leading order would allow one to determine $O(\Delta m^2/\xi)$ corrections to our solution.

We now focus on the meaning of Eq. (4.100). This equation is merely an equation for the kink in the CP^1 model (4.85). To see this, we write the Bogomol'nyi representation for kinks in the model (4.85). The energy functional can be rewritten as

$$E = \frac{\beta}{2}\int dz\{|\partial_z S^a - \Delta m(\delta^{a3} - S^a S^3)|^2 + 2\Delta m \partial_z S^3\}. \tag{4.101}$$

The above representation implies the first-order equation (4.100) for the BPS-saturated kink. It also yields $2\beta\Delta m$ for the kink mass.

Thus, we have demonstrated that the junction solution for the $S^a=(0,0,1)$ and $(0,0,-1)$ Z_2 strings is given by the non-Abelian string with a slowly varying orientation vector S^a. The variation of S^a is described in terms of the kink solution of the (1+1)-dimensional $CP(1)$ model with the twisted mass.

In conclusion, we want to match the masses of the four-dimensional monopole and two-dimensional kink. The string mass and that of the string junction is given by the first and last terms in the surface energy (4.94) (the second term vanishes). The first term reduces to

$$M_{\text{string}} = 2\pi\xi L, \tag{4.102}$$

i.e., proportional to the total string length L. Note that both $S^a=(0,0,1)$ and $(0,0,-1)$ Z_2 strings have the same tension (4.34). The third term should give the mass of the monopole. The surface integral in this term reduces to the flux of the $S^a=(0,0,-1)$ string at $z\to\infty$ minus the flux of the $S^a=(0,0,1)$ string at $z\to-\infty$. The F^{*3} flux of the $S^a=(0,0,-1)$ string is 2π while the F^{*3} flux of the $S^a=(0,0,1)$ string is -2π. Thus, taking into account Eq. (4.13), we get

$$M_m = \frac{4\pi}{g_2^2}\Delta m. \tag{4.103}$$

Note that although we discussed the monopole in the confinement phase at $|\Delta m| \ll \sqrt{\xi}$ (which is a junction of two strings in this phase), nevertheless the Δm and g_2^2 dependence of its mass coincides with the result (4.97) for the unconfined monopole on the Coulomb branch (i.e., at $\xi=0$). This is no accident—there is a theoretical reason explaining the validity of the unified formula. A change occurs only in passing to the highly quantum regime depicted in the right lower corner of Fig. 8. We discuss this regime briefly in the next subsection.

Now compare Eq. (4.103) with the kink mass in the effective CP^1 model on the string worldsheet. As mentioned, the surface term in Eq. (4.101) gives

$$M_{\text{kink}} = 2\beta\Delta m. \tag{4.104}$$

Now expressing the two-dimensional coupling constant β in terms of the coupling constant of the microscopic theory, see Eq. (4.55), we obtain

$$M_{\text{kink}} = \frac{4\pi}{g_2^2}\Delta m, \tag{4.105}$$

thus verifying that the four-dimensional calculation of M_m and the two-dimensional calculation of M_{kink} yield the same,

$$M_m = M_{\text{kink}}. \tag{4.106}$$

Needless to say, this is in full accordance with the physical picture that emerged from our analysis that the two-dimensional $CP(1)$ model is merely the macroscopic description of confined monopoles occurring in the four-dimensional microscopic Yang-Mills theory. Technically the coincidence of the monopole and kink masses is based on the fact that the integral in the definition (4.50) of the sigma-model coupling β is unity.

3. Strong-coupling limit

In this subsection, we consider the limit of small $\tilde{m}_A$ when the effective worldsheet theory enters the strong-coupling regime. For illustrative purposes, we consider the simplest case with $N=2$. The generalization to general N is straightforward.

As we further diminish $|\Delta m|$ approaching Λ_σ and then taking Δm to zero, we restore $SU(2)_{C+F}$ symmetry. In particular, on the Coulomb branch $SU(2)\times U(1)$ gauge symmetry is restored. The monopole becomes a truly non-Abelian object. In this limit, the size of the monopole grows, and, classically, it would explode. Moreover, the classical formula (4.103) shows that its mass goes to zero [see the discussion of "monopole clouds" by Weinberg (1980, 1982) for a review on what becomes of monopoles upon restoration of non-Abelian gauge symmetry]. Thus classically we would say that the monopole disappears.

This is where quantum effects on the confining string take over. As reviewed below they make the non-Abelian confined monopole a well defined stable state (Shifman and Yung, 2004a). From the point of view of the effective worldsheet theory this domain presents the regime of highly quantum worldsheet dynamics. While the thickness of the string (in the transverse direction) is $\sim\xi^{-1/2}$, the z-direction size of the kink representing the confined monopole in the highly quantum regime is much larger, $\sim\Lambda_\sigma^{-1}$; see the lower right corner in Fig. 8. Still it remains finite in the limit $\Delta m\to 0$ stabilized by nonperturbative effects in the worldsheet $CP(1)$ model. Remember that $CP(N-1)$ models develop a mass gap and no massless states are present in the spectrum; see Sec. IV.D.3. Moreover, the mass of the confined monopole [kink of the $CP(1)$ model] is also determined by the scale Λ_σ. This defines the notion of what is non-Abelian confined monopole. It is a kink of massless two-dimensional $CP(N-1)$ model (Shifman and Yung, 2004a).

We can have a more quantitative insight in the physics of worldsheet theory in strong coupling using an exact BPS spectrum of the $CP(N-1)$ model with twisted mass obtained by Dorey (1998). This was obtained by generalization of Witten analysis (Witten, 1993) as done for the massless case. The exact central charge of BPS states is given by

$$Z_{2D} = i\Delta m q + m_D T. \tag{4.107}$$

Here T is the topological charge of the kink under consideration, $T=\pm 1$, while the parameter q is

$$q = 0, \pm 1, \pm 2, \dots \;. \tag{4.108}$$

This U(1) charge of the "dyonic" states arises due the presence of the U(1) group unbroken in Eq. (4.85) by the twisted mass [the $SU(2)_{C+F}$ symmetry is broken down to U(1) by Δm].

The quantity m_D is introduced in analogy with a_D of Seiberg and Witten (1994a),

$$m_D = \frac{\Delta m}{\pi}\left[\frac{1}{2}\ln\frac{\Delta m + \sqrt{\Delta m^2 + 4\Lambda_\sigma^2}}{\Delta m - \sqrt{\Delta m^2 + 4\Lambda_\sigma^2}} - \sqrt{1 + \frac{4\Lambda_\sigma^2}{\Delta m^2}}\right], \tag{4.109}$$

where Δm is assumed to be complex. The two-dimensional central charge is normalized such that $M_{\text{kink}} = |Z_{2D}|$. The limit $|\Delta m|/\Lambda_\sigma\to\infty$ corresponds to the quasiclassical domain, while corrections of the type $(\Lambda_\sigma/\Delta m)^{2k}$ are induced by instantons.

What happens when one travels from the domain of large $|\Delta m|$ to small $|\Delta m|$? If $\Delta m=0$, we know, e.g., from the mirror representation (Hori and Yafa, 2000), that there are two degenerate two-dimensional kink supermultiplets, corresponding to the CFIV index=2 (Cecotti *et al.*, 1992; Fendley and Intriligator, 1992b; Cecotti and Vafa, 1993). They have quantum numbers $\{q,T\}=(0,1)$ and (1, 1). Away from the point $\Delta m=0$, the masses of these states are no longer equal; there is one singular point with one of the two states becoming massless (Shifman *et al.*, 2006). The region containing the point $\Delta m=0$ is separated from the quasiclassical region of large Δm by the curve of the marginal stability (CMS) on which the infinite number of other BPS states, visible quasiclassically, decay. Thus, the infinite tower of the $\{q,T\}$ BPS states existing in the quasiclassical domain degenerates in just two stable BPS states in the vicinity of $\Delta m=0$.

As outlined above, there are no massless states in the $CP(1)$ model at $\Delta m=0$. In particular, the kink (confined monopole) mass is

$$M_m=\frac{2}{\pi}\Lambda_\sigma, \tag{4.110}$$

as is clear from Eq. (4.109). On the other hand, in this limit both the last term in Eq. (4.94) and the surface term in Eq. (4.101) vanish for the monopole and kink masses, respectively. This puzzle is solved by the following observation: Anomalous terms in the central charges of both four-dimensional and two-dimensional SUSY algebras emerge. In two dimensions, it was obtained by Losev and Shifman (2003) and Shifman *et al.* (2006). In four dimensions, it was worked out by Shifman and Yung (2004a).

In the bulk theory, the central charge associated with the monopole has the following general form:

$$\{Q^f_\alpha Q^g_\beta\}=\varepsilon_{\alpha\beta}\varepsilon^{fg}2Z_{4\mathrm{D}}, \tag{4.111}$$

where $Z_{4\mathrm{D}}$ is an $\mathrm{SU}(2)_R$ singlet. It is most convenient to write $Z_{4\mathrm{D}}$ as a topological charge (i.e., the integral over a topological density),

$$Z_{4\mathrm{D}}=\int d^3x\,\zeta^0(x). \tag{4.112}$$

In this model,

$$\begin{aligned}\zeta^\mu=&\frac{1}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\bigg(\frac{i}{g_2^2}a^aF^a_{\rho\sigma}+\frac{i}{g_1^2}aF_{\rho\sigma}-\frac{i}{2\pi^2}a^aF^a_{\rho\sigma}\\&+\frac{1}{8\sqrt{2}\pi^2}[\lambda^a_{f\alpha}(\sigma_\rho)^{\alpha\dot\alpha}(\bar\sigma_\sigma)_{\dot\alpha\beta}\lambda^{af\beta}\\&+2g_2^2\tilde\psi_{A\alpha}(\sigma_\rho)^{\alpha\dot\alpha}(\bar\sigma_\sigma)_{\dot\alpha\beta}\psi^{A\beta}]\bigg).\end{aligned} \tag{4.113}$$

Note that the general structure of the operator in the square brackets is unambiguously fixed by dimensional arguments, the Lorentz symmetry, and other symmetries of the bulk theory. The numerical coefficient was found by Shifman and Yung (2004a) by matching monopole and kink masses at $\Delta m=0$. We also include the bosonic part of the anomaly term associated with magnetic field here (last term in the first line).

The anomalous term plays a crucial role in the Higgs phase for the confined monopole. On the Coulomb branch, it does not contribute to the mass of the monopole due to too fast fall off of fermion fields at infinity. On the Coulomb branch the bosonic anomalous terms become important. The relationship between the 't Hooft–Polyakov monopole mass and the $\mathcal{N}=2$ central charge has been analyzed by Rebhan *et al.* (2004a), which identified an anomaly in the central charge explaining a constant (i.e., nonlogarithmic) term in the monopole mass on the Coulomb branch. The result of Rebhan *et al.* (2004a) is in agreement with the Seiberg-Witten formula for the monopole mass.

F. 2D kink and 4D Seiberg-Witten exact solution

Why is the 't Hooft–Polyakov monopole mass (i.e., on the Coulomb branch at $\xi=0$) given by the same formula (4.97) as the mass (4.103) of the strongly confined large-ξ monopole (subject to condition $\sqrt{\xi}\gg\Delta m$)? This fact was noted in Sec. IV.E.2. Now we explain the reason behind this observation (Hanany and Tong, 2004; Shifman and Yung, 2004a). *En route*, we explain another striking observation made by Dorey (1998). A remarkably close parallel between the four-dimensional Yang-Mills theory with $N_f=2$ and the two-dimensional $CP(1)$ model was noted, by virtue of a comparison of the corresponding central charges. The observation was made on the Coulomb branch of the Seiberg-Witten theory, with unconfined 't Hooft–Polyakov-like monopoles and dyons. Valuable as it is, the parallel was quite puzzling since the solution of the $CP(1)$ model seemed to have no physics connection to the Seiberg-Witten solution. The latter gives the mass of the unconfined monopole in the Coulomb phase at $\xi=0$ while the $CP(1)$ model emerges only in the Higgs phase of the bulk theory.

Now we show that the reason for the correspondence mentioned above is that in the BPS sector (and only in this sector) the parameter ξ, in fact, cannot enter relevant formulas. Therefore, one can vary ξ at will, in particular, making it less than $|\Delta m|$ or even tending to zero, where $CP(1)$ is no longer the worldsheet model for our bulk theory. Nevertheless, the parallel expressions for the central charges and other BPS data in 4D and 2D, established at $|\Delta m|\ll\xi$, will continue to hold even on the Coulomb branch. The strange coincidence observed in Sec. IV.E.2 is no accident. We deal here with an exact relation that stays valid, including both perturbative and nonperturbative corrections.

Physically, the monopole in the Coulomb phase is different from the one in the confinement phase; see Fig. 8. In the Coulomb phase it is a 't Hooft–Polyakov monopole, while in the confinement phase it becomes related to a junction of two non-Abelian strings. However, the masses of these two objects are given by the same expression,

$$M_m^{\mathrm{Coulomb}}=M_m^{\mathrm{confinement}}, \tag{4.114}$$

provided that Δm and the gauge couplings are kept fixed. The superscripts refer to the Coulomb and monopole-confining phases, respectively.

Our point is that the mass of the monopole cannot depend on the FI parameter ξ. Start from the monopole in the Coulomb phase at $\xi=0$. Its mass is given by the exact Seiberg-Witten formula (Seiberg and Witten, 1994a)

TABLE II. The $U(1)_R$ charges of fields and parameters of the bulk theory.

Field/parameter	a	a^a	λ^α	q	ψ^α	m_A	Λ_{SU_N}	ξ
$U(1)_R$ charge	2	2	1	0	−1	2	2	0

$$M_m^{\rm Coulomb} = \sqrt{2}\left| a_D^3\left(a^3 = -\frac{\Delta m}{\sqrt{2}}\right)\right| = \left|\frac{\Delta m}{\pi}\ln\frac{\Delta m}{\Lambda_{SU(2)}} + \Delta m\sum_{k=1}^{\infty} c_k\left(\frac{\Lambda}{\Delta m}\right)^{2k}\right|, \tag{4.115}$$

where a_D^3 is the dual Seiberg-Witten potential for the SU(2) gauge subgroup, and we take into account that for $N_f=2$ the first coefficient of the β function is 2. Here $a^3=-\Delta m/\sqrt{2}$ is the argument of a_D^3, the logarithmic term takes into account the one-loop result (4.21) for the SU(2) gauge coupling at the scale Δm, while the power series is the expansion in instanton-induced corrections.

Now, if we introduce a small FI parameter $\xi\neq 0$ in the theory, on dimensional grounds we expect in Eq. (4.115) corrections to the monopole mass in powers of $\sqrt{\xi}/\Lambda_{SU(2)}$ and/or $\sqrt{\xi}/\Delta m$. These corrections are forbidden by the $U(1)_R$ charges. Namely, the $U(1)_R$ charges of $\Lambda_{SU(2)}$ and Δm are equal to 2 [and so is the $U(1)_R$ charge of the central charge under consideration] while ξ has a vanishing $U(1)_R$ charge. For convenience, the $U(1)_R$ charges of different fields and parameters of the microscopic theory are presented in Table II. Thus, neither $(\sqrt{\xi}/\Lambda_{SU(2)})^k$ nor $(\sqrt{\xi}/\Delta m)^k$ can appear.

By the same token, we could start from the confined monopole at large ξ, and study the dependence of the monopole (string junction) mass as a function of ξ as we reduce ξ. Again, the above arguments based on the $U(1)_R$ charges tell us that corrections in powers of $\Lambda_{SU(2)}/\sqrt{\xi}$ and $\Delta m/\sqrt{\xi}$ cannot appear. This leads us to Eq. (4.114).

Another way to arrive at the same conclusion is to observe that the mass of the monopole is determined by the central charge (4.112). This central charge is a holomorphic quantity and, thus, cannot depend on the FI parameter ξ, which is not holomorphic [it is a component of the $SU(2)_R$ triplet (Hanany *et al.*, 1998; Vainshtein and Yung, 2001)].

Now recall that the mass of the monopole in the confinement phase is given by the kink mass in the $\mathcal{N}=2$ CP^1 model; see Eq. (4.106). Thus, we obtain

$$M_m^{\rm Coulomb} \leftrightarrow M_m^{\rm confinement} \leftrightarrow M_{\rm kink}. \tag{4.116}$$

In particular, at the one-loop level the kink mass is determined by one-loop renormalization of the $CP(1)$ model coupling constant β, while the monopole mass on the Coulomb branch is determined by the renormalization of g^2. This leads to the relation $\Lambda_\sigma=\Lambda_{SU(2)}$ between the 2D and 4D dynamical scales, which we noted earlier as a strange coincidence, see Eq. (4.73). Now we know the physical reason behind it. Note that the first coefficient of the β function is equal 2 (N for the generic N case) for both theories.

The above relation can be generalized [cf. Dorey (1998) and Hanany and Tong (2004)] to theories with the $SU(N)\times U(1)$ gauge group and $N_f=N$ flavors on the four-dimensional side, and $CP(N-1)$ sigma models on the two-dimensional side.

This correspondence can be seen in more quantitative terms (Dorey, 1998; Hanany and Tong, 2004). The four-dimensional $\mathcal{N}=2$ QCD with the $U(N)$ gauge group and $N_f=N$ is described by the degenerate Seiberg-Witten curve,

$$y^2 = \frac{1}{4}\left[\prod_{i=1}^{N}(x+m_i) - \Lambda_{SU(N)}^N\right]^2 \tag{4.117}$$

in the special point (4.13) on the Coulomb branch, which becomes a quark vacuum upon the ξ perturbation. The periods of this curve give the BPS spectrum of the two-dimensional $CP(N-1)$ model (Dorey, 1998) [this spectrum is given for the $N=2$ case in Eqs. (4.107) and (4.109)].

In fact, Dorey (1998) has shown that BPS spectra of the $\mathcal{N}=2$ 2D $CP(N-1)$ model and 4D $SU(N)$ QCD coincide if one chooses a point on the Coulomb branch corresponding to the baryonic Higgs branch defined by $\Sigma m_A=0$ [in the SU(2) case this amounts to taking $m_1=-m_2$]. However, one can check that BPS spectra of massive states for SU(2) and U(2) theories coincide in the corresponding quark vacua upon identification of m_A of $SU(N)$ theory with $\tilde{m}_A$ of $U(N)$ theory [in the $N=2$ case one should identify $m_1=-m_2$ of SU(2) theory with $\Delta m/2$ of U(2) theory]. Note that vacuum (4.13) and (4.16) of $U(N)$ theory is an isolated vacuum rather than a root of any Higgs branch. There are no massless states in the $U(N)$ bulk theory in this vacuum; see Sec. IV.A.2.

Note that BPS spectra of both theories include not only monopole (kink) and dyonic states but also elementary excitations with $T=0$ as well. On the 2D theory side, they correspond to elementary fields n^ℓ in the large-$\tilde{m}_A$ limit. On the 4D side, they correspond to nontopological (i.e., $T=0$ and $q=\pm 1$) BPS excitations of the string with masses proportional to $\tilde{m}_A$ confined to the string. They can be interpreted as follows. Inside the string the squark profiles vanish, effectively bringing us toward the Coulomb branch ($\xi=0$) where the W bosons and quarks would become BPS-saturated states in the bulk. For $N=2$ on the Coulomb branch the W boson and off-diagonal quark mass would just equal Δm. Hence, the $T=0$ BPS excitation of the string is a wave of such W

bosons and/or quarks propagating along the string. One could call it a confined W boson (or quark). It is localized in the perpendicular but not in the transverse direction. What is important is that it has no connection with the bulk Higgs phase W bosons, which are not BPS and much heavier than Δm, nor do these nontopological excitations have a connection with the bulk quarks of our bulk model, which are also not BPS saturated.

To conclude, conformal theory with massless quarks and monopoles arising on the Coulomb branch of the four-dimensional $\mathcal{N}=2$ QCD upon special choice of mass parameters $\tilde{m}_A$ [Argyres-Douglas point (Argyres and Douglas, 1995)] was compared with the two-dimensional $CP(N-1)$ model with twisted mass (Tong, 2006a). The coincidence of monopole and kink masses explained above ensures that $CP(N-1)$ flows to a nontrivial conformal point at these values of $\tilde{m}_A$. It has been shown by Tong (2006a) that scaling dimensions of chiral primary operators in four- and two-dimensional conformal theories agree.

G. More quark flavors

In this section, we consider the theory (4.9) with more fundamental flavors, $N_F > N$. In this case, we have a number of isolated vacua like Eqs. (4.13) and (4.16), in which N quarks out of N_f develop VEV's, while adjoint VEV's are determined by masses of these quarks as in Eq. (4.13). Now consider the case of equal masses. In this case isolated vacua coalesce and a Higgs branch develops from the common root, whose location on the Coulomb branch is given by Eq. (4.13) (with equal masses). The dimension of this branch is $4N(N_f-N)$, (Argyres *et al.*, 1996; Marshakov and Yung, 2002). The Higgs branch is noncompact and has hyper-Kähler geometry (Seiberg and Witten, 1994b; Argyres *et al.*, 1996). It has a compact base manifold defined by

$$\bar{\tilde{q}}^{kA} = q^{kA}. \tag{4.118}$$

The dimension of this manifold is two times less than the total dimension of the Higgs branch, $2N(N_f-N)$, which gives 4 for $N_f=3$ and 8 for $N_f=4$ for the simplest $N=2$ case. BPS string solutions exist only on the base manifold of the Higgs branch [strings become non-BPS if we move in noncompact directions (Evlampier and Yung, 2003)], therefore we are interested in vacua that belong to the base manifold.

Strings in theories with many flavors (typically on Higgs branches) in many instances are not usual ANO strings; they become so-called semilocal strings [see Achucarro and Vachaspati (2000) for a review]. The simplest model in which they appear is the Abelian Higgs model with two complex flavors,

$$S_{\mathrm{AH}} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu q^A|^2 + \frac{g^2}{8}(|q^A|^2 - \xi)^2 \right\}, \tag{4.119}$$

where $A=1,2$ is the flavor index.

At nonzero ξ, scalar fields develop VEV's breaking U(1) gauge group. A photon becomes massive together with one real scalar. For the particular choice of quartic coupling made in Eq. (4.119), this scalar has the same mass as the photon, model (4.119) is a bosonic part of a supersymmetric theory, and vortices are BPS saturated. The topological reason for the existence of ANO vortices is that for the gauge group $U(1)$, $\pi_1[U(1)]=Z$. On the other hand, we can go to the low-energy limit in Eq. (4.119) integrating out massive photon and real massive scalar. This leads us to the four-dimensional sigma model on the vacuum manifold $|q^A|^2=\xi$. The vacuum manifold has dimension $4-1-1=2$, where we subtract one real condition mentioned above as well as one gauge phase. This represents the two-dimensional sphere S_2. Thus, the low-energy limit of theory (4.119) is the $O(3)$ sigma model. Now recall that $\pi_2[S_2] = \pi_1[U(1)]=Z$ and this is a topological reason for the existence of instantons in the two-dimensional $O(3)$ sigma model (Polyakov and Belavin, 1975). Lifted in four dimensions, they become stringlike objects (lumps).

Like an instanton of the O(3) sigma model, the semilocal string in the model (4.119) possesses two additional zero modes associated with its complex transverse size ρ. The semilocal string interpolates between the ANO string and the two-dimensional sigma model instanton lifted in four dimensions (lump). At zero ρ, we have the ANO string while at $\rho\to\infty$ it becomes a 2D instanton lifted in four dimensions. At nonzero ρ, the semilocal string has power falloff of the profile functions at infinity, instead of the exponential falloff for the ANO string at $\rho=0$.

Now if we return to our non-Abelian theory (4.9), we see that semilocal strings in this theory have, besides $2(N-1)$ orientational moduli n^ℓ, also size moduli. The total dimension of the moduli space of the semilocal string is (Hanany and Tong, 2003)

$$2N_f = 2 + 2(N-1) + 2(N_f - N), \tag{4.120}$$

where the first, second, and third terms correspond to translational, orientational, and size moduli.

The study of the moduli space geometry of the semilocal string was not carried out for some time due to infrared problems. It was known that size zero modes are logarithmically non-normalizable in the infrared (Ward, 1985; Leese and Samols, 1993), as is the case for the sigma model instanton. This problem was addressed by Shifman and Yung (2006a) for the case of non-Abelian strings in the theory with U(2) gauge group, and it was shown that the effective theory on the world of the string has the form

$$S^{(1+1)} = \beta M_W \int dt dz \left\{ \frac{\rho^2}{4}(\partial_k S^a)^2 + |\partial_k \rho_i|^2 \right\} \ln \frac{1}{|\rho|\delta m}, \tag{4.121}$$

where M_W is the mass of the W boson (4.18), $i=3,\ldots,N_f$, while ρ_i denotes (N_f-2) complex fields associated with size moduli. The parameter δm measures small quark mass differences. One has to introduce this parameter lifting slightly size moduli ρ_i in order to regularize the IR logarithmic divergence.

The metric (4.121) has been derived by Shifman and Yung (2006a) for large but not too large values of $|\rho|^2 \equiv |\rho_i^2|$ inside the window,

$$\frac{1}{M_W} \ll |\rho| \ll \frac{1}{\delta m}. \tag{4.122}$$

The first inequality refers to the limit in which the semilocal string becomes an O(3) sigma model lump, while the second one ensures the validity of the logarithmic approximation.

For ρ_i inside the window (4.122) with the logarithmic accuracy, one can introduce new variables

$$z_i = \rho_i \left[M_W^2 \ln \frac{1}{|\rho|\,\delta m} \right]^{1/2}. \tag{4.123}$$

It has been shown by Shifman and Yung (2006a) that in terms of these variables, the metric of the worldsheet theory (4.121) become flat. There are corrections to this flat metric in powers of both $1/M_W|\rho|$ and $1/\ln(1/|\rho|\delta m)$ that are have not been calculated so far within the field theory approach.

On the other hand, using brane-based arguments it was conjectured by Hanany and Tong (2003, 2004) [see also Eto *et al.* (2004)] that the effective theory on the worldsheet of a non-Abelian semilocal string is given by the strong-coupling limit ($e^2 \to \infty$) of two-dimensional gauge theory,

$$S = \int d^2x \Bigg\{ 2\beta|\nabla_k n^\ell|^2 + 2\beta|\nabla_k z_i|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k \sigma|^2 + 4\beta\left|\sigma - \frac{\tilde{m}_\ell}{\sqrt{2}}\right|^2 |n^\ell|^2 + 4\beta\left|\sigma - \frac{\tilde{m}_i}{\sqrt{2}}\right|^2 |z_i|^2 + 2e^2\beta^2(|n^\ell|^2 - |z_i|^2 - 1)^2 \Bigg\}, \tag{4.124}$$

where $\ell = 1, \ldots, N$, $i = N+1, \ldots N_f$, while z_i denotes $N_f - N$ complex fields associated with size moduli. Fields n^ℓ and z_i have charges +1 and −1 with respect to the U(1) gauge field in Eq. (4.124). This theory is similar to the model (4.88) for non-Abelian string in the theory with $N_f = N$.

There is another argument in favor of the above conjecture. As discussed in the preceding subsection, the BPS spectrum of dyons on the Coulomb branch of 4D theory should coincide with the BPS spectrum in 2D effective theory on the string worldsheet. We expect that this correspondence can be generalized to theories with $N_f > N$. The 2D theory (4.124) was studied by Dorey *et al.* (1999), and it was shown that its BPS spectrum agrees with the spectrum of U(N) four-dimensional QCD with N_f flavors.[21] In particular, the one-loop coefficient of the β function is equal to $2N - N_f$ for both theories. This leads to the identification of their scales; see Eq. (4.73).

This shows that 2D theory (4.124) is a promising candidate for the effective theory on the worldsheet of a semilocal string. In particular, the metric in Eq. (4.124) is asymptotically flat and variables z_i in Eq. (4.124) should be identified with the variables in Eq. (4.123) introduced within the field theory approach in Shifman and Yung (2006a). It is quite plausible that corrections to the flat metric in powers $1/M_W|\rho|$ are correctly reproduced by the worldsheet theory (4.124). As for logarithmic corrections, Shifman and Yung (2006a) have shown the approximate nature of the worldsheet theory (4.124). In particular, corrections at large ρ_i suppressed by the large infrared logarithm $1/\ln(1/|\rho|\delta m)$ are not captured by Eq. (4.124).

The most physically important consequence of emergence of semilocal strings is that we lose the monopole confinement (Evlampiev and Yung, 2003; Shifman and Yung, 2006a). To study monopole confinement upon quark condensation, we consider a string of a finite length L stretched between the heavy monopole and the antimonopole of the SU(N+1)/$S(N) \times$U(1) sector. The ANO string has size $1/g\sqrt{\xi}$, and if L is much larger than this size, the energy of this configuration is

$$V(L) = TL, \tag{4.125}$$

where T is the string tension. This linear potential ensures confinement of monopoles.

In the presence of semilocal strings, this picture changes drastically. Now the size of the string can be arbitrarily large. Imagine the configuration in which string size becomes much larger than L. Clearly, the problem now becomes three dimensional. The monopole flux is not trapped now in a narrow flux tube. Instead, it is freely spread over a large three-dimensional volume of size of order of L. This produces a Coulomb-type potential between a monopole and an antimonopole,

$$V(L) \sim 1/L. \tag{4.126}$$

The energy of this configuration is lower than that of the stringy configuration (4.125) at large L, so it is more preferable. We conclude that the semilocal string increases its size and effectively disintegrates, resulting in the Coulomb potential (4.126). In fact, lattice studies show that the semilocal string width always increases upon small perturbations (Leese, 1992).

We see that the formation of semilocal strings on Higgs branches leads to a dramatic physical effect, namely, deconfinement.

[21]In fact, as in Dorey (1998), Dorey *et al.* (1999) deal with SU(N) theory at the root of the baryon Higgs branch. However, as explained, the BPS spectra of massive states in these two 4D theories are the same.

H. Non-Abelian k-strings

In this section, we review the construction of multiple strings with winding number k. They can be considered as a bound state of k BPS elementary strings. The Bogomol'nyi representation (4.31) shows that the tension of the BPS k-string is determined by its total U(1) flux, which is $2\pi k$. This gives the tension of the k-string in the theory (4.9),

$$T_k = 2\pi k \xi, \tag{4.127}$$

and shows that elementary strings forming the composite k-string do not interact.

If we take elementary strings forming k-string at large separations, the total moduli space factorizes into k copies of moduli spaces of elementary strings. This suggests that the dimension of the moduli space is

$$2kN_f = 2k + 2k(N-2) + 2k(N_f - N); \tag{4.128}$$

see Eq. (4.120). The total dimension is written as a sum of dimensions of position, orientation, and size moduli spaces. This result was confirmed by the index theorem of Hanany and Tong (2003), which gives Eq. (4.128) at any separations. The moduli space of well separated k elementary strings forming the k-string, say for $N_f = N$, is

$$\frac{[C \times CP(N-1)]^k}{S_k}, \tag{4.129}$$

where S_k stands for permutations of positions of elementary strings.

The explicit solution for a non-Abelian 2-string at zero separations in the simplest bulk theory with $N = N_f = 2$ was constructed by Auzzi *et al.* (2006). It has a peculiar feature. If the orientation vectors of two strings S_1^a and S_2^a are opposite, the composite 2-string becomes an Abelian ANO string. It carries no non-Abelian flux. Therefore, the rotation of the $SU(2)_{C+F}$ group acts trivially on this string. This means that the internal moduli space of this string is singular (Hashimoto and Tong, 2005; Auzzi *et al.*, 2006). The section of the orientational moduli space corresponding to $S_1^a = -S_2^a$ degenerates to a point. Auzzi *et al.* (2006) argued that the internal moduli of a 2-string at zero separation is equivalent to $CP(2)/Z_2$. This differs by the discrete quotient from the result $CP(2)$ obtained by Hashimoto and Tong (2005). Results obtained by Eto, Konishi, *et al.* (2006) and Eto *et al.* (2007) confirm the $CP(2)/Z_2$ metric.

The metric on the moduli space of k-strings is not known. For Abelian k-strings, the exponential corrections to the flat metric were calculated by Manton and Speight (2003). Exponentially small corrections are natural since vortices have exponential falloff of their profile functions at large distances.

Hanany and Tong (2003, 2004) used brane construction to construct the metric of the k-string in terms of the Higgs branch of two-dimensional gauge theory; see Eqs. (4.88) and (4.124). The proposed theory is now an $\mathcal{N}=2$ supersymmetric U(k) gauge theory with N fundamental and $N_f - N$ antifundamental flavors n^ℓ ($\ell = 1, \ldots, N$) and ρ_i ($i = N, \ldots, N_f$) as well as the adjoint chiral multiplet Z. The D-term condition for this theory reads

$$\frac{1}{2\beta}[\bar{Z}, Z] + n^\ell \bar{n}_\ell - \rho_i \bar{\rho}^i = 1. \tag{4.130}$$

The metric defined by this Higgs branch has power corrections in separations between elementary strings to the factorized metric. Thus it has rather dramatic disagreement with field theory expectations. It is believed (Hanany and Tong, 2003, 2004; Tong, 2005; Eto *et al.*, 2006b) that it reproduces correctly some data protected by supersymmetry, like the BPS spectrum.

Eto *et al.* (2006c) used the so-called method of moduli matrix to extract the moduli of the general solution for the k-string. It was noticed that the last of the first-order equations (4.32) can be solved by the substitution

$$\varphi = S(z,\bar{z}) H_0(z), \quad A_1 + iA_2 = S^{-1} \bar{\partial}_z S, \tag{4.131}$$

where $z = x_1 + ix_2$ and H_0 is an $N \times N_f$ matrix with a holomorphic dependence on z. Then equations for gauge strength in Eq. (4.32) produce an equation on $S(z,\bar{z})$, which is difficult to solve in the general case. However, it was argued by Eto *et al.* (2006c) that S does not involve new moduli parameters. So all moduli parameters enter the moduli matrix $H_0(z)$. Finding $H_0(z)$ gives the moduli space, which agrees with the one given by the Higgs branch (4.130).

V. NON-ABELIAN STRINGS IN NONSUPERSYMMETRIC THEORIES

In this section, we review non-Abelian strings in nonsupersymmetric theories. We will see that although for BPS strings in supersymmetric theories the transition from quasiclassical to quantum regimes in the world-sheet theory on the string goes smoothly (see Sec. IV.D.4), for the non-Abelian strings in nonsupersymmetric theories these two regimes are separated by a phase transition.

In particular, we review works of Gorsky *et al.* (2005, 2006) that consider non-Abelian strings in nonsupersymmetric gauge theories. The theory studied by Gorsky *et al.* (2005) is the bosonic part of $\mathcal{N}=2$ supersymmetric QCD with gauge group $SU(N) \times U(1)$ described in Sec. IV in the supersymmetric setting. The action of this model has the form

$$S = \int d^4x \Bigg\{ \frac{1}{4g_2^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{g_2^2}|D_\mu a^a|^2 + |\nabla^\mu \varphi^A|^2 + \frac{g_2^2}{2}(\bar{\varphi}_A T^a \varphi^A)^2 + \frac{g_1^2}{8}(|\varphi^A|^2 - N\xi)^2 + \frac{1}{2}|(a^a T^a + \sqrt{2} m_A)\varphi^A|^2 + \frac{i\theta}{32\pi^2} F^a_{\mu\nu} F^{*a}_{\mu\nu} \Bigg\}, \tag{5.1}$$

where $F^{*a}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$. This model is the bosonic part of $\mathcal{N}=2$ supersymmetric theory (4.9) where instead of two squark fields q^k and $\tilde{q}_k$ only one fundamental scalar φ^k is introduced for each flavor $A=1,\dots,N_f$; see reduced model (4.24) in Sec. IV.B. We also consider the case $N_f=N$ and drop neutral scalar a present in Eq. (4.9) as it plays no role in string solutions. To keep the theory in the weak coupling, we consider large values of the parameter ξ in Eq. (5.1), $\xi \gg \Lambda_{\mathrm{SU}(N)}$.

We assume that

$$\sum_{A=1}^{N} m_A = 0. \tag{5.2}$$

Later on it will be convenient to make a specific choice of the parameters m_A, namely,

$$m^A = m \times \mathrm{diag}\{e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2(N-1)\pi i/N}, 1\}, \tag{5.3}$$

where m is a single common parameter, and the constraint (5.2) is automatically satisfied. We can assume m to be real and positive. We also introduce a θ term in the model (5.1).

The vacuum structure of the model (5.1) is the same as that of the theory (4.9); see Sec. IV.A. Moreover, solutions for Z_N strings are the same, given by Eq. (4.28). The adjoint field plays no role in this solution and is given by its VEV (4.13). The tensions of these strings are given classically by Eq. (4.34). However, in contrast to the case of supersymmetric theory, now tensions of Z_N strings acquire corrections in quantum theory.

If masses of the fundamental matter are zero in Eq. (5.1), this theory has unbroken $\mathrm{SU}(N)_{C+F}$ much in the same way as the theory (4.9). In this limit, Z_N strings acquire orientational zero modes and become non-Abelian. The solution for an elementary non-Abelian string is given by Eq. (4.40). We consider the two-dimensional effective low-energy theory on the worldsheet of the non-Abelian string. Its physics appears to be quite different, as compared with the one in the supersymmetric case.

A. Worldsheet theory

The derivation of the effective worldsheet theory for the non-Abelian string in the model (5.1) can be done in much the same way as for the supersymmetric case (Gorsky *et al.*, 2005), see Sec. IV.D. The worldsheet theory is now the two-dimensional nonsupersymmetric $CP(N-1)$ model (4.49). Its coupling constant β is given by the coupling constant g_2^2 of the bulk theory via Eq. (4.50). Classically, the normalization integral I is given by Eq. (4.51). Then it follows that $I=1$ as in the supersymmetric case. However, now we expect that quantum corrections modify this result. In particular, I can become a function of N in the quantum theory.

Now we discuss the impact of the θ term that is present in our bulk theory (5.1). At first glance, seemingly it cannot produce any effect because our string is magnetic. However, if one allows for slow variations of n^l with respect to z and t, one immediately observes that the electric field is generated via $A_{0,3}$ in Eq. (4.45). Substituting F_{ki} from Eq. (4.47) into the θ term in the action (5.1) and taking into account the contribution from F_{kn} times F_{ij} ($k,n=0,3$ and $i,j=1,2$), we get the topological term in the effective $CP(N-1)$ model (4.49) in the form

$$S^{(1+1)} = \int dt dz \Big\{ 2\beta[(\partial_\alpha n^* \partial_\alpha n) + (n^* \partial_\alpha n)^2] - \frac{\theta}{2\pi} I_\theta \varepsilon_{\alpha\gamma}(\partial_\alpha n^* \partial_\gamma n) \Big\}, \tag{5.4}$$

where I_θ is another normalizing integral given by

$$I_\theta = -\int dr \Big\{ 2f_{NA}(1-\rho)\frac{d\rho}{dr} + (2\rho-\rho^2)\frac{df}{dr} \Big\} = \int dr \frac{d}{dr}\{2f_{NA}\rho - \rho^2 f_{NA}\}. \tag{5.5}$$

As is seen, the integrand here reduces to a total derivative, and the integral is determined by the boundary conditions for the profile functions ρ and f_{NA}. Substituting Eqs. (4.46) and (4.48) and Eqs. (4.30) and (4.29), we get

$$I_\theta = 1, \tag{5.6}$$

independent of the profile function form. This latter circumstance is perfectly natural for the topological term.

The additional term in the $CP(N-1)$ model (5.4) just derived is the θ term in standard normalization. The result (5.6) could have been expected since the physics is 2π periodic with respect to θ both in the four-dimensional bulk gauge theory and in the two-dimensional worldsheet $CP(N-1)$ model. The result (5.6) is not sensitive to the presence of supersymmetry. It will hold in supersymmetric models as well. Note that the complex bulk coupling constant converts to the complex worldsheet coupling constant,

$$\tau = \frac{4\pi}{g_2^2} + i\frac{\theta}{2\pi} \to 2\beta + i\frac{\theta}{2\pi}.$$

Now we introduce small masses for the fundamental matter in Eq. (5.1). The diagonal color-flavor group $\mathrm{SU}(N)_{C+F}$ is now broken by adjoint VEV's down to $\mathrm{U}(1)^{N-1} \times Z_N$. The solutions for the Abelian (or Z_N) strings are the same as in Sec. IV.D.4 since the adjoint field does not enter these solutions. In particular, we have N distinct Z_N string solutions depending on what particular squark winds at infinity; see Sec. IV.D.4. The string solution with the winding last flavor is still given by Eq. (4.28).

What is changed with the color-flavor $\mathrm{SU}(N)_{C+F}$ explicitly broken by $m_A \neq 0$ is that the rotations (4.40) no longer generate zero modes. In other words, the fields n^ℓ become quasimoduli: a shallow potential (4.86) for the quasimoduli n^l on the string worldsheet is generated (Hanany and Tong, 2004; Shifman and Yung, 2004a; Gorsky *et al.*, 2005). Note that we can replace $\tilde{m}_A$ by m_A due to the condition (5.2). This potential is shallow as long as $m_A \ll \sqrt{\xi}$.

The potential simplifies if the mass terms are chosen according to Eq. (5.3),

$$V_{CP(N-1)} = 2\beta m^2 \left\{ 1 - \left| \sum_{\ell=1}^{N} e^{2\pi i \ell/N} |n^\ell|^2 \right|^2 \right\}. \tag{5.7}$$

This potential is obviously invariant under the cyclic Z_N substitution,

$$\ell \to \ell + k, \quad n^\ell \to n^{\ell+k}, \quad \forall\, \ell, \tag{5.8}$$

with k fixed. This property will be exploited below.

Now our effective two-dimensional theory on the string worldsheet becomes a massive $CP(N-1)$ model. As in the supersymmetric case, the potential (5.7) has N vacua at

$$n^\ell = \delta^{\ell\ell_0}, \quad \ell_0 = 1,2, \ldots ,N. \tag{5.9}$$

These vacua correspond to N distinct Abelian Z_N strings with $\varphi^{\ell_0\ell_0}$ winding at infinity; see Eq. (4.44).

B. Physics in the large-N limit

The massless nonsupersymmetric $CP(N-1)$ model (5.4) was solved long ago by Witten in the large-N limit (Witten, 1979b). The massive case with potential (5.7) was considered at large N by Gorsky *et al.* (2005, 2006) in relation to non-Abelian strings. Here we review this analysis.

As discussed in Sec. IV.D.4, the $CP(N-1)$ model can be understood as a strong-coupling limit of U(1) gauge theory. The action has the form

$$S = \int d^2x \left\{ 2\beta |\nabla_k n^\ell|^2 + \frac{1}{4e^2} F_{kp}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 - \frac{\theta}{2\pi} \varepsilon_{kp} \partial_k A_p + 4\beta \left| \sigma - \frac{\tilde{m}_\ell}{\sqrt{2}} \right|^2 |n^\ell|^2 + 2e^2\beta^2 (|n^\ell|^2 - 1)^2 \right\}, \tag{5.10}$$

where we also included the θ term. As in the supersymmetric case in the limit $e^2 \to \infty$, the σ field can be excluded via an algebraic equation of motion, which leads to the theory (5.4) with potential (4.86).

The Z_N cyclic symmetry (5.8) now takes the form

$$\sigma \to e^{i2\pi k/N} \sigma, \quad n^\ell \to n^{\ell+k}, \quad \forall\, \ell, \tag{5.11}$$

where k is fixed.

It turns out that the nonsupersymmetric version of the massive $CP(N-1)$ model (5.10) has two phases separated by the phase transition (Gorsky *et al.*, 2005, 2006). At large values of mass parameter m we have the Higgs phase, while at small m the theory is in the Coulomb-confining phase.

C. The Higgs phase

At large m, $m \gg \Lambda_\sigma$, the renormalization-group flow of the coupling constant β in Eq. (5.10) is frozen at the scale m. Thus, the model is at weak coupling and quasiclassical analysis is applicable. The potential (5.7) has N degenerate vacua that are labeled by the order parameter $\langle\sigma\rangle$, the vacuum configuration being

$$n^\ell = \delta^{\ell\ell_0}, \quad \sigma = \frac{\tilde{m}_{\ell_0}}{\sqrt{2}}, \quad \ell_0 = 1, \ldots ,N \tag{5.12}$$

as in the supersymmetric case; see Eq. (4.90). In each given vacuum, the Z_N symmetry (5.11) is spontaneously broken.

These vacua correspond to Abelian Z_N strings of the bulk theory. N vacua of worldsheet theory have strictly degenerate vacuum energies. From the four-dimensional point of view, this means that we have N strictly degenerate Z_N strings.

There are $2(N-1)$ elementary excitations. Here we count real degrees of freedom. The action (5.10) contains N complex fields n^ℓ. The common phase of n^{ℓ_0} is gauged away. The condition $|n^\ell|^2=1$ eliminates one more field. These elementary excitations have physical masses

$$M_\ell = |m_\ell - m_{\ell_0}|, \quad \ell \neq \ell_0. \tag{5.13}$$

There are also kinks (domain walls that are particles in two dimensions) interpolating between these vacua. Their masses scale as

$$M_\ell^{\rm kink} \sim \beta M_\ell. \tag{5.14}$$

The kinks are much heavier than elementary excitations at weak coupling. Note that they have nothing to do with Witten's n solitons (Witten, 1979b) identified as solitons at strong coupling. The point of phase transition separates these two classes of solitons.

As already discussed for the supersymmetric case (see Sec. IV.E), the flux of the Abelian 't Hooft–Polyakov monopole is equal to the difference of the fluxes of two neighboring strings, see Eq. (4.91). Therefore, the confined monopole in this regime is a junction of two distinct Z_N strings. It is seen as a quasiclassical kink interpolating between the neighboring ℓ_0th and (ℓ_0+1)th vacua of the effective massive $CP(N-1)$ model on the string worldsheet. A monopole can move freely along the string as both attached strings are tension degenerate.

D. The Coulomb-confining phase

Now we discuss the Coulomb-confining phase of the theory occurring at small m. As mentioned, at $m=0$ the $CP(N-1)$ model was solved by Witten in the large-N limit (Witten, 1979b). The model at small m is very similar to Witten's solution. (In fact, in the large-N limit it is the same.) Gorsky *et al.* (2006) presented a generalization of Witten's analysis to the massive case, which is then used to study the phase transition between the Z_N asymmetric and symmetric phases. Here we summarize Witten's results for the massless model.

The nonsupersymmetric $CP(N-1)$ model is asymptotically free (like its supersymmetric version) and devel-

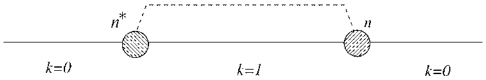

FIG. 9. Linear confinement of the $n-n^*$ pair. The solid straight line represents the string. The dashed line shows the vacuum energy density (normalizing $\mathcal{E}_0$ to zero).

ops its own scale Λ_σ. If $m=0$, classically the field n^ℓ can have an arbitrary direction; therefore, one might naively expect spontaneous breaking of SU(N) and the occurrence of massless Goldstone modes. This cannot happen in two dimensions. Quantum effects restore the full symmetry, making the vacuum unique. Moreover, the condition $|n^\ell|^2=1$ becomes in effect relaxed. Due to strong coupling, we have more degrees of freedom than in the original Lagrangian, namely all N fields n become dynamical and acquire masses Λ_σ.

This is not the end of the story, however. In addition, one gets another composite degree of freedom. The U(1) gauge field A_k acquires a standard kinetic term at the one-loop level,[22] of the form

$$N\Lambda^{-2}F_{kp}F_{kp}. \tag{5.15}$$

Comparing Eq. (5.15) with Eq. (5.10), we see that the charge of the n fields with respect to this photon is $1/\sqrt{N}$. The Coulomb potential between two charges in two dimensions is linear in separation between these charges. The linear potential scales as

$$V(R) \sim \frac{\Lambda_\sigma^2}{N}R, \tag{5.16}$$

where R is separation. The force is attractive for pairs $\bar{n}$ and n, leading to the formation of weakly coupled bound states (weak coupling is the manifestation of the $1/N$ suppression of the confining potential). Charged states are eliminated from the spectrum. This is the reason why the n fields were called quarks by Witten. The spectrum of the theory consists of $\bar{n}n$ mesons. The picture of confinement of n's is shown in Fig. 9.

The validity of the above consideration rests on large N. If N is not large, the solution (Witten, 1979b) ceases to be applicable. It remains valid in the qualitative sense, however. Indeed, at $N=2$ the model was solved exactly (Zamolodchikov and Zamolodchikov, 1979, 1992) [see also Coleman (1976)]. Zamolodchikovs found that the spectrum of the O(3) model consists of a triplet of degenerate states (with mass $\sim\Lambda_\sigma$). At $N=2$, the action (5.10) is built of doublets. In this sense, one can say that Zamolodchikovs's solution exhibits confinement of doublets. This is in qualitative accord with the large-N solution (Witten, 1979b).

Inside the $\bar{n}n$ mesons, we have a constant electric field; see Fig. 9. Therefore, the spatial interval between $\bar{n}$ and n has a higher energy density than the domains outside the meson.

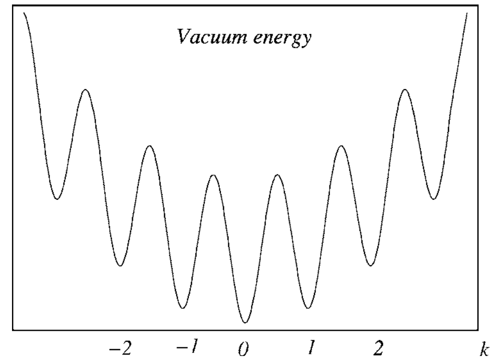

FIG. 10. The vacuum structure of the $CP(N-1)$ model at $\theta=0$.

Modern understanding of the vacuum structure of the massless $CP(N-1)$ model (Witten, 1998) [see also Shifman (1998)] allows one to reinterpret confining dynamics of the n fields in different terms (Gorsky *et al.*, 2005; Markov *et al.*, 2005). Indeed, at large N, along with the unique ground state, the model has $\sim N$ quasistable local minima, quasivacua, which become absolutely stable at $N=\infty$. The relative splittings between the values of the energy density in the adjacent minima are of the order of $1/N$, while the probability of the false vacuum decay is proportional to $N^{-1}\exp(-N)$ (Shifman, 1998; Witten, 1998). The n quanta (n quarks solitons) interpolate between the adjacent minima.

The existence of a large family of quasivacua can be inferred from studying the θ evolution of the theory. Consider the topological susceptibility, i.e., the correlation function of two topological densities

$$\int d^2x\langle Q(x), \quad Q(0)\rangle, \tag{5.17}$$

where

$$Q = \frac{i}{2\pi}\varepsilon_{kp}\partial_k A_p = \frac{1}{2\pi}\varepsilon_{kp}(\partial_k n^*_\ell \partial_p n^\ell). \tag{5.18}$$

The correlation function (5.17) is proportional to the second derivative of the vacuum energy with respect to the θ angle. From Eq. (5.18), it is not difficult to deduce that this correlation function scales as $1/N$ in the large-N limit. The vacuum energy by itself scales as N. Thus, we conclude that, in fact, the vacuum energy should be a function of θ/N.

On the other hand, the vacuum energy must be a 2π-periodic function of θ. These two requirements are seemingly self-contradictory. A way to reconcile the above facts is as follows. Assume that we have a family of quasivacua with energies

$$E_k(\theta) \sim N\Lambda_\sigma^2\left\{1 + \text{const}\times\left(\frac{2\pi k+\theta}{N}\right)^2\right\}, \tag{5.19}$$

where $k=0,\ldots,N-1$. A schematic picture of these vacua is given in Fig. 10. All these minima are entangled in the

[22]By loops here we mean perturbative expansion in $1/N$ perturbation theory.

θ evolution. If we vary θ continuously from 0 to 2π, the depths of the minima "breathe." At $\theta = \pi$, two vacua become degenerate, while for larger values of θ the former global minimum becomes local while the adjacent local minimum becomes global. It is obvious that for the neighboring vacua that are not too far from the global minimum,

$$E_{k+1} - E_k \sim \frac{\Lambda_\sigma^2}{N}. \tag{5.20}$$

This is also the confining force acting between n and $\bar{n}$.

One could introduce order parameters that would distinguish between distinct vacua from the vacuum family. An obvious choice is the expectation value of the topological charge. The kinks n^ℓ interpolate, say, between the global minimum and the first local one on the right-hand side. Then $\bar{n}$'s interpolate between the first local minimum and the global one. Note that the vacuum energy splitting is an effect suppressed by $1/N$. At the same time, kinks have masses that scale as N^0,

$$M_\ell^{\text{kink}} \sim \Lambda_\sigma. \tag{5.21}$$

The multiplicity of such kinks is N (Acharya and Vafa, 2001); they form an N-plet of SU(N). This is in full accord with the fact that the large-N solution of Eq. (5.10) exhibits N quanta of the complex field n^ℓ.

Thus we see that the $CP(N-1)$ model has a fine structure of vacua that are split, with the splitting of the order of Λ_σ^2/N. In four-dimensional bulk theory, these vacua correspond to elementary non-Abelian strings. Classically, these strings have the same tension (4.34). Due to quantum effects in the worldsheet theory, the degeneracy is lifted: the elementary strings become split, with tensions

$$T = 2\pi\xi + c_1 N \Lambda_\sigma^2 \left\{ 1 + c_2 \left(\frac{2\pi k + \theta}{N} \right)^2 \right\}, \tag{5.22}$$

where c_1 and c_2 are numerical coefficients. Note that (i) the splitting does not appear to any finite order in the coupling constants; (ii) since $\xi \gg \Lambda_\sigma$, the splitting is suppressed in both parameters $\Lambda_\sigma/\sqrt{\xi}$ and $1/N$.

Kinks of the worldsheet theory represent confined monopoles (string junctions) in the four-dimensional bulk theory. Therefore, kink confinement in the $CP(N-1)$ model can be interpreted as follows (Gorsky *et al.*, 2005; Markov *et al.*, 2005). The non-Abelian monopoles, in addition to the four-dimensional confinement (which ensures that monopoles are attached to the strings), acquire a two-dimensional confinement along the string: a monopole-antimonopole forms a mesonlike configuration, with necessity; see Fig. 9.

In summary, the $CP(N-1)$ model in the Coulomb-confining phase, at small m, has a vacuum family with a fine structure. For each given θ (except $\theta = \pi, 3\pi$, etc.) the true ground state is unique, but there are a large number of almost degenerate ground states. The Z_N symmetry is unbroken. The classical condition (4.43) is

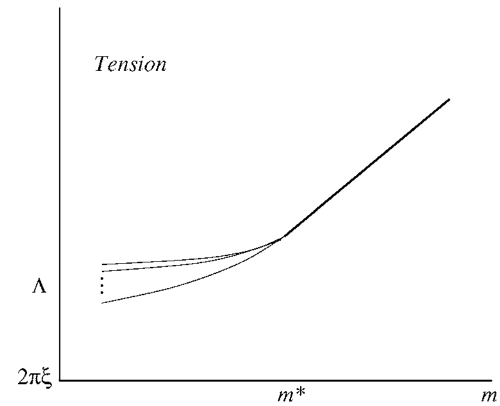

FIG. 11. Schematic dependence of string tensions on the mass parameter m. At small m in the non-Abelian confinement phase the tensions are split, while in the Abelian confinement phase at large m they are degenerative.

replaced by $\langle n^\ell \rangle = 0$. The spectrum of physically observable states consists of kink-antikink mesons that form the adjoint representation of SU(N).

Instead, at large m the theory is in the Higgs phase; it has N strictly degenerate vacua (5.12); the Z_N symmetry is broken. We have $N-1$ elementary excitations n^ℓ with masses given by Eq. (5.13). Thus we conclude that these two regimes should be separated by a phase transition at some critical value m_* (Gorsky *et al.*, 2005, 2006). This phase transition is associated with the Z_N symmetry breaking: in the Higgs phase, the Z_N symmetry is spontaneously broken, while in the Coulomb phase it is restored. For $N=2$, we deal with Z_2, which makes the situation akin to the Ising model.

In the worldsheet theory, this is a phase transition between the Higgs and Coulomb-confining phase. In the bulk theory, it can be interpreted as a phase transition between the Abelian and non-Abelian confinement. In the Abelian confinement phase at large m, the Z_N symmetry is spontaneously broken, all N strings are strictly degenerate, and there is no two-dimensional confinement of the 4D-confined monopoles. Instead, in the non-Abelian confinement phase occurring at small m, the Z_N symmetry is fully restored, all N elementary strings are split, and the 4D-confined monopoles combine with antimonopoles to form a mesonlike configuration on the string; see Fig. 9. We show schematically the dependence of the string tensions on m in these two phases in Fig. 11.

Gorsky *et al.* (2006) found the phase-transition point using large-N methods developed by Witten (1979b). It turns out that the critical point is

$$m_* = \Lambda_\sigma. \tag{5.23}$$

The vacuum energy is calculated in both phases and is shown to be continuous at the critical point. If one approaches the critical point, say from the Higgs phase, some composite states of the worldsheet theory (5.10) such as photons as well as kinks become light. One is tempted to believe that these states become massless at the critical point (5.23). However, this happen only in the very narrow vicinity of the phase transition point where the $1/N$ expansion fails. Thus the large-N approximation is not powerful enough to determine the critical behavior.

To conclude this section, we note that we encounter a crucial difference between the non-Abelian confinement in supersymmetric and nonsupersymmetric gauge theories. For BPS strings in supersymmetric theories, we do not have a phase transition separating the phase of the non-Abelian strings from that of the Abelian strings (Hanany and Tong, 2004; Shifman and Yung, 2004a). Even for small values of the mass parameters, supersymmetric theory strings are strictly degenerate, and the Z_N symmetry is spontaneously broken. In particular, at $m_A=0$ the order parameter for the broken Z_N, which differentiates N degenerate vacua of the supersymmetric $CP(N-1)$ model, is the bifermion condensate of two-dimensional fermions living on the string worldsheet of the non-Abelian BPS string; see Sec. IV.D.3.

Moreover, the presence of the phase transition between Abelian and non-Abelian confinement in nonsupersymmetric theories suggests a solution for the problem of enrichment of the hadronic spectrum mentioned in Sec. IV. In the phase of Abelian confinement, we have N strictly degenerative Abelian Z_N strings that give rise to too many hadron states, not present in the real world QCD. Therefore, Abelian Z_N strings can hardly play a role of prototypes of QCD confining strings. Although BPS strings in supersymmetric theories become non-Abelian as we turn mass parameters m_A to a common value, still there are N strictly degenerative non-Abelian strings and therefore still too many hadron states in the spectrum.

As reviewed in this section, the situation in nonsupersymmetric theories is quite different. As we make mass parameters m_A equal, we enter the non-Abelian confinement phase. In this phase, N elementary non-Abelian strings are split. At $\theta=0$, we have only one lightest elementary string producing a single two-particle meson with given flavor quantum numbers and spin, exactly as observed in nature. If N is large, the splitting is small, however if N is not so large, the splitting is of order of Λ_σ^2. Therefore, mesons produced by excited strings are unstable and may appear invisible experimentally.

VI. DOMAIN WALLS AS D-BRANE PROTOTYPES

D-branes are extended objects in string theory on which strings can end (Polchinski, 1995). Moreover, the gauge fields are the lowest excitations of open superstrings, with the end points attached to D-branes. SU(N) gauge theories are obtained as a field-theoretic reduction of a string theory on the worldvolume of a stack of N D-branes.

In recent years, solitonic objects of the domain wall and string type were extensively studied in supersymmetric gauge theories in 1+3 dimensions. First, it was observed (Dvali and Shifman, 1997) that there should exist critical (BPS-saturated) domain walls in $\mathcal{N}=1$ gluodynamics, with the tension scaling as $N\Lambda^3$. Here Λ is the scale parameter. The peculiar N dependence of the tension prompted (Witten, 1997) a D-brane interpretation of such walls. Ideas as to how flux tubes can end on the BPS walls were analyzed (Kogan *et al.*, 1998) at the qualitative level shortly thereafter. Later on, BPS-saturated domain walls and their junctions with strings were discussed (Abraham and Townsend, 1992a, 1992b; Gauntlett *et al.*, 2001) in a more quantitative aspect in $\mathcal{N}=2$ sigma models. Some remarkable parallels between field-theoretical critical solitons and the D-brane construction were discovered.

In this and the next sections, we review the parallel found between field-theoretical BPS domain walls in gauge theories and D-branes/strings. In other words, we discuss BPS domain walls that localize gauge fields on their worldvolume, in this sense becoming a D-brane prototypes in field theory.

Research on field-theoretic mechanisms of gauge field localization on the domain walls has attracted much attention in recent years. The only viable mechanism of gauge field localization was put forward by Dvali and Shifman (1997), where it was noted that if a gauge field is confined in the bulk and is unconfined (or less confined) on the brane, this naturally gives rise to a gauge field on the wall [for further developments, see Dubovsky and Rubakov (2001) and Dvali and Vilenkin (2003)]. Although this idea seems easy to implement, in fact it requires a careful consideration of quantum effects (confinement is certainly such an effect), which is hard to do at strong coupling. Building on these initial proposals, models with localization of gauge fields on the worldvolume of domain walls at weak coupling in $\mathcal{N}=2$ supersymmetric gauge theories were suggested by Shifman and Yung (2003, 2004a) and Sakai and Tong (2005). The basic idea is that the gauge group is completely Higgsed in the bulk while inside the wall charged scalar fields are almost zero and gauge fields can propagate freely. Then the dual field lives on the wall. Shifman and Yung (2003) considered domain walls in the simplest $\mathcal{N}=2$ QED theory, while Shifman and Yung (2004a) and Sakai and Tong (2005) dealt with domain walls in non-Abelian $\mathcal{N}=2$ gauge theories (4.9) with gauge group U(N). We review the results obtained in these papers.

The moduli space of multiple domain walls in $\mathcal{N}=2$ supersymmetric gauge theories and sigma models were studied by Gauntlett *et al.* (2001b), Tong (2002), Isozumi *et al.* (2004a, 2004b), Eto *et al.* (2005a), and Sakai and Yang (2006). Note also that domain walls can intersect (Gauntlett, 2001a; Kakimoto and Sakai, 2003; Eto *et al.*, 2007). In particular, in Eto *et al.* (2005b, 2006a) obtained honeycomb webs of walls in Abelian and non-Abelian gauge theories, respectively.

We start by discussing BPS domain walls as D-brane prototypes first in the simplest Abelian theory, $\mathcal{N}=2$ SQED with two flavors (Shifman and Yung, 2003). It supports both the BPS-saturated domain walls and the

BPS-saturated ANO strings if the Fayet-Iliopoulos term is added to the theory.

A. $\mathcal{N}=2$ supersymmetric QED

The field content of $\mathcal{N}=2$ supersymmetric QED consists of a U(1) vector $\mathcal{N}=2$ multiplet as well as N_f matter hypermultiplets. The bosonic part of the action of this theory is

$$S=\int d^4x\left\{\frac{1}{4g^2}F^2_{\mu\nu}+\frac{1}{g^2}|\partial_\mu a|^2+\bar{\nabla}_\mu\bar{q}_A\nabla_\mu q^A+\bar{\nabla}_\mu\tilde{q}_A\nabla_\mu\bar{\tilde{q}}^A+\frac{g^2}{8}(|q^A|^2-|\tilde{q}_A|^2-\xi)+\frac{g^2}{2}|\tilde{q}_A q^A|^2+\frac{1}{2}(|q^A|^2+|\tilde{q}^A|^2)|a+\sqrt{2}m_A|^2\right\}, \tag{6.1}$$

where

$$\nabla_\mu=\partial_\mu-\frac{i}{2}A_\mu,\quad \bar{\nabla}_\mu=\partial_\mu+\frac{i}{2}A_\mu. \tag{6.2}$$

Here ξ is the coefficient in front of the Fayet-Iliopoulos term; we consider the FI D term here while g is the U(1) gauge coupling. The index $A=1,\ldots,N_f$ is the flavor index. In this section we consider the case $N_f=2$. This is the simplest case that admits a domain wall interpolating between quark vacua.

The mass parameters m_1,m_2 are assumed to be real. In addition, we assume

$$\Delta m\equiv m_1-m_2\gg g\sqrt{\xi}. \tag{6.3}$$

Simultaneously, $\Delta m\ll(m_1+m_2)/2$. There are two vacua in this theory: in the first vacuum,

$$a=-\sqrt{2}m_1,\quad q_1=\sqrt{\xi},\quad q_2=0; \tag{6.4}$$

and in the second one,

$$a=-\sqrt{2}m_2,\quad q_1=0,\quad q_2=\sqrt{\xi}. \tag{6.5}$$

The VEV of the field $\tilde{q}$ vanishes in both vacua. Hereafter in search for domain-wall solutions, we stick to the ansatz $\tilde{q}=0$.

Now we discuss the mass spectrum of light fields in both quark vacua. Consider for definiteness the first vacuum, Eq. (6.4). The spectrum can be obtained by diagonalizing the quadratic form in Eq. (6.1). This was done by Vainshtein and Yung (2001); the result is as follows: one real component of field q^1 is eaten up by the Higgs mechanism to become the third components of the massive photon. Three components of the massive photon, one remaining component of q^1, and four real components of the fields $\tilde{q}_1$ and a form one long $\mathcal{N}=2$ multiplet (8 boson states+8 fermion states), with mass

$$m_\gamma^2=\tfrac{1}{2}g^2\xi. \tag{6.6}$$

The second flavor q^2, $\tilde{q}_2$ (which does not condense in this vacuum), forms one short $\mathcal{N}=2$ multiplet (4 boson states+4 fermion states), with mass Δm, which is heavier than the mass of the vector supermultiplet. The latter assertion applies to the regime (6.3). In the second vacuum, the mass spectrum is similar—the roles of the first and second flavors are interchanged.

If we consider the limit opposite to that in Eq. (6.3) and set $\Delta m\to 0$, the photonic supermultiplet becomes heavier than that of q^2, the second flavor field. Therefore, it can be integrated out, leaving us with the theory of massless moduli from q^2, which interact through a nonlinear sigma model with the Kähler term corresponding to the Eguchi-Hanson metric. Domain walls in this sigma model limit were considered by Gauntlett *et al.* (2001, 2001b) and Tong (2002).

B. Domain walls in $\mathcal{N}=2$ QED

A BPS domain wall interpolating between the two vacua of the bulk theory (6.1) was explicitly constructed by Shifman and Yung (2003). Assuming that all fields depend only on the coordinate $z=x_3$, it is possible to write the energy in the Bogomol'nyi form (Bogomol'nyi, 1976),

$$E=\int dx_3\left\{\left|\nabla_3 q^A\pm\frac{1}{\sqrt{2}}q^A(a+\sqrt{2}m_A)\right|^2+\left|\frac{1}{g}\partial_3 a\pm\frac{g}{2\sqrt{2}}(|q^A|^2-\xi)\right|^2\pm\frac{1}{\sqrt{2}}\xi\partial_3 a\right\}. \tag{6.7}$$

Requiring the first two terms above to vanish gives us the BPS equations for the wall. Assuming that $\Delta m>0$, we choose the upper sign in Eq. (6.7) and obtain

$$\nabla_z q^A=-\frac{1}{\sqrt{2}}q^A(a+\sqrt{2}m_A),$$

$$\partial_z a=-\frac{g^2}{2\sqrt{2}}(|q^A|^2-\xi). \tag{6.8}$$

These first-order equations should be supplemented by the following boundary conditions:

$$q^1(-\infty)=\sqrt{\xi},\quad q^2(-\infty)=0,\quad a(-\infty)=-\sqrt{2}m_1;$$

$$q^1(\infty)=0,\quad |q^2(\infty)|=\sqrt{\xi},\quad a(\infty)=-\sqrt{2}m_2, \tag{6.9}$$

which show that our wall interpolates between the two quark vacua. Here we use U(1) gauge rotation to make q^1 in the left vacuum real.

The tension is given by the total derivative term [the last term in Eq. (6.7)], which can be identified as the (1,0) central charge of the supersymmetry algebra,

$$T_w=\xi\Delta m. \tag{6.10}$$

Now we work out the solution to the first-order equations (6.8), assuming the conditions (6.9) to be satisfied. The range of variation of the field a inside the wall is of the order of Δm [see Eq. (6.9)]. The minimization of its kinetic energy implies this field to be slowly varying. Therefore, we may safely assume that the wall is thick; its size $R\gg 1/g\sqrt{\xi}$. This fact will be confirmed shortly.

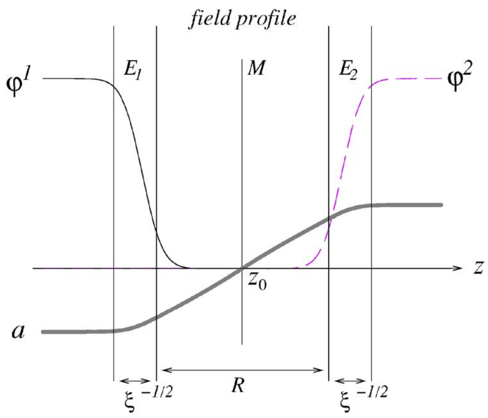

FIG. 12. (Color online) Internal structure of the domain wall: two edges (domains $E_{1,2}$) of the width $\sim(g\sqrt{\xi})^{-1}$ are separated by a broad middle band (domain M) of the width R; see Eq. (6.13).

We arrive at the following picture for the domain wall. The wall solution has a three-layer structure (Shifman and Yung, 2003); see Fig. 12: in the two outer layers (which have width $O[(g\sqrt{\xi})^{-1}]$), the squark fields drop to zero exponentially; in the inner layer, the field a interpolates between its two vacuum values.

Then to the leading order we can set the quark fields to zero in Eqs. (6.8) inside the inner layer. The second equation in Eqs. (6.8) tells us that a is a linear function of z. The solution for a takes the form

$$a = -\sqrt{2}\left(m - \Delta m\frac{z - z_0}{R}\right), \tag{6.11}$$

where the collective coordinate z_0 is the position of the wall center (and Δm is assumed positive). The solution is valid in a wide domain of z,

$$|z - z_0| < \frac{R}{2}, \tag{6.12}$$

except narrow areas of size $\sim 1/g\sqrt{\xi}$ near the edges of the wall at $z - z_0 = \pm R/2$.

Substituting the solution (6.11) into the second equation in Eqs. (6.8), we get

$$R = \frac{4\Delta m}{g^2\xi} = \frac{2\Delta m}{m_\gamma^2}. \tag{6.13}$$

Since $\Delta m/g\sqrt{\xi} \gg 1$ [see Eq. (6.3)], this result shows that $R \gg 1/g\sqrt{\xi}$, which justifies our approximation.

Furthermore, we can now use the first relation in Eqs. (6.8) to determine tails of the quark fields inside the wall. As mentioned above, we fix the gauge imposing the condition that q^1 is real at $z \to -\infty$; see the more detailed discussion by Shifman and Yung (2003).

Consider first the left edge (domain E_1 in Fig. 12) at $z - z_0 = -R/2$. Substituting the above solution for a in the equation for q^1, we get

$$q^1 = \sqrt{\xi}e^{-(m_\gamma^2/4)(z - z_0 + R/2)^2}, \tag{6.14}$$

where m_γ is given by Eq. (6.6). This behavior is valid in the domain M, at $(z - z_0 + R/2) \gg 1/g\sqrt{\xi}$, and shows that the field of the first quark flavor tends to zero exponentially inside the wall, as expected.

By the same token, we can consider the behavior of the second quark flavor near the right edge of the wall at $z - z_0 = R/2$. The first equation in Eqs. (6.8) for $A = 2$ implies

$$q^2 = \sqrt{\xi}e^{-(m_\gamma^2/4)(z - z_0 - R/2)^2 - i\sigma}, \tag{6.15}$$

which is valid in the domain M provided that $(R/2 - z + z_0) \gg 1/g\sqrt{\xi}$. Here σ is an arbitrary phase that cannot be gauged away. Inside the wall, the second quark flavor tends to zero exponentially too.

It is not difficult to check that the main contribution to the wall tension comes from the middle layer while edge domains produce contributions of the order of $\xi^{3/2}$, which makes them negligibly small.

Now we comment on the phase factor in Eq. (6.15). Its origin is as follows (Shifman and Yung, 2003). The bulk theory at $\Delta m \neq 0$ has U(1)×U(1) flavor symmetry corresponding to two independent rotations of two quark flavors. In both vacua, only one quark develops a VEV. Therefore, in both vacua only one of these two U(1)'s is broken. The corresponding phase is eaten by the Higgs mechanism. However, on the wall both quarks have nonvanishing values, breaking both U(1) groups. Only one of the corresponding two phases is eaten by the Higgs mechanism. The other one becomes a Goldstone mode living on the wall.

Thus we have two collective coordinates characterizing our wall solution, namely, the position of the center z_0 and the phase σ. In the effective low-energy theory on the wall, they become scalar fields of the worldvolume (2+1)-dimensional theory $z_0(t,x,y)$ and $\sigma(t,x,y)$, respectively. The target space of the second field is S_1.

This wall is a 1/2-BPS solution of the Bogomol'nyi equations. In other words, the soliton breaks four of eight supersymmetry generators of the $\mathcal{N}=2$ bulk theory. In fact, as was shown by Shifman and Yung (2003), the four supercharges selected by the conditions

$$\bar{\varepsilon}^2_{\dot{2}} = -i\varepsilon^{21}, \quad \bar{\varepsilon}^1_{\dot{2}} = -i\varepsilon^{22},$$

$$\bar{\varepsilon}^1_{\dot{1}} = i\varepsilon^{12}, \quad \bar{\varepsilon}^2_{\dot{1}} = i\varepsilon^{11} \tag{6.16}$$

act trivially on the wall solution. They become four supersymmetries acting in the (2+1)-dimensional effective worldvolume theory on the wall. Here $\varepsilon^{\alpha f}$ and $\bar{\varepsilon}^f_{\dot{\alpha}}$ are eight supertransformation parameters.

C. Effective field theory on the wall

In this subsection we review the (2+1)-dimensional effective low-energy theory of the moduli on the wall (Shifman and Yung, 2003). To do so, we make the wall collective coordinates z_0 and σ (together with their fermionic superpartners) slowly varying fields depending on x_n ($n=0,1,2$). For simplicity, we consider the bosonic fields $z_0(x_n)$ and $\sigma(x_n)$; the residual supersymmetry will allow us to readily reconstruct the fermion part of the effective action.

Because $z_0(x_n)$ and $\sigma(x_n)$ correspond to zero modes of the wall, they have no potential terms in the worldsheet theory. Therefore, in fact, our task is to derive kinetic terms, in much the same way as was done for strings; see Sec. IV.D. For $z_0(x_n)$, this procedure is very simple. Substituting the wall solution (6.11), (6.14), and (6.15) into the action (6.7) and taking into account the x_n dependence of this modulus, we get

$$\frac{T_w}{2}\int d^3x(\partial_n z_0)^2. \tag{6.17}$$

As far as the kinetic term for $\sigma(x_n)$ is concerned, more effort is needed. We start from Eqs. (6.14) and (6.15) for the quark fields. Then we have to modify our ansatz, introducing n components for the gauge field,

$$A_n = \chi(z)\partial_n\sigma(x_n). \tag{6.18}$$

We have introduced an extra profile function $\chi(z)$. It has no role in the construction of the static wall solution *per se*. It is unavoidable, however, in constructing the kinetic part of the worldsheet theory of the moduli. This new profile function is described by its own action, which will be subject to minimization. This is quite similar to the procedure of derivation of the worldsheet effective theory for non-Abelian string; see Sec. IV.D.

The gauge potential in Eq. (6.18) is not pure gauge. It does lead to a nonvanishing field strength. It is introduced in order to cancel the x dependence of the quark fields far from the wall (in the quark vacua at $z\to\infty$) emerging through the x dependence of $\sigma(x_n)$; see Eq. (6.15).

To ensure this cancellation, we impose the following boundary conditions for the function $\chi(z)$:[23]

$$\chi(z)\to 0, \quad z\to -\infty,$$

$$\chi(z)\to -2, \quad z\to +\infty. \tag{6.19}$$

Next, substituting Eqs. (6.14), (6.15), and (6.18) into the action (6.7), we arrive at

$$S^{\sigma}_{2+1} = \left[\int d^3x\frac{1}{2}(\partial_n\sigma)^2\right]\int dz\left\{\frac{1}{g^2}(\partial_z\chi)^2 + \chi^2|q^1|^2 + (2+\chi)^2|q^2|^2\right\}. \tag{6.20}$$

The expression in the integral is an action for the χ profile function.

Now to find the function χ, we have to minimize Eq. (6.20) with respect to χ. This gives

$$-\partial_z^2\chi + g^2\chi|q^1|^2 + g^2(2+\chi)|q^2|^2 = 0. \tag{6.21}$$

Note that the equation for χ is of the second order. This is because the domain wall is no longer a BPS state once we switch on the dependence of the moduli on the longitudinal variables x_n.

To the leading order in $g\sqrt{\xi}/\Delta m$, the solution of Eq. (6.21) can be obtained in the same manner as was done previously for other profile functions. We first discuss what happens outside the inner part of the wall. At $z-z_0\gg R/2$ the profile $|q^1|$ vanishes while $|q^2|$ is exponentially close to $\sqrt{\xi}$ and, hence,

$$\chi\to -2 + \text{const}\times e^{-m_\gamma(z-z_0)}. \tag{6.22}$$

At $z_0-z\gg R/2$, χ falls off exponentially to zero. Thus, outside the inner part of the wall, at $|z-z_0|\gg R/2$, the function χ approaches its boundary values with the exponential rate of approach.

Of most interest, however, is the inside part, the middle domain M (see Fig. 12). Here both quark profile functions vanish, and Eq. (6.21) degenerates into $\partial_z^2\chi=0$. As a result, the solution takes the form

$$\chi = -1 - 2\frac{z-z_0}{R}. \tag{6.23}$$

In narrow edge domains $E_{1,2}$, the exact χ profile smoothly interpolates between the boundary values [see Eq. (6.22)] and the linear behavior (6.23) inside the wall. These edge domains give small corrections to the leading term in the action.

Substituting the solution (6.23) into the χ action, in Eq. (6.20), we arrive at

$$S^{\sigma}_{2+1} = \frac{\xi}{\Delta m}\int d^3x\tfrac{1}{2}(\partial_n\sigma)^2. \tag{6.24}$$

As is well known (Polyakov, 1977), the compact scalar field $\sigma(t,x,y)$ can be reinterpreted to be dual to the (2+1)-dimensional Abelian gauge field living on the wall. The emergence of the gauge field on the wall is easy to understand. The quark fields almost vanish inside the wall. Therefore, the U(1) gauge group is restored inside the wall while it is Higgsed in the bulk. The dual U(1) is in the confinement regime in the bulk. Hence, the dual U(1) gauge field is localized on the wall, in full accordance with the general argument of Dvali and Shifman (1977). The compact scalar field $\sigma(x_n)$ living on the wall is a manifestation of this magnetic localization.

The result in Eq. (6.24) implies that the coupling constant of our effective U(1) theory on the wall is given by

$$e^2 = 4\pi^2\frac{\xi}{\Delta m}. \tag{6.25}$$

In particular, the definition of the (2+1)-dimensional gauge field takes the form

[23] Remember the electric charge of the quark fields is $\pm 1/2$.

$$F^{(2+1)}_{nm} = \frac{e^2}{2\pi}\varepsilon_{nmk}\partial^k\sigma. \tag{6.26}$$

This finally leads us to the following bosonic effective low-energy theory of the moduli fields on the wall:

$$S_{2+1} = \int d^3x\left\{\frac{T_w}{2}(\partial_n z_0)^2 + \frac{1}{4e^2}(F^{(2+1)}_{nm})^2\right\}. \tag{6.27}$$

The fermion content of the worldvolume theory is given by two three-dimensional Majorana spinors, as is required by $\mathcal{N}=2$ in three dimensions [four supercharges, see Eq. (6.26)]. The full worldvolume theory is a U(1) gauge theory in (2+1) dimensions, with four supercharges. The Lagrangian and corresponding superalgebra can be obtained by reducing four-dimensional $\mathcal{N}=1$ SQED (with no matter) to three dimensions.

The field z_0 in Eq. (6.27) is the $\mathcal{N}=2$ superpartner of the gauge field A_n. To make it more transparent we make a rescaling, introducing a new field

$$a_{2+1} = 2\pi\xi z_0. \tag{6.28}$$

In terms of a_{2+1}, the action (6.27) takes the form

$$S_{2+1} = \int d^3x\left\{\frac{1}{2e^2}(\partial_n a_{2+1})^2 + \frac{1}{4e^2}(F^{(2+1)}_{mn})^2\right\}. \tag{6.29}$$

The gauge coupling constant e^2 has dimension of mass in three dimensions. A characteristic scale of massive excitations on the worldvolume theory is of the order of the inverse thickness of the wall $1/R$; see Eq. (6.13). Thus the dimensionless parameter that characterizes the coupling strength in the worldvolume theory is e^2R,

$$e^2R = \frac{16\pi^2}{g^2}. \tag{6.30}$$

This can be interpreted as a feature of the bulk-wall duality: the weak-coupling regime in the bulk theory corresponds to strong coupling on the wall and vice versa (Shifman and Yung, 2003, 2006b). Of course, finding explicit domain-wall solutions and deriving the effective theory on the wall assumes a weak-coupling regime in the bulk, $g^2 \ll 1$. In this limit, the worldvolume theory is in the strong-coupling regime and is not very useful.

The fact that each domain wall has two bosonic collective coordinates—its center and the phase—in the sigma model limit was noted by Abraham and Townsend (1992a, 1992b) and Tong (2002).

To summarize, we showed that the worldvolume theory on the domain wall is $\mathcal{N}=2$ U(1) gauge theory (6.29). Thus the domain wall in the theory (6.1) presents an example of D-brane in field theory—it localizes a gauge field on its worldvolume. In string theory, gauge fields are localized on D-branes because fundamental open strings can end on a D-brane. It turns out that this is also true for field theory D-branes. We postpone to the next section reviewing 1/4-BPS solutions found for junctions of field theory strings (flux tubes) with domain walls (Gauntlett *et al.*, 2001; Shifman and Yung, 2003, 2004a). Below we consider generalizations of the effect of localization of gauge fields on a domain-wall worldvolume to non-Abelian theories.

D. Domain walls in U(N) gauge theories

In this subsection, we review domain walls in $\mathcal{N}=2$ QCD (4.9) with gauge group U(N). We assume that the number of quark flavors is $N_f > N$, so the theory has many vacua of type (4.13) and (4.16) depending on which N quarks out of N_f develop VEV's. We can denote different vacua as $(A_1, A_2, \ldots, A_N)$ specifying which quark flavors develop VEV's. First we assume that all quark masses are generically different.

1. Nondegenerate masses

We arrange quark masses as follows:

$$m_1 > m_2 > \cdots > m_{N_f}. \tag{6.31}$$

In this case, the theory (4.9) has

$$\frac{N_f!}{N!(N_f - N)!} \tag{6.32}$$

isolated vacua.

Domain walls interpolating between these vacua were classified by Sakai and Tong (2005). Below we review this classification. Bogomol'nyi representation of the action (4.9) leads to the first-order equations for wall solutions (Lambert and Tong, 2000), see also Shifman and Yung (2004a),

$$\partial_z\varphi^A = -\frac{1}{\sqrt{2}}(a_a\tau^a + a + \sqrt{2}m_A)\varphi^A,$$

$$\partial_z a^a = -\frac{g_2^2}{2\sqrt{2}}(\bar{\varphi}_A\tau^a\varphi^A),$$

$$\partial_z a = -\frac{g_1^2}{2\sqrt{2}}(|\varphi^A|^2 - 2\xi), \tag{6.33}$$

where we used ansatz (4.23) and introduce a single quark field φ^{kA} instead of two fields q^{kA} and $\tilde{q}_{Ak}$.

Tensions of the walls satisfying the above equations are given by the surface term

$$T_w = \sqrt{2}\xi\int dz\,\partial_z a. \tag{6.34}$$

They can be written as (Sakai and Tong, 2005)

$$T_w = \xi\vec{g}\vec{m}, \tag{6.35}$$

where we use Eq. (4.13) and $\vec{m} = (m_1, \ldots, m_{N_f})$ while

$$\vec{g} = \sum_{i=1}^{N_f} k_i\alpha_i. \tag{6.36}$$

Here k_i are integers while α_i are simple roots of SU(N_{N_f}) algebra,

$$\alpha_1 = (1,-1,0,\dots,0),$$

$$\alpha_2 = (0,1,-1,\dots,0),$$

$$\alpha_{N_f-1} = (0,\dots,0,1,-1). \tag{6.37}$$

Elementary walls arise when only one of k_i is equal to 1 while all others are zero. Their tensions are

$$T_w^i = \xi(m_i - m_{i+1}). \tag{6.38}$$

The ith elementary wall interpolates between vacua $(\dots,i,\dots)$ and $(\dots,i+1,\dots)$. All other walls can be considered as composite states of elementary walls.

Consider an example of the theory (4.9) with U(2) gauge group and N_f=4. Explicit solutions for elementary walls in the limit

$$(m_i - m_{i+1}) \gg g\sqrt{\xi} \tag{6.39}$$

were obtained by Shifman and Yung (2004a). They have the same three-layer structure as in the Abelian case; see Sec. VI.B. The elementary wall interpolating between vacua (1,2) and (1,3) has the following structure. In the left domain, quark φ^2 goes from its VEV $\sqrt{\xi}$ to zero exponentially, while in the right domain quark φ^3 goes from zero to its VEV $\sqrt{\xi}$. In the broad middle domain, fields a and a^3 linearly interpolate between their VEV's in two vacua. The novel feature of the domain-wall solution as compared to the Abelian case (see Sec. VI.B) is that quark q^1 is nonzero both outside and inside the wall.

The solution for the elementary wall has two real moduli as in the Abelian case: the position of the wall and the compact phase. The phase can be rewritten as a U(1) gauge field. Therefore, the effective theory on the elementary wall is of the type (6.29) as in the Abelian case. The physical reason for the localization of the U(1) gauge field on the worldvolume of the wall is easy to understand. Since quark q^1 is nonzero inside the wall, only one U(1) field (namely $A_\mu - A_\mu^3$) which does not interact with this quark can propagate freely inside the wall.

For composite domain walls in the case of generic quark masses, the effective worldvolume theory contains U(1) gauge fields associated with each elementary wall. However, the metric on the moduli space can be more complicated. For example, the metric for the $\alpha_1+\alpha_2$ composite wall is shown to have a cigarlike geometry (Tong, 2002; Isozumi *et al.*, 2003).

E. Degenerate masses

Now consider the case of degenerate quark masses. As we know from Sec. IV.A, the non-Abelian gauge group is not broken in this case by adjoint fields. It turns out that in this case certain composite domain walls can localize non-Abelian gauge fields (Shifman and Yung, 2004a).

Consider the following choice of quark mass parameters:

$$m_1 = m_2,$$

$$m_3 = m_4,$$

$$\Delta m \equiv m_1 - m_3 > 0, \tag{6.40}$$

assuming also that the condition $\Delta m \gg g\sqrt{\xi}$ is satisfied for both Abelian and non-Abelian coupling constants g_1 and g_2. For this degenerate choice of masses, four of six isolated vacua of the theory with nondegenerate masses coalesce.

This theory has two isolated vacua, namely (1,2) and (3,4) with unbroken $\mathrm{SU}(2)_{C+F}$ symmetry while four other vacua coalesce and a Higgs branch is developed from the common root. We will denote this Higgs branch as (A,B), where A=1, 2 and B=3, 4.

The elementary domain walls interpolating between (1,2) and (A,B) vacua [or between (A,B) and (3,4)] have the same structure as elementary walls described above for the theory with nondegenerate masses. Following Shifman and Yung (2004a), we are mostly interested in the composite $(1,2)\to(3,4)$ wall, which is a bound state of the two elementary walls mentioned above.

The solution of the first-order equations (6.33) for this wall also has a three-layer structure similar to solutions for elementary walls. Now all quark fields (nearly) vanish inside the wall. The solution for the a fields in the middle domain M is given by

$$a = -\sqrt{2}\left(m_1 - \Delta m\frac{z - z_0 + \tilde{R}/2}{\tilde{R}}\right),$$

$$a^3 = 0, \tag{6.41}$$

where we introduced the thickness $\tilde{R}$ of the composite wall, to be considered large, $\tilde{R} \gg 1/g\sqrt{\xi}$; see below. The equation for a^3 in Eqs. (6.33) is trivially satisfied, while the equation for a yields

$$\tilde{R} = \frac{2\Delta m}{g_1^2\xi}, \tag{6.42}$$

demonstrating that $\tilde{R} \gg 1/g\sqrt{\xi}$.

Substituting the above solutions in the first two equations in Eqs. (6.33), we determine the falloff of the quark fields inside the wall. Near the left edge at $(z - z_0 + R/2) \gg 1/g\sqrt{\xi}$,

$$\varphi^{kA} = \sqrt{\xi}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} e^{-(\Delta m/2\tilde{R})(z - z_0 + \tilde{R}/2)^2}, \quad A = 1,2, \tag{6.43}$$

while near the right one at $(R/2 - z + z_0) \gg 1/g\sqrt{\xi}$,

$$\varphi^{kB} = \sqrt{\xi}(\tilde{U})^{kB} e^{-(\Delta m/2\tilde{R})(z - z_0 - \tilde{R}/2)^2}, \quad B = 3,4, \tag{6.44}$$

where $\tilde{U}$ is a matrix from the U(2) global flavor group, which takes into account possible flavor rotations inside the flavor pair B=3,4. It can be represented as a product of a U(1) phase factor and a matrix U from SU(2),

$$\tilde{U} = e^{i\sigma_0}U. \tag{6.45}$$

This matrix is parametrized by four phases σ_0 plus three phases residing in the matrix U.

The occurrence of these four wall moduli—one related to U(1) and three to SU(2)—is quite similar to the occurrence of one U(1) phase σ for the domain wall in Abelian theory; see Sec. VI.B. Shifman and Yung (2004a) identified these four moduli with (2+1)-dimensional gauge fields living on the wall worldvolume. The phase σ_0 is identified with the U(1) gauge field while the SU(2) matrix U gives rise to the non-Abelian SU(2) gauge field.

Thus, we get four gauge fields localized on the wall. The physical interpretation of this result is as follows. The quark fields are condensed outside the $12 \to 34$ wall while inside they almost vanish. Therefore, both U(1) and SU(2) gauge fields of the bulk theory are Higgsed in the bulk while they can freely propagate inside the wall.

The bosonic part of worldvolume theory derived by Shifman and Yung (2004a) for the composite wall looks like

$$S_{2+1} = \int d^3x \left\{ \frac{1}{2e_{2+1}^2}(\partial_n a_{2+1})^2 + \frac{1}{2g_{2+1}^2}(D_n a_{2+1}^a)^2 + \frac{1}{4e_{2+1}^2}[F_{nm}^{(2+1)}]^2 + \frac{1}{4g_{2+1}^2}[F_{nm}^{(2+1)a}]^2 \right\}, \tag{6.46}$$

where (2+1)-dimensional couplings in terms of the parameters of the bulk theory are given by

$$e_{2+1}^2 = 2\pi^2 \frac{\xi}{\Delta m},$$

$$g_{2+1}^2 = 2\pi^2 \frac{g_1^2}{g_2^2}\frac{\xi}{\Delta m}. \tag{6.47}$$

The domain wall is a 1/2-BPS object, so it preserves four supercharges on its world volume. Thus, we must have the extended $\mathcal{N}=2$ supersymmetry, with four supercharges, in the (2+1)-dimensional worldvolume theory. This is in accord with Eq. (6.46) in which the U(1) and SU(2) gauge fields are combined with the scalars a_{2+1} and a_{2+1}^a to form the bosonic parts of $\mathcal{N}=2$ vector multiplets.

A few comments are in order. The first comment refers to four noncompact moduli a, a^a, which emerged in Eq. (6.46). We can use gauge transformation in the worldvolume theory to set two of them to zero, say $a_{2+1}^{1,2}=0$. The other two a_{2+1}^3 and a_{2+1} should be identified with (linear combinations of) two centers of the elementary walls comprising our composite wall. More exactly, as a_{2+1} has no interactions whatsoever, it should be identified with the center of mass of the composite wall,

$$a_{2+1} = \pi\xi(z_1 + z_2), \tag{6.48}$$

where z_1 and z_2 are positions of the elementary walls forming the composite wall, while a_{2+1}^3 can be identified with the relative separation between the elementary walls,

$$a_{2+1}^3 = \pi\xi\frac{g_1}{g_2}(z_1 - z_2). \tag{6.49}$$

The second comment is devoted to a technical element of the derivation of Eq. (6.46) in Shifman and Yung (2004a). In fact, this worldvolume action was obtained through a calculational procedure similar to the one described in Sec. VI.C for the Abelian case, only at the quadratic level (i.e., omitting non-Abelian nonlinearities). To recover nonquadratic (truly non-Abelian) terms in Eq. (6.46) and thus prove rigorously the non-Abelian nature of the worldvolume theory, one needs to go beyond the quadratic approximation.

Still there are rather convincing general arguments showing that the proposal made by Shifman and Yung (2004a) is correct. First, the number of fields matches. We have four compact phases and two noncompact centers. Upon dualization, they fit into a vector multiplet of 3D $\mathcal{N}=2$ theory with the $SU(2)\times U(1)$ gauge group. If the gauge group were $U(1)^4$ [as for the case of nondegenerate masses, see Sakai and Tong (2005)], we would have four phases and four noncompact coordinates.[24] Thus, the non-Abelian gauge symmetry of the worldvolume theory is supported by supersymmetry. Second, there are only two distinct coupling constants in Eq. (6.46) rather than four. This also indicates that three phases, upon dualization, should be united in the SU(2) gauge theory.

The 1/4-BPS solution for a junction of a non-Abelian string (discussed in Sec. IV) with the composite wall described above has been found by Shifman and Yung (2004a). Thus non-Abelian string can end on the composite $(1,2)\to(3,4)$ wall in the theory with degenerate masses; see Eq. (6.40). Since the non-Abelian string carries a non-Abelian flux with arbitrary direction of the magnetic flux inside the SU(2) gauge subgroup, this becomes another argument in favor of localization of non-Abelian gauge fields on the worldvolume of the composite wall.

VII. WALL-STRING JUNCTIONS

To make contact with this string-brane picture, one may address the question of whether solitonic ANO strings can end on a domain wall that localizes gauge fields. The answer to this question is yes. Moreover, the string end point plays the role of charge with respect to the gauge field localized on the wall surface. This issue was studied by Gauntlett *et al.* (2001) for the sigma model setup, and by Dvali and Vilenkin (2003) for gauge theories at strong coupling. A solution for the 1/4-BPS wall-string junction in $\mathcal{N}=2$ supersymmetric U(1) gauge theory at weak coupling was found by Shifman and

[24] For nondegenerate masses, the composite $(1,2)\to(3,4)$ wall is a bound state of four elementary walls; see Sakai and Tong (2005). However, two of them disappear in the limit (6.40).

Yung (2003) while Shifman and Yung (2004a) deal with its non-Abelian generalization. In the literature, the wall-string junction goes under the name of *boojum*. Further studies of the wall-string junctions were carried out by Isozumi *et al.* (2005), where all 1/4-BPS solutions of Eqs. (4.95) were obtained, by Sakai and Tong (2005) and Auzzi, Shifman, *et al.* (2005), where the energy associated with a wall-string junction (boojum) was calculated, and by Shifman and Yung (2006b) and Tong (2006b), where the quantum version of the effective theory in the domain-wall worldvolume taking into account charged matter (strings of the bulk theory) was worked out.

The simplest 1/4-BPS wall-string junction in $\mathcal{N}=2$ QED was obtained by Shifman and Yung (2003). In both vacua of this theory, the gauge field is Higgsed while it can spread freely inside the wall. This is the physical reason why ANO string, which carries magnetic flux, can end on the wall.

Assume that at large distances from the string end point at $r=0$, $z=0$ the wall is almost parallel to the (x_1,x_2) plane while the string is stretched along the z axis. As usual, we look for a static solution assuming that all relevant fields can depend only on x_n ($n=1,2,3$). The Abelian version of first-order equations (4.95) for various 1/4-BPS junctions for the theory (6.1) looks like (Shifman and Yung, 2003)

$$F_1^* - iF_2^* - \sqrt{2}(\partial_1 - i\partial_2)a = 0,$$

$$F_3^* - \frac{g^2}{2}(|q^A|^2 - \xi) - \sqrt{2}\partial_3 a = 0,$$

$$\nabla_3 q^A = -\frac{1}{\sqrt{2}} q^A (a + \sqrt{2} m_A),$$

$$(\nabla_1 - i\nabla_2) q^A = 0. \tag{7.1}$$

These equations generalize the first-order equations for the wall (6.8) and for Abelian ANO string.

Needless to say, the solution of first-order equations (7.1) for a string ending on the wall can be found only numerically, especially near the end point of the string where both the string and the wall profiles are heavily deformed. However, far away from the end point of the string, deformations are weak and one can find the asymptotic behavior analytically. This has been described in detail in the original publication. In this review, we limit ourselves to the issue of the wall-string junction energy.

There are two distinct contributions to the boojum energy (Auzzi, Shifman, *et al.*, 2005). The first contribution is due to the gauge field inside the wall,

$$E^G_{(2+1)} = \int \frac{1}{2e^2_{2+1}}(F_{0i})^2 2\pi r dr = \frac{\pi\xi}{\Delta m}\int \frac{dr}{r} = \frac{\pi\xi}{\Delta m}\ln(g\sqrt{\xi}L). \tag{7.2}$$

The integral $\int dr/r$ is logarithmically divergent in both the ultraviolet and infrared. It is clear that the uv divergence is cut off at the transverse size of the string $\sim 1/g\sqrt{\xi}$ and presents no problem. However, the infrared divergence is much more serious. We introduced a large size L to regularize it in Eq. (7.2).

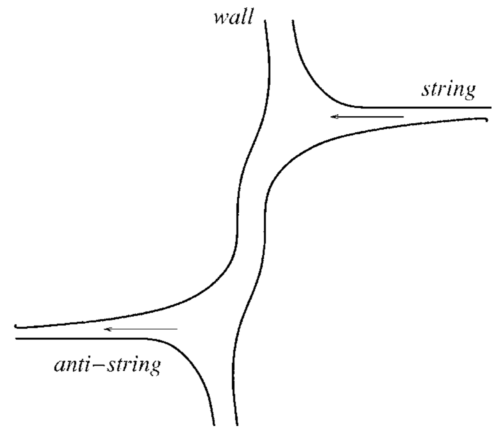

FIG. 13. String and antistring ending on the wall from different sides. Arrows denote the direction of the magnetic flux.

The second contribution, due to the z_0 field, is associated with the bending of the wall. It is proportional to $\int dr/r$ too,

$$E^H_{(2+1)} = \int \frac{T_w}{2}(\partial_r z_0)^2 2\pi r dr = \frac{\pi\xi}{\Delta m}\ln(g\sqrt{\xi}L). \tag{7.3}$$

Both contributions are logarithmically divergent in the infrared. Their occurrence is an obvious feature of charged objects coupled to massless fields in (2+1) dimensions due to the fact that the fields A_n and a_{2+1} do not die off at infinity, which means infinite energy.

The above two contributions are equal (with the logarithmic accuracy), even though their physical interpretation is different. The total energy of the string junction is

$$E^{G+H} = \frac{2\pi\xi}{\Delta m}\ln(g\sqrt{\xi}L). \tag{7.4}$$

Thus, the energy of an individual boojum is ill-defined. What can be done? A way out was suggested by Auzzi, Shifman, *et al.* (2005) and Shifman and Yung (2006b). For the infrared divergences to cancel, we consider strings and antistrings with incoming and outgoing fluxes, as well as strings coming to the wall from the right and from the left (see Fig. 13). Then the boojum-antiboojum pair will have a finite energy. Auzzi, Shifman, *et al.* (2005) have shown that the configuration depicted in Fig. 13 is a noninteracting 1/4-BPS configuration. All logarithmic contributions are canceled; the junction energy in this geometry is given by a finite negative contribution

$$E = -\frac{8\pi}{g^2}\Delta m. \tag{7.5}$$

[In fact, this energy was first calculated by Sakai and Tong (2005).] The procedure of how to separate this finite energy from logarithmic contributions described above was discussed in detail by Auzzi, Shifman, *et al.* (2005).

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REFERENCES

Abraham, E. R., and P. K. Townsend, 1991, Nucl. Phys. B **351**, 313.
Abraham, E. R., and P. K. Townsend, 1992a, Phys. Lett. B **291**, 85.
Abraham, E. R., and P. K. Townsend, 1992b, Phys. Lett. B **295**, 225.
Abrikosov, A., 1957, Sov. Phys. JETP **32**, 1442 [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984)], p. 356.
Acharya, B. S., and C. Vafa, 2001, e-print arXiv:hep-th/0103011.
Achucarro, A., and T. Vachaspati, 2000, Phys. Rep. **327**, 347.
Aharony, O., A. Hanany, K. A. Intriligator, N. Seiberg, and M. J. Strassler, 1997, Nucl. Phys. B **499**, 67.
Alford, M. G., K. Benson, S. R. Coleman, J. March-Russell, and F. Wilczek, 1991, Nucl. Phys. B **349**, 414.
Alvarez-Gaumé, L., and D. Z. Freedman, 1983, Commun. Math. Phys. **91**, 87.
Alvarez-Gaumé, L., and E. Witten, 1984, Nucl. Phys. B **234**, 269.
Argyres, P., and M. Douglas, 1995, Nucl. Phys. B **448**, 93.
Argyres, P., M. Plesser, and N. Seiberg, 1996, Nucl. Phys. B **471**, 159.
Armoni, A., and M. Shifman, 2003, Nucl. Phys. B **664**, 233.
Auzzi, R., S. Bolognesi, and J. Evslin, 2005, J. High Energy Phys. 0502, 046.
Auzzi, R., S. Bolognesi, J. Evslin, and K. Konishi, 2004a, Nucl. Phys. B **686**, 119.
Auzzi, R., S. Bolognesi, J. Evslin, K. Konishi, and H. Murayama, 2004b, Nucl. Phys. B **701**, 207.
Auzzi, R., S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, 2003, Nucl. Phys. B **673**, 187.
Auzzi, R., M. Shifman, and A. Yung, 2005, Phys. Rev. D **72**, 025002.
Auzzi, R., M. Shifman, and A. Yung, 2006, Phys. Rev. D **73**, 105012.
Bais, F. A., 1981, Phys. Lett. **98B**, 437.
Bardakci, K., and M. B. Halpern, 1972, Phys. Rev. D **6**, 696.
Bezerra de Mello, E. R., 1990, Mod. Phys. Lett. A **5**, 581.
Binosi, D., and T. ter Veldhuis, 2000, Phys. Lett. B **476**, 124.
Bogomol'nyi, E. B., 1976, Sov. J. Nucl. Phys. **24**, 861 [Sov. J. Nucl. Phys. **24**, 449 (1976)] [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 389].
Bolognesi, S., 2005, Nucl. Phys. B **719**, 67.
Carlino, G., K. Konishi, and H. Murayama, 2000, Nucl. Phys. B **590**, 137.
Carroll, S. M., S. Hellerman, and M. Trodden, 2000, Phys. Rev. D **61**, 065001.
Cecotti, S., P. Fendley, K. A. Intriligator, and C. Vafa, 1992, Nucl. Phys. B **386**, 405.
Cecotti, S., and C. Vafa, 1993, Commun. Math. Phys. **158**, 569.
Chatterjee, A. K., and P. Majumdar, 1984, Phys. Rev. D **30**, 844.
Chatterjee, A. K., and P. Majumdar, 1985, Phys. Lett. **159B**, 37.
Chibisov, B., and M. A. Shifman, 1997, Phys. Rev. D **56**, 7990; **58**, 109901(E) 1998.
Clark, T. E., O. Piguet, and K. Sibold, 1979, Nucl. Phys. B **159**, 1.
Coleman, S. R., 1976, Ann. Phys. (N.Y.) **101**, 239.
Coleman, S. R., 1983, in *The Magnetic Monopole: Fifty Years Later*, in *The Unity of the Fundamental Interactions, Proceedings of the 1981 International School of Subnuclear Physics, Erice, Italy*, edited by A. Zichichi (Plenum, New York).
D'Adda, A., and P. Di Vecchia, 1978, Phys. Lett. **73B**, 162.
D'Adda, A., R. Horsley, and P. Di Vecchia, 1978, Phys. Lett. **76B**, 298.
Davis, S. C., A. C. Davis, and M. Trodden, 1997, Phys. Lett. B **405**, 257.
de Azcarraga, J. A., J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, 1989, Phys. Rev. Lett. **63**, 2443.
de Boer, J., K. Hori, and Y. Oz, 1997, Nucl. Phys. B **500**, 163.
de Vega, H. J., and F. A. Schaposnik, 1976, Phys. Rev. D **14**, 1100 [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 389].
de Vega, H. J., and F. A. Schaposnik, 1986a, Phys. Rev. Lett. **56**, 2564.
de Vega, H. J., and F. A. Schaposnik, 1986b, Phys. Rev. D **34**, 3206.
Dirac, P. A. M., 1931, Proc. R. Soc. London, Ser. A **133**, 60.
Di Vecchia, P., and S. Ferrara, 1977, Nucl. Phys. B **130**, 93.
Dorey, N., 1998, J. High Energy Phys. 9811, 005.
Dorey, N., T. J. Hollowood, and D. Tong, 1999, J. High Energy Phys. 9905, 006.
Douglas, M. R., and S. H. Shenker, 1995, Nucl. Phys. B **447**, 271.
Dubovsky, S. L., and V. A. Rubakov, 2001, Int. J. Mod. Phys. A **16**, 4331.
Dvali, G. R., and Z. Kakushadze, 1999, Nucl. Phys. B **537**, 297.
Dvali, G. R., and M. A. Shifman, 1997, Phys. Lett. B **396**, 64; **407**, 452(E) (1997).
Dvali, G. R., and A. Vilenkin, 2003, Phys. Rev. D **67**, 046002.
Edelstein, J. D., W. G. Fuertes, J. Mas, and J. M. Guilarte, 2000, Phys. Rev. D **62**, 065008.
Edelstein, J. D., C. Nunez, and F. Schaposnik, 1994, Phys. Lett. B **329**, 39.
Eguchi, T., K. Hori, and C.-S. Xiong, 1997, Int. J. Mod. Phys. A **10**, 4203.
Eto, M., L. Ferretti, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, 2006, e-print arXiv:hep-th/0611313.
Eto, M., T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, and N. Sakai, 2007, Phys. Rev. D **75**, 045010.
Eto, M., K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi, and W. Vinci, 2007, Phys. Rev. Lett. **98**, 091602.
Eto, M., Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, and Y. Tachikawa, 2005a, Phys. Rev. D **71**, 105009.
Eto, M., Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, 2005b, Phys. Rev. D **72**, 085004.
Eto, M., Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, 2006a,

Phys. Lett. B **632**, 384.
Eto, M., Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, 2006b, J. Phys. A **39**, R315.
Eto, M., Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, 2006c, Phys. Rev. Lett. **96**, 161601.
Eto, M., K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, 2006, Phys. Rev. D **74**, 065021.
Eto, M., M. Nitta, and N. Sakai, 2004, Nucl. Phys. B **701**, 247.
Everett, A. E., and M. Aryal, 1986, Phys. Rev. Lett. **57**, 646.
Evlampiev, K., and A. Yung, 2003, Nucl. Phys. B **662**, 120.
Fateev, V. A., I. V. Frolov, and A. S. Schwarz, 1979a, Sov. J. Nucl. Phys. **30**, 590 [Yad. Fiz. **30**, 1134 (1979)].
Fateev, V. A., I. V. Frolov, and A. S. Schwarz, 1979b, Nucl. Phys. B **154**, 1.
Fayet, P., and J. Iliopoulos, 1974, Phys. Lett. **51B**, 461.
Fendley, P., and K. Intriligator, 1992a, Nucl. Phys. B **380**, 265.
Fendley, P., and K. Intriligator, 1992b, Nucl. Phys. B **372**, 533.
Fendley, P., S. D. Mathur, C. Vafa, and N. P. Warner, 1990, Phys. Lett. B **243**, 257.
Ferrara, S., and M. Porrati, 1998, Phys. Lett. B **423**, 255.
Fuertes, W. G., and J. M. Guilarte, 1998, Phys. Lett. B **437**, 82.
Gabadadze, G., and M. Shifman, 2000, Phys. Rev. D **61**, 075014.
Gates, S., Jr., M. Grisaru, M. Roček, and W. Siegel, 1983, *Superspace* (Benjamin/Cummings, Reading, MA).
Gates, S. J., 1984, Nucl. Phys. B **238**, 349.
Gates, S. J., C. M. Hull, and M. Roček, 1984, Nucl. Phys. B **248**, 157.
Gauntlett, J. P., R. Portugues, D. Tong, and P. K. Townsend, 2001, Phys. Rev. D **63**, 085002.
Gauntlett, J. P., D. Tong, and P. K. Townsend, 2001a, Phys. Rev. D **63**, 085001.
Gauntlett, J. P., D. Tong, and P. K. Townsend, 2001b, Phys. Rev. D **64**, 025010.
Georgi, H., and S. L. Glashow, 1972, Phys. Rev. Lett. **28**, 1494.
Gibbons, G. W., and P. K. Townsend, 1999, Phys. Rev. Lett. **83**, 1727.
Goldhaber, A. S., A. Rebhan, P. van Nieuwenhuizen, and R. Wimmer, 2004, Phys. Rep. **398**, 179.
Golfand, Y. A., and E. P. Likhtman, 1971, Zh. Eksp. Teor. Fiz. Pis'ma Red. **13**, 452 [JETP Lett. **13**, 323 (1971)] [reprinted in *Supersymmetry*, edited by S. Ferrara (North-Holland, Amsterdam, 1987), Vol. 1, p. 51].
Gorsky, A., and M. Shifman, 2000, Phys. Rev. D **61**, 085001.
Gorsky, A., M. Shifman, and Y. Tung, 2005, Phys. Rev. D **71**, 045010.
Gorsky, A., M. Shifman, and Y. Tung, 2006, Phys. Rev. D **73**, 065011.
Gorsky, A., M. Shifman, and A. Yung, 2007, Phys. Rev. D **75**, 065032.
Graham, N., and R. L. Jaffe, 1999, Nucl. Phys. B **544**, 432.
Haag, R., J. T. Lopuszanski, and M. Sohnius, 1975, Nucl. Phys. B **88**, 257 [reprinted in *Supersymmetry*, edited by S. Ferrara (North-Holland, Amsterdam, 1987), Vol. 1, p. 51].
Hanany, A., and K. Hori, 1998, Nucl. Phys. B **513**, 119.
Hanany, A., M. J. Strassler, and Z. Zaffaroni, 1998, Nucl. Phys. B **513**, 87.
Hanany, A., and D. Tong, 2003, J. High Energy Phys. 0307, 037.
Hanany, A., and D. Tong, 2004, J. High Energy Phys. 0404, 066.
Harvey, J. A., 1997, in *Magnetic Monopoles, Duality, and Supersymmetry*, edited by C. Efthimiou and B. Greene (World Scientific, Singapore), pp. 157–216.
Hashimoto, K., and D. Tong, 2005, J. Cosmol. Astropart. Phys. **0509**, 004.
Heo, J., and T. Vachaspati, 1998, Phys. Rev. D **58**, 065011.
Hinchliffe, I., (Particle Data Group), 2006, J. Phys. G **33**, 110.
Hindmarsh, M., 1989, Phys. Lett. B **225**, 127.
Hindmarsh, M., and T. W. B. Kibble, 1985, Phys. Rev. Lett. **55**, 2398.
Hloušek, Z., and D. Spector, 1992, Nucl. Phys. B **370**, 143.
Hori, K., and C. Vafa, 2000, e-print arXiv:hep-th/0002222.
Horsley, R., 1979, Nucl. Phys. B **151**, 399.
Hruby, J., 1980, Nucl. Phys. B **162**, 449.
Imbimbo, C., and S. Mukhi, 1984, Nucl. Phys. B **247**, 471.
Intriligator, K., and N. Seiberg, 1996a, Nucl. Phys. B, Proc. Suppl. **45BC**, 1.
Intriligator, K., and N. Seiberg, 1996b, in *Proceedings of the Conference on Future Perspectives in String Theory (Strings '95)*, edited by I. Bars, P. Bouwknegt, J. Minahan, D. Nemeschansky, K. Pilch, H. Saleur, and N. Warner (World Scientific, Singapore).
Isozumi, Y., M. Nitta, K. Ohashi, and N. Sakai, 2004a, Phys. Rev. Lett. **93**, 161601.
Isozumi, Y., M. Nitta, K. Ohashi, and N. Sakai, 2004b, Phys. Rev. D **70**, 125014.
Isozumi, Y., M. Nitta, K. Ohashi, and N. Sakai, 2005, Phys. Rev. D **71**, 065018.
Isozumi, Y., K. Ohashi, and N. Sakai, 2003, J. High Energy Phys. 0311, 060.
Jackiw, R., and C. Rebbi, 1976, Phys. Rev. D **13**, 3398 [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 331].
Kakimoto, K., and N. Sakai, 2003, Phys. Rev. D **68**, 065005.
Kaul, R. K., and R. Rajaraman, 1983, Phys. Lett. **131B**, 357.
Kneipp, M. A. C., 2003, Phys. Rev. D **68**, 045009.
Kneipp, M. A. C., 2004, Phys. Rev. D **69**, 045007.
Kneipp, M. A. C., and P. Brockill, 2001, Phys. Rev. D **64**, 125012.
Kogan, I. I., A. Kovner, and M. A. Shifman, 1998, Phys. Rev. D **57**, 5195.
Konishi, K., 1984, Phys. Lett. **135B**, 439.
Konishi, K., and K. Shizuya, 1985, Nuovo Cimento Soc. Ital. Fis., A **90**, 111.
Konishi, K., and L. Spanu, 2003, Int. J. Mod. Phys. A **18**, 249.
Kovner, A., M. A. Shifman, and A. Smilga, 1997, Phys. Rev. D **56**, 7978.
Lambert, N. D., and D. Tong, 2000, Nucl. Phys. B **569**, 606.
Leese, R. A., 1992, Phys. Rev. D **46**, 4677.
Leese, R., and T. M. Samols, 1993, Nucl. Phys. B **396**, 639.
Lopuszanski, J. T., and M. Sohnius, 1974, Karlsruhe Rep. Print-74-1269 (unpublished).
Losev, A., and M. Shifman, 2003, Phys. Rev. D **68**, 045006.
Losev, A., M. A. Shifman, and A. I. Vainshtein, 2001, Phys. Lett. B **522**, 327.
Losev, A., M. A. Shifman, and A. I. Vainshtein, 2002, New J. Phys. **4**, 21.
Mandelstam, S., 1976, Phys. Rep., Phys. Lett. **23**, 245.
Manton, N. S., and J. M. Speight, 2003, Commun. Math. Phys. **236**, 535.
Markov, V., A. Marshakov, and A. Yung, 2005, Nucl. Phys. B **709**, 267.
Marshakov, A., and A. Yung, 2002, Nucl. Phys. B **647**, 3.
Mottola, E., 1978, Phys. Lett. **79B**, 242; **80B**, 433(E) (1979).
Nastase, H., M. A. Stephanov, P. van Nleuwenhuizen, and A.

Rebhan, 1999, Nucl. Phys. B **542**, 471.
Nielsen, H., and P. Olesen, 1973, Nucl. Phys. B **61**, 45 [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 365].
Nishino, H., and S. J. J. Gates, 1993, Int. J. Mod. Phys. A **8**, 3371.
Novikov, V. A., M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, 1984, Phys. Rep. **116**, 103.
Oda, H., K. Ito, M. Naganuma, and N. Sakai, 1999, Phys. Lett. B **471**, 140.
Penin, A. A., V. A. Rubakov, P. G. Tinyakov, and S. V. Troitsky, 1996, Phys. Lett. B **389**, 13.
Polchinski, J., 1995, Phys. Rev. Lett. **75**, 4724 [see also the excellent text by J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, 1998), Vols. 1 and 2].
Polyakov, A., 1987, *Gauge Fields and Strings* (Harwood, New York).
Polyakov, A. M., 1974, Zh. Eksp. Teor. Fiz. Pis'ma Red. **20**, 430 [JETP Lett. **20**, 194 (1974)] [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 522].
Polyakov, A. M., 1975, Phys. Lett. **59B**, 79.
Polyakov, A. M., 1977, Nucl. Phys. B **120**, 429.
Polyakov, A. M., and A. A. Belavin, 1975, JETP Lett. **22**, 245.
Prasad, M. K., and C. M. Sommerfield, 1975, Phys. Rev. Lett. **35**, 760 [reprinted in *Solitons and Particles*, edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 530].
Preskill, J., and A. Vilenkin, 1993, Phys. Rev. D **47**, 2324.
Rebhan, A., and P. van Nieuwenhuizen, 1997, Nucl. Phys. B **508**, 449.
Rebhan, A., P. van Nieuwenhuizen, and R. Wimmer, 2004a, Phys. Lett. B **594**, 234.
Rebhan, A., P. van Nieuwenhuizen, and R. Wimmer, 2004b, Nucl. Phys. B **679**, 382.
Redlich, A. N., 1984a, Phys. Rev. Lett. **52**, 18.
Redlich, A. N., 1984b, Phys. Rev. D **29**, 2366.
Ritz, A., M. Shifman, and A. Vainshtein, 2002, Phys. Rev. D **66**, 065015.
Ritz, A., A. Vainshtein, and M. Shifman, 2004, Phys. Rev. D **70**, 095003.
Rouhani, S., 1981, Nucl. Phys. B **182**, 462.
Rubakov, V. A., 2002, *Classical Theory of Gauge Fields* (Princeton University Press, Princeton).
Sakai, N., and D. Tong, 2005, J. High Energy Phys. 0503, 019.
Sakai, N., and Y. Yang, 2006, Commun. Math. Phys. **267**, 783.
Schaposnik, F. A., 2006, e-print arXiv:hep-th/0611028.
Schaposnik, F. A., and P. Suranyi, 2000, Phys. Rev. D **62**, 125002.
Schmidt, J. R., 1992, Phys. Rev. D **46**, 1839.
Schönfeld, J. F., 1979, Nucl. Phys. B **161**, 1979.
Seiberg, N., 1995, in *The Power of Holomorphy: Exact Results in 4D SUSY Field Theories*, in *Proceedings of the VI International Symposium on Particles, Strings, and Cosmology (PASCOS 94)*, edited by K. C. Wali (World Scientific, Singapore).
Seiberg, N., and E. Witten, 1994a, Nucl. Phys. B **426**, 19; **430**, 485(E) (1994b).
Seiberg, N., and E. Witten, 1994b, Nucl. Phys. B **431**, 484.
Shifman, M., 1998, Phys. Rev. D **59**, 021501(R).
Shifman, M., and T. ter Veldhuis, 2000, Phys. Rev. D **62**, 065004.
Shifman, M., A. Vainshtein, and R. Zwicky, 2006, J. Phys. A **39**, 13005.
Shifman, M., and A. Yung, 2003, Phys. Rev. D **67**, 125007.
Shifman, M., and A. Yung, 2004a, Phys. Rev. D **70**, 025013.
Shifman, M., and A. Yung, 2004b, Phys. Rev. D **70**, 045004.
Shifman, M., and A. Yung, 2005, Phys. Rev. D **72**, 085017.
Shifman, M., and A. Yung, 2006a, Phys. Rev. D **73**, 125012.
Shifman, M., and A. Yung, 2006b, Phys. Rev. D **74**, 045006.
Shifman, M. A., and A. I. Vainshtein, 1988, Nucl. Phys. B **296**, 445.
Shifman, M. A., and A. I. Vainshtein, 1999, *Instantons Versus Supersymmetry: Fifteen Years Later, ITEP Lectures in Particle Physics and Field Theory* (World Scientific, Singapore), Vol. 2, pp. 485–648.
Shifman, M. A., A. I. Vainshtein, and M. B. Voloshin, 1999, Phys. Rev. D **59**, 045016.
Strassler, M., 1998, Prog. Theor. Phys. Suppl. **131**, 439.
Suranyi, P., 2000, Phys. Lett. B **481**, 136.
Taubes, C. H., 1980, Commun. Math. Phys. **72**, 277.
't Hooft, G., 1974, Nucl. Phys. B **79**, 276.
't Hooft, G., 1981, Nucl. Phys. B **190**, 455.
Tong, D., 2002, Phys. Rev. D **66**, 025013.
Tong, D., 2004, Phys. Rev. D **69**, 065003.
Tong, D., 2005, e-print arXiv:hep-th/0509216.
Tong, D., 2006a, J. High Energy Phys. 0612, 051.
Tong, D., 2006b, J. High Energy Phys. 0602, 030.
Townsend, P. K., 1988, Phys. Lett. B **202**, 53.
Townsend, P. K., 1999, in *Supermembranes and M-Theory*, edited by M. Duff (IOP, Bristol, UK), pp. 375–389.
Uchiyama, A., 1984, Nucl. Phys. B **244**, 57.
Uchiyama, A., 1986a, Prog. Theor. Phys. **75**, 1214.
Uchiyama, A., 1986b, Nucl. Phys. B **278**, 121.
Vainshtein, A. I., and A. Yung, 2001, Nucl. Phys. B **614**, 3.
van Holten, J. W., and A. Van Proeyen, 1982, J. Phys. A **15**, 3763.
Ward, R. S., 1985, Phys. Lett. **158B**, 424.
Weinberg, E. J., 1980, Nucl. Phys. B **167**, 500.
Weinberg, E. J., 1982, Nucl. Phys. B **203**, 445.
Wess, J., and J. Bagger, 1992, *Supersymmetry and Supergravity*, 2nd ed. (Princeton University Press, Princeton).
Witten, E., 1979a, Phys. Lett. **86B**, 283.
Witten, E., 1979b, Nucl. Phys. B **149**, 285.
Witten, E., 1982, Nucl. Phys. B **202**, 253.
Witten, E., 1985, Nucl. Phys. B **249**, 557.
Witten, E., 1993, Nucl. Phys. B **403**, 159.
Witten, E., 1997, Nucl. Phys. B **507**, 658.
Witten, E., 1998, Phys. Rev. Lett. **81**, 2862.
Witten, E., and D. I. Olive, 1978, Phys. Lett. **78B**, 97.
Yamagishi, H., 1984, Phys. Lett. **147B**, 425.
Yung, A., 1999, Nucl. Phys. B **562**, 191.
Yung, A., 2000, e-print arXiv:hep-th/0005088.
Zamolodchikov, A., and Al. Zamolodchikov, 1979, Ann. Phys. (N.Y.) **120**, 253.
Zamolodchikov, A. B., and A. B. Zamolodchikov, 1992, Nucl. Phys. B **379**, 602.