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Onsager and the theory of hydrodynamic turbulence

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Lars Onsager, a giant of twentieth-century science and the 1968 Nobel Laureate in Chemistry, made deep contributions to several areas of physics and chemistry. Perhaps less well known is his ground-breaking work and lifelong interest in the subject of hydrodynamic turbulence. He wrote two papers on the subject in the 1940s, one of them just a short abstract. Unbeknownst to Onsager, one of his major results was derived a few years earlier by A. N. Kolmogorov, but Onsager's work contains many gems and shows characteristic originality and deep understanding. His only full-length article on the subject in 1949 introduced two novel ideas—negative-temperature equilibria for two-dimensional ideal fluids and an energy-dissipation anomaly for singular Euler solutions—that stimulated much later work. However, a study of Onsager's letters to his peers around that time, as well as his private papers of that period and the early 1970s, shows that he had much more to say about the problem than he published. Remarkably, his private notes of the 1940s contain the essential elements of at least four major results that appeared decades later in the literature: (1) a mean-field Poisson-Boltzmann equation and other thermodynamic relations for point vortices; (2) a relation similar to Kolmogorov's 4/5 law connecting singularities and dissipation; (3) the modern physical picture of spatial intermittency of velocity increments, explaining anomalous scaling of the spectrum; and (4) a spectral turbulence closure quite similar to the modern eddy-damped quasinormal Markovian equations. This paper is a summary of Onsager's published and unpublished contributions to hydrodynamic turbulence and an account of their place in the field as the subject has evolved through the years. A discussion is also given of the historical context of the work, especially of Onsager's interactions with his contemporaries who were acknowledged experts in the subject at the time. Finally, a brief speculation is offered as to why Onsager may have chosen not to publish several of his significant results.

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CONTENTS

I. INTRODUCTION

Lars Onsager (see Fig. 1) is recognized as a giant of twentieth-century science. His deep contributions to many areas of physics and chemistry are widely appreciated. His founding work on the thermodynamics of irreversible processes and a key result, the reciprocal relations for linear transport coefficients (Onsager, 1931a, 1931b), won him the Nobel Prize in Chemistry in 1968. His exact solution for the partition function of the two-dimensional Ising model (Onsager, 1944) is recognized as a tour de force of mathematical physics, which helped to usher in the modern era of research in critical phenomena. Among the other celebrated contributions are his work on liquid helium, including quantization of circulation (Onsager, 1949a, 1949d) and off-diagonal long-range order (Penrose and Onsager, 1956), his semiclassical theory of the de Haas–van Alphen effect in metals (Onsager, 1952), his entropic theory of transition to nematic order for rod-shaped colloids (Onsager, 1949c), and the reaction field in his theory of dielectrics (Onsager, 1936). Of course, his largest body of research was on corrections and extensions of the Debye-Hückel theory

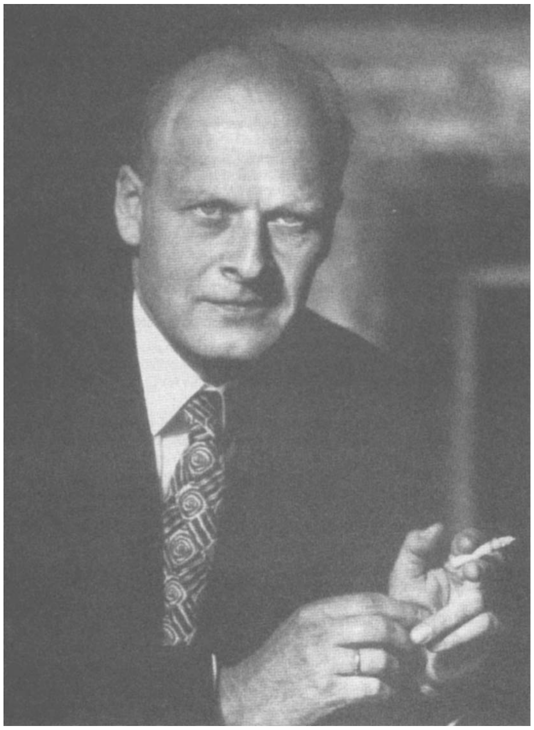


FIG. 1. Lars Onsager (1903–1976), circa 1945. Courtesy of Theoretical Physics Group, Norges Teknisk-Naturvitenskapelige Universitet (NTNU).

of electrolytes, from his first scientific paper at age 23 (Onsager, 1926) to those in the last year of his life (Hubbard and Onsager, 1977; Onsager and Chen, 1977; Onsager and Kim, 1977; Onsager *et al.*, 1977).

Perhaps less well known among physicists is Onsager's ground-breaking work and lifelong interest in the subject of hydrodynamic turbulence. In fact, classical fluid mechanics was a part of much of his research, including the work on electrolyte solutions (Onsager, 1926), viscosity of colloidal solutions (Onsager, 1932), and theory of liquid diffusion (Onsager, 1945d). Onsager's earliest recorded encounter with fluid turbulence was a paper in 1939 on convection in gases between concentric vertical cylinders (Onsager and Watson, 1939). This was an outgrowth of his theoretical work on isotope separation by thermal diffusion (Onsager, 1939b; Onsager *et al.*, 1939). Experiments on isotope separation in an apparatus with this geometry revealed the onset of turbulence at unexpectedly low Reynolds numbers, thus reducing the efficiency of the separation method. Onsager and Watson (1939) gave a simple theory for the scaling of the critical Reynolds number to explain this early onset of turbulence.

In the following decade Onsager published two seminal works on the subject of fully developed turbulence. The first in 1945 was just a short abstract (Onsager, 1945c) of a contributed talk, which he gave at a meeting of the Metropolitan Section of the American Physical Society held on November 9 and 10, 1945, at Columbia University. Here, Onsager predicted an energy spectrum for velocity fluctuations that rolls off as the $-5/3$ power of the wave number. The published abstract appeared a few years after, but entirely independently of, the now-famous trilogy of papers by Kolmogorov (1941a, 1941b, 1941c) proposing his similarity theory of turbulence. Therefore Onsager is often credited as a codiscoverer of the Kolmogorov theory along with Kolmogorov's student, Obukhov (1941a, 1941b), and with Heisenberg (1948) and von Weizsäcker (1948). The 1945 abstract was followed a few years later by Onsager's only full-length article on the subject of fluid turbulence (Onsager, 1949d). The paper is based on his address at the first International Union of Pure and Applied Physics (IUPAP) conference on statistical mechanics in Florence, Italy, held May 17–20, 1949. This conference is famous for Onsager's announcement—as discussion remarks—of two other spectacular results: the quantization of circulation in superfluid helium and the critical exponent for spontaneous magnetization in the two-dimensional (2D) Ising model. The first result was independently rediscovered by Feynman (1955), while a proof of the second was later published by Yang (1952). These events have passed into the folklore of statistical physics.

It may be less widely appreciated that Onsager's talk on statistical hydrodynamics at the Florence conference introduced two highly innovative ideas in the subject of fluid turbulence, in addition to lucidly reviewing the Kolmogorov theory. The first idea was a theory on the spontaneous formation of large-scale, long-lived vortices in 2D flows, explaining them as a consequence of equilibrium statistical mechanics at negative absolute temperature. The second was a theory on the anomalous rate of energy dissipation in three-dimensional (3D) turbulence based upon conjectured singularities of the incompressible Euler equation. These ideas did not seem to excite much attention during the conference, perhaps because turbulence was not foremost in the minds of most participants. Furthermore, Onsager's presentation had a typical spare elegance, dense with deep insights, and decidedly cryptic. Only over a period of several decades have his ideas been pursued, extended, and refined by various researchers. Onsager never published again on the subject.

However, Onsager made greater inroads into the theory of turbulence than he ever fully made public. Our studies of the historical sources presented here show that he obtained at least four results, new in the 1940s, which could have been made the basis of major publications. The documents containing these remarkable results are Onsager's typewritten letters to contemporaries and his own private, handwritten notes. For whatever reasons, the results in these documents were never published and were only rediscovered decades later by others. In addition, there are several rather mysterious and incompletely understood ideas sketched in Onsager's notes that may still bear some fruit.

It is the purpose of this article to review Onsager's work on turbulence. We shall discuss the contents of his two published articles and describe some of the later developments of his seminal ideas expounded there.

Our main focus, however, will be the unpublished sources. These give a fascinating view into the mind of a scientific genius at work and still have a pedagogical value in the subject, even after decades of subsequent development in the field. Our study should therefore be of interest not only to turbulence experts and to historians of science, but also to working physicists who are curious to learn some of the basic facts of this intriguing subject of classical physics. Those who wish to know more may consult the monograph of Frisch (1995) or a more recent mathematical review by Robert (2003). Readers interested in Onsager's broader scientific contributions will benefit from perusing the *Collected Works of Lars Onsager* (Hemmer *et al.*, 1996). Papers on the varied topics in this collection are introduced by different working scientists, those on turbulence by Chorin (1996). Attention must also be drawn to the special issue of *Journal of Statistical Physics* (Lebowitz, 1995) dedicated to Onsager's life and work—in particular to the delightful biographical essay by Longuet-Higgins and Fisher (1995).

II. SOURCE MATERIALS

Before we begin, we briefly remark on the historical sources. They are the letters that Onsager sent to his professional colleagues in 1945, and his own private, handwritten notes.

The letters that Onsager exchanged with L. Pauling, C.-C. Lin, and T. von Kármán (see Fig. 2) have been preserved in the Caltech archive, "Theodore von Kármán Papers, 1871–1963," under Personal Correspondence. In particular, Box 18, Folders 22 and 23, of that collection contains letters between von Kármán and his former Ph.D. student Lin, in the period 1942–1947, along with other related correspondence. Also, Folder 8 of Box 22 contains direct correspondence of Onsager with von Kármán in 1945. Relevant for our purposes are two remarkable documents that Onsager wrote, the first a two-page note to Pauling [Onsager (1945a); the "Pauling note," reproduced here as Appendix A] on March 15, 1945, and the second an 11-page account he sent to Lin [Onsager (1945b); the "Lin note," reproduced as Appendix B] in June 1945. Both notes were forwarded to von Kármán by the primary recipients and later were sent to him directly by Onsager on July 25, 1945. Both are organized along the lines of Onsager's later 1949 paper (Onsager, 1949d) discussing first two-dimensional point-vortex equilibria and then dissipative, three-dimensional turbulence. The Pauling note reveals a whimsical side to Onsager, who entitled the section on point vortices "The little vortices who wanted to play" and whose presentation, while technically sound, is in the humorous style of a nursery story. The Lin note, by contrast, is serious in tone.

The handwritten notes are preserved in the "Lars Onsager Archive" maintained at the Norges Teknisk-Naturvitenskapelige Universitet (NTNU) in Trondheim, Norway. Upon Onsager's death of heart failure in 1976, his widow, Margarethe, deposited most of his unpublished papers and research notes at the Sterling Memorial Library of Yale University. Several researchers (S. Machlup, P. A. Lyons, W. W. Watson, R. M. Fuoss, A. Patterson, Jr., and D. Leaist) helped to classify the material. The original collection was retrieved by the family in 1981 but a microfilm copy is available at Yale. The original documents were stored at the Onsager farm at Tilton, New Hampshire, until they were deposited in 1997 in Trondheim.

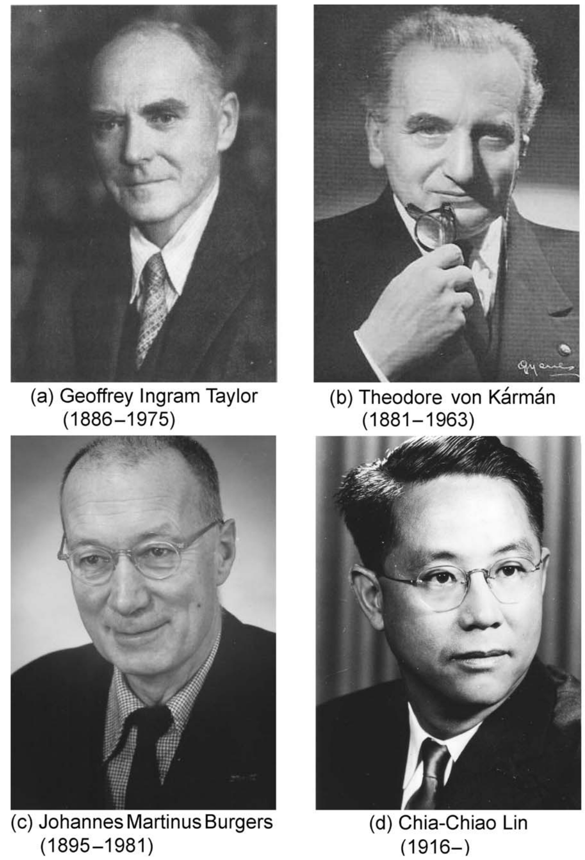


FIG. 2. Onsager's famous contemporaries who influenced his theories of turbulence. Figures courtesy of (a) Cambridge University Press; (b) the Archives, California Institute of Technology; (c) the JM Burgerscentrum; and (d) the MIT Museum and the Library of Congress.

Among the extensive materials stored at Trondheim are three folders, numbered as 11:129, 11:132, and 11:135. These folders contain Onsager's private research notes on hydrodynamic turbulence. The material in the first folder 11:129 can be reliably dated to the period 1940–1945. First, as internal evidence, the opening page of the folder contains a list of references to various papers on turbulence, including those of Taylor (1935, 1937, 1938), Taylor and Green (1937), von Kármán (1937), von Kármán and Howarth (1938), MacPhail (1940), Trubridge (1934), Tollmien (1933a, 1933b), and Burgers (1929a, 1929b, 1929c, 1933a, 1933b, 1933c, 1933d), and an unpublished 1931 reference of C. W. Oseen. The latest paper in the list is that of MacPhail in 1940, which sets a lower bound for the date of the folder. Second, almost the entire contents of the folder are either mentioned explicitly in the Lin note of June 1945, or at least alluded to therein. Thus 1940–1945 seems to be the likely period for Folder 11:129, and it is probably safe to say that all the material was worked out by Onsager before he published his full-length paper in 1949. On the other hand, several pages in Folder 11:132 are written on stationary from the 10th Coral Gables Conference on Fundamental Interactions, held during January 22–26, 1973, in Coral Gables, Florida. Thus Folder 11:132 is assuredly from after 1973. This is presumably one reason why it was labeled by S. Machlup, one of the compilers of the archives, with the notation “1975?” indicating his estimate of the date. The contents of Folder 11:135 build upon the results of 11:132 and likely follow it chronologically. Thus Folders 11:132 and 11:135 can be dated with some certainty to 1973–1975, a couple of years before Onsager’s death. It is interesting that Onsager returned to fluid turbulence at a time when, as is widely known (Longuet-Higgins and Fisher, 1995), most of his attention was on biology.

The entire contents of the three folders are listed briefly in our Appendix C. In the following discussion, we shall reproduce several pages of these handwritten notes, those that bear on our discussion most directly. Copies of the folder contents may be obtained by request from the Lars Onsager archive in Trondheim. Onsager’s notes on the wide range of topics that exercised his attention during his lifetime, in mathematics, physics, chemistry, and biology, are archived there and summarized on-line.[1] There are doubtless many surprises for scientists and for historians of science hidden in those folders and still waiting to be discovered.

III. STATISTICAL EQUILIBRIUM OF TWO-DIMENSIONAL FLUIDS

A. Onsager’s theory of point-vortex equilibria

The Pauling and Lin notes discuss, in their first half, the same subject as the first half of the published paper (Onsager, 1949d) entitled “Ergodic Motion of Parallel Vortices.” In all these writings Onsager discussed a simple Hamiltonian particle model of 2D ideal fluid flow, the point-vortex model of Helmholtz (1867) and Kirchhoff (1883), describing this motion for a system of N vortices in a plane, or of straight and parallel line vortices in three dimensions. If the planar coordinates of the ith vortex are $\mathbf{r}_i=(x_i,y_i)$ and if that vortex carries a net circulation κ_i, then the equations of motion are

$$\kappa_i\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i},\quad \kappa_i\frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}, \tag{1}$$

where H is the fluid kinetic energy. When there are no boundaries, H has the form

$$H=-\frac{1}{2\pi}\sum_{i<j}\kappa_i\kappa_j\ln(r_{ij}/L), \tag{2}$$

where r_{ij} is the distance between the ith and jth vortex and L is an (arbitrary) length scale. For confined flow the logarithm is replaced by a more general Green’s function of the Laplacian $G(\mathbf{r}_i,\mathbf{r}_j)$ with appropriate boundary conditions, and single-vortex terms are added to represent the interactions of each vortex with its own image charges and possibly with an external stream function. Onsager cited as his source for these equations the works of Lin (1941, 1943), who extensively studied their mathematical properties. In a letter of September 4, 1945, Lin wrote to von Kármán (Lin, 1945a):

> “I was asking him [Onsager] for some reprints of his paper on the statistical mechanics of crystal lattices on behalf of a friend of mine. He sent me the reprints, and asked, perhaps out of courtesy, for reprints of my papers. I sent him some, including my earlier work at Toronto on the motion of vortices. He was apparently struck by the Hamiltonian form of the differential equations satisfied by the coordinates of the vortices, and tried to develop a statistical mechanics for them.”

A good modern source for the point-vortex model is Marchioro and Pulvirenti (1994) in which it is proved that the model describes the motion of concentrated blobs of vorticity, evolving according to the 2D incompressible Euler equations, as long as the distance between the blobs is much greater than their diameters (Theorem 4.4.2).[2] Another result in the opposite direction (Theorem 5.3.1) states that a smooth solution of the 2D Euler equations $\omega(\mathbf{r},t)$ can be approximated as $N\to\infty$, over any finite time interval, by a sum $\omega_N(\mathbf{r},t)=\sum_{i=1}^{N}\kappa_i\delta(\mathbf{r}-\mathbf{r}_i(t))$, where $\kappa_i=\pm 1/N$ and $\mathbf{r}_i(t)$, $i=1,\ldots,N$ are the solutions of Eqs. (1).

With this model, Onsager proposed a theoretical explanation for a commonly observed feature of nearly two-dimensional flows: the spontaneous appearance of large-scale, long-lived vortices. Examples are the large, lingering storms in the atmospheres of the gas giants of the outer solar system, such as the Great Red Spot of Jupiter; see Fig. 3. Large vortices are also readily seen downstream of flow obstacles (von Kármán, 1911, 1912),

[1] http://www.ub.ntnu.no/formidl/hist/tekhist/tek5/

[2] In a footnote, Onsager (1949d) remarked that the model should work better for superfluids because “vortices in a suprafluid are presumably quantized; the quantum of circulation is h/m, where m is the mass of a single molecule.” In fact, these equations of motion have been formally derived from quantum many-body equations for parallel line vortices in superfluids (Fetter, 1966) and rigorously derived within the 2D Gross-Pitaevskii model in a limit where radiation into sound waves is negligible (Lin and Xin, 1999). In particular, if all drag and radiative effects are negligible, then vanishing of the net Magnus force requires that vortices move “quasistatically” with the local superfluid velocity. See Thouless *et al.* (1999) and Barenghi *et al.* (2001) for recent reviews of the still-active field of quantized vortex dynamics.

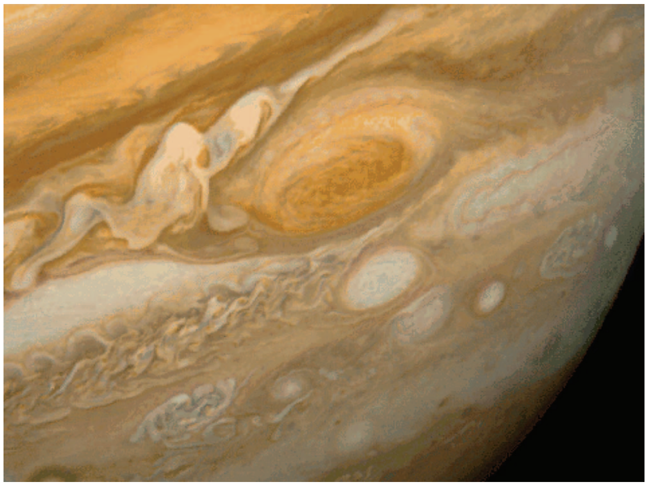


FIG. 3. (Color) The Great Red Spot, an anticyclonic vortex in the upper atmosphere of Jupiter. It has existed at least since it was observed in 1610 by Galileo Galilei with one of the first telescopes and measures 14 000 km north-south and 40 000 km east-west. Similar large-scale, long-lived vortices exist in the atmospheres of the other gas giant planets of our solar system. Features of "small-scale" turbulence can be clearly seen in the picture. Courtesy of NASA, Voyager 1, from the NSSDC Image Catalog.

often evolving out of the turbulent wake. The common appearance of such vortices was noted by other scientists, for example, H. Poincaré, who gave a theory based on stability considerations (Poincaré, 1893).

Onsager suggested a probabilistic treatment. In his own words (Onsager, 1949d):

> "The formation of large, isolated vortices is an extremely common, yet spectacular phenomenon in unsteady flow. Its ubiquity suggests an explanation on statistical grounds."

Onsager proposed an explanation of the phenomenon by an application of Gibbsian equilibrium statistical mechanics to the point-vortex model. His theory assumed that the generation of the large-scale vortices was a consequence of the inviscid Euler equations, which form a Hamiltonian system conserving total kinetic energy. This is rigorously true in two dimensions (as noted by Onsager, 1949d) due to the conservation of enstrophy. In particular, no sustained forcing is required to maintain the vortex in this theory as long as the dissipation by viscosity is weak. Onsager also assumed the validity of the point-vortex approximation, though with reservations. In the Lin note he wrote:

> "When we come to *volume* distributions of vorticity (still parallel), the approximate description by line vortices introduces fictitious possibilities because it makes us forget the restrictions imposed by the *incompressibility* of the fluid. More or less equivalent artificial restrictions might serve as a crude substitute."

Likewise, he wrote thus in Onsager (1949d):

> "When we compare our idealized model with reality, we have to admit one profound difference: the distributions of vorticity which occur in the actual flow of normal liquids are continuous…. As a statistical model in two-dimensions it is ambiguous: what set of discrete vortices will best approximate a continuous distribution of vorticity?"

Finally, Onsager assumed that the point-vortex dynamics is ergodic in phase space over the surface of constant energy, so that a microcanonical distribution is achieved at long times. "We inquire about the ergodic motion of the system," Onsager wrote to Lin.

It is worth noting that Onsager may have gotten the idea for this statistical treatment from a series of papers by Burgers (1929a, 1929b, 1929c, 1933a, 1933b, 1933c, 1933d), which are cited on the first page of Folder 11:129. In these articles Burgers attempts to apply statistical-mechanical maximum entropy ideas to turbulent flows. There is the following footnote in the first paper (Burgers, 1929a):

> "In the case of the motion of an ideal fluid of parallel rectilinear vortices, the diameters of which are small compared to their distances, canonical variables can be introduced according to a method developed by KIRCHHOFF and by LAGALLY [comp. M. LAGALLY, Sitz. Ber. Münch. Akad. p. 377, 1914]. For these coordinates LIOUVILLE's theorem can be proved. In applying statistical methods now the kinetic energy of the motion has to be given."

Thus the essential ingredients of Onsager's theory were already stated by Burgers, without, however, any suggestion of their relevance to the problem of large-vortex formation. It is interesting to observe that Onsager wrote in the Lin note:

> "That Liouville's theorem holds in configuration-space has no doubt been observed before, but it appears that certain possible effects of the conservation laws have been overlooked."

The really ingenious step in Onsager's theory was his realization that point vortices would yield states of negative absolute temperature, at sufficiently high energy, and that this result could explain the spontaneous appearance of large-scale vortices in two-dimensional flows.

The crucial feature of the point-vortex system which permits this conclusion is the fact that the total phase-space volume is finite. Since the x and y components of the vortices are canonically conjugate variables, the total phase-space volume is $\Phi(\infty)=A^N$, where A is the area of the flow domain and

$$\Phi(E) = \int \prod_{i=1}^{N} d^2\mathbf{r}_i \theta(E - H(\mathbf{r}_1, \ldots, \mathbf{r}_N)). \tag{3}$$

Here the Heaviside step function $\theta(x)=1$ for $x>0$ and $=0$ for $x<0$. Thus one can see that $\Phi(E)$ is a non-negative increasing function of energy E, with constant limits $\Phi(-\infty)=0$ and $\Phi(\infty)=A^N$. Therefore its derivative

$$\Omega(E)=\Phi'(E)=\int\prod_{i=1}^{N}d^2\mathbf{r}_i\delta(E-H(\mathbf{r}_1,\ldots,\mathbf{r}_N)) \tag{4}$$

is a non-negative function going to zero at both extremes, $\Omega(\pm\infty)=0$. Thus the function must achieve a maximum value at some finite E_m, where $\Omega'(E_m)=0$. For $E>E_m$, $\Omega'(E)$ will be negative. On the other hand, by Boltzmann's principle, the thermodynamic entropy is

$$S(E)=k_B\ln\Omega(E) \tag{5}$$

and the inverse temperature $1/\Theta=dS/dE$ is thus negative for $E>E_m$. This argument for the existence of negative absolute temperatures is the same as that published by Onsager (1949d) some two years prior to their introduction by Purcell and Pound (1951) for nuclear-spin systems. Onsager further pointed out that negative temperatures will lead to the formation of large-scale vortices by clustering of smaller ones. In his own words (Onsager, 1949d)

> "In the former case [when $1/\Theta>0$], vortices of opposite sign will tend to approach each other. However, if $1/\Theta<0$, then vortices of the same sign will tend to cluster—preferably the strongest ones—so as to use up excess energy at the least possible cost in terms of degrees of freedom. It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganized contributions to the flow."

The statistical tendency of vortices of the same sign to cluster in the negative-temperature regime is clear from a description by a canonical distribution $\propto e^{-\beta H}$, with $\beta=1/k_B\Theta$. Negative β corresponds to reversing the sign of the interaction, making like "charges" statistically attract and opposite "charges" repel.

B. Subsequent research on the point-vortex model

Onsager carried these considerations no further in his 1949 paper nor in any subsequent published work.[3] After his talk at the Florence meeting, there was no comment on the new theory of large-vortex formation from any conference participant. However, Onsager's work was not totally ignored. In a masterful review of theories of turbulence written in the same year, von Neumann (1949) took note of the point-vortex model and Onsager's statistical-mechanical theory. It led von Neumann to speculate about the limits of Kolmogorov's reasoning in three dimensions and to recognize the profound consequences of enstrophy conservation in two dimensions. These considerations were carried further by Lee (1951), who also extended the Gibbsian statistical-mechanical approach to Fourier-truncated Euler dynamics (Lee, 1952).

However, after this initial flurry of work, there was a comparatively long period in which Onsager's statistical theory was not further explored. In due course, however, it played a role in stimulating the development by Kraichnan (1967) of the concept of the *inverse energy cascade*, which describes the irreversible, dynamical process by which energy injected through an external force accumulates at large scales in two dimensions. The situation changed further in the early 1970s when a connection was made to the two-dimensional electrostatic guiding-center plasma. This system is one in which long filaments of charge are aligned parallel to a uniform magnetic field $\mathbf{B}$ and move under their mutual electric field $\mathbf{E}$ with the "guiding-center" velocity $\mathbf{E}\times\mathbf{B}/B^2$. The mathematical description of this system is identical to that of a set of point vortices in two-dimensional, incompressible Euler equations, as described above, in which the charge of a filament corresponds to the circulation of a vortex. In a pair of seminal papers on this subject, Joyce and Montgomery (1973) and Montgomery and Joyce (1974) returned to Onsager's theory and worked out a predictive equation for the large-scale vortex solutions conjectured by Onsager.

A brief review of the Joyce-Montgomery considerations, in the language of the 2D point-vortex system, is worthwhile here. These authors considered a neutral system, which we describe as consisting of N vortices of circulation $+1/N$ and N vortices of circulation $-1/N$. For this system, there are two vortex densities,

$$\rho_\pm(\mathbf{r})=\frac{1}{N}\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{r}_i^\pm), \tag{6}$$

where $\mathbf{r}_i^\pm$, $i=1,\ldots,N$, are the positions of the N vortices of circulation $\pm 1/N$, respectively. Note that the vorticity field is represented by

$$\omega(\mathbf{r})=\rho_+(\mathbf{r})-\rho_-(\mathbf{r}). \tag{7}$$

Then, by state-counting arguments similar to those used by Boltzmann in deriving his entropy function for kinetic theory, Joyce and Montgomery (1973) derived the following formula for the entropy (per particle) of a given field of vortex densities:

$$S=-\int d^2\mathbf{r}\rho_+(\mathbf{r})\ln\rho_+(\mathbf{r})-\int d^2\mathbf{r}\rho_-(\mathbf{r})\ln\rho_-(\mathbf{r}). \tag{8}$$

They next reasoned that the equilibrium distributions should be those which maximized the entropy subject to the constraints of fixed energy,

[3] In fact, by modern standards, referees might not have permitted the publication of these remarks, objecting that the author had not worked out sufficient details or derived concrete, testable consequences of the theory. This would not necessarily have been bad had such objections stimulated Onsager to expound the theory further. As we see below, he could indeed have said more on the subject. However, such an objection might also have had the undesirable consequence of causing Onsager to remove the discussion from his paper completely, with the loss to posterity of a creative idea.

$$E = \frac{1}{2}\int d^2\mathbf{r}\int d^2\mathbf{r}'\,G(\mathbf{r},\mathbf{r}')\omega(\mathbf{r})\omega(\mathbf{r}'), \tag{9}$$

with the unit normalization given by

$$\int d^2\mathbf{r}\rho_\pm(\mathbf{r}) = 1. \tag{10}$$

From here, it is straightforward to work out the variational equation

$$\rho_\pm(\mathbf{r}) = \exp\left[\mp\beta\int d^2\mathbf{r}'\,G(\mathbf{r},\mathbf{r}')\omega(\mathbf{r}') - \beta\mu_\pm\right], \tag{11}$$

where β and $\mu_\pm$ are Lagrange multipliers to enforce the constraints, having the interpretation of inverse temperature and chemical potentials, respectively. A closed equation is obtained by introducing the stream function

$$\psi(\mathbf{r}) = \int d^2\mathbf{r}'\,G(\mathbf{r},\mathbf{r}')\omega(\mathbf{r}') \tag{12}$$

and writing, via the inverse relation $-\Delta\psi = \omega$, that

$$-\Delta\psi(\mathbf{r}) = \exp\{-\beta[\psi(\mathbf{r}) - \mu_+]\} - \exp\{\beta[\psi(\mathbf{r}) + \mu_+]\}. \tag{13}$$

This is the final equation derived by Joyce and Montgomery. Its maximum-entropy solutions give exact, stable, stationary solutions of the 2D Euler equations and should describe the macroscopic vortices proposed by Onsager when $\beta < 0$. Montgomery and Joyce (1974) gave another independent derivation of the same equation by considering the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for the n-point correlation functions $\rho_n(\mathbf{r}_1,\kappa_1,\ldots,\mathbf{r}_n,\kappa_n)$, with $\kappa_i = \pm 1$ and using the mean-field approximation that $\rho_n(\mathbf{r}_1,\kappa_1,\ldots,\mathbf{r}_n,\kappa_n) = \prod_{i=1}^n \rho_1(\mathbf{r}_i,\kappa_i)$.

C. Onsager's unpublished work on vortex statistics

As Joyce and Montgomery themselves observed, their final mean-field equation for $\beta > 0$ is similar to the Poisson-Boltzmann equation in the Debye-Hückel theory of plasmas and electrolytes. Considering that Onsager was a world expert in the Debye-Hückel theory of electrolytes and had written extensively on the Poisson-Boltzmann equation in that context, e.g., Onsager (1933, 1964), it might be considered surprising if he had failed to recognize this connection. In fact, the Lin note discusses this explicitly. We quote Onsager in full:

> "Let us consider $\underline{n}$ parallel vortices of circulations $\underline{K}_1,\ldots,\underline{K}_n$ confined by a non-circular[4] boundary to a region of area $\underline{S}$. There exists a path-function $W(X_1,Y_1,\ldots,X_n,Y_n)$ and no other important integral of the motion. We inquire about the ergodic motion of the system.
>
> The effect of the restriction $W(X_1..Y_n)$=const. depends on whether the prescribed value is less than the average
>
> $$\bar{W} = S^{-n}\int W dX_1\ldots dY_n$$
>
> over configuration-space, or greater. In the critical case $W=\bar{W}$ we get on the average a random distribution of the vortices.
>
> For the case $\underline{W} < \underline{\bar{W}}$ we may approximate the micro-canonical distribution by a canonical distribution with a positive temperature. For small values of $\bar{W} - W$ we can develop a theory analogous to the Debye-Hückel theory of electrolytes. When $\bar{W} - W$ is not small, we get a pronounced tendency toward mutual association of vortices of opposite sign, and strong vortices will be squeezed against the boundary. These phenomena could be discussed by methods analogous to Bjerrum's treatment of weak electrolytes.[5] The process of neutralization and trapping by the boundary will release energy until the vortices which are still 'free' can move almost at random.
>
> The case $W > \bar{W}$ is quite different. We now need a *negative* temperature to get the required energy. The appropriate statistical methods have analogues not in the theory of electrolytes, but in the statistics of stars. In a general way we can foresee what will happen. Vortices of the *same* sign will tend to move together, more so the stronger the repulsion[6] between them. After this aggregation of the stronger vortices has disposed of the excess energy, the weaker vortices are free to roam at will."

These remarkable passages require a thorough discussion.

First, we find here a rather different argument than from Onsager (1949d) for the existence of a critical energy E_m at which negative temperatures appear. Furthermore, Onsager in this passage provides a concrete formula for the critical energy as a volume average of the Hamiltonian. This result was not mentioned four years later by Onsager (1949d) and, in fact, was never published by him. The simple idea underlying it is that the maximum of the Gibbs entropy $\mathcal{S}[f] =$

[4]For circular domains, there is a second conserved quantity, fluid angular momentum. For example, see Lundgren and Pointin (1977). This quantity must be considered for equilibrium statistical mechanics of fluids in circular enclosures, and it plays a crucial role in justifying the existence of negative temperatures for free flows in infinite two-dimensional domains.

[5]Many years later, J. McCauley, Jr., Onsager's last Ph.D. student at Yale, used such methods to study the kinetics of the Kosterlitz-Thouless vortex-unbinding transition in low-temperature helium films. See McCauley (1977) and the entertaining account in McCauley (1995).

[6]In the manuscript from the von Kármán archive, there is a handwritten question mark at this point, written presumably by either von Kármán or Lin. The writer may have been perplexed as to why repulsion should lead to clustering.

$-k_B \int dX_1 \cdots dY_n f \ln f$ is obtained for the constant n-particle distribution function $f(X_1, \ldots, Y_n) = 1/S^n$, with S the area of the flow domain. Therefore, assuming an equivalence of ensembles, the value $W = \bar{W}$ at which the microcanonical entropy $\ln \Omega(W)$ is maximized is just the average of W with respect to the uniform distribution over the flow domain. In this maximum-entropy state, the n vortices are all uniformly and independently distributed over the area. Much later, the problem of obtaining the critical energy of the point-vortex system was considered in the published literature. It was argued by Taylor (1972) and by Joyce and Montgomery (1973) that the critical energy $E_m = 0$ for the neutral vortex system with N vortices of circulation $+\kappa$ and N of $-\kappa$. It was not until 1993 that the above more general formula of Onsager was rediscovered and rigorously justified by Eyink and Spohn (1993).

The second factor to emerge is that Onsager did realize the connection with Debye-Hückel theory, as would be expected. There is more: When Onsager wrote that "we can develop a theory analogous to the Debye-Hückel theory," he was not just speaking rhetorically. There is, in fact, a remarkable set of six pages in Folder 11:129 of Onsager's unpublished notes, pp. 8–13, which develops exactly such a theory. These are the only pages in the folder which deal with the point-vortex model. Two of them (pp. 8 and 11; see Figs. 4 and 5) are reproduced here. The notations follow those of Lin (1941, 1943). On the first page (p. 8), Onsager sketches a derivation of the following mean-field Poisson-Boltzmann equation,

$$\nabla^2 \bar{\psi}_i + \frac{1}{A} \sum_j \kappa_j e^{\beta \kappa_j (\bar{\psi}_i - \chi)} = 0,$$

using a maximum-entropy argument. It is interesting that he refers to the Lagrange multiplier for the energy constraint as the "hydrodynamic temperature" (in fact, its negative). Note that $-\int f \ln f$ in Onsager's argument corresponds to the Gibbs' ensemble entropy, not the macroscopic Boltzmann entropy (8) used later by Joyce and Montgomery. Onsager's mean-field equation is actually more general than theirs because he allows for arbitrary circulations κ_i of the vortices. On the next page (p. 11 of the folder), Onsager works out the 2D analog of standard calculations in Debye-Hückel theory for small $|\beta|$ or for energies close to E_m, e.g., see Landau and Lifschitz (1980), Chaps. 78 and 79, pp. 239–245. Onsager considers only positive temperatures ($\beta < 0$) and introduces the analog of the (inverse) Debye screening length $-\xi^2 = (\beta/A) \Sigma_i \kappa_i^2$. His final result on this page is the thermodynamic relation between temperature and mean energy, $\gamma + \ln(\frac{1}{2}\xi) = 4\pi W / \Sigma_i \kappa_i^2$. Considering the definition of ξ in terms of inverse temperature $1/T = -k_B \beta$, this formula can be rewritten as

$$\frac{1}{T} = (\text{const}) \frac{A}{\sum_i \kappa_i^2} \exp\left[\frac{8\pi W}{\sum_i \kappa_i^2}\right]. \tag{14}$$

Taking into account that W is the negative of the energy and differences in normalization, this expression is closely similar to that obtained later by Edwards and Taylor (1974) from a microscopic calculation; see their Eq (29). Of course, Onsager's relation (14) was worked out for the subcritical regime and admits no negative temperature solutions.

Thus these pages of Onsager's private notes substantially carry out the program suggested to Lin for studying the regime of "small [positive] values of $\bar{W} - W$."[7] The remaining pages fill in some additional details. On p. 10, Onsager derives the formula for the critical energy as we did above, writing that "$\Omega(W) = \max$ for $W = \bar{W}$." On p. 13 he attempts to evaluate space integrals of the Green's function, apparently in order to develop more concrete expressions for the critical energy.

It is unclear from available evidence if Onsager also realized the validity of the mean-field Poisson-Boltzmann equation for the negative-temperature regime above the critical energy. However, there is his tantalizing reference in the Lin note to "the statistics of stars." The equilibrium statistical mechanics of gravitating systems, with its many peculiarities and anomalies, was indeed already extensively investigated in the 19th century by scientists such as J. K. F. Zöllner (1834–1882), Lord Kelvin (1824–1907), G. A. D. Ritter (1826–1908), J. H. Lane (1819–1880), and R. J. Emden (1862–1940). A good review of that work was available in Onsager's time through the German monograph of Emden (1907), while a more modern source would have been Chandrasekhar (1939). Among other results discussed in Emden's book is the "Lane-Emden equation," which governs the temperature or density profiles of polytropic gas spheres in convective thermal equilibrium under gravitational attraction. The equation for the isothermal spheres is the exact analog of the Joyce-Montgomery mean-field equation for point vortices if one assumes, as is appropriate for gravitation, that there is only one sign of the charge (e.g., see Messer and Spohn, 1982). It may be that Onsager was aware of the validity of the mean-field equation for equilibrium vorticity distributions at negative temperature. If so, then it is somewhat surprising that he never expanded the material in Folder 11:129 on statistical mechanics of point vortices into a full publication. On the other hand, as we shall see below, Onsager's private notes contain other remarkable results that he did not publish or expand.

D. Recent advances and applications

One issue that Onsager never addressed was the appropriate thermodynamic limit for the validity of his statistical theory of large-scale 2D vortices. The Debye-Hückel theory is valid in the standard thermodynamic

[7] Incidentally, p. 11 contains Onsager's sketches of what appear to be coherent vortices, including a cat's-eye vortex and a hexatic vortex.

$$W = \sum_{i=1}^{n} \kappa_i \psi_0(x_i, y_i) + \frac{1}{2} \sum_{(i \neq j)} \kappa_i \kappa_j \mathcal{G}(x_i, y_i; x_j, y_j)$$

$$+ \frac{1}{2} \sum_{1}^{n} \kappa_i^2 g(x_i, y_i; x_i, y_i)$$

$$\psi(x, y; x_1, y_1; \dots; x_n, y_n) = \psi_0(x, y) + \sum_{i=1}^{n} \kappa_i \mathcal{G}(x, y; x_i; y_i)$$

$$\overline{\psi}_i(x,y) = \int \psi(x, y; x_1, y_1; \dots; x_n, y_n) \prod_{j \neq i} dx_j\, dy_j\, f(x_1, y_1 \dots x_n, y_n)$$

$$\nabla^2 \overline{\psi_i} = \quad - \overline{\kappa}_i(x_i, y_i; x, y)$$

$$\int \{ f \log f + \beta W f \}\, dx_1 dy_1 \dots dx_n dy_n = \mathrm{Max}$$

$$f(x_1 \dots x_n) = e^{\alpha + \beta W}$$

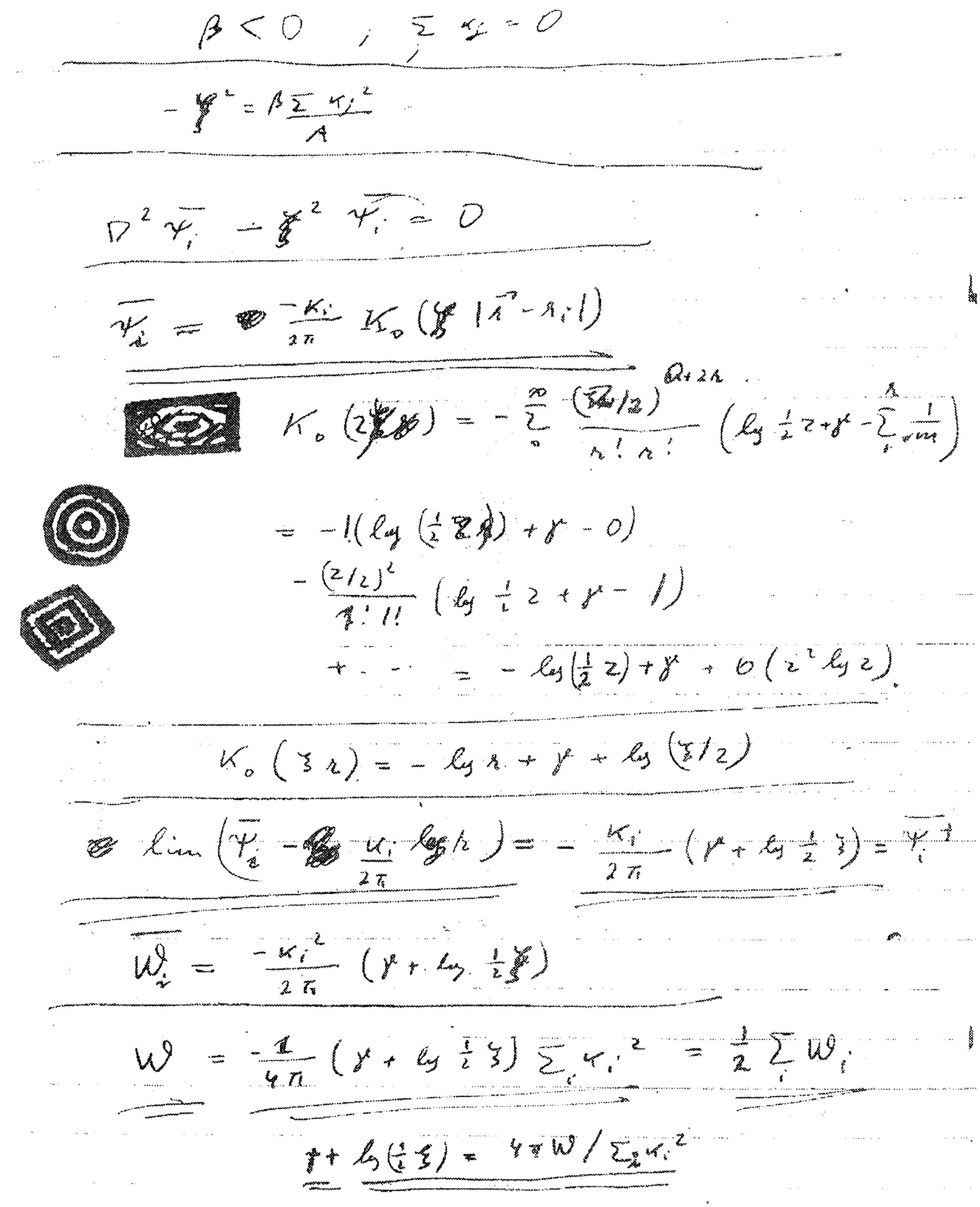
$(-\rho/\beta)$ = hydrodyn. temperature

$$\overline{K}_i = \frac{1}{A} \sum_j \kappa_j e^{\beta \kappa_j (\overline{\psi}_i - \chi)}$$

$$\nabla^2 \overline{\psi}_i + \frac{1}{A} \sum_j \kappa_j e^{\beta \kappa_j (\overline{\psi}_i - \chi)} = 0$$

$$\nabla^2 \overline{\psi}_i + \frac{1}{A} \sum_j \kappa_j e^{-\beta \chi \kappa_j} + \frac{\beta}{A} \sum_j \kappa_j^2 e^{-\beta \kappa_j \chi}\, \overline{\psi}_i = 0$$

FIG. 4. Page 8 of Folder 11:129 (Onsager, circa 1945). The first two lines, l.1 and l.2, give the Routh-Kirchhoff function W, which is $-H$, the negative of the Hamiltonian. l.3 is the net stream function $\psi(x,y;x_1,y_1;\dots;x_n,y_n)$ of the system of n point vortices. l.4 is the average of the stream function over the distribution function $f(x_1,y_1;\dots;x_n,y_n)$ of the vortices conditioned on the location of a distinguished vortex, the ith. l.5 is the Poisson equation for $\bar{\psi}_i$, whose source is the conditionally averaged vorticity. l.6 is the condition of maximum entropy at fixed energy. l.7 is the Gibbs canonical distribution, but where ϱ/β is the negative of the usual temperature. l.8 is the mean vorticity from the average over the Gibbs distribution. l.9 is the mean-field Poisson-Boltzmann equation and l.10 is its linearization for small $|\beta|$. Reproduced courtesy of the Onsager Archive, NTNU.



$$\beta < 0 \quad ; \quad \sum_i \kappa_i = 0$$

$$-\xi^2 = \beta \frac{\sum \kappa_i^2}{A}$$

$$\nabla^2 \bar{\psi}_i - \xi^2 \bar{\psi}_i = 0$$

$$\bar{\psi}_i = \frac{-\kappa_i}{2\pi} K_0(\xi |\vec{r} - r_i|)$$

$$K_0(z) = -\sum_0^\infty \frac{(z/2)^{2n}}{n!\,n!}\left(\log \tfrac{1}{2} z + \gamma - \sum_1^n \frac{1}{m}\right)$$

$$= -1\left(\log\left(\tfrac{1}{2} z\right) + \gamma - 0\right)$$

$$- \frac{(z/2)^2}{1!\,1!}\left(\log \tfrac{1}{2} z + \gamma - 1\right)$$

$$+ \cdots = -\log\left(\tfrac{1}{2} z\right) + \gamma + O(z^2 \log z).$$

$$K_0(\xi r) = -\log r + \gamma + \log(\xi/2)$$

$$\lim\left(\bar{\psi}_i - \frac{\kappa_i}{2\pi} \log r\right) = -\frac{\kappa_i}{2\pi}\left(\gamma + \log \tfrac{1}{2}\xi\right) = \bar{\psi}_i^+$$

$$\bar{W}_i = \frac{-\kappa_i^2}{2\pi}\left(\gamma + \log \tfrac{1}{2}\xi\right)$$

$$W = \frac{-1}{4\pi}\left(\gamma + \log \tfrac{1}{2}\xi\right) \sum_i \kappa_i^2 = \frac{1}{2}\sum_i W_i$$

$$\gamma + \log\left(\tfrac{1}{2}\xi\right) = 4\pi W / \sum_i \kappa_i^2$$

FIG. 5. Page 11 of Folder 11:129 (Onsager, circa 1945). l.1 is the condition of positive temperature ($\beta<0$) and of zero net circulation or neutrality $\Sigma_i \kappa_i = 0$. l.2 is the definition of the inverse screening length ξ. l.3 is the Helmholtz equation for $\bar{\psi}_i$. l.4 is the solution by a modified Bessel function of the second kind. l.5–9 is the asymptotic expansion of the Bessel function for small argument. l.10 defines $\bar{\psi}_i^+$, the remainder of the conditional mean stream function at the origin after subtracting the contribution of the ith vortex. l.11 is the mean energy $\bar{W}_i = \kappa_i \bar{\psi}_i^+$ in the atmosphere of the ith vortex. l.12 is the total mean energy from the formula $W = \frac{1}{2}\Sigma_i \bar{W}_i$ of electrostatics and l.13 is the mean energy-temperature relation. Reproduced courtesy of the Onsager Archive, NTNU.

limit in two dimensions, for which area $A \to \infty$ with the number of vortices $N \to \infty$ and energy $E \to \infty$ in such a way that $n=N/A$, $e=E/A$ tend to a finite limit. Further, the circulations κ_i are held fixed, independent of N, e.g., $\kappa_i = \pm 1$. These points are particularly clear from the derivation of Eq. (14) by Edwards and Taylor (1974). Note that the inverse temperature $1/T$ in Eq. (14) scales as A/N, since $\Sigma_i \kappa_i^2 \sim O(N)$, and approaches a finite limit in the thermodynamic limit. As E/N varies over all real values the temperature T stays positive. In fact, it has been rigorously proved by Fröhlich and Ruelle (1982) (see also Campbell and O'Neil, 1991; O'Neil and Redner, 1991) that the standard thermodynamic limit exists for the point-vortex model, but yields only positive temperatures. To obtain the negative temperature states proposed by Onsager, one must consider energies that are considerably higher, greater than the critical energy. It is an immediate consequence of Onsager's integral formula for the critical energy that if $\kappa_i \sim O(1)$ for all i, then $E_m \sim O(N^2)$, because of the sum over all vortex pairs. In other words, a nontrivial limit with energy $E \sim O(1)$ can only be obtained if one takes $\kappa_i \sim O(1/N)$, as in our discussion of the Joyce-Montgomery theory above. It has been rigorously proved that for the scaling $\kappa_i \sim O(1/N)$ in a fixed flow domain of finite volume the Joyce-Montgomery mean-field theory is valid. This was originally proved for the canonical ensemble by Caglioti *et al.* (1992) and Kiessling (1993) (see also Messer and Spohn, 1982) and later for the microcanonical ensemble by Eyink and Spohn (1993), Caglioti *et al.* (1995), and Kiessling and Lebowitz (1997). It is important to note that the equivalence of ensembles assumed by Onsager can break down if the specific heats become negative (Eyink and Spohn, 1993; Caglioti *et al.*, 1995; Kiessling and Lebowitz, 1997), as has in fact been observed in numerical studies and some experiments (Smith and O'Neil, 1990). In that case, it is the microcanonical ensemble which is physically correct and justified by the ergodicity assumption. The above results have put the Onsager theory and the Joyce-Montgomery mean-field equation on firm mathematical footing.

Nevertheless, as Onsager himself described the theory in the Lin note, "the simplifying hypothesis is rather bold." Onsager was very modest about what had been achieved and was clear about the tentative character of his two major assumptions: the point-vortex approximation and the ergodicity hypothesis. We close this section with a short report on the current status of Onsager's theory *vis à vis* the status of its founding assumptions and a comparison with later simulations and experiments.

Taking the point-vortex approximation first, we have already mentioned that there are rigorous results which show that any smooth 2D Euler solution $\omega(\mathbf{r},t)$ may be approximated arbitrarily well over a finite time interval $0<t<T$ by a sum of point vortices $\Sigma_{i=1}^{N} \kappa_i \delta(\mathbf{r}-\mathbf{r}_i(t))$ with $\kappa_i \sim c_i/N$, where c_i are constants as $N \to \infty$ (Marchioro and Pulvirenti, 1994, Theorem 5.3.1). However, this is not sufficient to justify equilibrium statistical mechanics at long times because the limits $T \to \infty$ and $N \to \infty$ need not commute. We mentioned earlier some of Onsager's own reservations about the point-vortex approximation, but he made an even more explicit criticism in Onsager (1949d):

> "…in two-dimensional convection the vorticity of every volume element of the liquid is conserved, so that convective processes can build vortices only in the sense of bringing together volume elements of great initial vorticity…. This digression will make clear that the present theory for the formation of large vortices does not apply to all cases of unsteady flow. As a matter of fact, the phenomenon is common but not universal. It is typically associated with separating boundary layers, whereby the initial conditions are not so very different from those contemplated in the theory: the vorticity is mostly concentrated in small regions, and the initial energy is relatively high."

These are the same issues that Onsager raised with Lin when he spoke about "the restrictions imposed by the *incompressibility* of the fluid." Onsager's concerns can be clearly understood by considering the initial condition of an ideal vortex patch, with a constant level of vorticity on a finite area. Because that area is conserved by incompressibility under the 2D Euler dynamics, it is not possible for the vorticity to concentrate or to intensify locally for this initial condition. However, this is not true if one were to approximate the patch by a distribution of point vortices at high energies. In that case, the mean-square distance between point vortices could decrease over time and the effective area covered could similarly decrease, leading to a more intense, localized vortex structure. Thus one expects discrepancies here between the continuum 2D Euler and the point-vortex model for long times.

A great step toward eliminating these defects was taken independently by Miller (1990) and Robert (1990). They both elaborated an equilibrium statistical-mechanical theory directly for the continuum 2D Euler equations, without making the point-vortex approximation. See Robert (1991), Robert and Sommeria (1991), and Miller *et al.* (1992) for further development and discussion. The basic object of both of these theories was a local distribution function $n(\mathbf{r},\sigma)$, the probability density that the microscopic vorticity $\omega(\mathbf{r})$ lies between σ and $\sigma+d\sigma$ at the space point $\mathbf{r}$. The picture here is that the vorticity field in its evolution mixes to very fine scales so that a small neighborhood of the point $\mathbf{r}$ will contain many values of the vorticity with levels distributed according to $n(\mathbf{r},\sigma)$. Thus n satisfies

$$\int d\sigma n(\mathbf{r},\sigma) = 1 \tag{15}$$

at each point $\mathbf{r}$ in the flow domain. Note that the microscopic vorticity $\omega(\mathbf{r})$ differs from the macroscopic vorticity obtained by averaging:

$$\bar{\omega}(\mathbf{r}) = \int d\sigma \sigma n(\mathbf{r},\sigma). \tag{16}$$

The latter is the vorticity that will be observed on a coarse-grained scale. Furthermore, the function $n(\mathbf{r},\sigma)$ records an infinite set of conserved quantities of the 2D incompressible Euler equations, namely, the area occupied by each level set of the initial vorticity. If $g(\sigma)d\sigma$ is the fraction of the total area A on which occur vorticities between σ and $\sigma+d\sigma$, then

$$\frac{1}{A}\int d^2\mathbf{r}\, n(\mathbf{r},\sigma) = g(\sigma). \tag{17}$$

By a Boltzmann counting argument, one can show that the entropy associated with a given distribution function $n(\mathbf{r},\sigma)$ is

$$S = -\int d^2\mathbf{r} \int d\sigma n(\mathbf{r},\sigma) \ln n(\mathbf{r},\sigma). \tag{18}$$

Maximizing this entropy subject to the constraints (15) and (17), as well as fixed energy

$$E = \frac{1}{2}\int d^2\mathbf{r} \int d^2\mathbf{r}' \int d\sigma \int d\sigma' \sigma\sigma' G(\mathbf{r},\mathbf{r}') \times n(\mathbf{r},\sigma) n(\mathbf{r},\sigma), \tag{19}$$

gives

$$n(\mathbf{r},\sigma) = \frac{1}{Z(\mathbf{r})} \exp\{-\bar{\beta}[\sigma\bar{\psi}(\mathbf{r}) - \mu(\sigma)]\}, \tag{20}$$

where $Z(\mathbf{r})$, $\mu(\sigma)$, $\bar{\beta}$ are Lagrange multipliers to enforce constraints (15), (17), and (19), respectively. The stream function satisfies the generalized mean-field equation

$$-\Delta\bar{\psi}(\mathbf{r}) = \frac{1}{Z(\mathbf{r})}\int d\sigma \sigma \exp\{-\bar{\beta}[\sigma\bar{\psi}(\mathbf{r}) - \mu(\sigma)]\}; \tag{21}$$

see Miller (1990) and Robert (1990). This theory is an application to 2D Euler of the method worked out by Lynden-Bell (1967) to describe gravitational equilibrium after "violent relaxation" in stellar systems.

The Robert-Miller theory solves the problems discussed by Onsager, in the passage quoted above, with respect to the point-vortex assumption. The new theory incorporates infinitely many conservation laws of 2D Euler, although in our opinion that is not the critical difference. In fact, the point-vortex model, in the generality considered by Onsager, also has infinitely many conserved quantities, i.e., the total number of vortices of a given circulation.[8] More importantly, the Robert-Miller theory includes information about the area of the vorticity level sets, which is lacking in the point-vortex model. As remarked by Miller *et al.* (1992), the Joyce-Montgomery mean-field equation is formally recovered in a "dilute-vorticity limit" in which the area of the level sets shrinks to zero keeping the net circulation fixed. This corresponds well with the conditions suggested by Onsager for the validity of the point-vortex model that "vorticity is mostly concentrated in small regions."

The second main assumption invoked in Onsager's theory is the ergodicity of the point vortex dynamics. This is a standard assumption invoked in justifying Gibbsian statistical theory. It has, however, proved to be false! Khanin (1982) showed that a part of the phase space of the system of N point vortices in the infinite plane consists of integrable tori. His proof used the fact that the three-vortex system is exactly integrable (Novikov, 1975). By adding additional vortices successively at further and further distances and using the fact that these additional vortices only weakly perturb the previous system, one can apply Kolmogorov-Arnold-Moser theory iteratively to establish integrability of the N-vortex system. Of course, statistical mechanics does not require strict ergodicity because macroscopic observables are nearly constant over the energy surface. Thus any reasonable mixing over the energy surface will suffice to justify the use of a microcanonical ensemble. Of more serious concern are the possible slow time

[8]Expanding upon this remark, we note that the Joyce-Montgomery mean-field theory for point vortices can be generalized to allow for any finite number of circulation values, and the resulting theory bears a striking resemblance to the Robert-Miller theory. For each of the possible circulations κ, one can introduce a density

$$\rho_\kappa(\mathbf{r}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\kappa_i,\kappa}\delta(\mathbf{r}-\mathbf{r}_i), \tag{22}$$

which satisfies

$$\int d^2\mathbf{r}\,\rho_\kappa(\mathbf{r}) = p_\kappa, \tag{23}$$

where p_κ is the fraction of the N vortices with circulation strength κ. Thus also

$$\sum_\kappa \int d^2\mathbf{r}\,\rho_\kappa(\mathbf{r}) = 1. \tag{24}$$

The Joyce-Montgomery theory can easily be generalized to this case, with the result that the equilibrium densities become

$$\rho_\kappa(\mathbf{r}) = \frac{1}{Z}\exp\{-\beta[\kappa\psi(\mathbf{r}) - \mu_\kappa]\}, \tag{25}$$

where

$$-\Delta\psi(\mathbf{r}) = \frac{1}{Z}\sum_\kappa \kappa \exp\{-\beta[\kappa\psi(\mathbf{r}) - \mu_\kappa]\} = \omega(\mathbf{r}). \tag{26}$$

Except for normalization, this equation is identical in form to the Robert-Miller equation for an initial condition consisting of a finite number of vortex patches with vorticity levels σ_κ and areas A_κ. It can be formally derived from that equation in a "dilute-vorticity limit" in which $\sigma_\kappa=\kappa/\Delta$ and $A_\kappa=p_\kappa\Delta$, while $\Delta\to 0$.

scales of this mixing. Onsager also worried about this point when he wrote to Lin that:

> "I still have to find out whether the processes anticipated by these considerations are rapid enough to play a dominant role in the evolution of vortex sheets, and just how the conservation of momentum will modify the conclusions."

Lundgren and Pointin (1977) performed numerical simulations of the point-vortex model with initial conditions corresponding to several local clusters of vortices at some distance from each other. The equilibrium theory predicts their final coalescence into a single large supervortex. Instead, it was found that the clusters individually reach a "local equilibrium," not coalescing over the time scale of the simulation. Lundgren and Pointin argued theoretically that the vortices will eventually reach the equilibrium, single-vortex state. Similar metastable states of several large vortices have been seen in experiments with magnetically confined, pure electron columns and dubbed "vortex crystals" by Fine *et al.* (1995) and Jin and Dubin (2000). These states have been explained by a *regional maximum-entropy theory* in which entropy is maximized assuming a fixed number of the strong vortices (Jin and Dubin, 1998, 2000). Clearly, Onsager's ergodicity hypothesis is nontrivial and open to question.

Despite these caveats, equilibrium theories of large-scale vortices have had some notable successes. Onsager himself considered decaying wake turbulence in an "infinite vortex trail," as he wrote to Lin. Indeed pp. 28–31 of Folder 11:129 contain detailed calculations,[9] similar to those in Lamb (1932), Chap. 156, pp. 224 and 225. Analytical solutions of the mean-field Poisson-Boltzmann equation for vortex street geometries were later discovered (Chow *et al.*, 1998; Kuvshinov and Shep, 2000). Final states of freely decaying 2D Navier-Stokes simulations at high Reynolds number, started from fully turbulent initial conditions, have also been found to be in remarkable agreement with the predictions of the Joyce-Montgomery or sinh-Poisson mean-field equation (Montgomery *et al.*, 1992, 1993). Similar simulations started from a single band of vorticity, periodically modulated to induce Kelvin-Helmholtz instability, show good agreement with the generalized Robert-Miller theory (Sommeria *et al.*, 1991). In the limit of a thin initial band, the original Joyce-Montgomery mean-field theory is found to give identical results and agrees well with the simulations. Furthermore, the process is much as Onsager anticipated when he wrote to Lin:

> "… the sheet will roll up and possibly contract into concentrated vortices in some places, and at the same time the remaining sections of the sheet will be stretched into feeble, more or less haphazard distributed discontinuities of velocity."

For further comparisons of mean-field equations with results of numerical simulations, see Yin *et al.* (2003). A number of natural phenomena have been tentatively described by equilibrium vortex models of the sort proposed by Onsager. A fascinating example that was mentioned earlier is the Great Red Spot of Jupiter. For some recent work on this topic, see Turkington *et al.* (2001) and Bouchet and Sommeria (2002).

IV. THREE-DIMENSIONAL TURBULENCE

A. Onsager's cascade theory

The second half of Onsager (1949d), titled "Turbulence," deals with three-dimensional and fully developed turbulence. The second halves of the Pauling and Lin notes also discuss 3D turbulence. The Gibbsian statistical theory discussed in the first half of these documents does not describe a turbulent cascade process. As Onsager wrote at the end of the first section of Onsager (1949d) on two dimensions,

> "How soon will the vortices discover that there are three dimensions rather than two? The latter question is important because in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available."

Of course, it is no surprise that equilibrium statistical mechanics is inapplicable to a dissipative, irreversible process such as turbulence. More startling is Onsager's conclusion that turbulent motion remains dissipative even in the limit as molecular viscosity tends to zero. In the Pauling note of March 1945, he had already made a similar assertion:

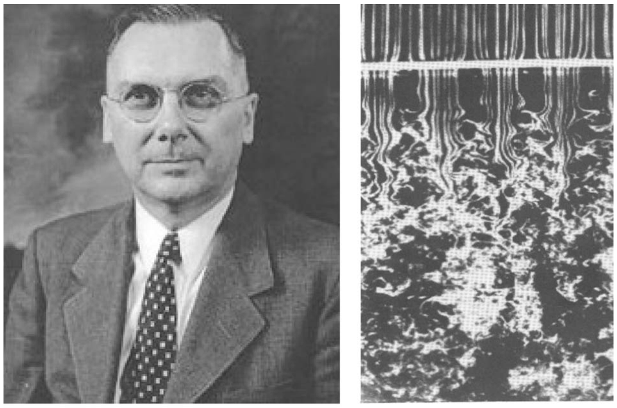
> "The energy is gradually divided up among ∞ degrees of freedom, only for sufficiently large $\underline{k}$ the viscosity disposes of it for good; but it does not seem to matter much just how large this $\underline{k}$ is."

This remark was repeated at greater length in the Lin note of June 1945 as well:

> "We anticipate a mechanism of dissipation in which the role of the viscosity is altogether secondary, as suggested by G. I. Taylor: a smaller viscosity is automatically compensated by a reduced micro-scale of the motion, in such a way that most of the *vorticity* will belong to the micro-motion, but only a small fraction of the energy."

Again, in the abstract of his APS talk in November, he wrote:

[9] Onsager's main object seems to have been to obtain analytical formulas for Lagrangian particle paths in the velocity field of a Kármán vortex street. He did this by introducing a suitable velocity at infinity, $-u_0$, to render the street stationary. He then used the constancy of the modified stream function $\chi = \psi - u_0 y$ along particle paths and the known formula for the stream function ψ of the Kármán street to deduce the equation of the curves corresponding to the particle paths.



(a) Hugh Latimer Dryden (1898–1965) (b) Grid turbulence

FIG. 6. (a) H. L. Dryden, the distinguished American aerodynamicist. His wind-tunnel experiments were a critical influence on Onsager's theories of 3D turbulence. (b) A smoke visualization of turbulent flow in a wind tunnel, where the grid is at the top and the mean flow is downward. (a) courtesy of the National Archives and Records Administration, and (b) with permission of Thomas Corke and Hassan Nagib.

> "In actual liquids this subdivision of energy is intercepted by the action of viscosity, which destroys the energy more rapidly the greater the wave number. However, various experiments indicate that the viscosity has a negligible effect on the primary process; hence one may inquire about the laws of turbulent dissipation in an ideal fluid."

For good measure, similar remarks were made no less than four times in the published paper (Onsager, 1949d). Considering the economy Onsager routinely prized in stating his results, it would appear that explaining the inviscid mechanism of energy dissipation in 3D turbulence was a chief preoccupation of Onsager's work on statistical hydrodynamics.

We can ask what evidence may have pushed Onsager in that direction. One reference in the 1949 paper was Dryden's review article (Dryden, 1943) on the statistical theory of turbulence. At the time, Dryden (see Fig. 6) was a researcher in aerodynamics at the National Bureau of Standards in Washington, D.C. Starting in 1929, he published a series of papers on the measurement of turbulence in wind tunnels. A problem he had studied was the decay of nearly homogeneous and isotropic turbulence behind a wire-mesh screen. Dryden used hot-wire anemometry techniques to take accurate measurements of turbulence levels v in the tunnel, where v denotes the velocity fluctuation away from the mean. This permitted him to determine the rate of decay of the turbulent kinetic energy[10] Q as $Q=-\frac{1}{2}(d/dt)\overline{v^2}$, where d/dt denotes the convective derivative. Let $V=(\overline{v^2})^{1/2}$ be the root-mean-square velocity fluctuation and L the spatial correlation length of the velocity, usually called the integral length scale. By simple dimensional analysis,

$$Q = A\frac{V^3}{L}, \tag{27}$$

where $A=A(\mathrm{Re})$ is a function only of the Reynolds number $\mathrm{Re}=VL/\nu$, with $\nu=\eta/\rho$ the kinematic viscosity of the fluid. Dryden found that $A(\mathrm{Re})$ tends to a constant at high Re. These and other data were collected by Batchelor (1953) several years later. The quality of the data in Batchelor's figure, certainly the data available in Onsager's day, was not compelling. In fact, several years later Saffman (1968) was led to remark: "The experimental evidence is far from convincing and would not rule out" weak dependencies on the Reynolds number.

However, modern experiments (Sreenivasan, 1984) have convincingly demonstrated that the turbulent energy dissipation in homogeneous and isotropic turbulence is independent of the molecular viscosity when the viscosity is small, or the Reynolds number large. Numerical solutions of the Navier-Stokes equations, these days an important tool, have confirmed this behavior (Sreenivasan, 1998). The recent numerical study of Kaneda *et al.* (2003) on a 4096^3 spatial grid has found that $A(\mathrm{Re})$ indeed asymptotes to a constant at high Reynolds numbers. The situation in shear flows is more complex and was summarized by Sreenivasan (1995). This complexity may perhaps be illustrated by citing the results of Cadot *et al.* (1997) in the Taylor-Couette flow. For smooth walls, distinctly different behaviors are observed in the bulk of the flow and at the boundary. Most of the dissipation is found to occur in a boundary layer at the walls of the apparatus, but this dissipation is a weakly decreasing function of the Reynolds number. On the other hand, the dissipation in the bulk obeys Eq. (27) with a coefficient that asymptotes to a constant at high Reynolds number.

Despite the inconclusive experimental evidence of the 1930s and 1940s, G. I. Taylor was also struck by the dramatic enhancement of dissipation in three-dimensional turbulence and proposed a dynamical explanation in terms of vortex stretching (Taylor, 1938). Onsager was well aware of Taylor's ideas, as he stated to Lin:

> "In terms of the Lagrangian description, the dissipation of energy in turbulent motion must be attributed to stretching of the vortex fibers, which generates vorticity more rapidly the more vigorous the motion and thus accelerates the final dissipation by viscosity (Taylor)."

Taylor's idea was again summarized by Onsager (1949d). The essential ingredients of G. I. Taylor's proposal are three facts of inviscid, incompressible fluid flow: (i) vortex lines are material lines, (ii) the volume of any material body is conserved, and (iii) the Kelvin-Helmholtz theorem, coupled with the reasonable assumption that

[10] Properly speaking, this quantity is the kinetic-energy dissipation per unit mass of fluid. It is now denoted in the turbulence literature, almost universally, by ε. We shall violate this convention in order to stay close to Onsager's notation in his papers and notes.

lines materially advected by a turbulent fluid will tend to lengthen (as well as become quite tangled and folded). Consider, then, a solid tube composed of a bundle of vortex lines. Because the lines are materially advected in a random way, they must, by the assumption, lengthen dramatically. Because the volume of the tube is conserved, its cross-sectional area dA must correspondingly shrink. Since the flux of vorticity along the tube ωdA is conserved in time, by the Helmholtz theorem the vorticity ω must be greatly magnified. Note that the energy dissipation itself may be related to the mean-square vorticity, or enstrophy, by

$$\frac{d}{dt}\int \frac{1}{2}v^2 d^3\mathbf{r} = -\nu \int \omega^2 d^3\mathbf{r}. \tag{28}$$

Thus the vortex-stretching process by which ω grows is a powerful engine of turbulence manufacturing prodigious amounts of energy dissipation (Taylor, 1938).

These considerations may have played some role in a mysterious discussion on pp. 2–6 at the beginning of Folder 11:129 of Onsager's unpublished notes. There he introduces the expression

$$I = \int_{\text{vortex}} \frac{d\ell}{|\omega|}, \tag{29}$$

where the integral is along a vortex line. On p. 4 Onsager gives a simple argument that this quantity is a material invariant. Indeed, $d\ell/|\omega| = d\ell dA/|\omega| dA$, where the numerator is a material invariant by incompressibility and the denominator by the Helmholtz theorem. Onsager then considers using this quantity for a phenomenology of turbulence. For example, on p. 2 he writes down a dimensionally correct expression for an "eddy viscosity" ν^*:

$$\nu^* = I^{1/2}V^{3/2}. \tag{30}$$

This is not a standard formula in the literature. Indeed, eddy viscosity can be defined by the formula $Q = \nu^*(V/L)^2$, which assumes that it accounts for the dissipation based on velocity gradients $\sim V/L$ at the large length scale L. In that case, using the earlier formula (27), one gets $\nu^* \sim VL$, which is the customary estimate. One only gets agreement with Onsager's formula (30) if one assumes that $I \sim L^2/V$. In fact, on pp. 5 and 6 Onsager uses such a formula to define a length scale L as a "Prandtl mixing length" for inhomogenous turbulence:

$$L^2 = IV. \tag{31}$$

But this leaves open several questions. First, how exactly is I defined? For example, is the integral along vortex lines attached to the boundary or along closed lines, like vortex rings? If it is along an infinite line, what is the range of integration? Second, if one takes the formula as defining a length scale L, then does it coincide (to within a factor) with the standard integral length scale? And, if the formula is correct, what is its special merit? Why is it important that I should be a material invariant? Some light may be shed by a remark in the Lin note:

> "The distribution law (19) [the $-5/3$ spectrum] is compatible with the hypothesis that the mean rate of stretching of vortex lines is given by the average rate of deformation in the liquid."

Onsager seems to have been searching for a phenomenological formulation of Taylor's view. Nevertheless, these pages of his notes are not intuitively clear.

In any case, Onsager had developed his own view of the process of turbulent energy dissipation. In the Pauling note he wrote:

> "Finally, the subdivision of the energy is a *stepwise* process (mostly) such that the wave-number increases typically by a factor between, say, 1 and 3, in each step, and the terms in which $\underline{k}$, $\underline{k}'$ and $(\underline{k} - \underline{k}')$ are of the same order of magnitude have to do most of it."

Again in the note to Lin:

> "The selection rule for the 'modulation' factor in each term of (8) suggests a 'cascade' mechanism for the process of dissipation, and also furnishes a dynamical basis for an assumption which is usually made on dimensional grounds only."

On an interesting historical note, this passage seems to contain the first use of the word "cascade" in the theory of turbulence. The same uniquely suggestive term was used again by Onsager in his 1945 abstract and in the 1949 article. For example, in Onsager (1949d) he wrote:

> "In order to understand the law of dissipation described by (11) [our Eq. (27)], which does not involve viscosity at all, we have to visualize the redistribution of energy as an accelerated cascade process."

Thus Onsager claims that this cascade is scale local, or between scales of the same order of magnitude, and accelerated. Both of these claims require some explanation.

Most of Onsager's considerations on the cascade are in wave-number space. Therefore we must say a bit about the Fourier transform of the Navier-Stokes equation

$$\frac{d}{dt}\mathbf{a}(\mathbf{k}) = -2\pi i \sum_{\mathbf{k}'} [\mathbf{a}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'] \times\left(\mathbf{a}(\mathbf{k}') - \frac{1}{k^2}[\mathbf{a}(\mathbf{k}')\cdot\mathbf{k}]\mathbf{k}\right) - \nu|2\pi\mathbf{k}|^2\mathbf{a}(\mathbf{k}). \tag{32}$$

Here $\mathbf{a}(\mathbf{k})$ are the Fourier coefficients of the velocity $\mathbf{v}(\mathbf{r})$ in a periodic box. Energy transfer between wave numbers is described by

$$\frac{d}{dt}|\mathbf{a}(\mathbf{k})|^2 = -2\nu|2\pi\mathbf{k}|^2|\mathbf{a}(\mathbf{k})|^2 + \sum_{\mathbf{k}'} Q(\mathbf{k},\mathbf{k}'), \tag{33}$$

where

$$Q(\mathbf{k},\mathbf{k}') = \pi i\{[\mathbf{a}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}'][\mathbf{a}(-\mathbf{k})\cdot\mathbf{a}(-\mathbf{k}')] + [\mathbf{a}(-\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}'][\mathbf{a}(\mathbf{k})\cdot\mathbf{a}(-\mathbf{k}')]\} + \text{c.c.} \tag{34}$$

The symbol c.c. denotes complex conjugate. The precise formula for Q is not that important at the moment, but what is important is the easily established identity

$$Q(\mathbf{k},\mathbf{k}') + Q(\mathbf{k}',\mathbf{k}) = 0. \tag{35}$$

The quantity Q represents the instantaneous transfer of energy out of wave number $\mathbf{k}$ and into wave number $\mathbf{k}'$, mediated by a third wave number $\mathbf{k}''=\pm\mathbf{k}\pm\mathbf{k}'$. The three wave numbers $\mathbf{k},\mathbf{k}',\mathbf{k}''$ are usually called a triad. The relation (35) expresses the fact that any energy leaving wave number $\mathbf{k}$ by the triadic interaction must appear in the wave number $\mathbf{k}'$. Note that the third, convective wave number $\mathbf{k}''$ plays a purely passive or catalytic role in the process and does not give or receive energy itself. The identity (35) is called *detailed energy conservation* for the triad and was observed by Onsager in the Lin note and published in the 1949 paper.

The concept of cascade can now be made more explicit, in Onsager's (1949d) own words:

> "…we note that according to (16a) [our Eq. (34)] the exchange of energy between wavenumbers $\pm\mathbf{k}$ and $\pm\mathbf{k}'$ depends only on the amplitudes $\mathbf{a}$ which belong to these wave-numbers and to their differences $(\pm\mathbf{k}\pm\mathbf{k}')$. If the latter, as well as $\mathbf{k}$ itself, are of the order $1/L$, then $\mathbf{k}'$ is at most of the order $2/L$. Similar reasoning may be applied to subsequent steps in the redistribution process, and we are led to expect a cascade such that the wave-numbers increase typically in a geometric series, by a factor of the order of 2 per step."

This is the key statement of scale locality: the essential interactions in the cascade are between wave numbers of similar magnitude. Therefore very distant scales are not involved in the transfer and the energy is passed in each successive step to a wave number higher by a factor of about 2. If this transfer process is also chaotic, the information about the low wave numbers will tend to be lost after many random steps, except for the constraints imposed by total conservation laws. This motivates the idea that the small scales of turbulence, or the high wave numbers, will have their statistical properties completely determined by the energy flux Q from the large scales.[11]

This was the basis for Onsager's announcement in 1945 of the energy spectrum $E(k)\sim\beta Q^{2/3}k^{-5/3}$, with a universal dimensionless constant β. Of course, Kolmogorov was led by similar considerations to the same conclusion a few years earlier (Kolmogorov, 1941a, 1941b, 1941c).[12]

These ideas can also be used to make an estimate of the time required for each successive cascade step. A dimensional reasoning similar to that above implies that the turnover time required for processing of energy through wave number k should be of the order

$$\tau(k) \sim Q^{-1/3}k^{-2/3}. \tag{36}$$

As one can see, the time becomes shorter as the wave number increases: this is why the cascade is called *accelerated*. The estimate (36) was already given in Eq. (24) of the Lin note, in Onsager's private notes (on p. 22 of Folder 11:129), and in Onsager (1949d). In all places, Onsager shows that the total time to go from wave number $k=1/L$ to wave number $k=\infty$ is finite:

$$\int_{1/L}^{\infty}\frac{dk}{k}\tau(k) < \infty. \tag{37}$$

Essentially, the steps in the cascade accelerate so quickly that—if not interrupted earlier by viscosity—it would require only a finite amount of time for energy to be passed via nonlinear interactions from a low wave number to an infinitely high wave number! This calculation of Onsager's has sometimes been used to argue for the idea that 3D incompressible Euler equations, started from smooth initial conditions, will develop singularities in finite time. As noted by Frisch (1995), however, this argument is not so clear because the validity of the formula (36) for the turnover time for any arbitrarily high wave number k presumes preexisting singularities.

B. Euler singularities and dissipative anomaly

Onsager did make a remarkable statement about 3D Euler singularities at the very end of the 1949 paper (Onsager, 1949d) relating them to the observed properties of turbulent energy dissipation:

> "It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the veloc-

[11] In fact, the small scales seem to remember the large length scale L, in addition to the number of cascade steps, due to a buildup of fluctuations in the course of the cascade. We discuss this "small-scale intermittency" in Sec. IV.C. Since $Q\sim V^3/L$ in homogeneous and isotropic turbulence, which occupied most of Onsager's attention, remembering L is the same as remembering V. In shear flows, the memory of V persists independently of L and can corrupt scaling (Sreenivasan and Dhruva, 1998). The principal effect of the shear often appears as a subleading term in scaling and can be taken into account with modest ingenuity (Arad *et al.*, 1998).

[12] As we discuss in detail below, Onsager was unaware of Kolmogorov's work until sometime after the end of June 1945. By the time of his Florence paper, Onsager had studied Kolmogorov's papers and had this to say: "…a promising start towards a quantitative theory of turbulence was achieved by Kolmogoroff… . For good measure, Kolmogoroff's main result was rediscovered at least twice… ." In the second sentence, he was referring to his own APS abstract (Onsager, 1945c) and to the papers of von Weizsäcker (1948) and Heisenberg (1948).

ity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

(26) $|\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')|<(\text{const.})r^n$

for any order n greater than $1/3$; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation (15) [our (32)] in terms of Fourier series will do. The detailed conservation of energy (17) [our (35)] does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected, and the double sum of $Q(\mathbf{k},\mathbf{k}')$ converges only conditionally."

These are the closing words of Onsager's paper and also his last published thoughts on the subject of singularities and dissipation for Euler equations.

What did Onsager mean? Clearly he proposed that there would be singularities of Euler equations, whether finite-time or otherwise, since he states that "the velocity field does not remain differentiable" in the inviscid limit as $\nu\to 0$. This is to be expected if, in that limit, a $k^{-5/3}$ spectrum develops all the way up to $k=\infty$, because such a slow decay in wave number implies that $\overline{|\boldsymbol{\nabla}\mathbf{v}|^2}=+\infty$ and that classical derivatives of the velocity can no longer exist. More remarkably, Onsager proposed that even for such a singular limit, the velocity field will satisfy the incompressible Euler equations in a suitable sense, e.g., Eqs. (32) for the Fourier coefficients will hold with $\nu=0$. Equivalently, the Euler equations will hold if the derivatives are taken in the sense of distributions. In that case, Onsager noted that there is a precise, minimal degree of singularity required to lead to dissipation in the ideal fluid. If the velocity satisfies a Lipschitz or Hölder condition with exponent $>1/3$, then energy is conserved. Therefore, to account for the observed dissipation in the inviscid limit, a Hölder singularity of exponent $\leq 1/3$ must appear at least at some points in the flow. In this picture, the turbulent velocity fields in the inviscid limit are continuous, nowhere differentiable functions, similar to ideal Brownian paths. What is especially remarkable about Onsager's claim is that Kolmogorov's scaling exponent of $1/3$ comes out of a dimensional argument for all statistical moments of velocity differences, but without direct use of the equations of motion. Here the exponent $1/3$ is shown to have dynamical significance.

After Onsager's talk at the Florence meeting, there was no comment regarding the $1/3$ Hölder singularity claim. In fact, very little note seems to have been taken of Onsager's remark for quite a long time. Even J. von Neumann in his authoritative review article (von Neumann, 1949) does not mention it. There was related but apparently completely independent work at about the same time by Burgers (1948), who proposed a simple model equation for turbulence which illustrated the possibility of singularity formation and its relation to energy dissipation. [See also the earlier work of Wiener (1938).] Burgers equation describes a 1D compressible fluid for which the singularities are simple shock discontinuities in the velocity profile. This work subsequently attracted a great deal of attention of both mathematicians and physicists (Lax, 1972; Frisch and Bec, 2001). However, Onsager's remark seems to have been nearly forgotten, except by a few experts in turbulence theory. Sulem and Frisch (1975) proved a related result for the 3D Euler equations, in Sobolev rather than Hölder spaces. Their result states that energy is conserved for 3D Euler equations if the energy spectrum of the solution is steeper than $k^{-8/3}$. This theorem neither implies nor is implied by the Onsager result. For example, a velocity field with Hölder regularity slightly greater than $1/3$ everywhere would have a spectrum just a little steeper than Kolmogorov's $k^{-5/3}$ and would conserve energy by Onsager's result, but not by that of Sulem and Frisch.

A result close to the one claimed by Onsager was first proved by Eyink (1994). The proof was based on the brief argument sketched by Onsager (1949d) using Fourier series. Total energy conservation for Euler equations naively follows from the calculation

$$\frac{d}{dt}\sum_{\mathbf{k}}|\mathbf{a}(\mathbf{k})|^2=\sum_{\mathbf{k}}\sum_{\mathbf{k}'}Q(\mathbf{k},\mathbf{k}')=\frac{1}{2}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\{Q(\mathbf{k},\mathbf{k}')+Q(\mathbf{k}',\mathbf{k})\}=0, \tag{38}$$

using the detailed conservation (35). However, this argument requires reordering the infinite summations over $\mathbf{k},\mathbf{k}'$, and that is inadmissable if the series are only conditionally convergent. In that case, the series can give results that depend on the order of summation. On the other hand, Onsager's claim seems to be that the series are absolutely convergent if the Hölder condition with exponent $n>1/3$ is valid. In terms of Fourier coefficients, the Hölder condition holds if

$$\sum_{\mathbf{k}}|\mathbf{k}|^n|\mathbf{a}(\mathbf{k})|<\infty; \tag{39}$$

see Zygmund (2002). If absolute convergence follows from this bound with $n>1/3$, then the formal calculation is correct and energy is conserved.

It is not hard to show that conservation of energy is also implied by a weaker condition on the *spectral energy flux*

$$\Pi(K)=-\sum_{|\mathbf{k}|<K}\sum_{\mathbf{k}'}Q(\mathbf{k},\mathbf{k}'), \tag{40}$$

a quantity which measures the flow of energy under the nonlinear interactions out of a sphere in Fourier space of radius K. If the double series in Eq. (38) is absolutely convergent (and thus equals zero), then

$$\lim_{K\to\infty}\Pi(K)=0. \tag{41}$$

This by itself is enough to conclude that energy is conserved, but the asymptotic energy flux may be zero even if the series converges only conditionally. In fact, Sulem and Frisch (1975) analyzed energy flux to prove their theorem. Now it is not hard to check that if condition (39) holds and, furthermore, if the local triads dominate in the sum (40) defining the flux, then

$$\Pi(K) = O(K^{1-3n}). \tag{42}$$

Therefore Onsager's conservation claim will be true if the local interactions indeed dominate in the energy flux. However, the matter is delicate. In Eyink (1994) a model velocity field was constructed as a counterexample which showed that, in some cases at least, $\Pi(K) \sim K^{1-2n}$ as $K\to\infty$. Thus the absolute convergence suggested by Onsager's 1949 remarks, literally speaking, does not hold true. The mechanism involved in this counterexample is the transfer by a small distance in Fourier space via highly nonlocal triads with one wave number slightly $<K$, one slightly $>K$, and a convective mode with very small wave number. Similar interactions have been seen to dominate in $\Pi(K)$ in numerical simulations of the Navier-Stokes equations (Domaradzki and Rogallo, 1990; Yeung and Brasseur, 1991).

Nonetheless, the essence of Onsager's claim is correct. For example, if one averages the spectral flux over an octave band, then local triads do dominate and

$$\bar{\Pi}(K) = \frac{1}{K}\int_K^{2K} dK'\,\Pi(K') = O(K^{1-3n}). \tag{43}$$

This observation yielded the proof by Eyink (1994) of Onsager's conservation claim. It was also shown there by an example that estimate (43) is sharp and cannot be improved. In particular, constant energy flux is possible for $n=1/3$. As a by-product of this analysis, it was also shown by Eyink (1994) that Onsager's assertion of locality holds for the averaged flux (43) whenever the velocity field satisfies Eq. (39) with $0<n<1$. This is exactly in agreement with Onsager's remark to Lin: "With a hypothesis slightly stronger than (14) the motion which belongs to wave-numbers of the same order of magnitude as $\underline{k}$ itself will furnish the greater part of the effective rate of shear." Onsager's Eq. (14) in the Lin note is just the condition that $\overline{v^2}<\infty$, $\overline{(\nabla\times v)^2}=\infty$; indeed, velocity fields satisfying Eq. (39) with $0<n<1$ are continuous but generally nondifferentiable.

The argument by Eyink (1994) still did not quite establish Onsager's claim because technically the Fourier condition (39) is sufficient but not necessary for Hölder continuity of index n (Zygmund, 2002). However, shortly thereafter, Constantin *et al.* (1994) found a proof of Onsager's precise statement and, in fact, proved a sharper result. We shall discuss their important theorem below. Here let us just note that their argument was given entirely in physical space and did not use Fourier methods. More recently, another similar proof was given by Duchon and Robert (2000). Of primary interest is the connection that they established between Onsager's theorem and another result of turbulence theory, the Kolmogorov 4/5 law (Kolmogorov, 1941c). This connection will also be discussed below.

It is remarkable that Onsager's claim is exactly correct, although the hints he gave using Fourier series do not yield quite the claimed result. It does not seem that Onsager ever worked through the details of a Fourier space proof. If so, then what was the basis of his accurate claim? The surprising answer seems to be that he had a valid proof in physical space and that it was virtually identical to that of Duchon and Robert (2000) 50 years later! The crucial result, which bears directly on this question, is Eq. (26) in the Lin note, which appears again on p. 14 of Folder 11:129 and whose derivation from the incompressible Euler equations is contained in pp. 15–20 of the Folder. The derivation is straightforward and need not concern us in detail (e.g., see Duchon and Robert, 2000). If $\Delta_{\mathbf{r}}\mathbf{v}(\mathbf{r}')=\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')$ is the velocity increment, then the crucial identity that Onsager derived was

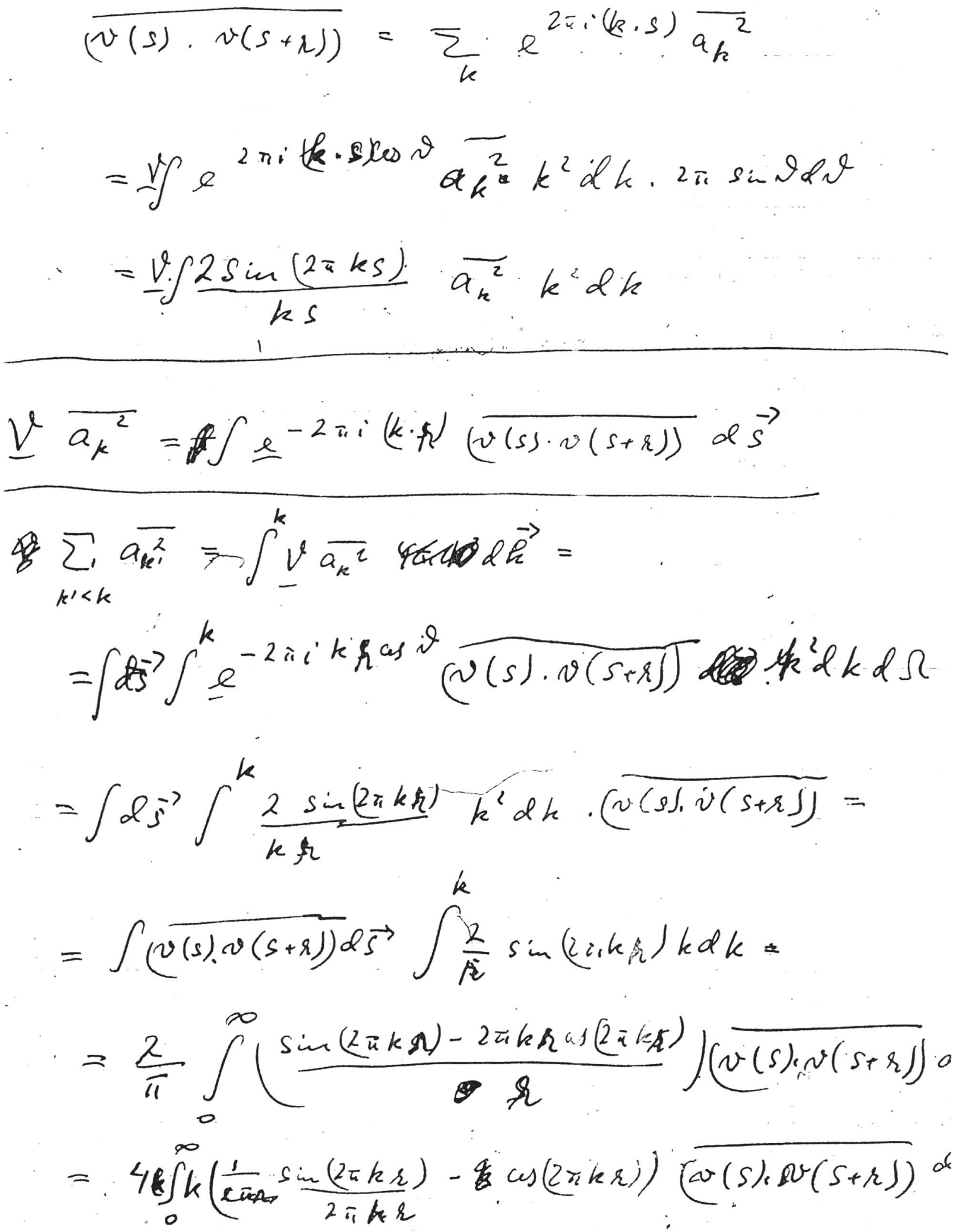
$$\frac{\partial}{\partial t}\int d^3\mathbf{r}F(r)\overline{\mathbf{v}(\mathbf{r}')\cdot\mathbf{v}(\mathbf{r}'+\mathbf{r})} = \frac{1}{2}\int d^3\mathbf{r}F'(r)\overline{(\hat{\mathbf{r}}\cdot\Delta_{\mathbf{r}}\mathbf{v})|\Delta_{\mathbf{r}}\mathbf{v}|^2}. \tag{44}$$

Here $F(\mathbf{r})$ is a spatial smoothing function assumed to be spherically symmetric. We have made a few minor simplifications of Onsager's notations for the sake of clarity. The motivation for this formula is made more clear by pp. 19 and 20 of Folder 11:129, reproduced here. On p. 20 (see Fig. 7) Onsager derived a formula expressing $\Sigma_{k'<k}|\mathbf{a}(\mathbf{k}')|^2$ in terms of an integral of the spatial velocity correlation. The time derivative of this quantity under the Euler dynamics is just (the negative of) what we called the energy flux $\Pi(k)$. However, because of the sharp cutoff in Fourier space, this results in a highly oscillatory integral, which is hard to control. Onsager solved that problem by using a smooth spectral cutoff function $f(\mathbf{k})$ with inverse Fourier transform $F(\mathbf{r})$. Thus the sum over low-wave-number Fourier modes is written on p. 19 (Fig. 8) as

$$\sum_{\mathbf{k}} f(\mathbf{k})|\mathbf{a}(\mathbf{k})|^2 = \int d^3\mathbf{r}F(\mathbf{r})\overline{\mathbf{v}(\mathbf{r}')\cdot\mathbf{v}(\mathbf{r}'+\mathbf{r})}, \tag{45}$$

where $\overline{(\ )}$ may be taken to be a spatial average over $\mathbf{r}'$ in the periodic box.[13] Notice that the oscillatory integral is now replaced by an integral with respect to the spatial smoothing function $F(\mathbf{r})$. The negative of the time derivative of Eq. (45) defines a scale-averaged energy flux similar to Eq. (43), and it is exactly this time derivative that appears on the left-hand side of Onsager's fundamental relation (44). Now, if one assumes that $\Delta_{\mathbf{r}}\mathbf{v} = O(r^n)$ and if F filters out scales $<\ell$, then the overall scaling of the right-hand side of Eq. (44) is $O(\ell^{3n-1})$ (where the -1 comes from the derivative on F), which

[13]Onsager's calculations started off with the assumption that this was an ensemble average. However, in the course of his proof of Eq. (44) he dropped this bar and replaced it with a spatial average only.



$$\overline{(v(s)\cdot v(s+r))} = \sum_k e^{2\pi i(k\cdot s)}\,\overline{a_k^2}$$

$$= V\int e^{2\pi i ks\cos\vartheta}\,\overline{a_k^2}\,k^2dk\cdot 2\pi\sin\vartheta\, d\vartheta$$

$$= V\int \frac{2\sin(2\pi ks)}{ks}\,\overline{a_k^2}\,k^2dk$$

$$V\,\overline{a_k^2} = \int e^{-2\pi i(k\cdot r)}\,\overline{(v(s)\cdot v(s+r))}\,d\vec{s}$$

$$\sum_{k'<k}\overline{a_{k'}^2} = \int^k V\,\overline{a_k^2}\,d\vec{k} =$$

$$= \int d\vec{s}\int^k e^{-2\pi i kr\cos\vartheta}\,\overline{(v(s)\cdot v(s+r))}\,k^2dk\,d\Omega$$

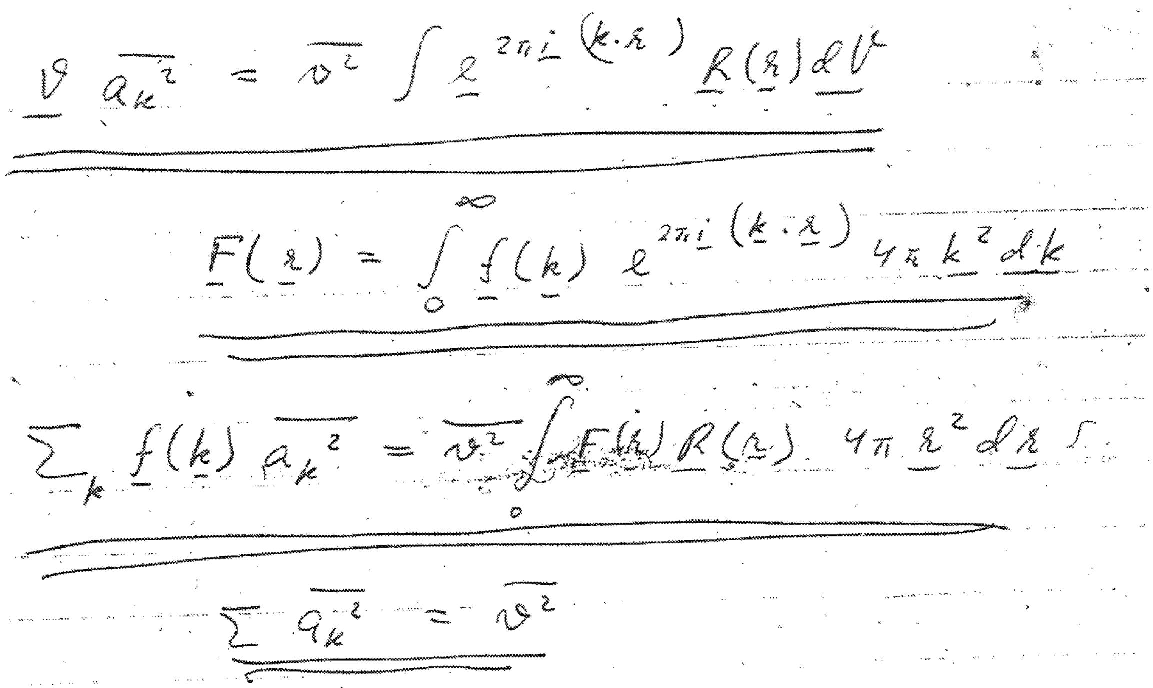
$$= \int d\vec{s}\int^k \frac{2\sin(2\pi kr)}{kr}\,k^2dk\cdot\overline{(v(s)\cdot v(s+r))} =$$

$$= \int\overline{(v(s)\cdot v(s+r))}\,d\vec{s}\int^k \frac{2}{r}\sin(2\pi kr)\,k\,dk =$$

$$= \frac{2}{\pi}\int_0^\infty\left(\frac{\sin(2\pi kr)-2\pi kr\cos(2\pi kr)}{r}\right)\overline{(v(s)\cdot v(s+r))}$$

$$= 4k\int_0^\infty k\left(\frac{\sin(2\pi kr)}{2\pi kr}-\cos(2\pi kr)\right)\overline{(v(s)\cdot v(s+r))}\,d$$

FIG. 7. Page 20 of Folder 11:129 (Onsager, circa 1945). Onsager divided the page by two lines. Above the top line the two-point correlation function of the velocity field is written as a Fourier transform of the energy spectrum (Wiener-Khinchin theorem). Between the two lines is the inverse relation, giving the energy spectrum in terms of the correlation function. Below the bottom line, Onsager used this result to derive a corresponding formula for the total energy in Fourier modes with wave-number magnitude less than k. Reproduced courtesy of the Onsager Archive, NTNU.



$$V\,\overline{a_k^2} = \overline{v^2} \int e^{2\pi i (k\cdot r)}\, R(r)\, dV$$

$$F(r) = \int_0^\infty f(k)\, e^{2\pi i (k\cdot r)}\, 4\pi k^2\, dk$$

$$\sum_k f(k)\,\overline{a_k^2} = \overline{v^2} \int_0^\infty F(r)\, R(r)\, 4\pi r^2\, dr$$

$$\sum \overline{a_k^2} = \overline{v^2}$$

FIG. 8. Page 19 of Folder 11:129 (Onsager, circa 1945). A smoothed version of the Wiener-Khinchin theorem employing a spectral filtering function $\mathbf{f}(\mathbf{k})$ with inverse Fourier transform $F(\mathbf{r})$ in physical space. Reproduced courtesy of the Onsager Archive, NTNU.

vanishes as $\ell \to 0$ if $n > 1/3$. This is exactly a space-integrated form of the argument used by Duchon and Robert (2000) to prove Onsager's stated result about $n = 1/3$. As we shall discuss later, virtually the same argument appears at the end of the note to Lin. It is hard to avoid the inference that this was the basis for the claim in Onsager (1949d).

The basic identity (44) is closely related to an expression derived by von Kármán and Howarth (1938), which is one of the references cited at the beginning of Folder 11:129. Moreover, Onsager's identity is an exact analog of the 4/5 law derived by Kolmogorov in the third of his 1941 papers (Kolmogorov, 1941c) making use of the earlier calculation of von Kármán and Howarth (1938). Kolmogorov assumed homogeneity and isotropy of turbulence, and also, crucially, that energy dissipation remains finite in the limit as viscosity tends to zero. A good derivation of the 4/5 law is given in the book by Frisch (1995) (which is essentially identical to the calculation of Onsager). He first derived a form of the law assuming homogeneity alone without isotropy, which in Frisch (1995) was called the Kolmogorov-Monin relation

$$\nabla_{\mathbf{r}} \cdot \overline{\Delta_{\mathbf{r}}\mathbf{v}|\Delta_{\mathbf{r}}\mathbf{v}|^2} = -4Q. \tag{46}$$

Here, as before, Q represents the energy dissipation per unit mass, which is assumed to remain positive in the inviscid limit. To get Onsager's identity—in a statistically averaged sense—one must simply integrate both sides of Eq. (46) with respect to $F(r)$ and use $\nabla_{\mathbf{r}} F(r) = F'(r)\hat{\mathbf{r}}$. The identity derived by Kolmogorov also used isotropy and was in the form

$$\overline{(\hat{\mathbf{r}} \cdot \Delta_{\mathbf{r}}\mathbf{v})^3} = -\frac{4}{5} Q r. \tag{47}$$

This is the classical statement of the 4/5 law (Kolmogorov, 1941c). It is noteworthy that Lin remarked to Onsager in his reply on June 26, 1945 that the "cascade process of the dissipation of energy in turbulence can also be seen from an equation derived rigorously from the Kármán-Howarth equation" (Lin, 1945b)—referring to an exact equation derived by Lin himself for the evolution of the energy spectrum—but did not point out its relation to Eq. (26) in Onsager's note.

The proof by Duchon and Robert (2000) in fact yielded an important generalization of Onsager's result and of the Kolmogorov-Monin relation. Under a very modest assumption—namely, that $\int_0^T dt \int d^3\mathbf{r} |\mathbf{v}(\mathbf{r},t)|^3 < \infty$—they proved that a singular solution of the incompressible Euler equations satisfies a local energy balance

$$\partial_t \frac{1}{2}|\mathbf{v}|^2 + \nabla \cdot \left[\left(\frac{1}{2}|\mathbf{v}|^2 + p \right) \mathbf{v} \right] = -D(\mathbf{v}). \tag{48}$$

The derivatives in this equation must be interpreted in the sense of distributions. If the Euler solutions are smooth, then the right-hand side will be zero, implying conservation of the energy. In general, however, that term need not vanish. Duchon and Robert established for it the identity

$$D(\mathbf{v}) = \lim_{\ell\to 0}\frac{1}{4}\int d^3\mathbf{r}\nabla_{\mathbf{r}}F_\ell(\mathbf{r})\cdot\Delta_{\mathbf{r}}\mathbf{v}|\Delta_{\mathbf{r}}\mathbf{v}|^2, \tag{49}$$

where $F_\ell(\mathbf{r})=(1/\ell^3)F(\mathbf{r}/\ell)$. Thus this term represents the energy flux asymptotically to zero length scale. If the Euler solution is obtained as a zero-viscosity limit of a Navier-Stokes solution, then it is also true that[14]

$$D(\mathbf{v}) = \lim_{\nu\to 0}\nu|\nabla\mathbf{v}|^2 \geqslant 0. \tag{50}$$

Combining the two expressions for $D(\mathbf{v})$ gives a space-time local form of the Kolmogorov-Monin relation. Duchon and Robert proved furthermore that

$$D(\mathbf{v}) = \lim_{r\to 0} -\frac{3}{4r}\langle(\hat{\mathbf{r}}\cdot\Delta_{\mathbf{r}}\mathbf{v})|\Delta_{\mathbf{r}}\mathbf{v}|^2\rangle_{\mathrm{ang}}, \tag{51}$$

where $\langle\,\rangle_{\mathrm{ang}}$ denotes a spherical average over the direction vector $\hat{\mathbf{r}}$. This is a local form of what is now called the "4/3 law" in turbulence theory. Eyink (2003) has shown that the term $D(\mathbf{v})$ can also be expressed as a local form of the original Kolmogorov 4/5 law.

In a paper on two-dimensional turbulence,[15] Polyakov (1993) pointed out an interesting analogy of the 4/5 law, and corresponding laws in 2D turbulence, to conservation-law anomalies in quantum field theory, e.g., the axial anomaly in quantum electrodynamics. In particular, the derivation of the 4/5 law from the dynamics of the two-point velocity correlation—as in Onsager's unpublished notes—was rediscovered by Polyakov who pointed out its similarity to Schwinger's derivation (Schwinger, 1951) of the axial anomaly by a (gauge-invariant) point-splitting regularization. In the case of the axial anomaly one obtains a local balance for axial charge as

$$\partial_\mu J_5^\mu = 2mJ_5 + D(A). \tag{53}$$

This equation is to be interpreted in the sense of the Heisenberg equations of motion for (renormalized) local composite field operators

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi, \quad J_5 = i\bar{\psi}\gamma_5\psi, \quad D(A) = \frac{\alpha}{2\pi}\tilde{F}_{\mu\nu}F^{\mu\nu}. \tag{54}$$

See Itzykson and Zuber (1980), for example, for relevant definitions and notations. If the mass $m=0$, then axial charge would be conserved if $D(A)$ were zero. The fact that it is not zero has physical consequences, such as the electromagnetic decay of the neutral pion $\pi^0\to\gamma\gamma$ (Adler, 1969; Bell and Jackiw, 1969).

The local balance equation (48) derived by Duchon and Robert (2000) for singular Euler equations is similar in structure to the anomalous conservation equation (53) in quantum gauge theory. Therefore we believe that Onsager's result on dissipative Euler solutions does have some analogy to anomalies in quantum field theory. In fact, the term $D(\mathbf{v})$ in Eq. (48) is nowadays often referred to in the turbulence literature as the *dissipative anomaly*. However, there are also some important differences between the dissipative anomaly $D(\mathbf{v})$ in turbulent solutions of Euler equations and the axial anomaly $D(A)$ in quantum gauge theory. For example, we have seen that the dissipative anomaly is always non-negative, $D(\mathbf{v})\geqslant 0$, whereas the axial anomaly can have either sign. The axial anomaly is also formally a total divergence $D=\partial_\mu K^\mu$ with $K^\mu=(\alpha/2\pi)\epsilon^{\mu\nu\sigma\rho}A_\nu F_{\sigma\rho}$. Its global integral is only nonzero for topologically nontrivial gauge fields A, related to the Atiyah-Singer index theorem (Jackiw and Rebbi, 1977; Nielsen and Schroer,

[14]Here we assume that there are no singularities in the solution of the Navier-Stokes equation at any small, but finite, viscosity. Otherwise the statement of this result must be slightly generalized (Duchon and Robert, 2000).

[15]A brief comment about 2D turbulence may be in order here. Onsager (1949d), in the first paragraph of the section on turbulence, pointed out that "the enhanced dissipation which takes place in turbulent motion cannot be explained by any mechanism of two-dimensional convection." After considering the energy balance relation (28) for incompressible hydrodynamics, he pointed out that "two-dimensional convection, which merely redistributes vorticity, cannot account for the rapid dissipation." Thus, as Onsager already observed, there can be no forward energy cascade and no dissipative anomaly for energy in two-dimensional flow. Related observations were made by von Neumann (1949), Lee (1951), Fjørtoft (1953), and Batchelor (1953), Sec. 8.4, all using the fact that enstrophy, or mean-square vorticity,

$$\Omega(t) = \frac{1}{2}\int d^2\mathbf{r}\,\omega^2(\mathbf{r},t), \tag{52}$$

is a conserved quantity for smooth 2D Euler flows. However, it was later observed by G. K. Batchelor, in work with his student R. W. Bray (1966), that there may be a forward enstrophy cascade and a corresponding dissipative anomaly for enstrophy in two-dimensional flow. They predicted a high-wave-number energy spectrum $\propto Z^{2/3}k^{-3}$ in such two-dimensional turbulence, where $Z=\langle\nu|\nabla\omega|^2\rangle$ is the mean rate of dissipation of enstrophy. This theory was published only somewhat later by Batchelor (1969) and, in the meantime, independently by Kraichnan (1967). Furthermore, Kraichnan in the latter paper pointed out that this cascade of enstrophy to high wave numbers should coexist with an inverse energy cascade to low wave numbers, corresponding to a Kolmogorov-type energy spectrum $\propto|Q|^{2/3}k^{-5/3}$ but now with energy flux $Q<0$. For more about this "dual cascade picture" of two-dimensional turbulence, see Sec. 9.7 of Frisch (1995) and references therein. It is now known that the 2D Euler equations have no finite-time singularities starting from smooth initial data (e.g., Rose and Sulem, 1978) and thus a dissipative anomaly for enstrophy in freely decaying 2D turbulence requires starting with sufficiently singular initial data (Eyink, 2001).

Onsager never considered such two-dimensional turbulent cascades in any of his published works. However, it is interesting to observe that p. 7 of Folder 11:129 of his unpublished notes contains the two-dimensional Navier-Stokes equation for the vorticity field in Fourier representation. This suggests that Onsager was considering the triadic interactions of the vorticity field that lead to enstrophy cascade. Of course, Onsager (1949d) had also pointed out the tendency of energy in two-dimensional flow to accumulate into large-scale vortex structures and this is one of the prior works which helped lead Kraichnan (1967) to his theory of the inverse energy cascade.

1977). The dissipative anomaly for Euler equations does not have such a topological interpretation.

C. Intermittency and anomalous scaling

We now discuss the strengthening of Onsager's theorem due to Constantin *et al.* (1994), although only insofar as it ties into our next subject, anomalous scaling. A physical interpretation of their method from the point of view of nonequilibrium thermodynamics is given by Eyink (1995b). What we emphasize here is that they gave the first proof of the theorem under Onsager's precise Hölder continuity condition and, in fact, under a sharper condition involving Besov spaces; for example, see Eyink (1995a). The Besov space B_p^s is very simply defined: it consists of functions that are Hölder continuous with index s, not pointwise but in the sense of spatial pth-order moments

$$\left[\int d^3\mathbf{r}'|\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')|^p\right]^{1/p} \leqslant (\text{const})r^s. \tag{55}$$

The result of Constantin *et al.* (1994) is that a singular solution of the Euler equations will conserve energy if the velocity field has Besov regularity with $s>1/3$ for any $p\geqslant 3$. Since the Besov spaces for $p=\infty$ coincide with the classical Lipschitz-Hölder spaces, the result of Constantin *et al.* includes that of Onsager as a special case. It is interesting to point out, however, that Onsager's own unpublished identity also suffices to derive the Besov result [or even a slight improvement, as noted by Duchon and Robert (2000)]. The theorem of Constantin *et al.* also includes the Sobolev space result of Sulem and Frisch (1975) since velocity fields in three-dimensional space with an energy spectrum steeper than $k^{-8/3}$ belong to the Besov space B_3^s for some $s>1/3$ by a standard embedding theorem (e.g., see Eyink, 1995a).

One of the interests of the Besov space improvement of Onsager's result has to do with the phenomenon of small-scale intermittency in turbulence. The original mean-field theory of Kolmogorov (1941a, 1941b, 1941c) and the others (Obukhov, 1941a, 1941b; Onsager, 1945c, 1949d; Heisenberg, 1948; von Weizsäcker, 1948) predicted that all pth-order moments should scale according to dimensional reasoning based on mean dissipation

$$\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^p} \sim (Qr)^{p/3}. \tag{56}$$

However, it was later hypothesized by Kolmogorov (Kolmogorov, 1962), and then confirmed subsequently by experiment (Anselmet *et al.*, 1984; Chen *et al.*, 2005), that these scaling laws also depend upon the large length scale L as

$$\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^p} \sim (Qr)^{p/3}(r/L)^{\xi_p} \sim V^p(r/L)^{\zeta_p}, \tag{57}$$

for $r\ll L$, with $\zeta_p=(p/3)+\xi_p$. Thus the ξ_p's are anomalous dimensions in the sense of quantum field theory or critical phenomena. From a physical point of view, $\xi_p\neq 0$ means that the statistics at the small scales have fluctuations growing in each cascade step and hence remember the total number of steps from large scale L to small scale r. A particular consequence is that the energy spectrum decays faster than $k^{-5/3}$ with

$$E(k) \sim Q^{2/3}k^{-5/3}(Lk)^{-\xi_2}, \quad \xi_2>0, \tag{58}$$

corresponding to the Fourier transform of Eq. (57) for $p=2$.

A modern interpretation of anomalous scaling of velocity increments is the *multifractal model* proposed by Parisi and Frisch (1985). According to this picture, the turbulent velocity field in the zero-viscosity limit remains Hölder continuous, as conjectured by Onsager. However, according to the multifractal model there is an entire spectrum of Hölder exponents $[h_{\min},h_{\max}]$ and the set of points $S(h)$ with exponent h, for each h in this interval, forms a fractal set with Hausdorff dimension $D(h)$. In that case, the probability of a velocity increment $\Delta_{\mathbf{r}}\mathbf{v}(\mathbf{r}')$ having $\mathbf{r}'$ in $S(h)$ scales as $(r/L)^{d-D(h)}$ in d dimensions, while at those points $|\Delta_{\mathbf{r}}\mathbf{v}(\mathbf{r}')|\sim V(r/L)^h$. Therefore a simple steepest descent calculation gives

$$\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^p} \sim V^p\int_{h_{\min}}^{h_{\max}} d\mu(h)\left(\frac{r}{L}\right)^{ph+[d-D(h)]} \sim V^p(r/L)^{\zeta_p}, \tag{59}$$

for $r\ll L$, with

$$\zeta_p = \min_{h\in[h_{\min},h_{\max}]}\{ph+[d-D(h)]\}. \tag{60}$$

For a more detailed discussion of the multifractal model, see Frisch (1995). In this language, Onsager's theorem on Euler equations is the statement that dissipation requires $h_{\min}\leqslant 1/3$. The Besov space improvement by Constantin *et al.* (1994) is the statement that dissipation also requires $\zeta_3\leqslant 1$. A precise formulation and mathematical proof of validity of the multifractal model remains, of course, an open question. For some relevant comments, see Yakhot and Sreenivasan (2004). For some recent investigations purely at the level of function spaces, without the use of the fluid equations, see Jaffard (2001).

It is remarkable that Onsager was led to similar views about local Hölder regularity of turbulent velocities based on relatively weak empirical evidence about energy dissipation and its explanation by the Euler fluid equations. His proposal was 40 years earlier than that of Parisi and Frisch, who were led to their views based on empirical evidence about anomalous scaling laws. On the other hand, Onsager gave no hint in any of his published works (Onsager, 1945c, 1949d) that he anticipated the phenomenon of intermittency and its potential to alter his proposed turbulence scaling laws. There is some irony in this since it was Onsager's exact solution of the 2D Ising model (Onsager, 1944) which gave conclusive evidence of anomalous scaling corrections to Landau's mean-field theory of critical phenomena (Landau, 1937a, 1937b). Of course, the physical role of fluctuations near the critical point was only widely appreciated after the work of Levanyuk (1959) and Ginzburg (1960) and not directly from the mathematical solution of the Ising model. Therefore it might be thought that Onsager did not realize that fluctuations could invalidate his own "mean-field theory" on turbulence scaling (Onsager, 1945c, 1949d).

Quite the contrary is the case, as shown by the Lin note. Onsager wrote on p. 16 as follows:

> "As far as I can make out, a more rapid decrease of $\overline{a_k^2}$ with increasing $\underline{k}$ would require a 'spotty' distribution of the regions in which the velocity varies rapidly between neighboring points."

Thus he clearly anticipated the faster spectral decay in our Eq. (58), and, moreover, understood its physical origin. This is made even more clear by Onsager's derivation of the 2/3 law in the Lin note, based upon his identity (26) there [our Eq. (44)]. We quote at length from this remarkable passage:

> "Now put
>
> $\underline{F}(\underline{r}) = 3/4\pi\underline{a}^3; \quad (\underline{r} < \underline{a})$
>
> $\underline{F}(\underline{r}) = 0; \quad (\underline{r} > \underline{a}).$
>
> Then for small $\underline{a}$ the left member of (26) is practically
>
> $\partial\overline{v^2}/\partial\underline{t}$
>
> and the right member is at most of the order
>
> $(\overline{|D_r(v)|^3}/r)_{r=a}$.
>
> Moreover, obviously,
>
> $\overline{|D_r(v)|^2} = 2\overline{v^2}(1 - R(r))$.
>
> Now the estimate (21) is a minimum hypothesis unless the mean cube of $D_r(v)$ consists mainly of contributions from exceptional regions of small aggregate volume."

We note that $F(\mathbf{r})$ is the spatial filtering function that appears in Eq. (26), $D_r(v)$ is Onsager's notation for the velocity increment $\Delta_{\mathbf{r}}\mathbf{v}$, $R(\mathbf{r})$ is the velocity correlation function, and Eq. (21) in the Lin note is the 2/3 law [our Eq. (56) with $p=2$]. Onsager chooses $F(\mathbf{r})$ to be a "box filter," uniform on the sphere of radius a. He then observes that for small enough a, the left side of his identity (26) will be $-2Q$, which corresponds to the condition of constant energy flux. But, in that case, $\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^3}$ can be no smaller than $O(|\mathbf{r}|)$. Onsager then refers to it as a "minimal hypothesis" that $\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^2} \sim |\mathbf{r}|^{2/3}$. This closely parallels one of the derivations of the 2/3 scaling that Kolmogorov presented in his third paper (Kolmogorov, 1941c) using the linear scaling from the 4/5 law. Kolmogorov's basic assumption was of self-similarity, which, with the linear scaling of the third-order structure function, implies a 2/3 power for the second-order structure function. Onsager's assumption leading to the 2/3 scaling was closely related, namely, that spatial fluctuations are negligible and that averages do not come "from exceptional regions of small aggregate volume."

Furthermore, Onsager continues to Lin:

> "You can get the formula suggested by G. I. Taylor:
>
> $1 - R(r) \sim r$
>
> if you make the extreme assumption that the vorticity is distributed in sheets of comparable intensity and finite total area. However, the discontinuities would give rise to oscillograms of a striking rectangular structure, rather unlike those which I have seen."

There is a related page of notes in Folder 11:129, p. 21, immediately following Onsager's derivation of Eq. (44). There he writes that it is a consequence of "dynamics" that $\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^3} = O(r)$ and of "Taylor" that $\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^2} = O(r)$. It is interesting that this page seems to originate from a period when Onsager was trying to determine the possible scaling exponents. The exponent 1 in the linear scaling laws was originally another value, apparently 2, which was then scratched out. The rest of this page seems to contain an investigation of the statistical realizability of the scaling relations at the top of the page. Onsager may have been wondering whether the linear scaling for $\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^2}$ proposed by Taylor (1938) could be consistent with the linear scaling for $\overline{|\Delta_{\mathbf{r}}\mathbf{v}|^3}$ that he had derived from Eq. (44). If so, he seems to have realized by the time he wrote to Lin that such scalings are indeed compatible if velocity increments of finite amplitude are supported entirely on sheets or shock surfaces in three-dimensional space. This is exactly what occurs for Burgers equation (Burgers, 1948). In fact, that model exhibits so-called bifractal statistics of increments in every space dimension d, with just two Hölder exponents: $h=0$ corresponding to shocks on sets of dimension $D(0)=d-1$ and $h=1$ corresponding to the rest of space where velocity is smooth with dimension $D(1)=d$ (Frisch, 1995; Frisch and Bec, 2001). This leads to linear scaling for structure functions of both second and third orders. However, Onsager notes that such scaling is unlikely for fluid turbulence based on the evidence from empirical time series.

Thus it is clear that Onsager realized already by 1945 that spatial fluctuations in the regularity of the velocity field could vitiate his proposed 2/3 scaling for the second-order structure function. Another hypothesis was needed, namely, that the region of large velocity increments is not "spotty" but instead uniform throughout space. It is quite surprising, again, that Onsager did not mention any of these considerations, contained in the Lin note, in his published paper (Onsager, 1949d) four years later. The first recognition of a possible correction to Kolmogorov 1941 scaling due to intermittency is often attributed to L. D. Landau in his famous remark at a 1942 meeting in Kazan and in a related remark that appeared as a footnote in the first 1944 edition of his textbook on fluid mechanics with Lifschitz (Landau and Lifschitz, 1987). However, Landau's remarks are very brief and open to different interpretation, and it is not clear that he was referring to intermittency for small-scale increments, despite the fact that Kolmogorov (1962) gave Landau considerable credit. For an excellent discussion of this issue, see Frisch (1995), Chap. 6.4. On the other hand, Onsager's statements are clearly and unambiguously about such intermittency and its effect on short-distance scaling laws of velocity increments.

To what extent did Onsager in the 1940s fully anticipate later work on anomalous scaling in turbulence? There are, of course, many elements of modern theories that he missed. Unlike Kolmogorov (1962) 20 years later, Onsager did not consider scaling of pth-order moments of velocity increments for general p but only for $p=2$ and 3. There is also nothing in any of Onsager's papers that we have seen at all suggestive of the "refined similarity hypothesis" proposed by Kolmogorov (1962). This hypothesis relates the anomalous scaling of velocity increments to intermittency of viscous energy dissipation and it has played a key role in many modern approaches to the problem of turbulent scaling laws (Stolovitzky *et al.*, 1992). For example, see Frisch (1995) for details. Onsager's ideas were much closer to those of the multifractal model of Parisi and Frisch (1985), formulated entirely in terms of velocity increments and not involving dissipation [whose multifractality was explored extensively—see Meneveau and Sreenivasan (1991)]. Furthermore, there was nothing fractal in any of Onsager's considerations. Even Onsager's Burgers-like bifractal example in the Lin note involves sets only of integer-valued dimension (d and $d-1$ in space dimension d). The first use of fractal (and multifractal) concepts in discussing turbulent intermittency was by Mandelbrot (1969, 1972, 1974) who cast the ideas of Kolmogorov (1962) and others of the Russian school in that framework. Nevertheless, while Onsager certainly did not foresee every element of our modern understanding, it is remarkable that he perceived so clearly and so early the possibility of spatial intermittency and its effect on turbulent scaling laws.

D. A closure for the energy spectrum

Folder 11:129 of Onsager's notes from the 1940s contains one more memorable result, which is not discussed in any of his letters or publications. In the remaining five pages of the notes from that folder, pp. 23–27, Onsager sketches very succinctly the derivation of a spectral energy closure, which is remarkably similar to the EDQNM (eddy-damped quasinormal Markovian) closure that was proposed by Orszag (1970, 1977). The EDQNM approximation was itself the end result of a long line of analytical closures which went back to the work of Proudman and Reid (1954) on the quasinormal closure, of Kraichnan on the direct-interaction approximation (Kraichnan, 1959) and the test-field model (Kraichnan, 1971), and of others as well.

We reproduce here two of these five pages from Folder 11:129, pp. 23 and 24. Page 23 (see Fig. 9) contains the main result, while the later pages contain supporting calculations. Onsager's basic idea is simple: he isolated the effect on energy transfer out of wave number $\mathbf{k}$ due to a single distinguished triad $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$. As he had already discussed in his Lin note and in the 1949 paper, the convective wave number in the triad, $\mathbf{k}''$, say, plays a purely passive role, simply catalyzing transfer between $\mathbf{k}$ and $\mathbf{k}'$. Therefore he froze the Fourier amplitude $\mathbf{a}(\mathbf{k}'')$ of that mode, leading to a linear equation for the other two modes. This is the 4×4 matrix system that he considered on p. 24 (see Fig. 10). As a first approximation, he ignored the contribution of the pressure term (with just a remark on p. 26 on how it may be included). On the pages which follow, Onsager diagonalized the 4×4 matrix and found the exact solution $\mathbf{u}(\mathbf{k},t)$ of the linear problem. The details of this need not concern us, just the final result for $|\mathbf{u}(\mathbf{k},t)|^2$, written at the top of p. 23. Onsager expanded this result to second order in time t, and used this to calculate an approximation to the time derivative

$$\frac{d}{dt}|\mathbf{a}(\mathbf{k})|^2 = (2\pi)^2 t\{|\mathbf{k}'\cdot\mathbf{a}(\mathbf{k}-\mathbf{k}')|^2 + |\mathbf{k}'\cdot\mathbf{a}(\mathbf{k}+\mathbf{k}')|^2\} \times [|\mathbf{a}(\mathbf{k}')|^2 - |\mathbf{a}(\mathbf{k})|^2] + O(t^2). \tag{61}$$

After averaging over a homogeneous ensemble and discarding fourth-order cumulants, as in the quasinormal closure (Proudman and Reid, 1954), the moments involving distinct wave numbers also factorize to leading order, e.g.,

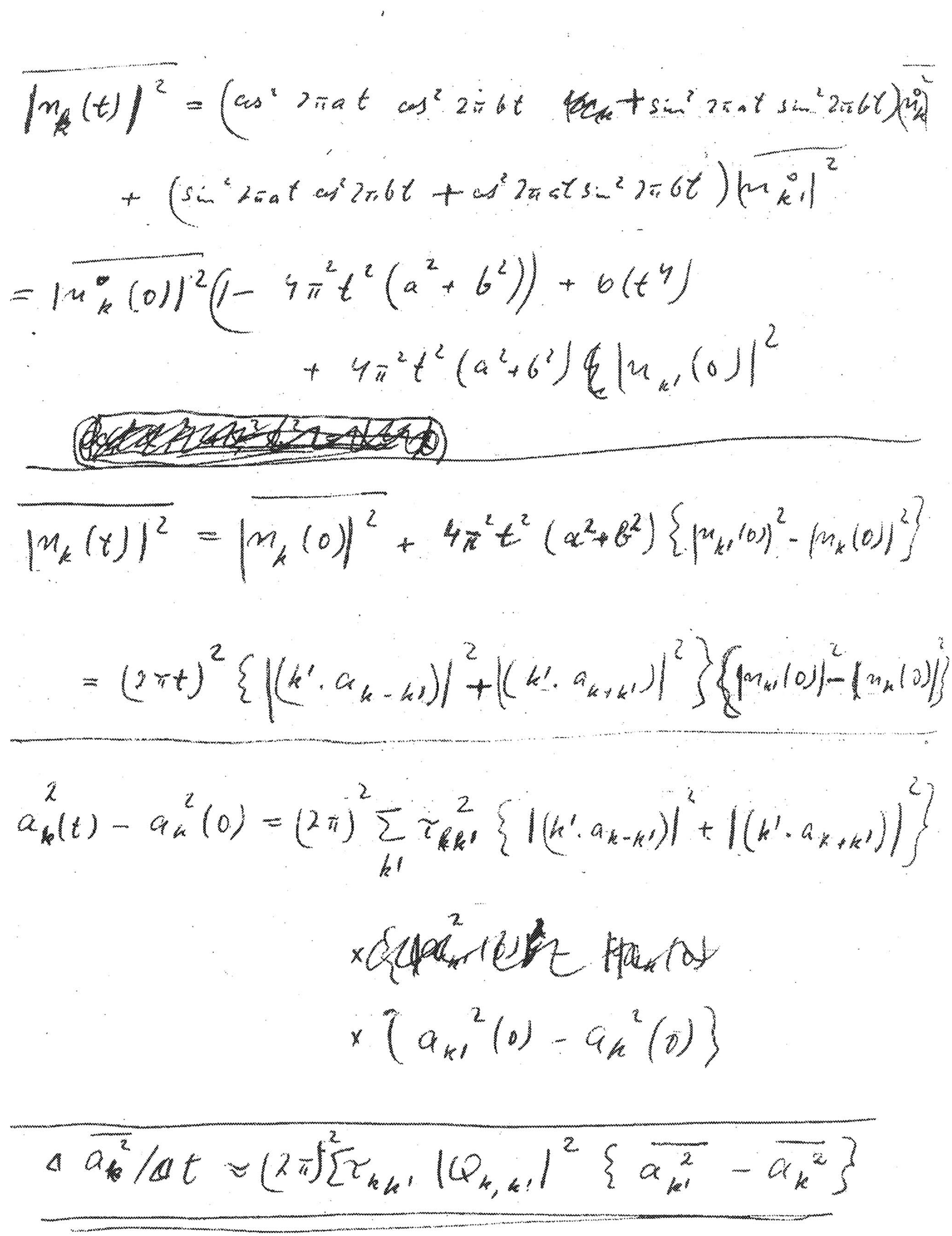
$$\overline{|\mathbf{a}(\mathbf{k}'')|^2|\mathbf{a}(\mathbf{k})|^2} = \overline{|\mathbf{a}(\mathbf{k}'')|^2}\;\overline{|\mathbf{a}(\mathbf{k})|^2}\,[1+O(1/V)]. \tag{62}$$

This is like the weak-dependence property used by Kraichnan (1959). Finally, Onsager made a bold approximation. Embedding the single triadic contribution in the sea of other triads, he assumed that the only effect was to replace the "bare" time t by an effective turnover time $\tau_{\mathbf{k},\mathbf{k}'}$, which depends upon the triad. The final result was

$$\frac{d}{dt}\overline{|\mathbf{a}(\mathbf{k})|^2} = (2\pi)^2\sum_{\mathbf{k}'}\tau_{\mathbf{k},\mathbf{k}'}[\overline{|\mathbf{k}'\cdot\mathbf{a}(\mathbf{k}-\mathbf{k}')|^2} + \overline{|\mathbf{k}'\cdot\mathbf{a}(\mathbf{k}+\mathbf{k}')|^2}] \times [\overline{|\mathbf{a}(\mathbf{k}')|^2} - \overline{|\mathbf{a}(\mathbf{k})|^2}]. \tag{63}$$

This equation is now closed in terms of the energy spectrum. If $\tau_{\mathbf{k},\mathbf{k}'}$ is symmetric in its dependence on the wave numbers' magnitudes k, k', k'', then this equation will exactly conserve energy because of a cancellation between "input" terms $\propto|\mathbf{a}(\mathbf{k}')|^2$ and "output" terms $\propto|\mathbf{a}(\mathbf{k})|^2$. The final result is very similar to EDQNM (Orszag, 1977) nearly 30 years before that closure would appear in the literature. Furthermore, the derivation itself is very close to one used by Fournier and Frisch (1978), Chaps. II and III, to obtain EDQNM. The argument and result would have been identical if Onsager had included the contribution of the pressure and also allowed the convective mode to evolve.

What is especially interesting from a historical point of view is that Onsager returned to this problem in the 1970s and worked to refine his closure. The relevant material is contained in Folders 11:132 and 11:135 from the Onsager Archive, which we have dated to the years 1973–1976. By that time, Onsager had left Yale for the University of Miami and was working mainly on the problem of the origin of life (Onsager, 1974b), statistical mechanics of water and ice (Onsager, 1973, 1974a; Chen *et al.*, 1974; Staebler *et al.*, 1978), and his lifelong favorite



$$\overline{|u_k(t)|^2} = \left(\cos^2 2\pi a t \cos^2 2\pi b t + \sin^2 2\pi a t \sin^2 2\pi b t\right)\overline{|u_k^0|^2}$$

$$+ \left(\sin^2 2\pi a t \cos^2 2\pi b t + \cos^2 2\pi a t \sin^2 2\pi b t\right)\overline{|u_{k'}^0|^2}$$

$$= \overline{|u_k^0(0)|^2}\left(1 - 4\pi^2 t^2 (a^2 + b^2)\right) + O(t^4)$$

$$+ 4\pi^2 t^2 (a^2 + b^2)\,\overline{|u_{k'}(0)|^2}$$

$$\overline{|u_k(t)|^2} = \overline{|u_k(0)|^2} + 4\pi^2 t^2 (a^2 + b^2)\left\{\overline{|u_{k'}(0)|^2} - \overline{|u_k(0)|^2}\right\}$$

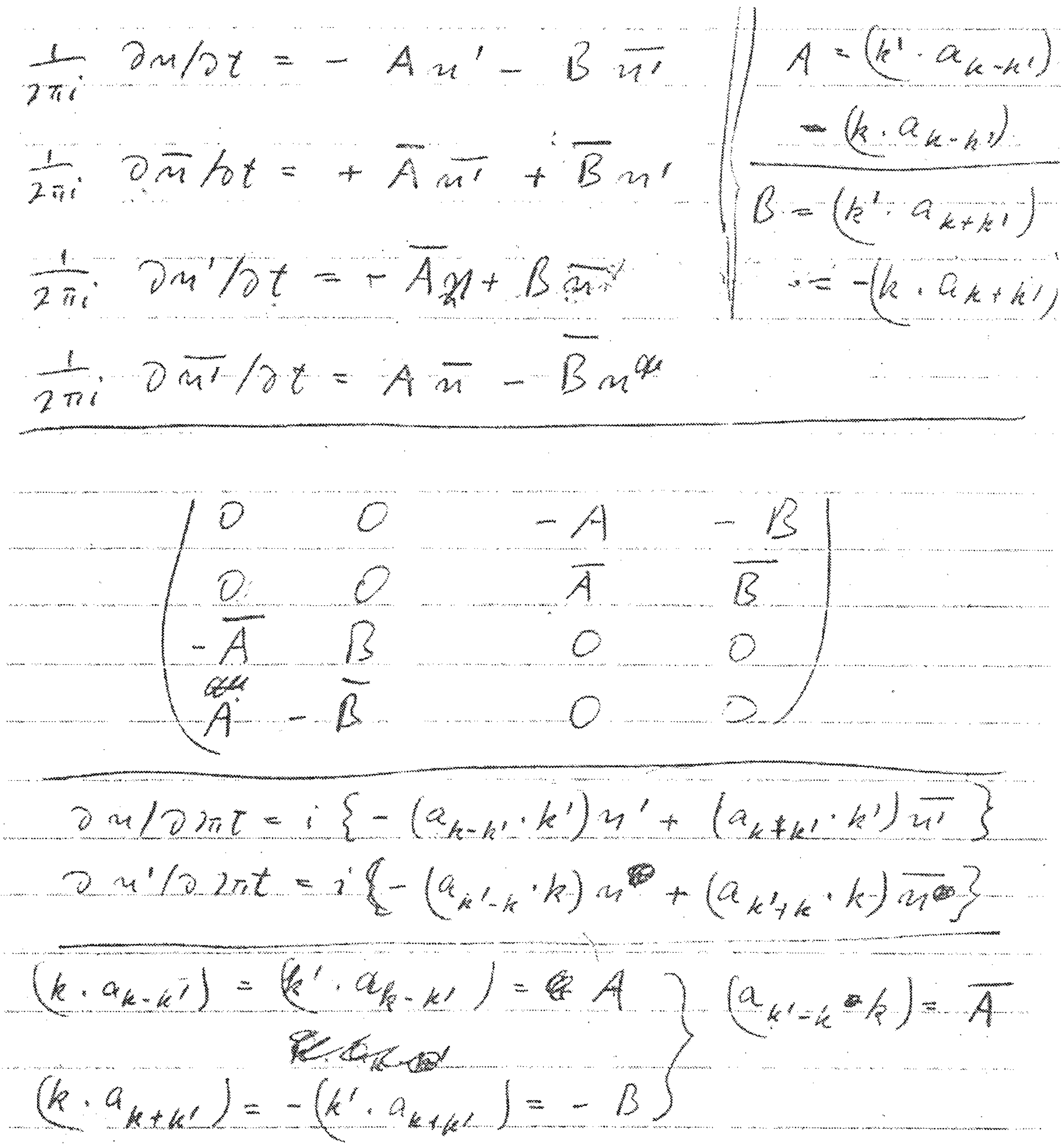
$$= (2\pi t)^2 \left\{ |(k' \cdot a_{k-k'})|^2 + |(k' \cdot a_{k+k'})|^2 \right\} \left\{ |u_{k'}(0)|^2 - |u_k(0)|^2 \right\}$$

$$a_k^2(t) - a_k^2(0) = (2\pi)^2 \sum_{k'} \tau_{kk'}^2 \left\{ |(k' \cdot a_{k-k'})|^2 + |(k' \cdot a_{k+k'})|^2 \right\}$$

$$\times \left\{ a_{k'}^2(0) - a_k^2(0) \right\}$$

$$\Delta \overline{a_k^2} / \Delta t \approx (2\pi)^2 \sum \tau_{kk'} |Q_{k,k'}|^2 \left\{ \overline{a_{k'}^2} - \overline{a_k^2} \right\}$$

FIG. 9. Page 23 of Folder 11:129 (Onsager, circa 1945). The first equation is the exact solution of the 4×4 system on p. 24. This is Taylor expanded in time, ensemble averaged, and differentiated to yield an expression for the evolution of the spectrum. The last equation is Onsager's closure equation, with the "bare" time t replaced by an effective "turnover" time $\tau_{\mathbf{k},\mathbf{k}'}$. Reproduced courtesy of the Onsager Archive, NTNU.



$$\frac{1}{2\pi i}\,\partial u/\partial t = -Au' - B\bar{u}' \qquad A = (k'\cdot a_{k-k'})$$
$$\frac{1}{2\pi i}\,\partial \bar{u}/\partial t = +\bar{A}\bar{u}' + \bar{B}u' \qquad = (k\cdot a_{k-k'})$$
$$\frac{1}{2\pi i}\,\partial u'/\partial t = -\bar{A}u + B\bar{u} \qquad B = (k'\cdot a_{k+k'})$$
$$\frac{1}{2\pi i}\,\partial \bar{u}'/\partial t = A\bar{u} - \bar{B}u \qquad = -(k\cdot a_{k+k'})$$

$$\begin{pmatrix} 0 & 0 & -A & -B \\ 0 & 0 & \bar{A} & \bar{B} \\ -\bar{A} & B & 0 & 0 \\ A & -\bar{B} & 0 & 0 \end{pmatrix}$$

$$\partial u/\partial \pi t = i\{-(a_{k-k'}\cdot k')u' + (a_{k+k'}\cdot k')\bar{u}'\}$$
$$\partial u'/\partial 2\pi t = i\{-(a_{k'-k}\cdot k)u + (a_{k'+k}\cdot k)\bar{u}\}$$

$$(k\cdot a_{k-k'}) = (k'\cdot a_{k-k'}) = A \qquad (a_{k'-k}\cdot k) = \bar{A}$$
$$(k\cdot a_{k+k'}) = -(k'\cdot a_{k+k'}) = -B$$

FIG. 10. Page 24 of Folder 11:129 (Onsager, circa 1945). The four linear equations at the top of the page give the evolution of the Fourier modes $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{k}')$ and their complex conjugates when the amplitude of the third leg of the triad $\mathbf{k}''$ is "frozen." The pressure force is omitted in this calculation. Reproduced courtesy of the Onsager Archive, NTNU.

topic, electrolyte theory (Hubbard and Onsager, 1977; Onsager and Chen, 1977; Onsager and Kim, 1977; Onsager *et al.*, 1977). However, as we see from the contents of the folders, he had also returned to the study of turbulence. In this new attempt, he correctly incorporated the pressure effects and permitted the evolution of the convective modes. However, the method of derivation was slightly changed. In his new approach Onsager directly studied the time derivative $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$, with $Q_{\mathbf{k},\mathbf{k}'}$ given by Eq. (34) above. Actually, that expression contains four terms, of which he studied just one representative term $-\pi i[\mathbf{a}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'][\mathbf{a}(\mathbf{k}')\cdot(-\mathbf{k})]$, which he called $Q_{\mathbf{k},\mathbf{k}'}$. Since the latter is cubic in Fourier amplitudes, its time derivative is quartic. The derivative can be divided into two contributions:

$$\frac{d}{dt}Q_{\mathbf{k},\mathbf{k}'} = -\pi i[\dot{\mathbf{a}}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'][\mathbf{a}(\mathbf{k}')\cdot\mathbf{a}(-\mathbf{k})] - \pi i[\mathbf{a}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}']\{[\dot{\mathbf{a}}(\mathbf{k}')\cdot\mathbf{a}(-\mathbf{k})] + [\mathbf{a}(\mathbf{k}')\cdot\dot{\mathbf{a}}(-\mathbf{k})]\}. \tag{64}$$

The first one contains the derivative of the passive convective mode, which was neglected before, while the second contains the derivative of the active modes involved in the transfer.

We show three key pages of Folder 11:132 which give the idea of Onsager's treatment (pp. 4–6; see Figs. 11 and 12). On pp. 4 and 5 he calculated the first term in Eq. (64) as a sum over all triads. However, he immediately singled out the term coming from the same triad $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$ as that represented in $Q_{\mathbf{k},\mathbf{k}'}$ itself and proposed that the rest of the triadic interactions may be treated as "random." This is very reminiscent of the direct-interaction approximation devised by Kraichnan (1959) in which only direct feedback loops of triads are retained. Eventually Onsager argued that the entire contribution from the derivative of passive modes was negligible (p. 21). On p. 6, he treated the second term in Eq. (64) from the derivative of active modes in a similar fashion, keeping only the contributions from "direct interactions." The result was quite similar to that in the 1940s notes except for two additional terms that came from pressure forces. Later in these notes (p. 17) Onsager evaluated the pressure terms and showed that they give a result of the same form. From this point the argument was similar to that of the notes in Folder 11:129—Onsager used the approximate formula for the time derivative to make an expansion of $Q_{\mathbf{k},\mathbf{k}'}$ to first order in time. Replacing the bare time t by an effective turnover time $\tau(\mathbf{k},\mathbf{k}')$ gave Onsager's final result on p. 16. It is equivalent, except for the prefactor, to the original formula from the 1940s:

$$Q_{\mathrm{ave}}(\mathbf{k},\mathbf{k}') = \frac{2\pi^2}{3}\tau(\mathbf{k},\mathbf{k}')\frac{k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')^2}{|\mathbf{k}-\mathbf{k}'|^2} \times|\mathbf{a}(\mathbf{k}-\mathbf{k}')|^2_{\mathrm{ave}}[|\mathbf{a}(\mathbf{k}')|^2-|\mathbf{a}(\mathbf{k})|^2]_{\mathrm{ave}}. \quad (65)$$

Here the subscript "ave" indicates ensemble averaging. Onsager also wrote down on p. 16 an expression for the effective turnover time as $\tau(\mathbf{k},\mathbf{k}')=\gamma Q^{-1/3}(k^{4/3}+k'^{4/3}+k''^{4/3})^{-1/2}$. This is very similar to the expression now usually adopted in EDQNM (Orszag, 1977).

A final topic treated in Folder 11:132 is wall-bounded flow. Onsager had long taken an interest in this subject, particularly through the work of the oceanographer R. B. Montgomery, whom he cited both in the Lin note and in Onsager (1949d). Montgomery experimentally investigated turbulent boundary layers, measuring both mean profiles and two-point correlations. Pages 9–12 of Folder 11:132 contain a very transparent construction of exact wave mode solutions of the linear Stokes equation for flow between two infinite, parallel plates, with exponential decay in time. The solutions are free plane waves along the horizontal direction and satisfy no-slip boundary conditions at the surface of each plate. The solutions are classified into even and odd sets under reflection about the center plane between the plates. For a given set of horizontal wave numbers α,β and decay constant k, the even solutions have the form

$$v_x = e^{i(\alpha x+\beta y)}\{a_x\cosh(\sqrt{\alpha^2+\beta^2}z) + b_x\cos(\sqrt{k^2-\alpha^2-\beta^2}z)\},$$

$$v_y = e^{i(\alpha x+\beta y)}\{a_y\cosh(\sqrt{\alpha^2+\beta^2}z) + b_y\cos(\sqrt{k^2-\alpha^2-\beta^2}z)\},$$

$$v_z = e^{i(\alpha x+\beta y)}\{a_z\sinh(\sqrt{\alpha^2+\beta^2}z) + b_z\sin(\sqrt{k^2-\alpha^2-\beta^2}z)\}, \quad (66)$$

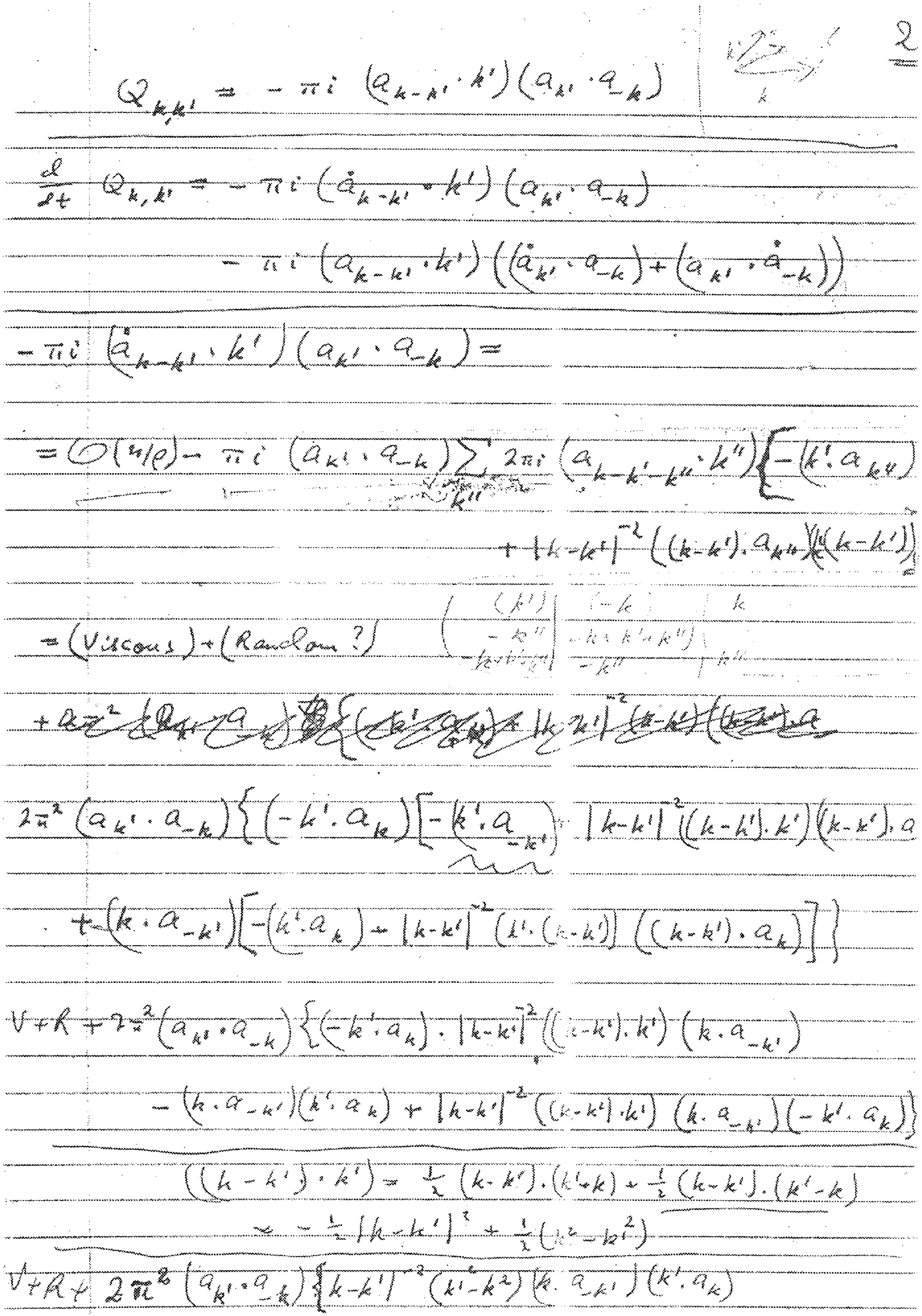
and the odd solutions exchange $\sin\leftrightarrow\cos$, $\sinh\leftrightarrow\cosh$. Onsager worked out a dispersion relation between the complex frequency k and the horizontal wave numbers α,β. He also observed a set of selection rules for triadic interactions between these modes, with only even-even-even and even-odd-odd as possible. Considering the context of the rest of the notes, it is possible that he was thinking of using these modes as the basis for an EDQNM-type analysis of turbulent channel flow.

In view of the effort that Onsager was expending on these notes in the 1970s, it appears that he may have been considering an article on the subject of his old spectral turbulence closure from his notes in the 1940s. Since that was the only part of his turbulence research which had never seen publication in any form, it might have been the part he felt most important to get out. If so, it is unlikely that he was aware of the closely parallel work of Orszag at about the same time (Orszag, 1970, 1977) or of the large body of related work by Kraichnan (Kraichnan, 1959, 1971) and others, which had appeared in the decades since he last worked on the subject. It was apparently Onsager's habit to first work out his own ideas and then to check the literature to see what others had done. It is remarkable that Onsager had already discovered the basic ideas of EDQNM on his own in the 1940s, but it is also clear that by the early 1970s the subject of spectral turbulence closures had surpassed his individual efforts.

V. CONCLUSIONS

A. Historical questions

We hope that a reader, if he is not an expert on turbulence, will have learned something interesting about this theoretically fundamental and practically important problem. As for the turbulence experts, we hope that they have learned some history of their subject that, probably, comes to them as a bit of a surprise. It certainly did to us. We believe that the evidence presented here convincingly shows that Onsager made four remarkable discoveries in the 1940s which, for some reason, he chose not to publish. First, we have shown that he carried out detailed calculations on the equilibrium statistics of 2D point vortices, deriving, in particular, the critical energy for onset of negative temperatures, the energy-temperature relation for large positive temperatures, and a mean-field Poisson-Boltzmann equation. Second, he derived a relation between the second- and third-order velocity structure functions related to the Kolmogorov 4/5 law, which is sufficient to prove his published claim on inviscid dissipation and 1/3 Hölder singularities. Third, he realized the possible violation of



2

$$Q_{k,k'} = -\pi i \left(a_{k-k'}\cdot k'\right)\left(a_{k'}\cdot a_{-k}\right)$$

$$\frac{d}{dt} Q_{k,k'} = -\pi i \left(\dot a_{k-k'}\cdot k'\right)\left(a_{k'}\cdot a_{-k}\right)$$

$$-\pi i \left(a_{k-k'}\cdot k'\right)\left(\left(\dot a_{k'}\cdot a_{-k}\right) + \left(a_{k'}\cdot \dot a_{-k}\right)\right)$$

$$-\pi i \left(\dot a_{k-k'}\cdot k'\right)\left(a_{k'}\cdot a_{-k}\right) =$$

$$= O(\nu/\rho) - \pi i \left(a_{k'}\cdot a_{-k}\right)\sum_{k''} 2\pi i \left(a_{k-k'-k''}\cdot k''\right)\Big\{-\left(k'\cdot a_{k''}\right)$$

$$+ |k-k'|^{-2}\left(\left(k-k'\right)\cdot a_{k''}\right)\left(k'\cdot\left(k-k'\right)\right)\Big\}$$

$$= (\text{Viscous}) + (\text{Random?}) \qquad \begin{array}{c|c|c} k' & -k & k \\ -k'' & -k+k'+k'' & \\ -k'-k'' & -k'' & k'' \end{array}$$

$$2\pi^2 \left(a_{k'}\cdot a_{-k}\right)\Big\{\left(-k'\cdot a_k\right)\Big[-\left(k'\cdot a_{-k'}\right) + |k-k'|^{-2}\left(\left(k-k'\right)\cdot k'\right)\left(\left(k-k'\right)\cdot a\right.$$

$$+ \left(k\cdot a_{-k'}\right)\Big[-\left(k'\cdot a_k\right) + |k-k'|^{-2}\left(k'\cdot\left(k-k'\right)\right)\left(\left(k-k'\right)\cdot a_k\right)\Big]\Big\}$$

$$V + R + 2\pi^2 \left(a_{k'}\cdot a_{-k}\right)\Big\{\left(-k'\cdot a_k\right)\cdot |k-k'|^{-2}\left(\left(k-k'\right)\cdot k'\right)\left(k\cdot a_{-k'}\right)$$

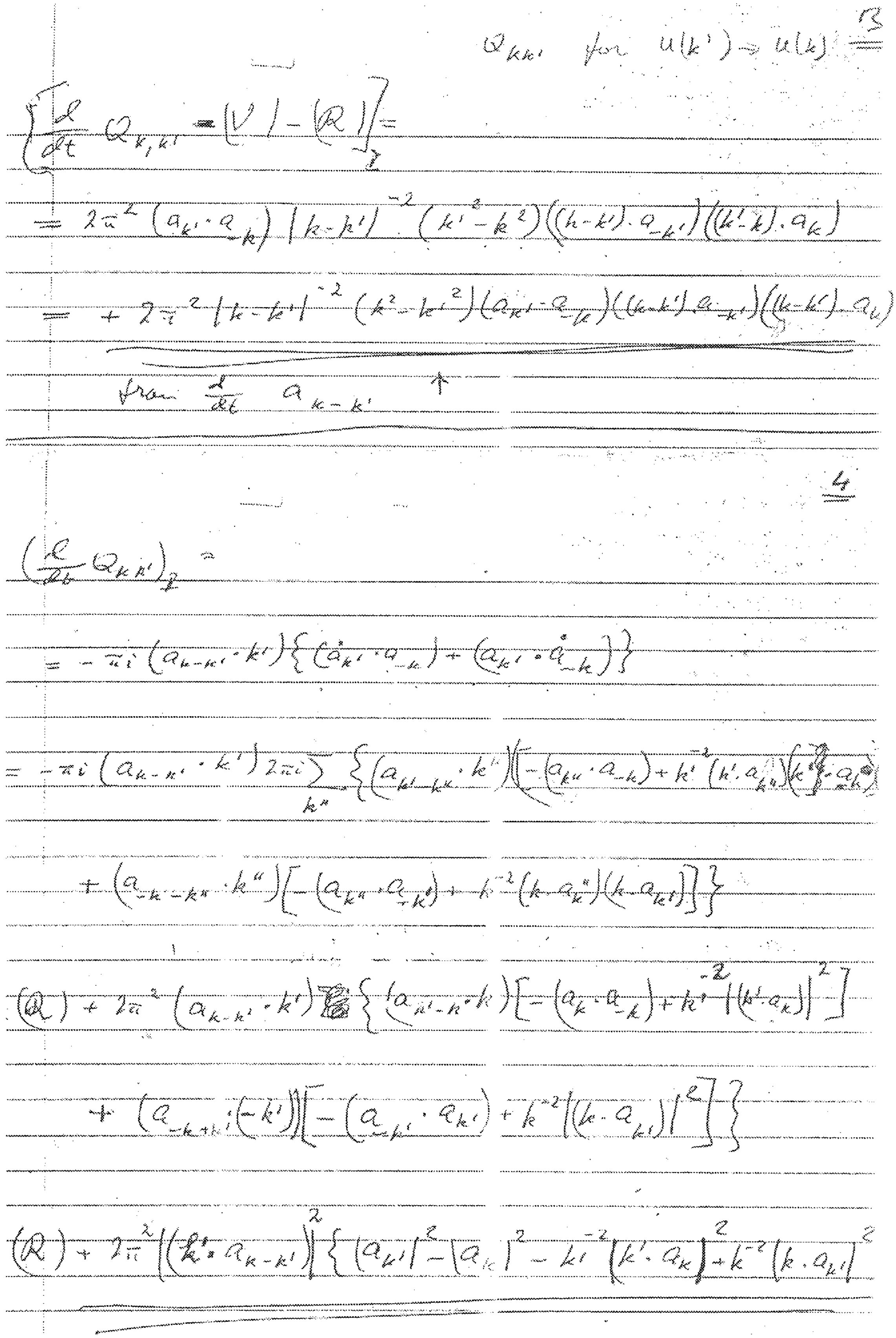
$$-\left(k\cdot a_{-k'}\right)\left(k'\cdot a_k\right) + |k-k'|^{-2}\left(\left(k-k'\right)\cdot k'\right)\left(k\cdot a_{-k'}\right)\left(-k'\cdot a_k\right)\Big\}$$

$$\left(\left(k-k'\right)\cdot k'\right) = \tfrac{1}{2}\left(k-k'\right)\cdot\left(k'+k\right) + \tfrac{1}{2}\left(k-k'\right)\cdot\left(k'-k\right)$$

$$= -\tfrac{1}{2}|k-k'|^2 + \tfrac{1}{2}\left(k'^2-k^2\right)$$

$$V + R + 2\pi^2 \left(a_{k'}\cdot a_{-k}\right)|k-k'|^{-2}\left(k'^2-k^2\right)\left(k\cdot a_{-k'}\right)\left(k'\cdot a_k\right)$$

FIG. 11. Page 4 of Folder 11:132 (Onsager, circa 1975). The top equation is the representative term of the transfer function $Q_{\mathbf{k},\mathbf{k}'}$ and the third equation is the part of its time derivative coming from the evolution of the passive, convective mode. In the fourth equation, Onsager selected out the direct interactions and lumped together the contributions of other triads as "(Random?)." The final result was truncated due to lack of space and rewritten on the following page. The numbers at the top right corner are his own pagination of this set of notes. Reproduced courtesy of the Onsager Archive, NTNU.



$Q_{kk'}$ for $u(k') \to u(k)$ 3

$$\left[\frac{d}{dt} Q_{k,k'} = (V) - (R)\right]_3 =$$

$$= 2\pi^2 (a_{k'}\cdot a_{-k})\, |k-k'|^{-2} (k'^2 - k^2)\big((k-k')\cdot a_{-k'}\big)\big((k'-k)\cdot a_k\big)$$

$$= +2\pi^2 |k-k'|^{-2}(k^2-k'^2)(a_{k'}\cdot a_{-k})\big((k-k')\cdot a_{-k'}\big)\big((k-k')\cdot a_k\big)$$

from $\frac{d}{dt}\, a_{k-k'}$ ↑

4

$$\left(\frac{d}{dt}Q_{kk'}\right)_2 =$$

$$= -\pi i (a_{k-k'}\cdot k')\{(\dot a_{k'}\cdot a_{-k}) + (a_{k'}\cdot \dot a_{-k})\}$$

$$= -\pi i (a_{k-k'}\cdot k')\, 2\pi i \sum_{k''} \{(a_{k'-k''}\cdot k'')[-(a_{k''}\cdot a_{-k}) + k'^{-2}(k'\cdot a_{k''})(k'\cdot a_{-k})]$$

$$+ (a_{-k-k''}\cdot k'')[-(a_{k''}\cdot a_{k'}) + k^{-2}(k\cdot a_{k''})(k\cdot a_{k'})]\}$$

$$(Q) + 2\pi^2 (a_{k-k'}\cdot k')\{(a_{k'-k}\cdot k)[-(a_k\cdot a_{-k}) + k'^{-2}|(k'\cdot a_k)|^2]$$

$$+ (a_{-k+k'}\cdot(-k'))[-(a_{-k'}\cdot a_{k'}) + k^{-2}|(k\cdot a_{k'})|^2]\}$$

$$(R) + 2\pi^2 |(k'\cdot a_{k-k'})|^2\{|a_{k'}|^2 - |a_k|^2 - k'^{-2}|k'\cdot a_k|^2 + k^{-2}|k\cdot a_{k'}|^2\}$$

FIG. 12. Pages 5 and 6 of Folder 11:132 (Onsager, circa 1975). The first two equations shown are the continuation of Onsager's formulas from the previous page in a short half page (numbered "3" in his own pagination). The next page contains a similar discussion as the preceding one, but for the part of the time derivative of $Q_{\mathbf{k},\mathbf{k}'}$ coming from the evolution of the active, convected modes. In the third equality, the contribution of the direct interactions was singled out and the remainder labeled as "$\mathcal{R}$" (presumably meaning "random"—see previous page). Reproduced courtesy of the Onsager Archive, NTNU.

his mean-field scaling laws for turbulent velocity increments due to small-scale intermittency and he foresaw aspects of the modern multifractal model. Fourth, he worked out a spectral energy closure closely related to EDQNM, using ideas and methods similar to those that were discovered later by others. These are in addition to Onsager's contributions in his published abstract and article for which he is already justly famous in the field. From a consideration of both the published and unpublished works, it is clear that Onsager anticipated several important theoretical developments in turbulence of the last 50 years.

One obvious question is: why did he not publish his four results? There are various possible answers.

First, Onsager was, in the 1940s and later, a very busy man. Consider that in the decade 1945–1955 alone Onsager published papers of fundamental importance on liquid diffusion (Onsager, 1945d), nematic order for rod-shaped colloids (Onsager, 1949c), correlations in the 2D Ising model (Onsager and Kaufman, 1947; Kaufman and Onsager, 1949), isotope separation (Watson *et al.*, 1949), the de Haas–van Alphen effect (Onsager, 1952, 1953a), the Gouy diffusion method (Gosting and Onsager, 1952), fluctuations in irreversible processes (Machlup and Onsager, 1953; Onsager and Machlup, 1953), liquid helium (Onsager, 1953b), and conductance of strong electrolytes (Fuoss and Onsager, 1955) in addition to his work on turbulence (Onsager, 1945c, 1949d)! Clearly he had very limited time to write up all his ideas. In fact, some of his most famous results of this period were not formally published, including quantization of circulation in superfluids (Onsager, 1949a) and spontaneous magnetization and long-range order in the 2D Ising model (Onsager, 1949b). Even at the best of times, Onsager was never quick to rush into print. The results of his work with Machlup on path functionals for fluctuations of time histories (Machlup and Onsager, 1953; Onsager and Machlup, 1953) were already announced in brief at the end of his second paper on reciprocal relations (Onsager, 1931b), as an "it is also possible to show" remark. Likewise, he wrote at the end of a paper (Onsager, 1939a) on electrostatic interactions of molecules "incidentally, it is possible to show" that Pauling's estimate of the residual entropy of ice is a rigorous lower bound, but he published the proof only in Onsager and Dupuis (1960). It is quite possible that Onsager planned to publish some of the four results on turbulence that we have uncovered—when he was ready.

Another factor which may have played a role in Onsager's reluctance to publish was the either cool or baffled reception that his work received in both the fluid dynamics and statistical-mechanics communities. We review here the essentials of what the sources reveal.

In Onsager's day, the top turbulence expert in the United States was T. von Kármán. As we have seen, Onsager first communicated with C.-C. Lin, who was von Kármán's student, because Lin's pioneering papers on point-vortex dynamics and stability of parallel flows played an important role in his ideas on the subject. A month later, on July 25, 1945, Onsager sent a letter directly to von Kármán. He enclosed both the Pauling and Lin notes, as well as two of his reprints. One of these was a review paper on concentrated electrolytes (Onsager, 1933) which discussed, among other things, Debye-Hückel theory and the Poisson-Boltzmann equation. He also included his 1931 papers on reciprocal relations (which, in the course of time, won Onsager the Nobel Prize). However, von Kármán was decidedly unimpressed. On August 23, 1945 he wrote a very brief letter to Lin, which we quote in full (von Kármán, 1945):

> "I received a letter and a kind of manuscript from a certain Mr. Lars Onsager. I find his letter somewhat 'screwy' so I would be glad to have your opinion whether the paper is worthwhile reading. Perhaps you could indicate to me in a few lines what the idea is, if any."

Lin replied to von Kármán with a three-page letter on September 4, 1945 (Lin, 1945a) summarizing and evaluating Onsager's work. First, Lin reported the results of his queries to physicist P. Epstein and others at Caltech that Onsager had a high reputation in statistical physics and that he had "many good things in his line (statistical mechanics, thermodynamics, etc.)." However, his evaluation of Onsager's ideas was unenthusiastic. On the equilibrium theory for point vortices he wrote that he was "rather inclined to think that his arguments are as yet not fully developed, if there is something to be found behind his idea." On the cascade theory for 3D and the $2/3$ law for the velocity correlation, Lin wrote that "his method for determining $F(n)$ [the energy spectrum as a function of wave number] for large values of n does not seem to be convincing." After this exchange with Lin, there is no record that we could find that von Kármán ever replied in any form to Onsager himself.

Lin wrote to Onsager, at least twice that we know. On June 26, 1945, Lin (1945b) replied to Onsager's long letter to him earlier in the month. The letter expressed polite interest in Onsager's note but also stated that Lin had "not yet had time to study it thoroughly." The letter pointed out to Onsager a number of developments, including current work on functional integration by mathematician C. Loewner, the spectral version of the Kármán-Howarth equation derived by Lin himself, and, of special interest here, the following (Lin, 1945b):

> "I would like to study in detail the way in which you arrived at definite results, for I never succeeded in getting any [on the spectrum]. I believe the way by which the Russians handled Kármán-Howarth equations is very ingenious. I reviewed a very recent paper of theirs in this line for the Mathematical Reviews. Enclosed is a carbon copy for your reference. This is a review of Loitziansky's invariant integral $\int_0^\infty f(r,t)r^4 dr$ and other discussions of Kolmogoroff. The author has some new ideas, which are very clever from a mathematical point of view, but which perhaps do not correspond to physical facts except when the 'Reynolds number of turbulence' is *low* [the emphasis added by us]."

As far as we can determine, this is the first occasion on which Onsager heard of Kolmogorov's work in turbulence. In fact, in the Lin note earlier that month, Onsager had written with prescience:

> "My tentative limiting formula for the correlation-function in isotropic turbulence is not so obvious that any one student could be expected to find it. However, it seemed very probable to me that somebody would have investigated the line of reasoning, which is not far fetched."

That "somebody" was, of course, Kolmogorov, as Onsager was to learn from Lin subsequently. It is not clear whether Lin enclosed carbon copies of his review of Kolmogorov's work or of Kolmogorov's papers; neither is it clear when Onsager actually received and read Lin's mail, but it is worth noting that this was just four months before the APS talk on which Onsager's published abstract (Onsager, 1945c) was based. Certainly by the time of the meeting in Florence four years later, Onsager had read Kolmogorov's works and acknowledged the priority.[16] Thus Lin's first letter to Onsager was quite informative. The second—and, as far as we know, the last—letter of Lin was on September 4, 1945 and was much shorter. After thanking Onsager for some of his reprints and reciprocating with reprints of his own, Lin wrote (Lin, 1945c):

> "I am sorry to say that I have not made much progress, except that I desire still more to see something done in this line to bring your ideas down to my level of understanding. And I certainly wish that I could have the happy chance of talking to you in person on this subject, and to learn some statistical mechanics from you some day."

That appears to be the end of their correspondence and interaction.

The last exchange that Onsager had with the fluid mechanics experts on his turbulence theories, of which we are aware, was an "account of his work" that he sent to G. K. Batchelor sometime before the latter's article in *Nature* on December 14, 1946 (Batchelor, 1946). This paper pointed out the remarkable simultaneous discovery of the 2/3 law for the velocity correlation by Kolmogorov, Onsager, Heisenberg, and von Weizsäcker. Along with Batchelor (1947), it was extremely influential in bringing the landmark work of Kolmogorov and Obukhov to the attention of the Western scientific community. From Batchelor's description, the "account" that Onsager had sent to him was close, if not identical, to that sent earlier to Lin. Although Onsager was given due credit in the 1946 article, it is noteworthy that Batchelor, in comparing the different approaches of the codiscoverers, wrote that "the neatest and most powerful formulation of the physical ideas is that of Kolmogoroff." It is probably fair to say that to this day Kolmogorov's work on turbulence is that most commonly remembered by the fluid mechanics community, largely overshadowing the distinctive contributions of Onsager and the other codiscoverers (see Fig. 13).

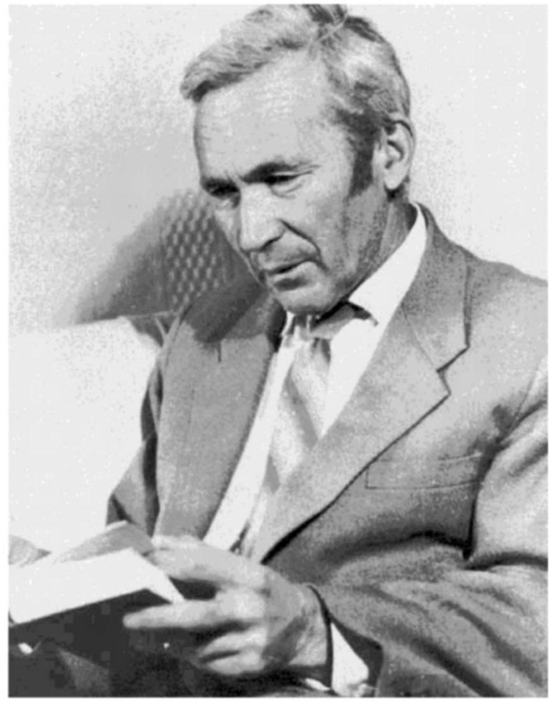


FIG. 13. The great Soviet mathematician Andrei Nikolaevich Kolmogorov (1903–1987). His works on fluid turbulence in the early 1940s and early 1960s, carried forward by several of his able disciples, among whom was Aleksandr Mikhaylovich Obukhov (1918–1989), have influenced the field enormously. Kolmogorov and Onsager were exact contemporaries, born in the same year, but there is no evidence that the two giants ever corresponded. Kolmogorov's work on turbulence was unknown to Onsager until sometime after June of 1945. Reprinted with permission of Albert W. Shiryaev.

Onsager had also communicated his ideas to his fellow chemist, L. Pauling, in March of 1945. However, the latter wrote back cordially but briefly on April 6, 1945 saying (Pauling, 1945):

> "Your work looks very interesting indeed to me, but it is too far over my head for me to appreciate it properly."

As we have already noted, Onsager tried again four years later at the IUPAP meeting on statistical mechanics in Florence, Italy, where he presented his paper on statistical hydrodynamics. However, the response was quite muted. No one made any remark about the novel concepts of negative absolute temperature for fluid vortices or of inviscid dissipation by singular Euler solutions. There is only one recorded question after Onsager's talk in Florence, by M. Born, who asked whether the new theories could predict the critical Reynolds number for transition to turbulence. Onsager replied. "No, the

[16] Perhaps this too played some role in Onsager's leaving the field. Few things in science can be more discouraging than to make a big advance and then to find that you were scooped by someone a short time earlier!

problem of the Reynold's number is more complicated. Consult recent work of C.-C. Lin." It is certainly not the case that Onsager was ill regarded by this audience, for his breakthroughs in nonequilibrium thermodynamics, the 2D Ising model, and superfluid helium were major subjects of the meeting and had created an enormous stir.[17] However, one surmises that Onsager's advanced ideas on turbulence were generally met there with polite incomprehension. We have already mentioned some of the reaction after the conference. von Neumann discussed Onsager's theory of vortex equilibria in his unpublished review article on turbulence for the Air Force Office of Scientific Research (von Neumann, 1949). In the following decade a number of famous physicists worked for a while on turbulence, for example, W. Heisenberg, T. D. Lee, R. P. Feynman, and S. Chandrasekhar. It is remarkable that none of these scientists either followed up on Onsager's hints or rediscovered the insights for themselves. Lee (1951, 1952) extended some of Onsager's equilibrium statistical-mechanics ideas to continuum Euler equations, but soon left the field for particle physics. Feynman is famously reported to have worked hard on turbulence in the 1950s but to have gotten nowhere and finally given up. With so little reaction and progress from others, it is perhaps not so surprising that Onsager moved off into other areas.

The situation in the subject really changed, so far as we can determine, with the entry of R. H. Kraichnan into the field through his publication of the direct-interaction approximation closure (Kraichnan, 1959) and his theory of the 2D dual cascades of inverse energy and direct enstrophy (Kraichnan, 1967). These publications opened up lines of work on analytical turbulence closure and 2D statistical hydrodynamics that were soon followed by others, such as Edwards (1964) and Frisch (1968) for closure and Joyce and Montgomery for 2D hydrodynamics (Joyce and Montgomery, 1973; Montgomery and Joyce, 1974). Kolmogorov's work on intermittency and anomalous scaling (Kolmogorov, 1962) also broke open new directions that saw a large influx of people and significant progress. Onsager's few cryptic hints in Onsager (1949d) on equilibrium statistics of vortices and on Euler singularities and dissipative anomalies, in the end, triggered significant work on those directions. Because Onsager never published his ideas on intermittency and spectral closure, he did not have any influence in those areas. However, it now seems that all of Onsager's key insights in turbulence—both published and unpublished—have been fully recovered and even advanced upon. It has taken the community only half a century to catch up!

B. The future of Onsager's ideas

Clearly, the two most original ideas that Onsager supplied to the field of turbulence are his theory of large-scale vortices in 2D flows and his theory of inviscid dissipation in three dimensions. What, if any, are their lasting significance for future development? This is a subjective question but worth an attempt at answering nevertheless.

As Onsager himself observed, the spontaneous appearance of large-scale and long-lived vortices is a frequent but striking occurrence in two-dimensional flow—particularly in planetary atmospheres. The mathematical foundations of Onsager's equilibrium theory are now largely explored and understood and also those of its generalization by Miller and Robert. A notable exception is the ergodicity or mixing properties of fluid dynamics sufficient for its validity over experimentally accessible times. The issue of time scales is understood only a little better than it was in Onsager's day. As far as the empirical confirmation of the equilibrium vortex theories is concerned, it must be admitted that while reasonable agreement has been obtained with a few numerical simulations and laboratory experiments, we

[17]Even these results of Onsager's, while better appreciated, took considerable time for the physics community to digest properly. Miller (1995) and Mazur (1996) in their historical articles on Onsager's irreversible thermodynamics both note that his fundamental papers (Onsager, 1931a, 1931b) received little or skeptical attention in the first decade after their publication. It was only about the time of the Florence meeting in 1949 that his ideas were becoming generally accepted, following the work of Meixner (1943), DeGroot (1945), and Prigogine (1947).

It is also true that Onsager's exact solution of the 2D Ising model in 1944, while creating an immediate sensation worldwide, was not fully appreciated until much later. In an engaging discussion of the history of statistical physics, C. N. Yang writes:

> "Young physicists today may find it surprising, even unbelievable, that in the 1950s the Ising model and similar problems were not deemed important by most physicists. They were considered arcane exercises, narrowly interesting, mathematically seducing, but of little real consequence. There was the phrase (Ref. 8), for example, of 'contracting the Ising disease.' " (Yang, 1996).

It was only in the 1960s that this situation changed, in part because of the experimental discovery of a logarithmic divergence in the specific heat of ^{4}He at the λ point (Fairbank *et al.*, 1958).

Likewise, Onsager's prediction at the Florence meeting of quantization of circulation in superfluids was not met with immediate understanding or acceptance. One of the conference participants, G. Careri, who later performed key experiments on visualizing the quantized vortex lines, writes that

> "However, on that occasion, he [Onsager] offered only a brief account of his ideas, and he was probably not understood by the distinguished audience, as emerges from the recorded questions and answers. Indeed, he spoke then as always like an oracle" (Careri, 2000).

Onsager's prediction of quantized circulation in superfluids—and, similarly, London's prediction of flux quantization in superconductors (London, 1948)—were not widely accepted until after the independent work of Feynman (1955). Thus the reception given to Onsager's work on turbulence seems to represent simply a more extreme example of a general tendency.

know of no really convincing verifications for flows in nature. In fact, comparison of the theory with natural flows may require extensions and corrections of various sorts: for finite Reynolds numbers, for three-dimensional effects, for driving forces from boundaries, and so forth. Only a few of these have been pursued seriously so far. Furthermore, most of them will carry the problem outside the domain of Gibbsian equilibrium theory proper and into the regime of nonequilibrium. Quite different methods may then be required.

The observed viscosity independence of turbulent energy dissipation is, first and foremost, a surprising physical phenomenon. It is as astonishing, in its own way, as quantum effects such as superfluid flow without apparent viscosity. In a certain sense, turbulent flows seem to be the opposite, i.e., "superdissipators," for which dissipation does not disappear even as viscosity approaches vanishingly small values. This is also the property of turbulence, which gives the phenomenon much of its practical importance. The scaling law (27) with constant A is equivalent to a drag force $\frac{1}{2}C_D\rho V^2 S$ with constant C_D for a fluid of density ρ and velocity V moving past a body with cross-sectional area S (Batchelor, 1953). The energy required to overcome such turbulence-enhanced drag in transport vehicles and in material transport by pipelines is enormously costly. The phenomenological law (27) is, furthermore, a bedrock assumption of all present-day theories of turbulence. Onsager's theorem is important because it gives a foundation to this basic experimental observation and provides insight into the dynamical mechanism producing it.

A fundamental physical issue still poorly understood is the relation of the Kolmogorov-Onsager cascade picture of turbulent dissipation with G. I. Taylor's Lagrangian picture based on chaotic stretching of vortex lines. However, some recent important progress has been made in a solvable turbulence model, the so-called Kraichnan model of a passive scalar advected by a Gaussian random velocity field that is white noise in time. For an excellent survey, see Falkovich *et al.* (2001). Here it has been shown that the singularity of the turbulent velocity field, first pointed out by Onsager, leads to the breakdown of uniqueness of Lagrangian particle paths. In this way the $1/3$ Hölder exponent of the velocity increments in space can be connected with the famous Richardson law predicting the t^3 growth in time of the square separation between two Lagrangian particles. Furthermore, the nonuniqueness of the Lagrangian trajectories has been shown to be responsible for the dissipative anomaly of the passive scalar intensity in the Kraichnan model. See Falkovich *et al.* (2001) for details. It remains a major challenge to carry over such insights from this simple toy model to the Lagrangian equations of motion of three-dimensional Euler equations.

We believe that Onsager's theoretical vision of an "ideal turbulence" described by inviscid fluid equations is a proper idealization for understanding high Reynolds number flows. Needless to say, in real physical turbulence there is viscosity, which is always positive. However, we regard the zero-viscosity limit for turbulence as quite analogous to the thermodynamic limit for equilibrium statistical mechanics. In any real physical system, the volume is finite not infinite. However, the thermodynamic limit is a useful idealization for equilibrium systems whose dimensions are large compared to the size of the constituent molecules. In the same way, the zero-viscosity limit, which supposes an infinite number of cascade steps, should be a good idealization for turbulence with a large but finite number of cascade steps, that is, a Reynolds number which is large but finite. The vindication of this belief, if it is true, must come from a set of calculational tools for the zero-viscosity limit, which will make it, in the end, a truly predictive device.

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APPENDIX A: MARCH 1945 NOTE TO L. PAULING

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<u>The little vortices who wanted to play</u>

Once upon a time there were <u>n</u> vortices of strengths $K_1, \ldots, K_n$ in a two-dimensional frictionless incompressible fluid. They were enclosed by a boundary but could play ring-around-the-rosy otherwise. The rule of that game was 1)

$$K_i\, dx_i/dt = -\partial W/\partial y_i \quad ; \quad K_i\, dy_i/dt = \partial W/\partial x_i$$

where $-\rho \underline{W}(x_1, y_1, \ldots, x_n, y_n)$ equals the energy apart from an additive constant (which is infinite on account of the self-energies). The function $\underline{W}$ is something like this:

$$W = \frac{1}{2\pi}\sum_{i,j}{}' K_i K_j \log(r_{ij}) + \text{(potential of image forces)}$$

and the image forces are finite except near the boundary, --- Now the vortices were very playful like I said and they liked to distribute themselves in completely random fashion but they could not do that because they had too much energy. You see they were not like molecules which have more room in momentum-space the more energy they have. The vortices had only a finite configuration-space. So when they had more energy than the average over that space, they could not play quite the way they wanted to. --- You can describe the ergodic distribution approximately by a canonical distribution

$$f(x_1, y_1, \ldots, x_n, y_n) = \exp((\psi + \rho W)/\Theta)$$

with an antitemperature $-\Theta > 0$. You will note that the phase-integral converges for one pair of vortices if and only if $(\rho/2\pi\Theta)K_iK_j > -2$

For a set of vortices there are further necessary conditions. You can figure out that there is no way to take care of much energy unless you let at least one pair of vortices of the same sign get close together. --- And now you know how the little vortices arranged it so that most of them could play just the way they wanted to. They just pushed the biggest vortices together until the big vortices had all the energy the little ones did not want, and then the little vortices played ring-around-the-rosy until you could not tell which was where, and it make no difference anyway.

(1) C. C. Lin, Proc. Nat. Acad. Sci. 27, 570, 577, (1941). Lin, Univ. of Toronto Press, Applied Math. Series #5.

Dissipation of energy by turbulence. A violet catastrophe.

In a volume V an incompressible fluid has $(8\pi V/3)\,k^3$ degrees of freedom of wave-number less than k. The energy is gradually divided up among ∞ degrees of freedom, only for sufficiently large k the viscosity disposes of it for good; but it does not seem to matter much just how large this k is. --- Now if the velocity u is represented by its Fourier series

$$\mathbf{u} \sim \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}} \exp(i\,\mathbf{k}\cdot\mathbf{r})$$

then with boundary conditions of periodicity (and $\mathbf{a}_{\mathbf{k}}^* = \mathbf{a}_{-\mathbf{k}}$) we have

$$\partial \mathbf{a}_{\mathbf{k}}/dt = -i\,\mathbf{k}\,\pi_{\mathbf{k}} + i\sum_{\mathbf{k}'} (\mathbf{a}_{\mathbf{k}-\mathbf{k}'}\cdot\mathbf{k}')\,\mathbf{a}_{\mathbf{k}'}$$

Each term of the expansion effects an exchange of energy between wave-numbers $\mathbf{k}$, $\mathbf{k}'$. The acceleration term can be ignored (Karman) for this purpose because

$$(\mathbf{k}\cdot\mathbf{a}_{\mathbf{k}}) = 0$$

For the same reason we have

$$(\mathbf{a}_{\mathbf{k}-\mathbf{k}'}\cdot\mathbf{k}') = (\mathbf{a}_{\mathbf{k}-\mathbf{k}'}\cdot\mathbf{k})$$

so that the order of magnitude of this factor is limited by the smaller of the wave-numbers $\mathbf{k}$, $\mathbf{k}'$. Finally, the subdivision of the energy is a stepwise process (mostly) such that the wave-number increases typically by a factor between, say, 1 and 3 in each step, and the terms in which $\mathbf{k}$, $\mathbf{k}'$ and $(\mathbf{k}-\mathbf{k}')$ are of the same order of magnitude have to do most of it. --- It is then plausible that the density of energy in high wave-numbers will depend mostly on the "instantaneous" rate of subdivision viz. dissipation. It is also plausible (cf. above & Dryden) that the subdivision through wave-number k is mostly due to the energy which belongs to wave-numbers of this same order of magnitude.

Then from dimensional considerations

$$Q = -(d\,\overline{u^2_{\text{total}}}/dt) = \alpha\, k\,(d\,\overline{u^2}/d\log k)^{3/2}$$

This leads to

$$k\,(\overline{du^2}/dk) = (Q/\alpha k)^{2/3}$$

and for the correlation-function of velocities

$$R(r) = 1 - (\text{const.})\, r^{2/3}$$

For very small r the latter is modified (linear in r^2) so that the viscous dissipation of energy equals Q; cf. G. I. Taylor.

APPENDIX B: JUNE 1945 NOTE TO C.-C. LIN

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DEPARTMENT OF CHEMISTRY
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To own work

Dr. Chia-Chiao Lin
Daniel Guggenheim Aeronautical Laboratory
California Institute of Technology
Pasadena, California

Dear Dr. Lin:

If my suggestions concerning the statistics of vortices are new to you, it is probably up to me to find out what they are good for, if anything. The simplifying hypothesis is rather bold, and I am not yet satisfied as to whether the results have any connection with experience.

That Liouville's theorem holds in configuration-space has no doubt been observed before, but it appears that certain possible effects of the conservation laws have been overlooked.

The analysis is simple enough; I have not gone through elaborate computations. I sent a very brief sketch to Dr. Pauling; you might be interested in a more complete presentation of the reasoning and what I think of it myself so far.

Let us consider n parallel vortices of circulations $K_1, \ldots K_n$ confined by a non-circular boundary to a region of area S. There exists a path-function $W(X_1, Y_1, \ldots, X_n, Y_n)$ and no other important integral of the motion. We inquire about the ergodic motion of the system.

The effect of the restriction $W(X_1 \ldots Y_n) =$ const. depends on whether the prescribed value is less than the average

$$\overline{W} = S^{-n} \int W \, dx_1 \ldots dy_n$$

over configuration-space, or greater. In the critical case $W = \overline{W}$ we get on the average a random distribution of the vortices.

For the case $W < \overline{W}$ we may approximate the micro-canonical distribution by a canonical distribution with a positive temperature. For small values of $(\overline{W} - W)$ we can develop a theory analogous to the Debye-Hückel theory of electrolytes. When $(\overline{W} - W)$ is not small, we get a pronounced tendency towards mutual association of vortices of opposite sign, and strong vortices will be squeezed against the boundary. These phenomena could be discussed by methods analogous to Bjerrum's treatment of weak electrolytes. The processes of neutralization and trapping by the boundary will release energy until the vortices which are still "free" can move almost at random.

Dr. Chia-Chiao Lin Page 2

The case $W > \bar{W}$ is quite different. We now need a negative temperature to get the required energy. The appropriate statistical methods have analogs not in the theory of electrolytes, but in the statistics of stars. In a general way we can foresee what will happen. Vortices of the same sign will tend to move together, more so the stronger the repulsion between them. After this aggregation of the stronger vortices has disposed of the excess energy, the weaker vortices are free to roam at will. (?)

These predicted effects carry some resemblance to familiar habits of vortex sheets. If the rolling up of vortices is to be explained thus on a statistical basis, we may describe it as a process of crystallization, which occurs in response to a prevailing negative "market price" for energy.

Assuming for an approximate analysis that a vortex sheet may be replaced by a series of line vortices, the manner of subdivision must be chosen by some arbitrary rule. We note that for a fine subdivision we are likely to find $W > \bar{W}$. We may vary the manner of subdivision greatly in other respects and still arrive at the same qualitative conclusion: the sheet will roll up and possibly contract into concentrated vortices in some places, and at the same time the remaining sections of the sheet will be stretched into feeble, more or less haphazardly distributed discontinuities of velocity.

I still have to find out whether the processes anticipated by these considerations are rapid enough to play a dominant role in the evolution of vortex sheets, and just how the conservation of momentum will modify the conclusions. In addition, an infinite vortex trail presents certain mathematical difficulties due to the conditional convergence of the sums involved. So you see, all of my effort is still a fumbling search for new methods, which may or may not bring results.

When we come to volume distributions of vorticity (still parallel), the approximate description by line vortices introduces fictitious possibilities because it makes us forget the restrictions imposed by the incompressibility of the fluid. More or less equivalent artificial restrictions might serve as a crude substitute.

For orientation, let us consider the possible effects of redistributing vorticity in certain simple continuous fields of flow.

In Couette flow the vortex density is constant, so that any redistribution of the vortex population is trivial. – In parabolic flow between parallel plates the vortex density is a linear function of the distance from one plate, and no redistribution of the vorticity is compatible with the conservation of energy and momentum. You have shown that the effects of viscosity will nevertheless render this flow unstable. – We might go on to consider convergent and divergent flow as well as boundary layers under various conditions, but you probably recognize a pair of well-worn ruts already.

My preoccupation with the old-fashioned problem of n parallel vortices may be repugnant to progressive minds among the students of statistical hydrodynamics. I find it so comfortable to deal with a distribution over points in a finite number of dimensions, which can be described completely in terms of a density-function. You probably know that distributions in function-space, where

Dr. Chia-Chiao Lin Page 3

the number of dimensions is infinite, do not admit of complete description in such terms. You may run into similar complications when you try a direct approach to certain problems in crystal statistics. I have a healthy respect for them. I am aware that N. Wiener has dealt with certain comparatively simple problems related to integration in function-space. Do you know whether anyone else has tackled such questions and put his ideas into print?

My tentative limiting formula for the correlation-function in isotropic turbulence is not so obvious that any one student could be expected to find it. However, it seemed very probable to me that somebody would have investigated the line of reasoning, which is not far fetched. The result may be right or wrong; but if it is wrong, then the reason for that would seem to invite rather interesting speculations.

The main innovation is that I consider the dissipation of energy in turbulence as a cascade process. This I think is strongly suggested by the equations of motion, although some of my estimates might well be questioned.

Now look out for the notation: A Roman letter will denote a scalar or the absolute magnitude of a real vector when underlined, otherwise a vector which will be real or complex according to context. A Greek letter denotes a scalar. The absolute square of a vector (a) I shall write simply

$$a^2,$$

and if the vector is complex that denotes the sum of the absolute squares of the components. Vectors may be used as subscripts.

In terms of the Lagrangian description, the dissipation of energy in turbulent motion must be attributed to stretching of the vortex fibers, which generates vorticity more rapidly. The more vigorous the motion and thus accelerates the final dissipation by viscosity (Taylor).

In terms of the Eulerian description, this process appears as a conversion of the energy to motion on ever smaller scale, whereby the viscosity destroys the energy more readily the smaller the scale. We may give a precise meaning to the concept of "scale" in terms of Fourier series (Dryden):

$$v(r) = \sum_k e^{2\pi \underline{i}(k\cdot r)} a_k \qquad (1)$$

$$a_{-k} = (a_k)\ \text{conjugate}$$

$$a_k^2 \equiv (a_k \cdot a_{-k})$$

Chia-Chiao Lin Page 4

The "scale" is then the reciprocal of the wave-number $\underline{k}$. With periodic boundary conditions

$$\begin{aligned} v(\underline{x}+\underline{X}, \underline{y}, \underline{z}) &= v(\underline{x}, \underline{y}+\underline{Y}, \underline{z}) \\ &= v(\underline{x}, \underline{y}, \underline{z}+\underline{Z}) = v(\underline{x}, \underline{y}, \underline{z}) \end{aligned} \tag{2}$$

the equation of motion

$$\partial v/\partial \underline{t} = \nabla \underline{P} - (v\cdot\nabla)v \tag{3}$$

takes the form

$$\underline{d}\, a_k / \underline{dt} = 2\pi \underline{i} \left(\underline{q}_k k - \sum_{k'} (a_{k-k'} \cdot k')\, a_{k'} \right). \tag{4}$$

The Fourier coefficients $\underline{q}_k$ of the pressure can be determined from the condition of incompressibility

$$(\nabla \cdot v) = 0 \;;\; (k \cdot a_k) = 0, \tag{5}$$

whereby (4) becomes

$$\frac{\underline{d}\, a_k}{\underline{dt}} = 2\pi \underline{i} \sum_{k'} (a_{k-k'} \cdot k')\left(-a_{k'} + \underline{k}^{-2} (a_{k'} \cdot k)\, k\right). \tag{6}$$

The exchange of energy between the various components of the motion is now governed by the equations

Dr. Chia-Chiao Lin Page 5

$$2\, d\, a_k^2 / dt = \sum_{k'} Q_{k,k'} \tag{7}$$

with the notation

$$Q_{k,k'} = Q_{k,-k'} = 2\pi i \Big\{ (a_{k+k'} \cdot k')(a_{-k} \cdot a_{-k'}) + (a_{k'-k} \cdot k')(a_k \cdot a_{-k'}) - (a_{k-k'} \cdot k')(a_{-k} \cdot a_{k'}) - (a_{-k-k'} \cdot k')(a_k \cdot a_{k'}) \Big\} \tag{8}$$

As a consequence of (5) we have the detailed conservation law for the energy

$$Q_{k,k'} + Q_{k',k} = 0 \tag{9}$$

(See (10) below). Thus we may say that $Q_{k,k'}$ is the rate at which the components of the motion belonging to $(\pm k)$ receive energy from the components which belong to $(\pm k')$. (In the sums of (7) each term occurs twice because $Q_{k,k'}$ is repeated as $Q_{k,-k'}$).

Besides a_k and $a_{k'}$ the formula for $Q_{k,k'}$ involves the amplitudes of the components which belong to the combination wave numbers $(k \pm k')$. In this sense we may describe the evolution of the velocity field as a continuous process of self-modulation.

The following estimates would apply at least equally well to a hypothetical case of "foreign" modulation by a velocity field which would resemble the actual velocity field in a statistical sense only.

The selection rule for the "modulation" factor in each term of (8) suggests a "cascade" mechanism for the process of dissipation, and also furnishes a

Dr. Chia-Chiao Lin Page 6

dynamical basis for an assumption which is usually made on dimensional grounds only. In view of (5) we have the relation

$$(a_{k-k'} \cdot k') = (a_{k-k'} \cdot k) \tag{10}$$

so that the order of magnitude of this factor is limited by the <u>smaller</u> of the wave numbers $\underline{k}$, $\underline{k'}$. Thus we may infer that the small scale motion will be relatively inefficient in modulating the large scale motion. The degradation of energy will proceed mostly by stages - not from large to very small scale in one step - and the kinetics of the motion on any scale will depend mainly on those components of the motion which belong to wave-numbers of the corresponding order of magnitude.

While the components a_k of Eq. (1) do not represent "degrees of freedom" in the sense of analytical dynamics, they are still "momentoids" in the sense of Boltzmam, and the theorem of equipartition would apply if their number were finite. Since this is not the case, we get a "violet catastrophe" instead.

As you know, for a volume $\underline{V}$ there are

$$\underline{dN} = \underline{V} \cdot 4\pi\, \underline{k}^2\, \underline{dk} \tag{11}$$

wave-numbers of absolute magnitude between $\underline{k}$ and $\underline{k} + \underline{dk}$, and twice that many independent real square terms in the expansion of the energy density

$$\begin{aligned} \overline{v^2} &= \sum_k a_k^2 \\ &= \underline{V} \int_0^\infty \overline{a_k^2}\; 4\pi\, \underline{k}^2\, \underline{dk} \end{aligned} \tag{12}$$

We also note that the mean vorticity equals

$$\begin{aligned} \overline{(\nabla \times v)^2} &= \sum_k k^2 a_k^2 \\ &= \underline{V} \int_0^\infty \overline{a_k^2}\; 4\pi\, \underline{k}^4\, \underline{dk} \end{aligned} \tag{13}$$

Dr. Chia-Chiao Lin Page 7

The viscous dissipation of energy is proportional to the vorticity. We anticipate a mechanism of dissipation in which the role of the viscosity is altogether secondary, as suggested by G. I. Taylor: A smaller viscosity is automatically compensated by a reduced micro-scale of the motion, in such a way that most of the vorticity will belong to the micro-motion, but only a small fraction of the energy. In the limit of vanishing viscosity we therefore require

$$\overline{v^2} < \infty \; ; \quad \overline{(\nabla \times v)^2} = \infty \, . \tag{14}$$

From dimensional considerations we are then led to the well-known formula for the over-all decay of turbulent motion

$$\underline{Q} = -\underline{d}\,\overline{v^2}/\underline{dt} = \underline{k}_0 \left(\overline{v^2}\right)^{3/2}, \tag{15}$$

where $1/\underline{k}_0$ is some kind of an effective scale of the "main" motion. We shall expect that most of the energy will belong to wave-numbers of the same order of magnitude as $\underline{k}_0$. This would be true for a distribution of energy such that all a_k^2 were of comparable magnitude for $\underline{k} < \underline{k}_0$, with a rapid decrease of a_k^2 towards greater wave-numbers.

The right member of (15) is the product of the mean square velocity itself and a rate of deformation.

$$V_0 = \underline{k}_0 \left(\overline{v^2}\right)^{1/2} \tag{16}$$

While the over-all rate of shear is infinite according to (14), we recognize only that part of it which belongs to wave-numbers not much greater than $\underline{k}_0$ as fully effective for the primary step of the dissipation. Cf. (10).

For the "reprocessing" of the energy through the level $\underline{k} > \underline{k}_0$ we similarly admit all the deformations which belong to wave-numbers less than $\underline{k}$ or not much greater. With a hypothesis slightly stronger than (14) the motion which belongs to wave-numbers of the same order of magnitude as $\underline{k}$ itself will furnish the greater part of the effective rate of shear. We put accordingly

$$v(\underline{k}) = \beta\, \underline{k} \left(\underline{k}\, d\overline{v^2}/d\underline{k}\right)^{1/2} \tag{17}$$

where β is some numerical constant.

This dimensional analysis is closely parallel to Dryden's (J. Aero. Sci. 4, 273, 1937). The new idea is that all the energy which has been handed down from the larger scale motion must be redistributed among the components of the motion on every smaller scale. Thus we are led to assume

$$
\begin{aligned}
-\partial \overline{v^2}/\partial t &= Q \\
&= \nu(k)\left(k\, d\overline{v^2}/dk\right) \\
&= \beta k \left(k\, d\overline{v^2}/dk\right)^{3/2}. \qquad (18)
\end{aligned}
$$

You get substantially Dryden's results if you replace Q in (18) by

$$-\frac{\partial}{\partial t}\left(k\, d\overline{v^2}/dk\right)$$

and make some further assumptions which have no counterparts in the present theory.

On the hypothesis which underlies (18) the amount of energy associated with large wave-numbers is determined by the over-all rate of dissipation. In a general way this is analogous to one established relation between total viscosity and viscous dissipation. The distribution of energy required by (18) is

$$
\begin{aligned}
k\, d\overline{v^2}/dk &= \left(Q/\beta k\right)^{2/3} \\
4\pi V\, \overline{a_k^2} &= \left(Q/\beta\right)^{2/3} k^{-11/3} \qquad (19)
\end{aligned}
$$

for $k \gg k_0$.

The local properties of a function are related to the asymptotic properties of its Fourier transform viz. coefficients. In the present case we have from (1):

$$
\begin{aligned}
\overline{v^2}\, R(r) &= \overline{v^2}\left(\tfrac{1}{3} R_1(r) + \tfrac{2}{3} R_2(r)\right) \\
&= \overline{\left(v(r')\cdot v(r'+r)\right)} \\
&= \sum_k e^{2\pi i (k\cdot r)}\, \overline{a_k^2}\,, \qquad (20)
\end{aligned}
$$

so that the asymptotic estimate (19) implies a corresponding local property of the correlation-function. The computation yields

$$\overline{v^2}\, \underline{R}(\underline{r}) = \overline{v^2} - \gamma\left(\underline{Q}\,\underline{r}\right)^{2/3} + \underline{o}\left(\underline{r}^{2/3}\right), \tag{21}$$

where

$$1/\gamma = 3\,\Gamma(2/3)\,\beta^{2/3}. \tag{21a}$$

With (16) this yields in turn

$$\underline{R}(\underline{r}) = 1 - \gamma\left(\underline{k}_0\,\underline{r}\right)^{2/3} + \underline{o}\left(\underline{r}^{2/3}\right) \tag{22}$$

as an asymptotic formula for small distances when Taylor's micro-scale vanishes.

We assumed at the outset that the first steps in the subdivision of energy would determine the over-all rate of the process. On the basis of our estimate (19) the total time required for the subsequent steps will indeed be relatively small.

By (16) the time-scale associated with the main motion is

$$\underline{t}_0 = 1/\nu_0 = \underline{k}_0\left(\overline{v^2}\right)^{-1/2}. \tag{23}$$

By (17) and (19) the corresponding characteristic time for motion on the scale $1/\underline{k}$ is

$$\underline{t}(\underline{k}) = 1/\nu(\underline{k}) = \left(\underline{k}_0/\beta\,\underline{k}\right)^{2/3}\underline{t}_0. \tag{24}$$

The total time required for infinite subdivision is therefore finite

$$\int_{k_0}^{\infty} \underline{t}(\underline{k})\,\underline{dk}/\underline{k} = \tfrac{3}{2}\,\beta^{-2/3}\,t_0 < \infty. \tag{25}$$

Dr. Chia-Chiao Lin Page 10

The distribution law (19) is compatible with the hypothesis that the mean rate of stretching of vortex lines is given by the average rate of deformation in the liquid. As far as I can make out, a more rapid decrease of a_k^2 with increasing k would require a "spotty" distribution of the regions in which the velocity varies rapidly between neighboring points.

A formula for the rate of change of a weighted average of the correlation-function is useful in this connection. Let $F(r)$ be any function such that

$$\int_0^\infty (1 + r^3)\, |dF(r)| < \infty .$$

Then with the abbreviation

$$D_r(v) = v(r' + r) - v(r')$$

the equations of motion (3) lead to the following relation (assuming isotropy):

$$\frac{\partial}{\partial t} \int_0^\infty \overline{v^2}\, R(r)\, F(r)\, 4\pi r^2\, dr$$

$$= \int_0^\infty \overline{(r \cdot D_r(v))\,(D_r(v))^2}\; 2\pi r\, dF(r). \qquad (26)$$

Now put

$$F(r) = 3/4\pi a^3 \quad ; \quad (r < a)$$

$$F(r) = 0 \quad ; \quad (r > a)$$

Then for small a the left member of (26) is practically

$$\partial \overline{v^2} / \partial t$$

and the right member is at most of the order

$$\left(\overline{|D_r(v)|^3} / r \right)_{r = a}$$

Dr. Chia-Chiao Lin

Page 11

Moreover, obviously

$$\overline{(D_r(v))^2} = 2\,\overline{v^2}\,(1 - \underline{R}(\underline{r})). \tag{27}$$

Now the estimate (21) is a minimum hypothesis unless the mean cube of $D_r(v)$ consists mainly of contributions from exceptional regions of small aggregate volume.

You can get the formula suggested by G. I. Taylor:

$$1 - \underline{R}(\underline{r}) \sim \underline{r} \tag{?}$$

if you make the extreme assumption that the vorticity is distributed in sheets of comparable intensity and finite total area. However, the discontinuities would give rise to oscillograms of a striking rectangular structure, rather unlike those which I have seen.

By the way, among the contributions to hydrodynamics which I consider food for thought is a paper presented by R. B. Montgomery at a conference of the N. Y. Acad. in March 1942. You probably have a good library, but if you are at all interested, I think I can get a copy of those papers for you.

Yours sincerely,

Lars Onsager

Lars Onsager

until July 5th: R. F. D. 1, ~~[illegible]~~
Tilton, New Hampsh[illegible]

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APPENDIX C: FOLDERS FROM THE ONSAGER ARCHIVE

Folder 11:129

p. 1: List of references

pp. 2–6: Formula for the Prandtl mixing length

p. 2: Dimensional analysis of eddy viscosity and of the integral I

p. 3: Calculations for wind-tunnel experiments

p. 4: Proof that I is a Lagrangian invariant

pp. 5 and 6: Formula for mixing length in terms of I

pp. 7–13: Two-dimensional fluids and point vortices

p. 7: Two-dimensional Navier-Stokes for the vorticity in Fourier representation

p. 8: The mean-field equation and its derivation

p. 9: Energy-temperature relation from Debye-Hückel theory

p. 10: Derivation of formula for the critical energy

p. 11: Calculations for the Debye-Hückel theory

p. 12: Formula for the Hamiltonian of point vortices

p. 13: Space integrals of Green's functions

pp. 14–21: Equation relating second- and third-order structure functions

p. 14: Statement of relation between second- and third-order structure functions

pp. 15–18: Derivation of the main relation

p. 19: Wiener-Khinchin relation between velocity correlation function and spectrum

p. 20: Energy in low wave numbers as an oscillatory integral of the velocity correlation

p. 21: Scaling of second- and third-order structure functions; realizability considerations

p. 22: Demonstration of a finite time for infinitely many cascade steps

pp. 23–27: Closure equation for the energy spectrum

p. 23: Main derivation of the equation from the Taylor expansion in time

p. 24: Four-dimensional linear equation for a wave-number triad with one leg "frozen"

pp. 25–27: Exact solution of the linear equation

pp. 28–31: Calculations for two-dimensional Kármán vortex streets

p. 28: Derivation of conserved quantity along Lagrangian particle paths; sketches of paths in a vortex street

p. 29: Analytical formulas for plane curves of particle paths

pp. 30 and 31: Real and complex stream functions and velocity fields

Folder 11:132

p. 1: Alternative definition of $Q_{\mathbf{k},\mathbf{k}'}$

p. 2: Equation for Fourier modes of the vorticity

pp. 3–6: New approach to derivation of spectral closure

p. 3: Time derivative of spectrum and formula for $Q_{\mathbf{k},\mathbf{k}'}$

pp. 4 and 5: "Direct-interaction" part of $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$, from evolution of passive mode

p. 6: "Direct-interaction" part of $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$, from evolution of two active modes

p. 7: Page reading "Hydrodynamics (Turbulence). Kinetic Energy and Dissipation function to sums of squares." Referring to previous pages?

p. 8: Velocity space correlation with exponential decay

pp. 9–12: Eigenbasis of the Stokes operator between infinite parallel plates

p. 9: Statement of results for eigenfunctions odd and even under reflection about the center plane; selection rules for allowed triadic interactions

pp. 10–12: Derivation of the eigenbasis

p. 13: Page reading "Turbulent Cascade." Referring to following pages?

pp. 14–21: Detailed calculations for the spectral closure

pp. 14 and 15: Evaluation of terms in the closure assuming K41 scaling

p. 16: Final form of average $Q(\mathbf{k},\mathbf{k}')$ in the closure; a proposed formula for the dynamical correlation time of a triad

p. 17: Another brief calculation of the direct interaction (DI) part of $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$ from evolution of two active modes

p. 18: Formula for $Q_{\mathbf{k},\mathbf{k}'}$; another proposal for the correlation time of a triad

p. 19: Wave-number integrals of products of fractional powers of the spectrum

p. 20: Evaluation of average $\overline{|\mathbf{a}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'|^2}$

p. 21: Evaluation of DI part of $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$ from evolution of convective mode, on average; equals zero?

Folder 11:135

p. 1: Definition of $Q_{\mathbf{k},\mathbf{k}'}$ *and exact evaluation of* $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$

p. 2: Equation for Fourier mode $\mathbf{a}(\mathbf{k})$; *all four terms of* $Q_{\mathbf{k},\mathbf{k}'}$

pp. 3–5: Another calculation of the DI contribution to $(d/dt)Q_{\mathbf{k},\mathbf{k}'}$ *from evolution of two active modes*

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*All materials referred to as PTvK, CIT come from the Caltech institute archive *Papers of Theodore von Karman, 1881–1963*, California Institute of Technology, Pasadena, California. 18.22 means Box 18, Folder 22.

†All materials referred to as LOA, NTNU come from *The Lars Onsager Archive*, Norges Teknisk-Naturvitenskapelige Universitet, Universitetsbiblioteket i Trondheim, Norge. 11.129 means Box 11, Folder 129.

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