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Colloquium: Weakly interacting, dilute Bose gases in 2D

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This colloquium surveys a number of theoretical problems and open questions in the field of two-dimensional dilute Bose gases with weak repulsive interactions. In contrast to three dimensions, in two dimensions the formation of long-range order is prohibited by the Bogoliubov-Hohenberg theorem, and Bose-Einstein condensation is not expected to occur. Nevertheless, experimental indications supporting the formation of a condensate in low-dimensional systems have recently been obtained. This unexpected behavior appears to be due to the nonuniformity introduced into a system by the external trapping potential. Theoretical predictions, made for homogeneous systems, require therefore careful reexamination. A number of popular theoretical treatments of the dilute weakly interacting Bose gas are presented and their regions of applicability are discussed. The possibility of Bose-Einstein condensation in a two-dimensional gas, the validity of the perturbative *t*-matrix approximation, and the diluteness condition are issues also discussed in detail.

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I. INTRODUCTION

A. Revival of interest in low-dimensional systems

Low-dimensional systems are interesting in general, as their low-temperature physics is governed by strong long-range fluctuations. These fluctuations inhibit the formation of true long-range order (LRO), which is a key concept of phase transition theory in three dimensions (3D). Thus a two-dimensional uniform system of interacting bosons does not undergo Bose-Einstein condensation at finite temperatures. However, this system turns superfluid below a certain temperature T_{KT}, identified by Berezinskii, Kosterlitz, and Thouless (BKT) in 1971–1973, signaling the presence of a so-called topological order. Elementary excitations of the superfluid phase are pairs of vortices with opposite winding numbers.

The experimental realization of such a system was for many years restricted to films of superfluid ^{4}He on surfaces, which is also an example of a strongly interacting system. The breakthroughs in experimental physics at the end of the last century have changed the situation drastically. The combination of laser cooling (S. Chu, C. Cohen-Tannoudji, and W. D. Phillips, Nobel Prize for Physics, 1997) with evaporative cooling and magneto-optical traps provided experimental systems of cold atoms, which were primarily used to observe the long-awaited phenomenon of Bose-Einstein condensation (E. A. Cornell, W. Ketterle, and C. E. Wieman, Nobel Prize for Physics, 2001). The full *tunability* of magnetic and optical traps opens an extraordinary opportunity to study in practice not only one- and two-dimensional Bose systems, but also dimensional crossovers influenced by the number of particles, size and shape of the system, interaction strength, and temperature. These new developments have triggered a revival of theoretical interest in low-dimensional systems, when old theoretical predictions are to be tested or carefully revised in order to address *finite-size* experimental systems, and a large field of new phenomena are to be explained.

While experimental indications of the BKT transition in weakly interacting Bose system have been recently obtained (Stock *et al.*, 2005), many questions remain unanswered. One of the most interesting is whether topological order survives under some conditions in the inhomogeneous trapped system, or is it dominated by the true LRO and Bose-Einstein condensation prevails? Can we control and directly observe the formation of vortex pairs in two-dimensional quantum gases? These and other problems serve as the main motivation for this Colloquium.

In the next section we present a succinct overview of the history of work with dilute Bose systems, outlining some of the important theoretical problems relevant to weakly interacting Bose gases.

B. Historical overview

The condensation of conserved particles that obey the same statistics as photons was predicted by Einstein (1924, 1925) even before the concept of Fermi statistics was introduced. Einstein's prediction was preceded by an ingenious conjecture of Bose, who realized that black-body radiation can be treated as a gas of *indistinguishable* photons. Einstein generalized ideas of Bose to material particles and published two famous papers, in which he developed what we now call Bose-Einstein statistics (Einstein, 1924, 1925).

The ideal gas of Bose particles is remarkably the only example of a *noninteracting* system in condensed-matter physics that undergoes a phase transition upon decreasing the temperature. However, experimental realization of ideal Bose-Einstein condensates is extraordinarily difficult, since realistic systems always involve interactions. Largely for this reason Einstein's ideas did not receive a wide recognition in the scientific community for many years as being devoid of any practical significance. The condensation phenomenon did not even appear in the textbooks, until London recognized the analogy between superfluidity of liquid ^{4}He, discovered by Kapitza (1938) and Allen and Misener (1938), and an ideal Bose gas and emphasized that Einstein's statement was "erroneously discredited" (London, 1938).

In support of London's phenomenological ideas, the first *microscopic* theory of superfluidity in a system of weakly interacting Bose particles was introduced by Bogoliubov (1947). Subsequent discussions about the connection between superfluidity and Bose-Einstein condensation (BEC) led Penrose and Onsager (1956) to formulate the generalized criterion for BE condensation. This line of research culminated in a paper by C. N. Yang, who extended this criterion to superfluidity and superconductivity and proposed the concept of off-diagonal long-range order (ODLRO) (Yang, 1962). The condensed phase is characterized by a nonvanishing asymptotic of a one-body density matrix at large distances.

During the decades which followed the work of Bogoliubov, successful field-theoretical approaches were developed and many important predictions about the thermodynamics of the interacting Bose system were made. However, apart from the successful observation of superfluidity in liquid-helium systems, the quest to create Bose-Einstein condensates proved unrewarding for several decades. Finally, in 1995 Bose-Einstein condensates were realized in a fascinating series of experiments on rubidium and sodium vapors (Ketterle *et al.*, 1999; Ketterle, 2001; Cornell and Wieman, 2002). The importance of this experimental achievement was recognized in the 2001 Nobel Prize for Physics, shared by E. A. Cornell, W. Ketterle, and C. E. Wieman.

The experimental realization of BEC has offered a unique opportunity to probe and control many interesting phenomena, not accessible or unstudied in the field of superfluidity, such as dimensional transitions, the crossover from Bose-Einstein condensation to BCS pair condensation, interference effects, and disorder effects. Exotic links to cosmology (Fedichev and Fischer, 2003), quantum optics (Recati *et al.*, 2005) (two-state atomic quantum dots within a condensate), and wetting phenomena (Indekeu and Van Schaeybroeck, 2004) have been recently proposed. The growing interest in Bose systems has resulted in more than 600 studies per year during the last decade and the list of references related to BEC now exceeds 200 pages.

The actual observation of condensation was hindered by enormous technical difficulties, so that even 15 years ago researchers dared not to believe that nature would ever provide them with the "right" system. The main problem to overcome is the condensation of most systems into a solid or liquid upon cooling to low temperatures, which bypasses the BEC transition. In particular, the formation of clusters or molecules is driven by three-body collisions. The hard task for an experimentalist was therefore the creation of a gaseous system, in which three-body collisions occur less frequently than two-body interactions.

The gas in which two-body interactions prevail is called dilute. Diluteness implies a very low density of the gas, so that the characteristic range a_s of the potential between Bose particles is small compared to the mean particle distance, proportional to $n^{-1/3}$ in three dimensions ($n=N/V$ being the density of the gas). The diluteness condition is therefore equivalent to the requirement that the gas parameter $n^{1/3}a_s$ be small,

$$n^{1/3}a_s \ll 1. \tag{1.1}$$

Ultradilute systems can condense only at extremely low temperatures (in the nanokelvin range), realization of which was another technical obstacle for the experimentalists. At low temperature the thermal velocity of the particles v_T, which is proportional to the inverse de Broglie wavelength

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}, \tag{1.2}$$

becomes very small ($v_T = \hbar/m\lambda_T \sim 1$ mm/sec) and at temperatures of the order of a few nK all the particles "jump" into a coherent ground state. Sufficient diluteness of the gas is therefore one of the crucial conditions for BEC to be observed in the experiment.

In order to reach the required temperature and density regimes, various cooling and trapping techniques have been developed (Ketterle *et al.*, 1999). Before being cooled, atoms are confined in an external potential created by an applied magnetic field. The finite extent of the condensate cloud and its inherent inhomogeneity introduce a number of important differences between BEC in a trap and uniform gas. For example, a trapped gas of Bose atoms exhibits a BEC transition not only in momentum space but in coordinate space as well (Dalfovo *et al.*, 1999). In practice however, condensates are so small that the literal observation of their size and shape is limited by the resolution of existing experimental equipment. Nevertheless, real-space Bose condensates provide a novel resource for exploring many interesting phenomena, such as quantum interference effects and frequency-dependent collective excitations.

The effect of a magnetic trap becomes more dramatic for lower dimensional systems. For example, in 2D a noninteracting trapped gas undergoes a BEC phase transition at finite temperature (Widom, 1968; Bagnato and Kleppner, 1991; Li *et al.*, 1999) in contrast to the two-dimensional uniform case, where condensation is possible only at zero temperature. This difference arises because the gas density of states is modified in the presence of a trap.

The description of an *interacting* system in a two-dimensional harmonic potential is not trivial. In the case of a uniform gas, long-range order does not develop because of long-wavelength phase fluctuations, inherent to low-dimensional systems. This can also be seen as an infrared divergence of the integral $\int N(\mathbf{p}) d^2p/(2\pi\hbar)^2$, where $N(\mathbf{p})$ is the number of particles out of the condensate with momentum $\mathbf{p}$. This divergence, on the other hand, is a consequence of the fact that the energy of the system depends only on the phase gradient, and not on the phase itself, because the latter is not a well-defined quantity (Lifshitz and Pitaevskii, 2004). The absence of long-range order in two-dimensional systems with a continuous symmetry is often referred to as the Bogoliubov k^{-2} or Hohenberg-Mermin-Wagner (BHMW) theorem [see works by Bogoliubov (1961, 1991), Mermin and Wagner (1966), Wagner (1966), and Hohenberg (1967)], and we discuss this issue in Sec. IV.B. Fisher and Hohenberg (1988) pointed out that a consequence of long-wavelength phase fluctuations is a drastic modification of the diluteness condition, so that the conventional low-density requirement for weakly interacting two-dimensional Bose gas, $n^{1/2}a_s \ll 1$, is replaced by an inequality

$$\ln \ln \frac{1}{na_s^2} \gg 1. \tag{1.3}$$

Taken literally, condition (1.3) rules out the possibility of experimentally realizing a two-dimensional dilute Bose system. However, this condition does not work away from the transition. One can show from the analysis of quantum fluctuations [see Petrov *et al.* (2004) for review] that in this case the diluteness criterion amounts to $1/\ln(1/na^2) \ll 1$, previously derived by Schick (1971).

It is also intuitively clear that the trapping potential introduces a lower bound for the momentum of excitations and thus prevents establishing long-range thermal fluctuations which destroy the condensate. Based on these arguments, Petrov *et al.* (2000) showed the existence of a true condensate in a quasi-two-dimensional system in a wide parameter range.

More generally, the BHMW approach is not suitable for a proper analysis of an inhomogeneous system, such as trapped atomic vapor, as pointed out by Fischer (2002, 2005). In his work Fischer (2002, 2005) obtained a geometrical equivalent of the BHMW theorem, independent of the system's Hamiltonian, and showed that in the marginal $d=2$ case true condensation is possible in an appropriately defined thermodynamic limit.

In support of theoretical estimations, experimental confirmations of macroscopic occupation of the harmonic oscillator ground state (Görlitz *et al.*, 2001; Rychtarik *et al.*, 2004) was done with sodium atom vapors, confined to optical and magnetic traps. Rapid progress in experimental techniques made it possible to increase the aspect ratio (anisotropy) of the trap from 79 (Görlitz *et al.*, 2001) to 700 (Smith *et al.*, 2005). This large anisotropy of new traps is sufficient to confine condensates with $\sim 10^5$ atoms in a quasi-two-dimensional regime (Smith *et al.*, 2005). Signs of local coherence were also observed in a two-dimensional gas of hydrogen atoms, absorbed on the liquid-^{4}He surface (Safonov *et al.*, 1998). Quasi-two-dimensional condensates have also been recently created by Stock *et al.* (2005) and interesting phase defects have been measured. The crossover from three-dimensional condensates to 2D and ultimately one-dimensional condensates can be observed by changing the aspect ratio of the trap.

As indicated in the previous section, recent progress in laser-based trapping techniques and creation of optical lattices has led to a new generation of remarkable experiments. With controllable interparticle interaction it is now possible to observe the transition from the superfluid state to a Mott insulator (Bloch, 2004). Optical lattices provide a way to investigate various intriguing aspects of low-dimensional systems as well. Interest in two-dimensional configurations of Bose particles has arisen in the context of high-temperature superconductivity and the fractional quantum Hall effect. All in all, ultracold atomic gases have the potential to impact a very broad range of physics.

In this Colloquium we discuss a selected number of issues related to two-dimensional weakly interacting neutral Bose gases. When necessary, three-dimensional problems are mentioned. We attempt to cover many references and otherwise refer the reader to numerous resources, such as several excellent theoretical reviews (Dalfovo *et al.*, 1999; Castin, 2001; Leggett, 2001; Fetter, 2002; Petrov *et al.*, 2004; Yukalov, 2004) and books (Pines, 1962; Griffin, 1993; Pethick and Smith, 2002; Pitaevskii and Stringari, 2003), a resource letter for BEC (Hall, 2003) and BEC web sites. Though a certain level of subjectivity is unavoidable, we aim to provide the necessary information about the field to those who feel lost after a preliminary contact with current literature but want to learn more about the main problems of Bose-Einstein condensates in 2D.

II. IDEAL BOSE GAS

Consider a macroscopic system of noninteracting Bose particles at finite temperature in the grand-canonical ensemble. The total number of particles in such a system is defined by

$$N = \sum_k n_B(\epsilon_k) = \int \rho(\epsilon) n_B(\epsilon) d\epsilon, \tag{2.1}$$

where $n_B(\epsilon_k)=1/[\exp \beta(\epsilon_k-\mu)-1]$ is the Bose-Einstein distribution function, $\beta=1/k_BT$, and $\rho(\epsilon)$ is the density of states.

The chemical potential μ of the Bose gas, being negative, increases as the temperature drops and vanishes at the critical temperature T_c, indicating the phase transition to a condensed state. The transition temperature is therefore defined by Eq. (2.1) with $\mu=0$. In d dimensions the density of states $\rho(\epsilon)=dN_\epsilon/d\epsilon \sim \epsilon^{(d-2)/2}$, and the particle density is proportional to the integral

$$n \equiv \frac{N}{V} \sim \int \frac{\epsilon^{(d-2)/2} d\epsilon}{\exp(\epsilon/T_c)-1}. \tag{2.2}$$

In 3D this integral converges and the Bose-Einstein condensation temperature has a finite value, $T_c^{3\mathrm{D}} \sim n^{2/3}$. This result can be also understood as a temperature scale at which the thermal wavelength becomes comparable with the average interparticle spacing $\lambda_T \sim l \sim n^{-1/3}$. As λ is proportional to $T^{-1/2}$, T_c is proportional to $n^{2/3}$.

One can also calculate the number of particles occupying the ground state,

$$N_0 = N\left(1-\frac{T}{T_0}\right)^{3/2}. \tag{2.3}$$

It is readily seen that N_0 increases as the temperature decreases. This phenomenon of macroscopic occupation by particles of the state with minimal energy at low temperatures is referred to as Bose-Einstein condensation. Note that the actual condensation occurs in momentum space.

In 2D, a constant density of states leads to an infrared divergent integral in Eq. (2.2) and condensation is not possible at any finite temperature.

We now discuss how this picture changes in the presence of a trap. The general treatment of this problem was considered by Bagnato and Kleppner (1991). They studied the possibility of the Bose-Einstein condensation of an ideal gas, confined by a one- or two-dimensional power-law trap: $V_{\mathrm{ext}} \sim x^\eta$. Bagnato and Kleppner (1991) showed that a two-dimensional system undergoes BEC for any finite value of η, moreover, $T_c^{2\mathrm{D}}$ has a broad maximum in the vicinity of $\eta=2$, i.e., for a trapping potential close to parabolic. (A one-dimensional system displays BEC only for $\eta<2$.)

Practically the confining trap is well approximated by a harmonic potential,

$$V_{\mathrm{ext}}(\mathbf{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{2.4}$$

For noninteracting particles we can write the many-body Hamiltonian as a sum of one-particle Hamiltonians $H_{\mathrm{MB}}=\Sigma_{i=1}^N H_{\mathrm{SP}}(i)$, whose eigenvalues are

$$\epsilon_{n_x n_y n_z} = (n_x+\tfrac{1}{2})\hbar\,\omega_x + (n_y+\tfrac{1}{2})\hbar\,\omega_y + (n_z+\tfrac{1}{2})\hbar\,\omega_z. \tag{2.5}$$

The lowest energy of the system in the trap is $\epsilon_{000}=\frac{3}{2}\hbar\,\bar{\omega}$, where for the sake of simplicity we introduced the average frequency $\bar{\omega}=(\omega_x+\omega_y+\omega_z)/3$.

Note that in the ground state all N particles occupy the level ϵ_{000} and the wave function of the "cloud" of these particles is easy to find,

$$\phi(\mathbf{r_1}\dots\mathbf{r}_\mathrm{N}) = \prod_i \varphi_0(\mathbf{r_i}),$$

$$\varphi_0(\mathbf{r}_i) = \left(\frac{m\omega_{ho}}{\pi\hbar}\right)^{3/4} \exp\left(-\frac{m}{\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)\right), \tag{2.6}$$

where

$$\omega_{h0} = (\omega_x\omega_y\omega_z)^{1/3}. \tag{2.7}$$

In this case the density distribution of the particles is position dependent,

$$n(\mathbf{r}) = N|\varphi_0(\mathbf{r})|^2, \tag{2.8}$$

and the first important length scale appearing in the problem is the size of the cloud,

$$a_{h0} = \sqrt{\frac{\hbar}{m\omega_{h0}}}, \tag{2.9}$$

which is just the average width of the Gaussian distribution (2.6) (Fig. 1). Experimentally a_{h0} is usually of order of 1 μm.

Since we are interested in low-dimensional effects, it is instructive to mention the experimental realization of a two-dimensional atomic trap. An axially symmetric harmonic potential can be written in the form $V_{\mathrm{ext}}(r) = \frac{1}{2}m\omega_\perp^2 r_\perp^2 + \frac{1}{2}m\omega_z^2 z^2 = \frac{1}{2}m\omega_\perp^2(r_\perp^2+\lambda^2 z^2)$, where $\lambda=\omega_z/\omega_\perp$ characterizes the degree of anisotropy. For $k_BT \ll \hbar\,\omega_z$ and $k_BT > \hbar\,\omega_\perp$ the motion of atoms along the z direction is frozen (particles only undergo zero-point oscillations), and kinematically the gas can be considered as

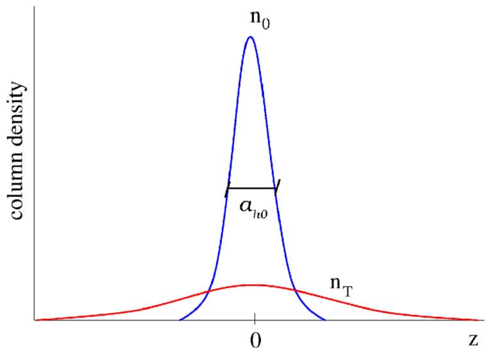

FIG. 1. (Color online) Column density of a cloud with trapped noninteracting bosons along the z direction. The total density is a superposition of condensate density n_0 and a thermal distribution of noncondensed particles n_T.

two dimensional. Thus by making one of the trap dimensions very narrow, oscillator states become widely separated, and an effective two-dimensional system is realized.

At finite temperature only a fraction of the particles N_0 occupies the lowest energy level and the others are thermally distributed over higher energy levels. However, we still can treat N_0 as a macroscopic number. Thermal excitations will cause the size of the atomic cloud to grow with temperature. In the *semiclassical* approximation $k_BT \gg \hbar\omega_{h0}$, where the relevant excitation energies are much larger than the interlevel spacing, it can be shown that the size of the cloud increases as a square root of temperature $R_T = a_{h0}\sqrt{k_BT/\hbar\omega_{h0}}$. The important conclusion of this short discussion is that, in harmonic traps, Bose condensation manifests itself as a sharp peak in the density distribution in real space. The appearance of such a peak in both coordinate and momentum space is a peculiar feature of trapped condensates, with significant impact on both theory and experiment. This is different from the uniform gas discussed above, where the condensation cannot be revealed in real space, for the condensate and uncondensed particles occupy the same volume.

The total number of particles in the trap is defined by

$$N = \sum_{n_x n_y n_z} \frac{1}{\exp[(\epsilon_{n_x n_y n_z} - \mu)/T] - 1}, \tag{2.10}$$

which is derived from Eq. (2.1) with a discrete energy spectrum (2.5). Note that in this case the chemical potential at the transition point acquires a nonzero value of the lowest energy level: $\mu(T \to T_c) \to \mu_c = (\frac{3}{2})\hbar\bar{\omega}$.

In the semiclassical approximation we can simplify Eq. (2.10) by replacing the summation with integration and straightforward solution for $\mu = \mu_c$ gives the Bose-condensation temperatures for the trapped gas in three and two dimensions,

$$T_c^{3D} = \frac{\hbar}{[\zeta(3)]^{1/3}} \omega_{ho} N^{1/3}, \tag{2.11}$$

$$T_c^{2D} = \frac{\hbar\sqrt{6}}{\pi} \omega_{ho} N^{1/2}. \tag{2.12}$$

The two-dimensional condensation temperature is now finite (nonzero). This is related to the density-of-states effect of the gas in the trap. Indeed, in the semiclassical approximation we can introduce a coordinate system defined by $\epsilon_{x,y,z} = \hbar n_{x,y,z}\omega_{x,y,z}$, where the surface of constant energy (2.5) is the plane $\epsilon = \epsilon_x + \epsilon_y + \epsilon_z$. Then the number of states $N(\epsilon)$ is proportional to the volume in the first octant bounded by this plane,

$$N_\epsilon = \frac{1}{\hbar^3\omega_{h0}^3}\int_0^\epsilon d\epsilon_x \int_0^{\epsilon-\epsilon_x} d\epsilon_y \int_0^{\epsilon-\epsilon_x-\epsilon_y} d\epsilon_z = \frac{\epsilon^3}{6\hbar^3\omega_{h0}^3}. \tag{2.13}$$

The density of states $\rho = dN_\epsilon/d\epsilon$ is then quadratic in energy $\rho^{3D} \sim \epsilon^2$ in three dimensions and linear in energy in two dimensions $\rho^{2D} \sim \epsilon$, in contrast to the constant density of states of a uniform two-dimensional gas, and the integral in Eq. (2.10) for $\mu = \mu_c$ is not infrared divergent until $d = 1$.

It is now straightforward to calculate the condensate fraction in 3D:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^{3D}}\right)^3 \tag{2.14}$$

and the total energy of the system and thermodynamic quantities. In 2D the condensate fraction is (Bagnato and Kleppner, 1991; Petrov *et al.*, 2004)

$$\frac{N_0}{N} \approx 1 - \left(\frac{T}{T_c^{2D}}\right)^2. \tag{2.15}$$

The sign "≈" Eq. (2.15) is related to the fact that at $T = T_c^{2D}$ the condensate fraction is not exactly zero, because there is a small correction due to the finite number of particles in the system (Petrov *et al.*, 2004). One should therefore be careful with the word "phase transition" in the context of trapped gases, because they are finite-size systems and the phase-transition notion is strictly defined only in the thermodynamic limit. It is better to say that at T_c there is a sharp crossover to the BEC state in the system. Note also that at T_c^{2D} the de Broglie wavelength λ_T becomes comparable with the mean interparticle separation $\sim\sqrt{T_c/Nm\omega_{h0}^2}$.

We end the section by remarking on the proper definition of the thermodynamic limit in the trapped case. It is well known that the transition temperature should be well defined in the thermodynamic limit. The usual definition when the ratio N/V is kept constant while the number of particles N and the volume V tend to infinity is apparently not suitable for the inhomogeneous situation. The appropriately defined limit is then obtained by letting $N \to \infty$ and $\omega_{h0} \to 0$, while keeping $N\omega_{h0}^3$ (or $N\omega_{h0}^2$ in 2D) constant. In this case the temperatures (2.11) and (2.12) are well defined.

TABLE I. Properties of the ideal gas. DOS stands for density of states ρ, TDL stands for thermodynamic limit. $T_{\rm BEC}$ is the critical temperature of Bose-Einstein condensation depending on the dimension. N is the number of particles, V is the volume of the system, and ω_{h0} is the geometric average of the oscillator frequencies: in 2D, $\omega_{h0}=(\omega_x\omega_y)^{1/2}$ and in 3D, $\omega_{h0}=(\omega_x\omega_y\omega_z)^{1/3}$.

Ideal gas property	Uniform	Trapped
DOS	$\rho_{3D}\sim\sqrt{\epsilon}$	$\rho_{3D}\sim\epsilon^2$
	$\rho_{2D}\sim$ const	$\rho_{2D}\sim\epsilon$
$T_{\rm BEC}$	$T_{c0}^{3D}\sim(\frac{N}{V})^{2/3}$	$T_{c0}^{3D}\sim\omega_{h0}N^{1/3}$
	$T_{c0}^{2D}\to 0$	$T_{c0}^{2D}\sim\omega_{h0}N^{1/2}$
TDL	$\lim\frac{N}{V}=$const	3D: $\lim\omega_{h0}N^{1/3}=$const
	$N\to\infty$, $V\to\infty$	2D: $\lim\omega_{h0}N^{1/2}=$const
		$\omega_{h0}\to 0, N\to\infty$

A comprehensive survey of various issues related to the behavior of the ideal Bose gas in a harmonic potential can be found in the paper by Mullin (1997).

The ideal-gas results are summarized in Table I.

III. GROUND STATE OF A WEAKLY INTERACTING BOSE GAS

A. Bogoliubov approximation

In his seminal paper "On the theory of superfluidity" (Bogoliubov, 1947) Bogoliubov introduced a microscopic description of the ground state of a uniform, weakly interacting Bose gas. The assumption about the uniformity of the unperturbed ground state is crucial to his results. To assure a uniform Bose gas, Bogoliubov considered the case of repulsive interactions and made use of periodic boundary conditions. The gas is also assumed to be dilute ($na^3\ll 1$), which permits us to simplify the many-body problem and account for interactions in a fundamental way. In contrast to the uniform case, the nonuniform ground state is very "sensitive" to interactions and makes the solution of the many-body problem highly nontrivial.

The standard Hamiltonian of an interacting Bose gas is

$$H=\frac{1}{2}\int\nabla\Psi^\dagger(r)\,\nabla\,\Psi(r)dr+\frac{1}{2}\int\Psi^\dagger(r)\Psi^\dagger(r')U(r-r')\Psi(r')\Psi(r)drdr', \tag{3.1}$$

where $U(r-r')$ is the interaction between particles. In momentum space this Hamiltonian reads

$$H=\sum_p\epsilon_p a_p^\dagger a_p+\frac{1}{2V}\sum_{pp'q}U_q a_p^\dagger a_{p'}^\dagger a_{p'-q}a_{p+q}, \tag{3.2}$$

$U_q=\int e^{-iqr}U(r)dr$ is a Fourier component of the interaction, the bosonic field operator $\Psi(x)=1/\sqrt{V}\Sigma_p e^{ipx}a_p$ (here x is a four-vector), and the boson creation and annihilation operators satisfy the usual commutation relations $[a_p,a_{p'}^\dagger]=\delta_{pp'}$.

Without interactions all N particles of the system occupy the state with zero energy and zero momentum. The number of condensed particles N_0 in this case is equal to the total number of particles N. When we switch on the interaction, two particles can scatter out of the condensate and occupy one of the many zero-total-momentum states with separate momenta $\mathbf{k}$ and $-\mathbf{k}$ (in the lowest-order perturbation theory) and N_0 naturally decreases.

For a dilute weakly interacting Bose gas one can assume that the total depletion of the condensate is small ($\delta N/N_0\ll 1$) and most particles remain in the condensate $N_0\gg 1$. The key observation of Bogoliubov is that in this case the second-quantized condensate operators can be simply replaced by the c-number $\sqrt{N_0}$,

$$\hat{a}_0,\hat{a}_0^\dagger\sim\sqrt{N_0}. \tag{3.3}$$

The drawback of this prescription is that it leads to a Hamiltonian which no longer conserves the number of particles. This problem can be partly resolved by working in the grand-canonical ensemble, in which additional terms $-\mu N_p$ ($N_p=\Sigma_{p\neq 0}a_p^\dagger a_p$) are introduced into the Hamiltonian (4.27). This secures the conservation of particles on the average. It is also worth mentioning that the Bogoliubov approximation is equivalent to neglecting dynamics in the condensed state.

In the weak-coupling limit the Hamiltonian (4.27) can be diagonalized by applying the Bogoliubov canonical transformation,

$$a_k=u_k\alpha_k-v_k\alpha_{-k}^\dagger,$$

$$a_k^\dagger=u_k\alpha_k^\dagger-v_k\alpha_{-k}, \tag{3.4}$$

and the resultant Hamiltonian describes the system of noninteracting quasiparticles with spectrum

$$\xi_k=\sqrt{n_0U_0\frac{k^2}{m}+\frac{k^4}{4m^2}}, \tag{3.5}$$

where $n_0=N_0/V$ is the density of condensed particles.

From this dispersion relation (3.5) it follows that in the long-wavelength limit the Bogoliubov quasiparticles behave as "phonons" with sound velocity $s=\sqrt{n_0U_0/m}$, and the low-temperature thermodynamics of a Bose-condensed system is governed by this phonon spectrum. In the opposite short-wavelength limit quasiparticles behave as free particles with an energy $k^2/2m$. By equating the kinetic energy and the "Hartree" interaction energy n_0U_0 one can straightforwardly find the "transition" wave vector $k_c=\sqrt{2mn_0U_0}\sim\sqrt{2}ms$, which separates the phononlike behavior of elementary excitations from the free-particle one. k_c introduces an important length scale into the system (Fig. 2),

$$\lambda_c=\hbar/k_c=\hbar/\sqrt{2mn_0U_0}, \tag{3.6}$$

over which coherence effects are important in the inter-

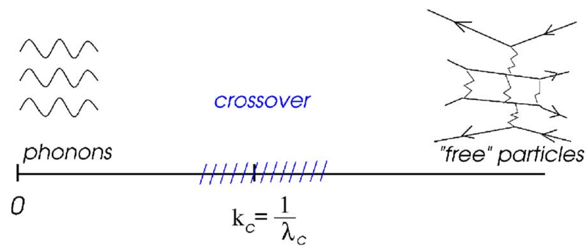

FIG. 2. (Color online) Length scale in the Bogoliubov problem: correlation length λ_c. $k_c \sim 1/\lambda_c$ separates the "free" particle behavior from the linear dispersion region. Zigzag lines denote the residual interaction between particles.

action between particles. It is usually called the *healing length* (in the context of trapped condensates), or sometimes the correlation or coherence length, and refers to correlations between excitations in the system. These correlations are distinct from long-range correlations, which lead to condensation in the $k=0$ mode.

One should also note that the Bogoliubov canonical transformation is equivalent to a summation over the most divergent terms in the perturbation-series expansion for the ground-state energy. Summation of such series is also equivalent to making the random-phase approximation (RPA).

It was important in the theory of superfluidity that the low-lying Bogoliubov quasiparticles follow a linear dispersion. This kind of behavior is fully consistent with the Landau criterion for superfluidity, i.e., that no excitation can be created in a liquid moving with a velocity v less than that of a sound ($v<s$). In case of noninteracting particles the dispersion is quadratic for all k and superfluidity is not possible.

B. Field-theoretical approaches: *t*-matrix approximation

To go beyond the Bogoliubov approximation, one needs to take both multiple-scattering diagrams and RPA contributions into account. That can be done, for example, by means of a pseudopotential method (Lee *et al.*, 1957), or by field-theoretical methods, first applied to the Bose gas of small density at $T=0$ by Beliaev (1958a, 1958b) and by Hugenholtz and Pines (1959).

The presence of the many-particle condensate in the ground state of the interacting Bose gas was the main obstacle to applying the usual technique of Feynman diagrams to this system. Consider, for example, the one-particle Green's function in the interaction representation

$$G(x-x') = -i\frac{\langle T\{\Psi(x)\Psi^\dagger(x')S\}\rangle}{\langle S\rangle}. \tag{3.7}$$

Here the average is taken over the ground state of N noninteracting Bose particles, which are in the condensate ($N_0=N$). The S matrix is expressed as usual,

$$S = T\left\{\exp\left(-\frac{i}{2}\int d^4x_1 d^4x_2 \times U(x_1-x_2)\Psi^\dagger(x_1)\Psi^\dagger(x_2)\Psi(x_2)\Psi(x_1)\right)\right\}, \tag{3.8}$$

where x_1 and x_2 are four-vectors and the interaction is $U(x_1-x_2)=U(r_1-r_2)\delta(t_1-t_2)$. In order to derive the diagram series for the Green's function, we need to expand the S matrix in powers of H_{int}. Usually the terms containing the odd number of annihilation operators vanish after averaging over the ground state, which unfortunately does not happen for a Bose gas due to the above-mentioned peculiarities of the ground state. The expectation value of the N product containing a_0 apparently does not vanish and the standard method of constructing diagrams cannot be applied to an interacting Bose gas.

This difficulty was successfully resolved by Beliaev in 1958. He noticed that for a large number of particles N the diagrammatic approach can be applied to particles with momenta $p\neq 0$, while the condensed phase (which does not disappear when interactions are turned on) can be described as a sort of external field. It is thus convenient to separate the operators a_0 and $a_0^\dagger$ (which act only on the ground state) from Ψ and $\Psi^\dagger$,

$$\Psi = \Psi' + a_0/\sqrt{V}, \quad \Psi^\dagger = \Psi'^\dagger + a_0^\dagger/\sqrt{V}. \tag{3.9}$$

The Green's function (3.7) is then divided into two parts, and the operations T and $\langle\cdots\rangle$ are represented as two successive operations, the former acting only on Ψ' and $\Psi'^\dagger$, and the latter acting only on a_0 and $a_0^\dagger$. The operators a_0 and $a_0^\dagger$, occurring in the S matrix, are treated as parameters, and the expectation values over Ψ', $\Psi'^\dagger$ ground state can now be calculated using standard techniques.

With these ideas in mind, Beliaev succeeded in deriving a general expression for the one-particle Green's function of the interacting system in terms of some effective self-energies Σ_{ik} and chemical potential μ. However, the exact calculation of the Green's functions proved to be very complicated, and approximate methods of summing the series of Feynman graphs were developed.

For simplicity, Beliaev considered a short-range, central interaction potential $U_{\mathbf{p}}=U_0$ for $p<1/a$ and $U_{\mathbf{p}}=0$ for $p>1/a$. In the low-density limit $n_0a^3 \ll 1$, where n_0 is the density of the particles in the condensate, he obtained a crucial result that the main contributions to the self-energies of the Green's function originate from ladder diagrams. In this case the real interaction U is replaced by an effective two-particle interaction Γ, representing the sum of contributions from all ladder-type Feynman graphs (Fig. 3). The integral equation for the vertex Γ, called the Bethe-Salpeter equation, is

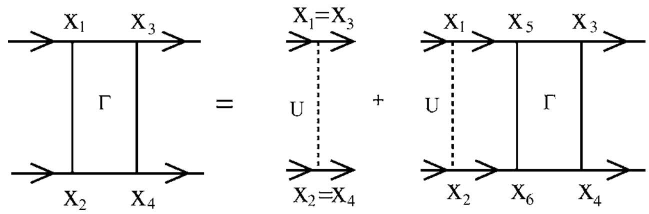

FIG. 3. Bethe-Salpeter equation for the two-particle scattering vertex Γ.

$$\Gamma(x_1,x_2;x_3,x_4) = U_{x_1-x_2}\delta(x_1-x_3)\delta(x_2-x_4) + i\int d^4x_5 d^4x_6 U_{x_1-x_2} G^0(x_1-x_5) \times G^0(x_2-x_6)\Gamma(x_5,x_6;x_3,x_4), \quad (3.10)$$

where $x\equiv(\mathbf{r},t)$. In momentum representation Eq. (3.10) reads

$$\Gamma(p_1,p_2;p_3,p_4) = U_{p_1-p_2} + i\int d^4p_5 d^4p_6 U_{p_1-p_5} \times G^0(p_5)G^0(p_6)\Gamma(p_5,p_6;p_3,p_4), \quad (3.11)$$

where the momentum conservation condition $p_1+p_2 = p_3+p_4=p_5+p_6$ is implied and $p_i\equiv(\mathbf{p}_i,p_0^i)$.

It is convenient to introduce relative and total momenta according to

$$p_1+p_2=P', \quad p_3+p_4=P$$
$$p_1-p_2=2p', \quad p_3-p_4=2p. \quad (3.12)$$

This transformation leads to the following:

$$t(p',p,P) = U(p'-p) + i\int \frac{d^4q}{2\pi^4} U(p'-q) \times G^0(P/2+q)G^0(P/2-q)t(q,p,P), \quad (3.13)$$

where we denote Γ in the center-of-mass representation by t, and the free-particle Green's function is $G^0(p) = (p^0 - p^2/2m + i\delta)^{-1}$.

A conventional t-matrix equation is obtained from Eq. (3.13) after integrating over q_0. In two dimensions this results in the following:

$$t(\mathbf{p}',\mathbf{p},P) = U_{\mathbf{p}'-\mathbf{p}} - \int \frac{d\mathbf{q}}{(2\pi)^2} U_{\mathbf{p}'-\mathbf{q}} \frac{t(\mathbf{q},\mathbf{p},P)}{k_0^2 - q^2/m + i\delta}, \quad (3.14)$$

where $k_0^2 = P^0 - \mathbf{P}^2/4m$. In scattering theory this equation is also known as the Lippmann-Schwinger equation. Physically the t matrix corresponds to the renormalization of the interaction by multiple scattering of one particle off another.

The standard way to treat the dilute Bose gas is to replace the real potential, which is usually singular, by the zero momentum t matrix generated from multiple two-particle scattering, represented by the infinite summation of the ladder diagrams described above.

The t matrix (3.14) cannot be solved explicitly, but in general its solution can be expressed in terms of the two-particle scattering amplitude in vacuum. The scattering amplitude $f(\mathbf{p}',\mathbf{p})$ for a transition from the initial relative wave vector $\mathbf{p}$ to a finite relative vector $\mathbf{p}'$ is defined by

$$f(\mathbf{p}',\mathbf{p}) = \int d\mathbf{q}\, U(\mathbf{p}'-\mathbf{q})\Psi_{\mathbf{p}}(\mathbf{q}), \quad (3.15)$$

where $\Psi_{\mathbf{p}}$ is a wave function of a scattering problem with potential U that satisfies the following Schrödinger equation in momentum representation:

$$(k^2-p^2)\Psi_k(\mathbf{p}) - \int d\mathbf{q}\, U(\mathbf{p}-\mathbf{q})\Psi_k(\mathbf{q}) = 0. \quad (3.16)$$

According to elementary scattering theory (Dalfovo *et al.*, 1999; Castin, 2001; Leggett, 2001; Fetter, 2002), at low energies s-wave scattering becomes dominant, and the scattering amplitude f_0 is approximated to leading order by

$$f_0 \simeq \frac{4\pi\hbar^2 a_s}{m}, \quad (3.17)$$

where the momentum dependence of the scattering amplitude can be ignored in the low-energy limit. Thus at low energies, in vacuum the only remaining parameter characterizing the interaction is the s-wave scattering length a_s.

In general, the t matrix (3.14) requires knowledge of the scattering amplitude for $k_0^2 \neq q^2/m$, known as the "off-the-energy-shell" t matrix. For two-particle scattering in vacuum, discussed above, only the on-shell t matrix is physically relevant. When three-body collisions become important, calculation of the off-shell t matrix is necessary (Fadeev, 1960). In the context of dilute Bose gases the off-shell t matrix arises in connection with the so-called many-body t-matrix approach (Stoof and Bijlsma, 1993; Bijlsma and Stoof, 1997; Proukakis *et al.*, 1998), which we discuss in the next section. The many-body t matrix takes into account the effect of the medium (mean field) in which collisions occur. At the low-energy limit the many-body t matrix is approximated by the off-shell two-body t matrix (Morgan *et al.*, 2002). The solution of the off-shell t-matrix was first proposed by Beliaev (1958b) and Galitskii (1958). An alternative approach based on the inhomogeneous Schrödinger equation, which allows us to treat the hard-sphere central potentials in one, two, and three dimensions, was considered by Morgan *et al.* (2002). Morgan *et al.* (2002) have shown for any dimension that, for potentials with a finite range, the long-wavelength limit of the off-shell t matrix depends only on energy and not on the initial and final relative momenta of the scattered particles. This result means that low-energy collisions can be represented by a contact potential.

Consider now the quasiparticle spectrum within the first-order Beliaev approach. It turns out one can reproduce the Bogoliubov result (3.5) with the only difference that instead of the potential U_0 the momentum independent scattering amplitude f_0 appears, for in the first order U_0 is equal to f_0. The healing length (3.6) can then be related to a scattering length

$$\lambda_c = \frac{1}{\sqrt{8\pi a_s n_0}}. \tag{3.18}$$

The second-order approximation does not modify the physical picture of the low-temperature behavior of the interacting Bose gas, but provides corrections to the sound velocity, and a damping proportional to p^5 related to the decay process of one phonon into two. Third-order corrections involve the solution of a three-particle problem, which to date has not been solved.

We now turn to the two-dimensional system. Following the methods developed by Beliaev, Schick (1971) examined a two-dimensional system of hard-disk bosons of diameter a at low densities and absolute zero [see also the recent study of Ovchinnikov (1993)]. The dimensionless expansion parameters are the interaction U_0 and the gaseous parameter na^2, which are small in the dilute limit. The application of Beliaev's method to two-dimensional systems is not as straightforward as it is for three-dimensional systems. In the three-dimensional case, the ladder diagrams are the only contributions which do not depend on na^3 and therefore it is natural to take them into account while calculating the first term in the density expansion of all quantities. In 2D contributions from the ladder diagrams depend logarithmically on na^2, in particular, the effective interaction, or t matrix, is proportional to $1/\ln(1/na^2)$,

$$f_0^{2D} \sim \frac{4\pi}{m \ln(1/na^2)}. \tag{3.19}$$

The key conclusion of Schick (1971) is that $1/\ln(1/na^2)$ plays the role of the small parameter in the two-dimensional dilute system at zero temperature and dominant contributions are derived from the first-order diagrams for this parameter. In this approximation he calculated the leading-order correction to the chemical potential,

$$\mu = -\frac{4\pi\hbar^2 n}{m \ln(na^2)}\{1 + O(1/\ln(na^2))\}, \tag{3.20}$$

and the quasiparticle excitation spectrum,

$$\xi_k = \sqrt{\mu\frac{k^2}{m} + \frac{k^4}{4m}} = \sqrt{\frac{k^4}{4m} + \frac{4\pi n}{m\ln(1/na^2)}k^2}. \tag{3.21}$$

In the long-wavelength limit quasiparticles behave as phonons with a speed of propagation $s = \sqrt{-4\pi n/m\ln(na^2)}$. The spectrum changes from phononlike to free-particle-like in the vicinity of the momentum k_c defined as

$$ka \ll k_c a \equiv -16\pi na^2[m\ln(na^2)]^{-1} \ll 1.$$

The ground-state energy per particle and the condensate fraction take the form (Schick, 1971)

$$E/N = -\frac{2\pi\hbar^2 n}{m\ln(na^2)}\{1 + O(1/\ln(na^2))\},$$

$$\frac{n_0}{n} = 1 + \frac{1}{\ln(na^2)} + O(1/[\ln(na^2)]^2). \tag{3.22}$$

C. Gross-Pitaevskii mean-field theory

The ground-state and thermodynamic properties of an interacting Bose system confined by an external potential $V_{\text{ext}}=\frac{1}{2}\hbar\omega_{h0}(r/a_{h0})^2$ [a_{h0} is the trap size (2.9)] can be directly calculated from the Hamiltonian

$$H = \int dr\psi^\dagger(r)\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(r)\right)\psi(r) + \frac{1}{2}\int drdr'\psi^\dagger(r)\psi^\dagger(r')U(r-r')\psi(r')\psi(r) \tag{3.23}$$

using numerical methods, such as quantum Monte Carlo. Nevertheless, for most experimentally relevant situations (when the number of atoms is large) the mean-field description of the system proves to be sufficient. In this case the macroscopic low-energy behavior of the system can be explored under the assumption that the order parameter varies over distances larger than the mean interparticle spacing.

Such a mean-field approximation was first developed by Gross and Pitaevskii. Their approach, which is valid in the dilute limit, is a straightforward generalization of Bogoliubov theory for the gas in the trap. One should bear in mind that the diluteness condition $n_{\max}a_s^3 \ll 1$ does not automatically secure the weakness of interactions. The interaction strength is specified by an extra parameter [see, in particular, the review of Dalfovo *et al.* (1999) and the paper by Fetter (1999)]. The interaction energy, which is of the order of gNn, is to be compared with the kinetic energy, proportional to Na_{h0}^{-2}. Since the average density of atoms $n \sim N/a_{h0}^3$, the interaction strength can be characterized by a dimensionless parameter $N|a_s|/a_{h0}$. When $Na_s \ll a_{h0}$, it means that the coherence length λ_c (3.6) is large in comparison with the size of the trap a_{h0} and the system is assumed to be nearly ideal and is described by a Gaussian distribution (2.6). In the opposite limit $Na_s \gg a_{h0}$, the coherence length is small and the dilute gas exhibits important nonideal behavior (Dalfovo *et al.*, 1999).

The mean-field Gross-Pitaevskii approximation is extensively presented in the literature [see, for instance, the review by Dalfovo *et al.* (1999), and the paper by Leggett (2003), and a review with an emphasis on experiment by Angilella *et al.* (2006)], therefore we only mention briefly the key concepts of its derivation. Gross and Pitaevskii's approach is based on the Bogoliubov prescription for the condensate (3.3), according to which the boson field operators ψ are written as a sum of a classical field ϕ, having the meaning of the order parameter, and a small perturbation ψ',

$$\psi(r,t) = \phi(r,t) + \psi'(r,t), \tag{3.24}$$

implying that the depletion of the condensate is small. As a side note, we mention that in principle the problem of the order-parameter definition in a finite inhomogeneous system arises in this case, but it turns out that the wave function of the condensate has a clear meaning, if determined through the diagonalization of the one-body density matrix in analogy with liquid-helium drops (Lewart *et al.*, 1988). This issue is also discussed in detail in a review by Leggett (2001).

One can expand the theory in the parameter ψ' and derive the equation for ϕ from either the standard Heisenberg equation or alternatively by taking the variation of the classical action S of the type

$$S = \int dt d\mathbf{r} \bar{\phi}\left[i\partial_t - \frac{\hbar^2}{2m}\nabla^2 - V_{\text{ext}} - \frac{g}{2}\bar{\phi}\phi \right]\phi$$

with respect to $\bar{\phi}$ (saddle-point approximation). The derivation of the Gross-Pitaevskii (GP) equation and next-order corrections within the Bosonic field theory can be found in the paper by Stenholm (1998).

The resulting Gross-Pitaevskii equation is

$$i\hbar \frac{\partial}{\partial t}\phi(r,t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + g|\phi(r,t)|^2 \right)\phi(r,t), \tag{3.25}$$

where we have approximated the potential by a δ function $V(r-r')=g\delta(r-r')$ (which we can do under the assumption that the interparticle spacing is much larger than the interaction range), and where $g=4\pi\hbar^2 a_s/m$ is the three-dimensional coupling constant. This coupling constant is equal to the zero-momentum limit of the scattering amplitude (3.17) discussed above.

In the limit $N\to\infty$ (Thomas-Fermi approximation) the kinetic energy contribution can be neglected and the Gross-Pitaevskii equation can be solved analytically. This classical Thomas-Fermi approximation breaks down in the vicinity of the condensate boundary, where the gradient of the condensate density is no longer small.

We discuss now the coupling constant of the two-dimensional Bose gas. It was first demonstrated by Lozovik in 1971 [see the review of Petrov *et al.* (2004)] that to zero order in perturbation theory the coupling constant $g^{2\text{D}}=\hbar^2 f_0^{2\text{D}}/m$, where $f_0^{2\text{D}}$ is the scattering amplitude at the energy of the relative motion $E=2\mu$.

This coupling constant can be treated as a parameter, as in the work by Bayindir and Tanatar (1998) (see also references therein), where the two-dimensional Bose gases described by the GP equation have been studied. For some range of interaction strength it was shown that interacting bosons behave similarly to the noninteracting case in a harmonic trap. For weak short-range interparticle interactions, a finite-temperature BEC phase transition was found to occur.

On the other hand, the coupling constant in 2D is expected to display a logarithmic dependence on density [cf. Eq. (3.19)] in accordance with estimations by Schick (1971) for f_0 in the case of a homogeneous gas. The precise choice of $g^{2\text{D}}$ has in fact been a controversial issue [see Lieb *et al.* (2001), and references therein]. For example, Kim *et al.* (1999) suggested $g^{2\text{D}}\sim 1/\ln(1/ka)$, where $0<ka\ll 1$ and k is the infrared cutoff introduced by the trap at $1/a_{h0}$, so that $g^{2\text{D}}\sim 1/\ln(a_{h0}/a)$. This kind of approximation may be reasonable when the size of the trap is much larger than other length scales in the problem.

Note, that for quasi-two-dimensional gas in a trap the coupling constant was derived by Petrov *et al.* (2000),

$$g^{\text{Q2D}} = \frac{2\sqrt{2\pi}\hbar^2}{m}\frac{1}{a_{h0}/a + (1/\sqrt{2\pi})\ln(1/\pi k^2 a_{h0}^2)}. \tag{3.26}$$

The rigorous derivation of the Gross-Pitaevskii functional for a two-dimensional interacting gas was provided by Lieb *et al.* (2001). Their analysis leads to the following expression for the coupling constant:

$$g^{2\text{D}} = \frac{1}{|\ln(\bar{n}a^2)|}, \tag{3.27}$$

where $\bar{n}$ is the average density of the particles, proportional to $\sqrt{N}$. The mean density is defined as $\bar{n}=(1/N)\int n^{\text{TF}}(r)^2 d^2r$, with the Thomas-Fermi density given by $n^{\text{TF}}(r)=[\mu^{\text{TF}}-V_{\text{ext}}(r)]/8\pi$, and μ^{TF} chosen so that the constraint $\int n^{\text{TF}}=N$ holds. The density expansion has been applied to the case of a two-dimensional Bose gas at zero temperature by Cherny and Shanenko (2001) in order to derive the Gross-Pitaevskii equation.

The modification of the GP equation due to the many-body renormalization of the scattering, mentioned in Sec. III.B, has been provided by Lee *et al.* (2002). The effective interparticle interaction in 2D is modeled by the off-shell two-body t matrix, that at low energies depends on the energy of the collision. The energy dependence of the effective interaction can be written in the density-dependent form and applied to the two-dimensional trapped gas. This leads to the GP equation, describing the condensate wave function that no longer has a cubic nonlinearity in Ψ, but instead goes as $(|\psi|^2/\ln|\psi|^2)\psi$ (Lee *et al.*, 2002).

It is also interesting to analyze the deviations from the mean-field behavior, since the experimental system is well controlled nowadays and different regimes can be realized. Corrections to the mean-field ground-state solution stem from quantum fluctuations, and their effect becomes more prominent with the growth of the gas parameter, as has been observed in Monte Carlo simulations. For calculating quantum corrections in a systematic way we refer the reader to the paper by Andersen and Haugerud (2002), and references therein. Many references on the GP approximation and beyond can be found elsewhere (Kolomeisky *et al.*, 2000; Angilella *et al.*, 2004). For the effects of a third spatial dimension and the self-consistent calculation of the coupling constant, see the paper by Cherny and Brand (2004), and references therein.

IV. FINITE-TEMPERATURE PROBLEMS

Zero-temperature techniques are not really suitable for controlling infrared (IR) thermal fluctuations, and new methods have to be devised to describe the interacting system at finite T. At the beginning of Sec. IV the generic properties of the two-dimensional XY models and the concept of quasi-long-range order is presented, which is the central concept in the phase-transition theory in 2D. The reason true long-range order cannot form in two-dimensional uniform system is discussed in detail in Sec. IV.B, and especially the way familiar concepts from two-dimensional phase-transition theory should be revised in the trapped case.

Section IV.C presents the theory of Popov, who pioneered the finite-temperature generalization of Beliaev's field-theoretic approach, and described the low-temperature superfluid state of the two-dimensional Bose gas. Section IV.D shows how the diluteness condition of Fisher and Hohenberg, discussed in the Introduction, arises as an applicability limit of the Popov's t-matrix approach. Section IV.E describes methods which generalized and/or improve the results of Popov, and also Monte Carlo simulations, which are to date the most reliable numerical calculations of the superfluid phase in a two-dimensional Bose system. Before concluding, we mention how unique the two-dimensional system with a contact interaction is, for it possesses an inherent symmetry, which leads to the birth of the special breathing modes, which in principle can be checked experimentally.

A. Introduction: Two-dimensional *XY* models

For our further analysis it is important to recognize that a uniform, interacting Bose system belongs to the XY universality class, characterized by a vector order parameter [for a comprehensive analysis see the book by Chaikin and Lubensky (1995)]. This means that the finite-temperature behavior of the two-dimensional Bose gas is determined by generic properties of the two-dimensional XY model.

We know that two-dimensional XY models are special, for long-range thermal fluctuations destroy the long-range order at finite temperatures (Bose-Einstein condensation in the case of a two-dimensional Bose gas). The existence of these long-wavelength modes in a two-dimensional Bose fluid was first pointed out by Bogoliubov in his k^{-2} theorem in 1961, and later confirmed by Hohenberg (1967) and by Mermin and Wagner (1966) (this issue is discussed in Sec. IV.B).

However, a special type of order—topological order—which gives rise to superfluidity, can develop in a two-dimensional Bose fluid below the Kosterlitz-Thouless temperature T_{KT}, as predicted by Kosterlitz and Thouless (1973) and Berezinskii (1970, 1971) using the renormalization-group method (RG). Below T_{KT} the continuous U(1) symmetry (rotations in a two-dimensional plane) is broken and the system acquires a finite rigidity, or phase stiffness ρ_s. The order-parameter

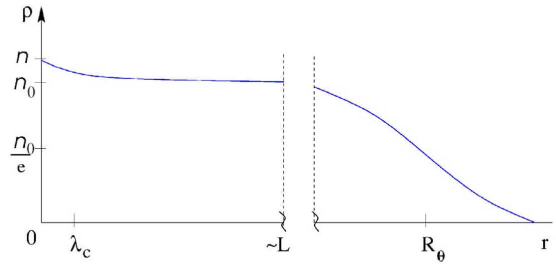

FIG. 4. (Color online) One-particle density matrix $\rho(r)=\langle\psi^\dagger(0)\psi(r)\rangle$ in two dimensions. Two characteristic length scales R_θ and λ_c are shown, $R_\theta \gg \lambda_c$. At large distances $\lambda_c \ll L \ll R_\theta$ the one-particle density matrix is equal to the condensate density n_0.

correlations decay algebraically (for any coupling of the XY model), and the average order parameter is zero. However, locally the order parameter can have a well-defined value. This unique situation is described in terms of quasi-long-range order (QLRO) (Chaikin and Lubensky, 1995). Important low-lying excitations of the QLRO phase are vortex pairs (two vortices with opposite winding numbers) whose fugacity decreases with distance, thus not destroying the connectivity of the state (therefore $\rho_s \neq 0$).

The phase transition to a disordered state (with $\rho_s=0$) is associated with a dissociation of the coupled vortex pairs. Above T_{KT} the vortex fluid can be treated as a kind of vortex plasma, where vortices play the role of mobile "charges," interacting via a Coulomb potential. In this language the state below T_{KT} can be described as an "insulating" state of bound charges. The mapping of the two-dimensional XY model onto the two-dimensional Coulomb gas is considered in detail in the review by Minnhagen (1987).

The rigidity or superfluid density ρ_s does not go continuously to zero at the critical temperature, but experiences a universal jump,

$$\frac{m^2 k_B T_{\mathrm{KT}}}{\hbar^2 \rho_s(T_{\mathrm{KT}\_})} = \frac{\pi}{2}, \tag{4.1}$$

first predicted by Nelson and Kosterlitz (1977) and successfully verified in experiments on superfluid ^{4}He films, absorbed on a substrate (Bishop and Reppy, 1978).

An interesting interpretation of Kosterlitz-Thouless physics in the context of bosonic systems was put forward by Kagan *et al.* (1987) almost 20 years ago. They propose that below T_{KT} the system forms a "quasicondensate," a condensed state achieved in a local sense. The introduction of the quasicondensate concept was motivated by a peculiar behavior of the one-particle density matrix $\rho(r)$ at large distances in 2D (Fig. 4).

There are two length scales associated with the behavior of $\rho(r)$: the aforementioned correlation length λ_c at which $\rho(r)$ relaxes from the value n at $r=0$ to n_0, and the characteristic radius of the phase fluctuations R_θ, which is rather large, $R_\theta \gg \lambda_c$. The appearance of large R_θ can be understood in the following way: at large distances ρ

falls off as a power law of r (Kane and Kadanoff, 1967) $\rho(r) \sim n_0(r/r^*)^{-\alpha}$, where $r \gg r^*$ and the coefficient α is proportional to the temperature and the Schick's parameter $\alpha \sim T/T^* \ln(1/na^2)$ $[T^* \sim (2m\lambda_c^2)^{-1}]$ and therefore is very small, $\alpha \ll 1$. As a result of this the density matrix ρ decays over a large length scale $R_\theta \sim r^* e^{1/\alpha}$ (Kagan *et al.*, 1987).

Conceptually, the system can be divided into blocks of size L, which is smaller than R_θ. In each block one can introduce the wave function of the condensate with a well-defined phase. The whole system is then described in terms of an ensemble of the block wave functions. Condensate wave functions within the ensemble corresponding to blocks separated by a distance greater than R_θ have uncorrelated phases, and it is impossible to define the condensate wave function for the entire system as a whole. The state of matter with a fluctuating phase is called a quasicondensate (Kagan *et al.*, 1987). See also the extention of Bogoliubov methods to quasicondensates by Mora and Castin (2003).

What happens to the XY universality class concepts in an experimentally realizable system of cold atoms confined in a trap remains a controversial issue. We will see in the next section that many issues should be crucially reformulated in order to address the physics of trapped cold gases.

B. Problems of long-range order formation in 2D

The notion that the development of the long-range order (LRO) is not possible in 2D dates back to the work of Peierls (1935), who argued that the thermal motion of low-energy phonons will ruin the LRO in a two-dimensional solid. A rigorous proof of Peierls's statement was provided later by Mermin (1968).

Subsequent work by Mermin and Wagner (1966) provided a proof that there is no spontaneous magnetization or sublattice magnetization in an isotropic Heisenberg model with finite-range interactions. At the same time Hohenberg (1967) succeeded in ruling out the existence of a conventional superfluid or superconducting ordering in one and two dimensions. It was also shown by Coleman (1973) that there are no Goldstone bosons in 2D, which is equivalent to saying that there is no LRO in 2D.

A rigorous proof of the Mermin-Wagner-Hohenberg results exploits the Bogoliubov and Schwartz inequalities (Appendix A) and leads to the following result for the average occupation number of $\mathbf{k}$ states:

$$\langle a_k^\dagger a_k \rangle \equiv n_k \geq -\frac{1}{2} + \frac{mTn_0}{k^2 n}. \tag{4.2}$$

Here n_0 is a condensate density and n is a total density. It is clear now that the appearance of the condensate (macroscopic occupation of a single state) in 2D for finite temperatures fails due to the fact that the function k^{-2} is not integrable at small momenta in two-dimensional k space. Physically, the long-range thermal fluctuations prevent the formation of a coherent condensate.

The same result can be obtained from the infrared asymptote of the one-particle Green's function at zero frequency,

$$G(\mathbf{k},0) \approx -\frac{n_0 m}{n_s k^2}, \tag{4.3}$$

and was first derived by Bogoliubov (1961). The derivation of the asymptotic behavior (4.3) in a functional integrals approach has been done by Popov (1983). Since the Green's function defines the average number of particles with momentum k, it can be readily seen that we arrive at the same result (4.2). The statement that the condensate does not appear in a two-dimensional interacting Bose system at any finite temperature is also known as the Bogoliubov k^{-2} theorem.

We have mentioned that in the context of modern condensed-matter theory the absence of the LRO in 2D is discussed in terms of general properties of the XY models. A respective direction or a phase of the d-dimensional XY order parameter is specified by an angle θ. The variance in the fluctuation of the order-parameter phase is given by

$$\langle \theta^2(\mathbf{r}) \rangle \sim \frac{T}{\rho_s} \int \frac{d^d q}{(2\pi)^d} q^{-2} = \frac{T\Lambda^{d-2}}{\rho_s (d-2)}, \tag{4.4}$$

where Λ is the wave-number cutoff (Chaikin and Lubensky, 1995). It can be readily seen that $d=2$ is the critical dimension of the XY universality class and fluctuations destroy long-range order in the two-dimensional XY model in accordance with the conclusions of Bogoliubov, Mermin, Wagner, and Hohenberg. Quasi-long-range order, discussed in the previous section, is nevertheless possible in 2D.

In the case of a trapped gas the Bogoliubov-Mermin-Wagner-Hohenberg (BMWH) theorem rules out BEC in 2D in the interacting system [see Mullin (1997)]. However, the question arises if one can actually apply BMHW theorem to a system confined within a harmonic potential. Is it still possible to unambiguously rule out the condensate formation in two-dimensional atom traps? The applicability of the BMWH theorem to the inhomogeneous case requires careful consideration, for the Bogoliubov-Hohenberg inequality was derived assuming an infinite uniform system. In this approximation, many features of practically realized condensates, such as their formation in real space, are excluded.

An alternative version of the Hohenberg inequality, suitable for experimentally realizable Bose systems, has been proposed by Fischer (2002, 2005). Taking the dimension of the trap to be an experimentally controlled parameter, Fischer addressed the issue of a spatially localized Bose condensate, with the question in mind of how far one could "stretch" the three-dimensional condensate cloud before coherence will be destroyed. Fischer derived an inequality which controls the size of the smallest possible condensate for a given condensate and density profile. In Appendix A we briefly sketch the underlying concepts of his derivation.

The resulting inequality reads

$$\frac{n-n_0}{n_0} \geqslant \frac{2\pi R_c^2}{n\lambda_{\mathrm{dB}}^2}C(\mathbf{k}) - \frac{1}{2n_0}[1-|\psi_0(\mathbf{k})|^2/V_0], \quad (4.5)$$

where $\psi_0(\mathbf{k})$ is the condensate wave function, R_c is the effective radius of the condensate wave function (effective radius of the curvature of the condensate),

$$R_c = \left((V_0/n) \int d^d r \psi_0(r)[-\Delta_r \psi_0^*(r)]n(r) \right)^{-1/2}, \quad (4.6)$$

and

$$C(\mathbf{k}) = \left| \int d^d r |\psi_0|^2 \exp(ikr) - \psi_0(k) \int d^d r \psi_0^*(r)|\psi_0(r)|^2 \right|^2. \quad (4.7)$$

Note that the only requirement on the Hamiltonian of the system that is needed to derive the inequality (4.5) is that it should not contain any explicit velocity dependence in the interaction and external potentials.

Since in 2D R_c^2 scales as n, this case can be considered as marginal and the condensate still can emerge even in an interacting system. This is because the usual log divergences inherent for 2D are cut off by a trap. The inequality (4.5) is a geometrical equivalent of the Bogoliubov-Hohenberg inequality, since it gives the lower bound for the ratio of the effective radius of the condensate to the de Broglie wavelength λ_{dB}. The second term on the right-hand side of Eq. (4.5) can be used to obtain an upper limit on the possible condensate fraction as a function of temperature. Concrete examples of the application of Eq. (4.5) to quasi-one-dimensional systems have been given by Fischer (2002).

One can also approach the problem of the condensate formation by directly analyzing phase fluctuations of the order parameter [for a review, see Hellweg *et al.* (2001)]. Phase fluctuations are caused by thermal excitations and are always present at finite temperatures. Note that at very low temperatures density fluctuations in equilibrium are suppressed due to their energetic cost and can therefore be ignored. This assumption is not valid in the vicinity of a vortex core, but at very low temperatures the vortex formation is negligible.

As an aside, we mention that the concept of phase in quantum systems, introduced by Dirac as a canonical conjugate observable to the number operator $\hat{n}$, remains a controversial issue in certain circles. Formally it is known that if $\hat{n}$ is an operator with a purely discrete spectrum (which is always true for the number operator), then there can exist no operator $\hat{\theta}$ such that the commutator $[\hat{n},\hat{\theta}]=i\hat{1}$ holds. Different versions of phase-related operators have been constructed in order to overcome this difficulty [see, for example, the review by Carruthers and Nieto (1968) and the textbook on quantum optics by Mandel and Wolf (1995)]. Alternatives to conventional symmetry-breaking approaches have even been proposed [see the paper of Stenholm (2002), and references therein]. An intriguing suggestion that interference patterns of two atomic condensates can be explained without ever evoking the notion of phase was put forward by Javanainen and Yoo (1996).

In the present Colloquium we adopt the conventional and certainly more convenient approach, according to which the bosonic field operator takes on the form

$$\psi(\mathbf{r}) = \sqrt{n_0(\mathbf{r})}\exp[i\theta(\mathbf{r})] \quad (4.8)$$

for the large number of particles. Here $\theta(\mathbf{r})$ is the operator of the phase fluctuations and $n_0(\mathbf{r})$ is the condensate density at $T=0$.

To proceed with calculations it is convenient to expand the phase operator in terms of the creation and annihilation operators for Bogoliubov quasiparticles [see Shevchenko (1992)]

$$\hat{\theta}(\mathbf{r}) = \frac{1}{2\sqrt{n_0(\mathbf{r})}}\sum_k [(u_k+v_k)\hat{a}_k + (u_k-v_k)\hat{a}_k^\dagger], \quad (4.9)$$

where a_k is the annihilation operator for the Bogoliubov excitation with energy ϵ_k, and u_k, v_k are excitation functions, determined by a bosonic equivalent of the Bogoliubov–de Gennes equations [for a general reference, see the book by de Gennes (1966)]. Equation (4.9) can be obtained in the formalism of Bogoliubov transformation generalized to an inhomogeneous case.

Phase fluctuations in a quasi-two-dimensional system can be analyzed within the formalism of the one-particle density matrix [see the works by Petrov and co-workers (2000, 2001)],

$$\langle \psi^\dagger(\mathbf{r})\psi(0)\rangle = \sqrt{n_0(\mathbf{r})n_0(0)}\exp\{-\langle[\Delta\theta(\mathbf{r})]^2\rangle/2\}. \quad (4.10)$$

One should mention that the quasi-two-dimensionality of the system implies that the scattering of particles acquires a three-dimensional character, while the kinetic properties of the gas remain two dimensional.

It is clear from Eq. (4.9) that the estimation of the phase fluctuations $\langle\Delta\theta(\mathbf{r})^2\rangle$ requires a knowledge of the Bogoliubov quasiparticle spectrum in inhomogeneous systems [see papers of Stringari (1996) and Önberg *et al.* (1997) and references in papers by Petrov and co-workers (2000, 2001)]. This spectrum is discrete for $T\ll\mu$ and for $T\gg\mu$ one can use the local-density approximation. In the Thomas-Fermi regime for $T\gg\mu$ one obtains the following approximation:

$$\langle\Delta\theta(\mathbf{r})^2\rangle \sim T\ln(R/\lambda_{\mathrm{dB}}). \quad (4.11)$$

Note that Eq. (4.11) does not depend on a precise expression for the repulsive coupling constant.

From Eq. (4.11) one can estimate the characteristic radius R_θ of phase fluctuations (the characteristic length at which phase changes by 2π) to be $R_\theta \simeq \lambda_{\mathrm{dB}}\exp(T_\theta/T)$ with $k_B T_\theta = N(\hbar\omega_\perp)^2/\mu$. We thus arrive at the conclusion that at low temperatures $T\ll T_\theta$ the characteristic radius of phase fluctuations is larger than the size of the trap $R_\theta \gg R_\perp$, so a true condensate exists.

The emergence of a true condensate is attributed to the weakening of phase fluctuations induced by a trap, which introduces a low momenta cutoff into the excitations in the system. At higher temperatures $T \gg T_\theta$ the system is characterized as a quasicondensate ($R_\theta \ll R_\perp$).

The crucial effect of a trap for two-dimensional Bose gases was also emphasized by Ho and Ma (1999). They pointed out that long-wavelength quantum fluctuations will be partially suppressed due to the gapped spectrum of collective modes (Stringari, 1996) and off-diagonal order will survive in 2D.

Since there is an experimental evidence in support of BEC existence in 2D, the discussion is not yet closed. Quantum Monte Carlo simulations for bosons in a two-dimensional harmonic trap do indeed show that a significant fraction of particles is still present in the lowest state at low energies (Heinrichs and Mullin, 1998).

C. Popov's approach

In this section we consider how Popov (1983) generalized field-theoretical methods developed by Beliaev to finite temperatures. It is curious that the method, suggested by Popov in 1965, is conceptually similar to the renormalization-group approach successfully applied in the 1970s to phenomena unaccessible to perturbative methods, such as the Kondo effect (Hewson, 1993).

As usual one starts with the introduction of the temperature Green's function,

$$G(x,\tau;x',\tau') = -\langle \psi(x,\tau)\bar{\psi}(x',\tau')\rangle = -\frac{\int e^S \psi(x,\tau)\bar{\psi}(x',\tau')d\psi d\bar{\psi}}{\int e^S d\psi d\bar{\psi}}, \quad (4.12)$$

where S is the classical action of the Bose gas,

$$S = \int_0^\beta d\tau \int d^3x \bar{\psi}(x,\tau)\partial_\tau \psi(x,\tau) - \int_0^\beta d\tau H(\tau), \quad (4.13)$$

and

$$H(\tau) = \int d^3x \bar{\psi}(x,\tau)\left(-\frac{\nabla^2}{2m} - \mu\right)\psi(x,\tau) + \frac{1}{2}\int d^3x d^3y U(x-y) \times \bar{\psi}(x,\tau)\bar{\psi}(y,\tau)\psi(y,\tau)\psi(x,\tau). \quad (4.14)$$

The next step is construction of the perturbation theory and corresponding diagrams arising from integrals of the type (4.12), by performing the usual trick of separating out the condensate operators (3.9). However, in the case of the Bose system the perturbation series converges poorly for small momenta and frequencies. In other words, the infrared asymptote of the Green's function is singular. In order to avoid these difficulties, Popov suggested the following modifications: the bosonic field ψ,

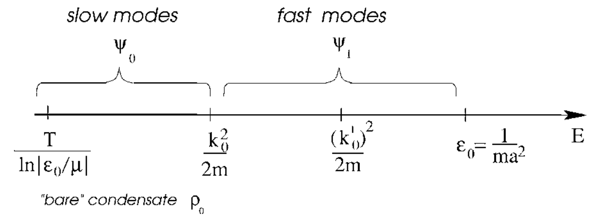

FIG. 5. Energy scales in Popov's approach: $T/\ln|\epsilon_0/\mu| \ll k_0^2/2m \sim T \ll (k_0')^2/2m \ll \epsilon_0 = 1/ma^2$.

$$\psi(x,\tau) = \sqrt{\frac{1}{\beta V}}\sum_{k,\omega} \exp[-i(kx-\omega\tau)]a(k,\omega), \quad (4.15)$$

is divided into a short-wavelength fast component ψ_1 and a long-wavelength slow component ψ_0 [$\psi(x,\tau) = \psi_0(x,\tau) + \psi_1(x,\tau)$] (see Fig. 5). The momentum k_0 which separates the slow modes from rapidly oscillating modes depends on the particular Bose system and only its order of magnitude can be estimated. The introduction of k_0 removes the divergences at small momenta, regularizing the perturbation theory.

A method of successive integration, first over rapid and then over slow fields, is then applied, using different schemes of perturbation theory at different stages of the integration [see Chap. 4 in Popov (1983)]. The fast modes see the slow modes as an effective condensate ("bare" condensate according to Popov) with a superfluid density $\rho_0 = |\psi_0|^2$. Appendix B gives a succinct derivation of main Popov's results.

This method of subsequent integration, developed by Popov, allows us to estimate the low-temperature asymptotic behavior of the one-particle Green's function, and to derive a power-law decay of $G(x,y) \sim |x-y|^{-\alpha}$ for $|x-y| \to \infty$ (in 1D and 2D) rather than the exponential decay that occurs at high temperatures. In 2D, as mentioned in Sec. IV.A, this signals the development of topological LRO at low temperatures.

The analysis is based on the t-matrix description of effective interactions, and the key property of the two-dimensional t matrix is that at low energies it vanishes, and at the high-energy cutoff the t matrix diverges (see Appendix B). This results in an extremely small critical temperature

$$T_c \simeq \frac{\mu \ln(\epsilon_0/\mu)}{4 \ln \ln(\epsilon_0/\mu)}, \quad (4.16)$$

where ϵ is a high-energy cutoff and μ is a chemical potential. Bear in mind that this is a mean-field derivation and the condition for the superfluid transition was assumed to be $\rho = \rho_n$, because Popov [as well as Berezinskii (1970, 1971)] thought that at the critical temperature T_c the superfluid density vanishes.

The applicability of Popov's mean-field description is based on the assumption of a very small exponent α. For large α the probability of creating quantum vortices becomes large and even this modified perturbation theory is invalid (see also the discussion in Sec. IV.D and the corrected many-body mean-field theory in Sec. IV.E).

The applicability of Popov's t-matrix description and the diluteness condition, derived by Fisher and Hohenberg, is the main subject of Sec. IV.D.

D. Diluteness condition and validity of t-matrix approximation

We have discussed that the perturbative treatment of the dilute weakly interacting Bose gas amounts to replacing the real potential by an effective two-particle t matrix, obtained by summing all ladder diagrams. From this point of view the diluteness condition determines the validity range of the t-matrix approximation.

An explicit form for the diluteness condition of two-dimensional interacting Bose gas at finite temperatures was first introduced by Fisher and Hohenberg (1988). They pointed out that singularities inherent to two-dimensional systems (vanishing of scattering t matrix at zero temperature and classical divergence of phase fluctuations) might lead to drastic modifications of the usual dilute gas expansion.

As discussed (see Sec. III.B), at zero temperature in 2D the diluteness condition $na^2 \ll 1$ is replaced by

$$\frac{1}{\ln(1/na^2)} \ll 1. \tag{4.17}$$

Popov's theory can be used to demonstrate that at finite temperature criterion (4.17) is replaced by an even more stringent inequality (Fisher and Hohenberg, 1988)

$$\ln \ln \frac{1}{na^2} \gg 1. \tag{4.18}$$

Fisher and Hohenberg provided a heuristic derivation of this result, based on the Bogoliubov quasiparticle picture. Their analysis is based on the simple observation that the usual Landau quasiparticle formula for the superfluid density,

$$\frac{\rho_s}{\rho} = 1 - \frac{\beta}{\rho d} \int \frac{d^d k}{(2\pi)^2} k^2 \frac{e^{\beta \xi_k}}{(e^{\beta \xi_k} - 1)^2}, \tag{4.19}$$

where d is the dimension, does not have any singularities for $d=2$, except in the case when the chemical potential is small. [μ is introduced in Eq. (4.19) via the Bogoliubov quasiparticle spectrum $\xi_k^2 = n_0 U_0 k^2/m + k^4/4m^2 \equiv \mu k^2/m + k^4/4m^2$. The validity of this approximation for μ is discussed in Beliaev (1958a, 1958b).] By introducing the infrared cutoff ($k_0 \sim \sqrt{\mu}$) via the ansatz

$$\mu \sim \frac{n}{|\ln(a^2 \mu)|}, \tag{4.20}$$

the regularization of the integral of Eq. (4.19) can be achieved, and one arrives at Popov's equations for superfluid and normal densities (B8) and (B9).

Analyzing the temperature dependence of the superfluid density allows one to separate out three characteristic regimes: (i) the low-temperature region, the physics of which is defined by phononic behavior of the quasiparticles, leading to a superfluid density which depends on temperature as $(1-\alpha T^3)$; (ii) a free-particle region,

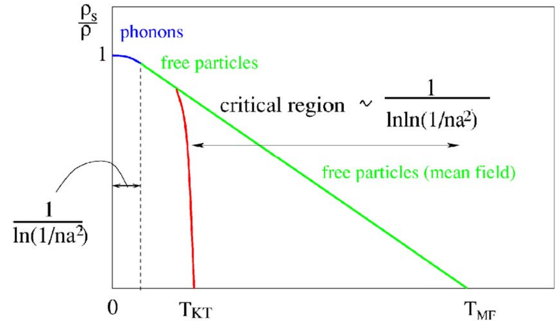

FIG. 6. (Color online) Schematic phase diagram of a uniform dilute Bose gas. ρ_s is the superfluid density, normalized by the mass density of the gas $\rho = mn$, T_{KT} is the critical temperature of the Kosterlitz-Thouless transition, T_{MF} is the mean-field temperature, calculated perturbatively. The size of the transition critical region is defined by a parameter $1/\ln\ln(1/na^2)$, where a is characteristic s-wave scattering length. The region, dominated by phononic quasiparticle behavior, is of the width $1/\ln(1/na^2)$.

where ρ_s behaves linearly with temperature; and (iii) a critical region, determined by the fluctuations around the critical temperature T_c (ρ_s vanishes at T_c) (see Fig. 6).

The diluteness criterion is determined by the condition that the critical region is small enough so that all three regimes can be well separated. The width of the first regime is in fact given by Schick's small parameter (4.17), while the size of the critical region is characterized by the double log (4.18). The problem, however, is that for all relevant situations, even for very small na^2, Fisher and Hohenberg's small parameter $1/\ln\ln(1/na^2)$ is still orders of magnitude greater than $1/\ln(1/na^2)$. This means that in practice the critical region associated with the Kosterlitz-Thouless transition is so large that mean-field based approaches do not give any reliable results. Note that the double log result was also reproduced by Fisher and Hohenberg in a more accurate way within a renormalization-group treatment of the same problem. They have also estimated the superfluid transition temperature, which reads

$$T_c \sim \frac{2\pi n}{m \ln \ln[1/(na_s)^2]}. \tag{4.21}$$

The results of Fisher and Hohenberg (1988) have been confirmed in other approaches, see, for example, the virial expansion of a dilute Bose gas by Ren (2004) or RG analysis by Kolomeisky and Straley (1992a, 1992b) and by Crisan *et al.* (2001). Pieri *et al.* (2001a, 2001b) demonstrated by analyzing the normal state with the standard diagram technique that the transition temperature (4.21) appears as a lower bound for the validity of the t matrix as a controlled approximation for the dilute Bose gas.

The Fisher-Hohenberg diluteness condition (4.18) is extremely stringent, and if straightforwardly applied to experimentally relevant situations (Görlitz *et al.*, 2001; Rychtarik *et al.*, 2004) would mean that systems observed to undergo a BEC phase transition in 2D are not actually dilute, and could never be so. This line of reasoning motivated Liu and Wen (2002) to come up with an exotic alternative scenario involving a two-dimensional strongly correlated spin liquid.

The extreme conclusions drawn from the diluteness criterion are, nevertheless, related to the general drawbacks of the Popov approximation. We show in the next section that in a more realistic model, which takes into account interactions in a self-consistent way, the diluteness condition becomes much weaker. Moreover, in view of the previous discussion about the inapplicability of arguments based on homogeneous systems in the thermodynamic limit to trapped gases, it would seem that the diluteness requirement is not relevant for the experimental situation of the Bose gas in a magnetic trap.

E. Other approaches: RPA, many-body *t* matrix, Monte Carlo

In this section we review a range of diagrammatic approaches that have built upon the early RPA and *t*-matrix approximation in order to improve the description of the superfluid of BKT transition and also the numerical methods, which allow us to directly probe the critical region of the two-dimensional transition.

The first finite-temperature generalization of Bogoliubov random-phase approximation (Sec. III.A) was introduced by Tserkovnikov (1964). He wanted to calculate the finite-temperature correction to the condensate density in three-dimensional dilute Bose gases with weak interactions. Tserkovnikov assumed that the average single-particle kinetic energy is small compared to the potential energy for temperatures below T_c^{3D}. He also remarked that this approximation does not meet the Landau superfluidity criterion and that more precise equations should be sought in future work.

The RPA method was further developed by Szepfalusy and Kondor (1974), whose main interest was investigating the dynamics of the second-order phase transition. Around the same time a large-N approach was applied to the Bose gas by Abe (1974) and Abe and Hikami (1974), who calculated the dynamical scaling for one-particle Green's function up to $O(1/N)$. Here the idea of the large-N approach is to expand the number of independent components of the Bose field from unity to $N/2$ using $1/N$ as an expansion parameter. To produce a controlled large-N limit, the interaction strength is scaled to be of order of $1/N$.

The RPA large-N method has been applied to two-dimensional Bose gas by Nogueira and Kleinert (2006). The interaction in their approach is approximated by a two-dimensional coupling constant, derived in the *t*-matrix approximation, considered by Popov and Schick (see Secs. III.B, IV.C, and IV.D). It is, however, known that in a large-N approach one cannot simultaneously account for both the particle-hole channel (RPA) and the particle-particle channel in a well-controlled fashion. Nevertheless, the authors claim that the diluteness condition leads first to the appearance of *t*-matrix diagrams, while the next class of diagrams are those from the particle-hole channel (Nogueira and Kleinert, 2006). This approximation results in the Bogoliubov quasiparticle dispersion containing a log correction due to low dimensionality

$$\xi_k = \sqrt{\epsilon_k^2 + 2g_{2D} n \epsilon_k \left[1 - \frac{Tm}{\pi n}\ln(ka)\right]},$$

so that excitations in the system exhibit a rotonlike minimum. Note that the excitation spectrum is calculated assuming that one-particle Green's function and density correlators share the same poles [this property was derived by Hohenberg and Martin (1965) in the case of a 3D condensed Bose system]. It would be interesting to check if these RPA results are confirmed in other approaches.

We now proceed to discuss the various generalizations of the two-body *t*-matrix approach. Though simple and elegant, the perturbative two-body *t*-matrix approach does have its drawbacks. The main problem is related to its inability to properly describe the critical region in low dimensions. For example, the *t*-matrix method does not predict the Nelson-Kosterlitz universal jump in the superfluid density. In 3D the two-body *t*-matrix approach leads to a first-order phase transition for the condensate density, which is the consequence of non-self-consistency of this first-order perturbative approximation [see also, Lee and Yang (1958), Reatto and Straley (1969), and Griffin (1988)].

Many of these problems can be solved if many-body corrections, arising due to the surrounding gaseous medium, are taken into account. This is the key idea in the many-body *t*-matrix approximation [see the comprehensive review by Shi and Griffin (1998), low-dimensional systems within the many-body *t*-matrix approach are analyzed in the papers by Stoof and Bijlsma (1993), Al Khawaja *et al.* (2002), Andersen *et al.* (2002), for a Hartree-Fock-Bogoliubov study of a two-dimensional gas see recent works by Gies and Hutchinson (2004) and Gies *et al.* (2004, 2005)].

Since many-body *t*-matrix methods are extensively discussed in the literature, here we restrict ourselves to a brief description providing all relevant references. The Bogoliubov-Hartree-Fock (BHF) approximation [see Griffin (1996) and the analysis of excitations in a trapped three-dimensional gas paper by Hutchinson *et al.* (1997)] has a Heisenberg equation of motion for a Bose field operator of the kind (3.24) as a starting point,

$$i\hbar \frac{\partial}{\partial t}\psi(r,t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) - \mu\right)\psi(r,t) + g\psi^\dagger(r,t)\psi(r,t)\psi(r,t). \tag{4.22}$$

A short-range interaction is assumed among atoms $U(\mathbf{r}-\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}')$. Treating the interaction term in Eq. (4.22) in the self-consistent mean-field approximation, one arrives at

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) - \mu\right)\phi(r)$$

$$+ g[n_0(r) + 2n'(r)]\phi(r) + gm'(r)\phi^*(r) = 0, \qquad (4.23)$$

where n_0 is the condensate density, $n'(r)=\langle\psi'^{\dagger}(r)\psi'(r)\rangle$, and $m'(r)=\langle\psi'(r)\psi'(r)\rangle$ (anomalous average). In order to describe excitations in the system one should also write down the equation of motion for ψ',

$$i\hbar\frac{\partial}{\partial t}\psi'(r) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) - \mu\right)\psi'(r,t)$$

$$+ 2gn(r)\psi'(r,t) + gm(r)\psi'^{\dagger}(r,t), \qquad (4.24)$$

with $n(r)=n_0+n'$ and $m(r)=\phi^2+m'$. Equations (4.23) and (4.24) correspond to the Bogoliubov-Hartree-Fock approximation: a bosonic analog of the finite-temperature Bogoliubov–de Gennes equations. These equations can also be reexpressed in terms of a Green's function formalism. Note that the appearance of anomalous averages in BHF formalism (which are not present in the Popov approach) leads to a gap in the quasiparticle excitation spectrum.

Many-body effects are also effectively treated with the variational method applied to dilute Bose gases by Bijlsma and Stoof (1997), and many-body t-matrix methods which have been improved in recent years [see Al Khawaja *et al.* (2002) and Andersen *et al.* (2002)]. A time-dependent BHF approximation has been developed by Proukakis *et al.* (1998). Here, Proukakis *et al.* (1998) claim that the pseudopotential approximation $U(\mathbf{r}-\mathbf{r}')=g\delta(\mathbf{r}-\mathbf{r}')$ should be imposed only after the effective interaction is expressed in terms of many-body t matrix. Both approaches, the one used by Griffin *et al.* and another developed by Stoof and collaborators, are qualitatively similar in that they treat interactions in many-body t-matrix approach, but they differ in some details, for instance, in selecting out the important diagrams.

Let us now discuss some of the results of the many-body t-matrix method for two-dimensional systems. More than a decade ago Stoof and Bijlsma (1993) demonstrated that the infrared divergences, appearing in the two-body t-matrix treatment, can be elegantly eliminated when surrounding gas effects are taken into account. With this approach the universal jump in superfluid density, predicted by Nelson and Kosterlitz, can be reproduced. A weaker diluteness condition, namely, that of Schick, Eq. (4.17), defines the applicability of this many-body t-matrix approximation, one that is satisfied experimentally in systems, such as spin polarized atomic hydrogen, absorbed on the ^{4}He surface (Stoof and Bijlsma, 1993). The application of many-body t-matrix to the scattering problem in 2D and density profiles of a two-dimensional Bose gas is also discussed by Rajagopal *et al.* (2004).

The conclusions of Petrov and co-workers (2000, 2001) have been confirmed in recent investigations by Andersen *et al.* (2002), Gies and Hutchinson (2004), and Gies *et al.* (2004, 2005). The theory of Andersen *et al.* (2002), free of infrared divergences in all dimensions, allows one to calculate the density profile of a (quasi-)condensate cloud of a gas for any aspect ratio of the trap (within local-density and Thomas-Fermi approximation). At very low temperatures, depending on the trapping geometry, the presence of a true condensate in the equilibrium state is found. Hutchinson and coworkers (Gies and Hutchinson, 2004; Gies *et al.*, 2004, 2005) also see within their HFB approach a macroscopic occupation of the ground state at low temperatures, implying the presence of a condensate state.

To conclude, the presence of the trap appears to stabilize the condensate against long-wavelength fluctuations and the BEC state can form at finite though very low temperatures, when the discrete nature of the energy spectrum is taken into account.

The most reliable description of the two-dimensional Bose gas to date is provided by Monte Carlo simulations (Kagan *et al.*, 2000; Prokof'ev *et al.*, 2001; Prokof'ev and Svistunov, 2002), because it allows one to study the critical region of the BKT transition, which is effectively very large and therefore unaccessible to perturbative methods. The numerical analysis is simplified by the fact that the critical properties of all XY models are the same (see Sec. IV.B). It suffices therefore to study the classical $|\Psi|^4$ model on the lattice within a Monte Carlo algorithm.

Consider, for instance, the temperature dependence of the particle density in the critical region of the BKT transition, which follows from perturbative analysis (Popov, 1983; Kagan *et al.*, 1987; Fisher and Hohenberg, 1988) of a weakly interacting system,

$$n = \frac{mT}{2\pi}\ln\frac{C}{mU_{\rm eff}}, \qquad (4.25)$$

where $U_{\rm eff}$ is an effective interaction, proportional to $f_0^{\rm 2D}$ (3.19), and C is a constant, which is not possible to evaluate within perturbative expansion in powers of $U_{\rm eff}$ (Prokof'ev *et al.*, 2001). Monte Carlo estimation gives $C=380\pm3$; this large value of C makes it virtually impossible to reach the limit of small $U_{\rm eff}$ for weakly interacting systems.

At the transition we obtain an accurate microscopic expression for the critical temperature of BKT transition (Prokof'ev *et al.*, 2001; Prokof'ev and Svistunov, 2002),

$$\frac{T_{\rm KT}}{n} = \frac{2\pi}{m\ln(C/mU_{\rm eff})}. \qquad (4.26)$$

It is interesting to compare this density n to the quasicondensate density n_q and superfluid density n_s in the critical region. It turns out that n_q/n is of order of unity, unless $mU_{\rm eff}$ is exponentially small, while the ratio n_q/n_s is of order of 2, which means that superfluid density is substantially smaller than quasicondensate density at the transition.

The temperature behavior of various densities, obtained with the Monte Carlo procedure, can be used for checking whether RG and perturbative approaches essentially overlap. Indeed, Monte Carlo simulations have been able to capture the crossover between the mean-field behavior and the critical fluctuation region described by the KT transition (Prokof'ev and Svistunov, 2002). Prokof'ev and Svistunov (2002) show that this crossover is characterized by a universal ratio of the superfluid and quasicondensate density. One can also see that the conventional mean-field result $n_s/n \sim 1-T/T_{\mathrm{KT}}$ is not valid anywhere, while the modified mean-field theory introduced by Prokof'ev and Svistunov (2002) can predict accurately the behavior of the quasicondensate density up to T_{KT}.

F. Breathing modes of two-dimensional systems

At the end of this section we consider the universal property of a two-dimensional gas with a *contact* interaction, confined in a harmonic potential. Pitaevskii and Rosch (1997) predicted that such a system develops oscillations or breathing modes, which can be probed experimentally or in simulations and thus can serve as a practical criterion of the two-dimensional nature of a system.

The appearance of breathing modes is related to a hidden Lorentz symmetry inherent in any two-dimensional Hamiltonian of the following general form:

$$H = H_0 + H_{\mathrm{ext}}, \tag{4.27}$$

where

$$H_0 = \sum_i \left(-\frac{1}{2m}\triangle_i\right) + \sum_{i<j} U(\mathbf{r}_i - \mathbf{r}_j), \tag{4.28}$$

and $H_{\mathrm{ext}}=\Sigma_i \frac{1}{2} m\omega_0^2 r_i^2$ is a harmonic potential.

It is readily seen that H_0 is scale invariant in the case of a local two-dimensional interaction,

$$U(\mathbf{r}_i - \mathbf{r}_j) = \frac{g}{2}\delta^2(\mathbf{r}_i - \mathbf{r}_j) \tag{4.29}$$

[in fact it is scale invariant for any potential with the property $U(l\mathbf{r}) = U(\mathbf{r})/l^2$]. The presence of a trap breaks the scale invariance of H_0. Note that in principle the scaling invariance of the Hamiltonian H_0 is broken in 2D, because then the scattering phase shift is energy dependent due to the logarithmic dependences characteristic of two dimensions [the phase shift is proportional to the coupling constant $g^{2\mathrm{D}}$ or to $1/\ln(ka)$]. The energy-dependent phase shift signals the breaking of scale invariance at the quantum level (Cabo *et al.*, 1998). But this symmetry breaking is explicit and is not attributed to any phase-transition physics.

In spite of scale invariance breaking, because of a special property of the harmonic oscillator, a powerful spectrum generating symmetry still exists. That can be seen from the commutator $[H_{\mathrm{ext}}, H] = i\omega_0^2 Q$, where $Q = \frac{1}{2}\Sigma_i(\mathbf{p}_i\mathbf{r}_i + \mathbf{r}_i\mathbf{p}_i)$ is the generator of scale transformations. One can check that $[Q, H_0] = 2iH_0$ and $[Q, H_{\mathrm{ext}}] = -2iH_{\mathrm{ext}}$. These results can be formulated within the well-known algebra of SU(1,1) or SO(2,1) symmetry groups, i.e., the two-dimensional Lorentz group.

Starting from the lowest energy state ϵ_0 one can produce higher-order states with energies $\epsilon_0 + 2n\omega_0$ $(n=1,2,\ldots)$ by applying one of the SO(2,1) group generators $L^+ = (L_1 + iL_2)/\sqrt{2}$ where $L_1 = (H_0 - H_{\mathrm{pot}})/2\omega_0$, $L_2 = Q/2$ [the corresponding annihilation operator is $L^- = (L_1 - iL_2)/\sqrt{2}$]. Excitations with energies $2n\omega_0$ are associated with the breathing, or pulsating, modes of the system.

As an example, Pitaevskii and Rosch (1997) considered the classical Gross-Pitaevskii equation and predicted the existence of undamped breathing modes in the condensate. The appearance of transverse breathing modes with a frequency equal to an integer multiple of the trap oscillation frequency was observed experimentally in an elongated condensate of ^{87}Rb atoms (Chevy *et al.*, 2002). Numerical simulations (exact diagonalization) seem to indicate the existence of dipole or breathing modes in a two-dimensional system even for a relatively small number of atoms (Haugset and Haugerud, 1998).

The Bogoliubov-Hartree-Fock study of a two-dimensional Bose gas by Gies and co-workers (Gies and Hutchinson, 2004; Gies *et al.*, 2004, 2005) has shown that at low temperatures the frequency of the lowest-lying excitation ($n=0$ mode) is precisely $2\omega_0$, independent of the interaction strength. At high temperatures the frequency of this mode shifts to a lower frequency region, modified by the addition of a potential from the static thermal cloud.

Nevertheless, it is important to note that a δ function is not well defined in two dimensions due to logarithmic UV divergences (Pitaevskii and Rosch, 1997) that are cut off by the finite range of interaction; whether it is a small or large effect should be investigated.

V. CONCLUSIONS AND OPEN QUESTIONS

We have surveyed a number of theoretical issues arising in the field of a weakly interacting uniform or confined in a trap dilute Bose system at low temperatures in 2D. The underlying physics of such a system depends on the size of the system, the degree of its inhomogeneity, and the temperature.

If the system is very large and uniform, one might expect realization of the BKT transition, characterized by the presence of a topological order below the critical temperature T_{KT} down to zero temperature when the true long-range order (BEC) forms. Perturbative approaches, based on a low-density approximation and pointlike or short-range interactions, surveyed in this Colloquium are not really suited to describing vortex excitations in the ordered phase of a BKT transition. However, these methods provide a good description of many physical properties. For example, a modified version of the mean-field theory, the many-body t-matrix approach, is able to capture the second-order nature of the phase transition in 3D and the Nelson-Kosterlitz universal jump of superfluidity in 2D.

In experiments the low-temperature regime of a quantum gas is achieved by confining the atomic system in an external potential. The proper description of a practically realized system requires therefore inclusion of the inhomogeneities, introduced by a trap. In 3D the effect of the trap is not very pronounced, and it is possible to calculate the correction to the critical temperature due to interactions perturbatively (Arnold and Tomasik, 2001), while in the uniform case the RG study is required (Ledowski *et al.*, 2004) [for details, see Pitaevskii and Stringari (2003)].

In 2D, as seen in the case of a gas without interactions, the presence of the trap dramatically modifies properties of the system (density of states) so that BEC becomes possible at finite temperature. The inclusion of interactions into the picture is a complicated task. First of all, the system is inhomogeneous and previously developed perturbative methods, such as the t-matrix approach, are strictly speaking not suitable for its analysis. In principle, one should solve the many-body scattering problem in a trap, taking into account the discrete spectrum and the finite range of a potential, which is extremely difficult. One can of course consider a simplified problem of a quasihomogeneous trap and adapt well-studied techniques for that case. As we have seen, the main effects of the trap are captured at least qualitatively within such a scheme.

Intuitively, it is clear that inhomogeneities would tend to suppress the universal jump of superfluidity, and would rather favor the true BEC state at low temperatures in a system with a finite number of particles (because long-range thermal fluctuations will be quenched by a trap).

Recently Holzmann *et al.* (2005) analyzed the behavior of a weakly interacting trapped system in the thermodynamic limit within the local-density approximation. They have shown that although the universal Nelson-Kosterlitz jump is indeed not present, the system does undergo a BKT transition at a temperature somewhat lower than T_c^{2D} (2.12).

Simula *et al.* (2005) predicted that both a BE condensed phase and a KT superfluid phase, separated by a first-order transition, will be present in a two-dimensional trapped gas. These authors arrived at their conclusions by comparing the Helmholtz free energy of the ground state, characterized by a condensate wave function and an excited state, containing a vortex pair in the ordering field. Based on entropy considerations, they find a critical temperature T_c above which a thermally activated transition to the state containing vortex-pair excitations becomes favorable. These conclusions require, however, solid confirmations from both theory and experiment.

A challenging problem is, however, the actual estimation of the BEC transition temperature in a two-dimensional interacting system and the relationship of the quasicondensate state to the KT vortex-pair plasma state, depending on the geometry of the system and number of particles. It is also important to develop numerical methods adapted for experiment. One of the promising developments in this direction has been made by Davis and co-workers (Davis, Ballagh, and Burnett, 2001; Davis, Morgan, and Burnett, 2001, 2002), who have proposed a projected Gross-Pitaevskii equation formalism, which allows investigation of finite-temperature properties of the equilibrium condensate state—even in the region where the Bogoliubov theory fails. An extension of this method to harmonically trapped condensates has been considered by Blakie and Davis (2005). Simula and Blakie (2006) have analyzed the two-dimensional Bose gas by classical field methods, adopted from quantum optics. They have demonstrated that two distinct superfluid phases, separated by thermal vortex-antivortex pair creation, exist in an experimentally producible quasi-two-dimensional Bose gas. Simula and Blakie (2006) provided strong evidence that a strange ("zipper") interference pattern observed in a recent experiment by Stock *et al.* (2005) can be explained by the presence of a vortex excitation in an experimental system.

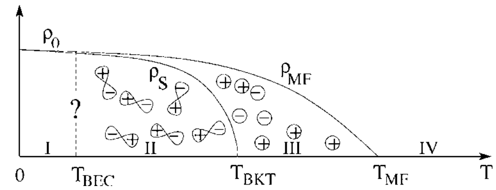

FIG. 7. Schematic phase diagram of a two-dimensional trapped weakly interacting dilute Bose gas. ρ_0 stands for the density of the condensate, ρ_s is the superfluid density, $\rho_{\rm MF}$ is the mean-field density, which can be estimated perturbatively. $T_{\rm BEC}$ is the crossover temperature from the superfluid regime to the true Bose-Einstein condensation phase. $T_{\rm BKT}$ is the critical temperature of the Kosterlitz-Thouless transition. $T_{\rm MF}$ marks the critical region of BKT transition.

One can thus tentatively identify the following phases in a trapped two-dimensional Bose gas (Fig. 7):

- phase I: low-temperature true BEC phase;
- phase II: KT vortex-antivortex pair superfluid, or quasicondensate, or condensate with a fluctuating phase, at the transition $T_{\rm KT}$ superfluid universal jump is almost suppressed;
- phase III: critical region of the BKT transition; vortex pair dissolve, above $T_{\rm KT}$ vortices are unbound and free;
- phase IV: above the mean-field temperature it is a normal fluid, no local order parameter or vortices exist.

The phase diagram of a real system will depend on many factors, as was stressed at the beginning of this section. Many aspects of the diagram, depicted in Fig. 7, require careful investigation, and reliable confirmation from both theory and experiment.

To end we attempt to summarize some of the open questions:

- The nature of superfluid phases in a two-dimensional weakly interacting Bose gas: What is the nature of the crossover to a superfluid phase? What is the explicit relation between superfluidity and the quasi-condensate state? Is there a crossover to the BEC state at low temperatures? If yes, under which conditions?

- Can we help experimentalists to see the vortex excitations in the superfluid state, to really identify the KT state? There is a clear need in good vortex detection methods. Can, for example, disorder help us to pin the vortices?
- What are, in general, measurable physical properties, which delineate between the coherent condensed state and superfluidity? Initial progress in this direction has been already made; Polkovnikov *et al.* (2006) have suggested how to identify the KT transition from experimentally measured interference pattern.
- Can we solve the many-body scattering problem in the trap? Does diluteness of the gas simplify this problem?
- Is local density a good approximation for describing the experimental systems? If yes, under which conditions?
- Could one justify a large-N approach which improves on existing methods by incorporating the t-matrix approximation?

We have also discussed the diluteness condition derived by Fisher and Hohenberg (1988) under certain conditions of the transition to the superfluid state. In a finite-sized experimental system this condition is not really applicable, and one should use the criterion of Schick, Eq. (4.17), which can be seen from the analysis of quantum fluctuations of the two-dimensional BEC at zero temperature (Petrov *et al.*, 2004).

We have considered only low-density approximations. When the gas is dense, approaches such as that of Gross and Pitaevskii are not applicable. In such high-density regimes, new strong coupling approaches are required. One of the possible solutions may be the slave-boson approximation, which is valid for hard-core bosons at any density (Ziegler and Shukla, 1997; Rajagopal, 2005).

We have not discussed in this Colloquium the role of disorder in the continuum Bose system, though it could be a subject of a separate review and opens up a lot of interesting perspectives. Recent Monte Carlo studies predict, for example, that for strong disorder the system enters an unusual regime, where the superfluid fraction is smaller than the condensate fraction (Astrakharchik *et al.*, 2002). Weak disorder can be treated within Bogoliubov theory (Huang and Meng, 1992; Giorgini *et al.*, 1994) and the striking result of this is that disorder is more active in reducing superfluidity than in depleting the condensate. These results suggest that the relation of superfluidity and Bose-Einstein condensation requires further theoretical and experimental investigation.

One cannot help but mention the increasing interest to cold gases with dipole-dipole interactions, which are responsible for a variety of phenomena in ultracold dipolar systems; see, for instance, Santos *et al.* (2000, 2003), Pedri and Santos (2005), Stuhler *et al.* (2005), Fischer (2006), and references therein.

Finally, the problem of measurable quantities is in fact one of the most important in the context of trapped Bose gases. Unlike an electron system, one cannot attach leads to the system and measure transport properties of a condensate cloud. One of the most pressing practical needs for theorists and experimentalists is therefore the development of controllable new methods to probe the trapped condensate.

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APPENDIX A

1. Sketch of the derivation of Mermin-Wagner-Hohenberg theorem

One should use the Bogoliubov inequality

$$\tfrac{1}{2}\langle\{A,A^\dagger\}\rangle\langle[[C,H],C^\dagger]\rangle \geq T|\langle[C,A]\rangle|^2, \tag{A1}$$

where the average $\langle X\rangle = \mathrm{Tr}(Xe^{-\beta H})/\mathrm{Tr}\, e^{-\beta H}$, and operators A and C are such that the ensemble averages on the left-hand side of (A1) exist. The inequality (A1) follows quite straightforwardly from the Schwartz inequality

$$(A,A)(B,B) \geq |(A,B)|^2, \tag{A2}$$

where a scalar product is defined by $(A,B) = T\int(d\omega/\pi)\times(1/\omega)\chi_{AB}(k\omega)$, where χ_{AB} is the Fourier transform of the response function $\chi_{AB}(rt,r't') = \langle(1/2\hbar)[A(rt), B(r't')]\rangle$ [see, for example, the textbook by Forster (1990)].

In the case of a Bose system the derivation of Hohenberg (1967) is based on Bogoliubov and Schwartz inequalities and the f sum rule

$$T\int\frac{d\omega}{\pi}\frac{1}{\omega}\chi_{AA^\dagger}(-k\omega) = \frac{k^2 n}{m}. \tag{A3}$$

2. Derivation by Fischer of geometrical analog of the Hohenberg inequality

The bosonic field operator is as usual decomposed into condensate and noncondensate parts,

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})a_0 + \delta\psi(\mathbf{r}). \tag{A4}$$

The key observation of Fischer (2002) is that the Bogoliubov prescription should be applied after implementing the commutation relation,

$$[\delta\psi(\mathbf{r}), \delta\psi^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}') - \psi_0(\mathbf{r})\psi_0^*(\mathbf{r}'), \tag{A5}$$

otherwise, the second term on the right-hand side of Eq. (A5), which turns out to be crucial for calculating the condensate fraction correctly, would vanish.

The operators A and C in the Bogoliubov inequality (A1) are chosen to be smeared excitation and total density operators,

$$\hat{A} = \int d^d r f_A(\mathbf{r})\delta\psi(\mathbf{r}), \quad \hat{C} = \int d^d r f_C(\mathbf{r})\delta\rho(\mathbf{r}), \tag{A6}$$

where f_A and f_C are carefully chosen smearing functions $[f_C(\mathbf{r}) \sim \psi_0^*(\mathbf{r}), f_A(\mathbf{r}) \sim e^{i\mathbf{kr}}]$. Next, the f-sum rule analogous to Eq. (A3), can be derived in coordinate space.

APPENDIX B

1. Popov's approach

To derive the phase-transition curve for a two-dimensional interacting Bose gas, one needs to explore the finite-temperature behavior of the t matrix (3.14). The Bethe-Salpeter equation (Fig. 3) in the Matsubara representation reads

$$\begin{aligned}\Gamma(p_1,p_2;p_3,p_4) = U_{\mathbf{k}_1-\mathbf{k}_3} - \frac{1}{\beta V}\sum_{q,i\omega_l} U_{\mathbf{q}} G^0(\mathbf{k}_1 - \mathbf{q}, i\omega_1 - i\omega_l)\\ \times G^0(\mathbf{k}_2 + \mathbf{q}, i\omega_2 + i\omega_l)\\ \times \Gamma(p_1 - q, p_2 + q; p_3, p_4),\end{aligned} \tag{B1}$$

where $\omega_j = 2\pi jT$ is an even Matsubara frequency, and the four-dimensional vector $p_j \equiv (\mathbf{k}_j, \omega_j)$ represents the momentum $\mathbf{k}_j$ and frequency ω_j of the particle before scattering ($j=1,2$) and after ($j=3,4$). Energy and momentum conservation requires $p_1+p_2=p_3+p_4$.

The main contribution to the sum over internal momenta in Eq. (B1) comes from $k \sim a^{-1}$ which is due to the potential discussed above. Since $a^{-1} \gg \sqrt{T} \gg \sqrt{\mu}$, the μ dependence in the Green's function can be safely neglected (Popov, 1983). Consequently, after integrating over frequencies Eq. (B1) is reduced to a t-matrix equation,

$$\begin{aligned}t(p_1,p_3,z) + \int \frac{dp'}{(2\pi)^2} U(p_1 - p')\frac{1}{p'^2/m - z} t(p',p_3,z)\\ = U(p_1 - p_3).\end{aligned} \tag{B2}$$

Schematically, the t-matrix equation can be expressed as $t_z + UR_z t_z = U$, with $R_z = 1/(p^2/m - z)$. The operator $1 + UR_z$ can be inverted and we get $U = t_z(1 - R_z t_z)^{-1}$. In this way the interaction is eliminated from the t-matrix equation and we arrive at the Hilbert identity $t_z - t_{z_0} = t_z(R_{z0} - R_z)t_{z_0}$. This last equation is readily integrable, since at low energies $(p_1, p_2 \ll 1/a, |z| \ll 1/ma^2)$ we can neglect the momentum dependence of the t matrix. The energy z_0 defines an arbitrary high-energy cutoff of the order $1/ma^2$, so that $t(z_0) \gg t(z)$, and we obtain the long-wavelength asymptotic of the t matrix in 2D,

$$t \simeq \frac{4\pi}{m \ln[\epsilon_0/(-z)]}. \tag{B3}$$

We see that in 2D the t matrix vanishes in the limit $p_1, p_2, z \to 0$ and in fact diverges at the high-energy cutoff $\epsilon_0 = |z_0|$.

Next we need to integrate out the high-energy modes with momenta $k > k_0'$ in our functional $\exp S$, (4.13). The cutoff k_0' is defined as

$$\max(|\mu|, T) \ll \frac{(k_0')^2}{2m} \ll \epsilon_0. \tag{B4}$$

The result of this integration is the reduced action

$$\begin{aligned}S' = \sum_{\omega,k<k_0'}\left(i\omega - \frac{k^2}{2m} + \mu\right)a^\dagger(p)a(p)\\ - \frac{1}{2\beta V}\sum_{p_1+p_2=p_3+p_4} t' a^\dagger(p_1)a^\dagger(p_2)a(p_3)a(p_4),\end{aligned} \tag{B5}$$

where all summations are cut off at $k = k_0'$ and the potential is replaced by a t matrix with

$$t' = t'(\omega) = \frac{4\pi}{m \ln(\epsilon_0/[k_0'^2/m - i\omega])}. \tag{B6}$$

Now the functional $\exp(S')$ is to be integrated over the variables $a^\dagger(p), a(p)$ within the momentum shell $k_0 < k < k_0'$, where k_0 is defined from the inequality $k_0^2/m \gg T/\ln|\epsilon_0/\mu|$ and serves to distinguish between slow and rapid particles. Variables with momenta smaller than k_0 are taken into account in the action by the transformation

$$a^\dagger(p), a(p) \to [\rho_0(k_0)\beta V]^{1/2}\delta_{p0}, \tag{B7}$$

where $\rho_0(k_0)$ is the density of slow particles and one can introduce the density of fast particles $\rho_1(k_0)$.

After the transformation (B7) one can make use of standard perturbation theory formalism and derive expressions for the densities ρ_0 and ρ_1. Their sum gives the total density $\rho = \rho_0 + \rho_1 = \rho_n + \rho_s$, which is independent of auxiliary momenta k_0 and k_0',

$$\rho = \frac{m\mu}{4\pi}(\ln \epsilon_0/\mu - 1) - \frac{1}{(2\pi)^2}\int d^2k \frac{k^2}{2m\epsilon(k)} n_B(k), \tag{B8}$$

where $n_B(k) = (e^{\beta\epsilon_k} - 1)^{-1}$, and the formulas for the normal and the superfluid component densities read

$$\rho_n = \frac{\beta}{(2\pi)^2}\int d^2k \frac{k^2}{2m} n_B(k)[1 + n_B(k)],$$

$$\rho_s = \frac{m\mu}{4\pi}(\ln \epsilon_0/\mu - 1) - \frac{1}{(2\pi)^2}\int \frac{d^2k}{2m}k^2 n_B(k) \times\left(\frac{1}{\epsilon_k} - \beta[n_B(k)+1]\right). \quad (B9)$$

Equations (B8) and (B9) define the thermodynamics of the system below the phase transition. Above the phase transition one can use the standard perturbation theory for calculating the Green's functions and thermodynamical quantities. The critical temperature is now derived from the condition $\rho = \rho_n$ at the transition.

REFERENCES

Abe, R., 1974, Prog. Theor. Phys. **52**, 1135.
Abe, R., and S. Hikami, 1974, Prog. Theor. Phys. **52**, 1463.
Al Khawaja, U., J. O. Andersen, N. P. Proukakis, and H. T. C. Stoof, 2002, Phys. Rev. A **66**, 013615.
Allen, J. F., and A. D. Misener, 1938, Nature (London) **141**, 75.
Andersen, J. O., U. Al Khawaja, and H. T. C. Stoof, 2002, Phys. Rev. Lett. **88**, 070407.
Andersen, J. O., and H. Haugerud, 2002, Phys. Rev. A **65**, 033615.
Angilella, G. G. N., S. Bartalini, F. S. Cataliotti, I. Herrera, N. H. March, and R. Pucci, 2006, in *Trends in Boson Research*, edited by A. V. Ling (Nova Science, New York).
Angilella, G. G. N., N. H. March, and R. Pucci, 2004, Phys. Rev. A **69**, 055601.
Arnold, P., and B. Tomasik, 2001, Phys. Rev. A **64**, 053609.
Astrakharchik, G. E., J. Boronat, J. Casulleras, and S. Giorgini, 2002, Phys. Rev. A **66**, 023603.
Bagnato, V., and D. Kleppner, 1991, Phys. Rev. A **44**, 7439.
Bayindir, M., and B. Tanatar, 1998, Phys. Rev. A **58**, 3134.
Beliaev, S. T., 1958a, Sov. Phys. JETP **7**, 289.
Beliaev, S. T., 1958b, Sov. Phys. JETP **7**, 299.
Berezinskii, V. L., 1970, Sov. Phys. JETP **7**, 493.
Berezinskii, V. L., 1971, Sov. Phys. JETP **34**, 610.
Bijlsma, M., and H. T. C. Stoof, 1997, Phys. Rev. A **55**, 498.
Bishop, D. J., and J. D. Reppy, 1978, Phys. Rev. Lett. **40**, 1727.
Blakie, P. B., and M. J. Davis, 2005, Phys. Rev. A **72**, 063608.
Bloch, I., 2004, Phys. World **17** (4), 25.
Bogoliubov, N., 1947, J. Phys. (Moscow) **11**, 23.
Bogoliubov, N. N., 1961, *Quasi-expectation Values in Problems of Statistical Machanics* (Gordon and Breach, New York).
Bogoliubov, N. N., 1991, *Selected Works, Part II: Quantum and Statistical Mechanics* (Gordon and Breach, New York).
Cabo, A., J. L. Lucio, and H. Mercado, 1998, Am. J. Phys. **66**, 240.
Carruthers, P., and M. M. Nieto, 1968, Rev. Mod. Phys. **40**, 411.
Castin, Y., 2001, in *Coherent Atomic Matter Waves*, edited by R. Kaiser C. Westbrook, and F. David, Lecture Notes of Les Houches Summer School (EDP Sciences/Springer-Verlag, Berlin), p. 1.
Chaikin, P. M., and T. C. Lubensky, 1995, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, UK).
Cherny, A. Yu., and J. Brand, 2004, Phys. Rev. A **70**, 043622.
Cherny, A. Yu., and A. A. Shanenko, 2001, Phys. Rev. E **64**, 027105.
Chevy, F., V. Bretin, P. Rosenbusch, K. W. Madison, and J. Dalibard, 2002, Phys. Rev. Lett. **88**, 250402.
Coleman, S., 1973, Commun. Math. Phys. **31**, 259.
Cornell, E. A., and C. E. Wieman, 2002, Int. J. Mod. Phys. B **16**, 4503.
Crisan, M., I. Tifrea, D. Bodea, and I. Grosu, 2001, Mod. Phys. Lett. B **15**, 837.
Dalfovo, F., S. Giorgini, L. P. Pitaevskii, and S. Stringari, 1999, Rev. Mod. Phys. **71**, 463.
Davis, M. J., R. J. Ballagh, and K. Burnett, 2001, J. Phys. B **34**, 4487.
Davis, M. J., S. A. Morgan, and K. Burnett, 2001, Phys. Rev. Lett. **87**, 160402.
Davis, M. J., S. A. Morgan, and K. Burnett, 2002, Phys. Rev. A **66**, 053618.
de Gennes, P. R., 1966, *Superconductivity of Metals and Alloys* (Benjamin, New York).
Einstein, A., 1924, Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl. **22**, 261.
Einstein, A., 1925, Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl. **23**, 3.
Fadeev, L. D., 1960, Zh. Eksp. Teor. Fiz. **39**, 1459 [Sov. Phys. JETP **12**, 1014 (1960)].
Fedichev, P. O., and U. R. Fischer, 2003, Phys. Rev. Lett. **91**, 240407.
Fetter, A. L., 2002, J. Low Temp. Phys. **129**, 263.
Fetter, Alexandr L., 1999, Int. J. Mod. Phys. **13**, 643.
Fischer, U. R., 2002, Phys. Rev. Lett. **89**, 280402.
Fischer, U. R., 2005, J. Low Temp. Phys. **138**, 723.
Fischer, U. R., 2006, Phys. Rev. A **73**, 031602(R).
Fisher, D. S., and P. C. Hohenberg, 1988, Phys. Rev. B **37**, 4936.
Forster, D., 1990, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Perseus Books, Reading, MA).
Galitskii, V. M., 1958, Sov. Phys. JETP **7**, 104.
Gies, C., and D. A. W. Hutchinson, 2004, Phys. Rev. A **70**, 043606.
Gies, C., M. D. Lee, and D. A. W. Hutchinson, 2005, J. Phys. B **38**, 1797.
Gies, C., B. P. van Zyl, S. A. Morgan, and D. A. W. Hutchinson, 2004, Phys. Rev. A **69**, 023616.
Giorgini, S., L. Pitaevskii, and S. Stringari, 1994, Phys. Rev. B **49**, 12938.
Görlitz, A., J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, and W. Ketterle, 2001, Phys. Rev. Lett. **87**, 130402.
Griffin, A., 1988, Physica C **156**, 12.
Griffin, A., 1996, Phys. Rev. B **53**, 9341.
Griffin, Allan, 1993, *Excitations in a Bose-Condensed Liquid* (Cambridge University Press, Cambridge, UK).
Hall, David S., 2003, Am. J. Phys. **71**, 649.
Haugset, T., and H. Haugerud, 1998, Phys. Rev. A **57**, 3809.
Heinrichs, S., and W. J. Mullin, 1998, J. Low Temp. Phys. **113**, 231.
Hellweg, D., S. Dettmer, P. Ryytty, J. J. Arlt, W. Ertmer, K. Sengstock, D. S. Petrov, G. V. Shlyapnikov, H. Kreutzmann, L. Santos, and M. Lewenstein, 2001, Appl. Phys. B: Lasers Opt. **73**, 781.
Hewson, A. C., 1993, *The Kondo Problem to Heavy Fermions* (Cambridge University Press, Cambridge, UK).
Ho, Tin-Lun, and Michael Ma, 1999, J. Low Temp. Phys. **115**, 61.
Hohenberg, P. C., 1967, Phys. Rev. **158**, 383.
Hohenberg, P. C., and P. C. Martin, 1965, Ann. Phys. (N.Y.) **34**,

291.
Holzmann, M., G. Baym, J.-P. Blaizot, and F. Laloë, 2005, e-print cond-mat/0508131.
Huang, K., and H. F. Meng, 1992, Phys. Rev. Lett. **69**, 644.
Hugenholtz, N. M., and D. Pines, 1959, Phys. Rev. **116**, 489.
Hutchinson, D. A. W., E. Zaremba, and A. Griffin, 1997, Phys. Rev. Lett. **78**, 1842.
Indekeu, J. O., and B. Van Schaeybroeck, 2004, Phys. Rev. Lett. **93**, 210402.
Javanainen, J., and S. M. Yoo, 1996, Phys. Rev. Lett. **76**, 161.
Kagan, Yu., V. A. Kashurnikov, A. V. Krasavin, N. V. Prokof'ev, and B. V. Svistunov, 2000, Phys. Rev. A **61**, 043608.
Kagan, Yu., B. V. Svistunov, and G. V. Shlyapnikov, 1987, Sov. Phys. JETP **66**, 314.
Kane, J. W., and L. P. Kadanoff, 1967, Phys. Rev. **155**, 80.
Kapitza, P., 1938, Nature (London) **141**, 74.
Ketterle, W., 2001, Rev. Mod. Phys. **74**, 1131.
Ketterle, W., D. S. Durfee, and D. M. Stamper-Kurn, 1999, in *Bose-Einstein Condensation in Atomic Gases*, edited by M. Inguscio, S. Stringari, and C. E. Wieman, Proceedings of the International School of Physics "Enrico Fermi" Course CXL (IOS Press, Amsterdam), p. 67.
Kim, S.-H., C. Won, S. D. Oh, and W. Jhe, 1999, eprint cond-mat/9904087.
Kolomeisky, E. B., T. J. Newman, J. P. Straley, and X. Qi, 2000, Phys. Rev. Lett. **85**, 1146.
Kolomeisky, E. B., and J. P. Straley, 1992a, Phys. Rev. B **46**, 11749.
Kolomeisky, E. B., and J. P. Straley, 1992b, Phys. Rev. B **46**, 13942.
Kosterlitz, J. M., and D. J. Thouless, 1973, J. Phys. C **6**, 1181.
Ledowski, S., N. Hasselmann, and P. Kopietz, 2004, Phys. Rev. A **69**, 061601.
Lee, M. D., S. A. Morgan, M. J. Davis, and K. Burnett, 2002, Phys. Rev. A **65**, 043617.
Lee, T. D., K. Huang, and C. N. Yang, 1957, Phys. Rev. **106**, 1135.
Lee, T. D., and C. N. Yang, 1958, Phys. Rev. **112**, 1419.
Leggett, A. J., 2001, Rev. Mod. Phys. **73**, 307.
Leggett, A. J., 2003, New J. Phys. **5**, 103.1.
Lewart, D. S., V. R. Pandharipande, and S. C. Pieper, 1988, Phys. Rev. B **37**, 4950.
Li, M., L. Chen, and C. Chen, 1999, Phys. Rev. A **59**, 3109.
Lieb, E. H., R. Seiringer, and J. Yngvason, 2001, Commun. Math. Phys. **224**, 17.
Lifshitz, E. M., and L P. Pitaevskii, 2004, *Statistical Physics: Part 2*, in *Course of Theoretical Physics*, edited by L. D. Landau and E. M. Lifshitz (Fizmatlit, Moskwa, Russia).
Liu, W. Vincent, and Xiao-Gang Wen, 2002, eprint cond-mat/0201187.
London, F., 1938, Phys. Rev. **54**, 947.
Mandel, L., and E. Wolf, 1995, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK).
Mermin, N. D., 1968, Phys. Rev. **176**, 250.
Mermin, N. D., and H. Wagner, 1966, Phys. Rev. Lett. **22**, 1133.
Minnhagen, Petter, 1987, Rev. Mod. Phys. **59**, 1001.
Mora, C., and Y. Castin, 2003, Phys. Rev. A **67**, 053615.
Morgan, S. A., M. D. Lee, and K. Burnett, 2002, Phys. Rev. A **65**, 022706.
Mullin, W. J., 1997, J. Low Temp. Phys. **106**, 615.
Nelson, D. R., and J. M. Kosterlitz, 1977, Phys. Rev. Lett. **39**, 1201.
Nogueira, F. S., and H. Kleinert, 2006, Phys. Rev. B **73**, 104515.
Önberg, P., E. L. Surkov, I. Tittonen, S. Stenholm, M. Wilkens, and G. V. Shlyapnikov, 1997, Phys. Rev. A **56**, R3346.
Ovchinnikov, A. A., 1993, J. Phys.: Condens. Matter **5**, 8665.
Pedri, P., and L. Santos, 2005, Phys. Rev. Lett. **95**, 200404.
Peierls, R. E., 1935, Ann. Inst. Henri Poincare **5**, 177.
Penrose, O., and L. Onsager, 1956, Phys. Rev. **104**, 576.
Pethick, C. J., and H. Smith, 2002, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, UK).
Petrov, D. S., D. M. Gangardt, and G. V. Shlyapnikov, 2004, J. Phys. IV **116**, 5.
Petrov, D. S., M. Holzmann, and G. V. Shlyapnikov, 2000, Phys. Rev. Lett. **84**, 2551.
Petrov, D. S., G. V. Shlyapnikov, and J. T. M. Walraven, 2001, Phys. Rev. Lett. **87**, 050404.
Pieri, P., G. C. Strinati, and I. Tifrea, 2001a, Eur. Phys. J. B **22**, 79.
Pieri, P., G. C. Strinati, and I. Tifrea, 2001b, Phys. Rev. B **64**, 052104.
Pines, D., 1962, *The Many Body Problem* (Benjamin, New York).
Pitaevskii, L. P., and A. Rosch, 1997, Phys. Rev. A **55**, R853.
Pitaevskii, L., and S. Stringari, 2003, *Bose-Einstein Condensation* (Clarendon, Oxford).
Polkovnikov, A., E. Altman, and E. Demler, 2006, Proc. Natl. Acad. Sci. U.S.A. **103**, 6125.
Popov, V. N., 1983, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, Holland).
Prokof'ev, N., O. Ruebenacker, and B. Svistunov, 2001, Phys. Rev. Lett. **87**, 270402.
Prokof'ev, N., and B. Svistunov, 2002, Phys. Rev. A **66**, 043608.
Proukakis, N. P., K. Burnett, and H. T. C. Stoof, 1998, Phys. Rev. A **57**, 1230.
Rajagopal, K. K., 2005, J. Phys. B **38**, 2257.
Rajagopal, K. K., P. Vignolo, and M. P. Tosi, 2004, Physica B **344**, 157.
Reatto, L., and J. P. Straley, 1969, Phys. Rev. **183**, 321.
Recati, A., P. O. Fedichev, W. Zwerger, J. von Delft, and P. Zoller, 2005, Phys. Rev. Lett. **94**, 040404.
Ren, Hai-cang, 2004, J. Stat. Phys. **114**, 481.
Rychtarik, D., B. Engeser, H. C. Nagerl, and R. Grimm, 2004, Phys. Rev. Lett. **92**, 173003.
Safonov, A. I., S. A. Vasilyev, I. S. Yasnikov, I. I. Lukashevich, and S. Jaakkola, 1998, Phys. Rev. Lett. **81**, 4545.
Santos, L., G. V. Shlyapnikov, and M. Lewenstein, 2003, Phys. Rev. Lett. **90**, 250403.
Santos, L., G. V. Shlyapnikov, P. Zoller, and M. Lewenstein, 2000, Phys. Rev. Lett. **85**, 1791.
Schick, M., 1971, Phys. Rev. A **3**, 1067.
Shevchenko, S. I., 1992, Sov. J. Low Temp. Phys. **18**, 223.
Shi, H., and A. Griffin, 1998, Phys. Rep. **304**, 1.
Simula, T. P., and P. B. Blakie, 2006, Phys. Rev. Lett. **96**, 020404.
Simula, T. P., M. D. Lee, and D. A. W. Hutchinson, 2005, Philos. Mag. Lett. **85**, 395.
Smith, N. L., W. H. Heathcote, G. Hechenblaikner, E. Nugent, and C. J. Foot, 2005, J. Phys. B **38**, 223.
Stenholm, S., 1998, Phys. Rev. A **57**, 2942.
Stenholm, S., 2002, Phys. Scr. **T102**, 89.
Stock, S., Z. Hadzibabic, B. Battelier, M. Cheneau, and J. Dalibard, 2005, Phys. Rev. Lett. **95**, 190403.
Stoof, H. T. C., and M. Bijlsma, 1993, Phys. Rev. E **47**, 939.

Stringari, S., 1996, Phys. Rev. Lett. **77**, 2360.
Stuhler, J., A. Griesmaier, T. Koch, M. Fattori, T. Pfau, S. Giovanazzi, P. Pedri, and L. Santos, 2005, Phys. Rev. Lett. **95**, 150406.
Szepfalusy, P., and I. Kondor, 1974, Ann. Phys. (N.Y.) **82**, 1.
Tserkovnikov, Yu. A., 1964, Dokl. Akad. Nauk SSSR **159**, 1023 [Sov. Phys. Dokl. **9**, 1095 (1965)].
Wagner, H., 1966, Z. Phys. **195**, 273.
Widom, A., 1968, Phys. Rev. **176**, 254.
Yang, C. N., 1962, Rev. Mod. Phys. **34**, 694.
Yukalov, V. I., 2004, Laser Phys. Lett. **1**, 435.
Ziegler, K., and A. Shukla, 1997, Phys. Rev. A **56**, 1438.