How generic scale invariance influences quantum and classical phase transitions

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(Published 5 July 2005)

This review discusses a paradigm that has become of increasing importance in the theory of quantum phase transitions, namely, the coupling of the order-parameter fluctuations to other soft modes and the resulting impossibility of constructing a simple Landau-Ginzburg-Wilson theory in terms of the order parameter only. The soft modes in question are manifestations of generic scale invariance, i.e., the appearance of long-range order in whole regions in the phase diagram. The concept of generic scale invariance and its influence on critical behavior is explained using various examples, both classical and quantum mechanical. The peculiarities of quantum phase transitions are discussed, with emphasis on the fact that they are more susceptible to the effects of generic scale invariance than their classical counterparts. Explicit examples include the quantum ferromagnetic transition in metals, with or without quenched disorder; the metal-superconductor transition at zero temperature; and the quantum antiferromagnetic transition. Analogies with classical phase transitions in liquid crystals and classical fluids are pointed out, and a unifying conceptual framework is developed for all transitions that are influenced by generic scale invariance.

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PREAMBLE

The theoretical understanding of classical, or thermal, phase transitions, which occur at a nonzero temperature, is very well developed. A characteristic feature of continuous phase transitions, or critical points, is that the free energy, as well as time correlation functions, obey generalized homogeneity laws. This leads to scale invariance, i.e., power-law behavior of various thermodynamic derivatives and correlation functions as functions of the temperature, external fields, wave number, time, etc. Quantum phase transitions, which occur at zero temperature as a function of some nonthermal control parameter such as composition or pressure, are not yet as well understood. One feature that has slowed down progress has turned out to be a connection between quantum phase transitions and the phenomenon known as *generic scale invariance*, which refers to power-law decay of correlation functions in entire phases, not just at an isolated critical point. It turns out that the critical behavior at some classical phase transitions is also heavily influenced by a coupling between generic and critical scale invariance, but these phenomena have usually not been cast in this language. The goal of this review article is to give a unifying discussion of the coupling between critical behavior and generic scale invariance, for both classical and quantum phase transitions and for both static and dynamic critical behavior. Accordingly, we limit our discussion to phase transitions in which this coupling is known or suspected to be important.

The structure of this paper is as follows. In Sec. I we give a brief introduction to phase transitions, both classical and quantum. In Sec. II we discuss the concept of generic scale invariance and illustrate it by means of various examples. These concepts are less well known than those pertaining to phase transitions, so our exposition is more elaborate. In Secs. III and IV we return to phase-transition physics and discuss the influence of generic scale invariance on classical and quantum transitions, respectively. We conclude in Sec. V with a summary and a discussion of open problems.

I. PHASE TRANSITIONS

Phase transitions are among the most fascinating phenomena in nature. They also have far-reaching implications. The liquid-gas and liquid-solid transitions in water, for instance, are common occurrences of obvious importance. The transition from a paramagnetic phase to a ferromagnetic one in the elements iron, nickel, and cobalt made possible the invention of the compass. The structural transition in tin from the β phase (white tin) to the α phase (gray tin) is responsible for the degradation of tin artifacts below a temperature of about 286 K, known as the tin pest. One could continue with a long list, but these three examples may suffice to demonstrate that phase transitions come in a wide variety of phenomenologies with no entirely obvious unifying features. Accordingly, early attempts at a theoretical understanding of phase transitions focused on particular examples. van der Waals (1873) developed the first example of what was later to become known as a *mean-field theory,* in this case to describe the liquid-gas transition. Weiss (1907)

gave a mean-field theory of ferromagnetism that was based on the concept of the mean field seen by each "elementary magnet" (spin had yet to be discovered at that time) that is produced by all other elementary magnets. A unification of all mean-field theories was achieved by Landau (1937a, 1937b, 1937c, 1937d; all of these papers, or their translations, are reprinted in Landau, 1965). He introduced the general concept of the *order parameter*, a thermodynamic observable that vanishes in one of the phases separated by the transition (the *disordered phase*[1]) and is nonzero in the other (the *ordered phase*). In the case of the ferromagnet, the order parameter is the magnetization; in the case of the liquid-gas transition, the order parameter is the density difference between the two phases. Landau theory underlies all later theories of phase transitions, and we therefore discuss it first.

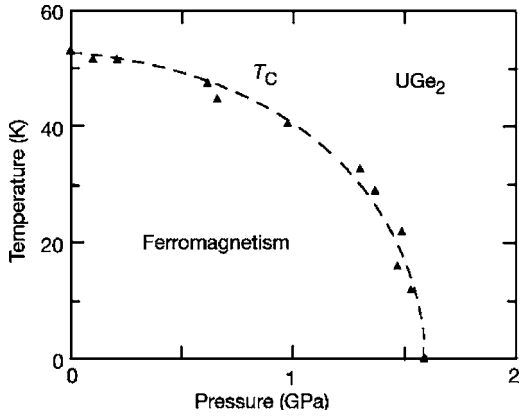

FIG. 1. Phase diagram of UGe_2. Adapted from Saxena *et al.*, 2000.

A. Landau theory

Landau theory is based on one crucial assumption, namely, that the free energy F is an analytic function of the order parameter[2] m, and hence can be expanded in a power series,

$$F \approx F_L(m) = rm^2 + vm^3 + um^4 + O(m^5). \tag{1.1}$$

Here r, v, u, etc., are parameters of the Landau theory that depend on all of the degrees of freedom other than m. F_L is sometimes referred to as the Landau functional, although it actually is just a function of the variable m. The physical value of m is the one that minimizes F_L.

Landau theory is remarkably versatile. For sufficiently large r, the minimum of F is always located at $m=0$, while for sufficiently small r it is located at some $m \neq 0$. If $v \neq 0$, the transition from $m=0$ to $m \neq 0$ occurs discontinuously, and the theory describes a *first-order transition*, with the liquid-gas transition, except at the liquid-gas critical point, being the prime example. If $v=0$, either accidentally or for symmetry reasons, it describes a *second-order transition*, or *critical point*, at $r=0$, provided $u>0$. Prime examples are the ferromagnetic transition in zero magnetic field and the liquid-gas transition at the critical point. Furthermore, Landau theory applies to both classical and quantum systems, including systems at zero temperature ($T=0$). In the latter case, the free energy $F=U-TS$ reduces to the internal energy U.[3] This is not an academic case. Consider a ferromagnet with a low Curie temperature T_c, e.g., UGe_2. By varying a non-thermal control parameter, e.g., hydrostatic pressure, one can suppress the Curie temperature to zero (see Fig. 1). The phase transition can then be triggered in particular by varying the pressure at $T=0$. Obviously, Landau theory predicts the same critical behavior in either case: The magnetization m vanishes as $m(r \to 0) = \sqrt{-r/2u}$, irrespective of how r is driven to zero. This is an example of the "superuniversality" inherent in Landau theory: It predicts the critical exponents to be the same for *all* critical points. For instance, the critical exponent β, defined by $m \propto |r|^\beta$, is predicted to have the value $\beta = 1/2$.

A complete description of a statistical-mechanics system requires, in addition to the thermodynamic properties encoded in the free energy, knowledge of time correlation functions. Of particular interest is the order-parameter susceptibility χ_m. As a critical point is approached from the disordered phase, the harbingers of the latter's instability are diverging order-parameter fluctuations, which lead to a divergence of χ_m at zero frequency and wave number. Within Landau theory, these fluctuations are described in a Gaussian approximation (Landau and Lifshitz, 1980). For the wave-number-dependent static order-parameter susceptibility $\chi_m(\boldsymbol{k})$ this yields the familiar Ornstein-Zernike form

$$\chi_m(\boldsymbol{k}) \propto \frac{1}{r + c\,\boldsymbol{k}^2}, \tag{1.2a}$$

with c a constant. In real space, this corresponds to exponential decay,

$$\chi_m(\boldsymbol{x}) \propto |\boldsymbol{x}|^{-1} e^{-|\boldsymbol{x}|/\xi}, \tag{1.2b}$$

where the correlation length ξ diverges, for $r \to 0$, according to $\xi \propto r^{-1/2}$.

The order-parameter fluctuations at criticality, $r=0$, are an example of a *soft mode*,[4] i.e., fluctuations that diverge in the limit of small frequencies and wave num-

[1] We shall use the term "disordered phase" in the sense of "phase without long-range order." This is not to be confused with the presence of quenched disorder, which we shall also discuss.

[2] In this section we consider a scalar order parameter for simplicity, but later we shall encounter more general cases.

[3] At $T>0$, a stable phase can have a higher internal energy U than an unstable one, as long as its entropy S is also higher. At $T=0$, the stable phase must represent a minimum of the internal energy.

[4] The terms "soft," "gapless," and "massless" are used interchangeably in this context, with the first and second most popular in classical and quantum statistical mechanics, respectively, and the third one borrowed from particle physics.

bers as illustrated by Eqs. (1.2a) and (1.2b). The soft-mode concept will play a crucial role in the remainder of this article. In addition to modes that are soft only at the critical point, we shall discuss, in Sec. II, modes that are soft in an entire phase. The coupling between such "generic soft modes" and the critical order-parameter fluctuations can have profound consequences for the critical behavior, as we shall see in Secs. III and IV. These consequences are the main topic of this review, but before we can discuss them we need to recall some additional aspects of critical behavior.

Equations (1.2a) and (1.2b) define three additional critical exponents, namely, the correlation length exponent ν, defined by $\xi \propto |r|^{-\nu}$, and the susceptibility exponents γ and η, defined by $\chi_m(\boldsymbol{k}=0) \propto r^{-\gamma}$ and $\chi_m(r=0) \propto |\boldsymbol{k}|^{-2+\eta}$. Landau theory universally predicts $\nu=1/2$, $\gamma=1$, and $\eta=0$.

Universality is actually observed in experiments, but it is weaker than the superuniversality predicted by Landau theory, and the observed values of the exponents are in general different from what Landau theory predicts. For instance, all bulk Heisenberg ferromagnets have a $\beta \approx 0.35$ (see, for example, Zinn-Justin, 1996), which is significantly smaller than the Landau value. Similarly, all bulk Ising ferromagnets have a common value of β, but that value, $\beta \approx 0.32$, is different from the one in Heisenberg systems. The critical exponents also turn out to be different for systems of different dimensionality, again in contrast to the prediction of Landau theory. For instance, in two-dimensional Ising ferromagnets, $\beta=1/8$ (Onsager, 1948; Yang, 1952).

B. Landau-Ginzburg-Wilson theory, the renormalization group, and scaling

The reason for the failure of Landau theory to correctly describe the details of the critical behavior is that it does not adequately treat the fluctuations of the order parameter about its mean value. The deviations of these fluctuations from a Gaussian character in general are stronger for lower dimensionalities and for order parameters with fewer components. This explains why the critical behavior of Ising magnets deviates more strongly from the Landau value than that of Heisenberg magnets and why the exponents of bulk systems are closer to the mean-field values than those of thin films. This observation suggests that Landau theory might actually yield the correct critical behavior in systems with a sufficiently high dimensionality d. Indeed, it turns out that in general there is an upper critical dimensionality, d_c^+, such that for $d > d_c^+$ fluctuations are unimportant for the leading critical behavior, and Landau theory gives the correct answer (this follows from the Ginzburg criterion; see, for example, Cardy, 1996, Sec. 2.4). For the ferromagnetic transition at $T>0$, $d_c^+=4$. For $d<d_c^+$, fluctuations need to be taken into account beyond the Gaussian approximation in order to obtain the correct critical behavior. This problem was solved by Wilson (Wilson and Kogut, 1974; see also, Ma, 1976; Fisher, 1983). He generalized the Landau functional, Eq. (1.1), by writing the partition function $Z=e^{-F/T}$ as a functional integral,[5]

$$Z = e^{-F/T} = \int D[\phi] e^{-S[\phi]}, \tag{1.3a}$$

where

$$S[\phi] = \frac{1}{TV} \int d\boldsymbol{x} [F_L(\phi(\boldsymbol{x})) + c(\boldsymbol{\nabla}\phi(\boldsymbol{x}))^2] \tag{1.3b}$$

is usually referred to as the *action*. Here V is the system volume, and ϕ is a fluctuating field whose average value with respect to the statistical weight $\exp(-S)$ is equal to m. This Landau-Ginzburg-Wilson (LGW) functional is then analyzed by means of the renormalization group (RG);[6] the lowest-order, i.e., saddle-point–Gaussian, approximation recovers Landau theory. Wilson's RG takes advantage of the fact that at a critical point there is a diverging length scale, namely, the *correlation length* ξ, which dominates the long-wavelength physics. By integrating out all fluctuations on smaller length scales, one can derive a succession of effective theories that describe the behavior near criticality. The RG made possible the derivation and proof of the behavior near criticality known as *scaling*, which previously had been observed empirically and summarized in the *scaling hypothesis*.[7] Due to scaling, all static critical phenomena are characterized by two independent critical exponents, one for the reduced temperature[8] $r \propto |T-T_c|$ and one for the field h that is conjugate to the order parameter. Let $\mu=(r,h)$ define the space spanned by these two parameters. Under RG iterations, the system moves away from criticality according to

$$\mu \to \mu(b) = (rb^{1/\nu}, hb^{y_h}), \tag{1.4a}$$

where $b>1$ is the RG rescaling factor. ν is the correlation length exponent defined above, and y_h is related to the susceptibility exponent η by

$$y_h = (d+2-\eta)/2. \tag{1.4b}$$

The exponents $1/\nu$ and y_h illustrate the more general concept of scale dimensions, which determine how parameters change under RG transformations. If p is some parameter with scale dimension $[p]$, then $p(b)=p_0 b^{[p]}$, with $p_0=p(b=1)$. Accordingly, $[r]=1/\nu$, and $[h]=y_h$. Parameter values μ^* with the property $\mu^*(b)=\mu^*$ constitute

[5] We use units such that Boltzmann's constant, Planck's constant, and the Bohr magneton are equal to unity.

[6] Pedagogical expositions of the Wilsonian RG have been given by Wilson and Kogut (1974); Ma (1976); Fisher (1983); Goldenfeld (1992); and Cardy (1996).

[7] The state of affairs just before the invention of the RG was reviewed by Kadanoff *et al.* (1967) and Stanley (1971).

[8] One needs to distinguish between the exact or fully renormalized value of r, which appears in formal scaling arguments, the bare value of r, which appears in Landau theory or in the LGW functional, and any partially renormalized values. We shall explicitly make this distinction when doing so is essential, but will suppress it otherwise.

a RG fixed point. Parameters with positive, zero, and negative scale dimensions with respect to a particular fixed point are called *relevant*, *marginal*, and *irrelevant parameters*, respectively. A critical fixed point is characterized by the presence of two and only two relevant parameters, r and h. The critical behavior of all thermodynamic quantities follows from a generalized homogeneity law satisfied by the free-energy density $f = -(T/V)\ln Z$,

$$f(r,h) = b^{-d} f(rb^{1/\nu}, hb^{y_h}). \tag{1.5}$$

The derivation of this relation was one of the major triumphs of the RG. In conjunction with the Wilson-Fisher ϵ expansion (Wilson and Fisher, 1972), it allows for the computation of the critical exponents in an asymptotic series[9] about d_c^+.

In addition to the diverging length scale set by ξ, there is a diverging time scale τ_ξ. Its divergence is governed by the dynamical critical exponent z, according to $\tau_\xi \propto \xi^z$, and it results in the phenomenon of "critical slowing down" (van Hove, 1954; see also Landau and Khalatnikov, 1954; a translation of the latter paper appears in Landau, 1965, p. 626), that is, the very slow relaxation towards equilibrium of systems near a critical point. The critical behavior of time correlation functions $C(\boldsymbol{k},\Omega;r,h)$ is given by

$$C(\boldsymbol{k},\Omega;r,h) = b^{x_C} C(\boldsymbol{k}b, \Omega b^z; rb^{1/\nu}, hb^{y_h}). \tag{1.6}$$

Here Ω denotes the frequency, which derives from a Fourier transform of the time dependence. x_C is an exponent that is characteristic of the correlation function C. This relation was first postulated as the "dynamical scaling hypothesis" (Ferrell *et al.*, 1967, 1968; Halperin and Hohenberg, 1967) and was later also derived by dynamical RG techniques (Hohenberg and Halperin, 1977).

C. Classical versus quantum phase transitions

In classical statistical mechanics, the dynamical critical exponent z is independent of the static critical behavior (Ma, 1976; Hohenberg and Halperin, 1977). The reason is as follows. Classically, the canonical partition function for a system consisting of N particles with f degrees of freedom is given as an integral over phase space of the Boltzmann factor,

$$\begin{aligned} Z &= \frac{1}{N!}\int dp\ dq\ e^{-\beta H(p,q)} \\ &= \frac{1}{N!}\int dp\ e^{-\beta H_{\rm kin}(p)} \int dq\ e^{-\beta H_{\rm pot}(q)} \\ &= \text{const} \times \int dq\ e^{-\beta H_{\rm pot}(q)}. \end{aligned} \tag{1.7}$$

Here $\beta = 1/T$ is the inverse temperature, p and q represent the generalized momenta and coordinates, respectively, H is the Hamiltonian, and $H_{\rm kin}$ and $H_{\rm pot}$ are the kinetic and potential energy, respectively.[10] Due to the factorization of the phase-space integral indicated in Eq. (1.7), one can integrate over the momenta and solve for the thermodynamic critical behavior without reference to the dynamics.

In quantum statistical mechanics, the situation is different. The Hamiltonian $\hat{H}$, and its constituents $\hat{H}_{\rm kin}$ and $\hat{H}_{\rm pot}$, are now operators, and $\hat{H}_{\rm kin}$ and $\hat{H}_{\rm pot}$ do not commute. Consequently, the grand canonical partition function,

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \mathrm{Tr}\, e^{-\beta(\hat{H}_{\rm kin}+\hat{H}_{\rm pot}-\mu\hat{N})}, \tag{1.8}$$

does not factorize, and one must solve for the dynamical critical behavior together with the thermodynamics. This becomes even more obvious if one rewrites the partition function as a functional integral (Casher *et al.*, 1968; Negele and Orland, 1988). We shall consider fermionic systems, in which case the latter is taken with respect to Grassmann-valued (i.e., anticommuting) fields $\bar{\psi}$ and ψ,[11]

$$Z = \int D[\bar{\psi},\psi] e^{S[\bar{\psi},\psi]}. \tag{1.9a}$$

The action S is determined by the Hamiltonian,

$$\begin{aligned} S[\bar{\psi},\psi] = &\int_0^\beta d\tau \int d\boldsymbol{x} \sum_\sigma \bar{\psi}_\sigma(\boldsymbol{x},\tau)(-\partial_\tau + \mu)\psi_\sigma(\boldsymbol{x},\tau) \\ &- \int_0^\beta d\tau \int d\boldsymbol{x}\ H(\bar{\psi}_\sigma(\boldsymbol{x},\tau), \psi_\sigma(\boldsymbol{x},\tau)). \end{aligned} \tag{1.9b}$$

Here $\boldsymbol{x}$ denotes the position in real space, σ is the spin index,[12] and the fields $\bar{\psi}_\sigma(\boldsymbol{x},\tau)$ and $\psi_\sigma(\boldsymbol{x},\tau)$ are, for each value of τ, in one-to-one correspondence with the creation and annihilation operators of second quantization,

[9] It is a popular misconception that the RG is useful *only* for dealing with critical phenomena. In fact, the technique is much more powerful and versatile, and can describe entire phases as well as the transitions between them; see Sec. II below. Although this was realized by the founding fathers of the RG (see Anderson, 1984, Chap. 5; Fisher, 1998), it has been exploited only recently (see, for example, Shankar, 1994). It is interesting to note that the systematic application of the Wilsonian RG to condensed-matter systems implements a program that in a well-defined way is the opposite of that in high-energy physics. In condensed-matter physics, the microscopic theory is known for all practical purposes, viz., the many-body Schrödinger equation. By applying the RG one derives effective theories valid at lower and lower energy scales. In high-energy physics, some effective theory valid at relatively low energies is known (say, the standard model), and the goal is to deduce a "more microscopic" theory valid at higher energies.

[10] We assume that there are no velocity-dependent potentials.

[11] For a thorough treatment of Grassmannian algebra and calculus, see Berezin, 1966.

[12] σ may also comprise other quantum numbers, e.g., a band index, depending on the model considered.

$\hat{\psi}^{\dagger}_{\sigma}(\boldsymbol{x})$ and $\hat{\psi}_{\sigma}(\boldsymbol{x})$, respectively. They obey antiperiodic boundary conditions, $\psi_{\sigma}(\boldsymbol{x},\tau=0)=-\psi_{\sigma}(\boldsymbol{x},\tau=\beta)$. The function H is defined such that $\hat{H}=\int d\boldsymbol{x}\, H(\hat{\psi}^{\dagger}_{\sigma}(\boldsymbol{x}),\hat{\psi}_{\sigma}(\boldsymbol{x}))$. μ is the chemical potential, and $\hat{N}$ in Eq. (1.8) is the particle number operator. A quantum-mechanical generalization of the LGW functional can be derived from Eq. (1.9b) by constraining appropriate linear combinations of products $\bar{\psi}\psi$ of fermion fields to a bosonic order-parameter field ϕ and integrating out all other degrees of freedom. Often one can also just write down a LGW functional based on symmetries and other general considerations.

Due to the coupling of statics and dynamics, the scaling relation (1.5) for the free energy must be generalized to (see, for instance, Sachdev, 1999)

$$f(r,h,T)=b^{-(d+z)}f(r\,b^{1/\nu},h\,b^{y_h},T\,b^{z}). \tag{1.10}$$

This relation reflects the fact that the temperature is necessarily a relevant operator at a $T=0$ critical point and that temperature and frequency are expected to scale the same way.[13]

Equation (1.9b) displays another remarkable property of quantum statistical mechanics. The auxiliary variable τ, usually referred to as *imaginary time*, acts effectively as an extra dimension. For nonzero temperature, $\beta=1/T<\infty$, this extra dimension extends only over a finite interval. If one is sufficiently close to criticality that the condition $\tau_{\xi}>1/T_c$ is fulfilled, the extra dimension will not affect the leading critical behavior. Rather, it will only lead to corrections to scaling that are determined by finite-size scaling effects (Barber, 1983; Cardy, 1996). We thus conclude that the asymptotic critical behavior at any transition with a nonzero critical temperature is purely classical. However, a transition at $T=0$ is described by a theory in an effectively different dimension, and will therefore in general be in a different universality class. This raises the question of how continuity is ensured as one moves to $T=0$ along the phase-separation line. The resolution is that, at low temperatures, the critical region, i.e., the region around the phase-separation line where critical behavior can be observed, is divided into several regimes. Asymptotically close to the transition the critical behavior is classical at any nonzero temperature,[14] but since this asymptotic classical regime is bounded by a crossover at $1/\tau_{\xi}\approx T$, it shrinks to zero as $T\to 0$. In the vicinity of the quantum critical point, quantum critical behavior is observed except in the immediate vicinity of the phase boundary.

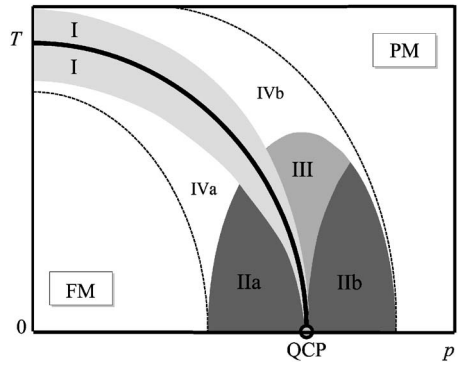

FIG. 2. Schematic phase diagram in the T-p plane, as in Fig. 1. The solid line is the phase-separation line, separating the ferromagnetic (FM) phase from the paramagnetic (PM) one, which ends in the quantum critical point (QCP). The critical region in the vicinity of the phase-transition line is bounded by the dashed lines, which are defined as the loci of points where the correlation length is a certain multiple of the microscopic length scale. The critical region is separated into four regimes that show different scaling behaviors: Region I (light shading) displays classical scaling governed by the classical fixed point. Regions IIa and IIb (dark shading) display static or temperature-independent quantum scaling governed by the zero-temperature fixed point, and region III (medium shading) displays dynamic or temperature quantum scaling. In regions IVa and IVb one has crossover scaling governed by both fixed points. The edges of the critical region, as well as the boundaries between the various scaling regimes, are not sharp. Notice that the dashed lines are not parallel to the phase-separation line, since both the exponents ν and the critical amplitudes associated with the classical and quantum fixed points, respectively, in general have different values.

This *quantum critical regime* is in turn divided into a region characterized by $T\lesssim r^{\nu z}$, where one sees static quantum critical behavior approximately independent of the temperature, and a region characterized by $T\gtrsim r^{\nu z}$, where one observes dynamic or temperature scaling. In the literature, the term "quantum critical regime" is often used for the latter region only. Finally, there is a regime inside the critical region but outside both the asymptotic classical and the quantum scaling regimes. This is characterized by crossover scaling governed by both the classical and quantum fixed points. These various regimes are shown schematically in Fig. 2.

Equation (1.9b) suggests that imaginary time or inverse temperature always scales like a length, i.e., that the dynamical critical exponent $z=1$ at any quantum phase transition. Indeed, early work generalizing the Wilsonian RG to quantum systems either dealt with systems in which $z=1$ (Suzuki, 1976) or assumed that $z=1$ universally (Beal-Monod, 1974). This is misleading, however. The point is that Eq. (1.9b) represents the bare microscopic action. Under renormalization, or even when integrating out degrees of freedom to derive a bare effective action, the relation between space and imaginary time can change. As a result, z can assume

[13] As we shall see later, the latter expectation can be violated, due to (1) the existence of multiple temperature and/or frequency scales and (2) the existence of dangerous irrelevant variables (Fisher, 1983).

[14] This includes the transitions at nonzero temperature in, say, superconductors or superfluids. Although the occurrence of the transition in these cases, and indeed the very existence of the order parameter, depend on quantum mechanics, the critical properties are described by classical physics.

any positive value, as was realized by Hertz (1976).[15] Hertz also noted the following interesting consequence of the space-time nature of quantum statistical mechanics, which holds independently of the exact value of z. Since the quantum-mechanical theory is effectively higher dimensional than the corresponding classical one, fluctuation effects are weaker, and mean-field theory will be more stable. Indeed, since Landau theory is valid classically for $d > d_c^+$, quantum mechanically it should be valid for $d > d_c^+ - z$. Since $d_c^+ = 4$ for many phase transitions, and usually $z \geqslant 1$, it seems to follow that most quantum phase transitions in the most interesting dimension, $d = 3$, show mean-field critical behavior. Hertz (1976) demonstrated this conclusion by means of a detailed LGW theory of the quantum phase transition in itinerant Heisenberg ferromagnets, in which $d_c^+ = 4$ and $z = 3$ in Hertz's theory.

D. The soft-mode paradigm

The above chain of arguments implies that a quantum phase transition is closely related to the corresponding classical phase transition in a higher dimension. Since this higher dimension will usually be above the upper critical dimension, the conclusion seems to be that quantum phase transitions generically show mean-field critical behavior and are thus uninteresting from a critical phenomena point of view. In this review we discuss one of the reasons why this is in general not correct. More generally, we shall discuss how the entire concept of a LGW theory in terms of a single order-parameter field can break down for either classical or quantum transitions, and why this breakdown is more common for the latter than for the former. The salient point is that a correct description of a phase transition, i.e., of phenomena at long length and time scales, must take into account *all* soft modes. If there are soft modes in addition to the order-parameter fluctuations, and if the former couple sufficiently strongly to the latter, then the behavior cannot be described in terms of a local field theory for the order-parameter fluctuations only. Away from the quantum phase transition, these extra soft modes, via interactions or nonlinear couplings, lead to power-law correlations in various physical correlation functions. This phenomenon is called *generic scale invariance* and we shall discuss it in Sec. II.

In a derivation of a LGW functional from a microscopic theory this manifests itself as follows. As mentioned above, such a derivation involves integrating out all degrees of freedom other than the order-parameter field. The parameters in the LGW functional are given in terms of integrals over the correlation functions of these degrees of freedom. If there are soft modes other than the order-parameter fluctuations, then these integrals may not exist, which can lead to diverging coefficients in the LGW theory. At the level of Landau or mean-field theory this manifests itself as a nonanalytic dependence of the free energy on the mean-field order parameter. We shall see later, in Secs. III.A.3 and IV.A.3, respectively, that both in liquid crystals and in quantum ferromagnets, extra soft modes lead to an effective Landau free energy

$$F = rm^2 + vm^4 \ln m^2 + um^4 + \cdots \tag{1.11}$$

as opposed to Eq. (1.1). The situation gets worse if order-parameter fluctuations are taken into account. Writing down a LGW theory based on symmetry considerations, or even deriving one in some crude approximation, will in general lead to erroneous conclusions if additional soft modes are present. A correct derivation, however, will lead to coefficients that are singular functions of space and imaginary time, which renders the theory useless. A more general approach, which keeps all of the soft modes on an equal footing, is thus called for. Getting in trouble by integrating out, explicitly or implicitly, additional excitations is not unprecedented in physics. For instance, the Fermi theory of weak interactions can be considered as resulting from integrating out the W gauge bosons in the Standard Model. This results in a theory that is not renormalizable (see, for example, LeBellac, 1991).

Before we enter into details, let us elaborate somewhat on the general aspects of this breakdown of LGW theory. Imagine a transition from a phase with an already-broken continuous symmetry to one with an additional broken symmetry. Then the Goldstone modes of the former will in general influence the transition. A good classical example of this mechanism is the nematic–smectic-A transition in liquid crystals (de Gennes and Prost, 1993), which we shall discuss in Sec. III.A. However, for the most obvious classical phase transitions, e.g., for the liquid-gas critical point, generic soft modes affect only the dynamics, i.e., the critical behavior of transport coefficients (Hohenberg and Halperin, 1977). Quantum mechanically, the concept is more important, for two reasons: (1) There are more soft modes at $T = 0$ than at $T > 0$. An example that will be important for our discussion is the particle-hole excitations in an electron fluid that are soft at $T = 0$ and acquire a mass proportional to T at $T > 0$. They couple strongly to the magnetization, with consequences for the quantum ferromagnetic transitions that will be discussed in Sec. IV below. They also provide a good example of the phenomenon of generic scale invariance, which we shall discuss in detail in Sec. II. (2) Quantum mechanically, the statics and the dynamics are coupled, as we have discussed above, and therefore effects that classically would affect only the dynamics influence the static

[15] One might object that the underlying microscopic theory must be Lorentz invariant, so $z = 1$ after all. Again, this argument ignores the fact that the RG derives an effective low-energy theory valid at small wave numbers and frequencies. In the critical theory, the relevant frequency, $1/\tau_\xi$, is zero. Another way to say this is that the speed of light has been renormalized to $c = \infty$. The critical theory is therefore *rigorously* nonrelativistic. As we have seen above, at any nonzero temperature, $\hbar$ renormalizes to zero, and the critical theory is also rigorously non-quantum-mechanical.

critical behavior as well. As a result of these two points, the original concept of mapping a quantum phase transition onto the corresponding classical transition in a higher dimensionality turns out to be mistaken. Rather, the mapping is in general onto a classical transition with additional soft modes.

In systems with generic scale invariance the correlation length, which diverges at the critical point, must be defined differently from Eq. (1.2b). The reason for this is that the order-parameter correlations decay as power laws both at and away from the critical point, albeit with different powers, the decay being slower at criticality. In this case, ξ is the length scale that separates the generic power-law correlations from the critical ones. Specifically, in the scaling region, and for $|\boldsymbol{x}|<\xi$, the correlations obey the critical power law, while for $|\boldsymbol{x}|>\xi$ the correlations show the generic power law.

In addition to this mechanism, there are other reasons why quantum phase transitions can be more complicated than one might naively expect. In low-dimensional systems, topological order can lead to very nontrivial effects. One example is the intricate behavior of spin chains (Haldane, 1982, 1983); another is the recent proposal of exotic quantum critical behavior in two-dimensional quantum antiferromagnets (Senthil *et al.*, 2004a, 2004b). In addition, the quantum phase transition may have no classical counterpart, and hence no classical upper critical dimension. Examples include metal-insulator transitions (Mott, 1990; Kramer and MacKinnon, 1993; Belitz and Kirkpatrick, 1994) and transitions between quantum Hall states (Sondhi *et al.*, 1990). These phenomena are beyond the scope of the present review.

Finally, we mention that there are also examples of quantum phase transitions in which none of the above complications arise and Hertz theory (Hertz, 1976) is valid. This class of transitions includes quantum metamagnetic transitions (Millis, Schofield, *et al.*, 2002; see Fig. 16 for an example) and the liquid-crystal-like transitions in high-T_c superconductors and quantum Hall systems (Du *et al.*, 1999; Emery *et al.*, 1999; Lilly *et al.*, 1999; Oganesyan *et al.*, 2001). One would also expect this class to include clean antiferromagnetic systems without local moments, although no convincing experimental examples have been found so far. A general classification of which quantum phase transition should be affected by generic soft modes has been attempted by Belitz *et al.* (2002). Although they are very interesting, we shall not further consider these "simple" quantum phase transitions in this review.

II. GENERIC SCALE INVARIANCE

Correlation functions characterize how fluctuations at one space-time point are correlated with fluctuations at another point (see, for example, Forster, 1975). The default expectation is that they decay exponentially for large distances in space or time; we expect fluctuations that are far apart to be weakly correlated. If they are, then there is a characteristic length or time scale associated with the decay, and the correlations are said to be of short range. It is well known that at special points in the phase diagram, at critical points in particular, certain correlations may decay only as power laws in space and/or time (see, for example, Ma, 1976). In this case, the correlations are called long ranged (in space) or long lived (in time); we shall call them long ranged regardless of whether the reference is to space or to time. Such correlation functions exhibit scale invariance: A scale change in space and/or time can be compensated by multiplying the correlation function by a simple scale factor. This property is conveniently expressed in terms of generalized homogeneous functions (see, for example, Chaikin and Lubensky, 1995); see, for example, Eqs. (1.5) and (1.6). In addition, systems can exhibit scale invariance in whole regions of the phase diagram, rather than only at special points. In this case one speaks of "generic scale invariance" (GSI; Law and Nieuwoudt, 1989; Nagel, 1992; Dorfman *et al.*, 1994). In this section we discuss various mechanisms for GSI in both classical and quantum systems, highlighting their similarities and differences. Before we go into detail, we list several mechanisms that can lead to this phenomenon.

(1) Spontaneous breaking of a global continuous symmetry leads to Goldstone modes that are massless everywhere in the broken-symmetry phase (Forster, 1975; Zinn-Justin, 1996). This mechanism is operative in both classical and quantum systems.

(2) Gauge symmetries lead to soft modes since a mass would be incompatible with gauge invariance (Ryder, 1985; Weinberg, 1996a, 1996b; Zinn-Justin, 1996). The only case we shall be interested in is the U(1) gauge symmetry that underlies the masslessness of the photon.

Both of these mechanisms, if operative, lead to soft modes that directly result, in obvious ways, from the symmetry in question, and hence lead to GSI that we shall refer to as *direct GSI*. What is usually less obvious is that these soft modes, via mode-mode coupling effects, can lead to other modes' becoming soft as well. We shall see various examples of these *indirect GSI* effects. Another source of direct generic scale invariance is

(3) conservation laws, which lead to power-law temporal decay of local time correlation functions (Forster, 1975). There are two ways in which conservation laws can lead to indirect GSI:

(3a) The conservation laws can appear in conjunction with mode-mode coupling effects, or nonlinearities in the equations of motion (Pomeau and Resibois, 1975; Boon and Yip, 1991). In classical systems, this leads to long-ranged time correlation functions; in quantum systems, it leads to long-ranged time correlation functions *and* thermodynamic quantities.

(3b) Conservation laws in conjunction with a nonequilibrium situation (Kirkpatrick *et al.*, 1982a, 1982b; Schmittmann and Zia, 1995) can lead to long-ranged time correlation functions and thermodynamic quantities even in classical systems.

In what follows we discuss specific examples, starting with classical systems.

A. Classical systems

We now illustrate the four mechanisms listed above by means of four examples that show how they lead to long-ranged correlations in classical systems. For each case we give a general discussion, followed by the specification of a suitable model and a demonstration of how explicit calculations and, where applicable, RG arguments yield generic scale invariance.

1. Goldstone modes

To illustrate the Goldstone mechanism for long-ranged spatial correlations in equilibrium, we choose as an example a generalized Heisenberg ferromagnet with an N-dimensional order parameter. $N=3$ represents the usual Heisenberg magnet, which will be relevant later in this review. Let $\boldsymbol{\phi}(\boldsymbol{x})$ be the fluctuating magnetization, with $\boldsymbol{m}=\langle\boldsymbol{\phi}(\boldsymbol{x})\rangle$ its average value, and let $\boldsymbol{h}$ be an external magnetic field conjugate to the order parameter. Goldstone's theorem (Goldstone, 1961; Goldstone *et al.*, 1962) states that, whenever there is a spontaneously broken global continuous symmetry, there will be massless modes. (This is not true for local, or gauge, symmetries; see Sec. II.A.2 below.) To understand this concept, suppose that the action is invariant under rotations of the vector field $\boldsymbol{\phi}$ provided that $\boldsymbol{h}=0$. This implies that the order parameter in zero field vanishes, $\boldsymbol{m}(\boldsymbol{h}=0)=0$, due to rotational invariance. However, in the limit $\boldsymbol{h}\to 0$ there are two possible behaviors, depending on the temperature, which determine whether the system is in the disordered phase or in the ordered phase, namely,

$$\boldsymbol{m}(\boldsymbol{h}\to 0)=0 \quad \text{(disordered phase)}, \tag{2.1a}$$

$$\boldsymbol{m}(\boldsymbol{h}\to 0)\neq 0 \quad \text{(ordered phase)}. \tag{2.1b}$$

The situation in the ordered phase is sometimes also characterized by saying that the action obeys the symmetry while the state does not, and it is referred to as a *spontaneously broken continuous symmetry*.[16] In the ordered phase, one still has invariance with respect to rotations that leave the vector $\boldsymbol{m}$ fixed, i.e., with respect to the subgroup $O(N-1)$ that is the little group of $\boldsymbol{m}$. Goldstone's theorem says that this results in as many soft modes as the quotient space $O(N)/O(N-1)$ has dimensions, i.e., there are dim $[O(N)/O(N-1)]=N-1$ Goldstone modes. These soft modes are *perpendicular* or *transverse* to the direction $\boldsymbol{m}$ of the spontaneous ordering. For example, if $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_N)$ and there is spontaneous ordering in the N direction, then one has, in the limit $\boldsymbol{h}\to 0$ and for asymptotically small wave numbers,

$$\langle\phi_i(\boldsymbol{k})\phi_j(-\boldsymbol{k})\rangle=\delta_{ij}/\zeta\,\boldsymbol{k}^2 \quad (i,j=1,\ldots,N-1), \tag{2.2a}$$

where ζ is called the *stiffness coefficient* of the Goldstone modes. In real space say, in, three dimensions, this implies that for large distances, $|\boldsymbol{x}_1-\boldsymbol{x}_2|\to\infty$,

$$\langle\phi_i(\boldsymbol{x}_1)\phi_j(\boldsymbol{x}_2)\rangle\propto 1/|\boldsymbol{x}_1-\boldsymbol{x}_2|. \tag{2.2b}$$

We see that Goldstone's theorem gives rise to power-law correlations, or GSI, in the entire ordered phase. In the terminology explained above, this is an example of direct GSI.

a. Nonlinear σ model

To illustrate the Goldstone mechanism more explicitly, and for later reference, let us consider the derivation of a nonlinear σ model (Gell-Mann and Lévy, 1960; Polyakov, 1975; Brézin and Zinn-Justin, 1976; Nelson and Pelcovits, 1977; Zinn-Justin, 1996) for our $O(N)$-symmetric Heisenberg ferromagnet. We specify the action by

$$S[\phi]=\int d\boldsymbol{x}[r\phi^2(\boldsymbol{x})+c(\boldsymbol{\nabla}\boldsymbol{\phi}(\boldsymbol{x}))^2+u\,\boldsymbol{\phi}^4(\boldsymbol{x}) - h\,\phi_N(\boldsymbol{x})]. \tag{2.3}$$

Here $\phi^2=\phi_i\phi^i$ and $(\boldsymbol{\nabla}\boldsymbol{\phi})^2=\partial_\alpha\phi_i\partial^\alpha\phi^i$, with $i=1,\ldots,N$ and $\alpha=1,\ldots,d$ in d dimensions. Summation over repeated indices is implied, and we have assumed an external field in the N direction. S determines the partition function via Eq. (1.3a). For $h=0$, it is obviously invariant under $O(N)$ rotations of the vector field ϕ. In the low-temperature phase, where the $O(N)$ symmetry is spontaneously broken,[17] it is convenient to decompose $\boldsymbol{\phi}$ into its modulus ρ and a unit vector field $\hat{\boldsymbol{\phi}}$,

$$\boldsymbol{\phi}(\boldsymbol{x})=\rho(\boldsymbol{x})\hat{\boldsymbol{\phi}}(\boldsymbol{x}), \quad \hat{\boldsymbol{\phi}}^2(\boldsymbol{x})=1. \tag{2.4}$$

$\hat{\phi}$ parametrizes the $(N-1)$ sphere, which is isomorphic to the coset space $O(N)/O(N-1)$. In terms of ρ and $\hat{\boldsymbol{\phi}}$, the action is[18]

$$S[\rho,\hat{\boldsymbol{\phi}}]=c^{(1)}\int d\boldsymbol{x}\;\rho^2(\boldsymbol{x})[\boldsymbol{\nabla}\hat{\boldsymbol{\phi}}(\boldsymbol{x})]^2+\int d\boldsymbol{x}[r\rho^2(\boldsymbol{x}) + c^{(2)}[\boldsymbol{\nabla}\rho(\boldsymbol{x})]^2+u\rho^4(\boldsymbol{x})]-h\int d\boldsymbol{x}\;\rho(\boldsymbol{x})\hat{\phi}_N(\boldsymbol{x}). \tag{2.5}$$

The bare values of $c^{(1)}$ and $c^{(2)}$ are equal to c. Notice that the field $\hat{\boldsymbol{\phi}}$ appears only in conjunction with two gradient operators. This implies that the $\hat{\boldsymbol{\phi}}$ fluctuations represent the $N-1$ Goldstone modes of the problem, while

[16] In this case, the continuous symmetry is characterized by the group $O(N)$, but the concept is valid for arbitrary Lie groups. A useful reference for Lie groups, or continuous groups, is Gilmore, 1974.

[17] Provided that $d>2$; no spontaneous symmetry breaking occurs in $d\leqslant 2$ (Mermin and Wagner, 1966).

[18] The change of variables from $\boldsymbol{\phi}$ to $(\rho,\hat{\phi})$ also changes the integration measure in Eq. (1.3a). If one eliminates σ in terms of π, Eqs. (2.6a) and (2.6b), the measure changes again; see Zinn-Justin, 1996.

the ρ fluctuations represent the massive mode. Assuming the system is in the ordered phase, and taking the order to be in the N direction, we parametrize $\hat{\phi}$ as follows:

$$\hat{\boldsymbol{\phi}}(\boldsymbol{x}) = (\boldsymbol{\pi}(\boldsymbol{x}), \sigma(\boldsymbol{x})), \tag{2.6a}$$

where the vector $\boldsymbol{\pi}$ represents the $N-1$ transverse directions, and

$$\sigma(\boldsymbol{x}) = [1 - \boldsymbol{\pi}^2(\boldsymbol{x})]^{1/2}. \tag{2.6b}$$

Next we split off the expectation value of the massive field by writing $\rho(\boldsymbol{x}) = M + \delta\rho(\boldsymbol{x})$, with $M = \langle\rho(\boldsymbol{x})\rangle$. Absorbing appropriate powers of M into the coupling constants, we can write the action

$$S[\rho, \boldsymbol{\pi}] = S_{\mathrm{NL}\sigma\mathrm{M}}[\boldsymbol{\pi}] + \delta S[\rho, \boldsymbol{\pi}]. \tag{2.7a}$$

Here

$$S_{\mathrm{NL}\sigma\mathrm{M}}[\boldsymbol{\pi}] = \frac{\zeta}{2}\int d\boldsymbol{x}[(\boldsymbol{\nabla}\pi(\boldsymbol{x}))^2 + (\boldsymbol{\nabla}\sigma(\boldsymbol{x}))^2] - h\int d\boldsymbol{x}\ \sigma(\boldsymbol{x}) \tag{2.7b}$$

is the action of the O(N) nonlinear σ model, with the bare value of ζ equal to $2cM^2$. δS is the part of the action that contains the $\delta\rho$ fluctuations as well as the coupling between $\delta\rho$ and $\hat{\phi}$,

$$\delta S = r\int d\boldsymbol{x}(\delta\rho(\boldsymbol{x}))^2 + O(\delta\rho^3, (\boldsymbol{\nabla}\delta\rho)^2, \sigma\delta\rho, \delta\rho(\boldsymbol{\nabla}\hat{\boldsymbol{\phi}})^2). \tag{2.7c}$$

This parametrization of the model accomplishes an explicit separation into modes that are soft in the broken-symmetry phase (i.e., the $\boldsymbol{\pi}$ fields), modes that are massive (i.e., the $\delta\rho$ field), and couplings between the two.

b. Generic scale invariance from explicit calculations

At the Gaussian level, the action reads

$$S_{\mathrm{G}} = \frac{\zeta}{2}\int d\boldsymbol{x}(\boldsymbol{\nabla}\pi(\boldsymbol{x}))^2 + \frac{h}{2}\int d\boldsymbol{x}(\boldsymbol{\pi}(\boldsymbol{x}))^2 + r\int d\boldsymbol{x}(\delta\rho(\boldsymbol{x}))^2 + c^{(2)}\int d\boldsymbol{x}(\boldsymbol{\nabla}\delta\rho(\boldsymbol{x}))^2. \tag{2.8}$$

Here we see explicitly that the Gaussian $\boldsymbol{\pi}$ propagators are soft as $h \to 0$,

$$\langle\pi_i(\boldsymbol{k})\pi_j(-\boldsymbol{k})\rangle = \delta_{ij}/(\zeta\boldsymbol{k}^2 + h). \tag{2.9}$$

That is, they have the form given by Eq. (2.2a), while the $\delta\rho$ propagator is massive. Explicit perturbative calculations (Nelson and Pelcovits, 1977) show that this does not change within the framework of a loop expansion. It is also interesting to consider the explicit perturbation theory for the magnetization $m = M\langle\sigma(\boldsymbol{x})\rangle$. To one-loop order, Eq. (2.6b) yields

$$m/M = 1 - \frac{1}{2}\int_{\boldsymbol{p}}\langle\pi_i(\boldsymbol{p})\pi_j(-\boldsymbol{p})\rangle = m(h=0)/M + \mathrm{const}h^{(d-2)/2}. \tag{2.10}$$

Here $\int_{\boldsymbol{p}} = \int d\boldsymbol{p}/(2\pi)^d$, and we show only the leading nonanalytic dependence of m on h.

c. Generic scale invariance from RG arguments

The next question is whether these perturbative results are generic, and valid independent of perturbation theory. Within a perturbative RG (Brézin and Zinn-Justin, 1976; Nelson and Pelcovits, 1977) this can be checked order by order in a loop expansion. The fact that Eqs. (2.9) and (2.10) are valid to all orders in perturbation theory, and are indeed exact properties independent of any perturbative scheme, hinges on the proof of the renormalizability of the nonlinear σ model (Brézin *et al.*, 1976).

Here we present a much simpler, if less complete, argument based on power counting (Belitz and Kirkpatrick, 1997). We employ the concept of Ma (1976), whereby one postulates a fixed point and then self-consistently checks its stability. Accordingly, we assign a scale dimension $[L] = -1$ to lengths L and look for a fixed point where the fields have scale dimensions

$$[\pi_i(\boldsymbol{x})] = (d-2)/2, \tag{2.11a}$$

$$[\delta\rho(\boldsymbol{x})] = d/2 \tag{2.11b}$$

in d dimensions. This ansatz is motivated by the expectation that the Gaussian approximation for the π correlation, Eq. (2.9), is indeed exact and that $\delta\rho$ is massive. Power counting shows that ζ and r are marginal, while all other coupling constants are irrelevant except for h, which is relevant with $[h] = 2$. In particular, all terms in δS that do not depend on h are irrelevant with respect to the putative fixed point, and so is the term $(\boldsymbol{\nabla}\sigma)^2$ in $S_{\mathrm{NL}\sigma\mathrm{M}}$. The fixed point is thus stable and describes the ordered phase. The fixed-point action plus the most relevant external-field term is Gaussian,

$$S_{\mathrm{FP}} = \frac{\zeta}{2}\int d\boldsymbol{x}(\boldsymbol{\nabla}\pi(\boldsymbol{x}))^2 + \frac{h}{2}\int d\boldsymbol{x}(\boldsymbol{\pi}(\boldsymbol{x}))^2 + r\int d\boldsymbol{x}(\delta\rho(\boldsymbol{x}))^2. \tag{2.12}$$

We see that the π-π correlation function for asymptotically small wave numbers is indeed given by Eq. (2.9), i.e., one has soft Goldstone modes, or GSI, everywhere in the ordered phase. Notice that the above RG arguments *prove* this statement, although the proof is not rigorous in a mathematical sense. Furthermore, the fixed-point value of the magnetization, $m = M\langle\sigma(\boldsymbol{x})\rangle$, is given by M, and the leading correction is given by the

π-π correlation function.[19] The scale dimensions of $\boldsymbol{\pi}$ and h then yield

$$m(h) = m(h=0) + \text{const} \times h^{(d-2)/2}, \tag{2.13a}$$

in agreement with the perturbative result, Eq. (2.10). This in turn implies that the longitudinal susceptibility

$$\chi_L = \partial m/\partial h \propto h^{(d-4)/2} \tag{2.13b}$$

diverges in the limit $h \to 0$ for all $d < 4$ (Brézin and Wallace, 1973). Alternatively, the zero-field susceptibility diverges in the homogeneous limit as

$$\chi_L(\boldsymbol{k} \to 0) \propto |\boldsymbol{k}|^{-(4-d)}. \tag{2.13c}$$

In real space this corresponds to a decay as $1/|\boldsymbol{x}|^{2(d-2)}$.

The above arguments assume that no structurally new terms are generated under renormalization which might turn out to be marginal or relevant; this is one of the reasons why the proof is not rigorous. With this caveat, they show that several properties of isotropic Heisenberg ferromagnets that are readily obtained by means of perturbation theory are indeed exact. They also illustrate how much more information can be extracted from simple power counting after performing a symmetry analysis and separating the soft and massive modes, as opposed to doing power counting on the original action, Eq. (2.3) (Ma, 1976). More importantly for our present purposes, they illustrate how soft modes in the presence of nonlinearities lead to slow decay of generic correlation functions. In the present case, the transverse Goldstone modes couple to the longitudinal fluctuations, which leads to the long-range correlations expressed in Eq. (2.13c). This is an example of indirect GSI due to mode-mode coupling effects. In Sec. II.A.3 we shall see that a very similar mechanism leads to long-ranged time correlation functions in classical fluids.

2. U(1) gauge symmetry

The second mechanism on our list is generic scale invariance caused by a gauge symmetry. Let us consider ϕ^4 theory again, Eq. (2.3), with $N=2$ or, equivalently, with a complex scalar field ϕ. Let us further require that the theory be invariant under *local* U(1) gauge transformations,

$$\phi(\boldsymbol{x}) \to \phi'(\boldsymbol{x}) = e^{i\Lambda(\boldsymbol{x})}\phi(\boldsymbol{x}), \tag{2.14a}$$

with an arbitrary real field Λ. It is well known (see, for example, Ryder, 1985) that this requirement forces the introduction of a gauge field $\boldsymbol{A}$ with components A_α ($\alpha = 1,2,3$ in three dimensions) that transforms as

$$\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}'(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \frac{1}{q}\boldsymbol{\nabla}\Lambda(\boldsymbol{x}), \tag{2.14b}$$

and a new action

$$S[\phi,\boldsymbol{A}] = \int d\boldsymbol{x}\Big(r|\phi(\boldsymbol{x})|^2 + c|[\boldsymbol{\nabla} - iq\boldsymbol{A}(\boldsymbol{x})]\phi(\boldsymbol{x})|^2 + u|\phi(\boldsymbol{x})|^4 + \frac{1}{16\pi\mu}F_{\alpha\beta}(\boldsymbol{x})F^{\alpha\beta}(\boldsymbol{x})\Big), \tag{2.15a}$$

where

$$F_{\alpha\beta}(\boldsymbol{x}) = \partial_\alpha A_\beta(\boldsymbol{x}) - \partial_\beta A_\alpha(\boldsymbol{x}). \tag{2.15b}$$

Here μ and q are coupling constants that characterize the gauge field $\boldsymbol{A}$ and its coupling to ϕ, respectively. The usual interpretation of this action is that of a charged particle, or excitation, with charge q, described by ϕ, that couples to electromagnetic fluctuations, or photons, described by the vector potential $\boldsymbol{A}$. We shall see in Sec. III.A.1, however, that it can describe other systems as well, at least within certain limits.

a. Symmetric phase

Let us first discuss the symmetric phase, in which both fields have zero expectation values. In this phase, we expect the Gaussian $\boldsymbol{A}$ propagator to be massless, as the $\boldsymbol{A}$ field appears only in conjunction with derivatives. This is a consequence of the local gauge invariance; a term of the form $m^2\boldsymbol{A}^2(\boldsymbol{x})$ would violate the latter. However, the action S, Eq. (2.15a), comes with the usual problems related to gauge fields. That is, the $\boldsymbol{A}$ propagator does not exist, since the $\boldsymbol{A}$ vertex has a zero eigenvalue. We deal with this problem by *gauge fixing*, i.e., we work in Coulomb gauge, $\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x}) = 0$, which we enforce by adding to the action a gauge-fixing term (Ryder, 1985, Chap. 7.1)

$$S_{GF} = \frac{1}{\eta}\int d\boldsymbol{x}(\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x}))^2, \tag{2.16}$$

with $\eta \to 0$. One finds from Eqs. (2.15a) and (2.16)

$$\langle A_\alpha(\boldsymbol{k})A_\beta(-\boldsymbol{k})\rangle = 4\pi\mu\frac{\delta_{\alpha\beta} - \hat{k}_\alpha\hat{k}_\beta}{\boldsymbol{k}^2}, \tag{2.17a}$$

so the photon is indeed soft. Local gauge invariance thus leads to GSI in an obvious way; this is another example of direct GSI. Notice that this mechanism is distinct from the one related to conservation laws, to be discussed in Sec. II.A.3 below: While local gauge invariance is sufficient for the conservation of the charge q, it is not necessary; a *global* U(1) suffices to make q a conserved quantity.

The ϕ field, however, has two massive components with equal masses. Writing $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, with ϕ_1 and ϕ_2 real, we have

$$\langle\phi_i(\boldsymbol{k})\phi_j(-\boldsymbol{k})\rangle = \delta_{ij}/(r + c\,\boldsymbol{k}^2) \quad (i,j = 1,2). \tag{2.17b}$$

We thus have two massive scalar fields and one massless vector field with two degrees of freedom.[20]

[19] A detailed analysis shows that δS does not contribute to the leading corrections to scaling at the stable fixed point.

[20] The vector field, or photon, has only two degrees of freedom, rather than three. In the gauge we have chosen this is obvious, since the propagator, Eq. (2.17a), is purely transverse.

b. The broken-symmetry phase

We now turn to the phase in which the local U(1) symmetry is spontaneously broken. Suppose the spontaneous expectation value of ϕ is in the ϕ_1 direction, so we have

$$\phi(\boldsymbol{x}) = v + [\phi_1(\boldsymbol{x}) + i\phi_2(\boldsymbol{x})]/\sqrt{2},$$

with v real and equal to $\sqrt{-r/2u}$ at tree level. If we expand about the saddle-point solution $\phi = v$, $\boldsymbol{A} = 0$, we find to Gaussian order

$$S = \int d\boldsymbol{x}\Bigg[2uv^2(\phi_1(\boldsymbol{x}))^2 + \frac{c}{2}(\boldsymbol{\nabla}\phi_1(\boldsymbol{x}))^2 + cq^2v^2(\boldsymbol{A}(\boldsymbol{x}))^2 + \frac{1}{16\pi\mu}F_{\alpha\beta}(\boldsymbol{x})F^{\alpha\beta}(\boldsymbol{x}) + \frac{c}{2}(\boldsymbol{\nabla}\phi_2(\boldsymbol{x}))^2 - \sqrt{2}c\,q\,v\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{\nabla}\phi_2(\boldsymbol{x})\Bigg]. \tag{2.18}$$

There are two interesting aspects of this action. First, the vector field has acquired a mass that is proportional to v^2. Second, the field ϕ_2, which is often called the Higgs field in this context, and which we would expect to form the Goldstone mode associated with spontaneously broken U(1) or O(2) symmetry, couples to the now-massive vector field. Indeed, ϕ_2 can be eliminated from the Gaussian action by shifting $\boldsymbol{A}$, $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{\nabla}\phi_2(\boldsymbol{x})/\sqrt{2}qv$. Notice that this shift just amounts to a change of gauge and hence does not change the physical nature of the vector field.[21] It does, however, give $\boldsymbol{A}$ a longitudinal component and thus increases the number of photon degrees of freedom from two to three. With this shift, the Gaussian action consists of only the first four terms in Eq. (2.18), which leads to the following propagators:

$$\langle A_\alpha(\boldsymbol{k})A_\beta(-\boldsymbol{k})\rangle = 4\pi\mu\frac{\delta_{\alpha\beta} + k_\alpha k_\beta/m^2}{m^2 + \boldsymbol{k}^2}, \tag{2.19a}$$

$$\langle \phi_1(\boldsymbol{k})\phi_1(-\boldsymbol{k})\rangle = \frac{1}{4uv^2 + c\boldsymbol{k}^2}, \tag{2.19b}$$

with $m^2 = 4\pi\mu c q^2 v^2$.

We see that now there are two massive fields, namely, a massive scalar field with one degree of freedom and a massive vector field with three degrees of freedom. In particular, there is *no* Goldstone mode, despite the spontaneously broken O(2) symmetry. Pictorially speaking, the gauge field has eaten the Goldstone mode and has become massive in the process. This phenomenon is commonly referred to as the *Higgs mechanism* (Anderson, 1963; Higgs, 1964a, 1964b). Notice that the situation is complementary, in a well-defined sense, to the case without a gauge field in Sec. II.A.1: Without local gauge invariance, one has two massive modes in the disordered phase and one massive mode and one soft Goldstone mode in the ordered phase. In the gauge theory, there is a massless mode (the photon) in the disordered phase, and no massless modes in the ordered phase. While there are no obvious examples of indirect GSI involving photons, we shall see in Sec. III.A.1 that the nature of the mass acquired by ϕ_1 in the ordered phase can have a drastic influence on the nature of the phase transition.

We close this subsection with one additional remark. In particle physics, local gauge invariance is an indispensable requirement, since it is necessary for Lorentz invariance. In the theory of phase transitions there is no such requirement, since the critical theory is rigorously nonrelativistic (see footnote 15). Nevertheless, it is important if an order-parameter field couples to photons, and also in some other cases. We shall elaborate on this in Sec. III.

3. Long-time tails

We now turn to the third mechanism on our list, namely, long-ranged correlations in time correlation functions due to conservation laws. Let us consider time correlation functions involving the local currents of conserved quantities, e.g., mass, momentum, or energy. Time integrals over the spatial averages of these correlation functions, referred to as *Green-Kubo expressions* (Green, 1954; Kubo, 1957, 1959), determine the transport coefficients of fluids. A simple example is the velocity autocorrelation function of a tagged particle. It is defined as

$$C_D(t) = \langle \boldsymbol{v}(t)\cdot\boldsymbol{v}(0)\rangle_{\rm eq}, \tag{2.20a}$$

where $\boldsymbol{v}(0)$ is the initial velocity of the tagged particle, $\boldsymbol{v}(t)$ its velocity at a later time t, and $\langle\cdots\rangle_{\rm eq}$ denotes an equilibrium ensemble average. The coefficient of self-diffusion in a d-dimensional system is then given by (see, for example, Boon and Yip, 1991)

$$D = \frac{1}{d}\int_0^\infty dt C_D(t). \tag{2.20b}$$

Analogous expressions determine the coefficients of shear viscosity η, bulk viscosity ζ, and heat conductivity λ, in a classical fluid (and, when appropriate, in a classical solid). In general we denote these current correlation functions by $C_\mu(t)$, where μ can stand for D, η, ζ, or λ.

In traditional many-body theories (e.g., the Boltzmann equation), the C_μ were always found to decay exponentially in time (Chapman and Cowling, 1952; Dorfman and van Beijeren, 1977; Cercignani, 1988), with a characteristic decay time on the order of the mean free time between collisions, and until the mid-1960s this was believed to be generally true. It thus came as a great

[21] By writing ϕ in terms of an amplitude and a phase, one can choose a gauge such that the entire action, not just the Gaussian part, is independent of ϕ_2. This is known as the "physical" or "unitary" gauge (Ryder, 1985, Chap. 8.3). Other choices, which retain ϕ_2 and lead to a different $\boldsymbol{A}$ propagator, are possible. It has been shown that all these formulations are indeed physically equivalent; the contributions from any nonzero ϕ_2 propagator cancel against pieces of the $\boldsymbol{A}$ propagator and thus do not contribute to any observable properties. See Ryder (1985), Weinberg (1996b), and Zinn-Justin (1996) for detailed discussions of this point.

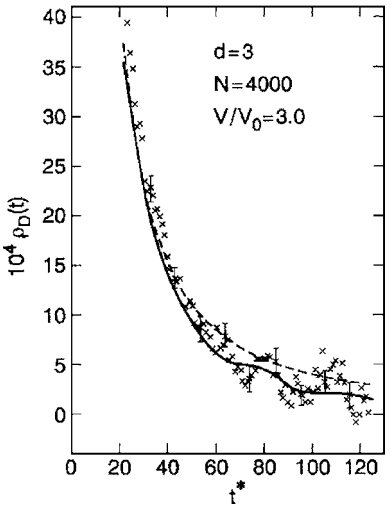

FIG. 3. Normalized velocity autocorrelation function $\rho_D(t) = C_D(t)/\langle \boldsymbol{v}^2(0)\rangle$ as a function of the dimension-less time $t^* = t/t_0$, where t_0 is the mean free time. The $\times$'s indicate computer results obtained by Wood and Erpenbeck (1975) for a system of 4000 hard spheres at a reduced density corresponding to $V/V_0=3$, where V is the actual volume and V_0 is the close-packing volume. The dashed curve represents the theoretical curve $\rho_D(t)=\alpha_D(t^*)^{-3/2}$. The solid curve represents a more complete evaluation of the mode-coupling formula with contributions from all possible hydrodynamic modes and with finite-size corrections included (Dorfman, 1981). From Dorfman *et al.*, 1994.

surprise when both numerical molecular-dynamics studies (Alder and Wainwright, 1967, 1968, 1970), and, shortly thereafter, more sophisticated theories (Dorfman and Cohen, 1970, 1972, 1975; Ernst *et al.*, 1970, 1971, 1976a, 1976b) showed that all of these correlations decay for asymptotically long times as $1/t^{d/2}$. This power-law decay in time is a type of GSI that is referred to as a *long-time tail*. In Fig. 3 we show results of both computer simulations and theoretical calculations for $C_D(t)$ for a hard-sphere fluid in three dimensions. The long-time tail is clearly visible.

a. Fluctuating hydrodynamics

The simplest and most general way to understand the long-time-tail mechanism in a classical fluid is via the equations of fluctuating hydrodynamics (Ernst *et al.*, 1971), i.e., the Navier-Stokes equations with appropriate Langevin forces added, although explicit many-body calculations, within the framework of a generalized kinetic theory, are also possible and give identical results for the long-time tails (Dorfman and Cohen, 1972, 1975).[22] The hydrodynamic approach gives the exact long-time behavior because the slowest-decaying fluctuations in a classical fluid are the fluctuations of the conserved variables, which are described by the hydrodynamic equations. The conserved variables are the mass density ρ, the momentum density $\boldsymbol{g}=\rho\boldsymbol{u}$, with $\boldsymbol{u}$ the fluid velocity, and the energy density ϵ or, alternatively, the entropy density s. In the long-wavelength and low-frequency limit the fluctuations of these variables are described exactly by the equations (Landau and Lifshitz, 1987)

$$\partial_t \rho + \partial_\alpha g^\alpha = 0, \tag{2.21a}$$

$$\partial_t g_\alpha + \partial_\beta g_\alpha u^\beta = -\partial_\alpha p + \partial_\beta \Big[\eta(\partial_\alpha u^\beta + \partial^\beta u_\alpha) + \Big(\zeta - \frac{(d-1)}{d}\eta\Big)\delta_{\alpha\beta}\partial_\gamma \mu^\gamma + P_{\alpha\beta}\Big], \tag{2.21b}$$

$$\rho T(\partial_t + u_\alpha \partial^\alpha)s = \partial_\alpha(\lambda \partial^\alpha T + q^\alpha). \tag{2.21c}$$

Here p denotes the pressure, and summation over repeated indices is implied. In Eq. (2.21c) we have neglected a viscous dissipation term that represents entropy production, since it is irrelevant to the leading long-time tails. The Langevin forces $P_{\alpha\beta}$ and q_α are uncorrelated with the initial hydrodynamic variables, and satisfy

$$\langle P_{\alpha\beta}(\boldsymbol{x},t)P_{\mu\nu}(\boldsymbol{x}',t')\rangle = 2T\Big[\eta(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}) + \Big(\zeta - \frac{(d-1)}{d}\eta\Big)\delta_{\alpha\beta}\delta_{\mu\nu}\Big] \times \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \tag{2.22a}$$

$$\langle q_\alpha(\boldsymbol{x},t)q_\beta(\boldsymbol{x}',t')\rangle = 2\lambda\, T^2\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \tag{2.22b}$$

$$\langle P_{\alpha\beta}(\boldsymbol{x},t)q_\mu(\boldsymbol{x}',t')\rangle = 0. \tag{2.22c}$$

The above equations can be derived in a number of ways (Fox and Uhlenbeck, 1970; Landau and Lifshitz, 1987) and are known to exactly describe the long-wavelength and low-frequency fluctuations in a fluid.

b. Generic scale invariance from explicit calculations

To illustrate how the long-time tails arise we choose a slightly different example than the self-diffusion coefficient discussed above, namely, the shear viscosity η. The appropriate time correlation function that enters the Green-Kubo formula in this case is the autocorrelation of the transverse velocity current, or the stress tensor (Ernst *et al.*, 1971). The basic idea is that η in the above equations is really a bare transport coefficient that gets renormalized by the nonlinearities and fluctuations. We shall present a simplified calculation of η (Kirkpatrick *et al.*, 2002) and then discuss more general results.

For simplicity, we assume an incompressible fluid. The mass conservation law then reduces to the condition

[22]Notice that the term "kinetic theory," which is often used synonymously with "Boltzmann theory," is used much more generally by these authors.

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x},t) = 0, \tag{2.23}$$

and the momentum conservation law is a closed partial differential equation for $\boldsymbol{u}$,

$$\partial_t u_\alpha + u_\beta \partial^\beta u_\alpha = -\partial_\alpha p/\rho + \gamma\nu\partial_\beta\partial^\beta u_\alpha + \partial_\beta P_{\alpha\beta}/\rho. \tag{2.24}$$

Here $\nu = \eta/\rho$ is the kinematic viscosity, which we assume to be constant. The pressure term serves only to enforce the condition of incompressibility. In fact, it can be eliminated by taking the curl of Eq. (2.24), which turns it into an equation for the transverse velocity. The cause of the long-term tails is the coupling of slow hydrodynamic modes due to the nonlinear term in Eq. (2.24); this is another example of the mode-mode coupling effects mentioned in the introduction to the current section. Since we shall treat this term as a perturbation, we have formally multiplied it by a coupling constant γ whose physical value is unity. We shall take the nonlinearity into account to lowest nontrivial order in γ.

Consider the velocity autocorrelation function tensor,

$$C_{\alpha\beta}(\boldsymbol{k},t) = \langle u_\alpha(\boldsymbol{k},t) u_\beta(-\boldsymbol{k},0)\rangle. \tag{2.25}$$

An equation for C can be obtained by Fourier transforming Eq. (2.24), multiplying by $u_\beta(-\boldsymbol{k},0)$, and averaging over the noise while keeping in mind that the noise is uncorrelated with the initial fluid velocity. In the case of an incompressible fluid we need consider only the transverse-velocity correlation function $C_\perp$. This is easily done by multiplying with unit vectors, $\hat{\boldsymbol{k}}_\perp^{(i)}$ ($i = 1,2,\ldots,d-1$), that are perpendicular to $\boldsymbol{k}$, which eliminates the pressure term. We obtain

$$(\partial_t + \nu\boldsymbol{k}^2) C_\perp(\boldsymbol{k},t) = -i\gamma k_\mu \hat{k}_{\perp\alpha}^{(i)} \hat{k}_{\perp\beta}^{(i)} \times \sum_{\boldsymbol{q}} \langle u^\mu(\boldsymbol{k}-\boldsymbol{q},t) u^\alpha(\boldsymbol{q},t) u^\beta(-\boldsymbol{k},0)\rangle. \tag{2.26}$$

Here we have used the incompressibility condition, Eq. (2.23), to write all gradients as external ones. To zeroth order in γ we find, with the help of the f-sum rule (Forster, 1975), $C_\perp(\boldsymbol{k},0) = T/\rho$,

$$C_\perp(\boldsymbol{k},t) = (T/\rho) e^{-\nu\boldsymbol{k}^2 t} + O(\gamma). \tag{2.27a}$$

This is the standard result obtained from linearized hydrodynamics (Chapman and Cowling, 1952), which predicts exponential decay for $\boldsymbol{k} \neq 0$.[23] Notice that it amounts to GSI in time space for the local time correlation function,

$$C_\perp(\boldsymbol{x}=0,t) \propto 1/t^{d/2}. \tag{2.27b}$$

This is an immediate consequence of the conservation law for the transverse momentum, and hence an example of direct GSI.

To calculate corrections due to the nonlinearity, we need an equation for the three-point correlation function in Eq. (2.26). The simplest way to obtain this is to use the time translational invariance properties of this correlation function to put the time dependence in the last velocity, and then use Eq. (2.24) again. The result is an equation for the three-point function in terms of a four-point one that is analogous to Eq. (2.26). To solve this equation, we note that, due to the velocity's being odd under time reversal, the equal-time three-point correlation vanishes. By means of a Laplace transform, one can therefore express the three-point function as a product (in frequency space) of the zeroth-order result for $C_\perp$, Eq. (2.27a), and the four-point function. To leading (i.e., zeroth) order in γ the latter factorizes into products of velocity autocorrelation functions. Upon transforming back into time space, and to quadratic order in the coupling constant γ, we obtain

$$(\partial_t + \nu\boldsymbol{k}^2) C_\perp(\boldsymbol{k},t) + \int_0^t d\tau\, \Sigma(\boldsymbol{k},t-\tau) C_\perp(\boldsymbol{k},\tau) = 0, \tag{2.28a}$$

with

$$\Sigma(\boldsymbol{k},t) = \gamma^2 \frac{\rho}{T} k_\mu k_\nu \sum_{\boldsymbol{q}} [C^{\alpha\beta}(\boldsymbol{q},t) C^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},t) + C^{\alpha\nu}(\boldsymbol{q},t) C^{\mu\beta}(\boldsymbol{k}-\boldsymbol{q},t)] \hat{k}_{\perp,\alpha}^{(i)} \hat{k}_{\perp,\beta}^{(i)} + O(\gamma^3). \tag{2.28b}$$

The self-energy Σ is proportional to $\boldsymbol{k}^2$ and thus provides a renormalization of the bare viscosity ν, or the time correlation function that determines the shear viscosity, $C_\eta(t)$. This correction to ν is time and wave number dependent.

The source of the long-time tails is now evident. In our model of an incompressible fluid, only the transverse component of $C_{\alpha\beta}$ is nonzero,

$$C_{\alpha\beta}(\boldsymbol{q},t) = (\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta) C_\perp(\boldsymbol{q},t). \tag{2.29}$$

Putting $\gamma = 1$, and defining $\delta C_\eta(t) = \rho \lim_{\boldsymbol{k}\to 0} \Sigma(\boldsymbol{k},t)/\boldsymbol{k}^2$, one obtains for asymptotically long times

$$\delta C_\eta(t) = T\rho \left(\frac{d^2-2}{d(d+2)}\right) \frac{1}{(8\pi\nu t)^{d/2}}. \tag{2.30a}$$

The correction to the static, or zero-frequency, shear viscosity is given by $\delta\eta \propto \int_0^\infty dt\ \delta C_\eta(t)$ [cf. Eq. (2.20b)]. We thus have

$$C_\eta(t\to\infty) = \delta C_\eta(t\to\infty) \propto t^{-d/2}. \tag{2.30b}$$

This is the well-known contribution of the transverse velocity modes to the long-time tail of the viscosity (Ernst *et al.*, 1976a, 1976b). Notice that C_η describes correlations of shear stress, which is *not* conserved. The long-time tail therefore is a result of mode-mode coupling effects and hence an example of indirect GSI. In a compressible fluid, a similar process coupling two longitudinal modes also contributes to the leading long-time tail. The other transport coefficients, e.g., λ and ζ, also have long-time tails proportional to $1/t^{d/2}$, and all of

[23] The reader might find it curious that this derivation did not make explicit use of the Langevin force correlations, Eqs. (2.22). However, it needs various equal-time correlation functions as input, which contain the same information as Eqs. (2.22).

them have less-leading long-time tails proportional to $1/t^{(d+1)/2}$ or weaker (Ernst *et al.*, 1976a, 1976b; Dorfman, 1981).

For the frequency-dependent kinematic viscosity, the nonexponential decay of $\delta\nu(t)$ implies a nonanalyticity at zero frequency. More generally, the algebraic $1/t^{d/2}$ long-time tails in the long-time limit imply, for the frequency or wave-number-dependent transport coefficients μ, a nonanalyticity at zero frequency Ω, or wave number $|\boldsymbol{k}|$,[24]

$$\mu(\Omega)/\mu(0) = 1 - c_d^{\mu}\Omega^{(d-2)/2}, \tag{2.31a}$$

$$\mu(\boldsymbol{k})/\mu(0) = 1 - b_d^{\mu}|\boldsymbol{k}|^{d-2}, \tag{2.31b}$$

where the prefactors c_d^{μ} and b_d^{μ} are positive, and only the leading nonanalyticities are shown. For the implications of these results in $d \leqslant 2$, see footnote 49 below.

4. Spatial correlations in nonequilibrium steady states

Finally, we consider a fluid in a nonequilibrium steady state; the equations of fluctuating hydrodynamics can be extended to this case (Ronis *et al.*, 1980). It has been known for some time that these systems in general exhibit generic scale invariance in both time correlation functions *and* thermodynamic quantities (see, for example, Dorfman *et al.*, 1994). The spatial correlations responsible for the GSI in thermodynamic susceptibilities are closely related to long-time tails of the equilibrium time correlation functions. We shall consider a fluid in a steady, spatially uniform, temperature gradient $\boldsymbol{\nabla} T$, but far from any convective instability. Further, we use a number of approximations that enable us to focus on the most interesting effects of such a gradient. For a justification of this procedure, as well as the underlying details, we refer the reader to the original literature (Kirkpatrick *et al.*, 1982a, 1982b).

We write the temperature $T = T_0 + \delta T$ as fluctuations δT about an average value T_0 and focus on the coupling between fluctuations of the transverse fluid velocity $\boldsymbol{u}_{\perp}$ and δT. If we neglect the nonlinearity in Eq. (2.21b), Eqs. (2.21) can be written

$$\partial_t u_{\perp,\alpha} = \nu\partial_\beta\partial^\beta u_{\perp,\alpha} + \frac{1}{\rho_0}(\partial^\beta P_{\alpha\beta})_{\perp}, \tag{2.32a}$$

$$\partial_t \delta T + u_\alpha \partial^\alpha T = D_T \partial_\alpha\partial^\alpha \delta T + \frac{1}{\rho_0 T_0 c_p}\partial_\alpha q^\alpha, \tag{2.32b}$$

where ρ_0 is the average mass density, c_p is the specific heat per mass at constant pressure, and $D_T = \lambda/\rho_0 c_p$ is the thermal diffusivity. These bilinear equations can be solved by means of Fourier and Laplace transformations. Focusing on static, or equal-time, correlations, one finds, for example,

$$\langle|\delta\rho(\boldsymbol{k})|^2\rangle = \rho T\left(\frac{\partial\rho}{\partial p}\right)_T + \frac{\rho T(\alpha_T \hat{\boldsymbol{k}}_{\perp}\cdot\boldsymbol{\nabla} T)^2}{D_T(\nu + D_T)\boldsymbol{k}^4}, \tag{2.33a}$$

$$\langle[\hat{\boldsymbol{k}}_{\perp}\cdot\boldsymbol{g}(\boldsymbol{k})]\delta\rho(-\boldsymbol{k})\rangle = \rho T\alpha_T\frac{\hat{\boldsymbol{k}}_{\perp}\cdot\boldsymbol{\nabla} T}{(\nu + D_T)\boldsymbol{k}^2}, \tag{2.33b}$$

where

$$\alpha_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p \tag{2.33c}$$

is the thermal expansion coefficient at constant pressure.

There are several remarkable aspects of these results. First, Eq. (2.33a) for the density correlations implies that the first term, which also exists in equilibrium, is delta correlated in real space, while the second term decays as const$-|\boldsymbol{x}|$ in three dimensions (Schmitz and Cohen, 1985; de Zarate *et al.*, 2001). Equation (2.33b) shows that the transverse-momentum–density correlation function decays as $1/|\boldsymbol{x}|$ in three dimensions. Both of these results show that spatial correlations in a nonequilibrium steady state exhibit GSI. Second, the right-hand side of Eq. (2.33b) is essentially the integrand of a long-time-tail contribution to the heat conductivity λ (Hohenberg and Halperin, 1977). This demonstrates the close connection between the long-time tails in equilibrium time correlation functions and the spatial GSI of equal-time correlation functions in nonequilibrium situations. As in the case of the equilibrium long-time tails, this is an example of indirect GSI.

These fluctuations can be directly measured by small-angle light-scattering experiments. Specifically, the dynamic structure factor $S_{\rho\rho}(\boldsymbol{k},t) = \langle\delta\rho(\boldsymbol{k},t)\,\delta\rho(-\boldsymbol{k},0)\rangle$, which is proportional to the scattering cross section, in a nonequilibrium steady state has the form (Dorfman *et al.*, 1994)

$$S_{\rho\rho}(\boldsymbol{k},t) = S_0[(1 + A_T)e^{-D_T\boldsymbol{k}^2 t} - A_\nu e^{-\nu\boldsymbol{k}^2 t}], \tag{2.34a}$$

where S_0 is the structure factor in equilibrium, and

$$A_T = \frac{c_p\nu}{TD_T(\nu^2 - D_T^2)}\frac{(\hat{\boldsymbol{k}}_{\perp}\cdot\boldsymbol{\nabla} T)^2}{\boldsymbol{k}^4}, \tag{2.34b}$$

$$A_\nu = \frac{c_p}{T(\nu^2 - D_T^2)}\frac{(\hat{\boldsymbol{k}}_{\perp}\cdot\boldsymbol{\nabla} T)}{\boldsymbol{k}^4}. \tag{2.34c}$$

For $t = 0$ one recovers the equal-time density correlation function, Eq. (2.33a). The amplitudes A_T and A_ν are proportional to $(\boldsymbol{\nabla} T)^2/\boldsymbol{k}^4$, which has been verified by experiments (see Fig. 4). Notice that the amplitude of the temperature fluctuations is enhanced by a factor of a hundred compared to the scattering by an equilibrium fluid. In real space, this strongly singular wave-number dependence corresponds to a decay as const$-|\boldsymbol{x}|$ in three dimensions.

[24]To avoid misunderstandings, we note that a static transport coefficient, $\mu(\boldsymbol{k},\Omega=0)$, is a time integral over a time correlation function; see the Green-Kubo formula, Eq. (2.20b). As such, it is long ranged in space, just as $\mu(\boldsymbol{k}=0,\Omega)$ is in time. A static susceptibility, however, is related to an *equal-time* correlation function via the fluctuation-dissipation theorem (Forster, 1975) and is *not* long ranged in space in classical equilibrium systems.

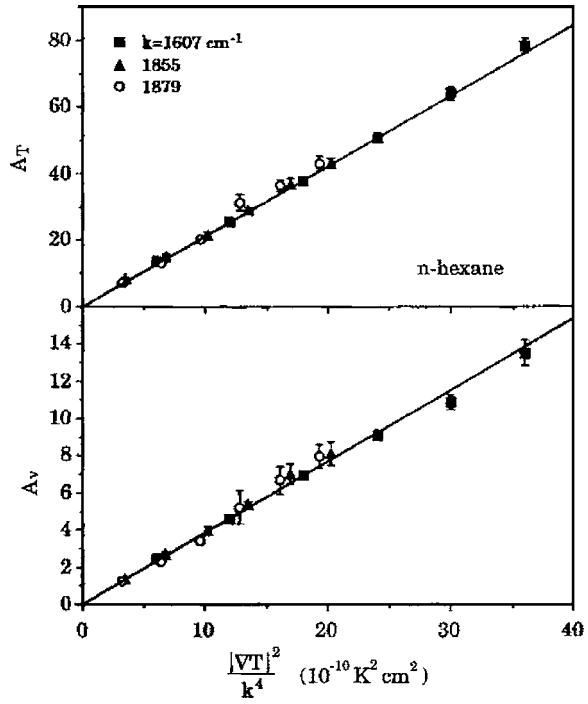

FIG. 4. Amplitudes A_T and A_v of the nonequilibrium temperature and transverse-momentum (viscous) fluctuations in liquid hexane at 25 °C as a function of $(\nabla T)^2/\boldsymbol{k}^4$. The symbols indicate experimental data. The solid lines represent the values predicted by Eqs. (2.34). Notice the excellent agreement with no adjustable parameters. From Li *et al.*, 1994.

B. Quantum systems

The same mechanisms discussed above for classical systems also lead to long-range correlations in space and time in quantum systems. The chief distinctions between quantum systems (most interesting at $T=0$) and classical systems are as follows.

(1) There are more soft, or gapless, modes in quantum systems. These extra modes are due to a Goldstone mechanism that is absent at $T\neq 0$. For example, particle-hole excitations across a Fermi surface are soft at $T=0$ but acquire a mass at $T\neq 0$.
(2) As discussed in Sec. I.C, there is a coupling between the statics and the dynamics in quantum statistical mechanics that is absent in the classical theory. In general, this means that long-ranged correlations in time correlation functions imply long-ranged spatial correlations in equal-time correlation functions, and thus in thermodynamic susceptibilities.

In this subsection we explain these points in some detail and lay the foundation for later sections in which we discuss the importance of generic scale invariance for understanding many quantum phase transitions. Our focus will be on itinerant interacting electron systems. To motivate our discussions and to make contact with the classical results discussed in the preceding section, we first give some of the results showing GSI in zero- or very-low-temperature electron systems, both disordered and clean.[25] We choose to discuss the disordered case first, because in this case the nonanalyticities that reflect the GSI are well known, although they usually have not been thought of as examples of GSI. In noninteracting electron systems they are usually referred to as "weak-localization effects,"[26] and in interacting ones as "interaction effects" or "Altshuler-Aronov effects"; we shall collectively refer to both classes as "GSI effects."[27] The fact that there are analogous effects in clean electron systems is much less well known. We then discuss the most important soft modes in electron fluids, paying particular attention to those that exist only at $T=0$. This is followed by an account of two distinct approaches to GSI in quantum systems: The first approach is via explicit many-body calculations, which is analogous to a generalized kinetic-theory approach to the classical fluid. The second approach relies on the concept of clean and disordered Fermi-liquid fixed points and uses very general RG arguments. This is analogous to the treatment of the classical Heisenberg ferromagnet discussed in Sec. II.A.1.c.

1. Examples of generic scale invariance in itinerant electron systems

a. Disordered systems in equilibrium

In the entire metallic phase of disordered interacting electron systems, various transport coefficients and thermodynamic quantities show nonanalytic frequency and wave-number dependencies that represent generic scale invariance. Since we are dealing with a quantum system, there also are corresponding nonanalytic temperature dependencies. For instance, for $2<d<4$, the electrical conductivity σ, the specific heat coefficient $\gamma=C_V/T$, and the static spin susceptibility χ_s, as functions of the wave vector $\boldsymbol{k}$, the frequency Ω, and the temperature T, are given by (for reviews, see Altshuler and Aronov, 1984; Lee and Ramakrishnan, 1985)

$$\sigma(\Omega\to 0,T=0)/\sigma(0,0)=1+c_d^{\sigma}\Omega^{(d-2)/2}, \tag{2.35a}$$

$$\sigma(\Omega=0,T\to 0)/\sigma(0,0)=1+\tilde{c}_d^{\sigma}T^{(d-2)/2}, \tag{2.35b}$$

$$\gamma(T\to 0)/\gamma(0)=1+c_d^{\gamma}T^{(d-2)/2}, \tag{2.35c}$$

[25] We shall refer to systems with and without quenched disorder as *disordered* and *clean*, respectively.

[26] The term "weak localization" is ill defined and is differently applied by different authors. The most restrictive meaning refers to the weakly nonmetallic weak-disorder regime in $d=2$, but we use it to denote the nonanalytic dependence of observables on the frequency or the wave number in noninteracting disordered electron systems in any dimension.

[27] Occasionally both classes have been collectively called "weak-localization effects" (as in Kirkpatrick *et al.*, 2002). Since most people associate weak localization with quenched disorder, at least to some degree, this use of the term tends to obscure the fact that precisely analogous effects are found in clean systems.

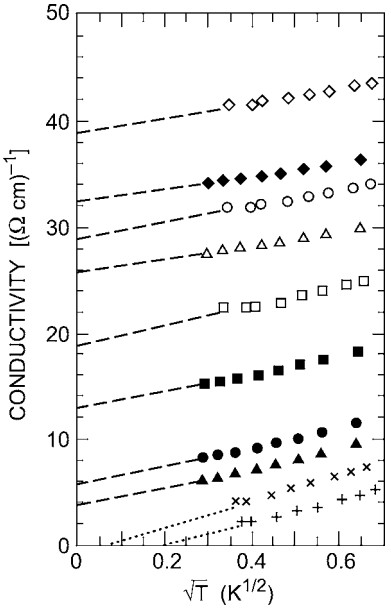

FIG. 5. Static low-T conductivity of nine Si:B samples in a magnetic field plotted vs $\sqrt{T}$. Adapted from Dai *et al.*, 1992.

$$\chi_s(\boldsymbol{k}\to 0, T=0)/\chi_s(0,0) = 1 - c_d^{\chi_s}|\boldsymbol{k}|^{d-2}. \tag{2.35d}$$

Here the $c_d^{\sigma,\gamma,\chi_s}$ are positive coefficients that depend on the disorder, the interaction strength,[28] and the dimensionality. In $d=2$ and $d=4$ the fractional powers in these equations are replaced by integer powers times logarithms. In the time domain, Eq. (2.35a) implies that current correlations decay as $1/t^{d/2}$, while Eq. (2.35d) implies that spatial spin correlations decay as $1/r^{2(d-1)}$.

All of the above effects are examples of indirect GSI, as will become clear from the derivations given below. There are other examples of long-ranged correlations in addition to the ones given above; which were chosen because of their experimental relevance. For instance, an experiment showing the $\sqrt{T}$ dependence of the conductivity expressed by Eq. (2.35b) is shown in Fig. 5. It is of interest to note, however, that not all obvious correlation functions have nonanalyticities and exhibit GSI. For example, explicit calculations show that the density susceptibility, or compressibility, has no $|\boldsymbol{k}|^{d-2}$ nonanalyticity, in contrast to the spin susceptibility, Eq. (2.35d). This has been discussed by Belitz *et al.* (2002).

b. Clean systems in equilibrium

For clean electronic systems, two interesting nonanalyticities that reflect GSI are, for $1<d<3$,

$$\gamma(T\to 0)/\gamma(0) = 1 + b_d^{\gamma} T^{d-1}, \tag{2.36a}$$

$$\chi_s(\boldsymbol{k}\to 0, T=0)/\chi_s(0,0) = 1 + b_d^{\chi_s}|\boldsymbol{k}|^{d-1}, \tag{2.36b}$$

with the b_d^{γ,χ_s} positive, interaction-dependent coefficients (for a recent overview, see Chubukov and Maslov, 2003; Chubukov *et al.*, 2005). In $d=1$ and $d=3$ the integer powers in these equations become multiplied by logarithms.

Let us explain the interesting sign difference between the nonanalytic corrections to the spin susceptibility in the clean and dirty cases, respectively, which will become important in Sec. IV. Quenched disorder is known to increase the effective spin-triplet interaction strength between the electrons (Altshuler and Aronov, 1984). This effect is strongest at zero wave number. $\chi_s(0,0)$ is therefore enhanced,[29] but the enhancement effect decreases with increasing wave number, which leads to the negative correction in Eq. (2.35d). In the clean case, however, the generic soft modes represent fluctuations that weaken the tendency towards ferromagnetism. This effect decreases $\chi_s(0,0)$ compared to the Pauli value (more precisely, the Pauli value with mean-field, or Hartree-Fock, corrections), and the wave-number-dependent correction is positive.

2. Soft modes in itinerant electron systems

Let us now consider the soft modes that are responsible for the results listed above. Apart from the soft modes due to the standard conservation laws, which are the same as in the classical case, the most interesting soft modes in itinerant electron systems are the Goldstone modes of a broken symmetry that is characteristic of quantum systems. To understand these new modes, we consider the formal action given by Eq. (1.9b) in more detail. Specifying the Hamiltonian, we consider a model action

$$S = S_0 + S_{\rm pot} + S_{\rm int}, \tag{2.37a}$$

with

$$S_0 = \int dx \sum_\sigma \bar{\psi}_\sigma(x)[-\partial_\tau + \boldsymbol{\nabla}^2/2m_e + \mu]\psi_\sigma(x), \tag{2.37b}$$

$$S_{\rm pot} = \int dx\, u(\boldsymbol{x})\bar{\psi}_\sigma(x)\psi_\sigma(x). \tag{2.37c}$$

Here $x\equiv(\boldsymbol{x},\tau)$ and $\int dx = \int d\boldsymbol{x}\int_0^\beta d\tau$, m_e is the electron mass, μ is the chemical potential, and $u(\boldsymbol{x})$ is a one-body potential. u can represent an electron-lattice interaction, but for the physical applications we are interested in this would not lead to qualitative deviations from the behavior of a simple "jellium" model of a parabolic band, and we shall therefore not pursue this possibility. We will, however, want to treat the interaction of electrons with

[28] Some of these nonanalyticities, e.g., the one in Eq. (2.35a), are present even in noninteracting electron systems (see, for example, Lee and Ramakrishnan, 1985; Kramer and MacKinnon, 1993), which are analogous to the classical Lorentz model (see, for example, Hauge, 1974). We shall be mainly interested in interacting electron systems, which are analogous to classical fluids.

[29] Note that this represents an effect of disorder that *enhances* the tendency towards ferromagnetism. It comes in addition to the more basic, and more obvious, opposite effect due to the dilution of the ferromagnet (Grinstein, 1985). The important point for our argument is that the former effect is wave-number dependent, while the latter is not.

quenched (i.e., static) random impurities (Edwards, 1958; Abrikosov *et al.*, 1963). In this case $u(\boldsymbol{x})$ is a random function. It is governed by a distribution that we take to be Gaussian with zero mean, and a second moment

$$\{u(\boldsymbol{x})u(\boldsymbol{y})\}_{\rm dis} = U_0\delta(\boldsymbol{x}-\boldsymbol{y}) \equiv \frac{\delta(\boldsymbol{x}-\boldsymbol{y})}{2\pi N_{\rm F}\tau_{\rm rel}}. \tag{2.38}$$

Here $\{\cdots\}_{\rm dis}$ denotes the disorder average, $\tau_{\rm rel}$ is the elastic relaxation time, and $N_{\rm F}$ is the density of states per spin at the Fermi surface. The factors in Eq. (2.38) are chosen such that the disordered average Green's function has a single-particle lifetime equal to $1/2\tau_{\rm rel}$. We will consider both clean systems, in which $U_0=0$, and disordered ones, in which $U_0>0$.

$S_{\rm int}$ in Eq. (2.37a) represents the Coulomb potential, but it is often advantageous to integrate out certain degrees of freedom to arrive at an effective short-range interaction. For our present purposes it is not necessary to specify the precise form of $S_{\rm int}$; it will suffice to postulate that the ground state of the interacting system is a Fermi liquid for $U_0=0$, or a disordered Fermi liquid for $U_0\neq 0$.[30] However, for later reference we write down a popular model (Abrikosov *et al.*, 1963) that describes the self-interaction of the electron number density, $n(x)$, and the electron spin density, $\boldsymbol{n}_{\rm s}(x)$, via pointlike, instantaneous interaction amplitudes $\Gamma_{\rm s}$ and $\Gamma_{\rm t}$, respectively,

$$S_{\rm int} = \frac{-\Gamma_{\rm s}}{2}\int dx\; n(x)n(x) + \frac{\Gamma_{\rm t}}{2}\int dx\; \boldsymbol{n}_{\rm s}(x)\cdot\boldsymbol{n}_{\rm s}(x). \tag{2.39a}$$

In terms of fermionic fields, n and $\boldsymbol{n}_{\rm s}$ are given by

$$n(x) = \sum_{\sigma}\bar{\psi}_{\sigma}(x)\psi_{\sigma}(x), \tag{2.39b}$$

$$\boldsymbol{n}_{\rm s}(x) = \sum_{\sigma,\sigma'}\bar{\psi}_{\sigma}(x)\boldsymbol{\sigma}_{\sigma,\sigma'}\psi_{\sigma'}(x). \tag{2.39c}$$

Here $\boldsymbol{\sigma}=(\sigma^x,\sigma^y,\sigma^z)$ represents the Pauli matrices. An underlying repulsive Coulomb interaction leads to $\Gamma_{\rm s}$, $\Gamma_{\rm t}>0$ in Eq. (2.39a).

It is useful to perform a Fourier representation from imaginary time to fermionic Matsubara frequencies $\omega_n = 2\pi T(n+1/2)$,

$$\psi_{n,\sigma}(\boldsymbol{x}) = \sqrt{T}\int_0^{\beta} d\tau\, e^{i\omega_n\tau}\psi_{\sigma}(x), \tag{2.40a}$$

$$\bar{\psi}_{n,\sigma}(\boldsymbol{x}) = \sqrt{T}\int_0^{\beta} d\tau\, e^{i\omega_n\tau}\bar{\psi}_{\sigma}(x). \tag{2.40b}$$

The noninteracting part of the action can then be written

$$\tilde{S}_0 = S_0 + S_{\rm pot} = \int d\boldsymbol{x}\sum_{\sigma,n}\bar{\psi}_{n,\sigma}(\boldsymbol{x})[i\omega_n + \boldsymbol{\nabla}^2/2m_{\rm e} + \mu + u(\boldsymbol{x})]\psi_{n,\sigma}(\boldsymbol{x}). \tag{2.41}$$

In the disordered case, it is further convenient to integrate out the quenched disorder by means of *the replica trick* (Edwards and Anderson, 1975; for a pedagogical discussion of this technique, see Grinstein, 1985). Accordingly, one introduces N identical replicas of the system, labeled by an index α, and integrates out $u(\boldsymbol{x})$. $S_{\rm pot}$ then gets replaced by

$$S_{\rm dis} = \frac{1}{4\pi N_{\rm F}\tau_{\rm rel}}\sum_{\alpha,\beta=1}^{N}\int d\boldsymbol{x}\sum_{n,m}\bar{\psi}^{\alpha}_{n,\sigma}(\boldsymbol{x})\psi^{\alpha}_{n,\sigma}(\boldsymbol{x}) \times \bar{\psi}^{\beta}_{m,\sigma'}(\boldsymbol{x})\psi^{\beta}_{m,\sigma'}(\boldsymbol{x}), \tag{2.42}$$

and physical quantities are obtained by letting $N\to 0$ at the end of calculations.

We now note two crucial features (Wegner and Schäfer, 1980). First, for $\omega_n=0$, the action $\tilde{S}_0$ is invariant under transformations of the fermionic fields that leave $\Sigma_n\bar{\psi}_n\psi_n$ invariant. That is, $\tilde{S}_0$ is invariant under a continuous rotation in frequency space.[31] Second, this symmetry is spontaneously broken whenever $N_{\rm F}\neq 0$. To see this, consider the order parameter

$$\bar{Q} = \lim_{\omega_n\to 0+}\langle\bar{\psi}_{n,\sigma}(\boldsymbol{x})\psi_{n,\sigma}(\boldsymbol{x})\rangle - \lim_{\omega_n\to 0-}\langle\bar{\psi}_{n,\sigma}(\boldsymbol{x})\psi_{n,\sigma}(\boldsymbol{x})\rangle. \tag{2.43}$$

$\bar{Q}$ is the difference between retarded and advanced Green's functions and is proportional to $N_{\rm F}$. The causal Green's function has a cut along the real axis for all frequencies at which $\bar{Q}\propto N_{\rm F}\neq 0$, which breaks the symmetry between frequencies with positive and negative imaginary parts or between retarded and advanced degrees of freedom. Since $S_{\rm int}$ cannot explicitly break this symmetry,[32] this will result in soft modes according to Goldstone's theorem. Technically, these soft modes are most conveniently discussed in terms of fluctuations of 4×4 matrices that are isomorphic to bilinear products of fermionic fields (Efetov *et al.*, 1980),

[30] A disordered analog of Landau's Fermi-liquid theory (see, for example, Baym and Pethick, 1991) has been developed by Castellani and Di Castro (1985, 1986); Castellani, Di Castro, *et al.* (1987); Castellani, Kotliar, and Lee (1987); Castellani *et al.* (1988).

[31] Because of the anticommuting nature of the fermion fields, the Lie group in question is symplectic; for a model with $2M$ Matsubara frequencies it is Sp $(2M)$.

[32] This follows generally from time translational invariance and can be checked explicitly for the interaction defined in Eq. (2.39a).

$$Q_{12} \cong \frac{i}{2}\begin{pmatrix} -\psi_{1\uparrow}\bar{\psi}_{2\uparrow} & -\psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\ -\psi_{1\downarrow}\bar{\psi}_{2\uparrow} & -\psi_{1\downarrow}\bar{\psi}_{2\downarrow} & -\psi_{1\downarrow}\psi_{2\downarrow} & \psi_{1\downarrow}\psi_{2\uparrow} \\ \bar{\psi}_{1\downarrow}\bar{\psi}_{2\uparrow} & \bar{\psi}_{1\downarrow}\bar{\psi}_{2\downarrow} & \bar{\psi}_{1\downarrow}\psi_{2\downarrow} & -\bar{\psi}_{1\downarrow}\psi_{2\uparrow} \\ -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\uparrow} & -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\downarrow} & -\bar{\psi}_{1\uparrow}\psi_{2\downarrow} & \bar{\psi}_{1\uparrow}\psi_{2\uparrow} \end{pmatrix}. \tag{2.44}$$

Here all fields are understood to be at the position $\boldsymbol{x}$, and $1 \equiv (n_1, \alpha_1)$, etc., comprises both frequency labels n and, for the disordered case, replica labels α. Since a nonzero frequency explicitly breaks the symmetry, one expects on the one hand, for $n_1 n_2 < 0$,

$$\lim_{\boldsymbol{k}\to 0} \langle Q_{n_1 n_2}(\boldsymbol{k}) Q_{n_1 n_2}(-\boldsymbol{k}) \rangle \propto \frac{N_F}{\omega_{n_1} - \omega_{n_2}}. \tag{2.45}$$

For $n_1 n_2 > 0$, on the other hand, one expects the correlation function to approach a finite constant in the limit of small wave numbers and frequencies. That is, Q_{nm} is soft if the frequencies ω_n and ω_m have opposite signs, and is massive if they have the same sign. For later reference, let us introduce a notation that distinguishes between the soft and massive components of Q. We write

$$Q_{nm}(\boldsymbol{x}) = \begin{cases} q_{nm}(\boldsymbol{x}) & \text{if } n > 0, m < 0 \\ q^{\dagger}_{nm}(\boldsymbol{x}) & \text{if } n < 0, m > 0 \\ P_{nm}(\boldsymbol{x}) & \text{if } nm > 0. \end{cases} \tag{2.46}$$

Explicit calculations confirm these expectations (Efetov *et al.*, 1980; Wegner and Schäfer, 1980; Belitz and Kirkpatrick, 1997). For technical reasons, it is convenient to expand the 4×4 matrix given by Eq. (2.44) in a spin-quaternion basis,

$$Q_{12}(\boldsymbol{x}) = \sum_{r,i=0}^{3} (\tau_r \otimes s_i)^i_r Q_{12}(\boldsymbol{x}), \tag{2.47}$$

with $\tau_0 = s_0$ the 2×2 unit matrix, and $\tau_j = -s_j = -i\sigma_j$ ($j=1,2,3$). In this basis, $i=0$ and $i=1,2,3$ describe the spin-singlet and spin-triplet degrees of freedom, respectively. $r=0,3$ corresponds to the particle-hole channel (i.e., products of $\bar{\psi}\psi$), while $r=1,2$ describes the particle-particle channel (i.e., products $\bar{\psi}\bar{\psi}$ or $\psi\psi$).

For small wave numbers and low frequencies, and for disordered noninteracting electrons, one finds (Efetov *et al.*, 1980)[33]

$$\langle {}^i_r q_{12}(\boldsymbol{k}) {}^j_s q_{34}(\boldsymbol{p}) \rangle = \frac{G}{8} \delta(\boldsymbol{k}+\boldsymbol{p}) \delta_{rs} \delta_{ij} \delta_{13} \delta_{24} \mathcal{D}_{n_1 n_2}(\boldsymbol{k}), \tag{2.48a}$$

with

$$\mathcal{D}_{n_1 n_2}(\boldsymbol{k}) = \frac{1}{\boldsymbol{k}^2 + GH|\omega_{n_1} - \omega_{n_2}|} \quad \text{(disordered)}. \tag{2.48b}$$

Here $H = \pi N_F/4$, and $G = 8/\pi\sigma_0 \propto 1/\tau_{rel}$, with σ_0 the conductivity in the Boltzmann approximation; see Eq. (2.56) below. $1/GH \equiv D$ is the *electron diffusion coefficient.* In the low-disorder limit, $D = v_F^2 \tau_{rel}/d$, with v_F the Fermi velocity. Note that this structure is consistent with Eq. (2.45), which was based on very the general arguments and Goldstone's theorem. For clean noninteracting systems, Eq. (2.48a) is still valid, but $G = 2/N_F v_F$, and $\mathcal{D}$ is given by[34]

$$\mathcal{D}_{n_1 n_2}(\boldsymbol{k}) = \frac{1}{|\boldsymbol{k}| + GH|\omega_{n_1} - \omega_{n_2}|} \quad \text{(clean)}. \tag{2.48c}$$

H is the same in either case. As expected, for disordered electrons $\mathcal{D}$ is diffusive, while for clean electrons it is ballistic.

For interacting systems, clean or disordered, the results are structurally the same (Finkelstein, 1983; Belitz and Kirkpatrick, 1997). The correlation functions remain diffusive and ballistic, respectively, but they are no longer the same in the spin-singlet ($i=0$) and spin-triplet ($i=1,2,3$) channels. Rather, instead of H in Eqs. (2.48b) and (2.48c), the combinations $H+K_s$ and $H+K_t$, respectively, appear in the two channels, with $K_{s,t} = \pi N_F^2 \Gamma_{s,t}/2$. We shall not give detailed correlation functions for the interacting case; we just mention the physical significance of these various coupling constants. H determines the specific-heat coefficient $\gamma = C_V/T$ via $\gamma = 8\pi H/3$ (Castellani and Di Castro, 1986)[35] and can be interpreted as a quasiparticle density of states (Castellani, Kotliar, and Lee, 1987). $H+K_s = \pi \partial n/\partial\mu$ and $H+K_t = \pi\chi_s$ are proportional to the thermodynamic density susceptibility $\partial n/\partial\mu$ and the spin susceptibility χ_s, respectively (Castellani *et al.*, 1984b, 1986; Finkelstein, 1984a). Notice that $\partial n/\partial\mu$, χ_s, and $3\gamma/2\pi^2$ all are equal to N_F for noninteracting electrons, but differ for interacting systems.

The above discussion assumes that the underlying continuous symmetry [see the discussion above Eq. (2.43)] is not broken explicitly. Such an explicit symmetry breaking occurs, for example, in the presence of a magnetic field, magnetic impurities, or spin-orbit scattering (Efetov *et al.*, 1980; Castellani *et al.*, 1984a; Finkelstein, 1984a, 1984b). In the presence of any of these symmetry breakers, some of the channels classified by the spin and particle-hole indices i and r of the matrix ${}^i_r Q$ become massive. For instance, magnetic impurities lead

[33] The normalizations of these propagators in the literature vary, depending on how the Q fields have been scaled. The normalization used in Eqs. (2.48) corresponds to a dimensionless $Q(\boldsymbol{x})$, in accord with the usual convention in the nonlinear σ model [see Eqs. (2.62) below], while the $Q(\boldsymbol{x})$ as defined by Eq. (2.44) is dimensionally a density of states.

[34] Equation (2.48c) is a schematic representation, which has the correct scaling properties, of a more complicated function. In contrast, Eq. (2.48b) represents the exact propagator in the limit of small frequencies and wave numbers.

[35] This reference established the relation between γ_V and H for disordered systems only. A later proof by means of Ward identities (Castellani *et al.*, 1988) can be generalized to apply to clean systems as well.

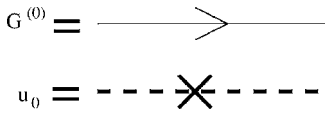

FIG. 6. Diagrammatic representation of the Green's function $G^{(0)}$ and the impurity factor u_0.

to a mass in both the particle-particle channel (r=1,2) and the spin-triplet channel (i=1,2,3). A summary of these effects has been given by Belitz and Kirkpatrick (1994). The nonanalyticities in various observables cited, in Sec. II.B.1 above are cut off accordingly, depending on which soft channel they rely on.

3. Generic scale invariance via explicit calculations

One way to obtain the results cited in Sec. II.B.1 is by explicit many-body, or Feynman diagram, calculations to lowest order in a small parameter (Abrikosov *et al.*, 1963; Fetter and Walecka, 1971). To illustrate this approach, we compute the electrical conductivity σ in a noninteracting, disordered, electronic system, to lowest nontrivial order in the disorder about the Boltzmann value. The calculation starts with the Kubo expression for $\sigma(\Omega)$ (Kubo, 1957),

$$\sigma(\Omega) = \frac{i}{i\Omega_n}\left[\pi(\boldsymbol{q}=0, i\Omega_n) + \frac{ne^2}{m}\right]_{i\Omega_n \to \Omega + i0}, \tag{2.49a}$$

with π the longitudinal current correlation function,

$$\pi(\boldsymbol{q}, i\Omega_n) = \frac{-1}{\boldsymbol{q}^2}\int_0^\beta d\tau\, e^{i\Omega_n \tau}\langle T_\tau \boldsymbol{q}\cdot\hat{\boldsymbol{j}}(\boldsymbol{q},\tau)\boldsymbol{q}\cdot\hat{\boldsymbol{j}}(\boldsymbol{q},0)\rangle. \tag{2.49b}$$

Here T_τ is the imaginary time-ordering operator and $\hat{\boldsymbol{j}}$ is the current operator,

$$\hat{\boldsymbol{j}}(\boldsymbol{q}) = \frac{1}{m}\sum_{\boldsymbol{k}} \boldsymbol{k}\hat{a}^\dagger_{\boldsymbol{k}-\boldsymbol{q}/2}\hat{a}_{\boldsymbol{k}+\boldsymbol{q}/2}, \tag{2.49c}$$

with $\hat{a}^\dagger$ and $\hat{a}$ electron creation and annihilation operators, respectively. Wick's theorem can be used to evaluate time-ordered correlation functions such as the one in Eq. (2.49b).

For our present purposes, the small parameter for a perturbative treatment is the impurity density n_i, or, alternatively, $1/k_F\ell$, with k_F the Fermi wave number and ℓ the mean free path between electron-impurity collisions. The averaging over the positions of the impurities can be performed using standard techniques (Edwards, 1958; Abrikosov *et al.*, 1963). The building blocks of the theory are the bare-electron Green's function,

$$G^{(0)}(\boldsymbol{q}, i\omega_n) = -\int_0^\beta d\tau\, e^{i\omega_n\tau}\langle T_\tau \hat{a}_{\boldsymbol{q}}(\tau)\hat{a}^\dagger_{\boldsymbol{q}}(0)\rangle_0, \tag{2.50}$$

and the impurity factor U_0, Eq. (2.38). The subscript 0 in Eq. (2.50) indicates that both the average and the time dependence of $\hat{a}$ are taken with the free-electron part of the Hamiltonian only. Diagrammatically, we denote $G^{(0)}$ by a directed straight line, and u_0 by two broken lines (one for each factor of the impurity potential) and a cross (for the factor of $n_i \sim 1/k_F\ell$); see Fig. 6. For our free-electron model we have

$$G^{(0)}(\boldsymbol{q}, i\omega_n) = (i\omega_n - \boldsymbol{q}^2/2m_e + \mu)^{-1}. \tag{2.51}$$

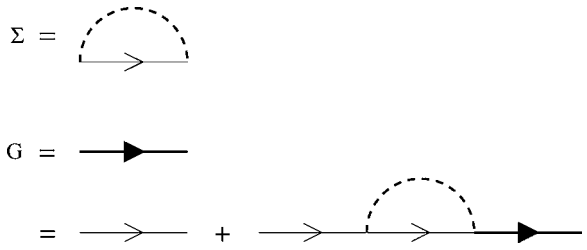

FIG. 7. The electronic self-energy and the Green's function in the Born approximation.

The exact disorder-averaged Green's function can be written in terms of a self-energy Σ as

$$G(\boldsymbol{q}, i\omega_n) = [i\omega_n - \boldsymbol{q}^2/2m_e + \mu + \Sigma(\boldsymbol{q}, i\omega_n)]^{-1}. \tag{2.52a}$$

The Born approximation for Σ and for G is shown diagrammatically in Fig. 7. Analytically, it is given by

$$\Sigma(\boldsymbol{q}, i\omega_n) = \frac{i}{2\tau_{\rm rel}}\mathrm{sgn}(\omega_n). \tag{2.52b}$$

In this approximation, single-particle excitations decay exponentially with a mean free time $\tau_{\rm rel}$. One commonly defines a (longitudinal) current vertex function Γ by writing π as

$$\pi(\boldsymbol{q}, i\Omega_n) = \int_{\boldsymbol{p}} T\sum_{i\omega_n} \Gamma_0(\boldsymbol{p},\boldsymbol{q}) G(\boldsymbol{p}-\boldsymbol{q}/2, i\omega_n - i\Omega_n/2) \times G(\boldsymbol{p}+\boldsymbol{q}/2, i\omega_n + i\Omega_n/2)\Gamma(\boldsymbol{p},\boldsymbol{q}; i\omega_n, i\Omega_n). \tag{2.53a}$$

Here $\int_{\boldsymbol{p}} = \int d\boldsymbol{p}/(2\pi)^d$ in d dimensions, and

$$\Gamma_0(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{p}\cdot\boldsymbol{q}/|\boldsymbol{q}| \tag{2.53b}$$

is the bare current vertex. It is often convenient to express Γ in terms of another vertex function, Λ, via the equation

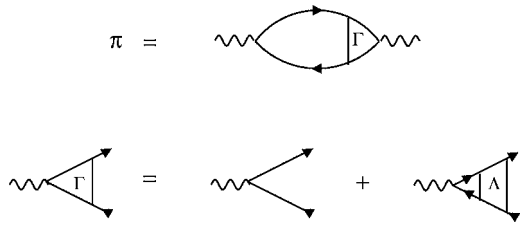

FIG. 8. Diagrammatic representation of the current correlation function and of the relation between the functions Γ and Λ.

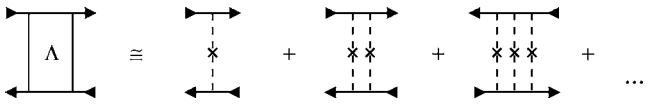

FIG. 9. The vertex function Λ in the Boltzmann approximation.

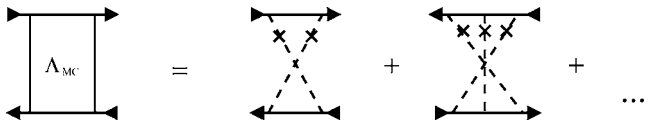

FIG. 10. Crossed-ladder contribution to Λ.

$$\Gamma(\boldsymbol{p},\boldsymbol{q};i\omega_n,i\Omega_n) = \Gamma_0(\boldsymbol{p},\boldsymbol{q}) + \int_{\boldsymbol{k}} \Gamma_0(\boldsymbol{k},\boldsymbol{q}) G(\boldsymbol{k}-\boldsymbol{q}/2,i\omega_n - i\Omega_n) \times G(\boldsymbol{k}+\boldsymbol{q}/2,i\omega_n + i\Omega_n/2)\Lambda(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q};i\omega_n,i\Omega_n). \quad (2.54)$$

Diagrammatically, these equations are illustrated in Fig. 8.

The Boltzmann approximation Λ_B for Λ (which is equivalent to solving the Boltzmann equation for σ) is shown diagrammatically in Fig. 9. Analytically, we find, for small Ω_n and $\boldsymbol{q}$,

$$\Lambda_B(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q};i\omega_n,i\Omega_n)|_{\boldsymbol{q}\to 0} = \Theta[-\omega_n(\omega_n+\Omega_n)] \times \frac{4\pi N_F u_0^2}{D\boldsymbol{q}^2 + |\Omega_n|}. \quad (2.55)$$

Using this in Eq. (2.54), and the result in Eqs. (2.53a) and (2.49a), we find that the Boltzmann approximation for σ is[36]

$$\sigma_0 = ne^2\tau_{rel}/m_e. \quad (2.56)$$

There are, of course, an infinite number of corrections to σ_0 of $O(1/k_F\ell)$ and higher. Of particular interest to us are the contributions that couple to the electronic soft modes, Eqs. (2.48). As in the classical long-time tail calculation (Sec. II.A.3.b), it is the coupling of these modes to the current fluctuations that lead to the slow decay of current correlations and to a singular dependence of the conductivity on the frequency. The relevant diagrams for this soft-mode contribution to σ are the so-called maximally crossed diagrams shown in Fig. 10 (Gorkov *et al.*, 1979). For these contributions to Λ, which we denote by Λ_{MC}, we find

$$\Lambda_{MC}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q};i\omega_n,i\Omega_n)|_{\boldsymbol{k}\approx\boldsymbol{p}} = \Theta[-\omega_n(\omega_n+\Omega_n)] \times \frac{4\pi N_F u_0^2}{D(\boldsymbol{k}+\boldsymbol{p})^2 + |\Omega_n|}. \quad (2.57)$$

Notice that this is just the Boltzmann result, Eq. (2.55), with $\boldsymbol{q}\to\boldsymbol{k}+\boldsymbol{p}$ (Vollhardt and Wölfe, 1980). For small real frequencies this contribution leads to the following result for the conductivity (Gorkov *et al.*, 1979):

$$\sigma(\Omega) = \sigma_0\left\{1 - \frac{1}{\pi N_F}\int_{\boldsymbol{q}} \frac{1}{-i\Omega + D\boldsymbol{q}^2}\right\}. \quad (2.58)$$

In $d=2$, this "correction" to σ_0 leads to a (negative) logarithmic divergence. This reflects the fact that, at least for noninteracting electrons, there is no metallic state in $d=2$ (Abrahams *et al.*, 1979). This phenomenon (the "weak localization" proper; see footnote 26) has generated a huge body of literature (for reviews, see Bergmann, 1984; Lee and Ramakrishnan, 1985; Kramer and MacKinnon, 1993). To the extent that it is caused by the long-time tail's not being integrable, this is analogous to the breakdown of classical hydrodynamics in $d\leqslant 2$ (see Sec. II.A.3.b), although the physics behind it is quite different.

In $d>2$, Eq. (2.58) predicts that $\sigma(\Omega)$ is a nonanalytic function of frequency, the small-frequency behavior for $2<d<4$ being

$$\sigma(\Omega\to 0)/\sigma(0) = 1 + \text{const}\times\Omega^{(d-2)/2}, \quad (2.59a)$$

with const>0. This particular correction is one of several that contribute to Eq. (2.35a); in the nomenclature explained at the beginning of Sec. II.B it is the weak-localization contribution. Another one is the interaction contribution that was found by Altshuler *et al.* (1980).[37] In the time domain this corresponds to a behavior of the current-current correlation function, Eq. (2.49b), at long (real) times t,

$$\pi(\boldsymbol{q}=0,t\to\infty) \sim -1/t^{d/2}. \quad (2.59b)$$

Since the current is not conserved in our model, this is an example of indirect GSI. Notice that the exponent is the same as for the corresponding long-time tail in a classical fluid, Eq. (2.31a), but the sign is different. Physically this means that, in the quantum case, the leading long-time tails decrease the diffusion coefficient, while in the classical fluid case they increase it. This is because the scattering in the electron-impurity model is due to

[36]Due to our pointlike impurity potential, Γ in the Boltzmann approximation is equal to Γ_0, and the conductivity is given in terms of the single-particle relaxation time τ_{rel}. For a more general scattering potential, the vertex corrections in Eq. (2.54) are nonzero, and the Boltzmann conductivity is given by Eq. (2.56) with τ_{rel} replaced by the transport relaxation time τ_{tr}. See Abrikosov *et al.* (1963), Sec. 39.2.

[37]Historically, the interaction effect predates weak localization. It was first discussed by Schmid (1974) for the electron inelastic lifetime and by Altshuler and Aronov (1979) for the density of states; a closely related effect was found by Brinkman and Engelsberg (1968).

static impurities, while in the classical fluid it is due to the other fluid particles.[38]

4. Generic scale invariance via renormalization-group arguments

The results given in Secs. II.B.1 and II.B.3 can also be obtained independent of perturbation theory, by general RG and scaling arguments, which establish these long-time tails as the exact leading behavior for long times or small frequencies (Belitz and Kirkpatrick, 1997). These authors showed that, in the disordered case, the GSI effects in $d>2$ can be understood as corrections to scaling at a stable zero-temperature fixed point that describes a disordered Fermi liquid. This also makes more explicit the connection between weak-localization and Altshuler-Aronov effects, on the one hand, and the Goldstone modes of the O(N) nonlinear σ model (Sec. II.A.1), or the hydrodynamic theory of GSI (Sec. II.A.4), on the other.[39] This is the first instance in which we see a close, if not entirely obvious, analogy between classical and quantum effects.

a. Disordered electron systems

The long-wavelength and low-frequency excitations in a noninteracting disordered electron system can be described by a nonlinear σ model (Wegner, 1979; Wegner and Schäfer, 1980; McKane and Stone, 1981). This approach has been generalized to interacting systems by Finkelstein (1983); a derivation in the spirit of Wegner and Schäfer (1980) has been given by Belitz and Kirkpatrick (1997) and Belitz *et al.* (1998). As for the simpler σ model derived and discussed in Sec. II.A.1, the basic idea is to construct an action solely in terms of the massless modes. The derivation of the effective theory then becomes a two-step process. First, one needs to identify the massless modes of the system (see Sec. II.B.2). Second, the microscopic theory, Eqs. (2.37)–(2.39), must be transformed into an effective one that keeps only the soft modes, while all other degrees of freedom are integrated out in some reasonable approximation. The result of this procedure is a generalized nonlinear σ model. Within this theory the partition function can be written as an integral over a matrix field $\hat{Q}$ that essentially contains the soft sectors of the field Q, Eq. (2.44),

$$Z = \int D[\hat{Q}] e^{\mathcal{A}[\hat{Q}]}. \tag{2.60}$$

The effective action $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{-1}{2G} \int d\boldsymbol{x}\ \mathrm{tr}(\boldsymbol{\nabla}\hat{Q}(\boldsymbol{x}))^2 + 2H \int d\boldsymbol{x}\ \mathrm{tr}(\Omega\hat{Q}(\boldsymbol{x})) + \mathcal{A}_{\mathrm{int}}. \tag{2.61a}$$

Here $\mathcal{A}_{\mathrm{int}} \sim O(T\Gamma\hat{Q}^2)$ is the interaction part of the action, i.e., S_{int} from Eq. (2.39a) expressed in terms of Q matrices. Schematically, leaving out the detailed structure of the frequency, replica, and spin-quaternion labels, it is of the form[40]

$$\mathcal{A}_{\mathrm{int}} = T\,\Gamma \int d\boldsymbol{x}\ \hat{Q}(\boldsymbol{x})\hat{Q}(\boldsymbol{x}). \tag{2.61b}$$

For our present purposes, Γ can stand for either Γ_s or Γ_t. Ω is a diagonal matrix with fermionic Matsubara frequencies as diagonal elements, and the coupling constants G and H were defined after Eq. (2.48b). $\hat{Q}$ is subject to the constraints

$$\hat{Q}^2(\boldsymbol{x}) = 1, \quad \hat{Q}^\dagger = \hat{Q}, \quad \mathrm{tr}\ \hat{Q}(\boldsymbol{x}) = 1. \tag{2.62a}$$

A standard way to enforce these constraints, analogous to Eqs. (2.6), is to write $\hat{Q}$ as a block matrix in frequency space,

$$\hat{Q} = \begin{pmatrix} (1-qq^\dagger)^{1/2} & q \\ q^\dagger & -(1-q^\dagger q)^{1/2} \end{pmatrix}, \tag{2.62b}$$

where the matrix q has elements q_{nm} with $n \geq 0$, $m < 0$.

Insight into the GSI effects in a disordered metal can be gained by an RG analysis that focuses on a stable fixed point of the above generalized σ model (Belitz and Kirkpatrick, 1997). This fixed point provides an RG description of the disordered Fermi-liquid ground state (see footnote 30), in analogy to Shankar's RG description of a clean Fermi liquid (Shankar, 1994). The procedure is analogous to that in Sec. II.A.1.c. We again define the scale dimension of a length L to be $[L]=-1$. The stable disordered Fermi-liquid fixed point is characterized by the choice

$$[q(\boldsymbol{x})] = -(d-2)/2 \tag{2.63}$$

for the scale dimension of the field q, which corresponds to diffusive correlations of the q. This choice is consistent with what one expects for the soft modes in a disordered metal; see Eq. (2.48b). In addition, the scale dimension of the frequency or temperature, i.e., the dynamical scaling exponent $z=[\Omega]=[T]$, is needed. In order for the fixed point to be consistent with diffusion, the frequency must scale as the square of the wave number, so we choose

[38] This makes the electron-impurity model more closely analogous to the classical Lorentz model, which shows a weaker $1/t^{(d+2)/2}$ long-time tail (with a negative sign) than the classical fluid (Hauge, 1974). The mode-mode coupling effects are stronger in the quantum system than in the classical one, which is why the electron model has a long-time tail with the same strength as the classical fluid but with the sign of the classical Lorentz model.

[39] We note in passing that it is possible to cast classical hydrodynamics in the form of a field theory Martin *et al.*, 1973). This technique has not been applied so far to long-term tails; doing so would make the analogy even closer.

[40] The complete expression (Finkelstein, 1983) will not be needed here.

$$z = 2. \tag{2.64}$$

Now we expand the action in powers of q. In a symbolic notation that leaves out everything not needed for power counting purposes, we have

$$\mathcal{A} = -\frac{1}{G}\int d\boldsymbol{x}(\nabla q)^2 + H\int d\boldsymbol{x}\ \Omega\, q^2 + \Gamma\ T\int d\boldsymbol{x}\ q^2 + O(\nabla^2 q^4, \Omega\, q^4, Tq^3). \tag{2.65}$$

Power counting shows that all of the coupling constants in Eq. (2.65) have vanishing scale dimensions with respect to our disordered Fermi-liquid fixed point,

$$[G] = [H] = [\Gamma] = 0. \tag{2.66}$$

Now consider the leading corrections to the fixed-point action, as indicated by Eq. (2.65). Power counting shows that all of these terms are irrelevant with respect to the disordered Fermi-liquid fixed point as long as $d > 2$. Furthermore, all of the terms that were neglected in deriving the generalized nonlinear σ model can be shown to be even more irrelevant than the ones considered here. The conclusion from these arguments is that the terms given explicitly in Eq. (2.65) constitute a stable fixed-point action and that the leading irrelevant operators (which we denote collectively by u) have scale dimensions

$$[u] = -(d-2). \tag{2.67}$$

These results can be used to derive the GSI effects from scaling arguments. We first consider the dynamical conductivity $\sigma(\Omega)$. Its bare value is proportional to G, and according to Eq. (2.66) its scale dimension is zero. We therefore have the homogeneity law (see Sec. I.B)

$$\sigma(\Omega, u) = \sigma(\Omega\, b^z, u\, b^{-(d-2)}). \tag{2.68a}$$

By putting $b = 1/\Omega^{1/z}$, and using $z = 2$ as well as the fact that $\sigma(1, x)$ is an analytic function of x, we find that the conductivity has a singularity at zero frequency, or long-time tail, of the form

$$\sigma(\Omega) = \sigma(\Omega = 0) + \text{const} \times \Omega^{(d-2)/2}. \tag{2.68b}$$

That is, we recover Eqs. (2.35a) and (2.59a). The present analysis proves (with the same caveats as in Sec. II.A.1.c) that the $\Omega^{(d-2)/2}$ is the exact leading nonanalytic behavior.

The specific-heat coefficient $\gamma = C_V/T$ is proportional to the quasiparticle density of states H, whose scale dimension vanishes according to Eq. (2.66). We thus have

$$\gamma(T, u) = \gamma(Tb^z, ub^{-(d-2)}), \tag{2.69}$$

which leads to the low-temperature behavior given by Eq. (2.35b).

Finally, we consider the wave-vector-dependent spin susceptibility $\chi_s(\boldsymbol{k})$. χ_s is a two-particle correlation and thus can be expressed in terms of a Q-Q correlation function. The leading correction to the finite Fermi-liquid value is obtained by replacing both of the Q's by q. We thus have structurally $\chi_s \sim T\int d\boldsymbol{x}\langle q^\dagger q\rangle$, with scale dimension $[\chi_s] = 0$. The corresponding homogeneity law is

$$\chi_s(\boldsymbol{k}, u) = \chi_s(\boldsymbol{k}b, ub^{-(d-2)}), \tag{2.70}$$

which leads to a nonanalytic dependence on the wave number, as given by Eq. (2.35c).

We conclude with an important caveat. Like all scaling arguments, those presented above are exact but very formal. There is no guarantee that the prefactors of the various nonanalyticities, i.e., the coefficients $c_d^{\sigma,\gamma,\chi_s}$, are nonzero. Indeed, explicit calculations show that both c_d^{γ} and $c_d^{\chi_s}$ are nonzero only in the presence of electron-electron interactions (see, for example, Altshuler and Aronov, 1984). Further, the same arguments would lead to the conclusion that the density susceptibility has the same form as Eq. (2.70) for the spin susceptibility, while perturbation theory finds that the corresponding prefactor vanishes even in the presence of interactions (see the discussion at the end of Sec. II.B.1.a). The situation is thus as follows: If explicit calculation shows that a nonanalyticity has a nonzero prefactor, then scaling arguments can be used to show that the perturbative result is indeed the leading nonanalytic behavior, which one can never conclude from perturbation theory alone. However, the scaling arguments by themselves can never establish the existence of a particular nonanalyticity.

b. Clean electron systems

Structurally identical arguments can be made for clean electronic systems (Belitz and Kirkpatrick, 1997). There are two motivating factors. First, while the weak-localization effects are clearly caused by disorder, disorder plays a much less crucial role in the explicit calculations for the Altshuler-Aronov effects which lead to the same type of nonanalyticities, and the scaling arguments presented above do not at all depend on disorder in any obvious way. Second, it is well known that in $d = 1$, a perturbative expansion in powers of the interaction breaks down due to logarithmic divergencies, and that this divergence is related to the breakdown of Fermi-liquid theory in this dimension (Dzyaloshinski and Larkin, 1971; Schulz, 1995). A natural question is, what happens to these singularities for $d > 1$? Explicit perturbation theory shows that both clean and disordered interacting electronic systems have related nonanalyticities that reflect generic scale invariance and that these singularities are closely related to the terms that cause the breakdown of Fermi-liquid theory in $d = 1$ (Belitz *et al.*, 1997; Chitov and Millis, 2001; Chubukov and Maslov, 2003, 2004; Galitski and Das Sarma, 2004).

The formal scaling arguments for these effects proceed as in Sec. II.B.4.a above. The only structural difference is that the two-point vertex in the disordered case, $\Gamma^{(2)}_{\text{disordered}} = \boldsymbol{k}^2 + GH\Omega$, is replaced by $\Gamma^{(2)}_{\text{clean}} = |\boldsymbol{k}| + GH\Omega$ in the clean case, and in the terms of higher order in q, the ∇^2 operator in the disordered case is effectively replaced

by a ∇ operator. In terms of scale dimensions, the net result of these substitutions, using the notation of the previous subsection, is

$$[q(\boldsymbol{x})] = -(d-1)/2, \tag{2.71a}$$

$$z = 1, \tag{2.71b}$$

$$[u] = -(d-1), \tag{2.71c}$$

with u denoting the least irrelevant variables. Note that technically the difference between the disordered and clean cases is that $(d-2)$ in the disordered case is replaced by $(d-1)$ in the clean case.

Using Eqs. (2.71) and arguments identical to those used in the disordered case, one obtains the results given by Eqs. (2.36). As for the disordered case, this reasoning shows that Eqs. (2.36) are exact.

5. Another example of generic scale invariance: Disordered systems in a nonequilibrium steady state

The electron-impurity model in a nonequilibrium steady state has been studied (Yoshimura and Kirkpatrick, 1996) by means of many-body diagrammatic techniques in conjunction with Zubarev's nonequilibrium statistical operator method (Zubarev, 1974). This work showed that effects analogous to those discussed in Sec. II.A.4 for classical nonequilibrium fluids also exist in disordered electron systems at $T=0$ and that quantum GSI effects due to nonequilibrium conditions are stronger, i.e., of longer range, than in classical systems.

The particular model studied was that defined by Eqs. (2.37) and (2.38), without the electron-electron interaction term $S_{\rm int}$, but in the presence of a chemical-potential gradient $\boldsymbol{\nabla}\mu$. Consider the nonequilibrium part of the electronic structure factor

$$C_2(\boldsymbol{x},\boldsymbol{y}) = \{\langle \delta n(\boldsymbol{x})\, \delta n(\boldsymbol{y})\rangle\}_{\rm dis}, \tag{2.72}$$

where $\langle\cdots\rangle$ denotes a nonequilibrium thermal average. As one might expect, the calculation leads to a result that is analogous to the one in the classical fluid, Eqs. (2.34), viz.,

$$C_2(\boldsymbol{k}) = \frac{N_{\rm F}\mu\,\tau_{\rm rel}}{6d\,\pi(D\boldsymbol{k}^2)^2}[25(\boldsymbol{\nabla}\mu)^2 - 12(\hat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}\mu)^2]. \tag{2.73}$$

As in the classical fluid, this corresponds to a decay of correlations in real space given by const$-|\boldsymbol{x}|$ in three dimensions.

In comparison, for a classical Lorentz gas (Hauge, 1974; see also footnotes 28 and 38) one finds instead

$$C_2(\boldsymbol{k}) \sim (\boldsymbol{\nabla}\mu)^2/\boldsymbol{k}^2. \tag{2.74}$$

That is, in both quantum and classical cases C_2 exhibits GSI, but the effect is stronger in the quantum case due to the existence of more soft modes.

6. Quantum Griffiths phenomena: Power laws from rare regions in disordered systems

In this section we discuss another mechanism for long-range correlations in time in disordered quantum systems, the so-called *quantum Griffiths phenomena*. In contrast to the examples above, the relevant soft modes are local in space.

Griffiths phenomena are nonperturbative effects of rare strong-disorder fluctuations. They were first discovered in the context of classical phase transitions in quenched disordered systems (Griffiths, 1969). Griffiths phenomena can be understood as follows: In general, impurities will decrease the critical temperature T_c from its clean value T_c^0. In the temperature region $T_c < T < T_c^0$ the system does not display global order, but in an infinite system one will find arbitrarily large regions that are devoid of impurities and hence show local order, with a small but nonzero probability that usually decreases exponentially with the size of the region. These static disorder fluctuations are known as *rare regions*, and the order-parameter fluctuations induced by them belong to a class of excitations known as *local moments*; sometimes they are also referred to as *instantons*. Since they are weakly coupled to one another, and flipping them requires changing the order parameter in a whole region, the rare regions have very slow dynamics. Griffiths (1969) was the first to show that they lead to a nonanalytic free energy everywhere in the region $T_c < T < T_c^0$, which is known as the Griffiths phase, or, more appropriately, *the Griffiths region*. In generic classical systems, the contribution of Griffiths singularities to thermodynamic (equilibrium) observables is very weak, since the singularity in the free energy is only an essential one. In contrast, the consequences for the dynamics are much more severe, with the rare regions dominating the behavior for long times (Randeria *et al.*, 1985; Bray, 1988; Dhar *et al.*, 1988). In the classical McCoy-Wu model, in which the disorder along a certain direction is correlated, the Griffiths singularities have drastic effects even for the statics (McCoy and Wu, 1968, 1969).

At quantum phase transitions, the quenched disorder is perfectly correlated in one of the relevant dimensions, viz., the *imaginary time dimension*. Therefore the quantum Griffiths effects are generically as strong as in the classical McCoy-Wu model (Thill and Huse, 1995; Rieger and Young, 1996). One of the most obvious realizations of quantum Griffiths phenomena can be found in random quantum Ising models,

$$H = \sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z + \sum_i h_i\sigma_i^x, \tag{2.75}$$

with Pauli matrices σ^x and σ^z, where the summation runs over nearest neighbors on a lattice. Both the Ising couplings J_{ij} and the transverse fields h_i are independent random variables. This model has been studied extensively in $d=1$ (Fisher, 1995; Young and Rieger, 1996; Young, 1997; Iglói and Rieger, 1998; Iglói *et al.*, 1999) and in $d=2$ (Rieger and Young, 1996; Pich *et al.*, 1998; Motrunich *et al.*, 2000), and the results are expected to

hold qualitatively in $d=3$ as well (Motrunich *et al.*, 2000). Let us consider a system that is globally in the disordered phase, with a rare region of volume V_R that is locally in the ordered phase. The probability density for such a region to occur is exponentially small, $p \propto \exp(-aV_R)$, with a a disorder-dependent coefficient. However, since the volume dependence of the local energy gap Δ is also exponential (Young and Rieger, 1996), $\Delta \sim \exp(-bV_R)$, the probability density $P(\Delta)$ for finding a gap of size Δ varies as Δ to a power. One thus has (Young, 1997)

$$P(\Delta) = p(V_R)|dV_R/d\Delta| \propto \Delta^{a/b-1}. \tag{2.76}$$

Thus we obtain a power-law density of low-energy excitations. Notice that the exponent a/b (which in the literature is often called an *inverse dynamical scaling exponent* and is denoted by $1/z$) is a continuous function of the disorder strength.

Many results follow from this. For instance, a region with a local energy gap Δ has a local spin susceptibility that decays exponentially in the limit of long imaginary times, $\chi_{\rm loc}(\tau\to\infty) \propto \exp(-\Delta\tau)$. Averaging by means of P, Eq. (2.76), yields

$$\chi^{\rm av}_{\rm loc}(\tau\to\infty) \propto \tau^{-a/b}. \tag{2.77a}$$

The temperature dependence of the static average susceptibility is then

$$\chi^{\rm av}_{\rm loc}(T) = \int_0^{1/T} d\tau\, \chi^{av}_{loc}(\tau) \propto T^{a/b-1}. \tag{2.77b}$$

If $a<b$, the local zero-temperature susceptibility diverges, even though the system is globally still in the disordered phase. The typical correlation function falls off much faster than the average one; for $d=1$, Young (1997) has found a stretched-exponential behavior,

$$\chi^{\rm typ}_{\rm loc}(\tau\to\infty) \propto \exp(-c\,\tau^{-a/(a+b)}), \tag{2.77c}$$

with c another coefficient. The large difference between the average and the typical values is characteristic of very broad probability distributions of the corresponding observables.

7. Quantum long-time tails from detailed balance

Finally, we discuss a very simple source of long-time tails, or temporal GSI, which is operative at zero temperature only. To be specific, consider a model with spin diffusion. The imaginary, or dissipative, part of the dynamical spin susceptibility as a function of wave number and real frequency is

$$\chi''(\boldsymbol{k},\Omega) = \chi(\boldsymbol{k})\frac{D\boldsymbol{k}^2\Omega}{\Omega^2 + (D\boldsymbol{k}^2)^2}, \tag{2.78}$$

with D the spin-diffusion coefficient and $\chi(\boldsymbol{k})$ the static spin susceptibility. $\chi''(\boldsymbol{k},\Omega)$ is an analytic function of frequency. Upon Fourier transformation, this yields an exponential decay of correlations in real time, $\chi(\boldsymbol{k},t) \propto \exp(-D\boldsymbol{k}^2 t)$. Now consider the dynamic structure factor, $S(\boldsymbol{k},\Omega)$, and the symmetrized fluctuation function, $\varphi(\boldsymbol{k},\Omega)$, which are defined by (Forster, 1975)

$$S(\boldsymbol{k},\Omega) = \int d\boldsymbol{x} \int_{-\infty}^{\infty} dt\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\Omega t}\langle \hat{M}(\boldsymbol{x},t)\hat{M}(\boldsymbol{0},0)\rangle, \tag{2.79a}$$

$$\varphi(\boldsymbol{k},\Omega) = \frac{1}{2}\int d\boldsymbol{x} \int_{-\infty}^{\infty} dt\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\Omega t} \times\langle \hat{M}(\boldsymbol{x},t)\hat{M}(\boldsymbol{0},0) + \hat{M}(\boldsymbol{0},0)\hat{M}(\boldsymbol{x},t)\rangle, \tag{2.79b}$$

where $\hat{M}$ is the magnetization operator. (We neglect the vector nature of $\hat{M}$ for simplicity.) The fluctuation-dissipation theorem (Callen and Welton, 1951; see also Forster, 1975) relates these two correlation functions to χ'' via

$$S(\boldsymbol{k},\Omega) = \frac{2}{1-e^{-\beta\Omega}}\chi''(\boldsymbol{k},\Omega), \tag{2.80a}$$

$$\varphi(\boldsymbol{k},\Omega) = \coth(\beta\Omega/2)\chi''(\boldsymbol{k},\Omega). \tag{2.80b}$$

For any nonzero temperature, these correlations are again analytic functions of the frequency, and their Fourier transforms therefore decay exponentially in time. However, in the zero-temperature limit, $\beta\to\infty$ (and for $\boldsymbol{k}\neq 0$), neither of these correlation functions is analytic at $\Omega=0$; that is, they are proportional to $\Omega\Theta(\Omega)$. This nonanalyticity leads to a power-law decay in real time (see, for example, Lighthill, 1958),

$$S(\boldsymbol{k},t\to\infty) \propto \varphi(\boldsymbol{k},t\to\infty) \propto 1/t^2. \tag{2.81}$$

Physically, the nonanalyticities in Eqs. (2.80) at $T=0$ reflect the absence of excitations in the ground state, as shown in the detailed balance relation

$$S(\boldsymbol{k},-\Omega) = e^{-\beta\Omega}S(\boldsymbol{k},\Omega), \tag{2.82}$$

which implies that $S(\boldsymbol{k},\Omega)$ at $T=0$ must vanish for $\Omega<0$. Note that these long-time tails are present at all wave numbers, not just in the long-wavelength limit. More generally, the above considerations imply that S and φ will display long-time tails for any zero-temperature system. For low but nonzero temperature there will be a preasymptotic long-time tail followed by an exponential asymptotic decay.

For later reference we also consider the spin susceptibility at imaginary Matsubara frequencies $i\Omega_n = i2\pi Tn$,

$$\chi(\boldsymbol{k},i\Omega_n) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\chi''(\boldsymbol{k},\omega)}{\omega - i\Omega_n} = \chi(\boldsymbol{k})\frac{D\boldsymbol{k}^2}{|\Omega_n| + D\boldsymbol{k}^2}, \tag{2.83}$$

which is a nonanalytic function of Ω_n. Accordingly, the imaginary time correlation function $\chi(\boldsymbol{k},\tau) = T\Sigma_n e^{-i\Omega_n\tau}\chi(\boldsymbol{k},i\Omega_n)$ at $T=0$ has a long-time tail for large imaginary time τ,

$$\chi(\boldsymbol{k},\tau\to\infty) \propto 1/\tau^2. \tag{2.84}$$

More generally, any imaginary time correlation function has a long imaginary time tail if the corresponding causal function is nonanalytic at zero frequency.

III. INFLUENCE OF GENERIC SCALE INVARIANCE ON CLASSICAL CRITICAL BEHAVIOR

We now combine the concept of generic scale invariance with critical phenomena. Given that critical phenomena are driven by soft modes, one expects that soft modes connected to GSI, provided they couple to the order-parameter fluctuations, will influence the critical behavior. This is indeed the case in both classical and quantum systems. In this section we shall discuss two classical examples. The first one deals with the nematic–smectic-*A* transition in liquid crystals, where the generic soft modes are the Goldstone modes of the nematic phase (Halperin *et al.*, 1974; Chen *et al.*, 1978). This turns out to be closely related to the normal-metal–superconductor transition, in which the generic soft modes are virtual photons. The second example considers phase separation in a binary liquid subject to shear, and the generic soft modes are due to the nonequilibrium situation (Onuki and Kawasaki, 1979). Another classical example, which we shall not dicuss here, involves the ferromagnetic transition in compressible magnets (Aharony, 1976; Bergman and Halperin, 1976). Quantum phase transitions will be discussed in Sec. IV below.

A. The nematic–smectic-*A* transition in liquid crystals

Nematic liquid crystals consist of rod-shaped molecules. In the nematic phase, there is directional order, i.e., the rod axes tend to be parallel to one another, while the centers of gravity of the molecules do not show long-range order (de Gennes and Prost, 1993). In the smectic-*A* phase, the molecules are additionally arranged in layers. Within each layer, the rod axes are on average aligned perpendicular to the layer, but the molecules still form a two-dimensional liquid. In either phase, there are Goldstone modes associated with the broken rotational invariance that is characteristic of the directional order. It turns out that these Goldstone modes couple to the order parameter for the smectic-*A* order and have an important influence on the critical behavior. Let us now discuss this effect, following Halperin *et al.* (1974), Chen *et al.* (1978), and de Gennes and Prost (1993).

1. Action, and analogy with superconductors

The layers of the smectic-*A* phase are described as a density modulation in, say, the z direction,

$$\rho(\boldsymbol{x}) = \rho_0 + \rho_1(\boldsymbol{x})\cos[qz + \varphi(\boldsymbol{x})]. \tag{3.1}$$

In the smectic-*A* phase, the fluctuating amplitude $\rho_1(\boldsymbol{x})$ has a nonzero mean value, q is the wave number associated with the smectic layer spacing, and φ is a phase. It is convenient to combine ρ_1 with φ to form a complex order parameter [see de Gennes and Prost (1993), and references therein]

$$\psi(\boldsymbol{x}) = \rho_1(\boldsymbol{x})e^{i\varphi(\boldsymbol{x})}. \tag{3.2}$$

One piece of the action for the smectic-*A* order then takes the form

$$S_{\mathrm{A}}[\psi] = \int d\boldsymbol{x}\{r_0|\psi(\boldsymbol{x})|^2 + c|\boldsymbol{\nabla}\psi(\boldsymbol{x})|^2 + u_0|\psi(\boldsymbol{x})|^4\}. \tag{3.3a}$$

This is the LGW functional for a complex scalar field or, equivalently, the O(2) version of Eq. (2.3).[41] r_0 and u_0 are bare parameters that will be renormalized later. This action is incomplete. One must add to it (1) a term describing the fluctuations of the nematic order parameter, and (2) a term describing the coupling between the two. The nematic rods are described by a unit vector called the *director* $\boldsymbol{n}$, and the director fluctuations are given by $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}) - \hat{z}$. To linear order in the fluctuations, $\boldsymbol{n}^2 = 1$ enforces $\boldsymbol{A} = (A_x, A_y, 0)$. The Gaussian contribution of $\boldsymbol{A}$ to the action takes the form of three gradient-squared terms (de Gennes and Prost, 1993), viz.,

$$S_{\mathrm{N}}[\boldsymbol{A}] = \int d\boldsymbol{x}\{K_1[\boldsymbol{\nabla}\cdot\boldsymbol{A}]^2 + K_2[\hat{z}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x}))]^2 + K_3[\partial\boldsymbol{A}(\boldsymbol{x})/\partial z]^2\}, \tag{3.3b}$$

where K_1, K_2, and K_3 are related to elastic constants. The coupling between ψ and $\boldsymbol{A}$ can be shown to be adequately represented by replacing the $\boldsymbol{\nabla}\psi$ in Eq. (3.3a) by

$$\boldsymbol{\nabla}\psi(\boldsymbol{x}) \to [\boldsymbol{\nabla} - iq\boldsymbol{A}(\boldsymbol{x})]\psi(\boldsymbol{x}). \tag{3.3c}$$

We see that the coupling between the order parameter and the director fluctuations takes the same form as the coupling between the order parameter and the vector potential in Eq. (2.15a). It is important to realize, however, that the action given by Eqs. (3.3), in contrast to Eqs. (2.15a), is not invariant under local gauge transformations due to the additional gradient terms in Eq. (3.3b). This has profound consequences for the ordered phase; for instance, it leads, via a Landau-Peierls instability, to the absence of true long-range order in a smectic (Lubensky, 1983; see also footnote 42 below). However, it has been shown that a simplified action obtained by neglecting K_1 and putting $K_2 = K_3 = 1/8\pi\mu$ has the same critical properties (Lubensky, 1983). Although the full action has properties that are in general very different from those of the gauge theory given in Eq. (2.15a), the former thus reduces to the latter for the purposes of determining the critical behavior. Keeping in mind that $\boldsymbol{A}$ lies in the xy plane, an appropriate effective action is

[41] The gradient terms parallel and perpendicular to the layers have different coefficients, but the action can be made isotropic by an appropriate real-space scale change.

$$S_{\mathrm{NA}}[\psi,\boldsymbol{A}] = \int d\boldsymbol{x}\Big\{ r_0|\psi(\boldsymbol{x})|^2 + c|[\nabla - iq\boldsymbol{A}(\boldsymbol{x})]\psi(\boldsymbol{x})|^2 + u_0|\psi(\boldsymbol{x})|^4 + \frac{1}{8\pi\mu}[\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x})]^2\Big\}. \quad (3.4)$$

As we have discussed in Sec. II.A.2, this action formally *is* gauge invariant.[42] We also recognize it as the action of a spin-singlet superconductor, with ψ the superconducting order parameter, $\boldsymbol{A}$ the vector potential, q the Cooper pair charge, and μ the magnetic permeability (see, for example, de Gennes, 1989). The nematic–smectic-*A* transition can thus be mapped onto a superconductor–normal-metal transition, even though the generic soft modes in the superconducting case, viz., the photons, have a very different origin from that of the director fluctuations: The latter are the Goldstone modes of a broken symmetry, while the former are the result of gauge invariance, as was discussed in Sec. II. This remarkable analogy (Halperin and Lubensky, 1974) is a good example of the versatility of effective-field theories. This becomes even more apparent upon the realization that the action (3.4) is closely related to the scalar electrodynamics problem studied by Coleman and Weinberg (1973), which demonstrated a mechanism for the spontaneous generation of mass in particle physics.

We make one final remark to put this section into context. In the disordered phase we have generic long-ranged correlations, or GSI, from the gauge-field fluctations [see Eq. (2.17a)]. In the following subsections we shall see how these generic long-ranged correlations modify the critical behavior compared to the one that would result from S_{A}, Eq. (3.3a), alone. In particular, see Eq. (3.7) below and the discussion following it.

2. Gaussian approximation

The simplest way to deal with the action S_{NA} is to treat it in Gaussian approximation, that is, to neglect the nonlinear coupling between $\boldsymbol{A}$ and ψ. The ψ^4 term needs to be kept for stability reasons, of course. In this approximation the model reduces to an *XY* model, which makes the liquid crystal analogous to a superfluid rather than to a superconductor, and it predicts a continuous transition in the *XY* universality class (de Gennes, 1972). As we know from Sec. II.A.2 and will see again below, this Gaussian approximation misses some crucial aspects of the full model.

[42]It is important to realize, however, that the only physical gauge is still the one in which $A_z=0$. Gauge transformations that take one to, e.g., the Coulomb gauge used in Sec. II.A.2 lead to a transformed order parameter that no longer represents a physical observable. For instance, the transformed order parameter exhibits true long-range order in the smectic phase, whereas, as mentioned in the text, the original one does not (Lubensky, 1983; de Gennes and Prost, 1993). Another way to say this is that the phase of the order parameter in a smectic is observable, while in a superconductor it is not.

3. Renormalized mean-field theory

The action (3.4) is characterized by two length scales.[43] In a superconducting language, they are the coherence length

$$\xi = \sqrt{c/|r|} \quad (3.5a)$$

and the London penetration depth

$$\lambda \equiv k_\lambda^{-1} = \sqrt{1/8\pi\mu c q^2 \langle|\psi(\boldsymbol{x})|^2\rangle}. \quad (3.5b)$$

Here r is the renormalized distance from the critical point (see below). The ratio $\kappa=\lambda/\xi=1/k_\lambda\xi$ is the Landau-Ginzburg parameter. Notice that κ is independent of r, since $\langle|\psi|\rangle\propto\sqrt{|r|}$. $\kappa<1$ characterizes type-I superconductors, in which the order-parameter coherence length is larger than the penetration depth. In a magnetic field, these materials form a Meissner phase that completely expels the magnetic flux.[44] In the extreme type-I limit, $\kappa\ll 1$, order-parameter fluctuations are negligible, $\psi(\boldsymbol{x})\approx\langle\psi(\boldsymbol{x})\rangle\equiv\psi$, and the part of the action that depends on the vector potential takes the form

$$\frac{1}{8\pi\mu}\int d\boldsymbol{x}[k_\lambda^2\boldsymbol{A}^2(\boldsymbol{x}) + (\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x}))^2].$$

We see that in the normal conducting phase the transverse photons are soft, as they should be according to Sec. II.A.2.a (these are the generic soft modes), while in the superconducting phase they acquire a mass proportional to $k_\lambda\propto\psi$, (see Sec. II.A.2.b). Choosing a gauge-fixing term, Eq. (2.16), with $\eta=1$,[45] one finds for the $\boldsymbol{A}$ propagator

$$\langle A_\alpha(\boldsymbol{x})A_\beta(\boldsymbol{y})\rangle = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})\frac{4\pi\mu}{k_\lambda^2+\boldsymbol{\nabla}^2}. \quad (3.6)$$

Since the vector potential no longer couples to any fluctuating fields, it can be integrated out to obtain an action entirely in terms of the superconducting order parameter (Halperin *et al.*, 1974),

[43]The full action for the liquid-crystal problem contains many more length scales. Hence more cases than the type I and type II of the superconductor need to be distinguished. These have not been fully classified.

[44]To avoid misunderstandings, we stress that the conclusions drawn below are valid in the absence of an external magnetic field, as the vector potential $\boldsymbol{A}$ can describe spontaneous electromagnetic fluctuations.

[45]Notice that the renormalized mean-field theory neglects all fluctuations of the field ψ, in contrast to the Gaussian theory discussed in Sec. II.A.2.b. No particular gauge needs to be chosen in order to eliminate the Higgs field. One should also keep in mind that the full action for the liquid crystal, which is not gauge invariant due to the additional gradient terms in Eq. (3.3b), has a different soft-mode spectrum in the ordered phase than the superconductor.

$$S_{\rm eff}[\psi] = \ln \int D[\boldsymbol{A}] e^{-S_{\rm NA}[\psi,\boldsymbol{A}]}$$

$$= V(r_0|\psi|^2 + u_0|\psi|^4) - \frac{1}{2}\mathrm{Tr}\,\ln(\boldsymbol{\nabla}^2 + k_\lambda^2). \qquad (3.7)$$

Here $S_{\rm NA}$ is given by Eq. (3.4), V is the system volume, and we have neglected a constant contribution to the action. The above procedure is exact to the extent that fluctuations of ψ can be neglected; otherwise, it is an approximation. It produces an action entirely in terms of ψ, but, since one has integrated out a soft mode, the price one pays is that this action is nonlocal.[46] The effect of the nonlocality is most conveniently studied by expanding the equation of state in powers of the order parameter ψ, and the free-energy density $f = T S_{\rm eff}/V$ can be obtained by integrating the result order by order. The details have been given by Chen *et al.* (1978). The result for the leading terms in three dimensions is

$$f/T = r|\psi|^2 - v_3|\psi|^3 + u|\psi|^4 - h\psi \quad (d=3). \qquad (3.8a)$$

Here we have added a field h conjugate to the order parameter. r and u are given by r_0 and u_0 with additive corrections proportional to μq^2 and $(\mu q^2)^2$, respectively. These changes do not affect the behavior of the theory. However, $v_3 > 0$ is a positive coupling constant proportional to $\sqrt{\mu q^2}$ that drives the transition first order. Notice that the new term generated by the generic soft modes is nonanalytic in the order parameter; such a term can never appear in a local Landau expansion. This provides a rather extreme example of generic soft modes influencing critical behavior: The critical point is destroyed, and a first-order transition takes place instead. This phenomenon is known as a *fluctuation-induced first-order transition* in condensed-matter physics[47] and as *the Coleman-Weinberg mechanism* in high-energy physics. Interestingly, it is not the soft mode *per se* that produces the first-order transition; it is the fact that a nonzero order parameter gives the soft mode a mass. This is an example of a more general principle that has been explored by Belitz *et al.* (2002). For later reference we also give here the free-energy density in four dimensions (Chen *et al.*, 1978), although this case is not of physical relevance for liquid crystals,

$$f/T = r|\psi|^2 + v_4|\psi|^4 \ln|\psi| + u|\psi|^4 - h\psi \quad (d=4), \qquad (3.8b)$$

with $v_4 > 0$.

Note that the free-energy functionals given by Eqs. (3.8a) and (3.8b) are nonanalytic in ψ and that this nonanalyticity has nothing to do with the phase transition. Indeed, they denote the proper free energy for the order parameter caused by a conjugate field far from the transition. Physically, the nonanalyticity reflects GSI in the disordered phase, and it is directly related to a nonanalytic wave-number dependence of the order-parameter susceptibility χ_ψ. This becomes obvious if one remembers that in any mean-field theory, $r = 1 - \Gamma\chi_\psi$, with Γ the appropriate interaction coupling constant, and that ψ scales like a wave number far from the phase transition, as can be seen from Eqs. (3.5b) and (3.7). In a scaling sense, Eq. (3.8a) thus corresponds to a wave-number dependence of χ_ψ given by

$$\chi_\psi(\boldsymbol{k})/\chi_\psi(0) = 1 + b^{\chi_\psi}|\boldsymbol{k}|, \qquad (3.9a)$$

with $b^{\chi_\psi} > 0$ a positive coefficient. Notice that this is analogous to Eqs. (2.36b) and (2.35d), although χ_ψ is not observable for the superconductor, and is related to an observable susceptibility in a complicated way for the liquid crystal. In Sec. IV we shall discuss examples in which the magnetic susceptibility, which is directly measurable, plays an analogous role. In real space, this behavior corresponds to

$$\chi_{\rm s}(r > 0, |\boldsymbol{x}| \to \infty) \propto 1/|\boldsymbol{x}|^4. \qquad (3.9b)$$

This manifestation of GSI is ultimately responsible for the failure of the Gaussian theory to correctly describe the phase transition.

The conclusion that the nematic–smectic-A transition and the BCS superconducting transition are of first order is inevitable if order-parameter fluctuations are negligible. Quantitatively, the effect turns out to be unobservably small in superconductors, as the first-order nature of the transition is predicted to manifest itself only within about one μK from the transition temperature. In liquid crystals, however, the numbers are much more favorable (Halperin *et al.*, 1974). After a long time of confusion, careful experiments have indeed confirmed the first-order nature of the transition in a variety of liquid crystals (Anisimov *et al.*, 1990; Lelidis, 2001; Yethiraj *et al.*, 2002).

4. Effect of order-parameter fluctuations

Interesting and technically difficult questions arise when order-parameter fluctuations are taken into account. First-order transitions are often thought of as unaffected by order-parameter fluctuations, almost by definition, since they preempt a fluctuation-driven critical point. However, if the first-order transition takes place within the critical region of an unrealized second-order transition, then the critical fluctuations associated with the latter can destabilize the mechanism that drives the transition first order. The resulting transition can then be continuous but described by a fixed point different from the one that destabilizes the fluctuation-induced first-order transition. An explicit example of such a "fluctuation-induced second-order transition" was given by Fucito and Parisi (1981). How this works technically becomes clear once one realizes that, within a RG treatment, a fluctuation-induced first-order transition comes about by the quartic coupling constant u flowing to negative values. Critical fluctuations can weaken this

[46] More precisely, the mode that has been integrated out has a small mass (for small $|\psi|$) that is given by the order parameter.

[47] Notice that the fluctuations in question are not the order-parameter fluctuations, but rather the generic soft modes.

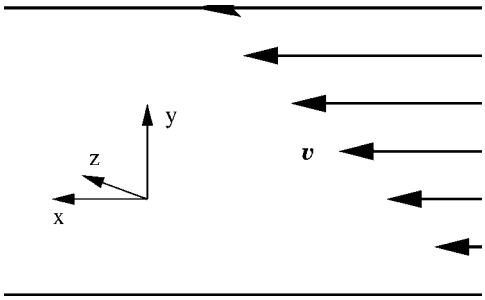

FIG. 11. Velocity field in a fluid subject to a steady, plane Couette flow.

tendency and lead to a positive fixed-point value of u after all. We shall discuss an explicit example of this in Sec. IV.A below.

For the superconducting/nematic–smectic-A transition problem, a RG analysis to one-loop order has been performed by Chen *et al.* (1978). To first order in $\epsilon=4-d$, these authors found no critical fixed point and concluded that the transition is always of first order, both for superconductors and for liquid crystals, and for both type-I and type-II materials. However, later work by Dasgupta and Halperin (1981), prompted by mounting experimental evidence that the transition is of second order in type-II materials, concluded that, as one enters into the type-II region, the first-order transition becomes weaker and weaker until the transition reverts to second order in the so-called inverted XY universality class. This has been confirmed by a sizable body of numerical and analytical evidence (see, for example, Herbut *et al.*, 2001, and references therein). Why the perturbative RG does not show this fixed point is not quite clear.

B. Critical behavior in classical fluids

Our second classical example deals with the critical behavior of a classical fluid, both in equilibrium (Kawasaki, 1976; Hohenberg and Halperin, 1977) and in a nonequilibrium situation created by applying a constant rate of shear (Onuki and Kawasaki, 1979). Let us consider a fluid confined between two parallel plates subject to a constant rate s of shear (see Fig. 11). The average local velocity is given by

$$\boldsymbol{v}(\boldsymbol{x}) = sy\boldsymbol{e}_x, \tag{3.10}$$

with $\boldsymbol{e}_x$ a unit vector in the x direction. This externally induced velocity has to be added to the Navier-Stokes equations (2.21). Close to the liquid-gas critical point, these equations can be replaced by the following set of simpler equations (Kawasaki, 1970; Onuki and Kawasaki, 1979):

$$\frac{\partial\psi}{\partial t} = -sy\frac{\partial}{\partial x}\psi - \rho\partial_\alpha\psi u^\alpha + \frac{\lambda}{2}\partial_\alpha\partial^\alpha\frac{\delta}{\delta\psi}S[\psi] + \theta, \tag{3.11a}$$

$$\frac{\partial u^\alpha}{\partial t} = -\frac{\rho}{2}\left(\psi\partial^\alpha\frac{\delta}{\delta\psi}S[\psi]\right)_\perp + \eta\partial_\beta\partial^\beta u^\alpha + \zeta_\perp^\alpha. \tag{3.11b}$$

Here the order parameter ψ represents the specific entropy, $\boldsymbol{u}$ is the deviation of the local velocity from its average $\boldsymbol{v}$, and λ and η are the bare thermal conductivity and the bare shear viscosity as in Eqs. (2.21).[48] $\boldsymbol{v}_\perp$ denotes the transverse part of a vector $\boldsymbol{v}$. S is a classical action, or free-energy functional, given by

$$S[\psi] = \int d\boldsymbol{x}[r\,\psi^2 + c(\boldsymbol{\nabla}\psi)^2 + u\,\psi^4], \tag{3.11c}$$

and θ and $\boldsymbol{\zeta}$ are Langevin forces that obey

$$\langle\theta(\boldsymbol{x},t)\,\theta(\boldsymbol{x}',t')\rangle = -2\lambda\boldsymbol{\nabla}^2\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t'), \tag{3.11d}$$

$$\langle\zeta_\alpha(\boldsymbol{x},t)\zeta_\beta(\boldsymbol{x}',t')\rangle = -2\,\delta_{\alpha\beta}\eta\boldsymbol{\nabla}^2\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t'). \tag{3.11e}$$

In the absence of shear, $s=0$, Eqs. (3.11) describe model H of Hohenberg and Halperin (1977), a time-dependent Ginzburg-Landau theory for the conserved order parameter ψ coupled to the conserved auxiliary field $\boldsymbol{u}$. Since the shear modes are soft, this coupling influences the critical behavior. We shall now discuss this influence, first in equilibrium and then for a system with $s\neq 0$.

1. Critical dynamics in equilibrium fluids

In Sec. II.B, when discussing long-time tails, we considered corrections to the bare kinematic viscosity ν in the GSI region far from criticality and mentioned that analogous results hold for other transport coefficients. Of particular interest for the critical dynamics near the liquid-gas critical point is a contribution to the correction to the thermal conductivity, $\delta\lambda$, from the coupling of transverse-velocity fluctuations to the entropy fluctuations described by Eqs. (3.11) for $s=0$. Before performing the wave-number integral, this particular contribution, which we denote by $\delta\lambda_{\perp s}$, is (Kawasaki, 1976; Hohenberg and Halperin, 1977)

$$\delta\lambda_{\perp s}(\boldsymbol{k},t) = \frac{1}{\rho^2}\sum_{\boldsymbol{p}}\chi(\boldsymbol{p})\sum_i(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}_\perp^{(i)})^2 e^{-(\nu p_-^2 + D_T p_+^2)t}. \tag{3.12}$$

Here $D_T=\lambda/\rho\,c_p$ is the thermal diffusivity in terms of c_p, the specific heat at constant pressure. χ is the order-parameter susceptibility for the phase transition, where we have anticipated that near the critical point we shall need the momentum-dependent χ, and $\boldsymbol{p}_\pm=\boldsymbol{p}\pm\boldsymbol{k}/2$. Setting $\boldsymbol{k}=0$ and carrying out the momentum integral leads to a $t^{-d/2}$ long-time tail. The correction to the thermal conductivity is obtained by integrating $\delta\lambda(t)$ over all

[48] Here we discuss a pure fluid for simplicity. For experimental purposes, binary fluids are preferred for technical reasons. This changes the interpretation of the various quantities in Eqs. (3.11), but the physics remains the same.

times, as in the case of the kinematic viscosity.[49]

By examining Eq. (3.12) one easily identifies a mechanism by which the long-time-tail effects can become even stronger. Consider a system with long-range static correlations, for instance due to Goldstone modes or to the vicinity of a continuous phase transition. In either case, some susceptibilities, e.g., the χ in Eq. (3.12), become long ranged, amplifying the lont-time-tail effect. Before we discuss the realization of this scenario in the vicinity of a phase transition we make one last point concerning long-time tails. So far we have stressed the leading tails that decay as $t^{-d/2}$, but there are numerous subleading tails as well. Most of them are uninteresting, but one becomes important near the critical point, via the mechanism discussed in the last paragraph. According to Eq. (3.12), a central quantity for determining the critical contribution to λ is the shear viscosity η [which enters ν; see the definition of ν after Eq. (2.24)]. It turns out that the contribution to η that is dominant near the critical point is a subleading long-time tail away from criticality. It involves a coupling of two heat or entropy modes and is given by (Kawasaki, 1976)

$$\delta\eta(\boldsymbol{k},t) = \frac{A}{\boldsymbol{k}^2}\sum_{\boldsymbol{p}} \chi(\boldsymbol{p})\chi(\boldsymbol{k}-\boldsymbol{p})\left(\frac{1}{\chi(\boldsymbol{p})} - \frac{1}{\chi(\boldsymbol{k}-\boldsymbol{p})}\right)^2 \times (\hat{\boldsymbol{k}}_\perp \cdot \boldsymbol{p})^2 e^{-D_T[\boldsymbol{p}^2+(\boldsymbol{k}-\boldsymbol{p})^2]t}, \tag{3.13}$$

with A a constant. Away from the critical point, this long-time tail decays as $t^{-(d/2+2)}$, so in fact it is a next-to-next leading long-time tail. Nevertheless, it is the dominant mode-coupling contribution to η near the critical point because of the two factors of χ in the numerator of Eq. (3.13).

Near continuous phase transitions, fluctuations grow and ultimately diverge at the critical point. For the liquid-gas critical point the order parameter is the difference between the density and the critical density, and the divergent fluctuations are the density fluctuations as described by the density susceptibility. The susceptibility χ in Eq. (3.13) is proportional to this divergent susceptibility. In the Ornstein-Zernike approximation it is given by Eqs. (1.2). Away from the critical point, $r \neq 0$, χ decays exponentially in real space; see Eq. (1.2b).

Carrying out the time integral, we find that the leading singular contribution to the static, wave-number-dependent thermal conductivity is (Hohenberg and Halperin, 1977)

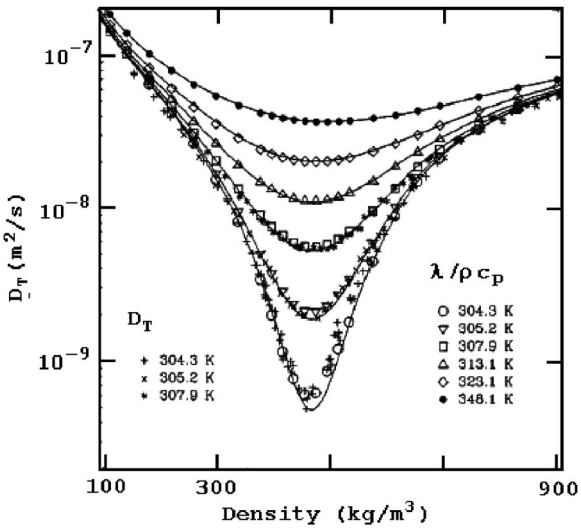

FIG. 12. The thermal diffusivity $D_T = \lambda/\rho\, c_p$ of carbon dioxide in the critical region as a function of density at various temperatures (T_c=304.12 K). The symbols indicate experimental data for D_T measured directly, and for $\lambda/\rho\, c_p$ deduced from thermal-conductivity data. The solid curves represent values calculated from the mode-coupling theory. Adapted from Luettmer-Strathmann *et al.*, 1995.

$$\delta\lambda(\boldsymbol{k}) = \frac{1}{\rho^2}\sum_{\boldsymbol{p}} \chi(\boldsymbol{p}) \frac{\sum_i (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}_\perp^{(i)})^2}{\nu\boldsymbol{p}_-^2 + D_T\boldsymbol{p}_+^2}. \tag{3.14}$$

Using Eq. (1.2a) in this equation we see that the homogeneous thermal conductivity is infinite at the critical point for all $d \leqslant 4$, diverging as $|r|^{-(4-d)/2}$. This is a result of the amplification of the long-time tail by the critical fluctuations.

By the same mechanism, we see that Eq. (1.2a) leads to a logarithmically singular contribution to $\delta\eta$, if we take into account that Eq. (3.14) implies that, at the critical point, $D_T(\boldsymbol{k}) \sim |\boldsymbol{k}|^{d-2}$.

The above conclusions result from one-loop calculations that use the Ornstein-Zernike susceptibility. To go beyond these approximations it is necessary to (1) use the correct scaling form for the susceptibility, and (2) use either a self-consistent mode-coupling theory (Kadanoff and Swift, 1968) or a RG approach (Forster *et al.*, 1977; Hohenberg and Halperin, 1977) to improve on the one-loop approximation. The result is that the thermal conductivity diverges as

$$\lambda \propto |r|^{-0.57} \tag{3.15a}$$

and that the thermal diffusivity vanishes as

$$D_T \propto |r|^{0.67}. \tag{3.15b}$$

This is in good agreement with experimental results, as shown in Fig. 12, which compares experimental and theoretical results for the thermal diffusivity of carbon dioxide in the critical region.

All of the above results are obtained by considering the effects on the dynamics (i.e., on transport coefficients) of the static critical behavior, as expressed by the

[49]These results imply that for $d \leqslant 2$, conventional hydrodynamics does not exist. Indeed, it is now known that for these dimensions the hydrodynamic equations are nonlocal in space and time. For a discussion of this topic, see Forster *et al.* (1977), and references therein. In smectic liquid-crystal phases this effect is stronger, since some susceptibilities behave, for certain directions in wave-vector space, as $1/\boldsymbol{k}^4$, amplifying the long-time-tail effect. This causes a breakdown of local hydrodynamics for all dimensions $d \leqslant 5$ (Mazenko *et al.*, 1983).

susceptibility χ, which has been determined independently. There is no feedback of the critical dynamics on the statics. This is an example of the missing coupling between the statics and the dynamics in classical equilibrium systems that we alluded to in the Introduction. As we shall see, this changes in a nonequilibrium situation.

2. Critical dynamics in a fluid under shear

Let us now consider Eqs. (3.11) in the presence of shear, $s \neq 0$. In this situation the entire fluid can still be at its critical temperature and density. One therefore still expects a sharp phase transition (in an infinitely large system), albeit a nonequilibrium one. This is in contrast to driving the system out of equilibrium by means of, e.g., a temperature gradient. Inspecting Eq. (3.11a), we make the following observations: (1) Shear will have a tendency to make the order-parameter susceptibility less soft. This is obvious since the operator $y\partial/\partial x$ in Eq. (3.11a), which scales like a mass by naive power counting, enters additively to the diffusive operator $\boldsymbol{\nabla}^2$. In addition, the susceptibility will become anisotropic. (2) $1/s$ sets a new time scale in the problem, which needs to be compared to the relaxation time τ. One therefore expects equilibrium critical behavior in the weak-shear region $s\tau<1$, and a crossover to different behavior in the strong-shear region $s\tau>1$. Alternatively, one can define a characteristic wave number k_s by $\tau(k_s)=1/s$. The strong-shear region is then given by $k_s\xi>1$, with ξ the order parameter correlation length.

It follows from point (1) above that the upper critical dimension cannot be greater than the one in the equilibrium case, which is equal to 4. At least for $d>4$, the nonlinearities in the stochastic equations (3.11) must therefore be irrelevant. In the linearized equations, ψ and $\boldsymbol{u}$ decouple, and the equation for the former becomes

$$\frac{\partial\psi}{\partial t} = -sy\frac{\partial}{\partial x}\psi + \lambda\boldsymbol{\nabla}^2(r-\boldsymbol{\nabla}^2)\psi + \theta. \tag{3.16}$$

The static order-parameter susceptibility,

$$\chi_\psi(\boldsymbol{k}) = \frac{1}{T}\int\frac{d\Omega}{2\pi}\int dt\; e^{i\Omega t}\langle\psi_{\boldsymbol{k}}(t)\psi_{-\boldsymbol{k}}(0)\rangle, \tag{3.17}$$

can be obtained from Eq. (3.16), Fourier transforming and using Eq. (3.11d). This yields the following differential equation for χ_ψ:

$$\left[\lambda\,\boldsymbol{k}^2(r+\boldsymbol{k}^2) - \frac{1}{2}sk_x\frac{\partial}{\partial k_y}\right]\chi_\psi(\boldsymbol{k}) = \lambda\boldsymbol{k}^2. \tag{3.18}$$

In the strong-shear region, the solution of this equation is adequately represented by (Onuki and Kawasaki, 1979)

$$\chi_\psi(\boldsymbol{k}) = 1/(r + \text{const}\times\; k_s^{8/5}|k_x|^{2/5} + \boldsymbol{k}^2), \tag{3.19a}$$

with the constant of O(1). This is the generalization of the Ornstein-Zernike susceptibility, Eq. (1.2a), to a sheared system. [Remember that χ in Eq. (3.14) is proportional to χ_ψ.] As expected, $\chi_\psi(\boldsymbol{k})$ is strongly anisotropic and less soft than for $s=0$. The nonequilibrium situation induces long-ranged static order-parameter correlations even away from criticality that manifest themselves in the nonanalytic wave-number dependence of χ_ψ. In real space in $d=3$, one finds power-law correlations at $r\neq 0$ (Onuki and Kawasaki, 1979),

$$\chi_\psi(r>0, |\boldsymbol{x}|\to\infty) \propto 1/|x|^{7/5} \tag{3.19b}$$

for $|\boldsymbol{x}_\perp| \ll k_s^{-1} \ll |x|$, where $\boldsymbol{x}=\boldsymbol{x}_\perp + x\hat{\boldsymbol{e}}_x$. For $|\boldsymbol{x}_\perp| \gg |x|$, however, χ_ψ decays exponentially. That is, χ_ψ exhibits GSI (and extreme anisotropy). At criticality, $r=0$, χ_ψ is of even longer range (Onuki and Kawasaki, 1979). If we use Eq. (3.19a) for $r=0$ in Eq. (3.14), we find

$$\delta\lambda(\boldsymbol{k}=0) \propto \int_0^{k_s} dp\; p^{d-3}k_s^{-8/5}|p_x|^{-2/5}$$

$$\propto k_s^{-8/5}\int_0^{k_s} dp\; p^{d-17/5}. \tag{3.20}$$

In contrast to the equilibrium situation, where $\delta\lambda(\boldsymbol{k}=0)$ diverged for all $d\leqslant 4$, we see that in the presence of shear the divergence occurs only for $d\leqslant 12/5$.

The above result suggests that the critical behavior of the thermal conductivity λ is mean-field-like, i.e., given by the time-dependent Ginzburg-Landau theory, for $d>12/5$, and in particular in $d=3$. The physical reason is that the long-range order-parameter correlations stabilize the mean-field critical behavior. It is not obvious that this conclusion is correct, though, as it has been derived by using the approximation (3.19a), which neglected the nonlinearities in the equations of motion. Onuki and Kawasaki (1979) have performed a RG analysis which shows that the simple argument given above is indeed correct. In the presence of shear, the upper critical dimensionality is $d_c^+=12/5$, and for $d>d_c^+$ there is a new simple critical fixed point where χ_ψ and $\delta\lambda$ are given by Eqs. (3.19a) and (3.20), respectively. The RG treatment shows that the flow equations for χ_ψ and the transport coefficients are coupled. This means that, contrary to the equilibrium case, it is *not* possible to solve for the static critical behavior independently of the dynamics. Onuki and Kawasaki (1979) also considered the equation of state, which they found to be of mean-field form for $d>12/5$. In particular, the critical exponent β has its mean-field value $\beta=1/2$.

Another theoretical prediction regards the suppression of the critical temperature by the shear. Consider the inverse correlation length ξ^{-1} as a function of r and s. Since s is an inverse time, it is expected to scale with the dynamical critical exponent z. Dynamical scaling then predicts the relation

$$\xi^{-1}(r,s) = b^{-1}\xi^{-1}(r\,b^{1/\nu}, s\,b^z) = r^\nu\xi^{-1}(1, s/r^{\nu z}), \tag{3.21}$$

with b an arbitrary scale factor, and ν and z the correlation length exponent and the dynamical critical exponent, respectively, at the equilibrium transition. For $s\neq 0$, the inverse correlation length thus vanishes not at $r=0$, but rather at $r=\text{const}\times s^{1/\nu z}$. The shear dependence

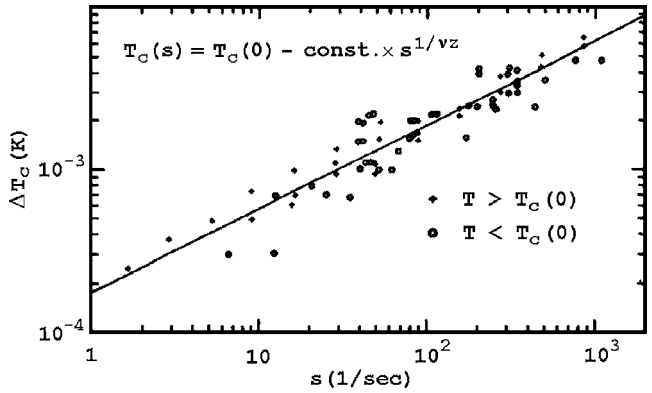

FIG. 13. Critical temperature vs shear in the binary fluid cyclohexane-aniline. Symbols represent the experimental data, and the solid line corresponds to the theoretical exponent $1/\nu z \approx 0.53$. Adapted from Beysens and Gbadamassi, 1980.

of the critical temperature, for small shear, is therefore

$$T_c(s) = T_c(0) - \text{const} \times s^{1/\nu z}. \tag{3.22}$$

At the equilibrium transition, $\nu \approx 0.63$, and $z \approx 3$ in $d=3$ (Hohenberg and Halperin, 1977). Many of these results have been confirmed experimentally in light-scattering experiments on binary fluids (Beysens and Gbadamassi, 1980). For the T_c suppression, the experimental results are shown in Fig. 13 together with the theoretical prediction.

The classical nonequilibrium phase transition described above is very much analogous to the equilibrium quantum phase transition in a ferromagnet with quenched disorder, which we shall discuss in Sec. IV.B. In that case, too, the order-parameter susceptibility reflects long-ranged correlations, although they are due to generic soft modes rather than a nonequilibrium situation. This in turn leads to the stabilization of a simple Gaussian critical fixed point, much like in the nonequilibrium classical fluid. An important difference is that in the quantum ferromagnetic case, the equation of state is *not* mean-field-like; see Eq. (4.24b) below. The reason for this difference is not known.

IV. INFLUENCE OF GENERIC SCALE INVARIANCE ON QUANTUM CRITICAL BEHAVIOR

We now turn to quantum systems. As we have discussed, the only qualitative difference compared to the classical case is the coupling between the statics and the dynamics in quantum mechanics, which leads to long-time tails affecting the critical behavior of thermodynamic quantities even in equilibrium. Otherwise, the phenomena are qualitatively very much analogous. However, there is an important quantitative difference. In any itinerant electron system, there are generic soft modes, viz., the particle-hole excitations that can be understood as Goldstone modes, as explained in Sec. II.B.2, which are soft *only* at zero temperature. Our first example for which these particle-hole excitations play the role of the generic soft modes is the case of the ferromagnetic transition in metals at $T=0$. We shall treat clean and disordered systems separately, since they turn out to behave quite differently and are qualitatively remarkably similar to the classical nematic–smectic-A transition discussed in Sec. III.A, on the one hand, and to the critical dynamics of a classical fluid under shear (Sec. III.B.2), on the other. Another example is the transition from a normal metal to a BCS superconductor $T=0$, which we cover in Sec. IV.C. The case of the quantum antiferromagnetic transition, which is very different and at this point rather incompletely understood, is discussed in Sec. IV.D.

A. Quantum ferromagnetic transition in clean systems

Let us consider the phase transition in a clean itinerant quantum Heisenberg ferromagnet as our first example of a quantum phase transition that is influenced by generic soft modes. In Sec. IV.B we shall discuss the effects of quenched disorder and also give some additional technical details.

Our exposition does not follow historical lines. The first detailed theory of the quantum ferromagnetic transition was given in an influential paper by Hertz (1976), who derived an effective action from a microscopic model and concluded that the critical behavior is mean-field-like in all dimensions $d>1$. As we shall show, this conclusion was the result of an approximation that ignored all but the most basic aspects of the generic soft modes.

1. Soft-mode action for itinerant quantum ferromagnets

The order parameter is the fluctuating magnetization field $\boldsymbol{M}(x)$ [with $x=(\boldsymbol{x},\tau)$ the space–imaginary-time coordinate, as in Sec. II.B.2], whose expectation value is proportional to the magnetization m. The generic soft modes in this system are the diffusive particle-hole excitations. They were parametrized in Sec. II.B.2 in terms of the soft sector q of the matrix field Q [see Eqs. (2.46) and (2.47), with the corresponding propagator given by Eq. (2.48c)]. The coupling between the two is provided by the fact that the magnetization couples linearly to the spin density, which can be expressed in terms of the Q. The action will thus consist of a part that depends only on the magnetization, a part that depends only on the generic soft modes, and a coupling between the two,

$$\mathcal{A}[\boldsymbol{M},Q] = \mathcal{A}_M + \mathcal{A}_q + \mathcal{A}_{M,q}. \tag{4.1}$$

$\mathcal{A}_M$ is a static, local, LGW functional for the magnetization fluctuations. It can be chosen static because, as we shall see in Sec. IV.A.2, the low-frequency dynamical part will be generated by the coupling to the ballistic modes. We thus write

$$\mathcal{A}_M[\boldsymbol{M}] = \int dx[\boldsymbol{M}(x)(r_0 - c\boldsymbol{\nabla}^2)\boldsymbol{M}(x) + u_0\boldsymbol{M}^4(x)]. \tag{4.2}$$

$\mathcal{A}_q$ must yield the ballistic propagator of the soft modes, Eqs. (2.48). The Gaussian part of the fermionic action will therefore have the form

$$\mathcal{A}_q^{(2)} = \frac{-4}{G}\int d\boldsymbol{x}\, d\boldsymbol{y} \sum_{1,2,3,4}\sum_{r,i} {}^i_r q_{12}(\boldsymbol{x})\,{}^i\Gamma^{(2)}_{12,34}(\boldsymbol{x}-\boldsymbol{y})\,{}^i_r q_{34}(\boldsymbol{y}). \tag{4.3a}$$

The particle-particle degrees of freedom are irrelevant for this problem, and we can therefore restrict the sum over the index r to $r=0,3$. The vertex function $\Gamma^{(2)}$ is most easily written in momentum space,

$$\begin{aligned}{}^i\Gamma^{(2)}_{12,34}(\boldsymbol{k}) = {}& \delta_{13}\delta_{24}\Gamma^{(2,0)}_{12}(\boldsymbol{k}) + \delta_{1-3,2-4}\delta_{i0}2\pi TGK_{\mathrm{s}} \\ & + \delta_{1-3,2-4}(1-\delta_{i0})2\pi TG\tilde{K}_{\mathrm{t}},\end{aligned} \tag{4.3b}$$

with

$$\Gamma^{(2,0)}_{12}(\boldsymbol{k}) = |\boldsymbol{k}| + GH\Omega_{1-2}. \tag{4.3c}$$

The inverse of $\Gamma^{(2,0)}$ yields the noninteracting propagator, Eqs. (2.48a) and (2.48c), the inverse of $\Gamma^{(2)}$, its generalization to interacting systems. K_{s} is the spin-singlet interaction amplitude defined after Eq. (2.48c), and $\tilde{K}_{\mathrm{t}}$ is a "residual" spin-triplet interaction.[50]

$\mathcal{A}_{M,q}$ originates from a term $\mathcal{A}_{M-Q}$ that couples $\boldsymbol{M}$ and Q. Such a term must be present, since in the presence of a magnetization the fermionic spin density will couple linearly to it. Using Eq. (2.44) to express the spin density in terms of Q, we thus obtain

$$\mathcal{A}_{M-Q} = 2c_1\sqrt{T}\int d\boldsymbol{x}\sum_n\sum_{i=1}^{3} M^i_n(\boldsymbol{x}) \times \sum_{r=0,3}(-1)^{r/2}\sum_m \mathrm{tr}[(\tau_r\otimes s_i)Q_{m,m+n}(\boldsymbol{x})], \tag{4.4a}$$

with a model-dependent coefficient c_1. Defining a symmetrized magnetization field by

$$b_{12}(\boldsymbol{x}) = \sum_{i,r}(\tau_r\otimes s_i)^i_r b_{12}(\boldsymbol{x}), \tag{4.4b}$$

with components

$${}^i_r b_{12}(\boldsymbol{x}) = (-)^{r/2}\sum_n \delta_{n,n_1-n_2}[M^i_n(\boldsymbol{x}) - (-)^r M^i_{-n}(\boldsymbol{x})] \tag{4.4c}$$

allows us to rewrite Eq. (4.4a) in a more compact form,

$$\mathcal{A}_{M-Q} = c_1\sqrt{T}\int d\boldsymbol{x}\ \mathrm{tr}(b(\boldsymbol{x})Q(\boldsymbol{x})). \tag{4.4d}$$

Using Eq. (2.46) in Eq. (4.4a) or Eq. (4.4d), and integrating out the massive P fluctuations, obviously leads to a series of terms coupling $\boldsymbol{M}$ and q, $\boldsymbol{M}$ and q^2, etc. We thus obtain $\mathcal{A}_{M,q}$ in the form of a series,

$$\mathcal{A}_{M,q} = \mathcal{A}_{M-q} + \mathcal{A}_{M-q^2} + \cdots. \tag{4.5a}$$

The first term in this series is obtained by just replacing Q with q and $q^\dagger$, respectively, in Eq. (4.4d):

$$A_{M-q} = 8c_1T^{1/2}\sum_{12}\int d\boldsymbol{x}\sum_r\sum_{i=1}^{3} {}^i_r b_{12}(\boldsymbol{x})\,{}^i_r q_{12}(\boldsymbol{x}). \tag{4.5b}$$

The next term in this expansion has an overall structure

$$\mathcal{A}_{M-q^2} \propto c_2\sqrt{T}\int d\boldsymbol{x}\ \mathrm{tr}(b(\boldsymbol{x})q(\boldsymbol{x})q^\dagger(\boldsymbol{x})), \tag{4.5c}$$

with c_2 another positive constant. The detailed structure (Kirkpatrick and Belitz, 2002) can be obtained by integrating out the P fluctuations in the tree approximation, in analogy to the derivation of the nonlinear σ model in the disordered case (Wegner and Schäfer, 1980; McKane and Stone, 1981). Terms of higher order in q in this expansion will turn out to be irrelevant for determining the behavior at the quantum phase transition.

Let us pause here and compare our soft-mode action with the one for the classical liquid-crystal transition in Sec. III.A.1. The order-parameter part of the action, Eqs. (3.3a) and (4.2), respectively, is a ϕ^4 theory in either case, and the generic soft modes are described by a Gaussian action, Eqs. (3.3b) and (4.3a), respectively. Finally, in either case there is a direct coupling between the order parameter and the generic soft modes. In the case of the liquid crystal, this coupling is between the square of the order parameter and the square of the soft-mode field, while in the case of the magnet the order parameter couples linearly to all powers of the soft-mode field, but this does not have major physical consequences, as we shall see. Apart from this, the main difference is that in the quantum case the fields depend

[50] One might argue that the spin-triplet degrees of freedom are included in $\boldsymbol{M}$, so also including them in $\mathcal{A}_q$ constitutes double counting. To see that this is not true, imagine deriving the effective action from a microscopic model, e.g., Eqs. (2.37)–(2.39). This introduces $\boldsymbol{M}$ by decoupling the spin-triplet interaction term in Eq. (2.39a) by means of a Hubbard-Stratonovich transformation (Stratonovich, 1957; Hubbard, 1959), which leaves spin-triplet degrees of freedom in the noninteracting part of the action S_0 and hence in $\mathcal{A}_q^{(2)}$.

A weaker version of the same objection is that $\tilde{K}_{\mathrm{t}}$ should be zero, since it represents the interaction that has been decoupled. However, this argument ignores the fact that a spin-triplet interaction will be generated from the spin-singlet one under renormalization, even if there is none in the bare action. The only restriction on $\tilde{K}_{\mathrm{t}}$ is therefore that it must not be so large as to induce a ferromagnetic instability in the electronic "reference system" described by $\mathcal{A}_q$ in the absence of a coupling to $\boldsymbol{M}$.

on time or frequency in addition to position or wave vector, which leads to a more complicated detailed structure of the terms in the action, the coupling terms in particular. Given these structural similarities, it is natural to analyze the quantum ferromagnetic transition in analogy to the liquid crystal one of Secs. III.A.2 and III.A.3.

2. Gaussian approximation

A strict Gaussian approximation neglects the term A_{M-q^2}, Eq. (4.5c). One can then integrate out the intrinsic soft modes q to obtain an action (Hertz, 1976)

$$\mathcal{A}_{\mathrm{H}}[\boldsymbol{M}] = \sum_k \boldsymbol{M}(k)\mathcal{M}_n^{-1}(\boldsymbol{k})\boldsymbol{M}(-k) + O(\boldsymbol{M}^4), \tag{4.6}$$

where $k=(\boldsymbol{k},\Omega_n)$ is a momentum-frequency four-vector, $d=4Gc_1^2/\pi$, and $\mathcal{M}^{-1}$ is the inverse of the paramagnon propagator,

$$\mathcal{M}_n(\boldsymbol{k}) = \frac{1}{r_0 + c\boldsymbol{k}^2 + d|\Omega_n|/(|\boldsymbol{k}| + GH|\Omega_n|)}. \tag{4.7a}$$

In the context of the full action, $\mathcal{M}$ gives the Gaussian $\boldsymbol{M}$ propagator,

$$\langle M_n^i(\boldsymbol{k})M_m^j(\boldsymbol{p})\rangle = \delta_{\boldsymbol{k},-\boldsymbol{p}}\delta_{n,-m}\delta_{ij}\frac{1}{2}\mathcal{M}_n(\boldsymbol{k}). \tag{4.7b}$$

$\mathcal{M}$ and the fermionic propagator $\mathcal{D}$, Eq. (2.48c), with the latter suitably generalized to allow for the interaction amplitudes K_{s} and $\tilde{K}_{\mathrm{t}}$, are the Gaussian propagators of the coupled field theory.[51] The $|\Omega_n|/|\boldsymbol{k}|$ structure of the dynamical piece of $\mathcal{M}$ is characteristic of the itinerant electron degrees of freedom that couple to the magnetic ones; it also manifests itself in the frequency dependence of the Lindhard function (see, for example, Pines and Nozières, 1989). Technically, it results from the inverse of the vertex $\Gamma^{(2)}$, Eq. (4.3b), which gets multiplied by a frequency due to frequency restrictions inherent in the definition of the q. This illustrates how the coupling to the generic soft modes generates the dynamics of the order parameter; see the remark above Eq. (4.2).

$\mathcal{A}_{\mathrm{H}}$ is Hertz's action, which predicts a continuous transition with mean-field critical behavior for all dimensions $d>1$. From our experience with the analogous classical transitions in Sec. III.A above, we suspect that the Gaussian approximation yields qualitatively incorrect results. Indeed, internal inconsistencies of Hertz's results have been discussed by Sachdev (1994) and Dzero and Gorkov (2003). More explicitly, the instability of Hertz's fixed point can be shown formally by means of arguments analogous to those we shall present in Sec. IV.B.4.b for the case of disordered magnets. In what follows, we show that the transition can be either of first order or of second order with non-mean-field critical behavior.

[51] There also are mixed propagators $\langle b\, q\rangle$, but they do not enter any diagrams that are important for the phase transition.

3. Renormalized mean-field theory

As a first step to improve upon the Gaussian approximation, it is natural to construct a renormalized mean-field theory in analogy to Sec. III.A.3. The relevant length scales are the magnetic coherence length

$$\xi = \sqrt{c/|r|} \tag{4.8a}$$

and a length

$$\lambda = 1/G\ H^2 c_2\sqrt{T}\langle M_n^3(\boldsymbol{x})\rangle = 1/\Lambda\ c_2 m, \tag{4.8b}$$

where m is the magnetization. $\Lambda = G\sqrt{\pi H^3/8}$ depends on microscopic length and energy scales only and ensures that m is dimensionally an inverse volume. In a magnetic phase, λ determines the mass in the transverse spin-triplet q propagator, just as the London penetration depth determines the mass of the transverse photon in a superconductor, or the mass of the director fluctuations in a smectic-A phase; see Eq. (3.6). The Ginzburg-Landau parameter,

$$\kappa = \lambda/\xi \propto \sqrt{u}/c_2\sqrt{c}, \tag{4.8c}$$

is again independent of m or $|r|$.

Let us now neglect the fluctuation of the magnetic order parameter, i.e., we put

$$M_n^i(\boldsymbol{x}) \approx \delta_{i3}\delta_{n0} m\Lambda/GH^2\sqrt{T}. \tag{4.9}$$

In the limit $\lambda \ll \xi$ this approximation becomes exact; in general, its validity will need to be investigated. q can then be integrated out, in exact analogy to the treatment of $\boldsymbol{A}$ in Sec. III.A.3. The result for the free-energy density f, in a magnetic field h and with $f_0=f(m=0)$, is

$$f = f_0 + r_0 m^2 + u_0 m^4 - hm + \frac{2}{V}\sum_{\boldsymbol{k}<\Lambda} T\sum_n \ln N(\boldsymbol{k},\Omega_n;m). \tag{4.10a}$$

Λ is an ultraviolet momentum cutoff, and

$$N(\boldsymbol{k},\Omega_n;m) = 16\ c_2^2 G^4 \tilde{K}_{\mathrm{t}}^2 m^2\Omega_n^2 + (|\boldsymbol{k}| + GH\Omega_n)^2 \times[|\boldsymbol{k}| + G(H+\tilde{K}_{\mathrm{t}})\Omega_n]^2. \tag{4.10b}$$

Minimizing f with respect to the magnetization gives the equation of state.

The integral in Eq. (4.10a) has been analyzed by Kirkpatrick and Belitz (2002). In $d=3$, the equation of state was found to take the form

$$h = 2rm + 4vm^3\ln(m^2+T^2) + m^3\left(4u_0 + 2v\frac{m^2}{m^2+T^2}\right). \tag{4.11a}$$

For the free-energy density at $T=h=0$, this implies (Belitz *et al.*, 1999)

$$f = f_0 + rm^2 + v_3 m^4 \ln m^2 + um^4 \tag{4.11b}$$

in $d=3$, and

$$f = f_0 + r\,m^2 - v_d m^{d+1} + u\,m^4 \tag{4.11c}$$

in generic dimensions. In these equations, f, m, and T are measured in terms of suitable microscopic quantities such that r, v_d, and u are all dimensionless. r and u are given by r_0 and u_0, respectively, plus additive renormalizations from the soft modes, in analogy to the classical example of renormalized mean-field theory in Sec. III.A.3. $v_d > 0$ is quadratic in c_1^2, so in strongly correlated systems v_d is larger than in weakly correlated ones. A comparison with Eq. (3.8b) shows that the free energy for the quantum ferromagnet in $d=3$ is precisely analogous to that of the classical liquid crystal in $d=4$ in the same approximation.

This free-energy functional is nonanalytic in m, in analogy to Eqs. (3.8) in our classical example, and the same discussion applies. Namely, the nonanalyticity reflects GSI in the paramagnetic phase, and it is directly related to the nonanalytic wave-number dependence of the spin susceptibility, Eq. (2.36b). In real space, the latter corresponds to

$$\chi_s(r > 0, |\boldsymbol{x}| \to \infty) \propto 1/|\boldsymbol{x}|^{2d-1}. \tag{4.12}$$

This manifestation of GSI is the ultimate physical reason behind the failure of Hertz theory to correctly describe the quantum critical behavior.

Note that the derivation of the above mean-field result for the free energy, Eqs. (4.11b) and (4.11c), implicity assumes that the nonanalytic wave-number dependence of χ_s given by Eq. (2.36b) or Eq. (4.12), which is exact in the paramagnetic phase, continues to hold in the critical region. This is a nontrivial assumption. In Sec. IV.A.4 below we shall discuss how critical fluctuations can modify this result. This point has recently been addressed by Chubukov *et al.* (2003).

The phase diagram predicted by these equations in $d=3$ is shown, in a more general context, in the first panel of Fig. 20 below. There is a tricritical point at

$$T = T_{tc} = e^{-u/2v}. \tag{4.13a}$$

At $T=0$, there is a first-order phase transition at $r=r_1$, with the magnetization changing discontinuously from zero to a value m_1. One finds

$$r_1 = v\,m_1^2, \quad m_1 = e^{-(1+u/v)/2}. \tag{4.13b}$$

In $d=2$, there is no finite-temperature magnetic phase transition. However, at zero temperature there is a quantum phase transition, which is predicted by Eqs. (4.10) to be discontinuous. In $d>3$ the nonanalytic terms produced by the soft modes are subleading, and the transition is described by ordinary mean-field theory. The generalized mean-field theory thus suggests an upper critical dimension $d_c^+ = 3$. As we shall see in the next subsection, a more sophisticated analysis confirms this result.

4. Effects of order-parameter fluctuations

The renormalized mean-field theory predicts that the quantum ferromagnetic transition in clean systems is *always* of first order. As we have already discussed, this conclusion is certainly valid in the limit $\lambda \ll \xi$. In general, however, the order-parameter fluctuations need to be taken into account. This can be done by a systematic RG analysis of the action, Eq. (4.1). In contrast to the liquid-crystal case, it turns out that a one-loop calculation predicts, under certain conditions, a critical fixed point that corresponds to a second-order transition.

The fixed point found by Kirkpatrick and Belitz (2002) has the property that G is marginal. To one-loop order, in $d=3$, and for $r=0$, the coupling constants u, H, c_2, and c obey the flow equations[52]

$$\frac{du}{d\ln b} = -2u - A_u c_2^2/H, \tag{4.14a}$$

$$\frac{dH}{d\ln b} = A_H/c, \quad \frac{dc}{d\ln b} = -A_c/H, \tag{4.14b}$$

$$\frac{dc_2}{d\ln b} = -c_2, \tag{4.14c}$$

with b the RG length rescaling factor, and A_u, A_H, and A_c positive constants. In a purely perturbative treatment, c_2 and H on the right-hand side of Eq. (4.14a) are constants, and u inevitably becomes negative at large scales ($b \to \infty$). A negative u is usually interpreted as signaling a first-order transition;[53] this is the RG version of the conclusion that the transition is always of first order. However, this perturbative argument is not consistent since, at the same level of the analysis, Eq. (4.14b) predicts a singular specific-heat coefficient H, which couples back into the coefficients of the Landau theory. This feedback is taken into account by solving the one-loop equations (4.14) self-consistently, whereby the one-loop correction to the u-flow equation decreases with increasing scale, due to a combination of c_2 being irrelevant and H increasing under renormalization. The conclusion that u flows to negative values is therefore no longer inevitable. Indeed, a solution of the above flow equations (Kirkpatrick and Belitz, 2002) shows that

$$u(b \to \infty) = [u_0 - (A_u/A_c)(c_{2,0})^2 c_0] b^{-2} \propto (\kappa^2 - \kappa_c^2) b^{-2}, \tag{4.15}$$

with $u_0 = u(b=1)$, etc., the bare coupling constants, κ the Ginzburg-Landau parameter from Eq. (4.8c), and $\kappa_c \propto \sqrt{A_u/A_c}$. We see that the RG analysis predicts an

[52] At this level of detail in the discussion it is not obvious that there are two time scales in the problem, one related to the order-parameter fluctuations and the other to the fermionic degrees of freedom. As a consequence, c_2, which couples the two, does not have a unique scale dimension and can be either marginal or irrelevant, depending on the context. The flow equation (4.14c) applies to its irrelevant incarnation. We shall discuss this point in more detail in Sec. IV.B.4.

[53] While this interpretation of u flowing to negative values is not always correct, in the current case it is bolstered by the renormalized mean-field theory.

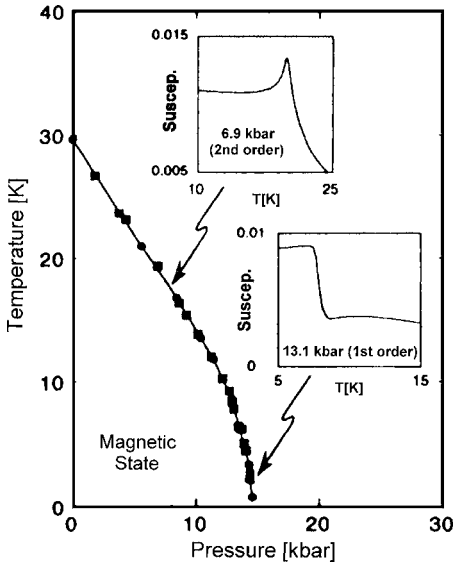

FIG. 14. Phase diagram of MnSi as measured by Pfleiderer *et al.* (1997). The insets show the behavior of the susceptibility close to the transition point. From Vojta *et al.*, 1999.

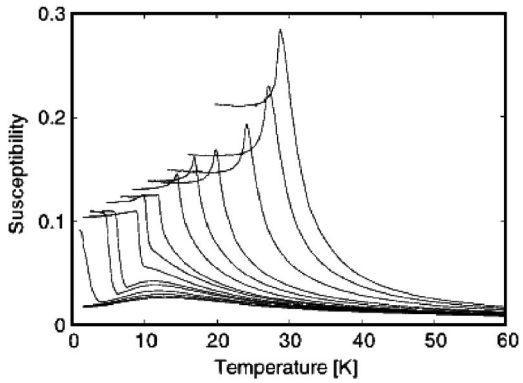

FIG. 15. Magnetic susceptibility (in SI units) vs temperature for MnSi at pressures corresponding to the data points in Fig. 14. Pressure values for the curves from right to left: 1.80, 3.80, 6.90, 8.60, 10.15, 11.25, 12.15, 13.45, 13.90, 14.45, 15.20, 15.70, and 16.10 kbar. The change from a sharply peaked susceptibility consistent with a second-order transition to a discontinuous one, as expected for a first-order transition, is apparent. From Pfleiderer *et al.*, 1997.

asymptotically negative value of u and hence a fluctuation-induced first-order transition, for small values of the Ginzburg-Landau parameter, $\kappa < \kappa_c$, in agreement with the renormalized mean-field theory and in analogy with the liquid-crystal case. For $\kappa > \kappa_c$, however, u stays positive and one finds a critical fixed point, in contrast to the liquid-crystal case. As mentioned in Sec. III.A.4 above, the mechanism for this "fluctuation-induced second-order transition" is very similar to the one discussed by Fucito and Parisi (1981) for a classical system. The critical behavior at this transition will be summarized in the next section.

There is experimental evidence that the quantum ferromagnetic transition in clean systems is of first order in some materials but continuous in others, consistent with the theoretical picture given above. Pfleiderer *et al.* (1997) have reported a continuous transition in MnSi at moderate hydrostatic pressures corresponding to relatively high values of T_c, while the transition becomes first order if T_c is driven to low values by higher values of the pressure; see Figs. 14 and 15.[54] Qualitatively the same magnetic phase diagram has been observed in UGe_2 (Huxley *et al.*, 2001); see Fig. 16. In $ZrZn_2$, however, the magnetic transition is of second order down to the lowest temperatures observed (Pfleiderer, Uhlarz, *et al.*, 2001). The latter two materials are of particular interest since they also have a phase of coexistent ferromagnetism and superconductivity. The superconductivity is very vulnerable to disorder, which testifies to the fact that the samples in question are very clean. Another material in which a continuous transition is observed down to very low temperatures is Ni_xPd_{1-x} (Nicklas *et al.*, 1999). While the substitutional nature of this system inevitably introduces some disorder, the quantum phase transition occurs at low Ni concentration ($x \approx 0.02$), and disorder is believed to play a minor role at the transition.

While the soft-mode theory is consistent with these observations, it is not the only possible explanation. Another obvious possibility is band-structure effects, which can change the sign of the coefficient u in an ordinary Landau theory [Eq. (4.10a) without the last term] and thus lead to first-order transitions in some materials and second-order transitions in others. Indeed, band-structure effects have been proposed as the source of the first-order transition, the superconductivity, and an observed metamagnetic transition within the ferromagnetic phase (Pfleiderer and Huxley, 2002; Sandeman *et al.*, 2003).

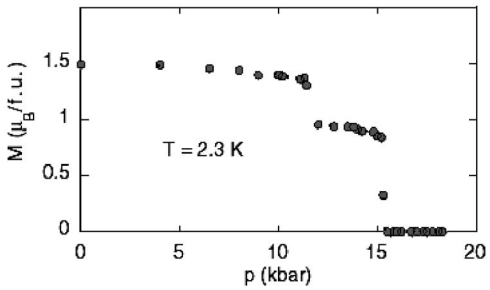

FIG. 16. Magnetization vs hydrostatic pressure at $T = 2.3$ K for UGe_2. The intial first-order transition at $p \approx 15$ kbar is followed by a metamagnetic transition at $p \approx 12$ kbar. Adapted from Pfleiderer and Huxley, 2002.

[54]Strictly speaking, MnSi is not a ferromagnet, but rather a weak helimagnet. It has been speculated that local remnants of the helical order are responsible for the non-Fermi-liquid properties observed in the paramagnetic phase (Doiron-Leyraud *et al.*, 2003; Pfleiderer *et al.*, 2004). The influence of the helical order on the quantum critical behavior has been studied by Vojta and Sknepnek (2001).

5. Critical behavior at the continuous transition

a. Critical behavior in d=3

A solution of the RG flow equations for the case $\kappa > \kappa_c$ (Kirkpatrick and Belitz, 2002) yields a wave-number-dependent coefficient c in the paramagnon propagator, Eq. (4.7b),

$$c(\boldsymbol{k} \to 0) \propto (\ln 1/|\boldsymbol{k}|)^{-1/26}, \tag{4.16}$$

where the exponent is determined by the ratio $A_H/A_c = 27$. Expressing the logarithmic corrections to power-law scaling in terms of scale-dependent exponents, the critical exponent η, which describes the wave-number dependence of the paramagnon propagator at criticality, is therefore given by

$$\eta = \frac{-1}{26} \ln \ln b / \ln b. \tag{4.17a}$$

The parameters r and d in the paramagnon propagator are not renormalized. The correlation length exponent ν, the susceptibility exponent γ, and the dynamical exponent z can therefore be directly read off from Eqs. (4.7b),

$$\nu = 1/(2-\eta), \quad z = 3-\eta, \quad \gamma = 1. \tag{4.17b}$$

The order-parameter exponents β and δ can be obtained from scaling arguments for the free energy. The result is

$$\beta = 1/2, \quad \delta = 3. \tag{4.17c}$$

Finally, one can generalize the definition of the specific-heat exponent α familiar from thermal phase transitions by defining $C_V \propto T^{-\alpha}$ at criticality. One finds

$$\alpha = -1 + (\ln \ln b / \ln b - \eta)/z. \tag{4.17d}$$

The result for η is valid to leading logarithmic accuracy; the values of γ, β, and δ, as well as the relations between η and ν, z, and α, respectively, are exact.

The theory thus predicts the critical behavior in $d=3$ to be mean-field-like with logarithmic corrections. Although the fixed point found by Hertz (1976) is unstable, it is only marginally so. Hertz's results, and their extension to nonzero temperatures by Millis (1993), should therefore apply apart from corrections that are too small to be detectable with current experimental accuracies.

Although the critical behavior has not been probed directly experimentally, a combination of various exponents determines the shape of the phase diagram at low temperatures. To see this, consider a homogeneity law for the magnetization, which can be obtained by differentiating Eq. (1.10) twice with respect to h and putting $h=0$:

$$m(r,T,u) = b^{-d-z+2y_h} m(rb^{1/\nu}, Tb^z, ub^{[u]}). \tag{4.18a}$$

Here we have included the coefficient u of the quartic term in the free energy, even though its scale dimension $[u]<0$ is negative and u is thus irrelevant. The reason is that u is a dangerous irrelevant variable with respect to the magnetization (Ma, 1976; Fisher, 1983), which influences the critical behavior of m. To see this, consider the equation of state, Eq. (4.11a). In the absence of the soft-mode corrections, one has $m \propto \sqrt{-r/u}$. m thus depends on r/u, and the homogeneity law can be written

$$\begin{aligned} m(r/u,T) &= b^{-d-z+2y_h} m((r/u)b^{1/\nu-[u]}, Tb^z) \\ &= (r/u)^{\beta} m(1, T(r/u)^{-\nu z/(1-\nu[u])}), \end{aligned} \tag{4.18b}$$

where $\beta = \nu(d+z-2y_h)/(1-\nu[u])$. The relation between T_c and r is now obtained from the requirement that m have a zero for $T=T_c$. This yields $r = \text{const} \times T^{(1-\nu[u])/\nu z}$ (Millis, 1993; Sachdev, 1997). $[u]=4-d-z$, as can be seen from Eq. (4.2) by power counting, and, neglecting the logarithmic corrections to scaling, we find from Eqs. (4.17) (Millis, 1993),

$$T_c \propto r^{3/4}. \tag{4.19}$$

This agrees well with the phase diagram measured in Ni_xPd_{1-x} and also with the portion of the phase diagram in MnSi where the transition is continuous; see Figs. 17 and 18.

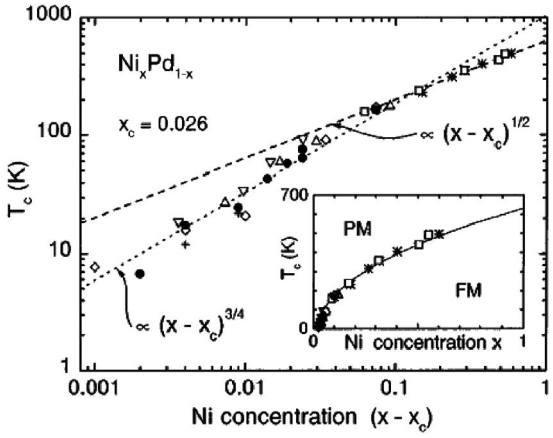

FIG. 17. Phase diagram of Ni_xPd_{1-x}. Equation (4.19) is obeyed over an x range of two decades close to the transition. From Nicklas *et al.*, 1999.

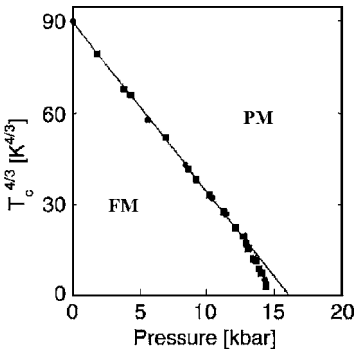

FIG. 18. Phase diagram of MnSi. These are the same data as in Fig. 14, scaled to display the relation given in Eq. (4.19). The tricritical point separating second- and first-order transitions coincides with the point at which the scaling breaks down. Adapted from Pfleiderer *et al.*, 1997.

b. Critical behavior in $d \neq 3$

In $d>3$, the RG analysis shows that Hertz's fixed point is stable, and all exponents have their mean-field values. This confirms the suggestion of the renormalized mean-field theory that the upper critical dimension is $d_c^+=3$.

In $d<3$, the critical behavior can be studied by means of an expansion in $\epsilon=3-d$. Kirkpatrick and Belitz (2002) have found a fixed point where G, H, c, and c_1 are marginal. c_2 can again be either marginal or irrelevant, depending on the context. A one-loop calculation of the exponent η and the specific-heat exponent α yields

$$\eta = -\epsilon/26, \quad \alpha = -d/(3-\eta). \tag{4.20}$$

ν, z, γ, β, and δ are still given by Eqs. (4.17b) and (4.17c).

B. Quantum ferromagnetic transition in disordered systems

If one adds quenched disorder to the problem considered in the previous subsection, the action changes relatively little. The nature of the generic soft modes changes; they are now diffusive rather than ballistic in nature, but this change by itself will clearly change only the upper critical dimensionality. Apart from this, the disorder needs to be averaged over, e.g., by means of the replica trick. It turns out that the disorder average leads to some terms in perturbation theory having a sign that is opposite from the corresponding one in the clean case. For instance, the one-loop correction to the quartic coupling constant u is positive, and as a result the phase transition is always of second order. Treating the generic soft modes in tree approximation, Hertz (1976) concluded that the critical behavior is mean-field-like for all $d>0$. This turns out not to be true, for reasons analogous to those that invalidate Hertz's theory in the clean case. It turns out, however, that the critical behavior can still be determined exactly for all dimensions $d>2$, although it is not mean-field-like. Historically, this extension of Hertz's theory for the disordered case was developed earlier than the corresponding treatment of the clean case (Kirkpatrick and Belitz, 1996; Belitz *et al.*, 2001a, 2001b). Here we explain the structure of this theory and summarize its results.

1. Soft-mode action

The action as given in Sec. IV.A.1 remains valid, with two modifications. First, all fields carry replica indices that need to be summed over. Specifically, $\boldsymbol{M}$ carries one replica index, while Q and q carry two replica indices, as explained in connection with Eq. (2.44). The symmetrized magnetization field b is diagonal in its two replica indices. Second, the fermionic vertex function $\Gamma^{(2)}$ changes; Eqs. (4.3b) and (4.3c) get replaced by

$$^{i}\Gamma^{(2)}_{12,34}(\boldsymbol{k}) = \delta_{13}\delta_{24}\Gamma^{(2,0)}_{12}(\boldsymbol{k}) + \delta_{1-3,2-4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3} \times 2\pi TG[\delta_{i0}K_s + (1-\delta_{i0})\tilde{K}_t] \tag{4.21a}$$

with replica indices α_i, and

$$\Gamma^{(2,0)}_{12}(\boldsymbol{k}) = \boldsymbol{k}^2 + GH\Omega_{n_1-n_2}. \tag{4.21b}$$

All other terms in the action remain unchanged, except for the addition of replica indices.

It is obvious from simple physical considerations that the quenched disorder must lead to additional terms in the action. For instance, consider the bare distance from the critical point, r in Eq. (4.2), a random function of space, and integrate out this "random mass" with respect to some distribution function. This will generate terms of higher order in $\boldsymbol{M}$, starting at $O(\boldsymbol{M}^4)$, that have a different imaginary-time structure than the $u\boldsymbol{M}^4$ term in Eq. (4.2). Such random-mass terms do indeed get generated by the basic disorder term in an underlying microscopic action, Eq. (2.37c) or (2.42), and thus need to be added to the effective action. However, they turn out to be irrelevant for the critical behavior in all dimensions except $d=4$, and we therefore neglect them.

2. Gaussian approximation

Keeping only terms that are bilinear in the fields yields the Gaussian approximation, as in the clean case. The paramagnon propagator now reads

$$\mathcal{M}_n(\boldsymbol{k}) = \frac{1}{r_0 + c\,\boldsymbol{k}^2 + d|\Omega_n|/(\boldsymbol{k}^2 + GH|\Omega_n|)}. \tag{4.22}$$

Formally this is the same as Eq. (4.7a) with $|\boldsymbol{k}|$ replaced by $\boldsymbol{k}^2$, but the coefficient G has a different physical interpretation, as was explained in connection with Eqs. (2.48). The basic fermionic propagator is given by Eq. (2.48b). As in the clean case, there is a mixed propagator that turns out not to be important for the determination of the critical behavior.

3. Renormalized mean-field theory

A renormalized mean-field theory can be constructed in exact analogy to the treatment of the clean case in Sec. IV.A.3 (Sessions and Belitz, 2003). The free energy in this approximation is again given by Eq. (4.10a), but the quantity N in Eq. (4.10b) is replaced by

$$N(\boldsymbol{k},\Omega_n;m) = \frac{[\boldsymbol{k}^2 + G(H+\tilde{K}_t)\Omega_n]^2 + 32\sqrt{\pi}c_2G^2m^2}{(\boldsymbol{k}^2 + GH\Omega_n)^2 + 32\sqrt{\pi}c_2G^2m^2}. \tag{4.23}$$

An analysis of the integral in Eq. (4.10a) with this expression for N yields a free energy at $T=0$ of the form

$$f = f_0 + rm^2 + wm^{(d+2)/2} + um^4 - h, \tag{4.24a}$$

or, equivalently, an equation of state

$$h = rm + wm^{d/2} + um^3. \tag{4.24b}$$

In Eqs. (4.24), $w>0$ is a positive coefficient that is proportional to the disorder, and r and u again represent additive renormalizations of r_0 and u_0, respectively.

Regarding the nonanalytic dependence of f on m, the same comments apply as in the clean case (Sec. IV.A.3), and an assumption analogous to that mentioned after Eq. (4.12) has been made. As far as the GSI is concerned, the only difference is that the decay of the spin susceptibility in real space far from the transition is given, not by Eq. (4.12), but by

$$\chi_s(r>0, |\boldsymbol{x}| \to \infty) \propto 1/|\boldsymbol{x}|^{2d-2}, \tag{4.25}$$

which is the Fourier transform of Eq. (2.35d). In addition, the magnetization now scales as a wave number squared, which explains the exponent of the nonanalyticity in Eq. (4.24a).

Despite these similarities, there is a crucial sign difference between the nonanalytic term in the clean case, Eqs. (4.11b), (4.11c), and in the one in Eqs. (4.24). Although this is not of much significance from a GSI point of view, it causes the quantum phase transition to be of first order in the clean case (at least at the level of the renormalized mean-field theory), while it is of second order in the disordered case. Physically, this can be understood by means of arguments very similar to those that explained the sign difference between Eqs. (2.35d) and (2.36b), respectively. In the clean case, the nonanalyticity is produced by fluctuation effects, represented by the generic soft modes, which weaken the tendency towards ferromagnetism. Accordingly, the constant contribution from the integral over $\ln N$ in Eq. (4.10a), which changes r_0 to r, is positive. The appearance of a nonzero magnetization gives the soft modes a mass and hence weakens these fluctuations, so the leading m-dependent correction to the constant contribution is negative. In the disordered case, however, the quenched disorder leads to diffusive motion of the electrons, which is much slower than the ballistic dynamics in the clean case. This increases the effective interaction strength between the electrons, which in turn favors ferromagnetism. Consequently r is smaller than r_0. A nonzero magnetization again weakens this effect, and therefore the leading nonanalytic m dependence of the free energy is positive.

A more general form of the free energy in the renormalized mean-field approximation, which is valid for $T \geqslant 0$ and allows the zero-disorder limit to be taken, has been considered by Belitz *et al.* (1999). The free-energy density discussed by these authors takes the form

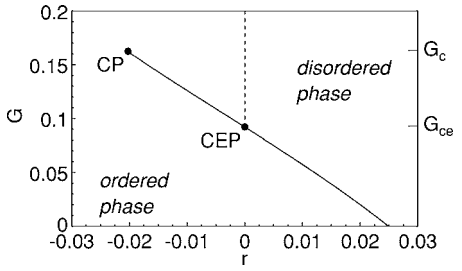

FIG. 19. Phase diagram in the G-r plane resulting from Eq. (4.26) at $T=0$ for $u=1$, $v=0.5$, $A=0.5$, $B_T=B_G=1$, showing a second-order transition (dashed line), a first-order transition (solid line), a critical end point (CEP), and a critical point (CP). From Belitz *et al.*, 1999.

$$f = f_0 + rm^2 + (G/\epsilon_F)Am^4[m^2 + (B_T T)^2]^{-3/4} + vm^4 \ln[m^2 + (T + B_G G)^2] + um^4 + O(m^6). \tag{4.26}$$

Here B_T and B_G are dimensionless parameters that measure the relative strengths of the temperature and the disorder dependence, respectively, in the two nonanalytic terms, and A is a measure of how strongly correlated the system is. They are all expected to be of order unity. For $G=0$ the resulting equation of state reduces to Eq. (4.11a). For sufficiently large disorder G, the logarithmic term is unimportant, and at $T=0$ one recovers Eqs. (4.24). This form of the free energy thus interpolates between the clean and disordered cases. It displays a rich phenomenology, as shown in Figs. 19 and 20. Notice that, for a disorder larger than a threshold value, the phase diagrams shown in these figures predict a second magnetic transition inside the ferromagnetic phase that is always of first order.

The critical exponents $\beta=2/(d-2)$ and $\delta=d/2$ (for $2<d<6$) can be read off from Eq. (4.24b), but a determination of the remaining exponents requires the consideration of order-parameter fluctuations. We shall discuss this in the next subsection, where we shall also see that the critical behavior predicted by the renormalized mean-field theory is very close to the exact one.

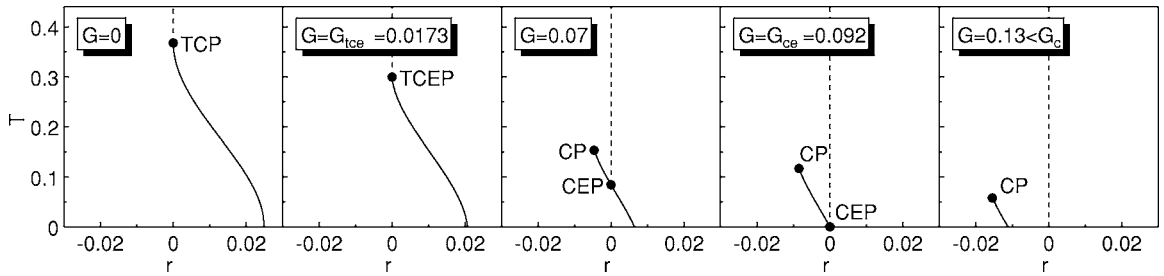

FIG. 20. Phase diagrams in the T-r plane resulting from Eq. (4.26) for $u=B_G=1$, $v=B_T=A=0.5$. TCP and TCEP denote a tricritical point and a tricritical end point, respectively. All other symbols are the same as in Fig. 19. From Belitz *et al.*, 1999.

4. Effects of order-parameter fluctuations

The effects of order-parameter fluctuations on the disordered ferromagnetic quantum phase transition have been studied at various levels. Hertz (1976) and Millis (1993) considered order-parameter fluctuations but integrated out the generic soft modes in a tree approximation that neglects all fermionic loops. Kirkpatrick and Belitz (1996) kept fermionic loops but still integrated out the fermions, which led to a nonlocal field theory in terms of the order parameter only, which they analyzed by means of power counting. Belitz *et al.* (2001a, 2001b) kept all of the soft modes explicitly and on an equal footing, in analogy to the theory for clean systems discussed in Sec. IV.A and to the classical theories covered in Sec. III. This theory contains the previous ones as approximations, and we shall sketch it in the remainder of this section.

a. Coupled field theory and power counting

In contrast to the clean case (Sec. IV.B.1), for disordered magnets the fermionic part of the soft-mode action is known in closed form, namely, the nonlinear σ model, Eqs. (2.61) and (2.62), with no spin-triplet interaction. The order-parameter part of the action is again given by Eq. (4.2), with a replicated $\boldsymbol{M}$ field as explained in Sec. IV.B.1, and the coupling between the two is given by Eq. (4.4a) , with Q replaced by $\hat{Q}$ and replica indices added as appropriate. This action can be written down on general principles or it can be derived from the complete nonlinear σ model, Eqs. (2.61), by performing a Hubbard-Stratonovich decoupling (Stratonovich, 1957; Hubbard, 1959) of the spin-triplet interaction and adding an $\boldsymbol{M}^4$ term.[55] Since the nonlinear σ model has been derived from a microscopic action, this also provides a derivation of the final effective action from a microscopic starting point, if that is desired.

We now expand this effective action in powers of $\boldsymbol{M}$ and q. In a schematic notation that omits everything not necessary for power counting, we have

$$\mathcal{A}_{\rm eff}[\boldsymbol{M},q] = \mathcal{A}_M + \mathcal{A}_q + \mathcal{A}_{M,q}, \tag{4.27a}$$

with

$$\mathcal{A}_M = -\int d\boldsymbol{x}\; M[r + c_{d-2}\partial_{\boldsymbol{x}}^{d-2} + c\partial_{\boldsymbol{x}}^2]M + O(\partial_{\boldsymbol{x}}^4 M^2, M^4), \tag{4.27b}$$

$$\mathcal{A}_q = -\frac{1}{G}\int d\boldsymbol{x}(\partial_{\boldsymbol{x}} q)^2 + H\int d\boldsymbol{x}\;\Omega q^2 + K_s T\int d\boldsymbol{x}\; q^2 - \frac{1}{G_4}\int d\boldsymbol{x}\;\partial_{\boldsymbol{x}}^2 q^4 + H_4\int d\boldsymbol{x}\;\Omega q^4 + O(Tq^3, \partial_{\boldsymbol{x}}^2 q^6, \Omega q^6), \tag{4.27c}$$

$$\mathcal{A}_{M,q} = \sqrt{T}c_1\int d\boldsymbol{x}\; Mq + \sqrt{T}c_2\int d\boldsymbol{x}\; Mq^2 + O(\sqrt{T}Mq^4). \tag{4.27d}$$

Here the fields are understood to be functions of position, frequencies, and replica labels. Only quantities that carry a scale dimension are shown. Accordingly, frequency and replica sums have been omitted, but appropriate powers of the temperature are shown. All of the terms except the second one in $\mathcal{A}_M$ can be obtained by combining Eqs. (4.2) and (4.3a), (4.4), (4.5), and (4.21). The bare value of the coefficient c_{d-2} is zero, but we shall see that such a term is generated under renormalization.

For a power-counting analysis of this effective action, we assign scale dimensions $[L]=-1$ and $[T]=[\Omega]=-z$ to lengths and temperatures or frequencies, respectively. It is important to note that z is unlikely to have a unique value: Since the critical paramagnon propagator will in general have a frequency-wave-number relation that is different from that of the fermionic q propagator, we expect at least two different time scales in the problem, hence two different values of z. This needs to be taken into account in the power-counting analysis.

b. Hertz's fixed point

The fixed point identified by Hertz (1976) is recovered from Eqs. (4.27) by requiring that the coefficients c and c_1 be dimensionless. The field q is expected to be diffusive, and we therefore choose its scale dimension to be $[q]=(d-2)/2$, in accord with Eqs. (2.63). From $\mathcal{A}_M$ we then obtain $[M]=(d-2)/2$ and $[t]=2$. r is thus relevant, as expected, and the correlation length exponent is $\nu = 1/[r]=1/2$. c_{d-2} is also relevant, of course, but since its bare value is zero we ignore it for now. The first term in $\mathcal{A}_{M,q}$ produces the dynamical part of the paramagnon, as was demonstrated in Sec. IV.A.2. The scale dimension of the $\sqrt{T}$ prefactor thus yields the paramagnon or critical dynamical exponent $z_c=4$. The frequency and the temperature in the second and third terms, respectively, in $\mathcal{A}_q$ carry the fermionic time scale $z_q=2$ that is consistent with diffusion. G, H, and K_s are then all marginal, and all higher-order terms in $\mathcal{A}_q$ are irrelevant. For the scale dimension of c_2 one finds

$$[c_2] = -(d+z-6)/2. \tag{4.28}$$

Due to the existence of two time scales, mentioned above, one needs to distinguish between two different variants of c_2, which differ with respect to the value of z that enters their scale dimension. With $z=z_c=4$, $c_2 \equiv c_2^-$ is irrelevant for all $d>2$. However, with $z=z_q=2$, $c_2 \equiv c_2^+$ is relevant with respect to the putative fixed point for $d<4$. This instability of Hertz's fixed point is indeed realized. Consider, for instance, the renormalization of the $\boldsymbol{M}^2$ vertex by means of the diagram shown in Fig. 21.

[55] The quartic and higher terms in $\boldsymbol{M}$ originate from massive modes in the underlying miscroscopic action, which have been dropped in deriving the nonlinear σ model.

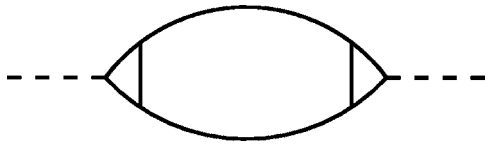

FIG. 21. One-loop renormalization of the M^2 vertex. Dashed lines denote M fields, solid lines denote q propagators, and the vertices carry a factor of c_2 each.

Since it is a pure fermion loop, the factors of $\sqrt{T}$ that come with the vertices and that are absorbed into the loop integral over the frequency indeed carry the fermionic time scale, and the relevant coupling constant is c_2^+. We note that this is the same perturbative contribution that leads to the nonanalytic wave-number dependence of the spin susceptibility deep in the paramagnetic phase, Eq. (2.35d). The term with coupling constant c_{d-2} is thus generated from the c_2 coupling. It is convenient to explicitly add this term to the bare action, as we have done in Eq. (4.27b), although the physics it represents is already contained in the c_2 coupling term. Indeed, with respect to Hertz's fixed point, $(c_2^+)^2$ and c_{d-2} have the same scale dimension, $[c_{d-2}]=2[c_2^+]=4-d$.

One thus finds that the generic soft modes render Hertz's fixed point unstable, and this first becomes apparent at one-loop order. This instability, although not its source, can also be deduced from the fact that the mean-field value for the correlation length exponent $\nu=1/2$ violates, for all $d<4$, the rigorous bound provided by the Harris criterion (Harris, 1974; Chayes *et al.*, 1986) for systems with quenched disorder, $\nu \geqslant 2/d$.

c. A marginally unstable fixed point

The behavior of the renormalized mean-field theory is recovered from the RG if one (1) takes the c_{d-2} term into account, and (2) drops the requirement that c_2 be marginal. We thus require only that c_1 be marginal, which implies $[M]=1+(d-z)/2$, and that $[q]=(d-2)/2$ and $z_q=2$ as before. At this point it is useful to consider an analogy between the present problem and that of a classical fluid under shear, discussed in Sec. III.B.2. In the latter case, long-ranged correlations between order-parameter fluctuations stabilized mean-field critical behavior in dimensions lower than the usual upper critical dimension. In the current problem, the $c_{d-2}\partial_{\boldsymbol{x}}^{d-2}$ term in the action represents long-range order-parameter correlations that decay as $|\boldsymbol{x}|^{-2(d-1)}$. By analogy with the classical fluid example, it is therefore natural to expect the stabilization of a Gaussian fixed point where c_{d-2} is marginal in the critical paramagnon. This implies $[M]=1$ and hence a critical time scale and an exponent η characterized by

$$z_c = 4 - \eta = d. \tag{4.29a}$$

r is relevant with $[r]=d-2$, which corresponds to a correlation length exponent

$$\nu = 1/(d-2). \tag{4.29b}$$

In contrast to the situation at Hertz's fixed point, this respects the Harris criterion.

Equation (4.29a) implies that the magnetic susceptibility at criticality decays in real space according to

$$\chi_s(r=0,|\boldsymbol{x}|\to\infty) \propto 1/|\boldsymbol{x}|^2. \tag{4.30}$$

In the relevant dimensionality range, $d>2$, these correlations are of longer range than the power-law correlations in the paramagnetic phase, Eq. (4.25). We see that, as in the case of the classical fluid, Sec. III.B.2, the long-ranged correlations or GSI in the disordered phase influence the critical behavior and lead to correlations with an even longer range at the critical point.

The results given above hold for $2<d<4$. For $d>4$, $\eta=0$ and $\nu=1/2$ have their mean-field values, and $z_c=4$. The exponents β and δ can be obtained either from scaling arguments for the free energy (Belitz *et al.*, 2001b) or from repeating the above counting arguments in the ferromagnetic phase (Sessions and Belitz, 2003). By either method one finds the same result as from Eq. (4.24b), viz.,

$$\beta = 2/(d-2), \quad \delta = d/2. \tag{4.31}$$

These relations hold for $2<d<6$, while for $d>6$ one has the mean-field values $\beta=1/2$, $\delta=3$. A detailed analysis reveals that Hertz's fixed point is actually stable for all $d>4$, but for $4<d<6$ the coefficient w in Eq. (4.24b) acts as a dangerous irrelevant variable with respect to the magnetization, which explains why β and δ lock into their mean-field values only for $d>6$.

What remains to be done is to investigate the stability of the Gaussian fixed point with the critical behavior described above. It turns out that it is marginally unstable and that the exact critical behavior is given by the power laws given above with additional logarithmic corrections to scaling. We present these results next.

d. Exact critical behavior

The results of the previous subsection characterize a Gaussian fixed point that was described by the theory given by Kirkpatrick and Belitz (1996). To check its stability, one needs to consider the scale dimensions of the remaining terms in the action. c_2 has a scale dimension

$$[c_2] = 1 - z/2 \tag{4.32}$$

with respect to the Gaussian fixed point, which implies that c_2^+ (i.e., c_2 with $z=z_q$) is marginal. This was to be expected, since c_{d-2}, which describes the same physics as c_2, is also marginal. However, it implies that the non-Gaussian c_2 term must be kept as part of the fixed-point action. All other terms are nominally irrelevant. However, it turns out that, for subtle reasons again related to the existence of multiple time scales, the terms of order q^4 in $\mathcal{A}_q$ can also be effectively marginal and must be kept as part of the fixed-point action. For all $d>2$, the

latter then consists of the terms shown explicitly in Eqs. (4.27).[56]

Since the fixed-point action is not Gaussian, the exact critical behavior cannot be obtained simply by power counting. However, it turns out that the problem can still be solved exactly, by means of an infinite resummation of perturbation theory (Belitz *et al.*, 2001b). For $2<d<4$, the result consists of logarithmic corrections to the exponents derived in the previous subsection. For instance, the correlation length depends on r via

$$\xi \propto r^{-1} e^{-\text{const}\times[\ln \ln(1/r)]^2}, \tag{4.33}$$

to leading logarithmic accuracy. Notice that the correction to the power law varies more slowly than any power of r, but faster than any power of $\ln r$. This behavior is conveniently expressed in terms of exponents that are scale dependent via a function g, whose asymptotic behavior for large arguments is

$$g(x \gg 1) \approx [2\ln(d/2)/\pi]^{-1/2} e^{[\ln(c(d)x)]^2/2\ln(d/2)}. \tag{4.34}$$

The dimensionality-dependent coefficient $c(d)$ goes to zero for $d \to 4$ and to a constant for $d \to 2$. In the case of ν, Eq. (4.33) corresponds to

$$1/\nu = d - 2 + \ln g(\ln b)/\ln b. \tag{4.35a}$$

For the other exponents one obtains

$$z_c = 4 - \eta = d + \ln g(\ln b)/\ln b, \tag{4.35b}$$

$$\delta = -\alpha/2d = z_c/2, \tag{4.35c}$$

$$\beta = 2\nu, \quad \gamma = 1, \tag{4.35d}$$

$$z_q = 2 + \ln g(\ln b)/\ln b, \tag{4.35e}$$

where the specific-heat exponent α is defined as in the context of Eq. (4.17d).

The critical behavior of various transport coefficients and relaxation rates, as well as the tunneling density of states, have also been determined (Belitz *et al.*, 2000, 2001b). In particular, these authors showed that the quasiparticle properties at criticality are those of a marginal Fermi liquid (Varma *et al.*, 1989). Here we quote just the result for the electrical conductivity. In $d=3$ at criticality, one finds a nonanalytic temperature dependence in addition to a noncritical background term,

$$\sigma(T \to 0) = \sigma(T=0) + \text{const} \times \left[Tg\left(\frac{1}{3}\ln(\epsilon_F/T)\right)\right]^{1/3}, \tag{4.36}$$

with $g(x)$ from Eq. (4.34).

[56] A more conventional approach would be to not include the c_{d-2} term in the action, but rather have its effects included in a renormalization of c while working in $d=4-\epsilon$. This can be done, but the loop expansion in this case does not lead to a controllable expansion in powers of ϵ since c_2 remains strictly marginal order by order. Nevertheless, such a procedure yields interesting technical insights (Belitz *et al.*, 2004).

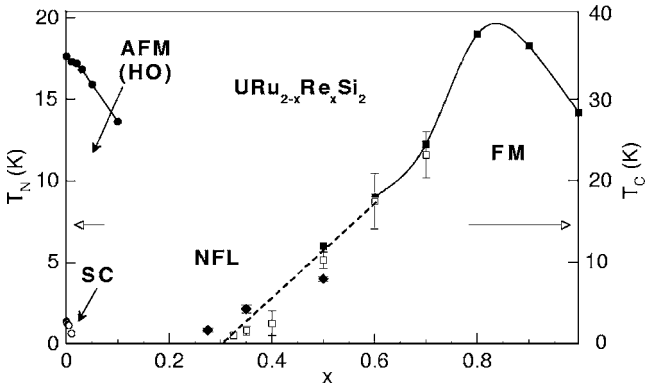

FIG. 22. Phase diagram of URuReSi, showing ferromagnetic (FM), antiferromagnetic/"hidden order" [AFM (HO)], superconducting (SC), and non-Fermi-liquid (NFL) phases. The different symbols refer to different methods for determining the phase boundaries. Adapted from Bauer, 2002 and Bauer *et al.*, 2005.

Experimentally, there are few systematic studies of the influence of quenched disorder on the ferromagnetic quantum phase transition. $URu_{2-x}Re_xSi_2$ shows a rich phase diagram in the T-x plane, with phases displaying ferromagnetic, antiferromagnetic, and superconducting order in addition to a paramagnetic non-Fermi-liquid ("strange metal") phase (Dalichaouch *et al.*, 1989; Stewart, 2001; Bauer, 2002; Bauer *et al.*, 2005); see Fig. 22. Near $x=0.3$ there is a quantum phase transition from the ferromagnetic phase to the strange-metal phase that has been studied in some detail by Bauer (2002) and Bauer *et al.* (2005). These authors found a value of $\delta=1.5$, in agreement with Eq. (4.35c) (ignoring logarithmic corrections to scaling). The specific-heat coefficient for $T \to 0$ is found to diverge logarithmically, or as a very small power of T, which is, within the experimental accuracy, consistent with Eq. (4.35c). However, the experimental values of two other exponents, $\beta=0.9$ and $\gamma=0.5$, do not agree with the theoretical prediction. Also, the logarithmic divergence of the specific-heat coefficient is observed to persist away from the quantum critical point in both the paramagnetic and the ferromagnetic phases. Similar non-Fermi-liquid properties have been found in the paramagnetic phase of other materials that display a quantum ferromagnetic transition, e.g., in MnSi (Pfleiderer, Julian, and Lonzarich, 2001). We shall come back to these observations in Secs. IV.D and V.

DiTusa *et al.* (2003) have investigated $Fe_{1-x}Co_xS_2$, which displays a $T=0$ insulator-metal transition at a very small value of x ($x<0.001$), followed by a quantum paramagnet-to-ferromagnet transition at $x \approx 0.032$. The strength of the quenched disorder is substantial in this system, with values of $k_F\ell$ ranging from 2 to 15 for x between 0.001 and 0.17. The authors have reported scaling of the conductivity near the ferromagnetic transition that is consistent with Eq. (4.36).

5. Effects of rare regions on the phase transition for an Ising system

We now turn to rare regions and nonperturbative disorder effects. In contrast to the previous subsections, which considered Heisenberg ferromagnets, we shall now discuss a special effect for magnetic order parameters with an Ising symmetry. In such systems, the rare-region effects are much stronger than the quantum Griffiths phenomena discussed in Sec. II.B.6 because of the coupling between the order parameter and the generic soft modes. To see this, consider a particular rare region devoid of impurities. Within such a region, the appropriate paramagnon propagator is given by Eq. (4.7a), and the coupling between the magnetization and the conduction electrons is reflected by the dependence on $|\Omega_n|$.[57] In imaginary time space, this corresponds to a long-time tail given by $1/\tau^2$ (see Sec. II.B.7). At $T=0$, every rare region thus maps onto a clean, classical $(d+1)$-dimensional Ising model that is finite in d dimensions and infinite in one dimension. The interactions between the spins are short ranged in the former and long ranged, proportional to $1/\tau^2$, in the latter.

This long-range interaction can have drastic consequences, namely, the sharp quantum phase transition can be destroyed by smearing. This can be understood as follows: A one-dimensional Ising model with a $1/r^2$ interaction is known to possess an ordered phase (Thouless, 1969; Cardy, 1981). A rare region can therefore develop a static order parameter independently from the rest of the system. This result is consistent with the one obtained by Millis, Morr, and Schmalian (2002), who directly calculated the tunneling rate of a Griffiths island in an itinerant system and found it to vanish for a sufficiently large island. Suppose a nonzero average order parameter first appears on a rare region, rather than in the bulk of the system. Since the order is truly static, the ordered region will effectively act like a small permanent magnet embedded in the system, and it will be energetically favorable for subsequent rare regions to align their order parameter with the first one (assuming that the effective interaction between the regions is also ferromagnetic). As a result, a finite magnetization appears as soon as a finite volume of rare regions start to order. This magnetization is exponentially small, and the correlation length does not diverge at this point. However, the ordered rare regions provide an effective magnetic field seen by the bulk of the system; therefore the latter cannot undergo a sharp quantum phase transition (Vojta, 2003). Notice that the number of ordered regions changes continuously with the control parameter, so the smeared transition is not given simply by the behavior of electrons in a fixed magnetic field. Rather, order develops in different parts of the system at different values of the control parameter, and the order parameter close to the smeared transition is very inhomogeneous in space. The same mechanism also destroys classical phase transitions in systems with planar defects (Sknepnek and Vojta, 2003; Vojta, 2003b).

The above arguments are valid at $T=0$, where the system is infinite in the imaginary-time direction. The behavior at finite temperatures is less well established. Since the rare regions are far apart, their interaction is very small. Therefore the relative alignment between the order parameter on different rare regions vanishes already at a temperature that is exponentially small in the density of the rare regions, i.e., double-exponentially small in the disorder strength. Above this temperature, the rare regions act as independent classical moments. Whether or not there is a second crossover at even higher temperatures to quantum Griffiths behavior similar to that in undamped systems (as discussed in Sec. II.B.6) is not fully understood; it appears to be a question of the microscopic parameter values (Castro Neto and Jones, 2000; Millis Morr, and Schmalian, 2002).

In systems with continuous order-parameter symmetry, the rare-region effects are weaker. Specifically, the quantum phase transition will remain sharp because the rare regions cannot develop static order. This can be seen by mapping each rare region onto a classical one-dimensional XY or Heisenberg model with $1/r^2$ interaction. In contrast to the corresponding Ising model, these models do not have an ordered phase (Kosterlitz, 1976; Bruno, 2001). The quantum Griffiths behavior in the vicinity of the transition has not yet been worked out in detail, but it is likely to be of a more conventional type.

C. Metal-superconductor transition

Sufficiently strong quenched nonmagnetic disorder[58] decreases the critical temperature T_c for the superconductor-metal transition in bulk conventional superconductors[59] (see Fig. 23). At a critical value of the disorder, T_c vanishes, and there is a quantum phase transition from a normal-metal phase to a superconducting phase in which the concepts of additional soft modes and GSI effects play a crucial role. Since the generic soft modes in question, particle-hole excitations again, are massive at nonzero temperature, this quantum phase transition requires a theoretical description that is qualitatively different from BCS theory and the theories that describe fluctuation corrections to it. Such a theory, whose main predictions should not be hard to check experimentally, is presented below. An alternative, and

[57] This is sometimes referred to as Landau damping, in analogy to the fate of the plasmon when it enters the particle-hole continuum. In the absence of such a coupling, e.g., in the pure spin model given by Eq. (2.75), one would have a propagating mode with a frequency dependence on Ω_n^2.

[58] *Weak* disorder can actually enhance T_c, sometimes substantially so. Aluminum, and many other low-T_c superconductors, are examples of this effect.

[59] Also very interesting are the analogous effects in thin superconducting films, where nonmagnetic disorder leads to a transition from a superconducting phase to an insulating one, with or without an intermediate metallic phase. This topic is beyond the scope of this review.

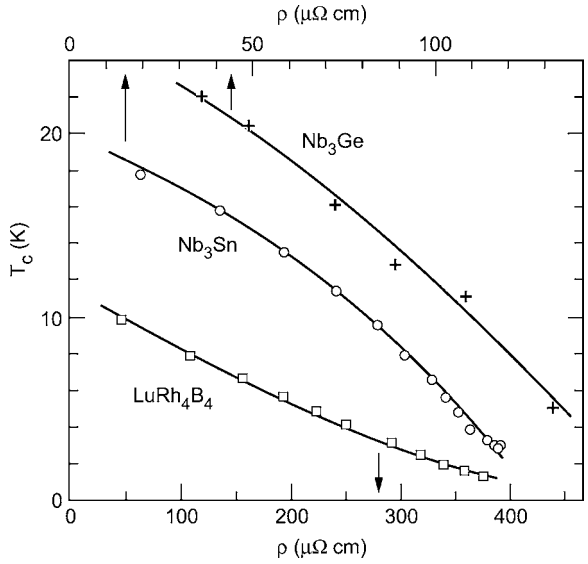

FIG. 23. Superconducting T_c vs normal-state resistivity for three superconducting materials. The data are taken from Rowell and Dynes (1980); the lines are fits to a theory for T_c degradation that is not important for the present discussion. From Belitz and Kirkpatrick, 1994.

physically different, theory has been proposed by Vishveshwara *et al.* (2000). These authors have proposed that the normal-metal–superconductor transition triggered by disorder leads generically to a gapless superconducting state, and they assert that the resulting gapless quasiparticle excitation spectrum will have consequences for the critical behavior. It remains to be worked out to what extent this assertion is correct, or to what extent the gaplessness is a property of the stable fixed point that describes the superconducting phase, rather than the critical fixed point. We shall comment further on this in Sec. V.B.2.b.

One can construct a theory for the metal-superconductor transition that is structurally very similar to that for the disordered ferromagnetic transition discussed in the previous section (Zhou and Kirkpatrick, 2004), so we shall keep the discussion brief. There are, however, a few crucial differences. The most important one is that the superconducting order-parameter field $\Psi(\boldsymbol{x})$, whose expectation value is closely related to the anomalous Green's function and the superconducting gap function, is related to the bilinear product of fermion fields at equal and opposite frequencies,

$$\Psi(\boldsymbol{x}) \sim \psi_{n,\sigma}(\boldsymbol{x})\psi_{-n,-\sigma}(\boldsymbol{x}), \tag{4.37}$$

which in turn can be expressed in terms of the matrix elements q of the matrix Q; see Eqs. (2.44) and (2.46). Since the q are soft modes [see Eqs. (2.48)], this means that the coupling of the order parameter to the generic soft modes is even stronger than in the ferromagnetic case, and therefore the effects of GSI are even more dramatic.

We write the action as

$$\mathcal{A}[\Psi,Q] = \mathcal{A}_\Psi + \mathcal{A}_q + \mathcal{A}_{\Psi,q}. \tag{4.38}$$

$\mathcal{A}_\Psi$ is analogous to $\mathcal{A}_M$ in Eq. (4.2). It is a static, local, LGW functional for a three-dimensional XY model. The order-parameter field one can take to be complex valued or as having two components $\Psi_n^r(\boldsymbol{x})$, with $r=1,2$. Structurally it is identical to Eq. (4.2) with $M_n^i(x)$ replaced by $\Psi_n^r(x)$. Since we are dealing with a disordered system, Ψ also carries a replica label. The fermionic part of the action, $\mathcal{A}_q$, is identical to that in the previous section, except now the particle-particle or Cooperon degrees of freedom, $r=1,2$, need to be taken into account. In particular, the Gaussian part of $\mathcal{A}_q$ is given by Eq. (4.3a) with Γ given by Eqs. (4.21) for $r=0,3$ and by

$$\sum_{r=1,2}{}^{i}\Gamma^{(2)}_{12,34}(\boldsymbol{k}) = \delta_{13}\delta_{24}\Gamma^{(2,0)}_{12}(\boldsymbol{k}) + \delta_{i0}\delta_{1+2,3+4}\delta_{\alpha_2\alpha_4}\delta_{\alpha_1\alpha_3}2\pi TG\tilde{K}_c \tag{4.39}$$

for $r=1,2$. Here $\Gamma^{(2,0)}$ is given by Eq. (4.21b), and $\tilde{K}_c$ is a repulsive Cooper channel interaction that is generated by the particle-hole channel electron-electron interactions K_s and K_t in analogy to the mechanism that generated $\tilde{K}_t$ in Eq. (4.3b). $\tilde{K}_c$ is also disorder dependent. As we shall see below, $\tilde{K}_c$ is responsible for driving T_c to zero. The bare attractive Cooper channel interaction that is due to, e.g., phonon exchange is denoted by $K_c = -|K_c| < 0$. The coupling between Ψ and q originates from a term $\mathcal{A}_{\Psi-Q}$ that can be written, in analogy to Eq. (4.4d),

$$\mathcal{A}_{\Psi-Q} = c_1 i\sqrt{T}\int d\boldsymbol{x}\,\mathrm{tr}(\beta(\boldsymbol{x})Q(\boldsymbol{x})). \tag{4.40a}$$

The functional form of this term becomes plausible if one realizes that the order-parameter field Ψ acts like an external field that couples to the particle-particle number density. To bilinear order this yields a contribution to $\mathcal{A}_{\Psi,q}$ given by

$$\mathcal{A}_{\Psi-q} = -8\,c_1\sqrt{T}\sum_{12}\int d\boldsymbol{x}\sum_{r=1,2}{}_r\beta_{12}(\boldsymbol{x})^0_r q_{12}(\boldsymbol{x}). \tag{4.40b}$$

Here $c_1 \propto |K_c|^{1/2}$, and

$${}_r\beta_{12}(\boldsymbol{x}) = \delta_{\alpha_1\alpha_2}\sum_n \delta_{n,n_1+n_2}\Psi_n^{r,\alpha_1}(\boldsymbol{x}). \tag{4.40c}$$

This structure is in exact analogy to Eqs. (4.4) and (4.5) for the magnetic case, only the factor of i in Eq. (4.40a), the sign in Eq. (4.40b), and the frequency structure in Eq. (4.40c) reflect the particle-particle channel. Again in analogy with the magnetic case, one can further expand $\mathcal{A}_{\Psi,q}$ in powers of q. The next term in this expansion has an overall structure

$$\mathcal{A}_{\Psi-q^2} \propto c_2\sqrt{T}\int d\boldsymbol{x}\,\mathrm{tr}(\beta(\boldsymbol{x})q(\boldsymbol{x})q^\dagger(\boldsymbol{x})). \tag{4.41}$$

It is now straightforward to determine the Gaussian propagators. In particular, the order-parameter correlation function reads

$$\langle \Psi_n^{r,\alpha}(\boldsymbol{x})\Psi_m^{s,\beta}(\boldsymbol{x})\rangle = \delta_{\boldsymbol{k},-\boldsymbol{p}}\delta_{rs}\delta_{\alpha\beta}\delta_{nm}\mathcal{N}_n(\boldsymbol{x}), \tag{4.42a}$$

with

$$\mathcal{N}_n(\boldsymbol{k}) = \frac{1}{r_0 - C(\boldsymbol{k},\Omega_n)}. \tag{4.42b}$$

Here

$$C(\boldsymbol{k},\Omega_n) = \frac{c_1^2 \ln[\Omega_0/(D\boldsymbol{k}^2 + |\Omega_n|)]}{1 + (\tilde{K}_c/H)\ln[\Omega_0/(D\boldsymbol{k}^2 + |\Omega_n|)]}, \tag{4.42c}$$

where Ω_0 is an ultraviolet frequency cutoff and r_0 is the coefficient of the Ψ^2 in $\mathcal{A}_\Psi$. For asymptotically small wave numbers and frequencies the critical propagator is given by

$$\mathcal{N}_n(\boldsymbol{k}) = \left[r + \frac{\text{const}}{\ln[\Omega_0/(D\boldsymbol{k}^2 + |\Omega_n|)]} \right]^{-1}, \tag{4.42d}$$

with const>0 and $r=r_0-c_1^2 H/\tilde{K}_c$ the Gaussian distance from the quantum critical point.

The Gaussian theory represented by Eqs. (4.42) has several interesting properties. First, for given $r_0>0$, H, and $\tilde{K}_c$ there is a quantum phase transition at a critical strength of $c_1^2 \propto |K_c|$, which yields $r=0$. Second, for fixed other parameters in the superconducting phase, an increase of $\tilde{K}_c$, which is an increasing function of disorder, will drive the system into the normal-metal phase. This is consistent with other theories that have identified the Coulomb pseudopotential as an important source of T_c degradation in disordered superconductors (Finkelstein, 1987; Kirkpatrick and Belitz, 1992; see Belitz and Kirkpatrick, 1994, for a discussion of earlier theories of this effect). Third, as in the ferromagnetic case there are long-ranged order-parameter correlations in the disordered phase away from criticality. Equation (4.42d) implies for the order-parameter susceptibility

$$\chi_\Psi(r>0, |\boldsymbol{x}| \to \infty) \propto 1/|\boldsymbol{x}|^d \ln|\boldsymbol{x}|. \tag{4.43}$$

A comparison with Eq. (4.25) shows that the correlations are of even longer range than in the ferromagnetic case for all $d>2$. This reflects the strong coupling of the superconducting order parameter to the generic soft modes mentioned in connection with Eq. (4.37).

Most of the critical behavior predicted by the Gaussian theory can simply be read off from Eqs. (4.42) (Kirkpatrick and Belitz, 1997). The correlation length depends exponentially on r, rather than as a power law,

$$\xi \sim e^{1/2|r|}. \tag{4.44a}$$

For the correlation length exponent this implies $\nu=\infty$. The exponent γ has its mean-field value,

$$\gamma = 1, \tag{4.44b}$$

and the exponents η and z for the order parameter (Ψ) and the fermionic (q) degrees of freedom, respectively, are

$$\eta_\Psi = 2, \quad \eta_q = 0, \quad z_\Psi = z_q = 2. \tag{4.44c}$$

The order-parameter susceptibility at criticality decays in real space as

$$\chi_\Psi(r=0, |\boldsymbol{x}| \to \infty) \propto \ln|\boldsymbol{x}|/|\boldsymbol{x}|^d. \tag{4.45}$$

As expected, the critical order-parameter fluctuations are of even longer range than those reflecting GSI in the disordered phase, Eq. (4.43).

These results also follow from a tree-level RG analysis of the field theory. The exponents η are related to the scale dimensions of the fields via

$$[q(\boldsymbol{x})] = -(d - 2 + \eta_q)/2, \tag{4.46a}$$

$$[\Psi(\boldsymbol{x})] = -(d - 2 + \eta_\Psi)/2. \tag{4.46b}$$

As in the magnetic case, Sec. IV.B.4.c, there is a critical fixed point at which c_1 is marginal and the fermions are diffusive, with exponents given by Eq. (4.44). However, in contrast to the magnetic case, the coupling constant c_2 of the term $\mathcal{A}_{\Psi-q^2}$ is RG irrelevant, and so are all higher terms in the expansion in powers of q. We therefore conclude that the Gaussian critical behavior is exact. The most obvious technical reason for this surprising result is that the time scales for the order-parameter fluctuations and the fermions, respectively, are the same, which renders inoperative the mechanism that led to the possibility of c_2's being marginal in Sec. IV.B.4. Physically, the very long range of the order-parameter fluctuations reflecting the GSI stabilizes the Gaussian critical behavior. This is in agreement with the fact that long-ranged order-parameter correlations in classical systems stabilize mean-field critical behavior (Fisher *et al.*, 1972).

The renormalized mean-field theory, the equation of state, and the critical exponent β can be discussed in analogy to the magnetic case (Kirkpatrick and Belitz, 1997; Zhou and Kirkpatrick, 2004).

The theory of the metal-superconductor transition presented above is assuming conventional s-wave spin-singlet superconductors. It is interesting to ask how the phase-transition scenario is modified by exotic (i.e., nonzero angular momentum, $\ell>0$) pairing. First of all, in contrast to conventional superconductivity, exotic superconductivity is rapidly destroyed by nonmagnetic disorder because Anderson's theorem (Anderson, 1959) does not hold. This has been observed, for example, in $ZrZn_2$ (Pfleiderer, Uhlarz, *et al.*, 2001), which is believed to be a p-wave spin-triplet superconductor, $\ell=1$. Recently, the resulting quantum phase transition between a dirty metal and an exotic superconductor has been studied within a Landau-Ginzburg-Wilson approach (Sknepnek *et al.*, 2002). It turns out that the nonzero order-parameter angular momentum suppresses the effects of GSI on this transition. The nonanalytic wave-number dependence in the Gaussian order-parameter propagator $\mathcal{N}_n(\boldsymbol{k})$ takes the form $\boldsymbol{k}^{2\ell}\ln \boldsymbol{k}$. Thus, compared to the s-wave case, Eq. (4.42d), the nonanalytic term is suppressed by a factor of $\boldsymbol{k}^{2\ell}$. This can be understood as follows: In the presence of nonmagnetic quenched disorder, the dominant electronic soft modes are those that

involve fluctuations of the number density, spin density, or anomalous density in the zero-angular-momentum channel, while the corresponding densities in higher-angular-momentum channels are not soft. Since the different angular momentum modes are orthogonal at zero wave number, the coupling between a finite angular momentum order parameter and the zero-angular-momentum soft modes must involve powers of the wave number $|\boldsymbol{k}|$.

In conclusion, for $\ell > 1$, the GSI-induced nonanalytic term is subleading compared to the conventional $\boldsymbol{k}^2$ term, and it will not influence the critical behavior. For $\ell = 1$, the nonanalytic term is marginally relevant. The ultimate fate of the transition has not yet been worked out, mainly because of (unrelated) complications stemming from disorder fluctuations similar to those in disordered itinerant antiferromagnets.

D. Quantum antiferromagnetic transition

Let us now consider a quantum antiferromagnetic transition, in analogy to the ferromagnetic one discussed in Sec. III.B.1. A crucial difference between these two cases is that in the latter, both the order-parameter field $\boldsymbol{M}$ (whose average is the magnetization) and the generic soft modes are soft at zero wave number, while in the former, the order-parameter field $\boldsymbol{N}$ (whose average is the staggered magnetization) is soft at a nonzero wave number $|\boldsymbol{p}|$. As a result, the hydrodynamic wave number $\boldsymbol{k}$ in the dynamical piece of the ferromagnetic Gaussian action, Eq. (4.6), gets replaced by $\boldsymbol{p}$, and one has (Hertz, 1976)

$$\mathcal{A}_{\mathrm{H}}[\boldsymbol{N}] = \sum_k \boldsymbol{N}(k)(r + c\,\boldsymbol{k}^2 + d|\Omega_n|)\boldsymbol{N}(k) + O(\boldsymbol{N}^4). \tag{4.47}$$

Here $\boldsymbol{k}$ is the wave vector measured from the reference wave vector $\boldsymbol{p}$. The missing inverse wave number in the frequency term reflects the much weaker coupling, compared to the ferromagnetic case, of the particle-hole excitations to the order parameter field.[60] Because of this weaker coupling, one expects the Gaussian approximation to be much better here than in the ferromagnetic case. Indeed, it is easy to show that the effects that needed to be taken into account for the ferromagnet are irrelevant with respect to the mean-field transition described by Eq. (4.47). One thus expects a continuous transition with mean-field critical behavior in all dimensions $d > 2$. The finite-temperature properties of this theory have been worked out in detail by Millis (1993). It is important to remember that Eq. (4.47) assumes that the only relevant soft modes at the quantum antiferromagnetic transition are the order-parameter fluctuations.

Experimental observations are not in agreement with this expectation, probably because most metallic materials that display easily accessible quantum antiferromagnetic transitions are far from being simple metals.[61] They fall into the class of heavy-fermion materials; an overview has been given by Coleman *et al.* (2001). The best-studied system is $CeCu_{6-x}Au_x$, which shows a quantum phase transition to an antiferromagnetic state at a critical gold concentration $x_c \approx 0.1$ (von Löhneysen *et al.*, 1994). There are experimental indications for the quantum critical fluctuations being two dimensional or quasi-two-dimensional in nature (Stockert *et al.*, 1998). A detailed phenomenological analysis of neutron scattering experiments (Schröder *et al.*, 2000) has shown that the magnetic susceptibility is well described by the form

$$\chi(\boldsymbol{k},\Omega) = a[(-i\Omega + b\,T)^{\alpha} + \theta(\boldsymbol{k})^{\alpha}]^{-1}. \tag{4.48}$$

Here a and b are constants, $\alpha = 0.75 \pm 0.05$, and $\theta(\boldsymbol{k})$ is the wave-vector-dependent Weiss temperature. This type of behavior is often referred to as "local quantum criticality," as it is believed (although not universally so; see below) to point to atomic-scale physics as the cause of the antiferromagnetism, as opposed to the Fermi-liquid or spin-density-wave description that underlies Eq. (4.47). The possibility of such a local quantum critical point was first pointed out by Si *et al.* (1999).

Perhaps the most obvious violation of the expected mean-field behavior is the occurrence of the exponent $\alpha \neq 1$. Other discrepancies between the observed behavior and the one expected from Hertz theory have been discussed by Coleman *et al.* (2001) and Si *et al.* (2001). A prominent one is that frequency and temperature are observed to scale in the same way; see Fig. 24. This observation is usually referred to as Ω/T *scaling*, and it is expected to hold at a quantum critical point that exhibits hyperscaling (Sachdev and Ye, 1992). By contrast, in Hertz theory Ω scales as $T^{3/2}$ (Millis, 1993).[62] Another discrepancy is the behavior of the specific-heat coefficient, which is observed to diverge logarithmically, while in Hertz theory it remains finite and shows a square-root cusp singularity (Millis, 1993). Schröder *et al.* (2000) have proposed the following physical picture to explain this violation. In CeCuAu, or any heavy-fermion material, the highly localized f orbitals of the lanthanide (in this case, Ce) or actinide provide local magnetic moments,

[60] This holds for a generic shape of the Fermi surface. Special geometric features of the Fermi surface can cause particle-hole excitations to be soft at the same wave vector as the order parameter even in an antiferromagnet; see Abanov and Chubukov (2000) and the discussion below.

[61] An interesting counterexample is the case of $Cr_{1-x}V_x$, which is a common transition metal yet displays properties, both at its antiferromagnetic quantum critical point and away from it, that are very similar to the exotic behavior shown by the heavy-fermion materials and high-T_c superconductors (Yeh *et al.*, 2002). We shall come back to this in Sec. V.

[62] Naively, one might expect Ω/T scaling simply to follow from the fact that the scale dimension of T is given by the dynamical critical exponent z and hence is the same as that of Ω. In general, this is not true due to the presence of (1) multiple temperature scales, and (2) dangerous irrelevant variables. See Millis (1993); Sachdev (1997).

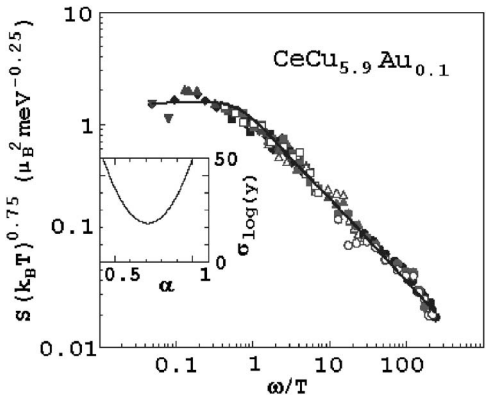

FIG. 24. The magnetic structure factor in $CeCu_{5.9}Au_{0.1}$ as measured by neutron scattering at various frequencies and temperatures, as a function of ω/T. The inset shows a measure of the scatter of the scaling plot for various values of the exponent α. Adapted from Schröder *et al.*, 2000.

while the more extended s, p, and d orbitals provide a Fermi surface. An increase in Au doping increases the hybridization between the localized and extended orbitals, which increases the coupling between the local moments via the conduction electrons and hence the Néel temperature T_N. If the hybridization becomes too strong, however, the f electrons are incorporated into the Fermi surface, and the antiferromagnetism disappears. As a function of doping, T_N thus goes through a maximum and disappears at a critical doping concentration; see Fig. 25. If this were all that happened, Hertz theory should apply. However, there is a second temperature scale besides T_N, namely, the Kondo temperature T_K. Only for $T<T_K$ do the local moments of the f orbitals become screened by the conduction electrons, and the hybridization becomes effective. Schröder *et al.* (2000) have proposed that T_K vanishes at the same critical concentration as T_N; see Fig. 25(b). In this picture, there is no heavy-electron Fermi surface at the quantum critical point, and Hertz theory is inapplicable. Notice that, according to this picture, there is nothing technically wrong with Hertz theory; the reason for its failure

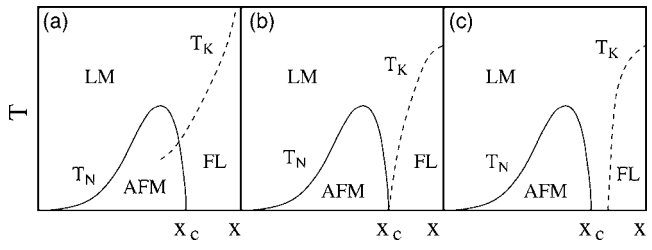

FIG. 25. Schematic phase diagram for $CeCu_{6-x}Au_x$. Shown are the antiferromagnetic (AFM) phase, the local-moments (LM) region, and the Fermi-liquid (FL) region. The Néel temperature T_N (solid lines) represents a true phase transition, while the Kondo temperature T_K (dashed lines) denotes a crossover. The three scenarios shown correspond to the three different physical situations discussed in the text. Adapted from Schröder *et al.*, 2000.

is that the model does not contain crucial slow degrees of freedom. Alternatively, one could imagine a scenario in which T_K remained nonzero into the antiferromagnetic phase [see Fig. 25(a)], but the experiments show that this is not the case in CeCuAu. Finally, it is conceivable, at least in principle, that the local-moment region might extend to zero temperature in an entire region of the phase diagram, as shown in Fig. 25(c), although there is currently no explicit theory that can explain why the unscreened local moments in heavy-fermion systems would not order at sufficiently low temperatures. Still, it is interesting to note that there are cases in which a non-Fermi-liquid phase is observed adjacent to an antiferromagnetic one, e.g., in $URu_{2-x}Re_xSi_2$; see Fig. 22.

Theoretically, the quantum antiferromagnetic transition is an open problem that is under very active consideration. A theoretical description of the above scenario involves the so-called *Kondo lattice problem*, i.e., the interaction of many localized spins with each other and with a band of conduction electrons. Si and collaborators have studied such a model within a dynamical mean-field approach (Si *et al.*, 2001; Si, 2003). In this approximation, these authors find what they call a local quantum critical point, which has many properties consistent with the observations. The exponent α is nonuniversal. More recently, Senthil *et al.* (2004) have proposed that the magnetic state is an unconventional spin-density wave in which spin-charge separation has taken place. Coleman *et al.* (2000) have developed a supersymmetric representation of spin operators that allows for a treatment of both magnetism and the Kondo effect in the context of a large-N expansion. These authors have speculated that, if applied to the Kondo lattice problem, this formalism could give rise to two different types of fixed points: A weak-coupling fixed point of Hertz type and a non-Fermi-liquid fixed point that displays spin-charge separation. There is, however, no consensus that the existence of a Fermi surface precludes an explanation of the observations. Rosch *et al.* (1997) have proposed an explanation in terms of three-dimensional conduction electrons coupling to two-dimensional ferromagnetic fluctuations. In a two-dimensional spin-fermion model, Abanov and Chubukov (2000) have found that special nesting properties of the Fermi surface lead to an exponent $\alpha \approx 0.8$, but their results do not display Ω/T scaling. A breakdown of the LGW expansion due to nesting has also been discussed by Lercher and Wheatley (2000); for a review, see Abanov *et al.* (2003). Sachdev and Morinari (2002) have found that similar "nonlocal" forms of the dynamic susceptibility are obtained in two-dimensional models where an order parameter couples to long-wavelength deformations of a Fermi surface.

Notice that all of these theories involve some coupling of soft modes, generic or otherwise, to the order parameter, although there currently is no consensus about their nature and origin. In the order of theoretical ideas listed above, they are

(1) slow local-moment fluctuations that serve as "Fermi surface shredders,"

(2) other fluctuations that destroy the Fermi surface,[63]

(3) ferromagnetic fluctuations,

(4) particle-hole fluctuations across a Fermi surface with special geometric features,

(5) volume and shape deformation of the Fermi surface.

We finally mention that antiferromagnetic quantum criticality, in particular in two dimensions, has received much attention in connection with high-T_c superconductivity, a topic that is beyond the scope of this article.

V. DISCUSSION AND CONCLUSION

In this section we summarize the main theoretical ideas we have presented, as well as the experimental situation. We also discuss a number of open problems in the field of quantum phase transitions and suggest a number of new experiments to address some of these.

A. Summary of review

Early work on quantum phase transitions suggested that most of them were conceptually quite simple, since they are related to corresponding classical phase transitions in a higher dimension. This led to the conclusion that quantum critical behavior generically would be mean-field-like. For a number of reasons, these conclusions have turned out not to be valid in general. In this review we have discussed one of the mechanisms that invalidates the mapping of a quantum phase transition onto a simple classical one in higher dimensions. The central idea behind this mechanism is as follows: The soft-mode spectrum of many-body systems is in general different at $T=0$ from the one at $T>0$, since there are soft modes at $T=0$ that develop a mass at nonzero temperature. These soft modes lead to long-ranged correlations in entire regions of the phase diagram, a phenomenon known as generic scale invariance. Physically, both the generic soft modes and the critical order-parameter fluctuations are equally important in the long-wavelength limit, and if the coupling between them is sufficiently strong, the former will influence the leading critical behavior. This can happen for classical phase transitions as well, but it is less common, since at $T>0$ there are fewer soft modes than at $T=0$. These observations led to the general paradigm that effects related to generic scale invariance are of fundamental importance for the theory of *generic* quantum critical points. As a result, quantum phase transitions are typically related to classical phase transitions with complications due to the presence of generic scale invariance, rather than to simple classical phase transitions.

In Sec. II we reviewed the concept of generic scale invariance in both classical and quantum systems. Although in both cases examples are plentiful, in the quantum case this is especially so because of additional Goldstone modes that exist at zero temperature. We distinguished between direct and indirect generic scale invariance effects, the former being immediate consequences of Goldstone's theorem, conservation laws, or gauge symmetries, while the latter arise from the former via mode-mode coupling effects. Classical examples discussed include Goldstone modes in Heisenberg ferromagnets or analogous systems, which lead to long-ranged susceptibilities everywhere in the magnetically ordered phase; local gauge invariance in superconductors or liquid crystals, in which context we have stressed interesting analogies between statistical mechanics and particle physics; long-time tails in time correlation functions, which determine the transport coefficients in a classical fluid in equilibrium; and long-ranged spatial correlations in a classical fluid in a nonequilibrium steady state. All the examples of generic scale invariance in the quantum case involved interacting electron systems, namely, weak-localization effects in disordered materials and the analogous effects in clean ones; and nonequilibrium effects analogous to those in classical fluids. Throughout this discussion we stressed the coupling between the statics and the dynamics in quantum systems, which leads to long-ranged spatial correlations in quantum systems even in equilibrium.

In Sec. III we discussed some classical phase transitions in which generic scale invariance plays a central role. Our first example was the nematic–smectic-A transition in liquid crystals, which maps onto the classical superconductor–normal-metal transition. In this case the relevant generic soft modes are the director fluctuations, which are Goldstone modes due to a broken rotational symmetry. They can drive the transition first order, even though in their absence one expects a continuous transition. The other classical phase transition discussed was the critical point in a classical fluid. Here the generic soft modes are due to conservation laws. In equilibrium, they influence the critical dynamics only. However, in a fluid that is subject to shear they also couple to the static critical behavior. In this case, long-ranged static correlations due to the generic soft modes stabilize mean-field critical behavior below the equilibrium upper critical dimension.

Section IV was devoted to four examples of quantum phase transitions. For the first three, namely, the quantum ferromagnetic transition in clean and disordered itinerant electron systems, respectively, and the normal-metal–superconductor transition at $T=0$, the generic soft modes are the particle-hole excitations that cause the weak-localization effects and their clean counterparts. The clean ferromagnetic case turned out to be analogous to the classical nematic–smectic-A transition in that the generic soft modes can lead to a fluctuation-induced first-order transition. The disordered quantum ferromagnetic transition is analogous to the classical fluid under shear, in the sense that static long-ranged

[63] These soft modes are generic in the sense of our definition only in the scenario depicted in Fig. 25(c).

correlations due to the generic soft modes stabilize a simple critical fixed point, and the critical behavior can be determined exactly even in $d=3$. In the case of a superconducting transition, this effect is even stronger, and the critical fixed point is Gaussian. Our fourth example dealt with the quantum antiferromagnetic transition, in which the critical behavior is currently not understood. There are many indications that, in the materials studied so far, generic soft modes related to local magnetic moments exist and couple to the critical order-parameter fluctuations, but a detailed theory of this effect remains to be worked out.

B. Open problems and suggested experiments

We conclude by listing a number of open questions, both theoretical and experimental, the answers to which would help shed light on some of the problems we have discussed. Our remarks are necessarily incomplete and speculative.

1. Generic scale invariance and soft modes

There are strong indications that the list of mechanisms for generic scale invariance in Sec. II is incomplete, even for relatively simple systems. Here we briefly mention two examples.

a. Generic non-Fermi-liquid behavior

As mentioned in Sec. IV.B.4, behavior not consistent with Landau's Fermi-liquid theory has been observed in a variety of materials far from any quantum critical point. While such exotic behavior may not come as too much of a surprise in strongly correlated systems involving rare earths or actinides, the well-documented case of MnSi (Pfleiderer, Julian, and Lonzarich, 2001) makes it likely that some more basic understanding is lacking. The observations clearly show that long-ranged correlations exist in a large region of parameter space and are not caused by any quantum phase transition. For instance, the resistivity has a $T^{3/2}$ asymptotic temperature dependence up to a factor of 2 in parameter space away from the quantum ferromagnetic transition. This generic long-time-tail behavior cannot be explained in an obvious way by any existing theory. Apart from being intrinsically interesting, this also raises the question of whether there are unknown generic soft modes that will be important for a complete understanding of the quantum phase transition. One obvious candidate would be slow fluctuations of local magnetic moments, which have been invoked for many mysterious effects, from hard-to-understand aspects of metal-insulator transitions (for a review see Sec. IX.B of Belitz and Kirkpatrick, 1994) to the critical behavior at the quantum antiferromagnetic transition (see Sec. IV.D). So far, however, there is no detailed theory that is capable of describing these soft modes.

b. Soft modes due to nested Fermi surfaces

Abanov and Chubukov (2000) have discussed nesting properties of Fermi surfaces that are important for quantum antiferromagnetic transitions. In the language of this review, the effect considered by these authors is likely a change of the properties of the generically soft particle-hole excitations. It would be useful to confirm this conjecture explicitly and to develop a general classification of the effects of Fermi-surface geometries on generic soft modes.

2. Aspects of the quantum ferromagnetic transitions

a. Disordered ferromagnets

A detailed experimental study of the critical behavior of both thermodynamic and transport properties near the quantum ferromagnetic transition in a disordered itinerant electron system would be very interesting. Some recent results on $Fe_{1-x}Co_xS_2$ (DiTusa *et al.*, 2003) appear to be consistent with the log-log-normal corrections to power-law scaling discussed in Sec. IV.B. A more precise determination of the critical behavior in this or other systems would be of great help in confirming or refuting the theoretical ideas.

Also of interest would be a systematic study of the destruction of the first-order ferromagnetic transition by nonmagnetic disorder that is predicted by the theory discussed in Sec. IV.B. For instance, the theory predicts that the tricritical point observed in MnSi and UGe_2 will move to lower and lower temperatures with increasing strength of quenched disorder, turn into a tricritical end point, then a critical end point, and finally the first-order transition should disappear (see Fig. 20). An experimental check of this prediction would be very valuable. For instance, it would give an indication of whether or not an understanding of the non-Fermi-liquid nature of the paramagnetic phase, mentioned in Sec. V.B.1, is important for describing the quantum phase transition.

b. Itinerant ferromagnets with magnetic impurities

An interesting quantum phase transition to study experimentally would be the ferromagnetic transition in an itinerant electron system with dilute magnetic impurities, which would fundamentally change the soft-mode structure in systems with or without additional nonmagnetic disorder. Such a study would provide an important check of the soft-mode paradigm, especially if it were possible to start with no magnetic impurities and study the crossover induced by their gradual introduction. In the absence of complications due to Kondoesque effects, one would expect a local LGW theory for the order-parameter fluctuations to be valid, since the magnetic impurities cut off the soft modes that strongly couple to the order-parameter fluctuations. Without nonmagnetic disorder, this would be Hertz's original theory, while in the nonmagnetically disordered case, rare-region effects could well play an important role.

It is likely, however, that Kondo screening effects and interactions between the impurity sites will lead to com-

plications not unlike those believed to be responsible for the observations in quantum antiferromagnets (Sec. IV.D), which are not understood. Generally, the physics of local moments, and their influence on the properties of conduction electrons, is one of the most important unsolved problems in condensed-matter physics.

c. Clean ferromagnets

The finite-temperature behavior of the observables near the continuous quantum ferromagnetic transition in clean itinerant electron systems has not been worked out. Given the existing RG description, which was briefly discussed in Sec. IV.A.4, and the existing theory of finite-temperature effects near Hertz's fixed point (Millis, 1993), this should be a relatively straightforward problem.

3. Metal-superconductor transition

a. Experimental study of critical behavior

It would be interesting to study experimentally the quantum phase transition between a disordered metal and a disordered conventional bulk superconductor. The theory reviewed in Sec. IV.C predicts that the quantum phase transition is governed by a Gaussian fixed point. If this is correct, there will be no crossover to mean-field behavior, and the asymptotic critical behavior will be observable, in contrast to the situation at the thermal transition. Since the predicted Gaussian critical behavior is radically different from mean-field critical behavior, this should be easy to observe.

b. Quantum phase transition to a gapless superconductor

The metal-superconductor transition described in Sec. IV.C assumed that the superconducting state was a conventional gapped state. It would be interesting to construct an analogous theory for the case of a gapless superconductor, especially in light of the suggestion by Vishveshwara *et al.* (2000) that the superconducting ground state in a disordered system will generically be gapless. On general grounds it is likely that the critical behavior will be the same in both cases if the transition is approached from the metallic side. However, the gapless superconducting state will have additional soft modes, and these modes may influence some of the critical behavior, such as the exponents governing the equation of state, when the transition is approached from the superconducting side.

4. Aspects of the quantum antiferromagnetic transitions

a. Phenomenological theory of the quantum antiferromagnetic problem

As discussed in Sec. IV.D, the quantum antiferromagnetic transition in real systems remains rather incompletely understood. In most materials studied so far, a basic ingredient of the problem seems to be itinerant electrons coupled to local moments. The latter are not screened since, at or near the quantum critical point, the Kondo temperature vanishes due to critical fluctuation effects. In the absence of a more fundamental approach to this very complicated problem, it would be very interesting to construct a purely phenomenological theory of these coupled fluctuations, including the soft modes associated with the slow temporal decay of local moments.

b. Simple quantum antiferromagnets

As we mentioned in Sec. IV.D, most of the work on the quantum antiferromagnetic transition has focused on heavy-fermion compounds and high-T_c superconductors, partly due to the general interest in these systems. These materials are far from being simple metals, and their behavior is heavily influenced by local-moment physics, among other complications. It would be interesting to find and study an itinerant electron system without local moments that has a quantum antiferromagnetic transition, provided such materials do indeed exist. The complicated behavior of $Cr_{1-x}V_x$ (see footnote 61) is discouraging in this respect, although at least some properties of this material have been explained as due to nesting properties of the Fermi surface (Norman *et al.*, 2003; Bazaliy *et al.*, 2004; Pépin and Norman, 2004). In the absence of disorder, the quantum phase transition in such a systems should be described by Hertz's theory. The introduction of nonmagnetic disorder would likely lead to a quantum phase transition where statistically rare events are important.

5. Nonequilibrium quantum phase transitions

In Sec. III.B.2 we discussed a classical nonequilibrium phase transition in which generic scale invariance plays a role. The general topic of classical phase transitions in driven systems has received a substantial amount of attention (Schmittmann and Zia, 1995). In Sec. II.B.5 we discussed how nonequilibrium situations at zero temperature can lead to correlations that are of even longer range than in equilibrium. Experimentally, however, the problem of nonequilibrium quantum phase transitions has received little attention so far. In quantum Hall systems, which we have not discussed in this review, electric-field effects have been studied (see Sondhi *et al.*, 1990), but in this case the electric field acts as a relevant operator with respect to the transition. Experiments that show an actual nonequilibrium quantum phase transition would be of interest.

6. Other quantum phase transitions in which generic scale invariance might play a role

In addition to the examples covered in this review, one can think of quantum phase transitions that have not been investigated so far either theoretically or experimentally, for which GSI effects are likely to play a central role. For example, the order parameter for the spin-triplet analog of the isotropic-to-nematic phase transition that has been proposed by Oganesyan *et al.* (2001) to occur in quantum Hall systems is expected to couple to generic soft modes. For this transition, in contrast to its spin-singlet analog, a simple LGW theory is therefore unlikely to be valid.

ACKNOWLEDGMENTS

We have benefited from discussions and correspondence with M. Aronson, E. D. Bauer, A. Chubukov, P. Coleman, N. G. Deshpande, W. Götze, A. Millis, C. Pepin, P. Phillips, S. Sachdev, Q. Si, and J. Toner. We would also like to thank our collaborators on some of the topics discussed in this review: J. R. Dorfman, F. Evers, M. T. Mercaldo, A. Millis, R. Narayanan, J. Rollbühler, J. V. Sengers, S. L. Sessions, R. Sknepnek, and Lubo Zhou. We are grateful to E. D. Bauer, J. R. Dorfman, M. Dzero, T. C. Lubensky, B. M. Maple, M. Norman, Q. Si, and J. Toner for comments on a draft version of the manuscript. Part of this work was performed at the Aspen Center for Physics and at the Max-Planck Institute for the Physics of Complex Systems in Dresden, Germany. We would like to thank both institutions for their hospitality. This work was supported by the NSF under Grant Nos. DMR-99-75259, DMR-01-32555, DMR-01-32726, and DMR-03-39147 and by the University of Missouri Research Board.

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