The plane-wave/super Yang-Mills duality

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This article reviews the plane-wave/super Yang-Mills duality, which states that strings on a plane-wave background are dual to a particular large R-charge sector of $\mathcal{N}=4$, $D=4$ superconformal $\mathrm{U}(N)$ gauge theory. This duality is a specification of the usual anti–de Sitter/conformed field theory (AdS/CFT) correspondence in the "Penrose limit." The Penrose limit of $\mathrm{AdS}_5\times S^5$ leads to the maximally supersymmetric ten-dimensional plane wave (henceforth "the" plane wave) and corresponds to restricting to the large R-charge sector, the Berenstein-Maldacena-Nastase (BMN) sector, of the dual superconformal field theory. After reviewing the necessary background, the authors state the duality and review some of its supporting evidence. They discuss the suggestion by 't Hooft that Yang-Mills theories with gauge groups of large rank might be dual to string theories and the realization of this conjecture in the form of the AdS/CFT duality. Plane waves as exact solutions of supergravity and their appearance as Penrose limits of other backgrounds are considered, followed by an overview of string theory on the plane-wave background, discussing the symmetries and spectrum. The article then makes precise the statement of the proposed duality and classifies the BMN operators. It examines the gauge theory side of the duality, studying both quantum and nonplanar corrections to correlation functions of BMN operators and their operator-product expansions. The important issue of operator mixing and the resultant need for rediagonalization is stressed. Finally, the article studies strings on the plane wave via light-cone string field theory and demonstrates agreement between the one-loop correction to the string mass spectrum and the corresponding quantity in the gauge theory. A new presentation of the relevant superalgebra is given.

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I. INTRODUCTION

In the late 1960s a theory of strings was first proposed as a model for the strong interactions describing the dynamics of hadrons. However, in the early 1970s, results from deep-inelastic scattering experiments led to the acceptance of the "parton" picture of hadrons, and this led to the development of the theory of quarks as basic constituents carrying color quantum numbers, and whose dynamics are described by quantum chromodynamics (QCD), which is an SU(N_c) Yang-Mills gauge theory with N_f flavors of quarks. According to the standard model of particle physics, $N_c=3$, $N_f=6$. With the acceptance of QCD as the theory of strong interactions the old string theory became obsolete. However, in 1974 't Hooft (1974a, 1974b) observed a property of SU(N_c) gauge theories which was very suggestive of a correspondence or "duality" between the gauge dynamics and string theory.

To study any field theory we usually adopt a perturbative expansion, generally in powers of the coupling constant of the theory. The first remarkable observation of 't Hooft was that the true expansion parameter for an SU(N) gauge theory (with or without quarks) is not the Yang-Mills coupling g_{YM}^2, but rather g_{YM}^2 dressed by N, in the combination λ, now known as the 't Hooft coupling:

$$\lambda = g_{\mathrm{YM}}^2 N. \tag{1.1}$$

The second remarkable observation 't Hooft made was that in addition to the expansion in powers of λ one may also classify the Feynman graphs appearing in the correlation function of generic gauge theory operators in powers of $1/N^2$. This observation is based on the fact that the operators of this gauge theory are built from simple $N\times N$ matrices. One is then led to expand any correlation function in a double expansion in powers of λ as well as $1/N^2$. In the $1/N^2$ expansion, which is a useful one for large N, the terms of lowest order in powers of $1/N^2$ arise from the subclass of Feynman diagrams that can be drawn on a sphere (a one-point compactification of the plane), once the 't Hooft double-line notation is used. These are called *planar* graphs. In the same spirit one can classify all Feynman graphs according to the lowest-genus surface on which they may be placed without any crossings. For genus h surfaces, with $h>0$, such diagrams are called *nonplanar*. The lowest-genus nonplanar surface is the torus with $h=1$. The genus h graphs are suppressed by a factor of $(1/N^2)^h$ with respect to the planar diagrams. According to this $1/N$ expansion, at large N, but finite 't Hooft coupling λ, the correlators are dominated by planar graphs.

The genus expansion of Feynman diagrams in a gauge theory resembles a similar pattern in string theory: stringy loop diagrams are suppressed by g_s^h where h is now the genus of the string worldsheet and g_s is the string coupling constant. The Feynman graphs in the large-N limit form a continuum surface which may be (loosely) interpreted as the string worldsheet. In Sec. I.A of the Introduction, we shall very briefly sketch the mechanics of the 't Hooft large-N expansion.

In the mid 1970s, string theory was promoted from an effective theory of strong dynamics to a theory of fundamental strings and put forward as a candidate for a quantum theory of gravity (Scherk and Schwarz, 1974). Much has been learned since then about the five different ten-dimensional string theories. In particular, by 1997, a web of various dualities relating these string theories, their compactifications to lower dimensions, and an as-yet-unknown, though more fundamental, theory known as M theory, had been proposed and compelling pieces of evidence in support of these dualities uncovered (Hull and Townsend, 1995; Witten, 1995). We do not intend to delve into the details of these dualities; for such matters the reader is referred to the various books and reviews, e.g., Johnson (2003) or Polchinski (1998a, 1999b).

Although our understanding of string and M theory had been much improved through the discovery of these various dualities, before 1997 the observation of 't Hooft had not been realized in the context of string theory. In other words, the 't Hooft strings and the "fundamental" strings seemed to be different objects. Amazingly, in 1997 a study of the near-horizon geometry of D3-branes

(Maldacena, 1998) led to the conjecture that *strings of type-IIB string theory on the* $AdS_5 \times S^5$ *background are the 't Hooft strings of an* $\mathcal{N}=4$, $D=4$ *supersymmetric Yang-Mills theory*.

$AdS_5 \times S^5$ is a 9+1-dimensional manifold composed of a 4+1 dimensional anti–de Sitter (AdS_5) space, a space of constant negative curvature and a five-dimensional sphere S^5 such that the radii of curvature of both AdS_5 and S^5 are equal. According to this conjecture any physical object or process in the type-IIB theory on an $AdS_5 \times S^5$ background can be equivalently described by $\mathcal{N}=4$, $D=4$ super Yang-Mills (SYM) theory, which is a four-dimensional super conformal field theory (Gubser *et al.*, 1998; Witten 1998; Aharony *et al.*, 2000). In particular, according to the Maldacena conjecture, the 't Hooft coupling (1.1) is related to the AdS radius R as

$$\left(\frac{R}{l_s}\right)^4 = g_{\rm YM}^2 N, \tag{1.2}$$

where l_s is the string scale. On the string theory side of the duality, l_s/R appears as the worldsheet coupling; hence when the gauge theory is weakly coupled the two-dimensional worldsheet theory is strongly coupled and nonperturbative, and vice versa. In this sense the anti–de Sitter/conformal field theory (AdS/CFT) duality (Witten, 1998; Aharony *et al.*, 2000) is a weak/strong duality. Due to the (mainly technical) difficulties of solving the worldsheet theory on the $AdS_5 \times S^5$ background,[1] our understanding of the string theory side of the duality has been mainly limited to the low-energy supergravity limit, and in order for the supergravity expansion about the AdS background to be trustworthy, we generally need to keep the AdS radius large. At the same time we must also ensure the suppression of string loops. As a result, most of the development and checks of the duality from the string theory side have been limited to the regime of large 't Hooft coupling and the $N \to \infty$ limit on the gauge theory side. A more detailed discussion of the AdS/CFT duality will be presented in Sec. I.B.

One might wonder if it is possible to go beyond the supergravity limit and perform real string theory calculations from the gauge theory side. We would then need to have similar results from the string theory side to compare with, and this seems notoriously difficult, at least at the moment.

The σ model for strings on $AdS_5 \times S^5$ is difficult to solve. However, there is a specific limit in which $AdS_5 \times S^5$ reduces to a plane wave (Gueven, 2000; Blau *et al.*, 2002a, 2002b, 2002c, 2003; Blau and O'Loughlin, 2003), and in this limit the string theory σ model becomes solvable (Metsaev, 2002; Metsaev and Tseytlin, 2002). In this special limit we then know the string spectrum, at least for noninteracting strings, and one might ask if we can find the same spectrum from the gauge theory side. For that we first need to understand how this specific limit translates to the gauge theory side. We then need a definite proposal for mapping the operators of the gauge theory to (single) string states. This proposal, following the work of Berenstein, Maldacena, and Nastase (2002), is known as the BMN conjecture. It will be introduced in Sec. I.C of the Introduction and is discussed in more depth in Sec. V. The BMN conjecture is supported by some explicit and detailed calculations on the gauge theory side. Spelling out different elements of this conjecture is the main subject of this review.

In Sec. II we review plane waves as solutions of supergravities that have a globally defined null Killing vector field, and we emphasize an important property of these backgrounds: they are exact solutions without α' corrections. We shall focus mainly on the ten-dimensional maximally supersymmetric plane-wave background. This maximally supersymmetric plane wave will be referred to as "the" plane wave to distinguish it from other plane-wave backgrounds. We study the isometries of this background, as well as the corresponding supersymmetric extension, and we show that this background possesses a $PSU(2|2) \times PSU(2|2) \times U(1)_- \times U(1)_+$ superalgebra. We also discuss the spectrum of type-IIB supergravity on the plane-wave background.

In Sec. III we review the procedure for taking the Penrose limit of any given geometry. We then argue that this procedure can be extended to solutions of supergravities to generate new solutions. As examples, we work out the Penrose limit of some $AdS_p \times S^q$ spaces. Moreover, we discuss how taking the Penrose limit manifests itself as a contraction at the level of the superalgebra. In particular, we show how to obtain the superalgebra of the plane wave, discussed in Sec. II, as a (Penrose) contraction of the supergroup $PSU(2,2|4)$ which is the superalgebra of the $AdS_5 \times S^5$ background.

Having established that plane-wave backgrounds form α'-exact solutions of supergravitiess, we find that they form particularly simple backgrounds for string theory. In Sec. IV we work out the σ-model action for type-IIB strings on the plane-wave background in the light-cone gauge. Formulating a theory in the light-cone gauge has the advantage that only physical (on-shell) degrees of freedom appear, and ghosts are decoupled (Polchinski, 1998a). For the particular case of strings on plane waves, due to the existence of the globally defined null Killing vector field, fixing the light-cone gauge has an additional advantage: the energies (frequencies) are conserved in this gauge, and as a result the well-known problem associated with nonflat spaces, namely, particle (string) production, is absent. Adding fermions is done using the Green-Schwarz formulation, and as usual redundant fermionic degrees of freedom arise from κ symmetry. After fixing the κ symmetry, we obtain the fully gauge-fixed action from which one can easily read off the spectrum of (free) strings on this background. We also present the representation of the plane-wave superalgebra in terms of stringy modes.

In Secs. V–VII, we return to the 't Hooft expansion, though in the BMN sector of the gauge theory, and de-

[1]For a recent work in the direction of solving this two-dimensional theory see Bena *et al.* (2003) and the references therein.

duce the spectrum of strings on the plane wave obtained in Sec. IV, from gauge theory calculations. In this sense these sections are the core of this review. In Sec. V we present the BMN or plane-wave/SYM duality conjecture.

In Sec. VI we present the first piece of supporting evidence for the duality, where we focus on the planar graphs. Reviewing the results of Constable *et al.* (2002a, 2002b), Gross *et al.* (2002), and Kristjansen *et al.* (2002), we show that the 't Hooft expansion is modified for the BMN sector of the gauge theory, and we are led to a new type of "'t Hooft expansion" with a different effective coupling. In Sec. VI.B we explain how and why anomalous dimensions of operators in the BMN sector correspond to the free-string spectrum obtained in Sec. IV.

In Sec. VII, we move beyond the planar limit and consider the contributions to the spectrum arising from nonplanar graphs, which correspond on the string theory side to inclusion of loops. We shall see that the genus counting parameter should also be modified in the BMN limit. Moreover, as we shall see, the suppression of higher-genus graphs with respect to the planar ones is not universal and in fact depends on the sector of the operators we are interested in. One of the intriguing consequences of the nonvanishing higher-genus contributions is the possible mixing between the original single-trace operators with double and in general multitrace operators. The mixing effects will force us to modify the original BMN dictionary of correspondences. After making the appropriate modifications, we present the results of the one-loop (genus one) corrections to the string spectrum.

After discussing the string spectrum on the plane wave at both planar and nonplanar order from the gauge theory side of the duality, we tackle the question of string interactions on the plane-wave background in Sec. VIII, with the aim of obtaining one-loop corrections to the string spectrum from the string theory side. This provides us with a nontrivial check of the plane-wave/SYM duality. From the string theory point of view, the presence of the nontrivial background, in particular the Ramond-Ramond form (Ramond, 1971), makes using the usual machinery for computing string scattering amplitude via vertex operators cumbersome, and one is led to develop the string field-theory formulation, a field theory that captures the dynamics of string theory and in which every string mode is represented by an appropriate field. From the gauge theory side, as we shall discuss in Sec. VII, the nature of the difficulties is different: it is not a trivial task to distinguish single, double, and in general multistring states. We work out light-cone string field theory on this background and use this setup to calculate one-loop corrections to the string spectrum. We shall show that the data extracted from nonplanar gauge theory correlation functions are in agreement with their string theoretic counterparts. In Sec. VIII.A we present some basic facts and necessary background regarding light-cone string field theory. In Sec. VIII.B we work out the three-string vertex in the light-cone string field theory on the plane-wave background. Then in Sec. VIII.C we consider higher-order string interactions and present one-loop corrections to the string mass spectrum.

Finally, in Sec. IX, we summarize the main points of the review and mention some interesting related ideas and developments in the literature. We also discuss some of the open questions in the formulation of strings on general plane-wave backgrounds and the related issues on the gauge theory side of the conjectured duality.

We have tried to make this review self-contained. However, we assume that the reader is familiar with supersymmetry at the level of Wess and Bagger (1992) and string theory at the level of Polchinski (1998a, 1998b). For standard reviews of the AdS/CFT, the reader is referred to Aharony *et al.* (2000a) and D'Hoker and Friedman (2002).

A. 't Hooft's large-N expansion

In an effort to understand strong dynamics in gauge theories, 't Hooft introduced a remarkable expansion for gauge theories with large gauge groups, with the rank of the gauge group $\sim N$ ('t Hooft, 1974a, 1974b). He suggested treating the rank of the gauge group as a parameter of the theory and expanding in $1/N^2$, which turns out to correspond to the genus of the surface onto which the Feynman diagrams can be mapped without overlap, yielding a topological expansion analogous to the genus expansion in string theory, with the gauge theory Feynman graphs viewed as "string theory" worldsheets. In this correspondence, the planar or nonplanar Feynman graphs may be thought of as tree or loop diagrams of the corresponding "string theory."

Asymptotically free theories, like SU(N) gauge theory with sufficiently few matter fields, exhibit dimensional transmutation, in which the scale-dependent coupling gives rise to a fundamental scale in the theory. For QCD, this is the confinement scale $\Lambda_{\rm QCD}$. Since this is a scale associated with physical effects, it is natural to keep this scale fixed in any expansion. This scale appears as a constant of integration when solving the β-function equation, and it can be held fixed for large N if we also keep fixed the product $g_{\rm YM}^2 N$ while taking $N \to \infty$. This defines the new expansion parameter of the theory, the 't Hooft coupling constant $\lambda \equiv g_{\rm YM}^2 N$.

To see how the expansion works in practice, let us consider the action for a gauge theory, for example, the $\mathcal{N}=4$ super Yang-Mills[2] theory written down in component form in Eq. (A1). All the fields in this action are in the adjoint representation. We have scaled our fields so that an overall factor of $1/g_{\rm YM}^2$ appears in front. We write this in terms of N and the 't Hooft coupling λ, using $1/g_{\rm YM}^2 = N/\lambda$. The perturbation series for this theory can be constructed in terms of Feynman diagrams built from propagators and vertices in the usual

[2]Supersymmetry is not consequential to this discussion, and we ignore it for now.

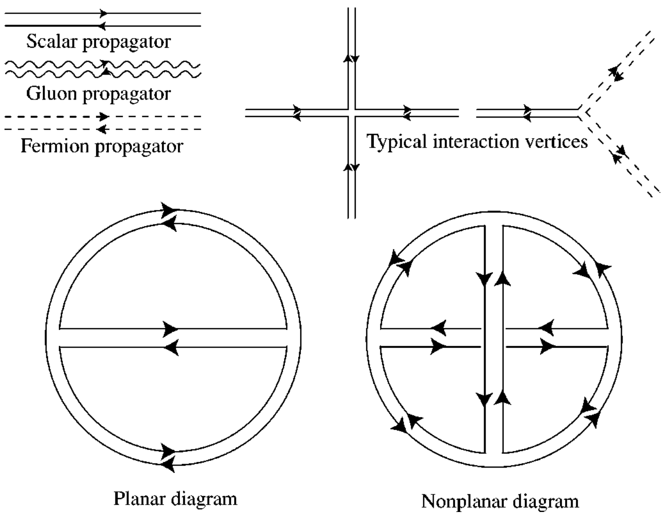

FIG. 1. Typical Feynman rules for adjoint fields and sample planar and nonplanar diagrams.

way. With our normalization, each propagator contributes a factor of λ/N, and each vertex a factor of N/λ. Loops in diagrams appear with group theory factors coming from summing over the group indices of the adjoint generators. These give rise to an extra factor of N for each loop. A typical Feynman diagram will be associated with a factor

$$\lambda^{P-V}N^{V-P+(L+1)} \tag{1.3}$$

if the diagram contains V vertices, P propagators, and L loops. These diagrams can be interpreted as simplicial complexes if we choose to draw them using the 't Hooft double-line notation. For U(N), the group index structure of adjoint fields is that of a direct product of a fundamental and an antifundamental. The propagators can be drawn with two lines showing the flow of each index, and with the arrows pointing in opposite directions (see Fig. 1).

The vertices are drawn in a similar way, with the directions of the arrows indicating the fundamental or antifundamental indices of the generators. In this diagrammatic presentation, the propagators form the edges, and the insides of loops are considered to be the faces. The one-point compactification of the plane then means that the diagrams give rise to closed, compact, and orientable surfaces, with Euler characteristic $\chi=V-P+F=2-2h$, where h is the genus of the surface. The number of faces is one more than the number of loops, since group theory always gives rise to an extra factor of N for the last trace. In the simplicial decomposition with the one-point compactification, the outside of the diagram becomes another face and can be interpreted as the last trace.

The perturbative expansion of the vacuum persistence amplitude takes the form of a double expansion,

$$\sum_{h=0}^{\infty} N^{2-2h}P_h(\lambda), \tag{1.4}$$

with h the genus and P_h some polynomial in λ, which itself admits a power-series expansion,

$$P_h(\lambda)=\sum_{n=0}^{\infty} C_{h,n}\lambda^n. \tag{1.5}$$

The basic idea is that all the diagrams generated for the vacuum correlation function can be grouped in classes based on their genera, and all the diagrams in each class will have varying dependences on the 't Hooft coupling λ. Collecting all the diagrams in a given class into groups sharing the same dependence on λ, we can extract the h- and n-dependent constant $C_{h,n}$. It is clear from Eq. (1.4) that, for large N, the dominant contributions come from diagrams of the lowest genus, the planar (or spherical) diagrams.

The double expansion (1.4) and (1.5) looks remarkably similar to the perturbative expansion for a string theory with coupling constant $1/N$ and with the expansion in powers of λ playing the role of the worldsheet expansion. The analogy extends to the genus expansion, with the Feynman diagrams loosely forming a sort of discretized string worldsheet. At large N, such a string theory would be weakly coupled. The string coupling measures the difference in the Euler character for worldsheet diagrams of different topology. This has long suggested the existence of a duality between gauge and string theory. We of course also have to account for the mapping of nonperturbative effects on the two sides of the duality.

So far we have considered only the vacuum diagrams, though the same arguments apply when considering correlation functions with insertions of the fields. The action appearing in the generating functional of connected diagrams must be supplemented with terms coupling the fundamental fields to currents, and these terms will enter with a factor of N. The planar[3] (leading) contributions to such correlation functions with j insertions of the fields will be suppressed by an extra factor of N^{-j} relative to the vacuum diagrams. The one-particle irreducible three- and four-point functions then come with factors of $1/N$ and $1/N^2$ relative to the propagator, suggesting that $1/N$ is the correct expansion parameter. The expansion (1.4) for these more general correlation functions still holds if we account for the extra factors of N coming from the insertions of the fields. The extra factor depends on the number of fields in the correlation function, but is fixed for the perturbative expansion of a given correlator.

The picture we have formed is of an oriented closed string theory. Adding matter in the fundamental representation would correspond to including propagators

[3]With the point at infinity identified, planar diagrams become spheres, and higher-genus diagrams become spheres with handles.

with a single line. These could then form the edges of the worldsheets, and so would correspond to a dual theory with open strings (with the added possibility of D-branes). Generalizations to other gauge groups such as O(N) and Sp(N) would lead to unorientable worldsheets, since their adjoint representations (which are real) would appear like products of fundamentals with fundamentals. This viewpoint has been applied to other types of theories, for example, nonlinear sigma models with a large number of fundamental degrees of freedom.

The new ingredient relevant to our discussion will be the following: for a conformally invariant theory such as $\mathcal{N}=4$ SYM, the β function vanishes for all values of the coupling $g_{\rm YM}$ (it has a continuum of fixed points). There is no natural scale in this theory that should be held fixed. This makes possible limits different from the 't Hooft limit, and we take advantage of such an opening via the so-called BMN limit, which we discuss at length in what follows. Not all such limits are well defined. In the BMN limit, we shall consider operators with large numbers of fields. If the number of fields is scaled with N, generically, higher-genus diagrams will dominate lower-genus ones, and the genus expansion will break down. The novel feature of the BMN limit is that these large numbers of fields combine in a way that makes it possible for diagrams of all genera to contribute without the relative suppression typical in the 't Hooft limit. In this sense, the BMN limit is the balancing point between two regions, one where diagrams of higher genus are suppressed and do not contribute in the limit, and the other where the limit is meaningless.

A concise introduction to the basic ideas underlying the large-N expansion can be found in the article of 't Hooft (2002), with a more detailed review being that of 't Hooft (1994). Applications to QCD are given by Manohar (1998). A review of the large-N limit in field theories and the relation to string theory was presented by Aharony *et al.* (2000), who discuss many issues related to anti–de Sitter spaces, conformal field theories, and the celebrated AdS/CFT correspondence.

B. String/gauge theory duality

't Hooft's original demonstration that the large-N limit of U(N) gauge theory is dual to a string theory, sparked many attempts to construct such a duality explicitly. One such attempt (Gross, 1993; Gross and Taylor, 1993) was to construct the dual to two-dimensional pure QCD as a map from two-dimensional worldsheets of a given genus into a two-dimensional target space. QCD_2 in two dimensions is almost a topological theory, with the correlation functions depending only on the topology and the area of the manifold on which the theory is formulated, making the theory exactly solvable. The partition function of this string theory sums over all branched coverings of the target space and can be evaluated by discretizing the target using a two-dimensional simplicial complex with an $N\times N$ matrix placed at each link. The partition function thus constructed can be evaluated exactly via an expansion in terms of group characters, giving rise to a matrix model, whose solution has been given by Kazakov *et al.* (1996), Kostov and Staudacher (1997), and Kostov *et al.* (1998). Zero-dimensional QCD was considered by Brezin *et al.* (1978) as a toy model that retains all the diagrammatic features but with trivial propagators, allowing the investigation of combinatorial counting in matrix models.

Another realization of 't Hooft's observation, this time via conventional string theory, is the celebrated AdS/CFT correspondence. The duality is suggested by the two viewpoints presented by D-branes. The low-energy effective action of a stack of N coincident $D3$-branes is given by $\mathcal{N}=4$ super Yang-Mills theory with gauge group U(N). While away from the brane the theory is type-IIB closed string theory, there exists a *decoupling limit* where the closed strings of the bulk are decoupled from the gauge theory living on the brane (Maldacena, 1998). $D3$-branes are solitons in the string theory, analogous to the solitons of Bogomolny, Prasad, and Sommerfeld (BPS; see Bogomolny, 1975; Prasad and Sommerfeld 1975). Because these D-branes break 16 of the supercharges of the type-IIB vacuum, they are sometimes referred to as "half BPS." The supercharges in the decoupling limit will be nonlinearly realized as the superconformal supercharges in the $\mathcal{N}=4$ worldvolume theory of the branes, a theory that exhibits superconformal invariance. For large N, the stack of D-branes will back-react, modifying the geometry seen by the type-IIB strings. In the low-energy description given by supergravity, the presence of the D-brane is seen in the form of the vacuum for the background fields like the metric and the Ramond-Ramond fields. These are two different descriptions of the physics of the stack of D-branes, and the ability to take these different viewpoints is the essence of the AdS/CFT duality. Type-IIB superstring-theory on the $\text{AdS}_5\times S^5$ background is dual to (or can be equivalently described by) $\mathcal{N}=4$, $D=4$ U(N) supersymmetric Yang-Mills theory with a prescribed mapping between string theory and gauge theory objects.

The specific prescription for the correspondence is suggested by the matching of the global symmetry groups and their representations on the two sides of the duality. The matching extends to the partition function of the $\mathcal{N}=4$ SYM on the boundary of AdS_5 ($R\times S^3$) and the partition function of type-IIB string theory on $\text{AdS}_5\times S^5$ (Witten, 1998),

$$\langle e^{\int d^4x\ \phi_0(x)\mathcal{O}(x)}\rangle_{\rm CFT} = \mathcal{Z}_{\rm string}[\ \phi|_{\rm boundary}=\phi_0(x)], \qquad (1.6)$$

where the left-hand side is the generating function of correlation functions of gauge-invariant operators $\mathcal{O}$ in the gauge theory (such correlation functions are obtained by taking derivatives with respect to ϕ_0 and setting $\phi_0=0$) and the right-hand side is the full partition function of (type-IIB) string theory on the $\text{AdS}_5\times S^5$ background with the boundary condition that the field $\phi=\phi_0$ on the AdS boundary (Aharony *et al.*, 2000). The dimensions of the operators $\mathcal{O}$ (i.e., the charge associated with the behavior of the operator under rigid coordinate scalings) correspond to the free-field masses of the bulk excitations. Every operator in the gauge theory can be put in one-to-one correspondence with a field propagating in the bulk of the AdS space, e.g., the gauge-invariant chiral primary operators and their descendents (D'Hoker and Freedman, 2002) on the Yang-Mills side can be put in a one-to-one correspondence with the supergravity modes of the type-IIB theory. In the low-energy approximation to the string theory, we have type-IIB supergravity, with higher-order α' corrections from the massive string modes. (Note, however, that the $AdS_5\times S^5$ background itself is an exact solution to supergravity with all α' corrections included.) The relation (1.2) between the radius of AdS_5 (and also S^5) shows that when the gauge theory is weakly coupled, the radius of AdS_5 is small in string units. In this regime, the supergravity approximation breaks down. Of course, to make the duality complete, we have to find the mapping for the nonperturbative objects and effects on the two sides.

This conjecture has been generalized and restated for string theories on many deformations of the $AdS_5\times S^5$ background, such as $AdS_5\times T^{1,1}$ (Klebanov and Witten, 1998), the orbifolds of $AdS_5\times S^5$ space (Gukov, 1998; Kachru and Silverstein, 1998; Lawrence *et al.*, 1998), and even nonconformal cases (Klebanov and Strassler, 2000; Polchinski and Strassler, 2000) and $AdS_3\times S^3\times M_4$ (Giveon *et al.*, 1998; Kutasov and Seiberg, 1999). For example, in the Klebanov-Witten case, the statement is that type-IIB strings on an $AdS_5\times T^{1,1}$ background are the 't Hooft strings of an $\mathcal{N}=1$ super conformal field theory and for the $AdS_3\times S^3$ case, 't Hooft strings of the $\mathcal{N}=(4,4)$, $D=2$ super conformal field theory are dual to strings on $AdS_3\times S^3\times T^4$, where T^4 is a four-dimensional torus. The latter have been made explicit by the Kutasov-Seiberg construction (Giveon *et al.*, 1998; Kutasov and Seiberg, 1999). In general it is a nontrivial task to determine the 't Hooft string picture of a given gauge theory.

C. Moving away from the supergravity limit; strings on plane waves

Although Witten's formula (1.6) is precise, from a practical point of view our calculational ability does not go beyond the large-N limit which corresponds to the supergravity limit on the string theory side (except for quantities protected by supersymmetry, the calculations on both sides of the duality beyond the large-N limit exhibit the same level of difficulty). However, one may still hope to go beyond the supergravity limit which corresponds to a restriction to some particular sector of the gauge theory.

In this section we recall some basic observations and facts which led Berenstein, Maldacena, and Nastase (2002) to their conjecture, as well as a brief summary of the results obtained based on and in support of the conjecture. These observations and results will be discussed in some detail in the main part of this review.

- Although so far we have not been able to solve the string σ model in the $AdS_5\times S^5$ background and obtain the spectrum of (free) strings, the Penrose limit (Penrose, 1976; Gueven, 2000) of $AdS_5\times S^5$ geometry results in another maximally supersymmetric background of type-IIB which is the plane-wave geometry. The corresponding σ model (in the light-cone gauge) is solvable, allowing us to deduce the spectrum of (free) strings on this plane-wave background.
- Taking the Penrose limit on the gravity side corresponds to restricting the gauge theory to operators with a large charge under one of its global symmetries [more precisely the R-symmetry charge associated with a $U(1)\subset SO(6)_R$] J, the BMN sector, and simultaneously taking the large-N limit.
- The BMN sector of $\mathcal{N}=4$, $D=4$ $U(N)$ SYM theory comprises operators with large scaling dimension Δ and large R-charge J, such that

$$\frac{1}{\mu}p^- \equiv \Delta - J = \text{fixed}, \tag{1.7a}$$

$$\alpha' \mu p^+ \equiv \frac{1}{2\sqrt{g_{YM}^2 N}}(\Delta + J) = \text{fixed}, \tag{1.7b}$$

together with

$$g_{YM} = \text{fixed}, \quad \frac{J^2}{N} = \text{fixed}, \quad N,J\to\infty. \tag{1.8}$$

In the above, $(1/\mu)p^-$ and $\alpha'\mu p^+$ are the corresponding string light-cone Hamiltonian and light-cone momentum, respectively. The parameter μ is a convenient but auxiliary parameter, the role of which will become clear in the following sections.

- In Eq. (1.7a), p^- should be understood as the full plane-wave light-cone string (field) theory Hamiltonian. Explicitly, one can interpret (1.7a) as an equality between two operators, the plane-wave light-cone string field-theory Hamiltonian H_{SFT} on one side and the difference between the dilatation and the R-charge operators on the other side, i.e.,

$$\frac{1}{\mu}H_{SFT} = \mathcal{D} - \mathcal{J}, \tag{1.9}$$

where $\mathcal{D}$ is the dilatation operator and $\mathcal{J}$ is the R-charge generator. Therefore according to the identification (1.9) which is the (improved form of the original) BMN conjecture, the spectrum of strings, which are the eigenvalues of the light-cone Hamiltonian $p^-=H_{SFT}$, should be equal to the spectrum of the dilatation operator, which is the Hamiltonian of the $\mathcal{N}=4$ gauge theory on $\mathbb{R}\times S^3$, restricted to the operators in the BMN sector of the gauge theory [defined in Eqs. (1.7) and (1.8)].

- As stated above, Eq. (1.9) sets up an equivalence between two operators. However, the second part of the BMN conjecture is about the correspondence between the Hilbert spaces that these operators act on; on the string theory side, it is the string (field) theory Hilbert space which is comprised of a direct sum of the zero-string, single-string, double-string, and multistring states, quite similarly to the flat-space case (Polchinski, 1998a). On the gauge theory side it is the so-called BMN operators, the set of $U(N)$ invariant operators of large R-charge J and large dimension in the free gauge theory, subject to Eqs. (1.7) and (1.8).
- According to the BMN proposal (Berenstein, Maldacena, and Nastase, 2002), single-string states map to certain single-trace operators in the gauge theory.[4] In particular, the single-string vacuum state in the sector with light-cone momentum p^+, $(\alpha' \mu p^+)^2 = (J^2/g_{\rm YM}^2 N)$, is identified with the chiral primary BPS operator

$$|0,p^+\rangle \leftrightarrow \mathcal{N}_J \mathrm{Tr}(Z^J)|\mathrm{vac}\rangle, \tag{1.10}$$

where $\mathcal{N}_J$ is a normalization constant that will be fixed later in Sec. VI. In the above, $Z=(1/\sqrt{2})(\phi^5 + i\phi^6)$, where ϕ^5 and ϕ^6 are two of the six scalars of the $\mathcal{N}=4$, $D=4$ gauge multiplet. The R-charge we want to consider, J, is the eigenvalue for a $U(1)$ generator, $U(1)\subset SU(4)_R$, so that Z carries unit charge and all the other bosonic modes in the vector multiplet, four scalars and four gauge fields, have zero R charge. Since all scalars have $\Delta=1$ (classically), for Z and hence Z^J, $\Delta - J=0$. The advantage of identifying string vacuum states with the chiral primary operators is twofold: (i) they have $(1/\mu)p^- = \Delta - J = 0$, and (ii) their anomalous dimension is zero, and hence the corresponding p^- remains zero to all orders in the 't Hooft coupling and even nonperturbatively; see, for example, D'Hoker and Freedman (2002).
- As for stringy excitations above the vacuum, Berenstein, Maldacena, and Nastase (2002) conjectured that we need to work with certain "almost" BPS operators, i.e., certain operators with large J charge and with $\Delta - J \neq 0$, but $\Delta - J \ll J$. In particular, single closed string states were (originally) proposed to be dual to single-trace operators with $\Delta - J = 2$. The exact form of these operators and a more detailed discussion regarding them will be presented in Sec. V. As we will see in Secs. VII.B.2 and VII.C, however, this identification of the single-string Hilbert space with the single-trace operators, because of the mixing between single-trace and multitrace operators, should be modified. Note that this mixing is present for both chiral primaries and "almost" primaries.
- As is clear from Eq. (1.8), in the BMN limit the 't Hooft coupling goes to infinity and naively any perturbative calculation in the gauge theory (except of course for chiral primary two- and three-point functions) is not trustworthy. However, the fact that we are working with "almost" BPS operators motivates the hope that, although the anomalous dimensions for such operators are nonvanishing, being close to primary and nearly saturating the BPS bound, some of the nice properties of primary operators might be inherited by the "almost" primary operators.
- We shall see in Sec. VII, as a result of explicit gauge theory calculations with the BMN operators, that the 't Hooft coupling in the BMN sector is dressed with powers of $1/J^2$. More explicitly, the effective coupling in the BMN sector is λ', rather than the 't Hooft coupling λ, where

$$\lambda' \equiv \frac{\lambda}{J^2} = g_{\rm YM}^2 \frac{N}{J^2} = (\alpha' \mu p^+)^{-2}. \tag{1.11}$$

The last equality is obtained using Eqs. (1.7) and (1.8).
- Moreover, we shall see that the ratio of nonplanar to planar graphs is controlled by powers of the genus counting parameter,

$$g_2 \equiv \frac{J^2}{N} = 4\pi g_s (\alpha' \mu p^+)^2, \tag{1.12}$$

which also remains finite in the BMN limit Eq. (1.8). Note that in Eq. (1.12) $g_s = e^{\phi}$, where ϕ is the value of the dilaton field,[5] is *not* the coupling for strings on the plane wave, although it is related to it.
- One can do better than simply finding the free-string mass spectrum; we can study real interacting strings, their splitting and joining amplitudes, and one-loop corrections to the mass spectrum. As we shall discuss in Secs. VII and VIII, the one-loop mass corrections compared to the tree level results are suppressed by powers of the "effective one-loop string coupling" [cf. Eq. (7.32)],

$$g_{\text{one-loop}}^{\rm eff} = \sqrt{\lambda' g_2^2} = g_{\rm YM} \frac{J}{\sqrt{N}} = 4\pi g_s \alpha' \mu p^+. \tag{1.13}$$

It has been argued that all higher-genus (higher-loop) results replicate the same pattern, i.e., g_2 always appears in the combination $\lambda' g_2^2$. This has been built into a quantum-mechanical model for strings on plane waves, the string-bit model (Vaman and Verlinde, 2002; Verlinde, 2002). However, *a priori* there is no reason why such a structure should exist and in principle g_2 and λ' can appear in any combination. Using another quantum-mechanical model constructed to capture some features of the BMN opera-

[4]The proposal as stated is only true for free strings. As we shall see in Sec. VII, this proposal should be modified once string interactions are included.

[5]Note that for the plane-wave background we are interested in, the dilaton is constant.

tor dynamics, it has been argued that at the g_2^4 level there are indeed $\lambda' g_2^4$ corrections to the mass spectrum (Beisert, Kristjansen, *et al.*, 2003b; Plefka, 2003).

- The above observations revive the hope that we might be able to do a full-fledged interacting string theory computation using perturbative gauge theory with (modified) BMN operators.

We should note that the BMN proposal has, since its inception, undergone many refinements and corrections. However, a full and complete understanding of the dictionary of strings on plane waves and the (modified) BMN operators is not yet at our disposal, and the field is still dynamic. Some of the open issues will be discussed in the main text and in particular in Sec. IX.

Finally, we would like to remind the reader that in this review we have tried to avoid many detailed and lengthy calculations, specifically in Secs. VI–VIII. In fact, we found the original papers on these calculations quite clear and useful, and for more details the reader is encouraged to consult with the references provided. We have made available a more extensive version of this review on the ArXiv (arxiv.org) as hep-th/0310119 (Sadri and Sheikh-Jabbari, 2003b). It includes a discussion of Penrose limits of AdS orbifolds and conifolds, a discussion of BMN operators with arbitrary numbers of impurities, the all-orders analysis in λ', operator-product expansions in the BMN sector, higher point functions, and more details on the calculations of one-loop mass corrections in light-cone string field theory.

II. PLANE WAVES AS SOLUTIONS OF SUPERGRAVITY

Plane-fronted gravitational waves with parallel rays, *pp waves*, are a general class of spacetimes and are defined as spacetimes that support a covariantly constant null Killing vector field v^μ,

$$\nabla_\mu v_\nu = 0, \quad v^\mu v_\mu = 0. \tag{2.1}$$

In the most general form, they have metrics which can be written as

$$ds^2 = -2dudv - F(u,x^I)du^2 + 2A_J(u,x^I)dudx^J + g_{JK}(u,x^I)dx^J dx^K, \tag{2.2}$$

where $g_{JK}(u,x^I)$ is the metric on the space transverse to a pair of light-cone directions given by u,v and the coefficients $F(u,x^I)$, $A_J(u,x^I)$, and $g_{JK}(u,x^I)$ are constrained by (super)gravity equations of motion. The *pp*-wave metric (2.2) has a null Killing vector given by $\delta/\delta v$ which is, in fact, covariantly constant by virtue of the vanishing of the Γ^v_{vu} component of the Christoffel symbol.

The most useful *pp* waves, and the ones generally considered in the literature, have $A_J = 0$ and are flat in the transverse directions, i.e., $g_{IJ} = \delta_{IJ}$, for which the metric becomes

$$ds^2 = -2dudv - F(u,x^I)du^2 + \delta_{IJ}dx^I dx^J. \tag{2.3}$$

As we shall discuss in the next subsection, the existence of a covariantly constant null Killing vector field guarantees the α'-exactness of these supergravity solutions (Horowitz and Steif, 1990).

A more restricted class of *pp* waves, plane waves, are those admitting a globally defined covariantly constant null Killing vector field. One can show that for plane waves $F(u,x^I)$ is quadratic in the x^I coordinates of the transverse space, but still can depend on the coordinate u, $F(u,x^I) = f_{IJ}(u)x^I x^J$, so that the metric takes the form

$$ds^2 = -2dudv - f_{IJ}(u)x^I x^J du^2 + \delta_{IJ}dx^I dx^J. \tag{2.4}$$

Here f_{IJ} is symmetric and by virtue of the only nontrivial condition coming form the equations of motion, its trace is related to the other field strengths present. For the case of vacuum Einstein equations, it is traceless.

There is an even more restricted class of plane waves, homogeneous plane waves, for which $f_{IJ}(u)$ is a constant, hence their metric is of the form

$$ds^2 = -2dudv - \mu_{IJ}^2 x^I x^J du^2 + dx^I dx^I, \tag{2.5}$$

with μ_{IJ}^2 being a constant.[6]

A. Plane waves as α'-exact solutions of supergravity

In this subsection we discuss a property of *pp* waves of the form (2.3) which makes them epecially interesting from the string theory point of view: they are α'-exact solutions of supergravity (Amati and Klimcik, 1988; Horowitz and Steif, 1990). Supergravities arise as low-energy effective theories of strings, and can receive α' corrections. Such corrections generically involve higher powers of curvature and form fields (Green *et al.*, 1987b). The basic observation made by Horowitz and Steif (1990) is that *pp*-wave metrics of the form (2.3) have a covariantly constant null Killing vector, $n_\mu = \partial/\partial v$, and their curvature is null (the only nonzero components of their curvature are R_{uIuJ}). Higher α' corrections to the supergravity equations of motion are in general comprised of all second-rank tensors constructed from powers of the Riemann tensor and its derivatives. (The only possible term involving only one Riemann tensor should be of the form $R^{\mu\alpha\nu\beta}{}_{;\mu\nu}$, which is zero by virtue of the Bianchi identity.) On the other hand, any power of the Riemann tensor and its covariant derivatives with only two free indices is also zero, because n_μ is null and $\nabla_\mu n_\nu = 0$ [Eq. (2.1)]. The same argument can be repeated for the form fields, noting that for *pp* waves that are solutions of supergravity these form fields should have zero divergence and be null. As a result, all the α' corrections for supergravity solutions with metric of the form (2.3) vanish, i.e., they also solve

[6]This use of the term "homogeneous" is not universal. For example, the term symmetric plane wave has been used by Blau and O'Loughlin (2003) for this form of the metric, reserving "homogeneous" for a wider subclass of plane waves.

α'-corrected supergravity equations of motion. This argument about α'-exactness does not hold for a generic *pp* wave of the form (2.2) with $g_{IJ}(u,x^I)\neq\delta_{IJ}$. The transverse metric g_{IJ} may itself receive α' corrections. However, there are no extra corrections due to the wave part of the metric (Fabinger and Hellerman, 2003).We would like to comment that *pp* waves are generically singular solutions with no event horizons (Hubeny and Rangamani, 2002); however, plane waves of the form (2.4) for which $f_{IJ}(u)$ is a smooth function of u, are not singular.

B. The maximally supersymmetric plane wave and its symmetries

Hereafter in this review we shall focus only on a very special plane-wave solution of 10-dimensional type-IIB supergravity which admits 32 supersymmetries and by “the plane wave” we shall mean this maximally supersymmetric solution. In fact, demanding a solution of 10- or 11-dimensional supergravity to be maximally supersymmetric is very restrictive. Flat space, $AdS_5\times S^5$, and a special plane wave in type-IIB theory in 10 dimensions and flat space, $AdS_{4,7}\times S^{7,4}$, and a special plane wave in 11 dimensions are the only possibilities (Figueroa-O’Farrill and Papadopoulos, 2003). Note that type IIA does not admit any maximally supersymmetric solutions other than flat space.

Here, we focus on the 10-dimensional plane wave, which is a special case of Eq. (2.5) with $\mu_{IJ}^2=\mu^2\delta_{IJ}$. This metric, however, is not a solution to source-free type-IIB supergravity equations of motion, and we need to add form fluxes. It is not hard to see that with $\mu_{IJ}^2=\mu^2\delta_{IJ}$ the only possibility is turning on a constant self-dual Ramond-Ramond five-form flux; moreover, the dilaton should also be a constant. As we shall see in Sec. III, this plane wave is closely related to the $AdS_5\times S^5$ solution. The (bosonic) part of this plane wave solution is then

$$ds^2=-2dx^+dx^--\mu^2(x^ix^i+x^ax^a)(dx^+)^2+dx^idx^i + dx^adx^a, \tag{2.6a}$$

$$F_{+ijkl}=\frac{4}{g_s}\mu\epsilon_{ijkl},$$

$$F_{+abcd}=\frac{4}{g_s}\mu\epsilon_{abcd}, \tag{2.6b}$$

$$e^{\phi}=g_s=\text{const}, \tag{2.6c}$$

where $i,j=1,2,3,4$, $a,b=5,6,7,8$. In the above, μ is an auxiliary but convenient parameter and can be easily removed by taking $x^+\to x^+/\mu$ and $x^-\to\mu x^-$ (which is in fact a light-cone boost).

Let us first check that the background (2.6) is really maximally supersymmetric. Note that this will ensure it is also a supergravity solution, because supergravity equations of motion are nothing but the commutators of the supersymmetry variations. For this we need to show that the gravitino and dilatino variations vanish for 32 independent (Killing) spinors, i.e.,

$$\delta_\epsilon\psi_\mu^\alpha\equiv(\hat{\mathcal{D}}_\mu)^\alpha_\beta\epsilon^\beta=0,$$

$$\delta_\epsilon\lambda^\alpha\equiv(\tilde{\mathcal{D}})^\alpha_\beta\epsilon^\beta=0, \tag{2.7}$$

with $\mu=0,1,\ldots,9$, $\alpha=1,2$. These equations have 32 solutions, where the dilatino λ^α, gravitinos ψ_μ^α, and Killing spinors ϵ^α are all 32-component 10-dimensional Weyl-Majorana fermions of the same chirality (for our notations and conventions see Appendix B.1), and the supercovariant derivative $\hat{\mathcal{D}}_\mu$ in the string frame is defined (Bena and Roiban, 2003; Cvetic *et al.*, 2003; Sadri and Sheikh-Jabbari, 2003a, 2003b) as

$$(\hat{\mathcal{D}}_\mu)^\alpha_\beta=\delta^\alpha_\beta\nabla_\mu+\frac{1}{8}(\sigma^3)^\alpha_\beta\Gamma^{\nu\rho}H_{\mu\nu\rho}+\frac{ie^\phi}{8}\bigg[(\sigma^2)^\alpha_\beta\Gamma^\nu\partial_\nu\chi - \frac{i}{3!}(\sigma^1)^\alpha_\beta\Gamma^{\nu\rho\lambda}F_{\nu\rho\lambda} + \frac{1}{2\times5!}(\sigma^2)^\alpha_\beta\Gamma^{\nu\rho\lambda\sigma\delta}F_{\nu\rho\lambda\sigma\delta}\bigg]\Gamma_\mu, \tag{2.8}$$

$$(\tilde{\mathcal{D}})^\alpha_\beta=\frac{1}{2}\delta^\alpha_\beta\Gamma^\nu\partial_\nu\phi-\frac{1}{4\times3!}(\sigma^3)^\alpha_\beta\Gamma^{\mu\nu\rho}H_{\mu\nu\rho} - \frac{i}{2}e^\phi\bigg[(\sigma^2)^\alpha_\beta\Gamma^\nu\partial_\nu\chi-\frac{i}{2\times3!}(\sigma^1)^\alpha_\beta\Gamma^{\nu\rho\lambda}F_{\nu\rho\lambda}\bigg], \tag{2.9}$$

with the spin connection $\omega_\mu^{\hat{a}\hat{b}}$ appearing in the covariant derivative $\nabla_\mu=\left(\partial_\mu+\frac{1}{4}\omega_\mu^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}}\right)$ and the hatted latin indices used for the tangent space. In these expressions ϕ is the dilaton, χ the axion, H the three-form field strength of the Neveu-Schwarz/Neveu-Schwarz (NS-NS) sector (Neveu and Schwarz, 1971), and the F’s represent the appropriate Ramond-Ramond (RR) field strengths.

For the background (2.6) $(\tilde{\mathcal{D}})^\alpha_\beta$ is identically zero, and $(\hat{\mathcal{D}})^\alpha_\beta$ takes a simple form,

$$(\hat{\mathcal{D}}_\mu)^\alpha_\beta=\delta^\alpha_\beta\left(\partial_\mu+\frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\right) + \frac{ig_s}{16\times5!}(\sigma^2)^\alpha_\beta\Gamma^{\nu\rho\lambda\sigma\delta}F_{\nu\rho\lambda\sigma\delta}. \tag{2.10}$$

In order to work out the spin connection $\omega_\mu^{\hat{a}\hat{b}}$ we need the vierbeins $e_\mu^{\hat{a}}$ which are

$$e_+^+=e_-^-=1,\quad e_i^j=\delta_i^j,$$

$$e_b^a=\delta_b^a,\quad e_-^+=\frac{1}{2}\mu^2(x_ix_i+x_ax_a), \tag{2.11}$$

and therefore

$$\omega^{+}_{-i}=\mu^2 x_i,\quad \omega^{+}_{-a}=\mu^2 x_a, \tag{2.12}$$

are the only nonvanishing components of $\omega_\mu^{\hat a\hat b}$.

The Killing spinor equation can now be written as

$$(\mathbf{1}\cdot\partial_\mu+\Omega_\mu)^\alpha_\beta\epsilon^\beta=0, \tag{2.13}$$

with

$$(\Omega_+)^\alpha_\beta=-\frac{1}{2}\mu^2x^I\Gamma^{+I}\delta^\alpha_\beta+\frac{i\mu}{4}(\Pi+\Pi')\Gamma^+\Gamma_+(\sigma^2)^\alpha_\beta,$$

$$\Omega_-=0,\quad (\Omega_I)^\alpha_\beta=\frac{i\mu}{4}\Gamma^+(\Pi+\Pi')\Gamma^I(\sigma^2)^\alpha_\beta. \tag{2.14}$$

In the above $I=\{i,a\}=1,2,\ldots 8$, $\Pi=\Gamma^{1234}$ and $\Pi'=\Gamma^{5678}$. The Ω's satisfy a number of useful identities such as

$$\Gamma^+\Omega_I=\Omega_I\Gamma^+=\Gamma^+\Omega_+=0,$$

$$\Omega_I\Omega_J=\Omega_I\Omega_+=0,$$

$$\Omega_+\Omega_I=-\frac{\mu^2}{4}(1+\Pi\Pi')\Gamma^{+I}\cdot\mathbf{1},$$

$$\Omega_+\Gamma^+=\frac{i\mu}{2}(\Pi+\Pi')\Gamma^+\cdot(\sigma^2)^\alpha_\beta. \tag{2.15}$$

We first note that the ($\mu=-$) component of Eq. (2.13) is simply $\partial_-\epsilon=0$, so all Killing spinors should be x^- independent. The $\mu=I$ component can be easily solved by taking

$$\epsilon^\alpha=(\mathbf{1}-x^I\Omega_I)^\alpha_\beta\chi^\beta, \tag{2.16}$$

where χ^β is an arbitrary x^I-independent fermion of positive 10-dimensional chirality. Plugging Eq. (2.16) into Eq. (2.13), using the identity $\Omega_I\Omega_+=0$ and the fact that $(\mathbf{1}+x^I\Omega_I)(\mathbf{1}-x^I\Omega_I)=\mathbf{1}$, the ($\mu=+$) component of the Killing spinor equation takes the form

$$[\mathbf{1}\cdot\partial_++\Omega_+(\mathbf{1}-x^I\Omega_I)]^\alpha_\beta\chi^\beta=0. \tag{2.17}$$

Equation (2.17) has an x^I-independent piece and a part that is linear in x^I. These two should vanish separately. Using the identities given in Appendix B.1 and after some straightforward Dirac matrix algebra, one can show that if $\Gamma^-\chi=0$, Eq. (2.17) simply reduces to $\partial_+\chi=0$. That is, any constant χ with $\Gamma^-\chi=0$ is a Killing spinor. These provide us with $2\times 8=16$ solutions. Now let us assume that $\Gamma^-\chi\neq 0$. Without loss of generality, all such spinors can be chosen to satisfy $\Gamma^+\chi=0$. For these choices of χ's the x^I-dependent part of Eq. (2.17) vanishes identically, and the x^I-independent part becomes

$$[\mathbf{1}\cdot\partial_++i\mu\Pi(\sigma^2)^\alpha_\beta]\chi^\beta=0,$$

where we have used the fact that $\Gamma^+\chi=0$ implies $\Pi\chi=\Pi'\chi$. This equation can be easily solved with (Blau *et al.*, 2002a)

$$\chi^\alpha=[\delta^\alpha_\beta\cos\mu x^+-i\Pi(\sigma^2)^\alpha_\beta\sin\mu x^+]\chi^\beta_0, \tag{2.18}$$

where χ_0^β is an arbitrary constant spinor of positive 10-dimensional chirality. We have shown that Eqs (2.7) have 32 linearly independent solutions and hence the background (2.6) is maximally supersymmetric. Note that Eq. (2.18) clearly shows the "wave" nature of our background (note the periodicity in x^+, the light-cone time), a fact that is not manifest in the coordinates we have chosen. This wave nature can be made explicit in the so-called Rosen coordinates (see Sec. III.A).

1. Isometries of the background

The background (2.6) has a number of isometries, some of which are manifest. In particular, the solution is invariant under translations in the x^+ and x^- directions. These translations can be thought of as two (noncompact) U(1)'s with the generators

$$i\frac{\partial}{\partial x^+}\equiv P_+=-P^-,\quad i\frac{\partial}{\partial x^-}\equiv P_-=-P^+. \tag{2.19}$$

Due to the presence of the $(dx^+)^2$ term, a boost in the (x^+,x^-) plane is not a symmetry of the metric. However, the combined boost and μ scaling,

$$x^-\to\sqrt{\frac{1-v}{1+v}}x^-,\quad x^+\to\sqrt{\frac{1+v}{1-v}}x^+,$$

$$\mu\to\sqrt{\frac{1-v}{1+v}}\mu, \tag{2.20}$$

is still a symmetry.

Obviously, the solution is also invariant under two SO(4)'s which act on the x^i and x^a directions. The generators of these SO(4)'s will be denoted by J_{ij} and J_{ab} where

$$J_{ij}=-i\left(x_i\frac{\partial}{\partial x^j}-x_j\frac{\partial}{\partial x^i}\right),\quad J_{ab}=-i\left(x_a\frac{\partial}{\partial x^b}-x_b\frac{\partial}{\partial x^a}\right). \tag{2.21}$$

Note that although the metric possesses SO(8) symmetry, because of the five-form flux this symmetry is broken to $SO(4)\times SO(4)$. There is also a $\mathbb{Z}_2$ symmetry which exchanges these two SO(4)'s, acting as

$$\{x^i\}\overset{\mathbb{Z}_2}{\leftrightarrow}\{x^a\}. \tag{2.22}$$

So far we have identified 14 isometries which are generators of a $U(1)\times U(1)\times SO(4)\times SO(4)\times\mathbb{Z}_2$ symmetry group. One can easily see that translations along the $x^I=(x^i,x^a)$ directions are not symmetries of the metric. However, we can show that if along with translation in x^I we also shift x^- appropriately, i.e.,

$$\begin{cases} x^I\to x^I+\epsilon^I_1\cos\mu x^+,\\ x^-\to x^--\epsilon^I_1\mu x^I\sin\mu x^+,\end{cases}$$

$$\begin{cases} x^I \to x^I + \epsilon_2^I \sin \mu x^+, \\ x^- \to x^- + \epsilon_2^I \mu x^I \cos \mu x^+, \end{cases} \tag{2.23}$$

where ϵ_1^I and ϵ_2^I are arbitrary but small parameters, the metric and the five-form remain unchanged. These 16 isometries are generated by the Killing vectors

$$L_I = -i\left(\cos \mu x^+ \frac{\partial}{\partial x^I} - \mu x^I \sin \mu x^+ \frac{\partial}{\partial x^-}\right),$$

$$K_I = -i\left(\sin \mu x^+ \frac{\partial}{\partial x^I} + \mu x^I \cos \mu x^+ \frac{\partial}{\partial x^-}\right), \tag{2.24}$$

satisfying the following algebra:

$$[L_I, L_J] = 0, \quad [K_I, K_J] = 0, \tag{2.25}$$

$$[L_I, K_J] = \mu \delta_{IJ} \frac{\partial}{\partial x^-} = i\mu P^+ \delta_{IJ},$$

$$[P^-, L_I] = i\mu K_I, \quad [P^-, K_I] = -i\mu L_I. \tag{2.26}$$

Equations (2.25) are in fact an eight-dimensional (or a pair of four-dimensional) Heisenberg-type algebra(s) with "$\hbar$" being equal to μP^+ (Das *et al.*, 2002). Note that P^+ commutes with the generators of the two SO(4)'s as well as K_I and L_I. In other words, P^+ is in the center of the isometry algebra which has 30 generators $(J_{ij}, J_{ab}, P^+, P^-, K_i, K_a, L_i, L_a)$. It is also easy to check that K_i, L_j and K_a, L_b transform as vectors (or singlets) under the corresponding SO(4) rotations. Altogether, the algebra of Killing vectors is $[\mathrm{h}(4) \oplus \mathrm{h}(4)] \oplus \mathrm{so}(4) \oplus \mathrm{so}(4) \oplus \mathrm{u}(1)_+ \oplus \mathrm{u}(1)_-$, where h(4) is the four-dimensional Heisenberg algebra.

In addition to the above 30 Killing vectors generating continuous symmetries, there are some discrete symmetries, one of which is the $\mathbb{Z}_2$ discussed earlier. There is also the *CPT* symmetry (Schwarz, 2002)

$$x^I \to -x^I, \quad x^\pm \to -x^\pm, \quad \mu \to -\mu \tag{2.27}$$

(note that we also need to change μ).

Finally, it is useful to compare the plane-wave isometries to that of flat-space, the 10-dimensional Poincaré algebra consisting of $P^+, P^-, P^I = -i(\partial/\partial x^I)$ and J^{+-} (light-cone boost), J^{+I}, J^{-I}, and J^{IJ} [the SO(8) rotations]. Among these 55 generators, P^+, P^- and J^{ij}, J^{ab} are also present in the set of plane wave isometries. However, as we have discussed, J^{+-} and J^{-I} are absent. As for rotations generated by J^{ia}, only a particular rotation, namely, the $\mathbb{Z}_2$ defined in Eq. (2.22), is present. From Eq. (2.24) it can readily be seen that K_I and L_I are a linear combination of P_I and J^{+I},

$$P_I = -i\frac{\partial}{\partial x^I}, \quad J^{+I} = x^+ P^I - x^I P^+, \tag{2.28}$$

and it is easy to show that $[P^-, J^{+I}] = -iP^I$, $[P^I, J^{+J}] = -i\delta_{IJ} P^+$. In summary, J^{+-}, J^{-I}, and J^{ia} (which arealtogether 25 generators) are not present among the Killing vectors of the plane wave, and therefore the number of isometries of the plane wave is 55−25=30, agreeing with our earlier results.

2. Superalgebra of the background

As we have shown, the plane-wave background (2.6) possesses 32 Killing spinors, and in Sec. II.B.1 we worked out all the isometries of the background. In this subsection we combine these two results and present the superalgebra of the plane-wave geometry (2.6). Noting the Killing spinor equations and their solutions, it is straightforward to work out the supercharges and their superalgebra (see, for example, Green *et al.*, 1987b).

As discussed earlier, the solutions to the Killing spinor equations are all x^- independent. This implies that supercharges should commute with P^+. However, as discussed in Sec. II.B.1, P^+ commutes with all the bosonic isometries and so is in the center of the whole superalgebra. We noted that Killing spinors fall into two classes, either $\Gamma^+\chi=0$ or $\Gamma^-\chi=0$. The former lead to *kinematical supercharges* $Q^{+\alpha}$ with the property that $\Gamma^+ Q^{+\alpha}=0$, while the latter lead to *dynamical supercharges* $Q^{-\alpha}$, which satisfy $\Gamma^- Q^{-\alpha}=0$. Since both sets of dynamical and kinematical supercharges have the same (positive) 10-dimensional chirality, the $Q^{+\alpha}$ are in the $\mathbf{8_s}$ and $Q^{-\alpha}$ in the $\mathbf{8_c}$ representation of the SO(8) fermions (for details of the conventions see Appendix B.1).

For the plane-wave background, however, it is more convenient to use the SO(4)×SO(4) decomposition instead of SO(8). The relation between these two has been worked out and summarized in Appendix B.2. We shall use $q_{\alpha\beta}$ and $q_{\dot\alpha\dot\beta}$ for the kinematical supercharges and $Q_{\dot\alpha\beta}$ and $Q_{\alpha\dot\beta}$ for the dynamical ones. Note that all q and Q are complex fermions.

The superalgebra in the SO(8) basis is presented by Blau *et al.* (2002b) and Metsaev (2002). Here we present it in the SO(4)×SO(4) basis:

- Commutators of bosonic generators with kinematical supercharges:

$$[J^{ij}, q_{\alpha\beta}] = \frac{1}{2}(i\sigma^{ij})_\alpha^\rho q_{\rho\beta}, \quad [J^{ij}, q_{\dot\alpha\dot\beta}] = \frac{1}{2}(i\sigma^{ij})_{\dot\alpha}^{\dot\rho} q_{\dot\rho\dot\beta},$$

$$[J^{ab}, q_{\alpha\beta}] = \frac{1}{2}(i\sigma^{ab})_\beta^\rho q_{\alpha\rho}, \quad [J^{ab}, q_{\dot\alpha\dot\beta}] = \frac{1}{2}(i\sigma^{ab})_{\dot\beta}^{\dot\rho} q_{\dot\alpha\dot\rho}, \tag{2.29}$$

$$[K^I, q_{\alpha\beta}] = [L^I, q_{\alpha\beta}] = 0, \quad [K^I, q_{\dot\alpha\dot\beta}] = [L^I, q_{\dot\alpha\dot\beta}] = 0, \tag{2.30}$$

$$[P^+, q_{\alpha\beta}] = [P^+, q_{\dot\alpha\dot\beta}] = 0, \tag{2.31}$$

$$[P^-, q_{\alpha\beta}] = +i\mu q_{\alpha\beta}, \quad [P^-, q_{\dot\alpha\dot\beta}] = -i\mu q_{\dot\alpha\dot\beta}. \tag{2.32}$$

- Commutators of bosonic generators with dynamical supercharges:

$$[J^{ij},Q_{\alpha\dot\beta}]=\frac{1}{2}(i\sigma^{ij})^{\rho}_{\alpha}Q_{\rho\dot\beta},\quad [J^{ij},Q_{\dot\alpha\beta}]=\frac{1}{2}(i\sigma^{ij})^{\dot\rho}_{\dot\alpha}Q_{\dot\rho\beta},$$

$$[J^{ab},Q_{\dot\alpha\beta}]=\frac{1}{2}(i\sigma^{ab})^{\rho}_{\beta}Q_{\dot\alpha\rho},\quad [J^{ab},Q_{\alpha\dot\beta}]=\frac{1}{2}(i\sigma^{ab})^{\dot\rho}_{\dot\beta}Q_{\alpha\dot\rho},\tag{2.33}$$

$$[K^{i},Q_{\alpha\dot\beta}]=\frac{\mu}{2}(\sigma^{i})^{\dot\rho}_{\alpha}q_{\dot\rho\dot\beta},\quad [K^{a},Q_{\alpha\dot\beta}]=-\frac{\mu}{2}(\sigma^{a})^{\rho}_{\dot\beta}q_{\alpha\rho},$$

$$[K^{i},Q_{\dot\alpha\beta}]=\frac{\mu}{2}(\sigma^{i})^{\rho}_{\dot\alpha}q_{\rho\beta},\quad [K^{a},Q_{\dot\alpha\beta}]=\frac{\mu}{2}(\sigma^{a})^{\dot\rho}_{\beta}q_{\dot\alpha\dot\rho},\tag{2.34}$$

$$[L^{i},Q_{\alpha\dot\beta}]=-\frac{\mu}{2}(\sigma^{i})^{\dot\rho}_{\alpha}q_{\dot\rho\dot\beta},\quad [L^{a},Q_{\alpha\dot\beta}]=\frac{\mu}{2}(\sigma^{a})^{\rho}_{\dot\beta}q_{\alpha\rho},$$

$$[L^{i},Q_{\dot\alpha\beta}]=\frac{\mu}{2}(\sigma^{i})^{\rho}_{\dot\alpha}q_{\rho\beta},\quad [L^{a},Q_{\dot\alpha\beta}]=\frac{\mu}{2}(\sigma^{a})^{\dot\rho}_{\beta}q_{\dot\alpha\dot\rho},\tag{2.35}$$

$$[P^{+},Q_{\alpha\dot\beta}]=0,\quad [P^{+},Q_{\dot\alpha\beta}]=0,\tag{2.36}$$

$$[P^{-},Q_{\alpha\dot\beta}]=0,\quad [P^{-},Q_{\dot\alpha\beta}]=0.\tag{2.37}$$

- Anticommutators of supercharges:

$$\{q_{\alpha\beta},q^{\dagger\rho\lambda}\}=2P^{+}\delta^{\rho}_{\alpha}\delta^{\lambda}_{\beta},\quad \{q_{\alpha\beta},q^{\dagger\dot\alpha\dot\beta}\}=0,$$

$$\{q_{\dot\alpha\dot\beta},q^{\dagger\dot\rho\dot\lambda}\}=2P^{+}\delta^{\dot\rho}_{\dot\alpha}\delta^{\dot\lambda}_{\dot\beta},\tag{2.38}$$

$$\{q_{\alpha\beta},Q^{\dagger\dot\rho\lambda}\}=i(\sigma^{i})^{\dot\rho}_{\alpha}\delta^{\lambda}_{\beta}(L^{i}+K^{i}),$$

$$\{q_{\alpha\beta},Q^{\dagger\rho\dot\lambda}\}=i(\sigma^{a})^{\dot\lambda}_{\beta}\delta^{\rho}_{\alpha}(L^{a}+K^{a}),$$

$$\{q_{\dot\alpha\dot\beta},Q^{\dagger\dot\rho\lambda}\}=i(\sigma^{a})^{\lambda}_{\dot\beta}\delta^{\dot\rho}_{\dot\alpha}(L^{a}-K^{a}),$$

$$\{q_{\dot\alpha\dot\beta},Q^{\dagger\rho\dot\lambda}\}=i(\sigma^{i})^{\rho}_{\dot\alpha}\delta^{\dot\lambda}_{\dot\beta}(L^{i}-K^{i}),\tag{2.39}$$

$$\{Q_{\alpha\dot\beta},Q^{\dagger\rho\dot\lambda}\}=2\delta^{\rho}_{\alpha}\delta^{\dot\lambda}_{\dot\beta}P^{-}+\mu(i\sigma^{ij})^{\rho}_{\alpha}\delta^{\dot\lambda}_{\dot\beta}J^{ij}+\mu(i\sigma^{ab})^{\dot\lambda}_{\dot\beta}\delta^{\rho}_{\alpha}J^{ab},$$

$$\{Q_{\alpha\dot\beta},Q^{\dagger\dot\rho\lambda}\}=0,$$

$$\{Q_{\dot\alpha\beta},Q^{\dagger\dot\rho\lambda}\}=2\delta^{\dot\rho}_{\dot\alpha}\delta^{\lambda}_{\beta}P^{-}+\mu(i\sigma^{ij})^{\dot\rho}_{\dot\alpha}\delta^{\lambda}_{\beta}J^{ij}+\mu(i\sigma^{ab})^{\lambda}_{\beta}\delta^{\dot\rho}_{\dot\alpha}J^{ab}.\tag{2.40}$$

Let us now focus on the part of the superalgebra containing only dynamical supercharges and SO(4) generators, i.e., Eqs. (2.33), (2.36), (2.37), and (2.40). Adding the two so(4) algebras to these, we obtain a superalgebra, which is of course a subalgebra of the full superalgebra discussed above. (We have another sub-superalgebra which contains only kinematical supercharges, $P^{\pm}$ and J's, but we do not consider it here.) The bosonic part of this sub-superalgebra is $U(1)_{+}\times U(1)_{-}\times SO(4)\times SO(4)\times\mathbb{Z}_2$, where $U(1)_{\pm}$ is generated by $P^{\pm}$ and $U(1)_{+}$ is in the center of the algebra. Next we note that the algebra does not mix $Q_{\alpha\dot\beta}$ and $Q_{\dot\alpha\beta}$. This sub-superalgebra is not a simple superalgebra, and it can be written as a semidirect product of two simple superalgebras. Noting that $spin(4)=SU(2)\times SU(2)$, we have four SU(2) factors, and $Q_{\alpha\dot\beta}$ and $Q_{\dot\alpha\beta}$ transform as doublets of two of the SU(2)'s, each coming from different SO(4) factors. In other words, the two SO(4)'s mix to give two $SU(2)\times SU(2)$'s. This superalgebra falls into Kac's classification of superalgebras (Kac, 1977) and can be identified as $PSU(2|2)\times PSU(2|2)\times U(1)_{-}\times U(1)_{+}$. [The bosonic part of the $PSU(n|n)$ supergroup is $SU(n)\times SU(n)$, while that of $SU(m|n)$ for $m\neq n$ is $SU(n)\times SU(m)\times U(1)$.] As mentioned earlier the two PSU(2|2) supergroups share the same U(1), $U(1)_{-}$, which is generated by P^{-}. The $\mathbb{Z}_2$ symmetry defined through Eq. (2.22) is still present and at the level of superalgebra exchanges the two PSU(2|2) factors. It is interesting to compare the 10-dimensional maximally supersymmetric plane wave superalgebra with that of the 11-dimensional one, which is SU(4|2) (Dasgupta *et al.*, 2002b). One of the main differences is that in our case the light-cone Hamiltonian P^{-} commutes with the supercharges [cf. Eq. (2.37)], and as a result, as opposed to the 11-dimensional case, all states in the same $PSU(2|2)\times PSU(2|2)\times U(1)_{-}$ supermultiplet have the same mass. Here we do not intend to study this superalgebra and its representations in detail. However, this is definitely an important question, which so far has not been addressed in the literature. For a more detailed discussion on $SU(m|n)$ supergroups and their unitary representations the reader is encouraged to look at the articles of Balantekin and Bars (1981); Dasgupta *et al.* (2002b); Motl *et al.* (2003), and for the $PSU(n|n)$ case Berkovits *et al.* (1999).

C. Spectrum of supergravity on the plane-wave background

The low-energy dynamics of string theory can be understood in terms of an effective-field theory in the form of supergravity (Green *et al.*, 1987b). In particular, the lowest-lying states of string theory on the maximally supersymmetric plane-wave background (2.6) should correspond to the states of (type-IIB) supergravity on this background. We are thus led to analyze the spectrum of modes in such a theory.

As in the flat space, the kinematical supercharges acting on different states would generate different "polarizations" of the same state, while dynamical supercharges would lead to various fields in the same supermultiplet. In the plane-wave superalgebra we discussed in the previous section, as could be seen from Eq. (2.32), kinematical supercharges do not commute with the light-cone Hamiltonian, and hence we expect different "polarizations" of the same multiplet to have different masses (their masses, however, should differ by an integer multiple of μ), as will be explicitly shown in this section. This should be contrasted with the flat-space case, in which the light-cone Hamiltonian commutes with all supercharges, both kinematical and dynamical. For the same reason, different states in the Clifford vacuum (Wess and Bagger, 1992), which are related by the action of kinematical supercharges, will carry different energies, and so this vacuum is nondegenerate. However, once a Clifford vacuum is chosen, the other states of the same multiplet are related by the action of dynamical supercharges and hence should have the same (light-cone) mass [see Eq. (2.37)].

As a warmup, let us first consider a scalar (or any bosonic) field ϕ with mass m propagating on such a background, with classical equation of motion

$$(\Box - m^2)\phi = 0, \tag{2.41}$$

with the d'Alembertian acting on a scalar given as

$$\begin{aligned}\Box &= \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu)\\ &= -2\partial_+\partial_- + \mu^2 x^I x^I \partial_-\partial_- + \partial_I\partial_I.\end{aligned} \tag{2.42}$$

Here, the index I corresponds to the eight transverse directions, and the repeated indices are summed. Then, Eq. (2.41) for the fields with $\partial_\pm\phi = ip^\mp\phi$ reduces to

$$[2p^+p^- - (\mu p^+)^2 x^I x^I + \partial_I\partial_I]\phi = 0, \tag{2.43}$$

which is nothing but a Schrödinger equation for an eight-dimensional harmonic oscillator, with frequency equal to μp^+. Therefore choosing the x^I dependence of ϕ through Gaussians times Hermite polynomials [the precise form of which can be found in the article of Bak and Sheikh-Jabbari (2003)], we obtain the spectrum of the light-cone Hamiltonian p^- as

$$p^- = \mu\left(\sum_{i=1}^{8} n_i + 4\right) + \frac{m^2}{2p^+} \tag{2.44}$$

for some set of positive or zero integers n_i. The spectrum is discrete for massless fields ($m=0$), in which case it is also independent of the light-cone momentum p^+. This means that for massless fields, we cannot form wave packets with nonzero group velocity ($\sim\partial p^-/\partial p^+$), and hence scattering of such massless states cannot take place.[7] The discreteness of the spectrum arises from the requirement that the wave function be normalizable in the transverse directions, and this is translated through a coupling of the transverse and light-cone directions in the equation of motion into the discreteness of the light-cone energy. The flat-space limit ($\mu\to 0$) is not well defined for these modes, but could be restored if we add the non-normalizable solutions to the equation of motion, in which case the flat-space limit would allow a continuum of light-cone energies. The case of vanishing light-cone momentum is not easily treated in the light-cone frame. For the massive case ($m\neq 0$) the light-cone energy does pick up a p^+ dependence, allowing us to construct proper wave packets for scattering.

These considerations can be applied to the various bosonic fields in supergravity. The low-energy effective theory relevant here is type-IIB supergravity, whose action (in the string frame) is (Polchinski, 1998)

$$\begin{aligned}S &= \frac{1}{2k_{10}^2}\int d^{10}x\sqrt{-\det g}(\mathcal{L}_{NS} + \mathcal{L}_R + \mathcal{L}_{CS}),\\ \mathcal{L}_{NS} &= e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right),\\ \mathcal{L}_R &= -\frac{1}{2}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right),\\ \mathcal{L}_{CS} &= -\frac{1}{2\sqrt{-\det g}}C_4\wedge H_3\wedge F_3,\end{aligned} \tag{2.45}$$

where $H_3 = dB^{NS}$ and $F_1 = d\chi$ are the Neveu-Schwarz Neveu-Schwarz (NS-NS) three-form and the Ramond-Ramond (RR) scalar field strengths, respectively, and

$$\tilde{F}_3 = F_3 - \chi\wedge H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}B_2^{\mathrm{RR}}\wedge H_3 + \frac{1}{2}B_2^{\mathrm{NS}}\wedge F_3,$$

and $F_3 = dB^{\mathrm{RR}}$ and $F_5 = df_4$. The equation of motion for F_5, which is nothing but the self-duality condition ($\tilde{F}_5 = {}^*\tilde{F}_5$), should be imposed by hand. We note here that the NS-NS and RR terminology in the supergravity action is motivated by the flat-space results of string theory, which as we point out later in Sec. IV.C.1, does not cor-

[7]The particles are confined in the transverse space by virtue of the harmonic-oscillator potential, but one can consider scattering in a two-dimensional effective theory on the (p^-, p^+) subspace (Bak and Sheikh-Jabbari, 2003).

respond to our SO(4)×SO(4) decomposition of states (see footnote 9). The mapping of the fields presented in this section and the string states will be clarified in Sec. IV.C.

To study the physical on-shell spectrum of supergravity on the plane-wave background with the nontrivial five-form flux (2.6), we linearize the supergravity equations of motion around this background and work in the light-cone gauge, by setting $\xi_{\mu\cdots\nu-}=0$, with $\xi_{\mu\cdots\nu-}$ generically any of the bosonic (other than scalar) tensor fields, considered as perturbations around the background. Then, the $\xi_{\mu\cdots\nu+}$ components are not dynamical and are completely fixed in terms of the other physical modes, after imposing the constraints coming from the equations of motion for the gauge-fixed components $\xi_{\mu\cdots\nu-}$. Therefore, in this gauge, we deal with only $\xi_{I\cdots J}$ modes, where $I,\dots,J=1,2,\dots,8$. Setting the light-cone gauge for fermions is accomplished by projecting out spinor components by the action of an appropriate combination of Dirac matrices (Metsaev and Tseytlin, 2002). The advantage of using the light-cone gauge is that in this gauge only the physical modes appear.

It will prove useful to first decompose the physical fluctuations of the supergravity fields in terms of SO(8) →SO(4)×SO(4) representations (Das *et al.*, 2002; Metsaev and Tseytlin, 2002). In the bosonic sector we have a complex scalar, combining the NS-NS dilaton and RR scalar, a complex two-form (again a combination of NS-NS and RR fields), a real four-form, and a graviton. Using the notation of Sec. II.B, we can decompose these into SO(4)×SO(4) representations. We label SO(8) indices by I,J,K,L, indices in the first SO(4) by i,j,k,l, and those of the second by a,b,c,d. The decomposition of the bosonic fields is given in Table I.

The fermionic spectrum consists of a complex spin-1/2 dilatino of negative chirality and a complex spin-3/2 gravitino with positive chirality. For the dilatino, 16 degrees of freedom survive the light-cone projection. For the gravitino, we note that removing the spin-1/2 component by projecting out the γ-transverse components leaves 112 degrees of freedom. The details of the decomposition of SO(8) fermions into representations of SO(4)×SO(4) can be found in Appendix B.2. Using the notation of the appendix, the dilatino is in $\mathbf{8_c}$ and the gravitino in $\mathbf{8_s}$. Fermions can be decomposed along the same lines, using the result of Appendix B.2.

The dilaton is decoupled, in the linear regime, from the four-form and is the simplest field to deal with. Its equation of motion is simply that of a complex massless scalar field (2.41). Its lowest energy state has $p^-=4\mu$, with discrete energy levels above it.

The graviton and four-form field are coupled in this background, leading to coupled equations of motion. The coupled Einstein and four-form potential equations of motion, after linearizing and going to light-cone gauge, and using the self-duality of the five-form field strength, imply the equation

TABLE I. SO(4)×SO(4) decomposition of bosonic supergravity fields. $\mathbf{3}^+$ and $\mathbf{3}^-$ are the self-dual and anti-self-dual projections of the $\mathbf{6}$ of SO(4). The complex scalar and two-form are defined as $\Phi=\chi+ie^{\phi}$ and $b=B^{\rm NS}+iB^{\rm RR}$, and we have also defined the pseudoscalar "trace" piece of the four-form potential $f=\epsilon^{ijkl}f_{ijkl}/6$, and $f_{ia}=\frac{1}{3}\epsilon_i^{jkl}f_{ajkl}$. The graviton h_{IJ} and four-form f_{IJKL} are fluctuations around a nontrivial plane wave background. $h=h_{ii}=-h_{aa}$ is the trace of the SO(4) "gravitons," and $\tilde{h}_{ij}=h_{ij}-\frac{1}{4}\delta_{ij}h_{kk}$.

Field	Components	SO(4)×SO(4)	D.o.f's
Complex scalar	Φ	(1,1)	2
Complex	b_{ij}	$(\mathbf{3}^+,\mathbf{1})\oplus(\mathbf{3}^-,\mathbf{1})$	12
two-form	b_{ab}	$(\mathbf{1},\mathbf{3}^+)\oplus(\mathbf{1},\mathbf{3}^-)$	12
	b_{ia}	$(\mathbf{4},\mathbf{4})$	32
Real	f_{ia}	$(\mathbf{4},\mathbf{4})$	16
four-form	f_{ijab}	$(\mathbf{3}^+,\mathbf{3}^+)\oplus(\mathbf{3}^-,\mathbf{3}^-)$	18
	f	$(\mathbf{1},\mathbf{1})$	1
	$\tilde{h}_{ij}$	$(\mathbf{9},\mathbf{1})$	9
Graviton	$\tilde{h}_{ab}$	$(\mathbf{1},\mathbf{9})$	9
	h_{ia}	$(\mathbf{4},\mathbf{4})$	16
	h	$(\mathbf{1},\mathbf{1})$	1

$$\Box h_{ij}-2\mu\delta_{ij}\partial_- f=0. \tag{2.46}$$

There is a similar expression for the other SO(4) projections of the metric and four-form. We see that the trace (which we have yet to separate out) of the SO(4) metric and four-form projections mix with each other. The equation of motion for the four-form, coupled to the metric through the covariant derivative, implies

$$\Box f+8\mu\partial_- h=0. \tag{2.47}$$

These are a pair of coupled equations which can be diagonalized by redefinition of the field and by using h defined above:

$$c=h_{ii}+if. \tag{2.48}$$

The equations governing the new fields are

$$\Box\tilde{h}_{ij}=0, \quad (\Box+i8\mu\partial_-)c=0, \tag{2.49}$$

together with the complex conjugate of the second. These are equations of motion for massive scalar fields. Fourier transforming as before, we can compare these equations to Eqs. (2.41) and (2.44), to arrive at the light-cone energy spectrum,

$$p^-(\tilde{h}_{ij})=\mu(n+4), \quad p^-(c)=\mu(n+8), \quad p^-(c^\dagger)=0, \tag{2.50}$$

with $n\in\mathbb{Z}^+$, and obviously similar results for the components along the other SO(4). Note that $c^\dagger$ is the only combination of fields whose light-cone energy is allowed to vanish. Similar reasoning leads, for the mixed [in terms of SO(4)×SO(4)] components of the metric and four-form, to

$$(\Box + 4i\mu\partial_-)\mathfrak{h}_{ia} = 0, \tag{2.51}$$

and its conjugate, where we have diagonalized the equations by defining

$$\mathfrak{h}_{ia} = \mathfrak{h}_{ia} + if_{ia}. \tag{2.52}$$

These lead to the light-cone energy for $\mathfrak{h}_{ia}$,

$$p^-(\mathfrak{h}_{ia}) = \mu(n+6), \quad p^-(\mathfrak{h}^\dagger_{ia}) = \mu(n+2). \tag{2.53}$$

Finally, for f_{ijab}, we can show that $p^- = \mu(n+4)$.

The complex two-form can be studied in the same way as the four-form and graviton, resulting in similar equations, but with different masses. The two-form can be decomposed into representations that transform as two-forms of each of the SO(4)'s, each of which can be further decomposed into self-dual and anti-self-dual components, with respect to the Levi-Cività tensor of each SO(4). The self-dual part will carry opposite mass from the anti-self-dual projection. The decomposition will also include a second rank tensor with one leg in each SO(4), which will obey a massless equation of motion [for the SO(4)×SO(4) decomposition see Table I]. The lowest light-cone energy for the physical modes of the two-form takes the values $p^-/\mu = 2,4,6$, with the middle value associated with the mixed tensor and the difference of energies between the self-dual and anti-self-dual forms equal to four.

The analysis of the fermion spectrum follows along essentially the same lines, with minor technical complications having to do with the spin structure of the fields (inclusion of spin connection and some straightforward Dirac algebra). These technicalities are not illuminating, and we merely quote the results. The interested reader is directed to Metsaev and Tseytlin (2002). For the spin-1/2 dilatino the lowest light-cone energies for the physical modes can take the values $p^-/\mu = 3,5$, while for the spin-3/2 gravitino the range is $p^-/\mu = 1,3,5,7$. It is worth noting that the lowest states of fermions/bosons are odd/even integers in μ units. This is compatible with what we expect from the superalgebra.

III. PENROSE LIMITS AND PLANE WAVES

As discussed in the previous section plane waves are particularly nice geometries with the important property of having a globally defined null Killing vector field. They are also special from the supergravity point of view because they are α' exact (see Sec. II.A). In this section we discuss a general limiting procedure known as the Penrose limit (Penrose, 1976), which generates a plane-wave geometry out of any given spacetime. This procedure has also been extended to supergravity by Gueven (2000), hence when applied to supergravity this limit is usually called the Penrose-Gueven limit (see, for example, Blau *et al.*, 2002a, 2002b). Although the Penrose limit can be applied to any spacetime, if we start with solutions of Einstein's equations (or more generally the supergravity equations of motion) we end up with a plane wave that is still a (super)gravity solution. In other words, the Penrose-Gueven limit is a tool to generate new supergravity solutions out of any given solution. In this section we first summarize the three steps for taking the Penrose limit, then apply this procedure to the interesting example of AdS spaces, and finally in Sec. III.B we study a contraction of the supersymmetry algebra corresponding to $AdS_5\times S^5$, PSU(2,2|4) (Aharony *et al.*, 2000), under the Penrose limit.

A. Taking Penrose limits

The procedure for taking the Penrose limit can be summarized as follows:

(i) Find a lightlike (null) geodesic in the given spacetime metric.

(ii) Choose the proper coordinate system so that the metric looks like

$$ds^2 = R^2\{-2du\,d\tilde{v} + d\tilde{v}[d\tilde{v} + A_I(u,\tilde{v},\tilde{x}^I)d\tilde{x}^I] + g_{JK}(u,\tilde{v},\tilde{x}^I)d\tilde{x}^J d\tilde{x}^K\}. \tag{3.1}$$

In the above R is a constant introduced to facilitate the limiting procedure, the null geodesic is parametrized by the affine parameter u, $\tilde{v}$ determines the distance between such null geodesics, and $\tilde{x}^I$ parametrizes the rest of the coordinates. Note that any given metric can be brought to the form (3.1).

(iii) Take the $R\to\infty$ limit together with the scalings

$$\tilde{v} = \frac{v}{R^2}, \quad \tilde{x}^I = \frac{x^I}{R}; \quad u,v,x^I = \text{fixed}. \tag{3.2}$$

In this limit the A_I term drops out and $g_{IJ}(u,d\tilde{v},\tilde{x}^I)$ now becomes only a function of u, therefore

$$ds^2 = -2du\,dv + g_{IJ}(u)dx^I dx^J. \tag{3.3}$$

This metric is a plane wave, though in the Rosen coordinates (Roen, 1937). Under the coordinate transformation

$$x^I \to h_{IJ}(u)x^J, \quad v\to v + \frac{1}{2}g_{IJ}h'_{IK}h_{JL}x^K x^L,$$

with $h_{IK}g_{IJ}h_{JL} = \delta_{KL}$ and $h'_{IJ} = (d/du)h_{IJ}$ the metric takes the more standard form of Eq. (2.4), the Brinkmann coordinates (Brinkmann, 1923; Hubeny *et al.*, 2002). The only nonzero component of the Riemann curvature of plane wave (2.4) is $R_{uIuJ} = f_{IJ}(u)$, and the Weyl tensor of any plane wave is either null or vanishes.

The above steps can be understood more intuitively. Let us start with an observer who boosts up to the speed of light. Typically such a limit in (general) relativity is singular; however, these singularities may be avoided by "zooming" onto a region infinitesimally close to the (lightlike) geodesic the observer is moving on, in the particular way given in Eq. (3.2), so that at the end of the day from the original spacetime point of view we remove all parts except a very narrow strip close to the geodesic. We then scale up the strip to fill the whole spacetime, which is nothing but a plane wave. The covariantly constant null Killing vector field of plane waves corresponds to the null direction of the original spacetime along which the observer has boosted. To demonstrate how the procedure works, we work out here the explicit example of $AdS_p\times S^q$ space.

Let us start with an $AdS_p\times S^q$ metric in the global AdS coordinate system (Aharony *et al.*, 2000)

$$ds^2 = R_a^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{p-2}^2) + R_s^2(\cos^2\theta d\phi^2 + d\theta^2 + \sin^2\theta d\Omega_{q-2}^2). \quad (3.4)$$

We then boost along a circle of radius R_s in S^q directions, i.e., we choose the lightlike geodesic along $\tau-(R_s/R_a)\phi$ direction at $\rho=\theta=0$. Next, we send $R_a,R_s\to\infty$ at the same rate, so that

$$\frac{R_s^2}{R_a^2} = k^2 = \text{fixed}, \quad (3.5)$$

and scale the coordinates as

$$x^+ = \frac{1}{2}\left(\tau + \frac{R_s}{R_a}\phi\right), \quad x^- = R_a^2\left(\tau - \frac{R_s}{R_a}\phi\right), \quad (3.6a)$$

$$\rho = \frac{x}{R_a}, \quad \theta = \frac{y}{R_s}, \quad (3.6b)$$

keeping x^+,x^-,x,y and all the other coordinates fixed. Inserting Eqs. (3.5) and (3.6) into (3.4) and dropping $\mathcal{O}(1/R_a^2)$ terms, we obtain

$$ds^2 = -2dx^+dx^- - (x^ix^i + k^2y^ay^a)(dx^+)^2 + dx^idx^i + dy^ady^a, \quad (3.7)$$

where $i=1,2,\ldots,p-1$ and $a=1,2,\ldots,q-1$. For the case of $(p,q)=(5,5)$ and $(3,3)$ $k=R_s/R_a=1$, $(4,7)$ $k=R_s/R_a=1/2$ and $(7,4)$ $k=R_s/R_a=2$ (Maldacena, 1998).

Since AdS_p and S^q are not Ricci flat, $AdS_p\times S^q$ geometries can be supergravity solutions only if they are accompanied by the appropriate fluxes; for the case of $AdS_5\times S^5$ that is a (self-dual) five-form flux of type-IIB. For $AdS_4\times S^7$ and $AdS_7\times S^4$ it is a four-form flux of 11-dimensional supergravity, and for $AdS_3\times S^3$ it is a three-form RR or NS-NS flux (Maldacena, 1998). Let us now focus on the $AdS_5\times S^5$ case and study the behavior of the five-form flux under the Penrose limit. The self-dual five-form flux on S^5 is proportional to $N=R_s^4/g_s$, explicitly (Aharony *et al.*, 2000b)

$$F_{S^5} = 4Nd\Omega_5, \quad F_{AdS_5} = {}^*F_{S^5}, \quad (3.8)$$

where $d\Omega_5$ is the volume form of a five-sphere of unit radius. The numeric factor 4 is just a matter of supergravity conventions, and we have chosen our conventions so that the 10-dimensional (super)covariant derivative is given by Eq. (2.8). Taking the Penrose limit we find that

$$F = \frac{4}{g_s}dx^+\wedge(dx^1\wedge dx^2\wedge dx^3\wedge dx^4 + dy^1\wedge dy^2\wedge dy^3\wedge dy^4).$$

Finally the metric can be brought to the form (2.6) through the coordinate transformation

$$x^+\to\mu x^+, \quad x^-\to\frac{1}{\mu}x^-.$$

We would like to note that, as we see from the analysis presented here, for the $AdS_5\times S^5$ case the x^i come from the AdS_5 and y^a from the S^5 directions. However, after the Penrose limit has been taken there is no distinction between the x^i or y^a directions. This leads to the $\mathbb{Z}_2$ symmetry of the plane wave [see Eq. (2.22)].

Starting with a maximally supersymmetric solution, e.g., $AdS_5\times S^5$, after taking the Penrose limit we end up with another maximally supersymmetric solution, the plane wave. In fact, generally speaking, under the Penrose limit we never lose any supersymmetries, and as we shall show in the next subsection we may even gain some. It has been shown that all plane waves, whether under the Penrose limit or not, preserve at least half of the maximal possible supersymmetries (i.e., 16 supercharges for the type-II theories). This gives rise to kinematical supercharges (for example, see Cvetic *et al.*, 2002). A class of these that may preserve more than 16 necessarily has a constant dilaton (Figueroa-O'Farrill and Papadopoulos, 2003).

B. Contraction of the superconformal algebra PSU(2,2|4) under the Penrose limit

In the previous subsection we showed how to obtain the plane wave (2.6) from the $AdS_5\times S^5$ solution. In this part we continue a similar line of logic and show that under the Penrose limit the isometry group of $AdS_5\times S^5$, $SO(4,2)\times SO(6)$ exactly reproduces the isometry group of the plane wave discussed in Sec. II.B.1. First, we note that $SO(4,2)\times SO(6)$ and the isometry group of Sec. II.B.1 both have 30 generators. In fact we shall show that this correspondence goes beyond the bosonic isometries and extends to the whole $AdS_5\times S^5$ superalgebra, PSU(2,2|4) (Minwalla, 1998). The contraction of PSU(2,2|4) superalgebra under the Penrose limit has been considered by Hatsuda *et al.* (2002).

1. Penrose contraction of the bosonic isometries

The bosonic part of the $AdS_5\times S^5$ isometries is comprised of the four-dimensional conformal group SO(4,2) times SO(6), the generators of which are

$$J_{\hat\mu\hat\nu}, \quad J_{\hat A\hat B}, \quad \hat\mu = -1,0,1,2,3,4, \quad \hat A = 1,2,\ldots,6.$$

Being $SO(4,2)\times SO(6)$ generators, they satisfy

$$[J_{\hat\mu\hat\nu},J_{\hat\rho\hat\lambda}] = i(\hat\eta_{\hat\mu\hat\rho}J_{\hat\nu\hat\lambda} + \text{permutations}), \quad (3.9a)$$

$$[J_{\hat{A}\hat{B}}, J_{\hat{C}\hat{D}}] = i(\delta_{\hat{A}\hat{C}} J_{\hat{B}\hat{D}} + \text{permutations}), \tag{3.9b}$$

where $\hat{\eta}_{\hat{\mu}\hat{\nu}}=\text{diag}(-,-,+,+,+,+)$. In order to take the Penrose limit, it is more convenient to decompose them as

$$J_{\hat{\mu}\hat{\nu}} = \left\{ J_{ij}, \mathcal{D} = J_{-1,0}, L_i = \frac{1}{R}(J_{-1,i} + J_{0i}), K_i = \frac{1}{R}(J_{-1,i} - J_{0i}) \right\}, \tag{3.10a}$$

$$J_{\hat{A}\hat{B}} = \left\{ J_{ab}, \mathcal{J} = J_{56}, L_a = \frac{1}{R}(J_{5a} + J_{6a}), K_a = \frac{1}{R}(J_{5a} - J_{6a}) \right\}, \tag{3.10b}$$

where i,j and a,b vary from 1 to 4, and we also redefine $\mathcal{D}$ and $\mathcal{J}$ as

$$\mathcal{D} = \mu R^2 P^+ + \frac{1}{2\mu} P^-, \tag{3.11a}$$

$$\mathcal{J} = \mu R^2 P^+ - \frac{1}{2\mu} P^-. \tag{3.11b}$$

Note that, in the above, R and μ are auxiliary parameters introduced to facilitate the procedure of taking the Penrose limit. In the above parametrization the Penrose limit (3.6) becomes $R \to \infty$ with J_{ij}, J_{ab}, K_i, L_i, K_a, L_a, and P^+, P^- held-fixed. It is straightforward to show that Eq. (3.9) goes over to the $[\text{h}(4)\oplus\text{h}(4)]\oplus\text{so}(4)\oplus\text{so}(4)\oplus\text{u}(1)_+\oplus\text{u}(1)_-$ discussed in detail in Sec. II.B.1.

2. Penrose contraction on the fermionic generators

The supersymmetry of $\text{AdS}_5\times S^5$ fits into the Kac classifications of the superalgerbas (Kac, 1977) and is $\text{PSU}(2,2|4)$ see (Dobrev and Petkova, 1985; Minwalla, 1998), meaning that the bosonic part of the algebra is $\text{su}(2,2)\oplus\text{su}(4)\simeq\text{so}(4,2)\oplus\text{so}(6)$. Usually in the literature this superalgebra is either written using so(3,1) notations (see D'Hoker and Freedman, 2002), or 10-dimensional type-IIB notations (see Metsaev and Tseytlin, 1998) for fermions. For our purposes, where we merely need the simplest form of the algebra, it is more convenient to use so(4,2) or so(6) spinors directly. The supercharges carry spinorial indices of both the SO(4,2) and SO(6) groups. First, we recall that spin(4,2) $=$su(2,2) and spin(6)$=$su(4). Therefore the supercharges should carry fundamental indices of su(2,2) and su(4) (see Appendix B.3), explicitly $Q_{\hat{I}J}$ where both $\hat{I}$ and J run from one to four, the hatted index is the su(2,2) spinorial index, and the unhatted one is that of su(4). In fact both of these indices are Weyl indices of the corresponding groups. Further details of these six-dimensional spinors are gathered in Appendix B.3. The fermionic part of $\text{PSU}(2,2|4)$ superalgebra in this notation is

$$[J_{\hat{\mu}\hat{\nu}}, Q_{\hat{I}J}] = \frac{1}{2}(i\gamma_{\hat{\mu}\hat{\nu}})_{\hat{I}}^{\hat{K}} Q_{\hat{K}J}, \tag{3.12a}$$

$$[J_{\hat{A}\hat{B}}, Q_{\hat{I}J}] = -\frac{1}{2}(i\gamma_{\hat{A}\hat{B}})_J^K Q_{\hat{I}K}, \tag{3.12b}$$

$$\{Q_{\hat{I}J}, Q^{\dagger\hat{K}L}\} = 2\delta_J^L (i\gamma^{\hat{\mu}\hat{\nu}})_{\hat{I}}^{\hat{K}} J_{\hat{\mu}\hat{\nu}} + 2\delta_{\hat{I}}^{\hat{K}} (i\gamma^{\hat{A}\hat{B}})_J^L J_{\hat{A}\hat{B}}. \tag{3.12c}$$

Writing the algebra in the above notation and using the decomposition (B23), we can readily take the Penrose limit, if together with Eqs. (3.10) and (3.11) we scale the supercharges as

$$Q_{\hat{I}J} \to \left(\sqrt{\mu} R q_{\alpha\beta}, \sqrt{\mu} R q_{\dot{\alpha}\dot{\beta}}, \frac{1}{\sqrt{\mu}} Q_{\alpha\dot{\beta}}, \frac{1}{\sqrt{\mu}} Q_{\dot{\alpha}\beta} \right), \tag{3.13}$$

where we have introduced proper scalings for the kinematical and dynamical supercharges (q and Q, respectively). Inserting Eq. (3.13) into Eq. (3.12), sending $R\to\infty$, and keeping the leading terms, it is straightforward to see that Eq. (3.12) contracts to the superalgebra of the plane wave studied in some detail in Sec. II.B.2.

IV. PLANE WAVES AS BACKGROUNDS FOR STRING THEORY

As discussed in Sec. II.A plane waves are α'-exact solutions of supergravity and hence provide us with nice backgrounds for string theory. In fact, noting the simple form of the metric (2.4) it can be seen that the bosonic part of the σ model action in this background in the light-cone gauge takes a very simple form (this is a direct manifestation of the globally defined null Killing vector field of the background). For $f_{IJ}=$const (Hyun and Shin, 2002; Metsaev, 2002; Russo and Tseytlin, 2002b; Sugiyama and Yoshida, 2002; Alishahiha *et al.*, 2003) and $f_{IJ}\propto u^{-2}$ (Papadopoulos *et al.*, 2003) and some more general cases (Blau and O'Loughlin, 2003) it is even exactly solvable. In this review, however, we shall focus on the maximally supersymmetric plane wave of Eq. (2.6) and work out the Green-Schwarz action for this background. Note that, due to the presence of the RR fluxes, the Ramond-Neveu-Schwarz formulation of string theory (Neveu and Schwarz, 1971; Ramond, 1971) cannot be used. The Green-Schwarz formulation of superstring theory on some other plane wave or *pp*-wave backgrounds has also been considered in the literature.[8]

[8] See, for example, Berkovits and Maldacena (2002); Fuji *et al.* (2002), Hikida and Sugawara (2002); Maldacena and Maoz (2002); Russo and Tseytlin (2002a); Cvetic *et al.* (2003); Gimon *et al.* (2003); Kunitomo (2003); Mizoguchi *et al.* (2003b); Sadri and Sheikh-Jabbari (2003a, 2003b); Walton and Zhou (2003).

A. Bosonic sector of type-IIB strings on the plane-wave background

The bosonic string σ-model action in the background (2.6), which has metric $G_{\mu\nu}$ and a vanishing NS-NS two-form, is (Polchinski, 1998a)

$$S=\frac{1}{4\pi\alpha'}\int d^2\sigma g^{ab}G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu$$
$$=\frac{1}{4\pi\alpha'}\int d^2\sigma g^{ab}(-2\partial_a X^+\partial_b X^- + \partial_a X^I\partial_b X^I-\mu^2X_I^2\partial_a X^+\partial_b X^+), \tag{4.1}$$

where g_{ab} is the worldsheet metric, $\sigma^a=(\tau,\sigma)$ are the worldsheet coordinates, and $I=1,2,\ldots,8$. Note that the RR background fluxes do not appear in the bosonic action. We first need to fix the two-dimensional gauge symmetry, which is partly done by choosing

$$\sqrt{-g}g^{ab}=\eta^{ab},\quad -\eta_{\tau\tau}=\eta_{\sigma\sigma}=1. \tag{4.2}$$

To fix the residual worldsheet diffeomorphism invariance, we note that the equation of motion for X^+, $(\partial_\tau^2-\partial_\sigma^2)X^+=0$, has a general solution of the form $f(\tau+\sigma)+g(\tau-\sigma)$. We choose $f(x)=g(x)=\frac{1}{2}\alpha' p^+x$, i.e.,

$$X^+=\alpha' p^+\tau,\quad p^+>0. \tag{4.3}$$

The choices (4.2) and (4.3) completely fix the gauge symmetry. This is the *light-cone gauge*. In this gauge, X^+ and X^- are not dynamical variables anymore and are completely determined by X^I's through the constraints resulting from Eq. (4.2) (Green *et al.*, 1987a),

$$\frac{\delta\mathcal{L}}{\delta g_{\tau\sigma}}=0,\quad \frac{\delta\mathcal{L}}{\delta g_{\tau\tau}}=\frac{\delta\mathcal{L}}{\delta g_{\sigma\sigma}}=0.$$

Using the solution (4.3) for X^+ and setting $-g_{\tau\tau}=g_{\sigma\sigma}=1$, these constraints become

$$2\alpha' p^+\partial_\tau X^-=\partial_\tau X^I\partial_\tau X^I+\partial_\sigma X^I\partial_\sigma X^I-(\mu\alpha' p^+)^2X^IX^I, \tag{4.4}$$

$$\alpha' p^+\partial_\sigma X^-=\partial_\sigma X^I\partial_\tau X^I. \tag{4.5}$$

We can now drop the first term in Eq. (4.1) and replace X^+ with its light-cone solution. After rescaling τ and σ by $\alpha' p^+$, we obtain the light-cone action

$$S_{\text{l.c.}}^{\text{bos.}}=\frac{1}{4\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma[\partial_\tau X^I\partial_\tau X^I-\partial_\sigma X^I\partial_\sigma X^I-\mu^2X_I^2]. \tag{4.6}$$

This action is quadratic in X^I's and hence it is solvable. The equations of motion for X^I,

$$(\partial_\tau^2-\partial_\sigma^2-\mu^2)X^I=0, \tag{4.7}$$

should be solved together with the closed string boundary conditions

$$X^I(\sigma+2\pi\alpha' p^+)=X^I(\sigma). \tag{4.8}$$

In fact, $X^\pm$ should also satisfy the same boundary condition. From Eq. (4.3) it is evident that X^+ satisfies this boundary condition. We shall come back to the boundary condition on X^- at the end of this subsection. The solutions to these equations are

$$X^I=x_0^I\cos\mu\tau+\frac{p_0^I}{\mu p^+}\sin\mu\tau+\sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\omega_n}}[\alpha_n^Ie^{-i\sigma_{n+}/\alpha' p^+}+\tilde{\alpha}_n^Ie^{-i\sigma_{n-}/\alpha' p^+}+\alpha_n^{I\dagger}e^{+i\sigma_{n+}/\alpha' p^+}+\tilde{\alpha}_n^{I\dagger}e^{+i\sigma_{n-}/\alpha' p^+}], \tag{4.9}$$

where

$$\omega_n=\sqrt{n^2+(\alpha'\mu p^+)^2},\quad n\geqslant 0, \tag{4.10}$$

$$\sigma_{n\pm}=\omega_n\tau\pm n\sigma, \tag{4.11}$$

and α and $\tilde{\alpha}$ correspond to the right- and left-moving modes. The case of $n=0$ has been included for later convenience. The canonical quantization conditions

$$[X^I(\sigma,\tau),P^J(\sigma',\tau)]=i\delta^{IJ}\delta(\sigma-\sigma'), \tag{4.12}$$

where $P^I=(1/2\pi\alpha')\partial_\tau X^I$, yield

$$[x_0^I,p_0^J]=i\delta^{IJ},\quad [\alpha_n^I,\alpha_m^{J\dagger}]=[\tilde{\alpha}_n^I,\tilde{\alpha}_m^{J\dagger}]=\delta^{IJ}\delta_{mn}. \tag{4.13}$$

Next, using the light-cone action we work out the light-cone Hamiltonian,

$$H_{\text{l.c.}}^{\text{bos.}}=\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+}d\sigma[(2\pi\alpha')^2P_I^2+(\partial_\sigma X^I)^2+\mu^2X_I^2]. \tag{4.14}$$

As we expect, the light-cone Hamiltonian density is the momentum conjugate to light-cone time X^+, $P^-=(2/\alpha' p^+)(\partial_\tau X^-+\mu^2X_I^2)$. Plugging the mode expansion (4.9) into Eq. (4.14), we obtain

$$H_{\text{l.c.}}^{\text{bos.}}=\frac{1}{\alpha' p^+}\left[\alpha'\mu p^+\alpha_0^{I\dagger}\alpha_0^I+\sum_{n=1}^{\infty}\omega_n(\alpha_n^{I\dagger}\alpha_n^I+\tilde{\alpha}_n^{I\dagger}\tilde{\alpha}_I^n)\right]+\frac{8}{\alpha' p^+}\left(\frac{1}{2}\alpha'\mu p^++\sum_{n=1}^{\infty}\omega_n\right), \tag{4.15}$$

where the last term contains the zero-point energies of bosonic oscillators (after normal ordering) and we have defined

$$\tilde{\alpha}_0^I\equiv\alpha_0^I=\frac{1}{\sqrt{2\mu p^+}}p_0^I-i\sqrt{\frac{\mu p^+}{2}}x_0^I. \tag{4.16}$$

It is easy to check that $[\alpha_0^I,\alpha_0^{J\dagger}]=\delta^{IJ}$. We shall see in the next subsection that this zero-point energy is canceled against the zero-point energy of the fermionic modes, a sign of supersymmetry.

Now let us check whether X^- also satisfies the closed string boundary condition $X^-(\sigma+2\pi\alpha' p^+)=X^-(\sigma)$. From Eq. (4.5) we learn that

$$\delta X^- = \int_0^{2\pi\alpha' p^+} d\sigma \partial_\sigma X^I \partial_\tau X^I$$
$$= \sum_{n=1}^{\infty} n(\alpha_n^{I\dagger}\alpha_n^I - \tilde{\alpha}_n^{I\dagger}\tilde{\alpha}_I^n) = 0, \quad (4.17)$$

where $\delta X^- = X^-(\sigma + 2\pi\alpha' p^+) - X^-(\sigma)$, and we have used the mode expansion (4.9). Equation (4.17) is the *level-matching condition*, which is in fact a constraint on the physical excitations of a closed string (Polchinski, 1998a).

The vacuum of the light-cone string theory, $|0,p^+\rangle$, is defined as a state satisfying

$$\tilde{\alpha}_I^n|0,p^+\rangle = \alpha_n^I|0,p^+\rangle = 0, \quad n \geqslant 0. \quad (4.18)$$

Note that this vacuum is specified with the light-cone momentum p^+, i.e., for different values of p^+ we have a different string theory vacuum state and hence a different Fock space built from it. As we see from Eq. (4.15), in the plane-wave background all the string modes, including the zero modes, are massive. In other words, all the supergravity modes (created by $\alpha_0^\dagger$) are also massive, in agreement with the discussion in Sec. II.C.

Before moving on to the fermionic modes, we would like to briefly discuss strings on compactified plane waves. Such plane waves may naturally arise in the Penrose limit of particular $AdS_5 \times S^5$ orbifolds (Mukhi *et al.*, 2002). Let us consider the compactification of X^- on a circle of radius R_-:

$$X^- \equiv X^- + 2\pi R_-. \quad (4.19)$$

As a result of this compactification the light-cone momentum p^+, which is the momentum conjugate to the X^- direction, should be quantized,

$$p^+ = \frac{m}{R_-}, \quad m \in \mathbb{Z} - \{0\}. \quad (4.20)$$

For fixed m, we are in fact studying the discrete light-cone quantization of strings on plane waves (Alishahiha and Sheikh-Jabbari, 2002; Mukhi *et al.*, 2002). After compactification, we might also have winding modes along the X^- direction. The X^- winding number w is related to X^I excitation modes through the constraint (4.5):

$$w = \frac{1}{2\pi R_-}\int_0^{2\pi\alpha' p^+} d\sigma \partial_\sigma X^- = \frac{\alpha'}{R_-}\int_0^{2\pi\alpha' p^+} d\sigma \partial_\sigma X^I P_I,$$

where $w \in \mathbb{Z}$. This equation together with Eq. (4.20) gives the "improved" level-matching condition for strings, which is $mw = \Sigma_{n>0}\, n(\alpha_n^{I\dagger}\alpha_n^I - \tilde{\alpha}_n^{I\dagger}\tilde{\alpha}_I^n)$. The string-theory vacuum state is now identified by two integers m and w. As for toroidal compactifications in the transverse directions and T duality for strings on plane waves, we shall not discuss them here. The interested reader is referred to the available literature; see, for example, Ideguchi and Imamura (2003); Michelson (2002); Mizoguchi *et al.* (2003a).

B. Fermionic sector of type-IIB strings on the plane-wave background

1. The Green-Schwarz superstring action

The fermionic sector of the Green-Schwarz superstring action for type-IIB strings is (Green *et al.*, 1987a; Cvetic *et al.*, 2000)

$$S_F = \frac{i}{4\pi\alpha'}\int d^2\sigma (\theta^\alpha)^{\mathsf{T}}(\beta^{ab})_{\alpha\rho}\partial_a X^\mu \Gamma_\mu (\hat{D}_b)^\rho_\beta \theta^\beta + \mathcal{O}(\theta^3). \quad (4.21)$$

In the above θ^α, $\alpha = 1,2$ are two fermionic worldsheet fields giving embedding coordinates of $\mathcal{N}=2$ type-IIB superspace, i.e., they are 32-component 10-dimensional Weyl-Majorana fermions of the same chirality,

$$(\beta^{ab})_{\alpha\rho} = \sqrt{-g}g^{ab}\delta_{\alpha\rho} - \epsilon^{ab}(\sigma^3)_{\alpha\rho}, \quad (4.22)$$

and $(\hat{D}_b)^\rho_\beta$ is the pullback of the supercovariant derivative (2.8) to the worldsheet, which for our background becomes

$$(\hat{D}_b)^\rho_\beta = \delta^\rho_\beta \partial_b + \partial_b X^\nu (\Omega_\nu)^\rho_\beta, \quad (4.23)$$

and Ω_ν is given in Eq. (2.14).[9] Our notations for 10-dimensional type-IIB fermions is summarized in Appendix B.1. By $(\theta^\alpha)^{\mathsf{T}}$ we mean the transposition in the fermionic indices. The ϵ_{ab} term in Eq. (4.22) is in fact coming from the Wess-Zumino term in the Green-Schwarz action.

2. Fixing κ symmetry and the fermionic spectrum

κ symmetry is a necessary fermionic symmetry for spacetime supersymmetry of the on-shell string modes. In fact, by fixing the κ symmetry, we remove half of the fermionic gauge (unphysical) degrees of freedom so that after gauge fixing we are left with 16 physical fermions, describing on-shell spacetime fermionic modes. This number of fermionic degrees of freedom is exactly equal to the number of physical bosonic degrees of freedom coming from the X^I modes after fixing the light-cone gauge (note that there are left and right modes).

It has been shown that the action (4.21) for the plane-wave background possesses the necessary k symmetry

[9]The $\mathcal{O}(\theta^3)$ terms come from the higher-order θ contributions to the supervielbein. Explicitly, the Green-Schwarz Lagrangian for a general background is

$$\mathcal{L} = g^{ab}\Pi_a^\mu \Pi_a^\nu G_{\mu\nu} + \mathcal{L}_{\rm wz},$$

with $\Pi_a^\mu = \partial_a Z^N E_N^\mu$, and where $Z^M = (X^\mu, \theta^{A\alpha})$ are the type-IIB superspace coordinates and E_N^M are the supervierbiens (see Metsaev, 2002). One can then show that after fixing the light-cone gauge for the plane-wave background all $\mathcal{O}(\theta^3)$ corrections to E_N^M vanish (Metsaev, 2002) and the action reduces to Eq. (4.21) without $\mathcal{O}(\theta^3)$ terms. These complications are beyond the scope of this paper, and the interested reader is referred to Metsaev (2002). A similar procedure for the $M2$-brane action in the 11-dimensional plane-wave background has been carried out by Dasgupta *et al.* (2002a).

(Metsaev, 2002), and to obtain the physical fermionic modes we need to gauge-fix it, which can be achieved by choosing

$$\Gamma^+\theta^\alpha=0, \quad \alpha=1,2. \tag{4.24}$$

As in the flat-space case (Green *et al.*, 1987a), the above suffices to fix the full κ symmetry of the plane-wave background (Metsaev, 2002). By imposing Eq. (4.24) as shown in Appendix B.1, we can reduce the 10-dimensional fermions to the SO(8) representations, and since the two θ^α have the same 10-dimensional chiralities, both of them end up being in the same SO(8) fermionic representation, which we have chosen to be $\mathbf{8}_s$.

To simplify the action we note that Eq. (4.24) implies

$$(\theta^\alpha)^{\mathsf{T}}\Gamma^I\theta^\beta=0\ \forall\ \alpha,\beta, \quad (\Omega_I)^\alpha_\beta\theta^\beta=0.$$

From the $\partial_a X^\mu\Gamma_\mu$ term in the action, only $\partial_a X^+\Gamma_+$ survives, and from the Ω_μ terms only Ω_+ survives, and hence

$$S^{\rm fer}_{\rm l.c.}=\frac{i}{4\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma\{(\theta^\alpha)^{\mathsf{T}}(\beta^{ab})_{\alpha\rho}$$
$$\times(\partial_a X^+\Gamma_+)[\delta^\rho_\beta\partial_b+\partial_b X^+(\Omega_+)^\rho_\beta]\theta^\beta\}.$$

Next, we use Eqs. (2.14) and (4.3) to further simplify the action; after some straightforward algebra we obtain

$$S^{\rm fer}_{\rm l.c.}=\frac{-i}{4\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma[\theta^\dagger\partial_\tau\theta+\theta\partial_\tau\theta^\dagger+\theta\partial_\sigma\theta$$
$$+\theta^\dagger\partial_\sigma\theta^\dagger-2i\mu\theta^\dagger\Pi\theta]. \tag{4.25}$$

Note that in the above we have replaced θ^1 and θ^2, which are now eight-component $\mathbf{8_s}$ fermions with their complexified version [see Appendix B.1, Eq. (B9)]. The last term in the action is a mass term resulting from the RR five-form flux of the background. As we see, after fixing the κ symmetry, the spin connection does not contribute to the action.

The above action takes a particularly nice and simple form if we adopt SO(4)×SO(4) representations for fermions (see Appendix B.2). In that case θ and $\theta^\dagger$ are replaced with $\theta_{\alpha\beta},\theta_{\dot\alpha\dot\beta}$ and their complex conjugates, where α and $\dot\alpha$ are Weyl indices of either of the SO(4)'s. In this notation the action is

$$S^{\rm fer}_{\rm l.c.}=\frac{-i}{4\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma[\theta^\dagger_{\alpha\beta}\partial_\tau\theta^{\alpha\beta}+\theta^{\alpha\beta}\partial_\tau\theta^\dagger_{\alpha\beta}$$
$$+\theta_{\alpha\beta}\partial_\sigma\theta^{\alpha\beta}+\theta^{\dagger\alpha\beta}\partial_\sigma\theta^\dagger_{\alpha\beta}-2i\mu\theta^\dagger_{\alpha\beta}\theta^{\alpha\beta}$$
$$+\alpha,\beta\to\dot\alpha,\dot\beta]. \tag{4.26}$$

As can be seen, $\theta_{\alpha\beta}$ and $\theta^{\dot\alpha\dot\beta}$ decouple from each other. The coupled equations of motion for the fermions are

$$(\partial_\tau+\partial_\sigma)(\theta_{\alpha\beta}+\theta^\dagger_{\alpha\beta})-i\mu(\theta_{\alpha\beta}-\theta^\dagger_{\alpha\beta})=0,$$
$$(\partial_\tau-\partial_\sigma)(\theta_{\alpha\beta}-\theta^\dagger_{\alpha\beta})-i\mu(\theta_{\alpha\beta}+\theta^\dagger_{\alpha\beta})=0. \tag{4.27}$$

The solution to the above is

$$\theta=\frac{1}{\sqrt{p^+}}\beta_0 e^{i\mu\tau}+\frac{\sqrt{\alpha'\mu}}{2}\sum_{n=1}^\infty\frac{1}{\sqrt{\omega_n}}\left[\left(\frac{1-\rho_{-n}}{\sqrt{\rho_{-n}}}\right)\beta_n e^{-i\sigma_{n+}/\alpha' p^+}\right.$$
$$+\left(\frac{1+\rho_{-n}}{\sqrt{\rho_{-n}}}\right)\beta^\dagger_n e^{+i\sigma_{n-}/\alpha' p^+}+\left(\frac{1-\rho_n}{\sqrt{\rho_n}}\right)\tilde\beta_n e^{-i\sigma_{n-}/\alpha' p^+}$$
$$\left.+\left(\frac{1+\rho_n}{\sqrt{\rho_n}}\right)\tilde\beta^\dagger_n e^{+i\sigma_{n-}/\alpha' p^+}\right], \tag{4.28}$$

where ω_n is defined in Eq. (4.10), $\sigma_{n\pm}$ in Eq. (4.11), and

$$\rho_{\pm n}=\frac{\omega_n\pm n}{\alpha'\mu p^+}. \tag{4.29}$$

In the above, since there was no confusion, we have dropped the fermionic indices. The $\theta^{\dot\alpha\dot\beta}$'s also satisfy a similar equation.

Imposing the canonical quantization conditions,

$$\{\theta^{\alpha\beta}(\sigma,\tau),\theta^\dagger_{\rho\lambda}(\sigma',\tau)\}=2\pi\alpha'\,\delta^\alpha_\rho\delta^\beta_\lambda\delta(\sigma-\sigma'), \tag{4.30}$$

leads to

$$\{\beta_0,\beta_0^\dagger\}=1, \quad \{\beta_n,\beta^\dagger_m\}=\{\tilde\beta_n,\tilde\beta^\dagger_m\}=\delta_{mn}, \tag{4.31}$$

where again we have suppressed the fermionic indices.

Using the light-cone action and the mode expansion (4.28), we work out the light-cone Hamiltonian:

$$H^{\rm fer.}_{\rm l.c.}=\frac{1}{\alpha' p^+}\left[\alpha'\mu p^+\beta_0^\dagger\beta_0+\sum_{n=1}^\infty\omega_n(\beta^\dagger_n\beta_n+\tilde\beta^\dagger_n\tilde\beta_n)\right]$$
$$-\frac{8}{\alpha' p^+}\left(\frac12\alpha'\mu p^++\sum_{n=1}^\infty\omega_n\right). \tag{4.32}$$

In the above we have used $\beta^\dagger_n\beta_n$ as a shorthand for $\beta^\dagger_{n\alpha\beta}\beta^{\alpha\beta}_n+\beta^\dagger_{n\dot\alpha\dot\beta}\beta^{\dot\alpha\dot\beta}_n$ for $n\geq0$.

In the full light-cone Hamiltonian, which is a sum of bosonic and fermionic contributions, the zero-point energies cancel and

$$\mathcal{H}^{(2)}_{\rm l.c.}=\frac{1}{\alpha' p^+}\left[\alpha'\mu p^+(\alpha_0^{I\dagger}\alpha_0^I+\beta_0^\dagger\beta_0)+\sum_{n=1}^\infty\omega_n(\alpha_n^{I\dagger}\alpha_n^I\right.$$
$$\left.+\tilde\alpha_n^{I\dagger}\tilde\alpha^n_I+\beta^\dagger_n\beta_n+\tilde\beta^\dagger_n\tilde\beta_n)\right]. \tag{4.33}$$

C. Physical spectrum of closed strings on the plane-wave background

Having worked out the Hamiltonian and the mode expansions, we are now ready to summarize and list the low-lying string states in the plane-wave background. First, we note that the level-matching condition (4.17) also receives contributions from fermionic modes. Again using the fact that $\delta(L_b+L_f)/\delta g_{\tau\sigma}=0$, we find that a term like $\theta^\dagger\theta$ should be added to the right-hand side of Eq. (4.5), and hence the improved level-matching condition in which the fermionic modes have been taken into account is

$$\sum_{n=1}^{\infty} n(\alpha_n^{I\dagger}\alpha_n^I + \beta_n^\dagger\beta_n - \tilde{\alpha}_n^{I\dagger}\tilde{\alpha}_I^n - \tilde{\beta}_n^\dagger\tilde{\beta}_n)|\Psi\rangle = 0, \tag{4.34}$$

with $|\Psi\rangle$ a generic physical closed string state.

As usual the *free* string theory Fock space $\mathbb{H}$ is (Polchinski, 1998a)

$$\mathbb{H} = |\text{vacuum}\rangle \oplus_{m=1}^{\infty} \mathbb{H}_m, \tag{4.35}$$

where $\mathbb{H}_m$, the m-string Hilbert space, is nothing but m copies of (or the direct product of m) single-string Hilbert spaces $\mathbb{H}_1$. The string theory vacuum state in the sector with light-cone momentum p^+, which will be denoted by $|v\rangle$, is the state that is annihilated by all α_n and β_n:

$$\alpha_n|v\rangle = \tilde{\alpha}_n|v\rangle = 0, \quad \beta_n|v\rangle = \tilde{\beta}_n|v\rangle = 0, \quad \forall\, n \geqslant 0. \tag{4.36}$$

Hereafter we shall use convention of suppressing the light-cone momentum in the vacuum state, and the light-cone momentum p^+ is implicit in $|v\rangle$. Again, we have defined $\tilde{\beta}_0 = \beta_0$ for later convenience.

This state is clearly invariant under $SO(4)\times SO(4)$ symmetry and has zero energy. However, it is possible to define some other "vacuum" states that are invariant under the full SO(8). These states all necessarily have higher energies. Two such vacua which have been considered in the literature are (Metsaev and Tseytlin, 2002; Spradin and Volovich, 2002)

$$|0\rangle \equiv \beta_{011}^\dagger\beta_{012}^\dagger\beta_{021}^\dagger\beta_{022}^\dagger|v\rangle, \quad \text{or}$$

$$|\dot{0}\rangle \equiv \beta_{0\dot{1}\dot{1}}^\dagger\beta_{0\dot{1}\dot{2}}^\dagger\beta_{0\dot{2}\dot{1}}^\dagger\beta_{0\dot{2}\dot{2}}^\dagger|v\rangle. \tag{4.37}$$

It is evident that both $|0\rangle$ and $|\dot{0}\rangle$ have energy equal to 4μ. The interesting and important property of $|0\rangle$ and $|\dot{0}\rangle$ is that they are SO(8) invariant, and hence it is natural to assign them positive $\mathbb{Z}_2$ eigenvalues. [Note that, as discussed in Sec. II.B.1, $\mathbb{Z}_2$ is a specific SO(8) rotation.] On the other hand, it is not hard to check that under $\mathbb{Z}_2$

$$\beta_{012} \leftrightarrow \beta_{021} \quad \text{and} \quad \beta_{0\dot{1}\dot{2}} \leftrightarrow \beta_{0\dot{2}\dot{1}}.$$

Therefore $|v\rangle$ and $|0\rangle$ should have opposite $\mathbb{Z}_2$ charges (Chu *et al.*, 2002); with the positive assignment for $|0\rangle$, $|v\rangle$ should have a negative $\mathbb{Z}_2$ eigenvalue. Giving negative $\mathbb{Z}_2$ charge to $|v\rangle$ at first sight may look strange, but this charge assignment is the more natural one when we note the arguments of Sec. II.C. The $|v\rangle$ vacuum state, which has zero energy (mass), in fact arises from a combination of metric and the five-form field excitations. On the other hand, since the full transverse metric is traceless, the traces of the SO(4) parts of the metric should have opposite signs, and hence we expect $|v\rangle$ to be odd under $\mathbb{Z}_2$. The vacua $|0\rangle$ and $|\dot{0}\rangle$ arise from excitations of the axion-dilaton field, which is an SO(8) scalar. Therefore the natural assignment is to choose them to be even under $\mathbb{Z}_2$ (Pankiewicz, 2003a).

Based on the vacuum state $|v\rangle$, we can build the single-string Hilbert space $\mathbb{H}_1$ by the action of pairs of right- and left-moving (bosonic or fermionic) modes on the vacuum. This would guarantee that the level-matching condition (4.34) is satisfied. Note that the above does *not* exhaust all the possibilities when we have zero-mode excitations. In fact if we only excite $n=0$ modes the level-matching condition (4.34) is fulfilled for any number of excitations. Therefore we consider generic n and $n=0$ cases separately.

1. Generic single-string states

These states are generically of the following forms:

bosonic modes:

$$\alpha_n^{i\dagger}\tilde{\alpha}_n^{j\dagger}|v\rangle, \alpha_n^{a\dagger}\tilde{\alpha}_n^{b\dagger}|v\rangle, \alpha_n^{i\dagger}\tilde{\alpha}_n^{a\dagger}|v\rangle, \alpha_n^{a\dagger}\tilde{\alpha}_n^{i\dagger}|v\rangle, \tag{4.38a}$$

$$\beta_{n\alpha\beta}^\dagger\tilde{\beta}_{n\rho\lambda}^\dagger|v\rangle, \beta_{n\dot{\alpha}\dot{\beta}}^\dagger\tilde{\beta}_{n\dot{\rho}\dot{\lambda}}^\dagger|v\rangle, \beta_{n\alpha\beta}^\dagger\tilde{\beta}_{n\dot{\rho}\dot{\lambda}}^\dagger|v\rangle, \beta_{n\dot{\alpha}\dot{\beta}}^\dagger\tilde{\beta}_{n\rho\lambda}^\dagger|v\rangle, \tag{4.38b}$$

fermionic modes:

$$\alpha_n^{i\dagger}\tilde{\beta}_{n\alpha\beta}^\dagger|v\rangle, \alpha_n^{a\dagger}\tilde{\beta}_{n\alpha\beta}^\dagger|v\rangle, \beta_{n\alpha\beta}^\dagger\tilde{\alpha}_n^{i\dagger}|v\rangle, \beta_{n\alpha\beta}^\dagger\tilde{\alpha}_n^{a\dagger}|v\rangle, \tag{4.39a}$$

$$\alpha_n^{i\dagger}\tilde{\beta}_{n\dot{\alpha}\dot{\beta}}^\dagger|v\rangle, \alpha_n^{a\dagger}\tilde{\beta}_{n\dot{\alpha}\dot{\beta}}^\dagger|v\rangle, \beta_{n\dot{\alpha}\dot{\beta}}^\dagger\tilde{\alpha}_n^{i\dagger}|v\rangle, \beta_{n\dot{\alpha}\dot{\beta}}^\dagger\tilde{\alpha}_n^{a\dagger}|v\rangle, \tag{4.39b}$$

with $n\neq 0$. All the above states have mass equal to $2\omega_n$, though they are in different $SO(4)\times SO(4)$ representations. The first line of Eq. (4.38) for which both the left- and right-movers are coming from bosonic modes, in the usual conventions, comprise the "NS-NS" sector, and the second line of Eq. (4.38) the "RR" modes.[10]

It is instructive to work out the $SO(4)\times SO(4)$ representations of these modes. Here we shall study only the bosonic modes, and the fermionic modes are left to the reader. First we note that

(i) $|v\rangle$ is $SO(4)\times SO(4)$ singlet,

(ii) $\alpha_n^{i\dagger}$ and $\alpha_n^{a\dagger}$ are, respectively, in $(\mathbf{4},\mathbf{1})$ and $(\mathbf{1},\mathbf{4})$ of $SO(4)\times SO(4)$, and

(iii) as discussed in Appendix B.2, $\beta_{n\alpha\beta}^\dagger$ and $\beta_{n\dot{\alpha}\dot{\beta}}^\dagger$ are in $((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$ and $((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))$, respectively.

[10]In the usual (flat-space) conventions NS-NS and RR modes come from the decomposition of two bosonic and two fermionic modes of SO(8), respectively (Green *et al.*, 1987a). It is worth noting that this classification does *not* hold in our case in the sense that two bosonic modes (or equivalently two bosonic insertions) give rise to a combination of the metric and the self-dual five-form, while two fermionic insertions give rise to two-forms, NS-NS and RR. So, as we can see, there is a mixture of the usual NS-NS and RR modes which appear from two bosonic or fermionic stringy modes. This is not surprising, recalling that in our case we are dealing with the plane-wave background and $SO(4)\times SO(4)$ representations instead of flat space and SO(8). Therefore in our notations we reserve "NS-NS" and "RR" (instead of NS-NS and RR) to distinguish this difference from flat space.

Therefore $\alpha_n^{i\dagger}\tilde{\alpha}_n^{j\dagger}|v\rangle$ is in the SO(4)×SO(4) representation

$$(\mathbf{4},\mathbf{1})\otimes(\mathbf{4},\mathbf{1})=(\mathbf{1},\mathbf{1})\oplus(\mathbf{9},\mathbf{1})\oplus(\mathbf{3}^+,\mathbf{1})\oplus(\mathbf{3}^-,\mathbf{1}), \quad (4.40)$$

where by $\mathbf{3}^{\pm}$ we mean the self-dual (or anti-self-dual) part of 6 of SO(4). Likewise $\alpha_n^{a\dagger}\tilde{\alpha}_n^{b\dagger}|v\rangle$ can be decomposed into $(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{9})\oplus(\mathbf{1},\mathbf{3}^+)\oplus(\mathbf{1},\mathbf{3}^-)$. Here $\alpha_n^{i\dagger}\tilde{\alpha}_n^{a\dagger}|v\rangle$ and $\alpha_n^{a\dagger}\tilde{\alpha}_n^{i\dagger}|v\rangle$ are both in $(\mathbf{4},\mathbf{4})$ because

$$(\mathbf{4},\mathbf{1})\otimes(\mathbf{1},\mathbf{4})=(\mathbf{4},\mathbf{4}). \quad (4.41)$$

Now let us consider the "RR" modes. For two $\beta^{\dagger}_{n\alpha\beta}$ or $\beta^{\dagger}_{n\dot{\alpha}\dot{\beta}}$ excitations we note that

$$((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))\otimes((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1})) =(\mathbf{1},\mathbf{1})\oplus(\mathbf{3}^+,\mathbf{3}^+)\oplus(\mathbf{3}^+,\mathbf{1})\oplus(\mathbf{1},\mathbf{3}^+), \quad (4.42a)$$

$$((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))\otimes((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2})) =(\mathbf{1},\mathbf{1})\oplus(\mathbf{3}^-,\mathbf{3}^-)\oplus(\mathbf{3}^-,\mathbf{1})\oplus(\mathbf{1},\mathbf{3}^-), \quad (4.42b)$$

and for one $\beta^{\dagger}_{n\alpha\beta}$ and one $\beta^{\dagger}_{n\dot{\alpha}\dot{\beta}}$ type excitations

$$((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))\otimes((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))=(\mathbf{4},\mathbf{4}). \quad (4.42c)$$

2. Zero-mode excitations

Now let us restrict ourselves to the excitations which only involve $\alpha_0^{\dagger}$ and $\beta_0^{\dagger}$ modes. Compared to the previous case, there are two specific features to note. One is that the left- and right-movers are essentially the same (e.g., there is no independent $\tilde{\alpha}_0^{\dagger}$ or $\tilde{\beta}_0^{\dagger}$) and second, any number of excitations is physically allowed [there are no restrictions imposed by the level-matching condition (4.35)].

Here we consider only strings with two excitations, i.e., those with mass equal to 2μ. These modes are very similar to Eqs. (4.38) and (4.39) after setting $n=0$. This means that the modes of the form $\alpha_0^{i\dagger}\alpha_0^{j\dagger}|v\rangle$ are symmetric in i and j indices. In other words, in the decomposition (4.40) only $(\mathbf{1},\mathbf{1})\oplus(\mathbf{9},\mathbf{1})$ survive. Similarly, $\alpha_0^{a\dagger}\alpha_0^{b\dagger}|v\rangle$-type states are in the $(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{9})$ representation. The $\alpha_0^{i\dagger}\alpha_0^{b\dagger}|v\rangle$ states, however, would lead to a single $(\mathbf{4},\mathbf{4})$ representation. In sum the 36 "NS-NS" zero modes are in $(\mathbf{1},\mathbf{1})\oplus(\mathbf{9},\mathbf{1})\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{9})\oplus(\mathbf{4},\mathbf{4})$.

In the decomposition of "RR" modes among Eqs. (4.2) we should keep modes that are antisymmetric. Explicitly, they are $\epsilon^{\alpha\rho}\beta^{\dagger}_{0\alpha\beta}\beta^{\dagger}_{0\rho\lambda}|v\rangle$ in $(\mathbf{1},\mathbf{3}^+)$, $\epsilon^{\beta\lambda}\beta^{\dagger}_{0\alpha\beta}\beta^{\dagger}_{0\rho\lambda}|v\rangle$ in $(\mathbf{3}^+,\mathbf{1})$, $\epsilon^{\dot{\alpha}\dot{\rho}}\beta^{\dagger}_{0\dot{\alpha}\dot{\beta}}\beta^{\dagger}_{0\dot{\rho}\dot{\lambda}}|v\rangle$ in $(\mathbf{1},\mathbf{3}^-)$, and $\epsilon^{\dot{\beta}\dot{\lambda}}\beta^{\dagger}_{0\dot{\alpha}\dot{\beta}}\beta^{\dagger}_{0\dot{\rho}\dot{\lambda}}|v\rangle$ in $(\mathbf{3}^-,\mathbf{1})$ of SO(4)×SO(4). Therefore altogether the 28 "RR" modes are in $(\mathbf{3}^+,\mathbf{1})\oplus(\mathbf{3}^-,\mathbf{1})\oplus(\mathbf{1},\mathbf{3}^+)\oplus(\mathbf{1},\mathbf{3}^-)\oplus(\mathbf{4},\mathbf{4})$ representations.

The above may be compared with the supergravity modes discussed in Sec. II.C. Clearly there is a perfect match. This basically indicates that there exists a *low-energy limit* in the plane-wave background so that the effective string dynamics are governed by the supergravity modes; in such a limit, the lowest modes of strings created by $\alpha_0^{\dagger}$ and $\beta_0^{\dagger}$ would decouple from the rest of the string spectrum. For such a decoupling to happen, two necessary conditions should be met: first, $\omega_n \gg \alpha'\mu p^+$ for any $n\geq 1$, and second, strings should be "weakly coupled," i.e., $g_s^{\rm eff}\ll 1$. The former is satisfied if $\alpha'\mu p^+\ll 1$.

D. Representation of the plane-wave superalgebra in terms of string modes

String theory on the plane-wave background in the light-cone gauge that we discussed earlier has the same supersymmetry as the background whose algebra was introduced in Sec. II.B. In this section we shall explicitly construct the representations of that algebra in terms of string modes.

1. Bosonic generators

As in the flat-space case (Green *et al.*, 1987a), to find the representation of 30 bosonic isometries of the plane-wave background in terms of string modes, we start with their representations in terms of coordinates and their derivatives and then replace them with string worldsheet fields and their momenta, respectively. Noting Eqs. (2.29), (2.33), and (2.37), we learn that some of these bosonic generators should also have a part that is quadratic in stringy fermionic modes. Putting this all together we have

$$P^+=p^+\mathbb{1}, \quad P^-=\mathcal{H}^{(2)}_{\rm l.c.}, \quad (4.43)$$

$$K^I=\int_0^{2\pi\alpha' p^+}d\sigma\left[\sin\mu\tau P^I+\frac{\mu}{2\pi\alpha'}X^I\cos\mu\tau\right],$$

$$L^I=\int_0^{2\pi\alpha' p^+}d\sigma\left[\cos\mu\tau P^I-\frac{\mu}{2\pi\alpha'}X^I\sin\mu\tau\right], \quad (4.44)$$

$$J^{ij}=\int_0^{2\pi\alpha' p^+}d\sigma\Bigg[(X^iP^j-X^jP^i)-\frac{i}{4\pi\alpha'}[\theta^{\dagger}_{\alpha\beta}(\sigma^{ij})^{\alpha}_{\rho}\theta^{\rho\beta} + \theta^{\dagger}_{\dot{\alpha}\dot{\beta}}(\sigma^{ij})^{\dot{\alpha}}_{\dot{\rho}}\theta^{\dot{\rho}\dot{\beta}}]\Bigg],$$

$$J^{ab}=\int_0^{2\pi\alpha' p^+}d\sigma\Bigg[(X^aP^b-X^bP^a) -\frac{i}{4\pi\alpha'}[\theta^{\dagger}_{\alpha\beta}(\sigma^{ab})^{\beta}_{\rho}\theta^{\alpha\rho}+\theta^{\dagger}_{\dot{\alpha}\dot{\beta}}(\sigma^{ab})^{\dot{\beta}}_{\dot{\rho}}\theta^{\dot{\alpha}\dot{\rho}}]\Bigg]. \quad (4.45)$$

Note that in the above P^+ is proportional to the identity operator, which is compatible with the observation in of Sec. II.B.2 that the U(1) generated by P^+ is in the center of the superalgebra. It is straightforward to check that these generators really satisfy the desired algebras.

2. Fermionic generators

As discussed in Sec. II.B.2, there are two classes of supercharges, the kinematical and dynamical ones. Let us first focus on the kinematical supercharges. From Eqs. (2.29)–(2.32) one can see that $q_{\alpha\beta}$ should be proportional to $\theta_{\alpha\beta}$, explicitly

$$q_{\alpha\beta} = \frac{\sqrt{2}}{2\pi\alpha'} \int_0^{2\pi\alpha' p^+} d\sigma \theta_{\alpha\beta}, \tag{4.46}$$

$$q_{\dot{\alpha}\dot{\beta}} = \frac{\sqrt{2}}{2\pi\alpha'} \int_0^{2\pi\alpha' p^+} d\sigma \theta_{\dot{\alpha}\dot{\beta}}.$$

As for the dynamical supercharges, we note that, unlike the q's, which are in the *complex* $\mathbf{8_s}$ of SO(8), they are in the *complex* $\mathbf{8_c}$. Next we note that if θ is in $\mathbf{8_s}$ then $\gamma^I\theta$ is in $\mathbf{8_c}$ [it has opposite SO(8) chirality]. We also expect the Q's to contain first-order X's and P's, so that their anticommutator would generate the Hamiltonian, which is quadratic in X's and P's. Putting these together and demanding the Q's satisfy Eqs. (2.33)–(2.37) fixes them to be

$$\begin{aligned} Q^{(0)}_{\alpha\dot{\beta}} = \frac{1}{2\pi\alpha'} \int_0^{2\pi\alpha' p^+} & d\sigma[(2\pi\alpha' P^i - i\mu X^i)(\sigma_i)_\alpha^{\dot{\rho}} \theta^\dagger_{\dot{\rho}\dot{\beta}} \\ & + (2\pi\alpha' P^a + i\mu X^a)(\sigma_a)_{\dot{\beta}}^{\rho} \theta^\dagger_{\alpha\rho} + i\partial_\sigma X^i (\sigma_i)_\alpha^{\dot{\rho}} \theta_{\dot{\rho}\dot{\beta}} \\ & + i\partial_\sigma X^a (\sigma_a)_{\dot{\beta}}^{\rho} \theta_{\alpha\rho}], \end{aligned} \tag{4.47a}$$

$$\begin{aligned} Q^{(0)}_{\dot{\alpha}\beta} = \frac{1}{2\pi\alpha'} \int_0^{2\pi\alpha' p^+} & d\sigma[(2\pi\alpha' P^i - i\mu X^i)(\sigma_i)_{\dot{\alpha}}^{\rho} \theta^\dagger_{\rho\beta} \\ & + (2\pi\alpha' P^a + i\mu X^a)(\sigma_a)_{\beta}^{\dot{\rho}} \theta^\dagger_{\dot{\alpha}\dot{\rho}} + i\partial_\sigma X^i (\sigma_i)_{\dot{\alpha}}^{\rho} \theta_{\rho\beta} \\ & + i\partial_\sigma X^a (\sigma_a)_{\beta}^{\dot{\rho}} \theta_{\dot{\alpha}\dot{\rho}}]. \end{aligned} \tag{4.47b}$$

The superscript (0) on the Q's emphasizes that they are only linear in X and P's. As we shall argue in Sec. VIII, however, when we consider interacting strings there are corrections to the Hamiltonian as well as the dynamical supercharges, and in fact both H and Q should be viewed as power-series expansions in the string coupling. At zeroth order they match with the $Q^{(0)}$ and $\mathcal{H}^{(2)}$ presented here.

One may also try to insert the mode expansions and express the generators of the superalgebra in terms of string creation-annihilation operators. Doing so, it is easy to see that the "kinematical" generators, K^I, L^I, $q_{\alpha\beta}$, and $q_{\dot{\alpha}\dot{\beta}}$, which have a linear dependence on the string worldsheet fields, depend only on the zero modes. The "dynamical" generators, J_{ij}, J_{ab}, $Q^{(0)}_{\alpha\dot{\beta}}$, $Q^{(0)}_{\dot{\alpha}\beta}$, and $\mathcal{H}^{(2)}$, however, are quadratic and hence they depend on all the stringy operators.

V. STATING THE PLANE-WAVE/SUPER YANG-MILLS DUALITY

In Sec. III we demonstrated that plane waves may generically arise as Penrose limits of given geometries, and in particular the maximally supersymmetric plane wave appears as the Penrose limit of $AdS_5 \times S^5$ geometry. On the other hand, as briefly discussed in the Introduction (Gubser *et al.*, 1998; Witten, 1998; Aharony *et al.*, 2000), type-IIB string theory on the $AdS_5 \times S^5$ background is dual to the $\mathcal{N}=4$, $D=4$ (superconformal) gauge theory. In this section we show the latter duality can be revived for type-IIB strings on the maximally supersymmetric plane wave.

The basic idea of the BMN proposal (Berenstein, Maldacena, and Nastase, 2002b) is to start with the usual AdS/CFT duality and find what procedure parallels the taking of the Penrose limit in the dual gauge theory side. As we argued in Sec. III, the process of taking the Penrose limit consists of finding a lightlike geodesic and rescaling the other lightlike directions, as well as all the other transverse directions, in the appropriate way given in Eq. (3.2). For the case of $AdS_5 \times S^5$, the geodesic was chosen as a combination of a direction in S^5 and the global time (3.6). The generator of translation along this lightlike geodesic, P^- is then a combination of translation along the global time and rotation along the S^1 inside S^5 [Eq. (3.6a)]. According to the AdS/CFT duality, however, translation along global time corresponds to the dilatation operator (or, equivalently, Hamiltonian operator in the radial quantization) of the $\mathcal{N}=4$ gauge theory on $\mathbb{R}^4$, while the rotation in the S^1 direction corresponds to a U(1) of the R symmetry. Explicitly, the dilatation operator $\mathcal{D}$ is the generator of $U(1)_D \in SU(2,2) \simeq SO(4,2)$ (the conformal group in four dimensions) and $\mathcal{J}$ is the generator of $U(1)_J \in SU(4) \simeq SO(6)$ R symmetry [see Eq. (3.10)].

As an initial step towards building the plane-wave/super Yang-Mills (SYM) duality we state the proposal in this section. As mentioned in the Introduction, Sec. I.C, this duality can be stated as the operator equality (1.9) supplemented with a correspondence between the Hilbert spaces on both sides, where the operators act. In the first part of this section we show how the $\mathcal{N}=4$ gauge theory fields fall into the $SO(4)\times SO(4)$ representations, which is the first step in making the correspondence with the string theory. Then in the later parts of this section we state the duality and introduce the BMN operators. Our conventions for the $\mathcal{N}=4$ gauge theory fields and the action of the theory are summarized in Appendix A.

A. Decomposition of $\mathcal{N}=4$ fields into $\mathcal{D},\mathcal{J}$ eigenstates

The matter content of the $\mathcal{N}=4$ gauge multiplet naturally falls into the representations of $SO(4,2)\times SO(6)$ (for more details see, for example, D'Hoker and Freedman, 2002). However, in order to trace the Penrose limit in the gauge theory and state the BMN proposal, we need to study their representations in the $SO(4) \times SO(4)\times U(1)\times U(1)$ subgroup of $SO(4,2)\times SO(6)$. The $\mathcal{N}=4$ gauge multiplet contains six real scalars ϕ_I, $I=1,\dots,6$, four gauge fields A_a, $a=1,2,3,4$, and eight complex Weyl fermions ψ^A_α, $\alpha=1,2$ and $A=1,2,3,4$ (Wess and Bagger, 1992; also see Appendix A). Here we are only interested in $U(N)$ gauge theories where scalars and fermions are both in the adjoint representation of the U(N), so they are $N\times N$ Hermitian matrices. A_a are not in the adjoint representation, however (but they do transform in the adjoint for global transformations), and as in any gauge theory one might consider the covariant derivative of the gauge theory,

$$D_a=\partial_a+iA_a, \tag{5.1}$$

which is in the adjoint of the local U(N). In all our arguments we shall consider Euclidean gauge theory on $\mathbb{R}^4$ so the a index of D_a is an O(4) index. We might, however, switch between field theories on $\mathbb{R}^4$ and its conformal map, $\mathbb{R}\times S^3$.

The eigenvalues of $\mathcal{J}$ will be denoted by J. Since $\mathcal{J}$ is the generator of a U(1) subgroup of the U(4) R-symmetry group, the gauge fields are trivial under it. That is,

$$[\mathcal{J},D_a]=0; \tag{5.2}$$

in other words, D_a has charge $J=0$. The scalars, however, decompose into two sets. We choose $\mathcal{J}$ to make rotations in the ϕ^5 and ϕ^6 plane, i.e.,

$$Z=\frac{1}{\sqrt{2}}(\phi^5+i\phi^6); \quad [\mathcal{J},Z]=+Z, \tag{5.3}$$

and hence $[\mathcal{J},Z^\dagger]=-Z^\dagger$. Therefore Z has $J=1$ (and $Z^\dagger$, $J=-1$). The other four scalars, which will be denoted by ϕ_i, $i=1,2,3,4$ commute with $\mathcal{J}$ and have $J=0$. The 16 fermionic fields also decompose into two sets of eight with $J=\pm\frac{1}{2}$.

The eigenvalue of $\mathcal{D}$ will be denoted by Δ. For fields in the $\mathcal{N}=4$ gauge multiplet at free-field-theory level, $\Delta=1$ for scalars and D_a and $\Delta=\frac{3}{2}$ for fermions. Hereafter we shall use Δ_0 to denote the dimension of operators at free-field level (the engineering dimensions) and Δ for the full interacting theory; explicitly for any operator $\mathcal{O}(x)$,

$$[\mathcal{D},O(0)]=[\Delta_0+\mathcal{O}(g^2_{\rm YM})]\mathcal{O}(0). \tag{5.4}$$

After taking out the two U(1) factors $(\mathcal{D},\mathcal{J})$ of the SO(4,2)$\times$SO(6) [or SU(2,2)$\times$SU(4)], the bosonic part of the four-dimensional superconformal group, we remain with an SO(4)$\times$SO(4) [one SO(4)$\in$SO(4,2) and the other SO(4)$\in$SO(6)] subgroup. We also need to find the SO(4)$\times$SO(4) representation of the fields. Obviously Z and $Z^\dagger$ are singlets of both SO(4)'s, the $(\mathbf{1},\mathbf{1})$ representation, ϕ_i are in (1,4), and the D_a are in $(\mathbf{4},\mathbf{1})$. The SO(4)$\times$SO(4) representation of fermions can be worked out noting the arguments of Sec. III.B and Appendix B.3. Explicitly, we first note that SO(4)$\simeq$SU(2)$\times$SU(2) and, as for the usual four-dimensional Euclidean Weyl fermions, they are in $(\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2})$ of each SO(4)'s (see Appendix B.2). The SO(4)$\times$SO(4)$\times$U(1)$\times$U(1) representations of all fields of the $\mathcal{N}=4$ gauge multiplet have been summarized in Table II. Note that for all the fields in Table II, bosonic and fermionic, Δ_0-J is integer valued.

TABLE II. SO(4)$\times$SO(4)$\times$U(1)$\times$U(1) representations of all fields of the $\mathcal{N}=4$ gauge multiplet. The dimensions are those of the free theory. For the J charge of fermions, note that $\psi_{\alpha\beta}$ and $\psi_{\dot\alpha\dot\beta}$ are related by CPT and hence have opposite J charge; similarly for the other two fermionic modes.

Field	Δ_0-J	Δ_0+J	$SO(4)\times SO(4)$
Z	0	2	$(\mathbf{1},\mathbf{1})$
$Z^\dagger$	2	0	$(\mathbf{1},\mathbf{1})$
ϕ_i	1	1	$(\mathbf{1},\mathbf{4})$
D_a	1	1	$(\mathbf{4},\mathbf{1})$
$\psi_{\alpha\beta}$	1	2	$((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$
$\psi_{\dot\alpha\dot\beta}$	1	2	$((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))$
$\psi_{\alpha\dot\beta}$	2	1	$((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))$
$\psi_{\dot\alpha\beta}$	2	1	$((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))$

B. Stating the Berenstein-Maldacena-Nastase proposal

Having worked out the SO(4)$\times$SO(4)$\times$U(1)$_D$ $\times$U(1)$_J$ representation of the $\mathcal{N}=4$ fields, we are ready to take the BMN limit, restricting to the operators with parameterically large R charge J, but finite Δ_0-J. In fact, starting with the AdS/CFT correspondence, the BMN limit on the gauge theory side parallels the Penrose limit on the gravity side, according to which

$$-i\frac{\partial}{\partial\phi}\leftrightarrow\mathcal{J}, \quad i\frac{\partial}{\partial\tau}\leftrightarrow\mathcal{D}. \tag{5.5}$$

Then, Eqs. (3.6) and (3.11) imply that

$$i\mu\frac{\partial}{\partial x^-}=\frac{i\alpha'}{2R^2}\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\phi}\right)\leftrightarrow\frac{1}{2\sqrt{g^2_{\rm YM}N}}(\mathcal{D}+\mathcal{J}), \tag{5.6a}$$

$$\frac{i}{\mu}\frac{\partial}{\partial x^+}=i\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\phi}\right)\leftrightarrow\mathcal{D}-\mathcal{J}, \tag{5.6b}$$

where in Eq. (5.6a) we have used Eq. (1.2). On the gravity (string theory) side, $i(\partial/\partial x^-)$ and $i(\partial/\partial x^+)$ are the light-cone momentum and the light-cone Hamiltonian, respectively. Taking the Penrose limit Eq. (3.6) is then equivalent to taking $g^2_{\rm YM}N$ and J to infinity while keeping $J^2/g^2_{\rm YM}N$ fixed [see Eqs. (1.7) and (1.8)]. According to Eq. (5.6a) the value of $J^2/g^2_{\rm YM}N$ is equal to the string light-cone momentum (squared) on the string theory side [see Eq. (1.7b)].

In summary, part one of the plane-wave/SYM duality can be stated as follows:

The light-cone string field theory Hamiltonian in the plane-wave background is equal to the difference between the dilatation operator $\mathcal{D}$ and the R charge operator $\mathcal{J}$:

$$\frac{1}{\mu}H_{\rm SFT}=\mathcal{D}-\mathcal{J}, \tag{5.7}$$

in the sector of the gauge theory consisting of gauge-invariant operators with parametrically large R charge, the BMN sector.

A more detailed discussion about the construction and form of the BMN operators and also correspondence between the Hilbert spaces on the string and gauge theory sides, i.e., part two of the plane-wave/SYM duality, will be presented in the next subsection.

C. The BMN operators

As mentioned earlier, in the plane-wave/SYM duality the relevant operators on the gauge theory side are those with large R charge J; these are the so-called BMN operators, through which $\mathcal{D}-\mathcal{J}$ acts. In this section we present such gauge-invariant operators. The BMN operators can be classified by the number of traces (over the $N\times N$ gauge theory indices) involved, and also the value of Δ_0-J. In fact, because of the BPS bound (D'Hoker and Freedman, 2002) $\Delta\geqslant J$ and when $\Delta=J$ the BPS bound is saturated. This can be seen from Table II and the fact that the value of Δ_0-J for composite operators is just the sum of Δ_0-J of the basic fields present in the composite operator. Besides the value of Δ_0-J and number of traces to completely specify the operator we need to identify its $SO(4)\times SO(4)$ representation.

1. BMN operators with $\Delta_0-J=0$

The first class of the BMN operators we consider are those with $\Delta_0-J=0$, in the usual $\mathcal{N}=4$ conventions these are chiral primary operators (D'Hoker and Freedman, 2002). According to Table II, such operators can only be composed of Z fields. Therefore they are necessarily $SO(4)\times SO(4)$ singlets, and hence this class of BMN operators is completely specified with the number of traces, the simplest of which is of course the single-trace operator

$$\mathcal{O}^J(x)=\frac{1}{\sqrt{JN_0^J}}\mathrm{Tr}Z^J(x),\quad N_0=\frac{1}{8\pi^2}g_{\mathrm{YM}}^2N. \tag{5.8}$$

The normalization is fixed so that the planar two-point function of $\mathcal{O}^J(x)$ and $\mathcal{O}^{J\dagger}(0)$ is equal to $1/|x|^{2J}$; we shall come back to this point in Sec. VI.A. We would like to stress that the point x where the above operator is defined is in $\mathbb{R}^4$. One can then define a state by acting Eq. (5.8) on the vacuum of the gauge theory on $\mathbb{R}^4$, which will be denoted by $|\mathrm{vac}\rangle$. In this way there is a natural one-to-one correspondence between BMN states and BMN operators. Hence in this review we shall not distinguish between BMN operators and BMN states, and they will be used interchangeably. According to the second part of the BMN proposal the above single-trace operator (or state) corresponds to a single-string state on the string theory side:

$$|v\rangle\leftrightarrow\mathcal{O}^J(0)|\mathrm{vac}\rangle, \tag{5.9}$$

where $|v\rangle$ is the single-string vacuum with the light-cone momentum p^+ [Eq. (4.36)].

The next state belonging to this class is the double-trace operator

$$\begin{aligned}\mathcal{T}^{J,r}&=(\mathcal{O}^{r\cdot J}\mathcal{O}^{(1-r)\cdot J})(x)\\&=\frac{1}{J\sqrt{r(1-r)N^J}}:\mathrm{Tr}Z^{J_1}(x)\mathrm{Tr}Z^{J-J_1}(x),\end{aligned} \tag{5.10}$$

where $J_1/J=r$ and J_1 ranges between 1 and $J-1$. Of course the above operator is a BMN operator if J_1 is of the order of J. In a similar way Eq. (5.10) was proposed to correspond to the double-string state with the total light-cone momentum p^+, with the partition $r\cdot p^+$ and $(1-r)\cdot p^+$. One can then straightforwardly generalize the above to multitrace operators.

We would like to point out that each of the $\mathcal{O}^J$ or $\mathcal{T}^{J,r}$ operators are chiral primaries. In other words, they are half BPS states of the four-dimensional superconformal algebra $PSU(2,2|4)$. Being chiral primary, these operators (states) are eigenstates of the dilatation operator and have $\Delta-J=0$ exactly (D'Hoker and Freedman, 2002). We should stress, however, that from the $PSU(2|2)\times PSU(2|2)\times U(1)_-$ superalgebra discussed in Sec. II.B.2, these operators form a complete supermultiplet, which in this case is in fact a singlet, and are still half BPS in the sense that all the dynamical supercharges $Q_{\alpha\dot\beta}$ and $Q_{\dot\alpha\beta}$ annihilate them.

2. BMN operators with $\Delta_0-J=1$

The next level of states are those with $\Delta_0-J=1$. In order to obtain such BMN states we should insert one of the fields in Table II which have $\Delta_0-J=1$ into Eq. (5.8) or (5.10). There are eight bosonic states (corresponding to insertions of ϕ_i or D_a) and eight fermionic states (corresponding to insertions of $\psi_{\alpha\beta}$ or $\psi_{\dot\alpha\dot\beta}$). Each of these insertions may be viewed as *impurities* in the line of Z's. Due to cyclicity of the trace it does not matter where in the sequence of Z's these impurities are inserted. These $8+8$ states complete a supermultiplet of $PSU(2|2)\times PSU(2|2)\times U(1)_-$ superalgebra. Here, we should emphasize that in the full superconformal $PSU(2,2|4)$ algebra representations, however, these states are descendents of chiral primaries and are in the same short supermultiplet as chiral primaries. From the $PSU(2|2)\times PSU(2|2)\times U(1)_-$ superalgebra point of view they are in different multiplets than chiral primaries with $\Delta-J=0$.

As examples we present two such single-trace operators:

$$\mathcal{O}_i^J=\frac{1}{\sqrt{N_0^{J+1}}}\mathrm{Tr}(\phi_iZ^J),\quad\mathcal{O}_a^J=\frac{1}{\sqrt{N_0^{J+1}}}\mathrm{Tr}(D_aZZ^{J-1}). \tag{5.11}$$

These operators correspond to $\alpha_0^{i\dagger}$ or $\alpha_0^{a\dagger}$ on the string theory side. Note that in the closed string theory a physical state should satisfy the level-matching condition (4.34) and is generically composed of equal energy excitations of left and right modes. The operators (5.11), however, correspond to "zero-momentum" string states and satisfy the level-matching condition.

In the same spirit as Eq. (5.10) the double-trace $\Delta_0 - J = 1$ BMN operators can be obtained by combining $\mathcal{O}^J$ with Eq. (5.11), e.g.,

$$\mathcal{T}_i^{J,r} = (\mathcal{O}_i^{r\cdot J}\mathcal{O}^{(1-r)\cdot J})(x) = \frac{1}{\sqrt{(1-r)\cdot JN_0^J}} :\mathrm{Tr}\,\phi_i Z^{J_1}(x)\mathrm{Tr}Z^{J-J_1}(x):, \tag{5.12}$$

where, as in Eq. (5.10), r is the ratio J_1/J; we shall use this notation throughout the rest of this paper.

We would like to note that all the operators of this class, e.g., those presented in Eqs. (5.11) and (5.12), are descendents of chiral primaries and are exact eigenstates of $\mathcal{D}-\mathcal{J}$, with $\Delta - J = 1$.

3. BMN operators with $\Delta_0 - J = 2$

To obtain BMN operators with $\Delta_0 - J = 2$ we can either have two insertions of fields with $\Delta_0 - J = 1$ or a single insertion of a $\Delta_0 - J = 2$ field from Table II into the sequence of Z's. For the case of two $\Delta_0 - J = 1$ insertions, the position of the insertions is important. However, due to the cyclicity of the trace only the relative positions of the insertions is relevant. We fix our conventions so that one of the impurity fields always appears at the beginning of the sequence. In the single $\Delta_0 - J = 2$ insertion, similar to the case of Sec. V.C.2, the insertion position is immaterial. For single-trace operators with two $\Delta_0 - J = 1$ insertions, there are $J+1$ choices, depending on the relative positions of the insertions, which we may use as their "discrete" Fourier modes. To begin, let us consider the case in which both of the insertions are of the ϕ_i form

$$\mathcal{O}_{ij,n}^J = \frac{1}{\sqrt{JN_0^{J+2}}}\left[\sum_{p=0}^{J} e^{2\pi i p n/J}\ \mathrm{Tr}(\phi_i Z^p \phi_j Z^{J-p}) - \delta_{ij}\mathrm{Tr}(Z^\dagger Z^{J+1})\right]. \tag{5.13}$$

As we shall show in Sec. VI, once we turn on the gauge-theory coupling, individual operators of the form

$$\tilde{\mathcal{O}}_p \equiv \mathrm{Tr}(\phi_i Z^p \phi_j Z^{J-p}) \tag{5.14}$$

are no longer eigenvectors of the dilatation operator $\mathcal{D}$. However, the $\mathcal{O}_{ij,n}^J$ operators, at *planar level* (and of course in the large-J limit) have definite $\mathcal{D}$ eigenvalue (scaling dimension). Using Eq. (5.13) it is easy to check that

$$\mathcal{O}_{ij,n}^J = \mathcal{O}_{ji,-n}^J. \tag{5.15}$$

One may then consider two D_a insertions or one ϕ_i and one D_a [the D_a insertions have been considered by Gursoy (2003) and Klose (2003)] :

$$\mathcal{O}_{ab,n}^J = \frac{1}{2}\frac{1}{\sqrt{JN_0^{J+2}}}\left[\sum_{p=0}^{J} e^{2\pi i p n/J}\mathrm{Tr}[(D_a Z)Z^p(D_b Z)Z^{J-p}] + \mathrm{Tr}[(D_a D_b Z)Z^{J+1}]\right], \tag{5.16}$$

$$\mathcal{O}_{ia,n}^J = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{JN_0^{J+2}}}\left[\sum_{p=0}^{J} e^{2\pi i p n/J}\mathrm{Tr}[\phi_i Z^p (D_a Z) Z^{J-p}] + \mathrm{Tr}[(D_a\phi_i)Z^{J+1}]\right]. \tag{5.17}$$

Note that in the above equation of motion, $D_a D_a Z = 0$ should also be imposed on the fields. The normalization of $\mathcal{O}_{ij,n}^J$ operators has been fixed so that the two-point function of these operators, in the planar free gauge theory limit, is of the form $\langle \mathrm{vac}|\mathcal{O}_{ij,n}^{\dagger J}(x)\mathcal{O}_{i'j',n}^J(0)|\mathrm{vac}\rangle = \delta_{ii'}\delta_{jj'}(1/|x|^{2(J+1)})$, and similarly for $\mathcal{O}_{ia,n}^J$ and $\mathcal{O}_{ab,n}^J$ operators. The difference in factors of 1, $\frac{1}{2}$, and $1/\sqrt{2}$ in the normalization is a consequence of our conventions in which $\langle \mathrm{vac}|Z^\dagger(x^\mu/|x|)Z(0)|\mathrm{vac}\rangle = 1$ and $\langle \mathrm{vac}|\phi_i^\dagger(x^\mu/|x|)\phi_j(0)|\mathrm{vac}\rangle = \delta_{ij}$, while $\langle \mathrm{vac}|(D_a Z)^\dagger(x^\mu/|x|)[D_b Z(0)]|\mathrm{vac}\rangle = 2\delta_{ab}$.

The second part of the plane-wave/SYM duality, which is a map between the string theory Hilbert space and BMN operators, can then be stated as follows:

The operators $\mathcal{O}_{ij,n}^J$, $\mathcal{O}_{ab,n}^J$, and $\mathcal{O}_{ai,n}^J$ correspond to the "NS-NS" modes of the single-string sector of free closed string theory on the plane-wave background (see Secs. IV.C.1 and IV.C.2). Explicitly,

$$\mathcal{O}_{ij,n}^J \leftrightarrow \alpha_{i,n}^\dagger\tilde{\alpha}_{j,n}^\dagger,$$

$$\mathcal{O}_{ab,n}^J \leftrightarrow \alpha_{a,n}^\dagger\tilde{\alpha}_{b,n}^\dagger, \quad \forall\, n \geqslant 0,$$

$$\mathcal{O}_{ia,n}^J \leftrightarrow \alpha_{i,n}^\dagger\tilde{\alpha}_{a,n}^\dagger, \tag{5.18}$$

where $\alpha_{i,n}^\dagger$ and $\tilde{\alpha}_{i,n}^\dagger$ are the left- and right-moving string modes defined in Eq. (4.9). The "RR" and Neveu-Schwarz/Ramond ("NSR" or "RNS") modes (note the comment in footnote 10) and all of the fermionic modes can be obtained in a similar way through insertions of fermionic ψ^A fields, two ψ^A fields for the bosonic modes and one ψ^A and one ϕ_i or D_a for fermionic modes. On the string theory side the inner product on the Hilbert space is the usual one in which m and n string states are orthogonal to each other unless $m = n$. On the gauge theory side, however, the inner product corresponds to the two-point function of the corresponding BMN operators.

We should warn the reader that identifying the inner product on the Hilbert space with the two-point functions on the gauge theory side already suggests that the correspondence (5.18) should be modified because the two-point functions of the single- and double-trace operators generically do not vanish. This will produce further complications, which will be discussed in detail in Sec. VII.C.

The operators $\mathcal{O}_{ij,n}^J$, $\mathcal{O}_{ab,n}^J$, and $\mathcal{O}_{ai,n}^J$ form the bosonic states of a $PSU(2|2)\times PSU(2|2)\times U(1)_-$ supermultiplet. Note that from the superconformal $PSU(2,2|4)$ algebra point of view they are only a part of the bosonic states of a supermultiplet.[11] In general since the supercharges of the $PSU(2|2)\times PSU(2|2)\times U(1)_-$ commute with the Hamiltonian, P^- [see Eq. (2.37)] all the states in the same supermultiplet must have the same energy or mass. This should be contrasted with the superconformal algebra of $PSU(2,2|4)$ in which states with different $\Delta-J$ appear in the same multiplet. For example, chiral primaries and their descendents, Eqs. (5.8) and (5.11), and $\Delta_0-J=2$ with $n=0$ BMN operators discussed earlier, fall into the same $PSU(2,2|4)$ supermultiplet (Beisert, 2003).

These operators, as they are written, are not in irreducible representations of $SO(4)\times SO(4)$. Following the discussions of Sec. IV.C [Eq. (4.40)] one can decompose $\mathcal{O}^J_{ij,n}$ into $\frac{1}{2}\Sigma_{i=1}^4\mathcal{O}^J_{ii,n}$ in $(\mathbf{1},\mathbf{1})$, $\mathcal{O}^J_{(ij),n}=\frac{1}{2}(\mathcal{O}^J_{ij,n}+\mathcal{O}^J_{ji,n})$ in $(\mathbf{9},\mathbf{1})$, and $\frac{1}{2}\big(\mathcal{O}^J_{[ij],n}\pm\frac{1}{2}\epsilon_{ijkl}\mathcal{O}^J_{[kl],n}\big)$, where $\mathcal{O}^J_{[ij],n}=\frac{1}{2}(\mathcal{O}^J_{ij,n}-\mathcal{O}^J_{ji,n})$, in $(\mathbf{3}^\pm,\mathbf{1})$ representations of $SO(4)\times SO(4)$. Similar decompositions can be made for $\mathcal{O}^J_{ab,n}$ states. Noting Eq. (4.41), we see that the $\mathcal{O}^J_{ai,n}$ states form a $(\mathbf{4},\mathbf{4})$ of $SO(4)\times SO(4)$. For the cases in which we have two fermionic ψ-field insertions the decomposition can be carried out using Eq. (4.42a) if we have two $\psi_{\alpha\beta}$ insertions or Eq. (4.42b) if we have two $\psi_{\dot\alpha\dot\beta}$ insertions. We might also have one $\psi_{\alpha\beta}$ and one $\psi_{\dot\alpha\dot\beta}$ insertion, whose decomposition could than be read from Eq. (4.42c).

The $n=0$ case, i.e., $\mathcal{O}^J_{ij,0}$, $\mathcal{O}^J_{ab,0}$, and $\mathcal{O}^J_{ai,0}$, corresponds to the supergravity modes of the strings in the plane-wave background. At first it might seem that we should not expect to find supergravity modes and the results of Sec. II.C from gauge theory, because the truncation of stringy excitations to the supergravity modes only makes sense when all the other excitations are much heavier than the lowest modes, which, noting Eq. (4.10), is when $\alpha'\mu p^+\ll 1$. As we shall see in the next section, this is the limit where the "improved" 't Hooft coupling (1.11) is very large and one cannot trust the gauge theory analysis. However, one should note that from the superalgebra point of view these states are short (BPS) multiplets of the $PSU(2,2|4)$ superconformal algebra (Beisert, 2003) as well as the plane-wave superalgebra $PSU(2|2)\times PSU(2|2)\times U(1)_-$, and hence it is natural to expect them to be protected by supersymmetry. Noting Eq. (5.15), we see that the $(\mathbf{3}^+,\mathbf{1})$, $(\mathbf{3}^-,\mathbf{1})$, $(\mathbf{1},\mathbf{3}^+)$, and $(\mathbf{1},\mathbf{3}^-)$ representations are absent in these supergravity modes. These representations, which correspond to the fluctuations of type-IIB NS-NS or RR two-form fields (see Sec. II.C), can arise from two fermionic insertions. Note that for supergravity modes (the $n=0$ case), due to the fact that fermions anticommute, only the totally antisymmetric representations of Eqs. (4.42a) and (4.42b) remain which are $(\mathbf{3}^+,\mathbf{1})$, $(\mathbf{3}^-,\mathbf{1})$ and $(\mathbf{1},\mathbf{3}^+)$, $(\mathbf{1},\mathbf{3}^-)$. Then the two $(\mathbf{4},\mathbf{4})$ representations arising from $\mathcal{O}^J_{ai,0}$ and $\psi_{\alpha\beta}$, $\psi_{\dot\alpha\dot\beta}$ insertions form the 32 modes of metric and self-dual five-form fluctuations. This is compatible with the results of Secs. II.C and IV.C.2 These $n=0$ operators are descendents of chiral primaries (they are in fact 1/4 BPS), and hence we expect them to be exact eigenstates of $\mathcal{D}-\mathcal{J}$ with $\Delta-J=2$.

We may also build double-trace operators with $\Delta_0-J=2$. Clearly there are two possibilities, a combination of Eq. (5.8)-type operators and Eq. (5.13)-type or two Eq. (5.11)-type operators:

$$\mathcal{T}^{J,r}_{ij,n}=:\mathcal{O}^{r\cdot J}_{ij,n}\mathcal{O}^{(1-r)\cdot J}:,$$
$$\mathcal{T}^{J,r}_{ij}=:\mathcal{O}^{r\cdot J}_{i}\mathcal{O}^{(1-r)\cdot J}_{j}:. \tag{5.19}$$

These operators are conjectured to correspond to double-string states. As we shall see in Sec. VII, once the string coupling is turned on and we have the possibility of strings joining and splitting, because of operator mixing effects, there is a mixture of single-trace, double-trace, and multitrace operators which correspond to string states diagonalizing the string field-theory Hamiltonian. We remind the reader that, as stated in Sec. I.A, string loop diagrams correspond to nonplanar graphs in the gauge theory.

Finally we would also like to note that the set of BMN operators we have introduced in this subsection is invariant under the action of $\mathbb{Z}_2$ which exchanges the two SO(4) factors. One can similarly extend the above construction to BMN operators with an arbitrary number of impurities.

VI. SPECTRUM OF STRINGS ON PLANE WAVES FROM GAUGE THEORY I: FREE STRINGS

In this section we consider planar results in the $\mathcal{N}=4$ gauge theory, which, according to the BMN correspondence, should connect with the string theory side at zero string coupling. Higher-genus corrections will be postponed until Sec. VII, where a new complication arising from the need to rediagonalize the basis of BMN operators, at each order in the genus expansion, will be discussed. We start this section by studying the two-point functions of BMN operators with their conjugates, in the free-field-theory limit, and use the results to set the normalization of these operators. We then move on to discuss the quantum corrections to the scaling dimensions, i.e., the anomalous dimensions. We first present a very brief but general overview of the scaling behavior of correlation functions and the appearance of anomalous dimensions through the renormalization-group equation. While this discussion provides the physical context in which anomalous dimensions are normally encountered in quantum field theory, the main point of this section is the actual calculation of anomalous dimensions in the interacting theory at planar one-loop level. An important concept in the renormalization of compos-

[11]The operators we have presented here are given in the BMN ($J\to\infty$) limit. However, as discussed by Beisert (2003), there is a generalization of such operators for finite J, based on supersymmetry. The form of such operators is slightly different from that of the BMN ones, differing in the Fourier phase factor, where in $2\pi inp/J$, J should be replaced with $J+3$. These operators are in fact "generalized" Konishi operators, interpolating between the usual Konishi operators (at $J=0$ or 1) (Konishi, 1984) and $J=\infty$, the BMN operators.

ite operators, operator mixing, appears when loop corrections are taken account of. Operator mixing, together with the requirement that BMN operators have a well-defined scaling dimension, are used to motivate the choice of BMN operators. As a stringent test of the BMN correspondence, we compare the calculations of the corrected scaling dimensions to the masses on the string theory side and find agreement.

Another key point of this section is the appearance of the new modified 't Hooft coupling λ' [Eq. (1.11)], which will first be seen when taking the BMN limit of the one-loop anomalous dimension.

A. Normalization of BMN operators

The propagator for the scalars in the $\mathcal{N}=4$ supermultiplet, which transform in the adjoint of U(N), are

$$\langle \phi_i^{ab}(x)\phi_j^{cd}(0)\rangle_0 = \frac{g_{\rm YM}^2\delta_{ij}}{8\pi^2|x|^2}\delta^{ad}\delta^{bc}, \tag{6.1}$$

where we explicitly display the matrix indices on the fields. We denote correlation functions in the free theory with a subscript 0, as above. With the convention (5.3) for the fields carrying the $\mathrm{U}(1)_J$ charge, the propagator for them is

$$\langle Z^{ab}(x)(Z^\dagger)^{cd}(0)\rangle_0 = \frac{g_{\rm YM}^2}{8\pi^2|x|^2}\delta^{ad}\delta^{bc}. \tag{6.2}$$

Using these propagators, we can demonstrate a set of rules which facilitate the evaluation of correlation functions involving traces over algebra-valued fields (which we denote by Tr). We assume that the composite operators we work with are normal ordered, so no contractions between fields in the same operator (i.e., at the same spacetime point) will appear. Such contractions would lead to infinite renormalizations of the operator. We start with the simplest such structures, evaluated in the free theory. We have the following fission rules:

$$\mathrm{Tr}[:\phi_i\mathcal{A}::\phi_j\mathcal{B}:] \sim \delta_{ij}:\mathrm{Tr}[\mathcal{A}]::\mathrm{Tr}[\mathcal{B}]:,$$

$$\mathrm{Tr}[:\phi_i::\phi_j\mathcal{A}:] \sim \delta_{ij}N:\mathrm{Tr}[\mathcal{A}]:, \tag{6.3}$$

where for clarity we have dropped some obvious prefactors arising from the propagators, remembering that the rank of U(N) is N. Clearly the second identity is a special case of the first (with one of the operators taken to be the identity matrix in the space of color indices). We have explicitly kept the normal-ordering symbols here for clarity. Caution must be used when applying these rules not to allow contractions between fields at the same spacetime point (appearing in the same normal-ordering). In the second identity, we can take $\mathcal{A}=1$, which gives $\mathrm{Tr}[\phi_i\phi_j]\sim\delta_{ij}N^2$. We also have the fusion rule,

$$:\mathrm{Tr}[\phi_i\mathcal{A}]::\mathrm{Tr}[\phi_j\mathcal{B}]: \sim \delta_{ij}\mathrm{Tr}[:\mathcal{A}::\mathcal{B}:]. \tag{6.4}$$

In the future, we shall drop the normal-ordering symbol, but all calculations are implicitly assumed to account for their presence.

Consider now the normalization of the operator (5.14) in the free theory and at planar level. We assume that the vacuum of the theory leaves the SU(4) R symmetry unbroken, as is the case for the superconformal points in the moduli space of $\mathcal{N}=4$ SYM. The correlation function of any set of operators then vanishes if they do not form an SU(4) singlet.

Keeping the planar contributions, as is usual with 't Hooft expansions, amounts to keeping the leading-order contribution in $1/N^2$. The normalization of the operator $\tilde{\mathcal{O}}$ (5.15) is fixed by requiring $|x|^{2(J+1)}\times\langle\tilde{\mathcal{O}}_p(x)\tilde{\mathcal{O}}_p(0)\rangle_0=1$ at planar level. The two-point functions provide a natural notion of an inner product on the space of BMN operators, and in the BMN correspondence are the analog of the inner product between string states (see discussions of Sec. V.C.3).

We work with ($g_{\rm YM}=0$) free theory and use Wick contractions to write the correlation function as sums of products of scalar propagators. We first write out the traces explicitly,

$$\begin{aligned}\langle\tilde{\mathcal{O}}_p(x)\bar{\tilde{\mathcal{O}}}_q(0)\rangle_0 &= \langle\mathrm{Tr}(Z^p\phi_jZ^{J-p}\phi_i)(x)\\ &\quad\times\mathrm{Tr}(\phi_i\bar{Z}^{J-q}\phi_j\bar{Z}^q)(0)\rangle\\ &= \langle[Z^p_{ab}(\phi_j)_{bc}Z^{J-p}_{cd}(\phi_i)_{da}](x)\\ &\quad\times[(\phi_i)_{ef}\bar{Z}^{J-q}_{fg}(\phi_j)_{gh}\bar{Z}^q_{he})](0)_0,\end{aligned} \tag{6.5}$$

having used the cyclicity of the trace and defining $\bar{Z}\equiv Z^\dagger$. A sum over repeated U(N) color indices $a\cdots h$ is implied. The normal-ordering symbols can be safely dropped in this correlation function if we assume that $i\neq j$ (since then ϕ's at the same point could not be contracted, as is also the case for the Z's and $Z^\dagger$'s). We shall make this assumption since it also simplifies some of the combinatorics. Repeatedly taking Wick contractions on the ϕ's and Z's that are nearest to each other using Eqs. (6.1) and (6.2), we arrive at

$$\langle\tilde{\mathcal{O}}_p(x)\bar{\tilde{\mathcal{O}}}_q(0)\rangle_0 = \left(\frac{g_{\rm YM}^2N}{8\pi^2|x|^2}\right)^{J+2}\delta_{p,q}. \tag{6.6}$$

The requirement that these operators be normalized as $|x|^{2(J+1)}\langle\tilde{\mathcal{O}}_p(x)\bar{\tilde{\mathcal{O}}}_q(0)\rangle=\delta_{p,q}$ can be satisfied by taking $\tilde{\mathcal{O}}_p\to(8\pi^2/g_{\rm YM}^2N)^{(J+2)/2}\tilde{\mathcal{O}}_p$. Similar reasoning gives the normalization of the other BMN operators. For example, the normalization of the BMN operator with $\Delta_0-J=2$ in Eq. (5.13) is fixed by the normalization we have just considered, but an extra factor of $1/\sqrt{J+1}$ enters from the $J+1$ terms appearing in the sum.

B. Anomalous dimensions

In a conformal field theory such as $\mathcal{N}=4$ super Yang-Mills, the content of the theory can be extracted via the correlation functions of gauge-invariant operators and is embodied in their scaling dimensions, how they mix amongst each other under renormalization, and the coefficients in their operator-product expansions. We now present a brief overview of the first two topics, leaving the discussion of the operator-product expansion for a later section. Two discussions of these points in general quantum field theory are those of Zinn-Justin (1989) and Peskin and Schroeder (1995).

We consider a bare correlation function built of n bare fields ϕ_b and the renormalized correlation function, built in the same way, but using renormalized fields,[12]

$$\Gamma_n^{(\mathrm{bare})}(\{x_i\},\lambda^{(\mathrm{bare})},\Lambda) = \langle \phi^{(\mathrm{bare})}(x_1)\cdots\phi^{(\mathrm{bare})}(x_n)\rangle,$$

$$\Gamma_n^{(\mathrm{ren})}(\{x_i\},\lambda^{(\mathrm{ren})},\mu) = \langle \phi^{(\mathrm{ren})}(x_1)\cdots\phi^{(\mathrm{ren})}(x_n)\rangle. \tag{6.7}$$

The bare correlation functions depend implicitly on a set of bare parameters defined at the cutoff scale Λ of the theory, while the renormalized ones depend on the renormalized parameters defined at the renormalization scale μ. The renormalized fields are proportional to the bare fields, via the wave-function renormalization, $\phi^{(\mathrm{ren})}(x)=Z_\phi^{-1/2}(\mu)\phi^{(\mathrm{bare})}(x)$. The dependence of the field strength of the renormalized field on the renormalization scale μ is the source of the anomalous dimension.

A simple consequence of Eq. (6.7) is that the bare and renormalized n-point functions are related by powers of the wave-function renormalization,

$$\Gamma_n^{(\mathrm{ren})}(\{x_i\},\lambda^{(\mathrm{ren})},\mu) = Z_\phi^{-n/2}(\mu)\Gamma_n^{(\mathrm{bare})}(\{x_i\},\lambda^{(\mathrm{bare})},\Lambda). \tag{6.8}$$

The renormalization scale dependence enters the renormalized n-point function via the wave-function renormalization Z_ϕ and the renormalized parameters $\lambda^{(\mathrm{ren})}$ of the theory, which are defined at that scale, but not the bare n-point functions, hence

$$\frac{\partial}{\partial \ln \mu}\Gamma_n^{(\mathrm{bare})}(\{x_i\},\lambda^{(\mathrm{bare})},\Lambda) = 0. \tag{6.9}$$

The chain rule then gives

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\lambda^{(\mathrm{ren})}\frac{\partial}{\partial\lambda^{(\mathrm{ren})}} + n\gamma\lambda^{(\mathrm{ren})}\right)\Gamma_n^{(\mathrm{ren})}(\lambda^{(\mathrm{ren})},\mu) = 0. \tag{6.10}$$

For a single coupling massless theory (like $\mathcal{N}=4$ SYM), we have written this relation in terms of the dimensionless functions β and γ, which take account of shifts in the field strength, and coupling constants that compensate for changes in the renormalization scale to keep the bare correlation functions constant. They are defined as[13]

$$\beta(\lambda^{(\mathrm{ren})}) = \left.\mu\frac{\partial\lambda^{(\mathrm{ren})}(\mu)}{\partial\mu}\right|_{\lambda^{(\mathrm{bare})}},$$

$$\gamma(\lambda^{(\mathrm{ren})}) = \mu\frac{\partial \ln Z_\phi(\mu)}{\partial\mu}{}_{\lambda^{(\mathrm{bare})}}. \tag{6.11}$$

For a small change in the renormalization scale $\mu\to\mu+\delta\mu$, as a result of which the coupling and fields change as $\lambda\to\lambda+\delta\lambda$ and $\phi\to(1+\delta\eta)\phi$, the change in the field strength is related to the anomalous dimension via $\delta\eta=(\delta\mu/2\mu)\gamma$.

The renormalization-group equation (6.10) is a highly nontrivial statement about the behavior of correlation functions in a quantum field theory, with deep implications (for example, the running of couplings and masses). The scale dependence introduced into the renormalized theory in the guise of the renormalization scale μ generically breaks any classical scale invariance that might be present in a massless theory with dimensionless couplings. However, there may exist fixed points of the renormalization group (special values of the parameters λ_*) at which the β function vanishes.[14] At these fixed points, the classical scale invariance of the renormalized theory is restored. The classical scaling of the fields and the correlation functions might be modified by the presence of anomalous scaling dimensions, with the scaling dimension of the field becoming $\Delta=\Delta_0+\gamma$. Unlike the classical scaling dimensions, the anomalous dimensions may take on a continuum of values, which, however are constrained by the conformal algebra.[15] At a fixed point, the behavior of the correlation functions reflects the dependence on the nontrivial scaling:

$$\Gamma_n^{(\mathrm{ren})}(sx_i,\lambda_*,s^{-1}\mu) = s^{-n\Delta}\Gamma_n^{(\mathrm{ren})}(x_i,\lambda_*,\mu). \tag{6.12}$$

We shall encounter composite operators which are local monomial products of fields. The process of renormalization of a given composite operator might generate new divergences which are proportional to other composite operators, requiring their introduction as counterterms, and leading to a mixing of operators under renormalization. In general a composite operator may mix under renormalization with any operator of equal or lower dimension that carries the same quantum numbers. For a massless theory with no dimensionful parameters, only operators of the same classical dimension mix. If we choose as a basis for these local gauge-invariant operators a set, which we shall label $\{\mathcal{O}_i\}$, then multiplicative renormalization occurs in the form of matrix multiplication,

[12]Generally, a number of different types of fields may enter into a correlation function: however, we are most concerned with the scaling behavior of such correlators, and for the $\mathcal{N}=4$ SYM theory of interest to us, supersymmetry implies that all fields in a supermultiplet receive the same anomalous dimensions. We therefore simplify our notation and write only one type of field.

[13]The β function and anomalous dimension γ are universal in the sense that they are the same for all correlation functions in a given renormalizable theory.

[14]There is of course always the trivial fixed point for which the couplings vanish, and hence so do the anomalous dimensions. For $\mathcal{N}=4$ SYM, there is in fact a line of fixed points, and the β function vanishes at all values of the couplings.

[15]For example, for unitary representations, the dimensions are bounded from below (Minwalla, 1998). Also, as a result of supersymmetry, in $\mathcal{N}=4$ SYM, all fields in the same $\mathcal{N}=4$ multiplet receive the same anomalous dimensions.

$$\mathcal{O}_i^{(\mathrm{bare})}(x) = \sum_j Z_{ij}\mathcal{O}_j^{(\mathrm{ren})}(x). \tag{6.13}$$

The statement that the operator mixes only with those of lower or equal classical dimensions implies that the matrix Z_{ij} can be cast in triangular form when the basis is arranged in order of dimensions of the operators. Correlation functions with insertions of composite operators also satisfy a renormalization-group equation, generalizing Eq. (6.10), with a new anomalous-dimension matrix,

$$\gamma_{ij}(\lambda^{(\mathrm{ren})}) = \mu\frac{\partial \ln Z_{ij}(\mu)}{\partial \mu}_{\lambda^{(\mathrm{bare})}}. \tag{6.14}$$

A few final comments are in order concerning general properties of conformal field theories, which clarify some of the points we shall encounter in later sections. It is believed that unitary interacting scale-invariant quantum field theories generally exhibit a larger symmetry containing scale invariance, the group of conformal transformations. Conformal invariance turns out to be restrictive enough to completely fix the dependence of two- and three-point functions on the spacetime coordinates (in a suitable basis); those of higher point functions, while not completely fixed, are restricted by the requirement that they depend on certain special combinations of the coordinates, the conformal ratios (Di Francesco *et al.*, 1997). In a unitary conformally invariant quantum field theory, we can choose a basis of operators with definite scaling dimensions (eigenstates of the dilatation operator). These are the quasiprimary operators. In each multiplet of the conformal (or superconformal) algebra, the operators of lowest dimension[16] are the conformal (or superconformal) primaries. Two quasiprimary operators are correlated if and only if they have the same scaling dimensions, and the two-point correlation function takes the form (dropping normalization factors)

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{\Delta_i,\Delta_j}}{|x_{12}|^{2\Delta_i}}, \tag{6.15}$$

with $x_{12}\equiv x_1-x_2$. Here Δ_i is the full (engineering plus anomalous) scaling dimension of the operator $\mathcal{O}_i$. The three-point functions are similarly constrained and satisfy (Di Francesco *et al.*, 1997)

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{\Delta_i,\Delta_j,\Delta_k}(g_{\mathrm{YM}}^2,N)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}. \tag{6.16}$$

For the two-point functions, quantum corrections can enter only through anomalous dimensions for the operator, while for three-point functions there is the more general possibility that the coefficient $C_{\Delta_i,\Delta_j,\Delta_k}(g_{YM}^2,N)$ may also receive corrections at higher loops. When computing the anomalous dimension of an operator in perturbation theory, we have a power-series expansion $\gamma=\gamma_1+\gamma_2+\cdots$, and γ_n includes the nth power of the 't Hooft coupling λ^n. The dependence of the two-point function on the positions of the operators, when computed in perturbation theory, will take the form

$$\frac{1}{|x|^{2\Delta}} \approx \frac{\mu^{2\gamma_1}}{|x|^{2\Delta_0}}(1-\gamma_1\ln|x\mu|^2) \tag{6.17}$$

to one-loop order, with the renormalization scale entering to keep the argument of the log dimensionless. This approximation is valid so long as $\gamma_1 \ll \ln(x\mu)^{-2}$. While this expression suggests that scale invariance has been broken, the scale μ will drop out when it is re-summed to all orders in perturbation theory, to reproduce the left-hand side of the expression. The scale μ is merely an artifact of perturbation theory.

In the next two sections we move on to a practical calculation of the anomalous dimension of composite BMN operators, first at one loop, and then to all orders in perturbation theory.

C. Anomalous dimensions of the BMN operators, first order in g_{YM}^2

The goal of this section is to compute the anomalous dimension of a class of BMN operators to first-loop order on the gauge theory side and to compare the result to the appropriate computation of the string theory masses. This will provide a first check of the BMN correspondence stated in Sec. V. In this section we concentrate on anomalous dimensions only at the planar level; we shall revisit the issue at nonplanar level in Sec. VII.B.2.

Consider a local gauge-invariant operator of the form

$$\tilde{\mathcal{O}}_p^J(x) = \mathrm{Tr}[\phi_i Z^p \phi_j Z^{J-p}(x)] \tag{6.18}$$

with engineering dimension $\Delta_0=J+2$. Such a generic operator would not remain an eigenstate of the dilatation operator after renormalization, as a result of the operator mixing discussed in the previous section, and would therefore not have a well-defined scaling dimension. This means that after computing loop corrections, even at planar level, the two-point function of the operators (6.18) would not remain diagonal. Quantum effects induce a mixing with operators in the same SU(4) representation with Dynkin labels $(2,J-2,2)$ and the same engineering dimension. Any operator of the form (6.18) for any $0\leqslant p\leqslant J$ satisfies the mixing criteria, and in general any operators with this charge and dimension would take the form (6.18) for some p. The interaction term in the Hamiltonian which connects only the scalars takes the form $H_{\mathrm{int}}\sim :g_{\mathrm{YM}}^2\Sigma_{ij}\mathrm{Tr}([\phi_i,\phi_j][\phi^i,\phi^j]):$.[17] It therefore

[16]These are operators that are annihilated by the generator of special conformal transformations (or the superconformal supercharges).

[17]We work with a normal-ordered Hamiltonian, which amounts to discarding all self-contractions in a given insertion of the Hamiltonian in perturbation theory. Contractions across different insertions are not removed by normal ordering.

contains terms that exchange the order of $U(1)_J$ charged and neutral fields ϕ_i and Z, and also two different ϕ_i fields. We shall refer to such exchanges as "hopping." The interactions mix operators $\tilde{\mathcal{O}}_p^J$ and $\tilde{\mathcal{O}}_q^J$, both of the form (6.18), but with $p\neq q$. For each additional loop, the mixing would extend to operators with the insertions of the impurities shifted by one more position. For example, at one-loop order, these interactions generate diagrams with no hopping, and those which have one hop, either forward or backward.

Since the operators (6.18) do not have well-defined scaling dimensions, they cannot be put in a simple correspondence with the string theory side of the BMN conjecture. One of the main goals of this section will be to construct operators with well-defined renormalized scaling dimensions at planar level, and hence diagonal two-point functions, which can be put in one-to-one correspondence with string theory objects.

In the free theory ($g_{\rm YM}=0$), operators with $p\neq q$ do not mix at planar level, and the two-point function remains diagonal. We can write the nondiagonal contributions at higher-orders as

$$\langle \tilde{\mathcal{O}}_p^J(x)\tilde{\mathcal{O}}_q^J(y)\rangle \propto \sum_{l=0}^{\infty}\lambda^l M_{p,q}^{(l)}(x-y)\equiv M_{p,q}(x-y), \tag{6.19}$$

where we have dropped proportionality constants coming from the normalization of the tree-level two-point function. The zeroth-order term is simply the identity $M_{p,q}^{(0)}=\delta_{p,q}$. The matrices $M_{p,q}^{(l)}$ are proportional to lth powers of logs of the separation $(x-y)$ of the two operators, $M_{p,q}^{(l)}(x)=[\ln(x\mu)^2]^l\mathcal{M}_{p,q}^{(l)}$, coming from perturbation theory at l loops. The matrices $\mathcal{M}_{p,q}^{(l)}$ are symmetric in p,q, because for each insertion of the Hamiltonian which generates a hop to the right, there is one generating a hop to the left. The hopping can be exhibited more explicitly by separating $\mathcal{M}_{p,q}^{(l)}$ into "hopping" matrices $\mathfrak{m}_j^{(l)}$,

$$\mathcal{M}_{p,q}^{(l)}=\sum_{j=-l}^{l}\delta_{p,q+j}\mathfrak{m}_j^{(l)}, \tag{6.20}$$

with the interpretation that $\mathfrak{m}_j^{(l)}$ captures all the effects at loop l coming from j hops (j can be positive or negative), and $\mathfrak{m}_j^{(l)}=\mathfrak{m}_{-j}^{(l)}$ because forward and backward hops are governed by essentially the same term in the Hamiltonian. We were able to extract a p- and q-independent term $\mathfrak{m}_j^{(l)}$ here because in the interaction Hamiltonian, the commutator terms which generate the various hops all enter with precisely the same coefficient. Equation (6.20) makes it explicit that the range of allowed hops is set by the number of loops (or insertions of the Hamiltonian) which are included, a point we noted earlier. Using Eq. (6.17), we can read the l-loop anomalous dimensions directly from $\mathcal{M}_{p,q}^{(l)}$.

The sum (form all $p=0,\dots,J$) of the operators in (6.18) is protected by a BPS condition, and this gives the relation among the coefficients

$$\sum_{p=0}^{J}\mathcal{M}_{p,q}^{(l)}=0,\quad \forall\, l,q>0. \tag{6.21}$$

As mentioned previously, to specify precisely the correspondences between the gauge theory and string theory sides of the duality, we need to find a basis of operators with well-defined scaling dimensions. Such a basis would contain operators formed as linear combinations of the above,

$$\mathcal{O}_n^J(x)=\sum_{p=0}^{J}\mathcal{F}_{np}(J)\tilde{\mathcal{O}}_p^J(x), \tag{6.22}$$

for some $\mathcal{F}$ to be determined by the condition that Eq. (6.22) have a well-defined scaling dimension. We can think of $\mathcal{F}$ as a change of basis on the vector space of operators $\tilde{\mathcal{O}}_p^J$. We also impose an additional constraint on the expansion coefficients, requiring $\mathcal{F}_{0p}(J)=1$, which is another statement of the BPS condition.

A few comments are in order regarding the range of the summation in Eq. (6.22). The end points $p=0$ and $p=J$ correspond, for $i\neq j$ in (6.18), to the case where the positions of ϕ_i and ϕ_j are reversed. Both orderings must be included since the interaction Hamiltonian will generate such exchanges, and in principle these terms can mix with each other. In addition, for the BPS condition to hold when $n=0$ in Eq. (6.22), the summation must include both arrangements. Lastly, if we drop either of $p=0$ or $p=J$, we will compute an anomalous dimension with a finite piece in the BMN limit, and one that scales as λ and hence diverges in the double scaling limit. The divergent piece is exactly canceled when the missing term is included (Constable *et al.*, 2002; Kristjansen *et al.*, 2002).

We are now ready to determine the form of the matrix $\mathcal{F}_{np}(J)$, which at each order in perturbation theory acts on the operators (6.18), after which the transformed operators are diagonal, and hence their two-point functions have perturbative expansions in λ of the form

$$\langle \mathcal{O}_m^J(x)\bar{\mathcal{O}}_n^J(y)\rangle=\delta_{m,n}\sum_{l=0}^{\infty}\lambda^l f_l(x-y)\equiv\delta_{m,n}f_m(x-y) \tag{6.23}$$

and $f_m(x-y)$ can be different for each $\mathcal{O}_m^J$. We have the similarity transformation

$$(\mathcal{F}M\mathcal{F}^{\dagger})_{m,n}=\delta_{m,n}f_m(x-y) \tag{6.24}$$

with $\mathcal{F}$ admitting a power-series expansion in J. A suitable, though not unique, choice for $\mathcal{F}$ is

$$\mathcal{F}_{np}=e^{2\pi inp/J}, \tag{6.25}$$

which diagonalizes the above operators up to order $\mathcal{O}(1/J^2)$ for any order in perturbation theory, where the λ dependence appears in f_m. In the BMN limit where J

$\to\infty$, the correction terms vanish and the diagonalization, at planar level, is exact. Note that at planar level, the quantum corrections do not induce mixing between operators with different numbers of traces. When we come to consider the nonplanar corrections in Sec. VII, this lack of mixing will no longer be the case, and the mixing between operators with different numbers of traces will have to be dealt with also. In fact, even in the free theory, the single-trace BMN operators will mix among themselves at nonplanar level. The significance of this second type of mixing and its role in the duality will be the central theme of Sec. VII.

The statement that $\mathcal{O}_n^J$ has a well-defined scaling dimension can be translated into the requirement that, after renormalization, the bare and renormalized quantities be related by an overall scaling and not a matrix that connects it to other operators as in Eq. (6.13). Then,

$$\mathcal{O}_n^{J(\mathrm{bare})} = \mathcal{Z}_n(\lambda,\mu)\mathcal{O}_n^{J(\mathrm{ren})}, \tag{6.26}$$

with the renormalization constant generically a function of the coupling (going to the identity for $\lambda=0$), and the renormalization scale μ, or alternatively $\epsilon=2-D/2$ in dimensional regularization. The rescaling $\mathcal{Z}$ depends on the composite operator renormalization $Z_{\mathcal{O}}$ of the operator $\mathcal{O}_n^J$ in addition to the usual wave-function renormalizations Z_Z and Z_ϕ for the fields Z and ϕ, and takes the form

$$\mathcal{Z}_n = Z_{\mathcal{O}_n} Z_\phi (Z_Z)^{J/2}, \tag{6.27}$$

since there are J fields charged under the $\mathrm{U}(1)_J$ and two neutral fields.

The anomalous dimension γ_n of the operator $\mathcal{O}_n$ can be computed order by order in perturbation theory, and has a power-series expansion in the 't Hooft coupling λ,

$$\gamma_n(\lambda) = \sum_{l=1}^{\infty} \lambda^l c_l^{(n)}, \tag{6.28}$$

where the $l=0$ term vanishes since the anomalous dimension appears as a quantum correction to the classical scaling dimension. The coefficients of this expansion can be Fourier transformed,

$$c_l^{(n)} = \sum_{h=-l}^{l} c_{l,h}^{(n)} e^{-2\pi i n h/J}, \tag{6.29}$$

with a natural interpretation that, as we shall see below, $c_{l,h}^{(n)}$ represents the portion of the anomalous dimension of the operator $\mathcal{O}_n$ that arises at loop l, from the sum of diagrams with h hops. By convention, positive h will correspond to hops to the right. We now compute the Fourier coefficients $c_{l,h}^{(n)}$ at one loop, working at the planar level.

Our goal is to compute the counterterms necessary to absorb the divergences generated by insertion of the composite operator (6.22) and the wave-function renormalizations of the Z and ϕ scalar fields, and use these to derive the anomalous dimension of the composite operator, via Eq. (6.27). Here we shall only consider BMN operators with two *nonidentical* scalar impurities, which can be in $(\mathbf{9},\mathbf{1})$ or $(\mathbf{3}^{\pm},\mathbf{1})$ $\mathrm{SO}(4)\times\mathrm{SO}(4)$ representations [the explicit calculations regarding the singlet case $(\mathbf{1},\mathbf{1})$ have been made by Gomis *et al.* (2003b)]. All the other BMN operators should have the same anomalous dimensions, due to the supersymmetry (see Sec. V.C.3). We work in position space and use dimensional regularization. In dimensional regularization, the anomalous dimension becomes

$$\gamma(\lambda) = \epsilon \frac{\lambda}{\mathcal{Z}_n}\frac{\partial \mathcal{Z}_n}{\partial \lambda}. \tag{6.30}$$

Consider a two-point function of the operators (6.22) with the choice (6.25) for the diagonalizing matrix. We can expand this correlation function of sums of operators into a double sum of correlation functions of individual composite operators. In a generic quantum field theory, a correlation function of ordinary operators with an insertion of a single composite operator has divergences which can be removed, in addition to the usual counterterms, with a wave-function renormalization of the composite operator. Insertions of additional composite operators will in general produce additional divergences requiring subtractions. However, for a conformal field theory, the form of the two-point function is fixed, as shown in Eq. (6.15), and the wave-function renormalization (6.26) suffices to absorb all divergences coming from the composite operators.

The correlation function will then include the overlap of all operators of the form (6.18) with the appropriate exponential factors, in other words, the sum of the correlators of all pairs of operators (6.18), with some exponential coefficient. At one-loop order, where we have a single insertion of the interaction Hamiltonian, there will in general be two classes of diagrams: (i) those in which the correlator receives contributions from two-point functions with the same p, corresponding to diagrams with no exchange of ϕ and Z, and (ii) those in which one exchange of ϕ and Z takes place. The corresponding diagrams are presented in Fig. 2.

The first diagram arises from contractions where one ϕ field has "hopped" past a Z field, in this case to the left. The exponential factor appearing in front of this term is $\exp(2\pi i n/J)$, since the amplitude for this term is

$$e^{2\pi i p n/J} e^{-2\pi i (p-1)n/J} \langle \tilde{\mathcal{O}}_p^J(x) \tilde{\mathcal{O}}_{p-1}^J(y) \rangle. \tag{6.31}$$

There will also be a contribution from a diagram in which a ϕ field hops to the right, and it will be associated with a factor $\exp(-2\pi i n/J)$, with the amplitude otherwise the same.

We shall now compute the amplitude for this diagram at planar level, but only keep track of the divergent parts which determine the counterterm structure and eventually the anomalous dimension. In position space, this diagram consists of $J+2$ fields located at spacetime position x, interacting with $J+2$ fields located at y. The divergence arises from the loop at the center of the diagram, which corresponds to the integration over all

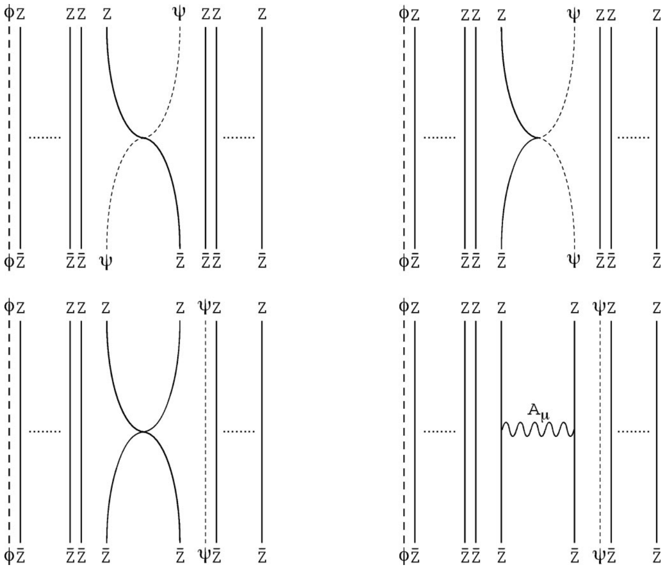

FIG. 2. Feynman diagrams containing a single insertion of the interaction Hamiltonian. The two different impurities are labeled ϕ and ψ.

spacetime (i.e., $\int d^4w$) of one insertion of the Hamiltonian and four propagators. The loop integral will contribute, beyond the tree-level result,

$$\lambda e^{2\pi i n/J}\int \frac{d^D w}{64\pi^4}\frac{1}{|w-x|^4|w-y|^4} \sim \frac{1}{16\pi^2|x-y|^4}\frac{\lambda}{\epsilon}e^{2\pi i n/J}, \tag{6.32}$$

where we have continued to $D=4-2\epsilon$ dimensions to regulate the ultraviolet divergence coming from $x\to w$ and $y\to w$, which now appears as a pole in ϵ. We have the 't Hooft coupling appearing here because a factor of $g_{\rm YM}^2$ combines with a factor of N at planar level when the first contractions across the traces are taken. We drop the contributions from the part of the diagram outside the interaction, since these do not modify the counterterm structure we are seeking. We see the appearance of the combination λ/ϵ appropriate to one-loop order. We ignore the issue of infrared divergences when the external momenta vanish; these do not affect the anomalous dimension.

There are also diagrams in which the ϕ and Z fields interact, but which nonetheless do not lead to hopping. The hopless diagrams in which ϕ and Z fields interact arise in two ways. The first such diagram is similar to the one we considered above, but with a different ordering of the fields in the interaction term. There is also a diagram in which the interaction between the scalar ϕ and Z fields is due to gluon exchange. These two diagrams contain the same divergences in their loops, but with opposite sign, and so their sum is finite. We ignore finite contributions since they do not give rise to anomalous dimensions.

The action (A1) contains an interaction term in which only Z fields interact with each other and a term in which Z fields interact with gluons and clearly lead to no hopping. Such interactions give rise to diagrams in which the four scalar Z fields interact directly and diagrams in which their interaction is a result of gluon exchange. Both these diagrams contribute equal divergences with the same sign. The divergence part of these is the same as in Eq. (6.32), but since there is no hopping, the exponential prefactor is missing. At planar level, there are $J-2$ possible ways the Z fields can interact among each other.

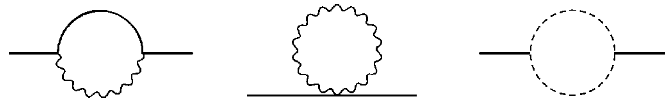

FIG. 3. Diagrams contributing to the scalar wave-function renormalization at one loop. The first is a scalar tadpole, the second a gauge-boson loop, and the third a fermion loop.

The ultraviolet divergences in these diagrams can be removed by the addition of counterterms to the action to absorb the divergences. Computing the correlation function above, to one-loop order, with an insertion of the composite operator, and including the counterterms appropriate to this order, we find the finite renormalized $Z_{\mathcal{O}}$, whose value is

$$Z_{\mathcal{O}_n} = 1 - \frac{\lambda}{8\pi^2\epsilon}[2e^{2\pi i n/J} + 2e^{-2\pi i n/J} + (J-2)], \tag{6.33}$$

where the first two terms absorb the divergences from the diagrams with one hop to the left or right, respectively, with a factor of 2 multiplying the exponential due to the hopping, since we are considering composite operators with two impurities. The last term absorbs the divergences from the two diagrams that do not result in a hop, come from the interactions of Z fields alone, and contribute $J-2$ such counterterms.

We are now almost ready to compute the anomalous dimension of the operator $\mathcal{O}_n$. The only remaining piece left to compute is the wave-function renormalizations of the individual fields which enter into the correlation function, as seen in Eq. (6.27). There are three types of diagrams which modify the scalar propagators at one loop. The wave-function renormalizations, which are the only kind of renormalization to the bare $\mathcal{N}=4$ propagators, are generated by diagrams in which a closed loop is constructed as in Fig. 3, and arise from gauge boson, fermion, and scalar loops. Their computation is straightforward, and the resultant one-loop wave-function renormalization is

$$Z_\phi = Z_Z = 1 + \frac{1}{4\pi^2}\frac{\lambda}{\epsilon}. \tag{6.34}$$

The factor of λ arises because there are two interaction vertices, each contributing $g_{\rm YM}$, and a factor of N enters due to the traces over color indices from the closed loop. Putting these together we have to first-loop order

$$\mathcal{Z}_n = 1 - \frac{\lambda}{4\pi^2\epsilon}(e^{2\pi i n/J} + e^{-2\pi i n/J} - 2), \tag{6.35}$$

which yields the anomalous dimension

$$\gamma_n = -\frac{\lambda}{4\pi^2}(e^{2\pi i n/J} + e^{-2\pi i n/J} - 2). \tag{6.36}$$

The 2 is a direct result of supersymmetry, since, for $n=0$, we have a BPS operator, which is protected against receiving quantum corrections. This is a manifestation of the BPS condition in the form (6.21). Incidentally, we can decompose this result into the $c_{l,h}^{(n)}$ we met in Eq. (6.29), whence $c_{1,1}^{(n)}=c_{1,-1}^{(n)}=1$ and $c_{1,0}^{(k)}=-2$.

We mention also that, had we separated the interaction Lagrangian into F and D terms, at first loop we would have found that only the F terms contributed, and the sum of all the diagrams with insertions of D terms would have vanished for two- and three-point functions (Constable *et al.*, 2002; Kristjansen *et al.*, 2002).

The anomalous dimension in Eq. (6.36) has been computed for finite J. In the BMN limit, when J is taken large, the anomalous dimension becomes

$$\gamma_n = n^2\lambda' \tag{6.37}$$

and we see explicitly the appearance of the new effective coupling $\lambda'=\lambda/J^2$ because the 't Hooft coupling $\lambda=g_{\rm YM}^2N$ has combined with a $1/J^2$ from the expansion of the exponentials. We see that in the BMN limit the anomalous dimensions of BMN operators are finite, since $g_{\rm YM}^2$ is held fixed while N and J are scaled such that λ' remains finite. Contrast this with a normal 't Hooft expansion, in which the expansion parameter is λ, and this diverges in the BMN limit. This is a key result, since it tells us that in the double scaling limit BMN operators will have finite, and hence well-defined, scaling dimensions, which can be compared to the string side of the duality. Recall that the exponentials entered as the diagonalizing matrix transforming the original basis of operators (6.18) to one with well-defined scaling dimension, which we took to define one set of BMN operators. In turn, the precise structure of this matrix originated in the hopping behavior embodied in the interaction Hamiltonian.

We can compare this result for anomalous dimensions of single-trace operators with two impurities to the string theory calculation of the mass spectrum for single-string states (6.10), with excitations of the left- and right-moving oscillators at level n in the plane-wave background. As we discussed in Sec. V.B, the BMN correspondence states a relationship between the effective coupling in the gauge theory and in the BMN limit, and string theory parameters on the plane-wave background, which was stated in Eqs. (1.11)–(1.13). We noted earlier in Sec. II.B.1 that p^+ is a central charge of the supersymmetry algebra of the plane-wave background, since its generator commutes with all the other generators of the algebra. As such, its value specifies a sector of the string theory, unmixed by actions of the isometry or string interactions. This is in distinct contrast to flat space, where the light-cone boosts can change p^+. Therefore it makes sense to think of $\alpha'\mu p^+$ (or equally μ in a sector of fixed p^+) as an expansion parameter on the string side. The effective gauge theory expansion parameter λ' is related to the light-cone momentum, which is held fixed in the BMN double scaling limit, via Eq. (1.11). When the gauge theory is weakly coupled and λ' is small, the light-cone momentum $\alpha'\mu p^+$ is large. This implies that the tension term in the light-cone string theory action (4.6) dominates the gradient terms (since we have taken μ large), and the quantum mechanics of the string becomes that of a collection of massive particles. This has motivated the string bit model (Vaman and Verlinde, 2002; Verlinde, 2002). Under these conditions, the mass spectrum (4.10) can be expanded to first order, with the result that

$$\omega_n \approx \alpha'\mu p^+\left(2 + \frac{n^2}{(\alpha'\mu p^+)^2}\right). \tag{6.38}$$

We use the relation $\mathcal{D}-\mathcal{J}=\Delta_0+\gamma-J$, which for the operator we have considered gives $\mathcal{D}-\mathcal{J}=2+\gamma$. We then have $2\omega_n/\alpha'\mu p^+=\mathcal{D}-\mathcal{J}$, after using Eqs. (6.37) and (1.11). The comparison is valid so long as $\lambda'\ll 1/n^2$, or equivalently, when $|\alpha'\mu p^+|\gg n$. For any finite n, we are free to choose λ' or $\alpha'\mu p^+$ so that these conditions hold. This is our first direct test of the BMN conjecture, and it has passed with flying colors.

In this section, we computed anomalous dimensions of a class of BMN operators to first order in $g_{\rm YM}^2$, and found that it reproduces the string theory calculation, giving a first test of the BMN conjecture. We may wonder whether this result extends to higher loops. The investigation of this question will be the focus of the next section, and we shall show that the result indeed holds to all orders in perturbation theory.

The above one-loop results have been extended to all orders in λ' using superspace techniques (Santambrogio and Zanon, 2002; see also the ArXiv version of this review, Sadri and Sheikh-Jabbari, 2003b, for more details).

VII. STRINGS ON PLANE WAVES FROM GAUGE THEORY II: INTERACTING STRINGS

Having carefully considered the planar structure of BMN operators, we are now ready to move on and examine nonplanar corrections to the quantities we have been studying in Sec. VI, first considering higher-genus corrections to two-point correlation functions of chiral primary operators (these receive no loop, i.e., λ', corrections). The BMN limit of these correlators is examined, showing that in the double scaling limit, certain higher-genus contributions survive. This result distinguishes the BMN limit from the standard 't Hooft limit, wherein all contributions from higher-genus diagrams are seen to vanish. This consideration will demonstrate explicitly the appearance of the genus counting parameter in the BMN limit. We next look at correlators of BMN (near-BPS) operators, first in the free field theory limit but with first nonplanar contributions, and then after turning on interactions, computing the first nontrivial contributions in both the genus counting parameter and the modified 't Hooft coupling λ'. Mixing between single- and multiple-trace BMN operators, and the requisite rediagonalization of the basis, leading to the so-called "improved BMN operators," will play a central role in the precise formulation of the correspondence between gauge theory operators and string states. We collect the above results in an elegant form suggested by Constable *et al.* (2002a, 2002b; see also Gomis *et al.*, 2002; Gross *et al.*, 2003).

A. Nonplanar contributions to correlators of chiral primary operators

We review the expansion to all genera of the two-point functions of chiral primary operators, which are protected against quantum corrections by virtue of being BPS. Hence the results we present can be calculated in the free theory, but extend to all values of the coupling.

To gain some insight into the genus expansion, consider the simplest correlation function that receives contributions from higher-genus diagrams, the two-point function of chiral primary operators (5.8) with $J=3$ (the case of $J=2$ receives only planar corrections)

$$\langle \mathcal{O}^J(x)\bar{\mathcal{O}}^J(0)\rangle_{J=3} = \frac{1}{3N_0^3}\langle Z_{ab}Z_{bc}Z_{ca}\bar{Z}_{de}\bar{Z}_{ef}\bar{Z}_{fd}\rangle. \quad (7.1)$$

There are six possible ways of applying Wick contractions. Three of these lead to a factor of N^3 from contractions (leaving aside for now the prefactor coming from the normalization). These correspond to the planar diagrams. Planar diagrams always generate the highest power of N, and hence are the ones that dominate a large-N expansion (for finite J). Planar diagrams are those which can be drawn on a sphere (a one-point compactification of the plane) without any lines crossing. There are also three (that this number equals J is a coincidence) nonplanar diagrams of genus one. These are diagrams that cannot be drawn on a sphere without crossing, but can be placed on a torus without crossing. They contribute a single power of N. One can see the structure more clearly by the following trick (Kristjansen *et al.*, 2002). Imagine that each trace corresponds to a loop on which we place beads corresponding to the individual fields Z and $\bar{Z}$, white beads depicting Z's and black ones for the conjugate fields $\bar{Z}$. The beads are free to move on the loop, but cannot be pushed past each other (their order is significant). Changing the ordering of two nearby beads corresponds to crossing or uncrossing the lines connecting them. For the case $J=3$, reversing the order of the beads on one of the loops while keeping the other loop's ordering fixed exchanges planar and nonplanar diagrams, showing how the ordering of the beads is relevant. The cyclicity of the trace is reflected in the fact that rotating the beads around the loop results in an identical loop. One of the possible nonplanar contractions is depicted in Fig. 4 (upper panel).

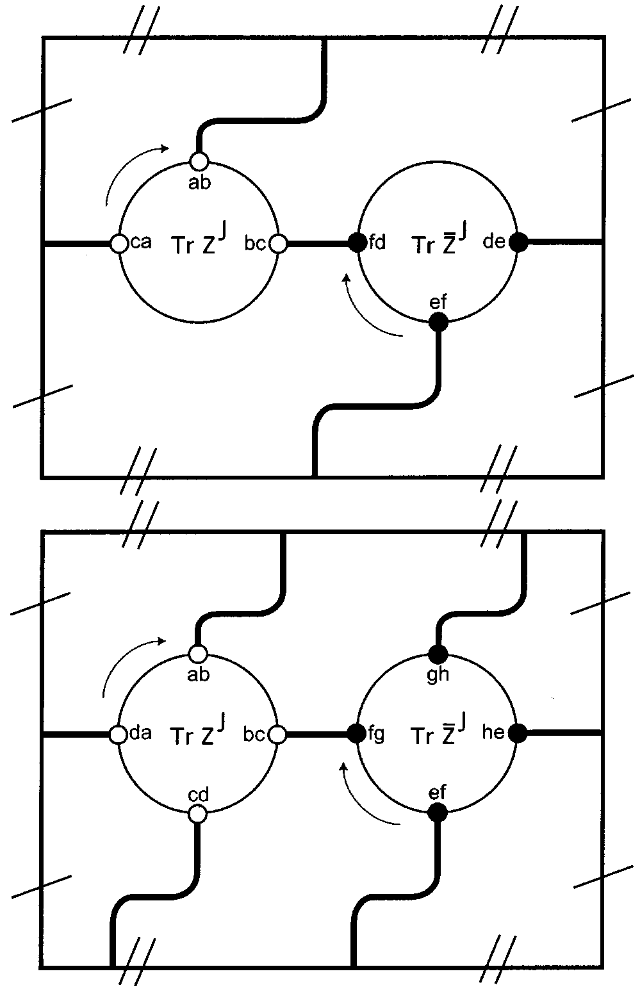

FIG. 4. Irreducible toroidal diagrams contributing to $\langle \mathrm{Tr}Z^J\mathrm{Tr}\bar{Z}^J\rangle$. The arrows indicate the direction in which traces are taken.

For $J=3$, the maximum genus contributing is the torus. This trick can be generalized to higher J and genus. First we need the notion of an irreducible diagram. We replace all lines in a diagram that are topologically parallel (calling these reducible) with a single line (irreducible). The resulting diagram built only from irreducible lines is itself irreducible. Diagrams can be grouped into equivalence classes, where the equivalence is defined as follows: two diagrams are considered equivalent if they both collapse to the same irreducible diagram. For $J=4$ there are diagrams which reduce to the one we have already considered for $J=3$, and new ones which reduce to the one depicted in the lower panel of Fig. 4. For higher J, all toroidal diagrams can be reduced to the two already considered. More generally, at genus h, the set of irreducible diagrams consists of those where the number of irreducible lines l ranges between $l=2h+1$ and $l=4h$, which for genus one gives $l=3,4$ and for genus two the range is $l=5\cdots 8$ (Kristjansen *et al.*, 2002).

At genus one, for arbitrary $J\geqslant 3$, there are $J!/[(J-3)!3!]$ ways of grouping the beads into three sets (the three irreducible lines in Fig. 4) while maintaining the order associated with the operator, and for for $J \geqslant 4$ the number of such groupings into sets of four is $J!/[(J-4)!4!]$. We denote the number of inequivalent irreducible diagrams with l irreducible lines at genus h by $n_{h,l}$. The calculation of this number is the trickiest part of working out the combinatorics. For the cases we have already considered, $n_{1,3}=1$ and $n_{1,4}=1$, while $n_{1,j}=0$ for $j>4$. However, for higher genera, there exist $n_{h,k}$ greater than one. The total number of diagrams in an equivalence class with l irreducible lines for fixed J can be found as follows: given a set of J elements, place the elements into l ordered distinct sets, maintaining the same overall cyclic ordering among all the elements. The number of possible ways of doing this is $J!/[(J-l)!l!]$. The total number of diagrams at genus h with l irreducible lines for fixed J is $n_{h,l}J!/[(J-l)!l!]$. At fixed genus, to arrive at the total number of graphs we must sum up the contribution from graphs in all equivalence class for all allowed l. For the torus, this gives

$$n_{1,3}\binom{J}{3} + n_{1,4}\binom{J}{4} \approx \frac{J^4}{4!}, \tag{7.2}$$

where in the last step we have shown the scaling in the large-J limit. Notice that sums of this form are always N independent. The N dependence in the combinatorics arise from traces over indices of Kronecker deltas appearing in the propagators (6.1) and (6.2) after all the Wick contractions are applied (of course keeping only diagrams at a fixed genus), and this dependence defines the genus order via the standard 't Hooft argument, where the suppression factor at any genus relative to the next lower genus goes like $1/N^2$ (or $1/N^{2h}$ relative to planar diagrams). As a result, in the BMN double scaling limit (1.8) (as opposed to the usual 't Hooft limit), diagrams at all genera contribute to correlation functions, giving rise to a new effective expansion parameter $g_2^2=(J^2/N)^2$, which is fixed at an arbitrary but finite value and measures the relative contribution of each genus in perturbation theory. Contributions from diagrams at genus h scale as g_2^{2h}. For the planar diagrams, there is an overall suppression by a factor of J due to the normalization of the operators in Eq. (5.8), but a compensating enhancement by the same factor arising from the cyclicity of the trace (which amounts to the rotation of the beads on one of the loops relative to the other one). Putting together these observations, we arrive at the planar-plus-toroidal contribution to the two-point function of chiral primary operators,

$$\langle \mathcal{O}^J(x)\bar{\mathcal{O}}^J(0)\rangle = \frac{1}{|x|^{2J}}\left(1+\frac{g_2^2}{4!}+\mathcal{O}(g_2^4)\right). \tag{7.3}$$

The normalization of the operator (5.8) is chosen to remove the overall dependence of the above two-point function on N as well as the coupling $g_{\rm YM}^2$ and factors of $8\pi^2$. Here we see the appearance of the parameter g_2^2 which organizes the expansion by genus. The planar diagrams contribute at order g_2^0 and the toroidal diagrams at order g_2^2. The new observation for the BMN double scaling limit is that the operators considered receive contributions from a number of diagrams which grow as J^{4h} at genus h, but these are suppressed by $1/N^{2h}$, and the J and N dependence combine into the new effective expansion parameter g_2^2, appearing at genus h as g_2^{2h}.

We shall now describe a method for establishing the all-orders (in g_2^2) result. We earlier mentioned two-dimensional QCD as a realization of 't Hooft's idea, and its exact solution via a matrix model (Kostov and Staudacher, 1997; Kostov *et al.*, 1998). It turns out that many of the correlation functions we are interested in can be reduced to correlation functions in this matrix theory. Higher-genus correlation functions in the complex matrix model, using loop equations, have been computed by Ambjorn *et al.* (1992). An alternative method for evaluating statistical ensembles of complex (or real) matrices was presented by Ginibre (1965) and Mehta (1990). We shall need only the most rudimentary results from matrix theory, which we collect here. Consider $N\times N$ complex matrices Z_{ij}, with i,j running from 1 to N, and define the measure $dZd\bar{Z}$ as

$$dZd\bar{Z} = \prod_{ij}\frac{1}{\pi}d(\mathrm{Re}Z_{ij})d(\mathrm{Im}Z_{ij}). \tag{7.4}$$

The partition function over these matrices is defined as the above measure weighted by a Gaussian function

$$Z = \int dZd\bar{Z}e^{-\mathrm{Tr}(Z\bar{Z})}. \tag{7.5}$$

The measure and the weight (and hence the partition function) are $U(N)\times U(N)$ invariant, representing independent multiplications on the left and the right. Correlation functions in this matrix model are defined as usual in quantum field theory:

$$\langle \mathcal{O}(Z,\bar{Z})\rangle_{\rm MM} = \int dZd\bar{Z}e^{-\mathrm{Tr}(Z\bar{Z})}\mathcal{O}(Z,\bar{Z}). \tag{7.6}$$

The normalization of the measure is chosen so that $\langle 1\rangle=1$.

The correlation functions we study are not invariant under the full symmetry, but only under those generated by the diagonal subgroup, acting in the adjoint representation. For correlators built out of traces which do not mix Z and $\bar{Z}$, the solution can be given by using character-expansion techniques, expanding the correlation function in terms of group characters. These characters are orthogonal, with a proportionality constant that can be evaluated from group-theory. The expansion coefficients are similarly computed from Young-diagram considerations. We summarize the relevant result for two-point functions:

$$\langle \mathrm{Tr} Z^J \mathrm{Tr} \bar{Z}^J \rangle_{\mathrm{MM}} = \sum_{k=1}^{J} \prod_{i=1}^{k} (N-1+i) \prod_{m=1}^{J-k} (N-m) = \frac{1}{J+1}\left(\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)} \right), \tag{7.7}$$

where have assumed $0<J<N$ in the last step. In the above, N is the rank of the group $\mathrm{U}(N)$ (we have kept N finite thus far). Up to this point the results are exact.

Let us now return to the correlation function (7.3). As we discussed in Sec. VI.B the spacetime dependence of this two-point function is completely fixed by conformal invariance. Moreover, being chiral primary, the scaling dimension is also fixed by supersymmetry to the free field theory engineering dimension. These have already been made manifest in (7.3). The remaining problem in computing Eq. (7.3) is that of computing the dependence on factors of J and N arising from the combinatorics of all the Wick contractions. Separating out the spacetime dependence, and also the numerical and coupling-constant factors in the scalar field propagators, we can rewrite the correlation function in terms of a correlation function in the matrix model we have described, which captures the combinatorics from evaluating all the traces over $\mathrm{U}(N)$ color indices (producing both planar and nonplanar contributions), as well as the combinatoric dependences on J,

$$\langle \mathrm{Tr}[Z^J(x)] \mathrm{Tr}[\bar{Z}^J(0)] \rangle = \left(\frac{g_{\mathrm{YM}}^2}{8\pi^2 |x|^2} \right)^J \langle \mathrm{Tr} Z^J \mathrm{Tr} \bar{Z}^J \rangle_{\mathrm{MM}}, \tag{7.8}$$

making use of the matrix model result (7.7). We are interested in the large-J limit of Eq. (7.8), and hence that of Eq. (7.7). We can expand it as

$$\langle \mathrm{Tr} Z^J \mathrm{Tr} \bar{Z}^J \rangle_{\mathrm{MM}} = J N^J \left[1 + \sum_{h=1}^{\infty} \sum_{k=2h+1}^{4h} \binom{J}{k} \frac{n_{h,k}}{N^{2h}} \right] \approx J N^J \left[1 + \sum_{h=1}^{\infty} \frac{n_{h,4h}}{(4h)!} \left(\frac{J^4}{N^2} \right)^h + \cdots \right], \tag{7.9}$$

where in the last expression we have taken the large J limit, and $\cdots$ denotes terms which vanish in the large-J and large-N limit if we scale $J \sim \sqrt{N}$. We see that the genus counting parameter $g_2^2 = J^4/N^2$ makes a natural appearance in this limit. We shall see in the next section when we come to consider non-BPS operators that this continues to be the case. In fact, this is another way to view the BMN limit: the limit is chosen precisely to ensure that the terms involving $n_{h,4h}$ in this limit remain finite and so we receive contributions from all genera. We can explicitly evaluate Eq. (7.8) using Eq. (7.9) in the BMN limit, giving for the chiral primary operators

$$\langle \mathcal{O}^J(x) \bar{\mathcal{O}}^J(0) \rangle = \frac{1}{|x|^{2J}} \cdot \frac{\sinh\left(\frac{g_2}{2}\right)}{\frac{g_2}{2}}. \tag{7.10}$$

Expanding this to first order in g_2^2 reproduces Eq. (7.3).

B. Nonplanar contributions to BMN correlators

In this section we move on to the non-BPS ("almost-BPS") BMN operators and compute the g_2^2 order nonplanar contributions to their two-point functions, first at free field theory and then at first order in λ'.

1. Correlators of BMN operators in free gauge theory to first nontrivial order in g_2

Having studied the two-point function of chiral primary operators to all orders, we are now ready to discuss the inclusion of phases in the more general BMN operators. We shall concentrate on operators of the form (5.13) for $i \neq j$, and choose the notation $\phi_i = \phi$ and $\phi_j = \psi$. In this section we study the correlator in the free theory, postponing consideration of interactions to the next section. The correlator we are interested in is

$$\langle \mathcal{O}^J_{ij,m}(x) \bar{\mathcal{O}}^J_{ij,n}(0) \rangle_0. \tag{7.11}$$

The calculation of the torus-level contribution to the two-point function of BMN operators in the free gauge theory has been carried out along two different lines, using matrix model technology (Kristjansen *et al.*, 2002) and via direct computation taking account of the combinatorics (Constable *et al.*, 2002a, 2002b). We shall see that the scaling with N and J, in the BMN limit, is the same as for the chiral primary operators, and $g_2^2 = (J^2/N)^2$ will appear again as the genus counting parameter. We follow closely the presentation of Constable *et al.* (2002a, 2002b)

To count the number of Feynman diagrams that contribute to a two-point function at genus h, we draw a polygon with $4h$ sides, then place one operator at the center and divide the other operator among the $4h$ vertices. We then pairwise identify all the sides and identify the vertices. All allowed diagrams are then generated by connecting the two operators via propagators, but without allowing the diagram to be collapsed to lower genus by shrinking homology cycles where no propagators have been placed. At genus h, the irreducible diagrams are those with $2h+1$ to $4h$ groups of lines. The number of ways of dividing J lines into $4h$ sets is

$$\binom{J}{4h} = \frac{J!}{(J-4h)!\,(4h)!} \approx \frac{J^{4h}}{(4h)!}, \tag{7.12}$$

where the last expression gives the behavior at large J. A similar counting applies to the diagrams where we group the lines into $4h-1$ sets and so on, down to $2h+1$, but the number of such groupings is suppressed relative to the $4h$ case. For example, the case $4h-1$ yields

$$\binom{J}{4h-1} = \frac{4h}{J-4h+1}\binom{J}{4h} \tag{7.13}$$

as the number of ways of distributing J lines into $4h-1$ sets, and their contributions relative to the $4h$ groupings vanish in the BMN limit. This is the same behavior we saw in the previous section at genus one, and it generalizes to arbitrary genus and for any finite number of impurities.

We can open up Fig. 4 for the torus diagrams with four groups of lines consisting of J scalar fields Z charged under $\mathrm{U}(1)_J$ and two different scalar impurities we shall label ϕ and ψ. Using the cyclicity of the trace, we can always place one of the impurities, say ϕ, as the first field in each operator before applying contractions. This simplifies the counting since the position of the ϕ field is fixed, and we only have to worry about placing the ψ field. The diagram can then be drawn as in Fig. 5.

Now there are five groups of fields, in which the first one begins with the ϕ field. Let J_i denote the number of fields, with $i=1,\dots,5$ (with no ψ field yet). We can place the ψ field in any of these groups, and there are J_i ways of doing so for the ith group. Let us consider first the case in which $m=n$ in Eq. (7.11), so the two operators have similar phase structures. The two impurities may appear in the same group, in which case when we contract the fields in the two operators, the relative positions of ϕ and ψ will remain fixed, and these diagrams will not contribute a phase factor. If the impurities are placed in different groups, then their relative positions in the two operators can in principle change, and the contractions will then be associated with a phase. For example, if ψ is placed in the second group, then it will contract with a field in the conjugate operator where its relative position to the other conjugate scalar will have shifted by J_3+J_4 places, and this introduces a relative phase of $\exp[2\pi i n(J_3+J_4)/J]$. Summing over all ways of placing ψ, we have for the two-point function the following expression:

$$\langle \mathcal{O}^J_{ij,m}(x)\bar{\mathcal{O}}^J_{ij,m}(0)\rangle_0^{\mathrm{torus}} = \frac{1}{JN^{J+2}}\left(\frac{1}{|x|^2}\right)^{J+2} N^J \times \sum_{J_1+\cdots+J_5=J+1}\sum_{k=1}^{5} J_k e^{2\pi i m\theta_k/J},$$

with the phases defined as $\theta_1=\theta_5=0$, $\theta_2=J_3+J_4$, $\theta_3=J_4-J_2$, and $\theta_4=J_2+J_3$. In performing the sum, we must impose the condition that $\Sigma_{i=1}^5 J_i=J+1$. The first term on the right-hand side is due to the normalization of the operators in Eq. (5.13). The next term arises from the propagators, with the normalization of the operators and propagators conspiring to remove the coupling and numerical factors. The last term comes from all the color index contractions at torus level. This expression is awkward, but can be turned into an integral representation in the large-J limit, with a delta function imposing the constraint, which can be evaluated explicitly. To see this, define $J_i=J\cdot j_i$. Then, in the large-J limit, we can rewrite the two-point functions as

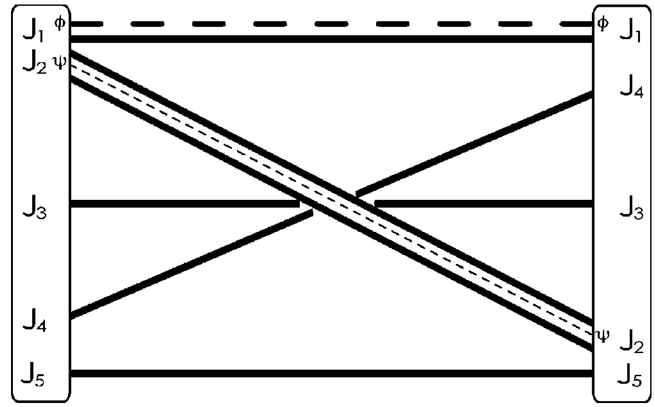

FIG. 5. Diagram depicting the phase shift in a torus diagram with no interactions. The solid lines represent an arbitrary number of Z fields, and the dashed lines represent the contraction between two ϕ's or ψ's.

$$\langle \mathcal{O}^J_{ij,m}(x)\bar{\mathcal{O}}^J_{ij,m}(0)\rangle_0^{\mathrm{torus}} = \left(\frac{1}{|x|^2}\right)^{J+2}\left(\frac{J^2}{N}\right)^2 \times \prod_{k=1}^{5}\int_0^1 dj_k j_k e^{2\pi i m\theta_k/J} \times \delta\left(1-\sum_{l=1}^{5} j_l\right),$$

and we again see the appearance of $J^2/N \equiv g_2$ which is held fixed in the BMN limit. The integral can be evaluated in a straightforward way. The construction when $m\neq n$ follows along the same lines, with the added complication that the position at which ψ is inserted in each group becomes relevant, since the two operators have different phase structures. This more complicated situation has been considered by Constable *et al.* (2002a, 2002b) and Kristjansen *et al.* (2002); we present only the result. The final expression for the torus two-point function of BMN operators of the type we have been considering is

$$\langle \mathcal{O}^J_{ij,m}(x)\bar{\mathcal{O}}^J_{ij,n}(0)\rangle_0 = \left(\frac{1}{|x|^2}\right)^{J+2}(\delta_{mn}+g_2^2 M^1_{mn}), \tag{7.14}$$

for $i\neq j$ and where $g_2=J^2/N$, and with the matrix M^1_{mn} symmetric, i.e., $M^1_{mn}=M^1_{nm}$, and is defined as

$$M^1_{mm} = \begin{cases} 0, & m=0, n\neq 0 \text{ or } m\neq 0, n=0; \\ \dfrac{1}{24}, & m=n=0; \\ \dfrac{1}{60} - \dfrac{1}{24\pi^2 m^2} + \dfrac{7}{16\pi^4 m^4}, & m=n\neq 0; \\ \dfrac{1}{48\pi^2 m^2} + \dfrac{35}{128\pi^4 m^4}, & m=-n\neq 0; \\ \dfrac{1}{4\pi^4 (m-n)^2}\left(\dfrac{\pi^2}{3} + \dfrac{1}{m^2} + \dfrac{2}{n^2} - \dfrac{3}{2mn} - \dfrac{1}{2(m-n)^2}\right), & \text{all other cases.} \end{cases} \tag{7.15}$$

The case of $m=0$ or $n=0$ corresponds to a two-point function with one of the composite operators being BPS; the $m=n=0$ gives the two-point function of BPS operators, while if $m\neq n$, but one of m or n zero, we see that the single-trace BPS operators do not mix with the non-BPS ones, at torus level. We expect the two-point function for $m=n=0$ to be exact to all orders in g^2_{YM}, since these operators are protected against receiving any anomalous dimensions. The non-BPS cases will receive g^2_{YM} corrections, and we will discuss these corrections in Sec. VII.B.2. The other cases show explicitly that in the free theory, single-trace non-BPS operators generically mix with each other, and this mixing begins at order g_2^2, where g_2^2 is the genus counting parameter. The discussion above can be generalized in an obvious way to higher-genus diagrams in the free theory, with the genus h contributions coming in at order $(g_2^2)^h$.

2. Correlators of BMN operators to first order in λ' and J^2/N

We have already computed the planar anomalous dimension to order λ' is Sec. VI.C. We are now ready to move beyond planar level, but will work only to first order in λ'. The duality would then put the result in correspondence with the string theory masses with loop corrections, giving a highly nontrivial test of the correspondence and a step beyond what has been possible in the standard AdS/CFT correspondence.

The result will be proportional to $\lambda' g_2^2$, showing that g_2^2 will continue to play the role of the genus counting parameter even with interactions switched on, and the role of the effective quantum loop counting parameter is still played by λ' in the BMN limit. The computation mirrors that of the previous section, but now taking account of the insertions of interaction terms. We only present an overview of the calculations; technical details can be found in the article of Constable *et al.* (2002a). At this order, only flavor-changing interactions contribute, and therefore the only interactions of relevance are the so-called F terms, which appear as the square of the commutator of scalars in different $\mathcal{N}=1$ chiral multiplets. We are considering two scalar impurity operators which are in $(\mathbf{1},\mathbf{9})$ of $SO(4)\times SO(4)$. They are symmetric in i,j indices. Therefore the F term, which involves the commutator of the two impurities, being antisymmetric, does not contribute. The two impurities can therefore be considered separately, since they do not simultaneously enter into interactions and only enter into interactions that are quadratic in the charged fields Z. These observations greatly reduce the number of possible diagrams that must be considered at this order.

There are three classes of Feynman diagrams to consider, involving nearest-neighbor, semi-nearest-neighbor and non-nearest-neighbor.[18] Nearest-neighbor diagrams are the ones where two lines alongside each other are connected through an interaction term. One of these lines will always be an impurity. There are four possible interaction types coming from squaring the commutator in the interaction, all with equal weight, with those that switch the order of the impurity and charged field contributing a minus sign relative to those which do not. We must sum over all ways of building such diagrams by inserting a single interaction into the free diagrams, taking care with the phases from exchanges and the phase of the free diagram. The phase considerations parallel our discussion in the previous section. Summing all nearest-neighbor diagrams, we find that the result (7.14) of the previous section is simply modified by a logarithmic correction which merely changes the scaling dimension we computed at planar level, since the result does not involve g_2. The other two types of diagrams will, however, involve honest toroidal corrections.

The class of semi-nearest-neighbor diagrams are those in which the fields entering an interaction are nearest neighbors in one of the composite operators, but not the other. These contribute to the two-point function only when $m\neq n$. For $m=n$ there are cancellations among semi-nearest-neighbor diagrams. The number of such diagrams is suppressed relative to the nearest-neighbor ones by $1/J$, but this is countered by an enhancement by

[18] These can be classified according to the possible combinations of contractible and noncontractible homology cycles on a torus, corresponding to the two propagator loops connecting to the interaction vertex. The diagrams with two noncontractible cycles on the torus do not enter at this order in g^2_{YM} because they involve interactions other than F terms.

a factor of J because these diagrams have a different phase structure, which in the large-J limit is larger by a factor J than the nearest-neighbor diagrams.

The non-nearest-neighbor contributions do introduce logarithmic corrections whether or not $m \neq n$. These diagrams are rarer than the nearest-neighbor one by a factor of $1/J^2$, but we again have an enhancement that compensates for this, due to the phase structure. When we sum over all contributions from the above graphs, we must also consider the phase associated to the diagram from the placement of the second impurity, as we had to do when considering the two-point function in the free theory.

The final result for the two-point function of the single-trace BMN operators we have been considering is

$$\langle \mathcal{O}^J_{ij,m}(x)\bar{\mathcal{O}}^J_{ij,n}(0)\rangle = \left(\frac{1}{|x|^2}\right)^{J+2}\Bigg\{\delta_{mn}(1+\lambda' L m^2) + g_2^2\Bigg[M^1_{mn} + \lambda' L\Bigg(mnM^1_{mn} + \frac{\mathcal{D}^1_{mn}}{8\pi^2}\Bigg)\Bigg]\Bigg\}, \tag{7.16}$$

with $L=-\ln(|x|^2\Lambda^2)$ and the matrix M^1_{mn} given in Eq. (7.15). This result holds for $i\neq j$. The matrix $\mathcal{D}^1_{mn}$ is

$$D^1_{mm} = \begin{cases} 0, & m=0 \text{ or } n=0; \\ \dfrac{2}{3}+\dfrac{5}{\pi^2 n^2}, & m=\pm n\neq 0; \\ \dfrac{2}{3}+\dfrac{2}{\pi^2 m^2}+\dfrac{2}{\pi^2 n^2}, & \text{all other cases.} \end{cases} \tag{7.17}$$

The next question of interest, the significance of which will become clear in the next subsection, is the correlation function of a single-trace operator and a double-trace one, and two-point functions of double-trace operators. The double-trace operators have been defined in Eqs. (5.10), (5.12), and (5.19). The double-trace operators (5.19) contain two scalar impurities, and, as discussed in Sec. V.C.3, can be in $(\mathbf{1},\mathbf{9})$, $(\mathbf{1},\mathbf{3}^{\pm})$, or $(\mathbf{1},\mathbf{1})$ tensor representations of $SO(4)\times SO(4)$. BPS operators do not occur in the antisymmetric representation of $\mathcal{T}^{J,r}_{ij,n}$, since $\mathcal{O}^J_{[ij],n}=-\mathcal{O}^J_{[ij],-n}$. The correlators of nonsinglets have been computed (Beisert, Kristjansen, *et al.*, 2003a), with the result

$$\langle \mathcal{T}^{J,r}_{ij}(x)\bar{\mathcal{T}}^{J,s}_{ij}(0)\rangle = \left(\frac{1}{|x|^2}\right)^{J+2}\delta_{rs},$$

$$\langle \mathcal{T}^{J,r}_{ij,m}(x)\bar{\mathcal{T}}^{J,s}_{ij,n}(0)\rangle = \left(\frac{1}{|x|^2}\right)^{J+2}\delta_{rs}\delta_{mn}\left(1+\lambda' L\frac{m^2}{r^2}\right),$$

$$\langle \mathcal{T}^{J,r}_{ij,m}(x)\bar{\mathcal{T}}^{J,s}_{ij}(0)\rangle = 0 \tag{7.18}$$

$(i\neq j)$, up to order λ' and g_2. We see that the double-trace operators are diagonal to order g_2. Their mixing, like the single-trace operators, begins at order g_2^2. We are considering here the SO(4) nonsinglet operators, but not making a distinction between the $\mathbf{9}$ and $\mathbf{3}^{\pm}$ representations. The results for the singlet representations are complicated by the inclusion of the $\mathrm{Tr}(Z^{\dagger}Z^{J+1})$ term in the definition (5.13). The $\mathbf{9}$, $\mathbf{3}^{\pm}$, and $\mathbf{1}$ representations all receive the same anomalous dimensions, and this degeneracy is a result of supersymmetry (see discussions of Sec. V.C.3). Starting from any one of these representations, we may reach the others by transformations generated by combinations of supercharges, and these combinations commute with the dilatation operator.

The single- and double-trace operators mix at order g_2, with the overlaps, at first order in λ', being

$$\langle \mathcal{T}^{J,r}_{ij,m}(x)\bar{\mathcal{O}}^J_{ij,n}(0)\rangle = \left(\frac{1}{|x|^2}\right)^{J+2}\frac{g_2}{\sqrt{J}}\frac{r^{3/2}\sqrt{1-r}\sin^2(\pi n r)}{\pi^2(m-nr)^2}\times\left(1+\frac{\lambda' L(m^2-mnr+n^2r^2)}{r^2}\right),$$

$$\langle \mathcal{T}^{J,r}_{ij}(x)\bar{\mathcal{O}}^J_{ij,n}(0)\rangle = \left(\frac{1}{|x|^2}\right)^{J+2}\frac{g_2}{\sqrt{J}}\left(\delta_{n,0}r-\frac{\sin^2(\pi n r)}{\pi^2 n^2}\right)\times(1+\lambda' L n^2). \tag{7.19}$$

Do not be alarmed by the appearance of $1/\sqrt{J}$ in these expressions. When we come to rediagonalize the single-trace operators in the next section, we shall see that the $1/J$ terms are compensated by sums (over r), and the two-point functions of the rediagonalized single-trace operators will receive contributions from such terms.

Extracting an overall power of g_2 in expressions like (7.19), we can arrange the remaining terms into an expansion in powers of g_2^2, i.e., in terms of planar and nonplanar diagrams.

At this order in g_2, there are nonzero overlaps between double- and triple-trace operators, and at order g_2^2 even overlaps between single-trace and triple-trace operators. More generally, the overlap of a single-trace operator with any t-trace operator begins at order g_2^{t-1}. We have ignored these corrections since they do not affect the anomalous dimensions of single-trace operators at order g_2^2.

C. Operator mixings and an improved BMN conjecture

The results of the previous two sections have been computed and presented in the BMN basis. These results are to be compared to those on the string theory side of the duality according to the identification (5.7), in which we are instructed to compare the eigenvalue spectrum of the string field-theory Hamiltonian to the spectrum of the dilatation operator minus the R charge in gauge theory. This is a basis-independent comparison. Alternatively, we may compare the matrix elements of the operators on the two sides of the duality. The two sides involve different Hilbert spaces, and the mapping between the bases of these distinct Hilbert spaces is part of the statement of the duality. We denote the basis on the gauge theory side by $\{|\mathfrak{a}\rangle_{\text{gauge}}\}$ and on the string theory side by $\{|\tilde{\mathfrak{a}}\rangle_{\text{string}}\}$, with $\mathfrak{a}$ labeling gauge theory states, and $\tilde{\mathfrak{a}}$ the labels on the string side. We need an isomorphism between the states of the two theories:

$$\{|\mathfrak{a}\rangle_{\text{gauge}}\} \leftrightarrow \{|\tilde{\mathfrak{a}}\rangle_{\text{string}}\}, \tag{7.20}$$

under the condition that the inner products on both sides agree,

$$_{\text{gauge}}\langle\mathfrak{a}|\mathfrak{b}\rangle_{\text{gauge}} = {}_{\text{string}}\langle\tilde{\mathfrak{a}}|\tilde{\mathfrak{b}}\rangle_{\text{string}}. \tag{7.21}$$

The duality, in the proper basis, holds between these matrix elements:

$$_{\text{gauge}}\langle\mathfrak{a}|(\mathcal{D}-\mathcal{J})|\mathfrak{b}\rangle_{\text{gauge}} = {}_{\text{string}}\left\langle \tilde{\mathfrak{a}} \left| \frac{H}{\mu} \right| \tilde{\mathfrak{b}} \right\rangle_{\text{string}}. \tag{7.22}$$

In Sec. V.C.3 the text around Eq. (5.18) (the second part of the SYM/plane-wave duality), we introduced a specific mapping between the Hilbert spaces on either side of the duality; however, we warned the reader that Eq. (7.21) does not hold for the identification (5.18) (more precisely, it only holds at g_2^0 level). In this section we intend to refine the correspondence between gauge theory and string theory Hilbert spaces, taking account of higher g_2 orders.

On the string theory side, there is a natural basis, the one which diagonalizes the free string theory Hamiltonian. We shall refer to this basis as the free-string basis. In this basis, m-string states are orthogonal to n-string states for $m \neq n$, and in fact this basis is orthonormal (see discussions of Sec. IV.C). The interactions induce mixings between these states; this basis does not diagonalize the full string field-theory Hamiltonian. For example, at order g_s, the cubic string field-theory Hamiltonian will cause transitions between one- and two-string states.

On the gauge theory side we start with the BMN basis, but if we are interested in the full scaling dimensions, including the anomalous dimensions, then we should choose a basis that diagonalizes the dilatation operator. This basis is referred to as the Δ-BMN basis (Georgiou *et al.*, 2003). Incidentally, in this basis the operators are conformal primaries, and this is the basis in which the two- and three-point functions take the forms (6.15) and (6.15) required by conformal invariance. This basis would correspond to the one on the string theory side that diagonalizes the full string field-theory Hamiltonian, and is not the free-string basis we defined above. The basis of BMN operators we have been working with above are neither of these. They have well-defined scaling dimensions at planar level, but nonplanar corrections induce nondiagonal mixings between the single-trace operators and between single-trace and multitrace operators in general. For example, at toroidal level, the classical ($\lambda'=0$) scaling dimensions are no longer well defined because of order g_2^2 mixings. This can be easily seen by noting that single-trace and double-trace BMN operators overlap at order g_2, as in Eq. (7.19), and show up at order g_2^2 in two-point functions of single-trace operators as in Eq. (7.16).

The results of Sec. VII.B.1 can be cast in the form

$$|x|^{2\Delta_0}\langle\mathcal{O}_{\mathfrak{a}}(x)\bar{\mathcal{O}}_{\mathfrak{b}}(0)\rangle = G_{\mathfrak{a}\mathfrak{b}} - \lambda'\Gamma_{\mathfrak{a}\mathfrak{b}}\ln(|x|^2\Lambda^2) + \mathcal{O}(\lambda'^2), \tag{7.23}$$

written in the BMN basis. We have introduced a notation whereby the indices $\mathfrak{a}$ range over single-, double-, and in general n-trace operators, and the operators within each such class. This expression is written up to first order in λ', with the remaining terms of higher order in λ'. Δ_0 is the classical (nonanomalous) scaling dimension. The matrix $G_{\mathfrak{a}\mathfrak{b}}$ is the inner product on the Hilbert space of states created by the BMN operators, and $\Gamma_{\mathfrak{a}\mathfrak{b}}$ is the matrix of anomalous dimensions.

The free-string basis can be constructed on the gauge theory side by taking linear combinations of the original BMN operators

$$|\mathfrak{a}\rangle_{\text{gauge}} = \mathcal{U}_{\mathfrak{a}\mathfrak{b}}\mathcal{O}_{\mathfrak{b}}(0)|0\rangle_{\text{gauge}}, \tag{7.24}$$

with the BMN operator $\mathcal{O}_{\mathfrak{a}}$ acting on the gauge theory vacuum. When g_2 vanishes, this basis coincides with the original BMN basis. Therefore, at order g_2^0, the change of basis matrix $\mathcal{U}$ is simply the identity.

Perturbative corrections in powers of g_2 result in a mixing between BMN operators with different numbers of traces, and we must rediagonalize this set of operators at each order in g_2 to maintain orthonormality of the inner product $G_{\mathfrak{a}\mathfrak{b}}$, to preserve the isomorphism with the free-string basis. The change of basis is chosen such that

$$\mathcal{U}G\mathcal{U}^\dagger = 1, \tag{7.25}$$

leading to

$$\begin{aligned} _{\text{gauge}}\langle\mathfrak{a}|(\mathcal{D}-\mathcal{J})|\mathfrak{b}\rangle_{\text{gauge}} &= [\mathcal{U}(\Delta_0 - J)G\mathcal{U}^\dagger + \mathcal{U}\Gamma\mathcal{U}^\dagger]_{\mathfrak{a}\mathfrak{b}} \\ &= n\delta_{\mathfrak{a}\mathfrak{b}} + \tilde{\Gamma}_{\mathfrak{a}\mathfrak{b}}, \end{aligned} \tag{7.26}$$

with n counting the number of impurities in the operator $\mathcal{O}_{\mathfrak{a}}(0)$ creating the state $|\mathfrak{a}\rangle_{\text{gauge}}$, and $\tilde{\Gamma}$ the anomalous-dimension matrix in the free-string basis. In the basis where the inner product G is diagonal, the anomalous-dimension matrix is symmetric. The matrix elements (7.26) are to be compared to the matrix elements of the string field-theory Hamiltonian in the free-string basis. We shall return to this in Sec. VIII.

The change of basis implemented by $\mathcal{U}$ is not unique, but all such choices are related by orthogonal transformations. We may make a unique choice, with one subtlety involving BPS operators which we mention momentarily, by requiring that the matrix $\mathcal{U}$ implementing the change of basis (7.25) be a real symmetric matrix. This turns out to be the choice for which the matrix elements of the rediagonalized operators can be matched to the matrix elements on the string side in the free-string basis. As we have already pointed out, in the BMN basis, single-trace BPS operators do not mix with single-trace non-BPS operators, and likewise for pairs of double-trace operators, but they may mix with each other. However, this mixing does not involve λ' corrections, since both operators are BPS. This mixing will not affect the anomalous dimensions. A similar pattern occurs in the string field theory, where the sums over intermediate BPS states do not alter the string masses. The correspondence between the string and gauge theory sides of the duality then seems to contain ambiguities for the BPS operators and their corresponding string states (Beisert, Kristjansen, *et al.*, 2003a); for example, we are unable to distinguish between single-string and double-string vacuum states, as well as single- and double-graviton states. We shall comment on this point briefly in Sec. IX. Mixing between BPS and non-BPS operators can be dealt with by choosing a basis in which BPS operators do not mix with non-BPS operators, regardless of the number of traces; however, the degeneracy in the BPS subspace remains.

We expand the diagonalizing matrix, the inner-product matrix, and the matrix of anomalous dimensions, to order g_2:

$$\mathcal{U} = 1 + g_2\mathcal{U}^{(1)} + \mathcal{O}(g_2^3),$$

$$G = 1 + g_2G^{(1)} + \mathcal{O}(g_2^3),$$

$$\Gamma = \Gamma^{(0)} + g_2\Gamma^{(1)} + \mathcal{O}(g_2^3). \tag{7.27}$$

$\mathcal{U}$ and G are the identity at zeroth order in g_2 since the BMN operators start mixing among each other only at order g_2 for single-trace and double-trace overlaps, and at order g_2^2 for single-trace overlaps with single-trace with double-trace intermediate channels, while the non-vanishing of $\Gamma^{(0)}$ to this order captures the first order (in λ') anomalous dimensions of the unmixed BMN operators.

Inserting the expansions (7.27) into Eq. (7.25), we find that the change of basis matrix $\mathcal{U}$, to order g_2 involves the term of the same order in the expansion of the inner-product matrix, and since $\mathcal{U}$ is unitary, we have

$$\mathcal{U}^{(1)} = -\frac{1}{2}G^{(1)}. \tag{7.28}$$

We may also solve for $\tilde{\Gamma}$ to first order in g_2, using Eq. (7.26) and expanding $\tilde{\Gamma}$ as above in Eq. (7.27), with $\tilde{\Gamma}^{(0)} = \Gamma^{(0)}$. This yields

$$\tilde{\Gamma}^{(1)} = \Gamma^{(1)} - \frac{1}{2}\{G^{(1)},\Gamma^{(0)}\}. \tag{7.29}$$

We then have for the order g_2 rediagonalized matrix of anomalous dimensions

$$\tilde{\Gamma}^{(1)} = \begin{pmatrix} 0 & \tilde{\Gamma}^{(1)}_{n,qs} & \tilde{\Gamma}^{(1)}_{n,s} \\ \tilde{\Gamma}^{(1)}_{pr,m} & 0 & 0 \\ \tilde{\Gamma}^{(1)}_{r,m} & 0 & 0 \end{pmatrix}. \tag{7.30}$$

In writing this matrix, we have chosen to discard the entries corresponding to the BPS operators $\mathcal{O}^J_{ij,n=0}$ and the combination $\sqrt{r}\mathcal{T}^{J,r}_{ij,n=0}+\sqrt{1-r}\mathcal{T}^{J,r}_{ij}$. This combination is chosen because it is orthogonal to $\mathcal{O}^J_{ij,n}$, $n\neq 0$, which can be easily seen from Eq. (7.19). The sub-matrix involving these BPS operators can be diagonalized using the freedom we mentioned in the discussion following Eq. (7.26). The remaining basis elements are chosen to correspond to the non-BPS single- and double-trace BMN operators given in $\mathcal{O}^J_{ij,n}, \mathcal{T}^{J,r}_{ij,n}$ $(n\neq 0)$, and $\sqrt{1-r}\mathcal{T}^{J,r}_{ij,n=0} - \sqrt{r}\mathcal{T}^{J,r}_{ij}$, in order. The entries of Eq. (7.30) in this basis can be read off from Eq. (7.19), and are

$$\tilde{\Gamma}^{(1)}_{n,r} = \tilde{\Gamma}^{(1)}_{r,n} = -\frac{\sin^2(\pi nr)}{\sqrt{J}2\pi^2}, \tag{7.31a}$$

$$\tilde{\Gamma}^{(1)}_{n,pr} = \tilde{\Gamma}^{(1)}_{pr,n} = \frac{\sqrt{1-r}}{\sqrt{Jr}}\frac{\sin^2(\pi nr)}{2\pi^2}. \tag{7.31b}$$

This procedure can be continued to higher-orders in g_2 in an obvious way.

To read off the anomalous dimensions, we must choose a basis that diagonalizes the dilatation operator. This basis would simultaneously diagonalize both the matrices $G_{\mathfrak{ab}}$ and $\Gamma_{\mathfrak{ab}}$. That such a diagonalization is possible (i.e., that these two matrices commute), can be argued from conformal invariance, since it implies that a basis of operators with definite scaling dimensions (classical plus anomalous) can be chosen. This choice of basis has been presented by Constable *et al.* (2002a, 2002b) and Beisert, Kristjansen, *et al.* (2003a). Going to the Δ-BMN basis, we find for the scaling dimension of single-trace BMN operators with two impurities, at order g_2^2,

$$\Delta = \Delta_0 + \lambda'\left[n^2 + g_2^2\left(\frac{1}{48\pi^2} + \frac{35}{128\pi^4n^2}\right)\right], \tag{7.32}$$

for $n\neq 0$ and with $\Delta_0=J+2$ for two impurities. For $n=0$, the classical scaling dimension is protected against quantum corrections by virtue of supersymmetry. Equation (7.32) is the main (basis-independent) result of this section, to be directly compared with the corresponding string field-theory results of Sec. VIII.

Up to this point, our focus has been on the calculation of two-point functions of BMN operators. One may wonder whether three and higher point functions have any relevance to the duality. The answer turn out to be negative. In the ArXiv version of this review (Sadri and Sheikh-Jabbari, 2003b), we address three and even higher point functions, and some pathology in their behavior is noted.

VIII. PLANE-WAVE LIGHT-CONE STRING FIELD THEORY

As a theory which is described by a two-dimensional σ model plus vertex operators, string theory is a first-quantized theory (Polchinski, 1998a) in the sense that all its states are always on-shell states and can only be found as external "particles" of an S matrix. However, one may ask if we can have a theory allowing (describing) off-shell string propagation. Such a theory, which is necessarily a field theory (as opposed to first-quantized quantum mechanics), is called *string field theory*. The on-shell part of "Hilbert space" of string field theory should then, by definition, match with the spectrum of string theory. There have been many attempts to formulate a superstring field theory. See, for example, Witten (1986a, 1986b), Berkovits *et al.* (2000), and for a review Siegel (1988). However, the final formulation has not yet been achieved. One of the major places where a string field-theory description becomes useful and necessary is when the vacuum (or background) about which we are expanding our string theory is not a true, stable vacuum. Such cases generally have the pathology of having tachyonic modes. This line of research has attracted a lot of attention (Berkovits *et al.*, 2000). In this section, we study a simpler question, string field theory after fixing the light-cone gauge, the light-cone string field theory, in the plane-wave background. Being a light-cone field theory, light-cone string field theory in the zero-coupling limit only describes on-shell particles. Therefore the "Hilbert space" of light-cone string field theory, where the corresponding operators act, is exactly the same as the one discussed in Sec. IV.C. The light-cone string field theory in flat space for bosonic closed and open strings was developed even before two-dimensional conformal-field-theory techniques were available (Mandelstam, 1974; Arfaei, 1975, 1976) and then generalized to supersymmetric open (Green and Schwarz, 1983) and closed (Green *et al.*, 1983) strings. A comprehensive discussion of light-cone string field theory on flat space can be found in Chapter 11 of Green *et al.* (1987b).

Here we first very briefly review the basic tools and concepts needed to develop light-cone closed superstring field theory and then focus on the plane-wave background. Using the symmetries, including supersymmetry, we fix the form of the cubic string vertices and then in Sec. VIII.C study second-order terms (in string coupling) in the light-cone string field-theory Hamiltonian.

A. General discussion of the light-cone string field theory

The fundamental object in light-cone string field theory is the string field operator Φ, which creates or destroys complete strings, i.e.,

$$\Phi:\mathbb{H}_m \rightarrow \mathbb{H}_{m\pm1}, \tag{8.1}$$

where $\mathbb{H}_m$ is the m-string Hilbert space (see Sec. IV.C). In the light-cone theory, Φ is a function of x^+ and p^+ (light-cone time and momentum), as well as string worldsheet fields $X^I(\sigma)$, $\theta_{\alpha\beta}(\sigma)$, and $\theta_{\dot\alpha\dot\beta}(\sigma)$, where $X^I(\sigma)=X^I(\sigma,\tau=0)$ and likewise for the other fields. Of course it is also possible to consider the "momentum"-space representation, in which Φ is a function of $P^I(\sigma)$, $\lambda_{\alpha\beta}(\sigma)$, and $\lambda_{\dot\alpha\dot\beta}(\sigma)$, with λ equal to $-i$ times the momentum conjugate to θ, i.e.,

$$\lambda_{\alpha\beta} = \frac{1}{2\pi\alpha'}\theta^\dagger_{\alpha\beta}, \quad \lambda_{\dot\alpha\dot\beta} = \frac{1}{2\pi\alpha'}\theta^\dagger_{\dot\alpha\dot\beta}. \tag{8.2}$$

Here we mainly consider the momentum-space representation. Noting the commutation relations (4.12) and (4.30), we find that

$$X^I(\sigma) = i\frac{\delta}{\delta P_I(\sigma)}, \quad \theta_{\alpha\beta}(\sigma) = i\frac{\delta}{\delta\lambda^{\alpha\beta}(\sigma)}. \tag{8.3}$$

As in any light-cone field theory, the light-cone dynamics of Φ is governed by the nonrelativistic Schrödinger equation

$$\mathcal{H}_{\text{SFT}}\Phi = i\frac{\partial}{\partial x^+}\Phi, \tag{8.4}$$

where $\mathcal{H}_{\text{SFT}}$ is the light-cone string field-theory Hamiltonian. In principle, in order to study the dynamics of the theory, we should know the Hamiltonian, and obtaining the Hamiltonian is the main goal of this section. As usual we assume that $\mathcal{H}_{\text{SFT}}$ has an expansion in powers of string coupling and at the free string theory limit it should be equal to the Hamiltonian coming from the string theory σ model. In our case this is $\mathcal{H}^{(2)}_{\text{l.c.}}$ [cf. Eq. (4.33)] :

$$\mathcal{H}_{\text{SFT}} = \mathcal{H}^{(2)}_{\text{l.c.}} + g_s\mathcal{H}^{(3)} + g_s^2\mathcal{H}^{(4)} + \cdots. \tag{8.5}$$

Our guiding principle for obtaining g_s corrections to the Hamiltonian is to use all the symmetries of the theory, both bosonic and fermionic, to restrict the form of such corrections. In the case of flat space these symmetries are so restrictive that they completely fix the form of $\mathcal{H}^{(3)}$ and all the higher-order corrections (Green and Schwarz, 1983; Green *et al.*, 1983). In the plane-wave case, as we discussed in Sec. II.B, the number of symmetry generators is less than in flat space. Nevertheless, as we shall see, the number of symmetry generators is large enough to determine $\mathcal{H}^{(3)}$ up to an overall p^+-dependent factor.

Let us now come back to Eq. (8.4) and try to solve it for free strings. This will give some idea of what the free-string fields Φ look like. Let us first consider the bosonic strings with the Hamiltonian (4.14). We shall work in the momentum basis.

Hereafter we shall set $\alpha'=2$; instead of p^+ we shall use $\alpha\equiv\alpha' p^+$ and $e(x)\equiv\text{sign}(x)=|x|/x$. If necessary, powers of α' can be recovered on dimensional grounds.

Since in the Hamiltonian there are $\partial_\sigma X$ terms, it is more convenient to use Fourier modes of $X^I(\sigma)$ and $P^I(\sigma)$, i.e., we use Eq. (4.9) at $\tau=0$. However, in order to match our conventions with those of the literature (Sprandlin and Volovich, 2002, 2003a; Pankiewicz, 2003a) we need to redefine the α_n and $\tilde\alpha_n$ modes:

$$X^I(\sigma) = x_0^I + \frac{1}{\sqrt{2}}\sum_{n\neq0}[x^I_{|n|} - ie(n)x^I_{-|n|}]e^{in\sigma/\alpha},$$

$$P^I(\sigma) = \frac{1}{2\pi\alpha}\left[p_0^I + \frac{1}{\sqrt{2}}\sum_{n\neq0}[p^I_{|n|} - ie(n)p^I_{-|n|}]e^{in\sigma/\alpha}\right], \tag{8.6}$$

where $x_n^I - ix^I_{-n} = \sqrt{2/\omega_n}(\tilde\alpha_n + \alpha_n^\dagger)$, $ip_n^I + p^I_{-n} = \sqrt{\omega_n/2}(\tilde\alpha_n - \alpha_n^\dagger)$, $n>0$. Using these x_n and $p_n (n\in\mathbb{Z})$, one can introduce another basis for creation-annihilation operators which is usually used in the light-cone string field theory (Spradlin and Volovich, 2002) and whose indices range from $-\infty$ to $+\infty$:

$$a_n = \frac{1}{\sqrt{2}i}(\alpha_n + \tilde{\alpha}_n), \quad a_{-n} = \frac{1}{\sqrt{2}}(\tilde{\alpha}_n - \alpha_n), \quad n > 0, \quad (8.7)$$

where $a_0 = \alpha_0$. Likewise for fermions,

$$b_n = \frac{1}{\sqrt{2}i}(\beta_n + \tilde{\beta}_n), \quad b_{-n} = \frac{1}{\sqrt{2}}(\tilde{\beta}_n - \beta_n), \quad n > 0, \quad (8.8)$$

and $b_0 = \beta_0$. It can be readily seen that

$$[a_n, a_m^\dagger] = \delta_{mn}, \quad \{b_n, b_m^\dagger\} = \delta_{mn}, \quad n \in \mathbb{Z},$$

where all the bosonic and fermionic indices have been suppressed. The light-cone Hamiltonian (4.33) in this basis is $\mathcal{H}^{(2)}_{\text{l.c.}} = \Sigma_{n\in\mathbb{Z}} \omega_n (a_n^\dagger a_n + b_n^\dagger b_n)$.

Since $[x_n^I, p_m^J] = i\delta^{IJ}\delta_{mn}$ or equivalently $x_n^I = i(\delta/\delta p_n^I)$, the Hamiltonian (4.14) written in terms of these Fourier modes becomes

$$\mathcal{H}^{(2)} = \frac{1}{\alpha}\sum_{n=-\infty}^{+\infty}\left[p_n^2 + \frac{1}{4}\omega_n^2 x_n^2\right] = \frac{1}{\alpha}\sum_{n=-\infty}^{+\infty}\left[p_n^2 - \frac{1}{4}\omega_n^2\left(\frac{\delta}{\delta p_n^I}\right)^2\right],$$

and hence the eigenfunctions of the Schrödinger equation (8.4) are products of (an infinite number of) momentum eigenfunctions $\psi_{N_n}(p_n)$, where N_n is the excitation number of the nth oscillator with frequency ω_n/α. Being a momentum eigenstate, $(\sqrt{\omega_n}/2)(a_n^\dagger + a_n)\psi(p_n) = p_n\psi(p_n)$ implies that

$$\psi(p_n) = \left(\frac{2}{\pi\omega_n}\right)^{1/4}\exp\left[-\frac{1}{\omega_n}p_n^2 + \frac{2}{\sqrt{\omega_n}}p_n a_n^\dagger - \frac{1}{2}a_n^\dagger a_n^\dagger\right]. \quad (8.9)$$

The string field Φ is a linear combination of these modes, i.e.,

$$\Phi[p_n] = \sum_{\{N_n\}} \phi_{\{N_n\}} \prod_{n=-\infty}^{+\infty} \psi_{\{N_n\}}(p_n). \quad (8.10)$$

To quantize the string field theory, as we do in any field theory, we promote $\phi_{\{N_n\}}$ to operators acting on the string Fock space where it destroys or creates a complete string with excitation number $\{N_n\}$ at $\tau=0$. Explicitly $\phi_{\{N_n\}}: \mathbb{H}_m \to \mathbb{H}_{m\pm 1}$ and $\phi_{\{N_n\}}|\text{vacuum}\rangle = |\{N_n\}\rangle$. Next we promote all the superalgebra generators to operators acting on the string field-theory Hilbert space. We shall generically use hatted letters to distinguish string field-theory representations from those of first-quantized string theory. As for the generators in the plane-wave superalgebra, as discussed in Sec. IV.D, the kinematical ones depend only on the zero modes of strings and the dynamical ones are quadratic in the string creation-annihilation operators. Therefore, at the free string theory limit (zeroth order in g_s), the dynamical $PSU(2|2)\times PSU(2|2)\times U(1)_-$ superalgebra generators, $\hat{J}_{ij}$, $\hat{J}_{ab}$, $\hat{Q}^{(0)}_{\alpha\dot{\beta}}$, $\hat{Q}^{(0)}_{\dot{\alpha}\beta}$, and $\hat{\mathcal{H}}^{(2)}$, should be quadratic in the string field Φ, for example,

$$\hat{\mathcal{H}}^{(2)} = \frac{1}{2}\int \alpha d\alpha D^8 p(\sigma) D^8\lambda(\sigma)\Phi^\dagger \mathcal{H}^{(2)}_{\text{l.c.}}\Phi, \quad (8.11)$$

with $D^8 p(\sigma) = \Pi_{n=-\infty}^{\infty} dp_n$ and $D^8\lambda(\sigma) = \Pi_{n=-\infty}^{\infty} d\lambda_n^{\alpha\beta} \times d\lambda_n^{\dot{\alpha}\dot{\beta}} d\lambda_n^{\dagger\alpha\beta} d\lambda_n^{\dagger\dot{\alpha}\dot{\beta}}$. Note that all these operators preserve the string number; i.e., they map $\mathbb{H}_m$ onto $\mathbb{H}_m$.

Now let us use the supersymmetry algebra (cf. Secs. II.B.1 and II.B.2) to restrict and obtain the corrections to supersymmetry generators once string interactions are turned on. The kinematical sector of the superalgebra, as well as P^+, are not corrected by the string interactions, because they depend only on the zero modes (or center-of-mass modes) of the strings and do not have the chance to mix with other string modes. Among the dynamical generators, $\hat{J}_{ij}, \hat{J}_{ab}$, being generators of a compact $SO(4)\times SO(4)$ group, cannot receive corrections, because their eigenvalues are quantized and cannot vary cotinuously (with g_s). Therefore only $\hat{Q}$ and $\hat{\mathcal{H}}$ can receive g_s corrections. We have parametrized the corrections to $\hat{\mathcal{H}}$ as in Eq. (8.5) and similarly the $\hat{Q}$'s can be expanded as

$$\hat{Q}_{\alpha\dot{\beta}} = \hat{Q}^{(0)}_{\alpha\dot{\beta}} + g_s\hat{Q}^{(3)}_{\alpha\dot{\beta}} + g_s^2\hat{Q}^{(4)}_{\alpha\dot{\beta}} + \cdots, \quad (8.12)$$

where the superscripts (3) and (4) in Eqs. (8.5) and (8.12) show that they are cubic and quartic in the string field Φ; more precisely,

$$\hat{\mathcal{H}}^{(3)}, \hat{Q}^{(3)}_{\alpha\dot{\beta}}, \hat{Q}^{(3)}_{\dot{\alpha}\beta}: \mathbb{H}_m \to \mathbb{H}_{m\pm 1}, \quad (8.13a)$$

$$\hat{\mathcal{H}}^{(4)}, \hat{Q}^{(4)}_{\alpha\dot{\beta}}, \hat{Q}^{(4)}_{\dot{\alpha}\beta}: \mathbb{H}_m \to \mathbb{H}_m \cup \mathbb{H}_{m\pm 2}. \quad (8.13b)$$

These g_s corrections, however, should be such that $\hat{\mathcal{H}}$ and $\hat{Q}$ still satisfy the superalgebra. This, as we shall show momentarily, will impose strong restrictions on the form of these corrections. From Eqs. (2.24), (2.26), (2.34), and (2.35), and the fact that the algebra should hold at any x^+, we learn that

$$[\hat{\mathcal{H}}^{(n)}, \hat{X}^I] = 0, \quad [\hat{Q}^{(n)}, \hat{X}^I] = 0, \quad (8.14a)$$

$$[\hat{\mathcal{H}}^{(n)}, \hat{P}^I] = 0, \quad [\hat{Q}^{(n)}, \hat{P}^I] = 0, \quad n > 2. \quad (8.14b)$$

Note in particular that Eq. (8.14b) means that the interaction parts of $\hat{\mathcal{H}}$ and $\hat{Q}$ are translationally invariant, while the quadratic parts of $\hat{\mathcal{H}}$ and $\hat{Q}$ do not have this symmetry [cf. Eqs. (2.26) and (2.34)]. Similarly Eqs. (2.32) and (2.39) imply that

$$[\hat{\mathcal{H}}^{(n)}, \hat{q}_{\alpha\beta}] = [\hat{\mathcal{H}}^{(n)}, \hat{q}_{\dot{\alpha}\dot{\beta}}] = 0, \quad (8.15a)$$

$$[\hat{Q}^{(n)}, \hat{q}_{\alpha\beta}] = [\hat{Q}^{(n)}, \hat{q}_{\dot{\alpha}\dot{\beta}}] = 0, \quad n > 2, \quad (8.15b)$$

and finally since $\hat{P}^+$ commutes with all generators,

$$[\hat{\mathcal{H}}^{(n)},\hat{P}^{+}]=0,\quad [\hat{Q}^{(n)},\hat{P}^{+}]=0\quad n>2. \tag{8.16}$$

B. Three-string vertices in the plane-wave light-cone string field theory

Let us now focus on the three-string vertex. We shall be working in the sector with light-cone momentum $p^{+}\neq 0$. Hereafter we shall relax the positivity condition on p^{+} [see Eq. (4.3)] and take the incoming states to have $p^{+}>0$ and the outgoing states $p^{+}<0$ (Sprandlin and Volovich, 2002). Without loss of generality we can assume that string one and string two are incoming and string three is outgoing. The physical quantities, such as P^{I} and $\lambda^{\alpha\beta}$ of the rth string ($r=1,2,3$) will be denoted by $P^{I}_{(r)}$ and $\lambda^{\alpha\beta}_{(r)}$. In order to guarantee Eqs. (8.14b), (8.15), and (8.16), which are nothing but the local momentum conservations of bosonic and fermionic fields, $\hat{\mathcal{H}}^{(3)},\hat{Q}^{(3)}$ must be proportional to

$$\Delta^{8}\left[\sum_{r=1}^{3}P^{I}_{(r)}(\sigma)\right]\Delta^{8}\left[\sum_{r=1}^{3}\lambda^{\alpha\beta}_{(r)}(\sigma)\right]\Delta^{8}\left[\sum_{r=1}^{3}\lambda^{\dot{\alpha}\dot{\beta}}_{(r)}(\sigma)\right]\delta\left(\sum_{r=1}^{3}\alpha_{(r)}\right)$$

where the Δ functionals are products of (an infinite number of) δ functions of the corresponding argument at different values of σ. In sum, so far we have shown that

$$\hat{\mathcal{H}}^{(3)}=\int d\mu_3 H_3\Phi(1)\Phi(2)\Phi(3),$$

$$\hat{Q}^{(3)}=\int d\mu_3 Q_3\Phi(1)\Phi(2)\Phi(3), \tag{8.17}$$

where $\Phi(r)$ is the string field of the rth string, $H_3,Q_3=H_3,Q_3(\alpha_{(r)},P_{(r)},X_{(r)},\theta_{(r)},\lambda_{(r)})$ are to be determined later using the dynamical part of the superalgebra, and

$$d\mu_3=\left(\prod_{r=1}^{3}d\alpha_{(r)}D^{8}P_{(r)}(\sigma)D^{8}\lambda_{(r)}(\sigma)\right)\Delta^{8}\left[\sum_{r=1}^{3}P^{I}_{(r)}(\sigma)\right]\times\Delta^{8}\left[\sum_{r=1}^{3}\lambda^{\alpha\beta}_{(r)}(\sigma)\right]\Delta^{8}\left[\sum_{r=1}^{3}\lambda^{\dot{\alpha}\dot{\beta}}_{(r)}(\sigma)\right]\delta\left(\sum_{r=1}^{3}\alpha_{(r)}\right). \tag{8.18}$$

We note that Eq. (8.14a) and its fermionic counterpart [which is a combination of Eqs. (8.15a) and (8.15b)] should still be imposed on $\hat{\mathcal{H}}^{(3)}$ and $\hat{Q}^{(3)}$. Since Eqs. (8.14)–(8.16) are exactly the same as their flat-space counterparts (Green and Schwarz, 1983; Green *et al.*, 1983), much of the analysis of Green *et al.* (1983; Green and Schwarz, 1983) carries over to our case.

1. Number-operator basis

Since in the string scattering processes we generally start and end up with states that are eigenstates of the number operator N_n (i.e., they have definite excitation number) rather than the momentum eigenstates, it is more convenient to rewrite Eq. (8.17) in the number-operator basis; in fact this is what is usually done in the light-cone string field-theory literature (for example, see Green *et al.*, 1987b, Chap. 11).

Since H_3 and Q_3 do not depend on the string field, we can simply ignore them for the purpose of converting the basis to a number-operator basis and focus on the measure $d\mu_3$ and $\Phi(r)$. For this change of basis we need to write down explicitly $\psi_{\{N_n\}}(p_n)$'s [cf. Eq. (8.10)] and perform the momentum integral. To identify $\hat{\mathcal{H}}^{(3)}$ and $\hat{Q}^{(3)}$ it is enough to find their matrix elements between two incoming strings and one outgoing string [see Eq. (8.13)]. However, it is more convenient to work with $|H^{(3)}\rangle,|Q^{(3)}\rangle\in\mathbb{H}_3$ where

$$\langle 1|\otimes\langle 2|\mathcal{H}^{(3)}|3\rangle\equiv\langle 1|\otimes\langle 2|\otimes\langle 3'|H^{(3)}\rangle \tag{8.19}$$

and similarly for $|Q^{(3)}\rangle$. In the above $\langle 3'|$ and $|3\rangle$ are related by worldsheet time reversal, in other terms $\langle 3'|=\langle v|\Phi(3)^{\dagger}$ while $\Phi(3)|v\rangle=|3\rangle$ (for more details see Green *et al.*, 1987b). Then, when we define $|V_3\rangle$ as

$$|V_3\rangle=\left[\int d\mu_3\prod_{r=1}^{3}\prod_{n=-\infty}^{\infty}\psi(p_n)\right]|v\rangle_3, \tag{8.20}$$

where a $|v\rangle_3$ is a three-string vacuum, $|H^{(3)}\rangle$ and $|Q^{(3)}\rangle$ take the form

$$|H^{(3)}\rangle=H_3|V_3\rangle,\quad |Q^{(3)}\rangle=Q_3|V_3\rangle. \tag{8.21}$$

H_3 and Q_3 are operators acting on three-string Hilbert space $\mathbb{H}_3$ and as we shall state in the next subsection Q_3 is linear and H_3 is quadratic in bosonic string creation operators. $|V_3\rangle$ itself maybe decomposed into a bosonic part $|E_a\rangle$ and a fermionic part $|E_b\rangle$ (Green *et al.*, 1987b; Spradlin and Volovich, 2002)

$$|V_3\rangle=|E_a\rangle\otimes|E_b\rangle\delta\left(\sum_r\alpha_r\right). \tag{8.22}$$

The notation a and b for bosons and fermions stems from our earlier notation in which the bosonic and fermionic creation operators where denoted by $a_n^{\dagger}$ and $b_n^{\dagger}$, $n\in\mathbb{Z}$.

As mentioned earlier, Eq. (8.14a) and a part of Eq. (8.15) should still be imposed on $|H^{(3)}\rangle$ and $|Q^{(3)}\rangle$. These are nothing but the *worldsheet continuity conditions*,

$$\sum_{r=1}^{3}e(\alpha_r)X_{(r)}(\sigma)|H^{(3)}\rangle=0,$$

$$\sum_{r=1}^{3}e(\alpha_r)\theta^{\alpha\beta}_{(r)}(\sigma)|H^{(3)}\rangle=0,$$

$$\sum_{r=1}^{3}e(\alpha_r)\theta^{\dot{\alpha}\dot{\beta}}_{(r)}(\sigma)|H^{(3)}\rangle=0, \tag{8.23}$$

and similarly for $|Q^{(3)}\rangle$.

We omit all the details of the computations and only present here the final result. More details may be found in the ArXiv version of this review (Sadri and Sheikh-

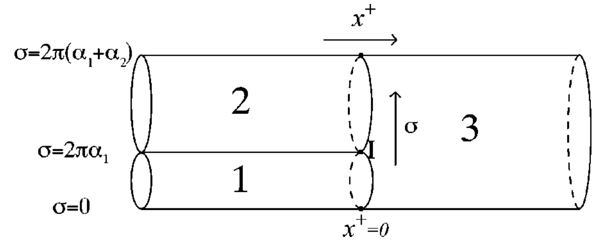

FIG. 6. Three-string interaction vertex in the light-cone gauge. Note that, due to closed string boundary conditions, $\sigma=0$, $\sigma=2\pi\alpha_1$, and $\sigma=2\pi(\alpha_1+\alpha_2)$ are identified and I is the interaction point.

Jabbari, 2003b), as well as in Spradlin and Volovich (2002) and Pankiewicz and Stefanski (2003b).

2. Interaction-point operator

In this section we use the dynamical $PSU(2|2)\times PSU(2|2)\times U(1)_-$ superalgebra to determine the "prefactors" H_3 and Q_3 [cf. Eqs. (8.17) or (8.21)]. To do this we expand both sides of Eqs. (2.37) and (2.40) in powers of g_s and note that the equality should hold at any order in g_s. At first order in g_s we obtain

$$[\hat{\mathcal{H}}^{(3)},\hat{Q}^{(0)}_{\alpha\dot{\beta}}]+[\hat{\mathcal{H}}^{(2)},\hat{Q}^{(3)}_{\alpha\dot{\beta}}]=0, \tag{8.24a}$$

$$\{\hat{Q}^{(3)}_{\alpha\dot{\beta}},(\hat{Q}^{(0)})^{\dagger}_{\rho\dot{\lambda}}\}+\{\hat{Q}^{(0)}_{\alpha\dot{\beta}},(\hat{Q}^{(3)})^{\dagger}_{\rho\dot{\lambda}}\}=2\epsilon_{\alpha\rho}\epsilon_{\dot{\beta}\dot{\lambda}}\hat{\mathcal{H}}^{(2)}. \tag{8.24b}$$

The equations for $\hat{Q}_{\dot{\alpha}\beta}$ are quite similar and hence we do not present them here. In fact, as in the flat-space case, one can show H_3 and Q_3 as a function of the worldsheet coordinate σ should only be nonzero at the interaction point (see Fig. 6) $\sigma=2\pi\alpha_1$ (Green *et al.*, 1987b, Chap. 11). These equations, being linear in Q_3 and H_3, only allow us to determine $\hat{\mathcal{H}}^{(3)}$ and $\hat{Q}^{(3)}$ up to an overall μ-dependent (or more precisely $\alpha'\mu p^+$-dependent) factor. This should be contrasted with the flat-space case, in which besides the above there is an extra condition coming from the boost in the light-cone directions (generated by J^{+-} in the notations of Sec. II.B.1; Green *et al.*, 1983). In the plane-wave background, however, this boost symmetry is absent and this overall factor should be fixed in some other way, e.g., by comparing the string field-theory results with their gauge theory correspondents (which are valid for $\alpha'\mu p^+\gg 1$) or by the results of supergravity on the plane-wave background (which are trustworthy for $\alpha'\mu p^+\ll 1$). Here we skip the detailed analysis and present only the final result [a more elaborate discussion of how to solve Eqs. (8.20) and (8.24) as well as the expression for $|E_a\rangle$ can be found in the arXiv version of this review, Sadri and Sheikh-Jabbari, 2003b, and references therein]:

$$|H^{(3)}\rangle=\frac{f(\mu)}{4\pi}|\alpha_3|^3\beta(\beta+1)\sum_{r=1}^{3}\delta\left(\sum_{r=1}^{3}\alpha_r\right)\sum_{n\in\mathbb{Z}}\frac{\omega_{n(r)}}{\alpha_r}\times(a^{i\dagger}_{n(r)}a^{i}_{-n(r)}-a^{a\dagger}_{n(r)}a^{a}_{-n(r)})|E_a\rangle, \tag{8.25}$$

where $\omega_{n(r)}=\sqrt{n^2+\mu^2\alpha_r^2}$ and $f(\mu)$ [or more precisely $f(\alpha'\mu p^+)$] is an overall factor that is not fixed through the superalgebra requirements.

We would like to note the $\mathbb{Z}_2$ behavior of $|H^{(3)}\rangle$. This $\mathbb{Z}_2$, as discussed in Sec. II.B.1, exchanges the two SO(4)'s of $SO(4)\times SO(4)$ isometry. From Eq. (8.25) it is evident that $a^{i\dagger}_{n(r)}a^{i}_{-n(r)}-a^{a\dagger}_{n(r)}a^{a}_{-n(r)}$ is odd under $\mathbb{Z}_2$. However, as we argued in Sec. IV.C the vacuum $|v\rangle$ is odd under $\mathbb{Z}_2$, therefore altogether $|H^{(3)}\rangle$ is $\mathbb{Z}_2$ even. Of course with a little bit of work, one can show that this property is also true for the full expression of H_3.

Before closing this subsection we should warn the reader that in most of the plane-wave string field-theory literature (e.g., Spradlin and Volovich, 2002, 2003c) SO(8) fermionic representations together with an SO(8)-invariant vacuum $|0\rangle$ or $|\dot{0}\rangle$ [see Eq. (4.37)] have been used. In the SO(8) notation, unlike our case, this $\mathbb{Z}_2$ symmetry is not manifest. It has been shown that the $SO(4)\times SO(4)$ formulation we presented here and the SO(8) one are indeed equivalent (Pankiewicz and Stefanski, 2003a). In the SO(8) notation it is very easy to observe that, as one would expect, this formulation in the $\mu\to 0$ limit goes over to the well-known flat-space result; this point can be (and in fact has been) used as a cross check for the calculations.

C. One-loop corrections to the string spectrum; plane-wave, light-cone string field-theory analysis

In this section we shall test the plane-wave/SYM duality at $\mathcal{O}(g_2^2)$ by working out the one-loop corrections to the single-string spectrum. Explicitly, we run the machinery of quantum-mechanical time-independent perturbation theory with the Hilbert space $\mathbb{H}$ and Hamiltonian $\hat{\mathcal{H}}$. One might also try to use time-dependent perturbation theory starting with string wave packets to study string scattering processes. This possibility will not be pursued here; we shall only make some comments about it later on in this section and also in Sec. IX.

It is easy to see that, at first order in g_s, time-independent perturbation theory gives a vanishing result for energy shifts, i.e., $\langle\psi|\hat{\mathcal{H}}^{(3)}|\psi\rangle=0$ for any $|\psi\rangle\in\mathbb{H}_1$ (of course one should consider degenerate perturbation theory; nevertheless, this result is obviously still true). Therefore we should consider the second-order corrections. For that, however, we need to work out $\hat{\mathcal{H}}^{(4)}$. So, in this section we shall first continue the analysis of Sec. VIII.B and work out the needed parts of $\hat{\mathcal{H}}^{(4)}$. As we shall see, to compare the gauge theory results of Sec. VII with the string field theory side we do not need to have the full expression for $\hat{\mathcal{H}}^{(4)}$, which considerably simplifies the calculation.

1. Four-string vertices

The procedure of finding $\hat{\mathcal{H}}^{(4)}$ and $\hat{Q}^{(4)}$ is essentially a direct continuation of the procedure of the previous section; i.e., solving the continuity conditions (8.14), (8.15), and (8.16) together with the constraints coming from the dynamical supersymmetry algebra, which are

$$[\hat{\mathcal{H}}^{(3)},\hat{Q}^{(3)}_{\alpha\dot{\beta}}]+[\hat{\mathcal{H}}^{(2)},\hat{Q}^{(4)}_{\alpha\dot{\beta}}]+[\hat{\mathcal{H}}^{(4)},\hat{Q}^{(0)}_{\alpha\dot{\beta}}]=0, \qquad (8.26a)$$

$$\{\hat{Q}^{(3)}_{\alpha\dot{\beta}},(\hat{Q}^{(3)})^{\dagger}_{\rho\dot{\lambda}}\}+\{\hat{Q}^{(0)}_{\alpha\dot{\beta}},(\hat{Q}^{(4)})^{\dagger}_{\rho\dot{\lambda}}\}+\{\hat{Q}^{(4)}_{\alpha\dot{\beta}},(\hat{Q}^{(0)})^{\dagger}_{\rho\dot{\lambda}}\}$$
$$=2\epsilon_{\alpha\rho}\epsilon_{\dot{\beta}\dot{\lambda}}\hat{\mathcal{H}}^{(4)}. \qquad (8.26b)$$

The important point to be noted is that $\hat{\mathcal{H}}^{(4)}$ contains two essentially different pieces, one that does not change the string number and the other that changes string number by two [see Eq. (8.13b)]. In fact, in our analysis to find mass corrections to single-string states, we need to calculate $\langle\psi|\hat{\mathcal{H}}^{(4)}|\psi\rangle, |\psi\rangle\in\mathbb{H}_1$. We then note that $\hat{Q}^{(4)}$ is quartic in the string field Φ and that $\hat{Q}^{(0)}$ maps $\mathbb{H}_1$ onto $\mathbb{H}_1$. Therefore the terms in Eq. (8.26b) involving $\hat{Q}^{(4)}$ do not contribute to the energy shift of single-string states at the g_s^2 level. This in particular means that we need not calculate $\hat{Q}^{(4)}$, and therefore we have all the necessary ingredients for calculating the one-loop string corrections to the string mass spectrum.

2. One-loop corrections to the string spectrum

In this subsection we compute the mass shift to the string state in the $(\mathbf{9},\mathbf{1})$ representation of SO(4) $\times$SO(4) (see Sec. IV.C), i.e.,

$$|(ij),n\rangle=\frac{1}{\sqrt{2}}\left(\alpha_n^{i\dagger}\tilde{\alpha}_n^{j\dagger}+\alpha_n^{j\dagger}\tilde{\alpha}_n^{i\dagger}-\frac{1}{2}\delta^{ij}\alpha_n^{k\dagger}\tilde{\alpha}_n^{k\dagger}\right)|v\rangle, \qquad (8.27)$$

where it is easy to show that

$$\langle(kl),m|(ij),n\rangle=\delta_{mn}\left(\delta^{jk}\delta^{il}+\delta^{il}\delta^{jk}-\frac{1}{2}\delta^{ij}\delta^{kl}\right). \qquad (8.28)$$

For other states, as a direct result of the superalgebra, we expect to see the same mass shift.

The corrections to the mass at order g_s^2 receive contributions from second-order perturbation theory with $\hat{\mathcal{H}}^{(3)}$ and first-order perturbation with $\hat{\mathcal{H}}^{(4)}$:

$$\delta E_n^{(2)}=g_s^2\left(\sum_{1,2\in\mathbb{H}_2}\frac{1}{2}\frac{|\langle 1,2|\hat{\mathcal{H}}^{(3)}|(ij),n\rangle|^2}{E_n^{(0)}-E_{1,2}^{(0)}}\right.$$
$$\left.+\frac{1}{8}\langle(ij),n|\{\hat{Q}^{(3)}_{\alpha\dot{\beta}},\hat{Q}^{(3)\dagger\alpha\dot{\beta}}\}|(ij),n\rangle\right). \qquad (8.29)$$

The extra factor of $\frac{1}{2}$ in the first term comes from the fact that this term arises from a second-order perturbation theory, $e^{S+\delta S}=e^{S}\left[1+\delta S+\frac{1}{2}(\delta S)^2\right]$ or, in other words, it is due to the reflection symmetry of the one-loop light-cone string diagram (Roiban *et al.*, 2002) while the factor of $\frac{1}{8}$ in the second term is obtained noting Eq. (8.26b) after taking the trace over $\alpha\rho$ and $\dot{\beta}\dot{\lambda}$ indices. Note that, since the Hamiltonian is a singlet of SO(4)$\times$SO(4)$\times\mathbb{Z}_2$, and also following our superalgebra arguments, we expect states in different *irreducible* SO(4)$\times$SO(4) representations not to mix and hence we use nondegenerate perturbation theory. Here we skip the details of the computations and the interested reader is referred to the arXiv version of this review (Sadri and Sheikh-Jabbari, 2003b) or Pankiewicz (2003b). The one-loop contribution to the single-string mass spectrum is (Roiban *et al.*, 2002; Pankiewicz, 2003a)

$$\delta E_n^{(2)}=\mu\frac{\lambda' g_2^2}{4\pi^2}\left(\frac{f(\mu)}{2\pi\mu^2\alpha_3^2}\right)\left(\frac{1}{12}+\frac{35}{32\pi^2n^2}\right). \qquad (8.30)$$

Choosing $f(\mu)=2\pi\mu^2\alpha_3^2$ for large μ, this result is in precise agreement with the gauge theory result of Eq. (7.32). In fact it is possible to absorb $f(\mu)$ into g_s, the string field-theory expansion parameter, i.e., the effective string coupling is

$$g_s^{eff}=g_s f(\alpha'\mu p^+)\sim g_s(\alpha'\mu p^+)^2=g_2^2, \qquad (8.31)$$

where $\sim$ in the above shows the large-μ limit.

3. Discussion of the string field-theory one-loop result

Here we would like to discuss briefly some of the issues regarding the large-μ expansion and the string field-theory one-loop result (8.30). As we discussed, Eq. (8.30) was obtained by allowing only the "impurity-conserving" intermediate string states in the sums (8.29). However, at the same order one can have contributions from string states that change impurity by two. For the impurity-nonconserving channel, the matrix elements of the first term of Eq. (8.29) are of order μ^2 while they are of order μ in the impurity-conserving channel (Pearson *et al.*, 2003; Spradlin and Volovich, 2003a, 2003c). Moreover, the energy difference denominator in the impurity-changing channel is of order μ, while it is of order μ^{-1} in the impurity-conserving channel. Therefore, altogether the contributions of the impurity-conserving and impurity-nonconserving channels are of the same order, and from the string theory side it is quite natural to consider both of them. However, the available gauge theory calculations are only in the impurity-conserving channel; this remains an open problem to be addressed.

The other point, which should be taken with a grain of salt, is the large-μ expansion. In fact, as we see in Eq. (8.29), sums may contain energy excitations ranging from zero to infinity. On the other hand, to obtain the large-μ expansion generically it is assumed that $\omega_n=\sqrt{n^2+(\alpha'\mu p^+)^2}$ can be expanded as $\alpha'\mu p^+ + n^2/\alpha'\mu p^+ +\cdots$; this expansion is obviously problematic when n is very large. In other words, the large-μ expansion and the sum over n do not commute. In fact, it has been shown that if we do the large-μ expansion first, we will get contributions that are linearly divergent (they grow like μ; Roiban *et al.*, 2002), leading to energy corrections of the order $\mu g_2^2\sqrt{\lambda'}$. However, if we do the sum first and then perform the large-μ expansion, we get a finite result for any finite value of μ. This is expected if the results are going to reproduce the flat-space results in the $\mu \to 0$ limit. This divergent result from the gauge theory point of view, being proportional to $\sqrt{\lambda'}$ seems like a nonperturbative effect (Klebanov *et al.*, 2002; Spradlin and Volovich, 2003a).

IX. CONCLUDING REMARKS AND OPEN QUESTIONS

In this review we have presented a new version of the string/gauge theory correspondence, the plane-wave/SYM duality, and spelled out the correspondence between various parameters and quantities on the two sides. As evidence for this duality we reviewed the gauge theory calculations leading to the spectrum of free strings on the plane wave as well as one-loop corrections to this spectrum, showing strong support for the duality. There have been many related problems pursued in the literature, which are interesting in their own right but which are beyond the scope of a pedagogical review. However, we would like to mention some of these topics:

- *Plane-wave/SYM duality for open strings*

The plane-wave/SYM duality we discussed in this review was constructed for (type-IIB) closed strings. The extension of the duality to the case of open strings has been studied, for example, by Berenstein, Gava, *et al.*, 2002; Gomis *et al.*, 2003a; Imamura, 2003; Lee and Park, 2003; Skenderis and Taylor, 2003; and Stefanski, 2003.

- *String bit model and quantum-mechanical model for BMN gauge theory*

In the large-μ limit one can readily observe that in Eq. (4.6) we can drop the $(\partial_\sigma X)^2$ term against the mass term $\mu^2 X^2$. This in particular implies that in such a limit strings effectively become a collection of some number of massive particles, the string bits. Hence it is quite natural to expect the large-μ dynamics of strings on the plane-wave background to be governed by a string bit model (Vaman and Verlinde, 2002; Verlinde, 2002; Zhou, 2003) in which the effects of string tension and interactions are introduced as interaction terms in the string bit Lagrangian. The proposed string bit model consists of J string bits of mass μ, with permutation symmetry and, more importantly, $\mathrm{PSU}(2|2)\times\mathrm{PSU}(2|2)\times\mathrm{U}(1)_-$ symmetry built into the model. The action for the string bit model, besides the kinetic (quadratic) term, has cubic and quartic terms, but terminates at the quartic level, as dictated by supersymmetry. The model has been constructed (or engineered) so that it gives the free-string mass spectrum. Moreover, one of the basic predictions of the string bit model is that the genus counting parameter g_2 would always appear through the combination $\lambda' g_2^2$ [see Eq. (1.13)]. This result, however, has been challenged by yet another quantum-mechanical model of the BMN gauge theory, constructed to capture the dynamics of BMN operators. The Hamiltonian for this quantum-mechanical model is the dilatation operator of the $\mathcal{N}=4$ SYM, and its Hilbert space is the set of BMN states with two impurities (Eynard and Kristjansen, 2002; Beisert, Kristjansen, *et al.*, 2003b; Kristjansen, 2003; Spradin and Volovich, 2003b).

- *D-branes in plane-wave backgrounds*

Here we have studied only strings on the plane-wave background. However, type-IIB string theory on this background also has D-brane solutions. Similar to the flat-space case, D-branes on the plane-wave background can be studied by introducing open strings in the type-II theory and imposing Dirichlet boundary conditions on them (Polchinski, 1995), or equivalently by giving the closed-string description through the boundary-state formulation (Callan *et al.*, 1996). Both approaches have been pursued for D-branes in the plane-wave background; see Billo and Pesando, 2002; Dabholkar and Parvizi 2002; and Bergman *et al.*, 2003 for examples.

In general, D-branes in the plane-wave background can be classified into two sets, those which are "parallel," meaning that they include x^- along their worldvolume, and those which are "transverse," in which the x^- direction is transverse to the worldvolume. It has been shown that in the plane-wave background we can have (half-supersymmetric) "parallel" D_p-branes for $p=3,5,7$, which are localized at the origin of the space transverse to the brane (Dabholkar and Parvizi, 2002). "Parallel" D_p-branes in plane-wave backgrounds, other than the maximally supersymmetric one, have been under intensive study.[19] Topics studied include their supersymmetric intersections, D-brane interactions, their worldvolume theory, and the corresponding supergravity solution.

As for the transverse D-branes, one can in fact show that the only half-supersymmetric brane solution of the maximally supersymmetric plane-wave background is a spherical three-brane, which is a giant graviton (McGreevy *et al.*, 2000). The role of these giant gravitons in the context of plane-wave/SYM duality has not been explored in detail. However some useful preliminary analyses have been made by Balasubramanian *et al.* (2002) and Metsaev (2003).

- *T duality on plane-wave backgrounds*

One of the other interesting directions in which the literature has grown is the question of extending the usual T or S dualities, which are generally studied for flat-space backgrounds, to plane waves. T duality is closely tied with compactification. Compactification is possible along directions that have translational symmetry (or along the Killing vectors). In the coordinates we have adopted for plane waves [see Eq. (2.4)] such isom-

[19]See, for example, Alishahiha and Kumar, 2002; Biswas *et al.*, 2002; Ganor and Varadarajan, 2002; Kumar *et al.*, 2002; Michishita, 2002; Skenderis and Taylor, 2002; Bain, Meessen, and Zamaklar, 2003; Bain, Peters, and Zamaklar, 2003; Chandrasekhar and Kumar, 2003; Gaberdiel *et al.*, 2003; Hyun *et al.*, 2003; Kim *et al.*, 2003; Ohta *et al.*, 2003; Sadri and Sheikh-Jabbari, 2003; Sarkissian and Zamaklar, 2003.

etries are not manifest. However, as we have extensively discussed, there are a pair of eight spacelike Killing vectors [L_I's and K_I's in Eq. (2.24)], and hence by a suitable coordinate transformation we could make them manifest. Such a coordinate transformation would necessarily involve using a "rotating frame" (Michelson, 2002). (Of course the possibility of lightlike compactification along the x^- direction always exists.) Upon compactification, in the fermionic sector we need to impose nontrivial boundary conditions on the (dynamical) supercharges and we may generically lose some supersymmetries. That is, T duality may change the number of supercharges. One can also study T duality and the Narain lattice at the level of string theory. However, on the plane-wave background the T-duality group is generally smaller than its flat-space counterpart; studies of compactification and T duality on the plane-wave background include those of Michelson (2002); Alishahiha *et al.* (2003); Bertolini *et al.* (2003); Mizoguchi *et al.* (2003a); and Sadri and Sheikh-Jabbari (2003a).

- "*Semiclassical*" *quantization of strings in the* $AdS_5 \times S^5$ *background*

The BMN sector of the $\mathcal{N}=4$ gauge theory is defined as a sector with large spin R-charge J. One may ask whether it is possible to make similar statements about the states with large spin S. It has been argued that the string σ model on the $\mathrm{AdS}_5\times S^5$ background takes a particularly simple form for strings with large spin, and one could quantize them semiclassically (Gubser *et al.*, 2002). This has opened a new line of explorations of the AdS/CFT correspondence. Among many useful references we mention Alishahiha and Mosaffa, 2002;. Frolov and Tseytlin, 2002; Mandal *et al.*, 2002; Russo, 2002; Arutyunov *et al.*, 2003; Beisert, Frolov, *et al.*, 2003; Beisort, Minahan, *et al.*, 2003; Minahan, 2003; Tseytlin, 2003.

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APPENDIX A: CONVENTIONS FOR $\mathcal{N}=4, D=4$ SUPERSYMMETRIC GAUGE THEORY

There are various formulations of the $\mathcal{N}=4$ supersymmetric Yang-Mills theory, three of which are most commonly encountered in the literature. One is based on dimensional reduction of the 10-dimensional component formulation of SYM theory, another realized by writing the Lagrangian in terms of $\mathcal{N}=1$ superspace gauge theory coupled to a set of chiral multiplets, and the third a formulation of the $\mathcal{N}=4$ SYM theory based on $\mathcal{N}=2$ harmonic superspace. The first one is more useful when actually computing Feynman diagrams and studying the combinatorics which lead to the double expansion characteristic of the double scaling limit proposed by Berenstein, Maldacena, and Nastase, and it is the one we have used performing the computations of Sec. VI.

We use the mostly minus metric convention, $g_{\mu\nu}=\mathrm{diag}(+,-,-,-)$. The Lagrangian (and field content) of the $\mathcal{N}=4$ super Yang-Mills theory can be deduced by dimensionally reducing the 10-dimensional $\mathcal{N}=1$ SYM theory (with 16 supercharges) on T^6 (which preserves all supersymmetries). There is a single vector, four Weyl fermions, and six real scalars, all in the adjoint representation of the gauge group. The reduced Lagrangian, in component form, is

$$\mathcal{L} = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\Bigg(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + \sum_{i=1}^{6} D_\mu\phi^i D^\mu\phi^i + \sum_{A=1}^{4} i\bar{\Psi}^A\Gamma^\mu D_\mu\Psi_A + \frac{1}{2}\sum_{i,j=1}^{6}[\phi^i,\phi^j]^2 + \sum_{A=1}^{4}\sum_{i=1}^{6}\bar{\Psi}^A\Gamma^i[\phi^i,\Psi_A]\Bigg). \tag{A1}$$

Decomposing the 10-dimensional Dirac matrices yields four- (Γ^μ) and six- (Γ^i) dimensional ones. This Lagrangian is manifestly invariant under a $\mathrm{U}(N)$ gauge symmetry. The generators of $\mathrm{U}(N)$ are chosen with the (nonstandard) normalization

$$\mathrm{Tr}(t^A t^B) = \delta^{AB}$$

($A,B=1,\ldots,N^2$), and satisfy the Lie algebra commutator relation and the appropriate completeness relation

$$[t^A,t^B] = if^{ABC}t^C,$$

$$\delta_{AB}(t^A)^a_b(t^B)^c_d = \delta^a_d\delta^c_b, \tag{A2}$$

$a,b=1,\ldots,N$, since these are the generators in the adjoint representation. The fields take values in the $\mathrm{U}(N)$ algebra

$$\chi(x) = \chi^A(x)t^A,$$

with χ any of the fields in the $\mathcal{N}=4$ multiplet. The sums above are taken over the N^2-1 generators of $\mathrm{SU}(N)$ and the single generator of the $\mathrm{U}(1)$ factor in $\mathrm{U}(N)$. The covariant derivative is defined as $D_\mu\chi=\partial_\mu - i[A_\mu,\chi]$. When diagrams are computed, the Feynman gauge is chosen to simplify calculations, taking advantage of the similarity between scalar and vector propagators in this gauge. There is also a global $\mathrm{SU}(4)\sim\mathrm{SO}(6)$ R symmetry, under which the scalars ϕ^i transform in the fundamental of $\mathrm{SO}(6)$, and the fermions Ψ_A in the fundamental of $\mathrm{SU}(4)=\mathrm{spin}(6)$. The vectors are singlets of the R symmetry. The θ term counts contributions from nontrivial instanton backgrounds, which is ignored when one assumes the trivial vacuum.

APPENDIX B: CONVENTIONS FOR 10-DIMENSIONAL FERMIONS

We briefly review our conventions for the representations of Dirac matrices in ten dimensions. We use the mostly plus metric. As for the 10-dimensional indices, mainly used in Sec. IV, we use greek indices $\mu,\nu,\ldots$ to range over the curved (target-space) indices, while hatted latin indices $\hat{a},\hat{b},\ldots$ denote tangent-space indices, and $I,J=1,2,\ldots,8$ label coordinates on the space transverse to the light-cone directions. In the plane-wave background, it is more convenient to decompose I,J indices into i,j and a,b, each ranging from one to four. In this review, unless explicitly stated otherwise, the a,b indices will denote these four directions. The curved-space gamma matrices are then defined via contraction with vierbeins, as usual, $\Gamma^\mu=e^\mu_{\hat{a}}\Gamma^{\hat{a}}$.

We may rewrite the two Majorana-Weyl spinors in 10-dimensional type-IIA and -IIB theories as a pair of Majorana spinors χ^α, $\alpha=1,2$, subject to the chirality conditions appropriate to the theory,

$$\Gamma^{11}\chi^1=+\chi^1,\quad \Gamma^{11}\chi^2=\pm\chi^2, \tag{B1}$$

where for the second spinor we choose − for nonchiral type-IIA and + for chiral type-IIB theories, and treat the index α labeling the spinor as an $SL(2,\mathbb{R})$ index. Type-II string theories contain two Majorana-Weyl gravitinos ψ^α_μ and two dilatinos λ^α, $\alpha=1,2$, which are of the same chirality in IIB and of opposite chirality in IIB.

1. Ten-dimensional fermions in SO(8) representations

The Dirac matrices in ten dimensions obey

$$\{\Gamma^\mu,\Gamma^\nu\}=2g^{\mu\nu}. \tag{B2}$$

A convenient choice of basis for 32×32 Dirac matrices, which we denote by Γ^μ, can be written in terms of 16×16 matrices γ^μ such that

$$\Gamma^+=i\begin{pmatrix}0&\sqrt{2}\\0&0\end{pmatrix},\quad \Gamma^-=i\begin{pmatrix}0&0\\\sqrt{2}&0\end{pmatrix},$$

$$\Gamma^I=\begin{pmatrix}\gamma^I&0\\0&-\gamma^I\end{pmatrix},\quad \Gamma^{11}=\begin{pmatrix}\gamma^{(8)}&0\\0&-\gamma^{(8)}\end{pmatrix}, \tag{B3}$$

and the γ^I satisfy $\{\gamma^I,\gamma^J\}=2\delta_{IJ}$ with δ_{IJ} the metric on the transverse space. Choosing a chiral basis for the γ's, we have $\gamma^{(8)}=\mathrm{diag}(1_8,-1_8)$. The above matrices satisfy

$$(\Gamma^+)^\dagger=-\Gamma^-,\quad (\Gamma^-)^\dagger=-\Gamma^+,\quad (\Gamma^+)^2=(\Gamma^-)^2=0,$$

$$\{\Gamma^{11},\Gamma^\pm\}=0,\quad \{\Gamma^{11},\Gamma^I\}=0,\quad [\Gamma^\pm,\Gamma^{IJ}]=0, \tag{B4}$$

and $\Gamma^\pm\Gamma^I\cdots\Gamma^J\Gamma^\pm=0$ if the same signs appear on both sides.

We define light-cone coordinates $x^\pm=(x^0\pm x^9)/\sqrt{2}$ and likewise for the lightlike gamma matrices $\Gamma^\pm=(\Gamma^0\pm\Gamma^9)/\sqrt{2}$, and also define antisymmetric products of γ matrices with weight one, $\gamma^{IJ\cdots KL}\equiv\gamma^{[I}\gamma^J\cdots\gamma^K\gamma^{L]}$.

We may choose our 10-dimensional, 32-component Majorana fermions ψ to satisfy

$$\Gamma^+\psi^+=0,\quad \Gamma^-\psi^-=0. \tag{B5}$$

Noting Eq. (B3), it can easily be seen that

$$\psi^+=\begin{pmatrix}\psi^+_\alpha\\0\end{pmatrix},\quad \psi^-=\begin{pmatrix}0\\\psi^-_\alpha\end{pmatrix},\quad \alpha=1,2,\ldots,16, \tag{B6}$$

where $\psi^\pm_\alpha$ can be thought of as SO(8) Majorana fermions, and the real γ^I matrices as 16×16 SO(8) Majorana gamma matrices. Moreover, we have

$$\Gamma^{11}\psi^+=\begin{pmatrix}\gamma^{(8)}\psi^+_\alpha\\0\end{pmatrix},\quad \Gamma^{11}\psi^-=\begin{pmatrix}0\\-\gamma^{(8)}\psi^-_\alpha\end{pmatrix}, \tag{B7}$$

i.e., the 10-dimensional chirality is related to 8-dimensional SO(8) chirality as indicated in Eq. (B7).

Now let us focus on the type-IIB theory where the maximally supersymmetric plane wave is defined. In this case we start with fermions of the same 10-dimensional chirality. Then, as stated in Eq. (B7), $\psi^\pm_\alpha$ should have ±SO(8) chirality. Explicitly, we have

$$(\gamma^{(8)}\psi^\pm)_\alpha=\pm\psi^\pm_\alpha. \tag{B8}$$

Therefore, in the type-IIB theory +/− can also be understood as SO(8) chirality. The above equation, however, can easily be solved with the choice $\gamma^{(8)}=\mathrm{diag}(\mathbf{1_8},-\mathbf{1_8})$, where

$$\psi^+_\alpha=\begin{pmatrix}\psi^+_a\\0\end{pmatrix},\quad \psi^-_\alpha=\begin{pmatrix}0\\\psi^-_{\dot{a}}\end{pmatrix},\quad a,\dot{a}=1,2,\ldots,8.$$

ψ^+_a and $\psi^-_{\dot{a}}$ are then Majorana-Weyl SO(8) fermions, usually denoted by $\mathbf{8_s}$ and $\mathbf{8_c}$, respectively (Green *et al.*, 1987a). The gamma matrices can also be reduced to 8×8 representations, $\gamma^I_{a\dot{a}}$ and $\gamma^I_{\dot{a}a}$, where the 16×16 γ^I matrices are

$$\gamma^I=\begin{pmatrix}0&\gamma^I_{a\dot{a}}\\\gamma^I_{\dot{a}a}&0\end{pmatrix},\quad I=1,2,\ldots,8,\quad a,\dot{a}=1,2,\ldots,8.$$

The fermionic coordinates of the IIB superspace consist of two same-chirality 10-dimensional Majorana-Weyl fermions, θ^1 and θ^2, and after fixing the light-cone gauge,

$$\Gamma^+\theta^{1,2}=0,$$

and as explained above, we end up with two SO(8) Majorana-Weyl fermions both in the $\mathbf{8_s}$ representation, θ^1_a and θ^2_a, $a=1,2,\ldots,8$. We may then combine these two real eight-component fermions into a single complex eight-component fermion,

$$\theta_a=\frac{1}{\sqrt{2}}(\theta^1_a+i\theta^2_a),\quad \theta^\dagger_a=\frac{1}{\sqrt{2}}(\theta^1_a-i\theta^2_a). \tag{B9}$$

As for the 32 supercharges, the 16 kinematical supersymmetries are in the complex $\mathbf{8_s}$ representation while the 16 dynamical ones are in the complex $\mathbf{8_c}$ representation. Note that this statement is true both in flat space and in the plane-wave background we are interested in.

2. Ten-dimensional fermions in SO(4)×SO(4) representations

In the plane-wave background, due to the presence of the RR five-form flux, the SO(8) symmetry is broken to SO(4)×SO(4). Therefore for the purpose of this review it is more convenient to make this SO(4)×SO(4), which is already manifest in the bosonic sector, explicit in the fermionic sector by choosing SO(4)×SO(4) representations instead of complex SO(8) $\mathbf{8_s}$ and $\mathbf{8_c}$ fermions. *Unless explicitly stated otherwise, we shall use this* SO(4) ×SO(4) *notation for fermions and gamma matrices*.

First, we note that an SO(4) Dirac fermion λ can be decomposed into two Weyl fermions λ_α and $\lambda_{\dot\alpha}$, $\alpha,\dot\alpha =1,2$. As usual for the SU(2) fermions, these Weyl indices are lowered and raised using the ϵ tensor,

$$\lambda_\alpha=\epsilon_{\alpha\beta}\lambda^\beta. \tag{B10}$$

We have defined $\theta^\dagger_{\alpha\beta}=(\theta_{\alpha\beta})^*$. Therefore the SO(4) ×SO(4) fermions are labeled by two SO(4) Weyl indices, i.e., $\lambda_{\alpha\beta'}$, $\lambda_{\alpha\dot\beta'}$, $\lambda_{\dot\alpha\beta'}$, $\lambda_{\dot\alpha\beta'}$ and $\lambda_{\dot\alpha\dot\beta'}$, where the "primed" indices, such as β' and $\dot\beta'$, correspond to the second SO(4). We may drop this prime whenever there is no confusion and then simply use, e.g., $\lambda_{\alpha\beta}$ where the first or second Weyl index corresponds to the first or second SO(4) factor. In fact, as explained in the main text in Sec. II.B.1, there is a $\mathbb{Z}_2$ symmetry which exchanges these SO(4) factors, and hence the theory should be symmetric under the exchange of the first and second Weyl indices.

To relate these SO(4)×SO(4) fermions to those of SO(8) (complex $\mathbf{8_s}$ and $\mathbf{8_c}$), we note that in our conventions $\mathbf{8_s}$ has positive SO(8) chirality, while $\mathbf{8_c}$ has nagative chirality. On the other hand, if we denote the two SO(4) "$\gamma^{(5)}$'s" by Π and Π', i.e.,

$$\Pi=\gamma^{1234},\quad \Pi'=\gamma^{5678}, \tag{B11}$$

then it is evident that

$$\gamma^{(8)}=\Pi\Pi'. \tag{B12}$$

Therefore for $\mathbf{8_s}$ fermions, the two SO(4)'s should have the same chirality while for $\mathbf{8_c}$ they should have opposite chirality. Explicitly

$$\psi_a\to\psi_{\alpha\beta'}\quad\text{and}\quad\psi_{\dot\alpha\dot\beta'},$$

$$\psi_{\dot a}\to\psi_{\alpha\dot\beta'}\quad\text{and}\quad\psi_{\dot\alpha\beta'}. \tag{B13}$$

We would like to emphasize that by $\mathbf{8_s}$ and $\mathbf{8_c}$ we mean the complex SO(8) fermions defined in Eq. (B9).

Noting that SO(4)≃SU(2)×SU(2), a Weyl SO(4) fermion can be represented as $(\mathbf{2},\mathbf{1})$ for λ_α and $(\mathbf{1},\mathbf{2})$ for $\lambda_{\dot\alpha}$ and hence an SO(4)×SO(4) fermion $\lambda_{\alpha\beta'}$ may be expressed as $((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$, and similarly for the others. In this notation, Eq. (B13) can be written as

$$\mathbf{8}_s\to((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))\oplus((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2})),$$

$$\mathbf{8}_c\to((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))\oplus((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1})). \tag{B14}$$

As the last step we need to choose a proper SO(4) ×SO(4) basis for the $\gamma^I_{a\dot a}$ matrices. Following the notation we have adopted in the review (see Sec. II), we denote the first four SO(4) directions by i,j and the other four by a,b:

$$\gamma^I_{a\dot a}=(\gamma^j_{a\dot a},\gamma^a_{a\dot a}),$$

where

$$\gamma^j_{a\dot a}=\begin{pmatrix}0 & (\sigma^j)_{\alpha\dot\beta}\delta^{\dot\beta'}_{\alpha'}\\ (\sigma^j)^{\dot\alpha\beta}\delta^{\beta'}_{\dot\alpha'} & 0\end{pmatrix},$$

$$\gamma^j_{\dot a a}=\begin{pmatrix}0 & (\sigma^j)_{\alpha\dot\beta}\delta^{\dot\beta'}_{\dot\alpha'}\\ (\sigma^j)^{\dot\alpha\beta}\delta^{\beta'}_{\alpha'} & 0\end{pmatrix}, \tag{B15}$$

and

$$\gamma^a_{a\dot a}=\begin{pmatrix}-\delta^\beta_\alpha(\sigma^a)_{\alpha'\dot\beta'} & 0\\ 0 & \delta^{\dot\beta}_{\dot\alpha}(\sigma^a)^{\dot\alpha'\beta'}\end{pmatrix},$$

$$\gamma^a_{\dot a a}=\begin{pmatrix}-\delta^\beta_\alpha(\sigma^a)^{\dot\alpha'\beta'} & 0\\ 0 & \delta^{\dot\beta}_{\dot\alpha}(\sigma^a)_{\dot\alpha'\beta'}\end{pmatrix}, \tag{B16}$$

with

$$(\sigma^j)_{\alpha\dot\alpha}=(\mathbf{1},\sigma^1,\sigma^2,\sigma^3)_{\alpha\dot\alpha}, \tag{B17}$$

and similarly for σ^a, where (σ^1, σ^2, σ^3) are the Pauli matrices. In the above

$$(\sigma^j)_{\alpha\dot\alpha}=\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}(\sigma^j)^{\dot\beta\beta}. \tag{B18}$$

In this basis, Π [see Eq. (B11)], is given by

$$\Pi_{ab}=\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}=\mathrm{diag}(\mathbf{1_4},-\mathbf{1_4}). \tag{B19}$$

As usual one can show that

$$(\sigma^i)_{\alpha\dot\beta}(\sigma^j)^{\dot\beta\gamma}+(\sigma^j)_{\alpha\dot\beta}(\sigma^i)^{\dot\beta\gamma}=2\delta^{ij}\delta^\gamma_\alpha,$$

$$(\sigma^i)^{\dot\alpha\beta}(\sigma^j)_{\beta\dot\gamma}+(\sigma^j)^{\dot\alpha\beta}(\sigma^i)_{\beta\dot\gamma}=2\delta^{ij}\delta^{\dot\gamma}_{\dot\alpha}. \tag{B20}$$

The generators of SO(4) rotations, $\gamma^{ij}=\frac{1}{2}[\gamma^i,\gamma^j]$, can be easily worked out in terms of σ^{ij}. They are

$$(\gamma^{ij})_{ab}=\begin{pmatrix}(\sigma^{ij})_{\alpha\beta}\delta^{\beta'}_{\alpha'} & 0,\\ 0 & (\sigma^{ij})^{\dot\alpha\dot\beta}\delta^{\dot\beta'}_{\dot\alpha'}\end{pmatrix}, \tag{B21}$$

where

$$(\sigma^{ij})_{\alpha\beta}=\frac{1}{2}[(\sigma^i)^{\dot\gamma}_\alpha(\sigma^j)_{\beta\dot\gamma}-(\sigma^j)^{\dot\gamma}_\alpha(\sigma^i)_{\beta\dot\gamma}]=(\sigma^{ij})_{\beta\alpha},$$

$$(\sigma^{ij})^{\dot\alpha\dot\beta}=\frac{1}{2}[(\sigma^i)^{\dot\alpha}_\gamma(\sigma^j)^{\dot\beta\gamma}-(\sigma^j)^{\dot\alpha}_\gamma(\sigma^i)^{\dot\beta\gamma}]=(\sigma^{ij})^{\dot\beta\dot\alpha}. \tag{B22}$$

3. SO(6) and SO(4,2) fermions

Here we briefly present the spin(6) and spin(4,2) fermion conventions used in Sec. II.B.2. Let us first consider the spin(6) spinors, i.e., six-dimensional Euclidean fermions (more details may be found in Polchinski, 1998b). In six dimensions we deal with $2^{6/2}=8$ component Dirac fermions. The so(6) 8×8 Dirac matrices satisfy

$$\{\Gamma^{\hat{A}},\Gamma^{\hat{B}}\}=2\delta^{\hat{A}\hat{B}},\quad \hat{A},\hat{B}=1,2,\ldots,6.$$

As usual (and by definition), the commutator of these Γ matrices, which is denoted by $\Gamma^{\hat{A}\hat{B}}=\frac{1}{2}[\Gamma^{\hat{A}},\Gamma^{\hat{B}}]$, forms an 8×8 representation of so(6). The eight-component so(6) Dirac fermions, however, may be decomposed into two four-component (complex) Weyl spinors. Explicitly, ψ_A, where $A=1,\ldots,8$, can be decomposed into ψ_I and $\psi_{\dot{I}}$ where $I,\dot{I}=1,2,3,4$ can be thought of as fundamental (antifundamental) su(4) indices. The Dirac matrices $\Gamma^{\hat{A}}$, similarly to Eq. (B3), can be decomposed into $\Gamma^{\pm}$ and γ^I, where now the γ's are 4×4 matrices and act on the Weyl spinors. Each of these so(6) Weyl spinors in turn can be decomposed into two four-dimensional [i.e. so(4)] Weyl spinors, though with opposite chiralities,

$$\psi_I\rightarrow(\psi_\alpha,\psi_{\dot{\alpha}}),$$

where $\alpha,\dot{\alpha}=1,2$. Since the arguments closely parallel those of Appendix B.1 [where we explained how to reduce SO(9,1) fermions into the SO(8) fermions], we do not repeat them here. In fact, a similar result is also true for so(4,2) fermions, and a Weyl so(4,2) fermion can be decomposed into two so(4) Weyl fermions of opposite chirality; if we denote the so(4,2) Weyl index by $\hat{I}$ ($I=1,2,3,4$), this means

$$\psi_{\hat{I}}\rightarrow(\psi_\alpha,\psi_{\dot{\alpha}}).$$

The $SO(4,2)\times SO(6)$ fermions naturally carry spinorial indices of both of the groups. Therefore in general we can have four different fermions depending on the chirality of the fermions under either of the groups. In our case the spinors that we deal with (those appearing in the $AdS_5\times S^5$ superalgebra), should have the same chirality under both groups. This comes from the fact that we are working with type-IIB theory in which both of the fermions have the same 10-dimensional chirality. So a general $AdS_5\times S^5$ fermion would carry two indices, which are fundamentals of su(2,2) and su(4), e.g., $\psi_{\hat{I}J}$ or $\psi_{\hat{\dot{I}}\dot{J}}$. (The choice of $\psi_{\hat{I}J}$ or $\psi_{\hat{\dot{I}}\dot{J}}$ fermions is related to the sign of the self-dual five-form flux on the S^5 of the $AdS_5\times S^5$ geometry. Here we have chosen the positive case and hence we are dealing with $\psi_{\hat{I}J}$ fermions.) Note that, since these are complex fermions, this spinor has 32 degrees of freedom. This fermion can be decomposed as an $SO(4)\times SO(4)$ fermion using the above decompositions:

$$\psi_{\hat{I}J}\rightarrow(\psi_{\alpha\beta},\psi_{\alpha\dot{\beta}},\psi_{\dot{\alpha}\beta},\psi_{\dot{\alpha}\dot{\beta}}). \tag{B23}$$

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