Colloquium: Magnetohydrodynamic turbulence and time scales in astrophysical and space plasmas

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Magnetohydrodynamic (MHD) turbulence has been employed as a physical model for a wide range of applications in astrophysical and space plasma physics. This Colloquium reviews fundamental aspects of MHD turbulence, including spectral energy transfer, nonlocality, and anisotropy, each of which is related to the multiplicity of dynamical time scales that may be present. These basic issues are discussed based on the concepts of sweeping of the small scales by a large-scale field, which in MHD occurs due to effects of counterpropagating waves, as well as the local straining processes that occur due to nonlinear couplings. These considerations give rise to various expected energy spectra, which are compared to both simulation results and relevant observations from space and astrophysical plasmas.

CONTENTS

I. INTRODUCTION

Much of the matter in space and astrophysical systems is in the plasma state, an electrically conducting gas or fluid that evolves in response to both mechanical and electromagnetic forces. These plasmas are usually found to be in complex motion, involving structure across a wide range of spatial scales. Hydrodynamic turbulence, a long studied but incompletely understood fundamental physical process, is clearly a first step in examining these systems, but in view of the pronounced role that magnetic fields play in astrophysical plasmas one is quickly led to the related but more complex study of magneto-hydrodynamic (MHD) turbulence. In particular, MHD turbulence differs from its hydrodynamic antecedent in that large-scale magnetic fields play a significant role, even in influencing much smaller scale turbulence processes. Related to this is the greater number of time scales that influence the dynamics of MHD turbulence. For these reasons the MHD "cascade," which transfers energy between structures of various sizes due to nonlinear dynamical couplings, is much more complex and can exist in more forms than the simpler hydrodynamic cascade. The goal of the present Colloquium is to discuss the various forms that the MHD energy cascade can take, due to the large-scale magnetic fields and the multiplicity of time scales that enter into the basic physics of the problem.

The focus will be mainly on the concepts and ideas behind MHD turbulence, the cascades and time scales involved, without going into too much detail regarding

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the computations. Frequent references will be made to applications in astrophysical and space plasmas as well as to numerical simulations, to illustrate the concepts discussed.

We shall briefly review in Sec. II some basic concepts of fluid turbulence and introduce the ideas of straining and sweeping in that context. We then proceed in Sec. III to examine the more complex subject of MHD turbulence. We shall argue that the key to understanding MHD turbulence in its various possible forms is to identify the relevant time scales, how they are influenced by anisotropy associated with a large-scale magnetic field, and how a balance is struck between nonlinear distortions and the sweep-like dynamics associated with wave propagation. Section IV contains conclusions and discussion. For completeness, more detailed applications of these ideas about MHD turbulence time scales are provided in two Appendixes. Different regimes of MHD turbulence are discussed in Appendix A and the form of the energy spectrum in each of them is addressed. Appendix B discusses the time correlations of MHD turbulence.

II. FUNDAMENTAL CONCEPTS IN FLUID TURBULENCE

A turbulent flow satisfies the Navier-Stokes equation (Batchelor, 1970) which is the momentum evolution of an element of fluid,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \tag{1}$$

Here $\mathbf{u}$ is the velocity field, which is a fluctuating quantity in time t and space $\mathbf{x}$, ∇ is the gradient with respect to $\mathbf{x}$, ρ is the density, p is the pressure, and ν is the kinematic viscosity (molecular viscosity/density). We consider an incompressible flow with constant density, so the continuity equation reduces to a divergence-free condition $\nabla \cdot \mathbf{u} = 0$, and so the pressure is determined by the constraint that emerges by computing the divergence of Eq. (1).

The macroscopic Reynolds number $R = uL/\nu$, where u is a typical flow velocity [the root-mean-square (rms) of the fluctuating velocity field] and L a typical (large) scale, is a measure of the strength of the nonlinear convective term $\mathbf{u} \cdot \nabla \mathbf{u}$ against the dissipative term $\nu \nabla^2 \mathbf{u}$ in Eq. (1). Turbulent flows are characterized by high Reynolds numbers: For example, pipe flow becomes turbulent around $R = 2000$, while for meteorological flows, typically $R \gg 10^{10}$.

In the absence of viscosity, the flow conserves the global kinetic energy (here $u^2 = \langle |\mathbf{u}|^2 \rangle$ is twice the energy per unit mass, where $\langle \cdots \rangle$ denotes a volume average). However, even with a small value of the viscosity the energy will decay, and the properties of decaying turbulent flows are particularly distinctive.

A. Global decay of energy

The global decay of incompressible homogeneous isotropic turbulence was considered in detail by Taylor (1935, 1938) and by von Karman and Howarth (1938) prior to Kolmogorov's (1941a, 1941b) ground-breaking work on the smaller-scale inertial range. The energy changes in response to viscous effects according to $du^2/dt = -\nu \langle |\nabla \times \mathbf{u}|^2 \rangle$. Based upon empirical results, Taylor understood that the decay of energy behaves as $du^2/dt \propto u^3$, and von Karman and Howarth (1938) provided the first theoretical justification of the simple decay law $du^2/dt = -\alpha u^3/\lambda$, which has long enjoyed empirical support in hydrodynamics (see, for example, the reviews by Batchelor, 1970; George, 1992; Speziale and Bernard, 1992; Zhou and Speziale, 1998).

A crucial element is λ, the *similarity scale* or *energy-containing scale*,[1] which behaves as $d\lambda/dt = \beta u$. The constants α and β are both of the order of unity, and specific values can be adopted based upon such physical assumptions as permanence of large eddies (Kolmogorov, 1941c), constant turbulent Reynolds number (von Karman and Lin, 1949), and others (see Orszag, 1970; Matthaeus *et al.*, 1996). This phenomenology of the energy-containing eddies gives a reasonable approximate picture of global energy decay and makes clear how the energy reservoir at the large scales ($\sim \lambda \sim L$) controls the process. Usually one defines the eddy turnover or nonlinear time scale as $\tau_{eddy} = \lambda/u$ (Rose and Sulem, 1978) so that the energy decay occurs at the rate $du^2/dt \sim -u^2/\tau_{eddy}$. The eddy turnover time scale is the fundamental time scale in turbulence, and its role in global hydrodynamic decay foreshadows the role of nonlinear time scales in the energy cascades of both hydrodynamics and MHD.

B. Locality of energy transfer and Kolmogorov spectrum

For a high-Reynolds-number fluid turbulence, the assumption regarding the triadic interaction and energy transfer process leads to the famous Kolmogorov $-5/3$ scaling law (Kolmogorov, 1941a; Batchelor, 1970). Briefly, Kolmogorov assumed that both the energy transfer and interacting scales are local.[2] This picture of the energy transfer process can be viewed as follows: force is applied to a fluid flow at a large-scale L, inject-

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[1] The similarity scale is often associated with the outer scale or correlation length of turbulence in observational work. See Batchelor (1970).

[2] The energy content at (or near) a scale ℓ may be defined more or less formally. For example, energy per unit mass at scale ℓ might be expressed by $u_\ell^2 \sim \langle |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}+\ell)|^2 \rangle$. Alternatively in wave-number k space one can compute first the correlation function $R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \rangle$. Then the spectral tensor $S_{ij}(\mathbf{k})$ is the Fourier transform of R_{ij}, and, summing over directions of $\mathbf{k}$ to get the omnidirectional spectrum, for isotropic turbulence, $E(k) = 4\pi k^2 S_{ii}(k)$. Finally the energy per unit mass associated with the scale $\ell \sim 1/k$ is $u_k^2 \sim \sqrt{kE(k)}$. See Batchelor (1970).

ing energy into the flow. The fluid motion at scale L becomes unstable and loses its energy to neighboring smaller scales without directly dissipating it into heat (local energy transfer). This process repeats itself until one reaches a dissipation scale l_d (the Kolmogorov scale), where the energy transfer is directly dispersed into heat by viscous action. The rate of energy input at the large scales and that of the energy dissipation (denoted ϵ) at the Kolmogorov scale are on average equal to each other and thus so is the energy transfer rate across the spectrum at intermediate scales. The range of intermediate scales is commonly called the *inertial range* in turbulence terminology. The anisotropy and inhomogeneity at large scales are thought to diminish with decreasing scales, so that scales far smaller than L become statistically isotropic and homogeneous (however, see Zhou *et al.*, 1996).

Although the energy-containing eddies exert a dominant control over the rate of transfer in decaying turbulence, this control is indirect, and excitations in the energy-containing range do not directly affect energy transfer within the inertial range. Therefore the average rate of energy dissipation is identified with the rate of spectral energy transfer and the rate of energy production ϵ. In order to infer the form of the inertial-range spectrum, it is necessary to estimate the magnitude of the transfer function correlations (the so-called "triple correlations" involving triple products of velocity components), which are responsible for inducing energy transfer. The time scale for decay of the transfer function correlations, $\tau_T(k)$, may depend on any relevant turbulence parameters and the wave number k (see footnote 2 for the concepts of wave number and length scales in turbulence). On general theoretical grounds (Batchelor, 1970; Monin and Yaglom, 1975; see Sec. II.D below) one may argue that the energy transfer flux $\Pi(k)$ is explicitly proportional to $\tau_T(k)$ and depends on the wave number and on the power of the omnidirectional energy spectrum $E(k)$. In the inertial range, because energy is conserved by the nonlinear interactions and a local cascade has been assumed, the energy flux Π becomes independent of the wave number k (see Zhou, 1993a, 1993b for numerical studies). By dimensional arguments one can obtain

$$\epsilon = \bar{C}\tau_T(k)k^4 E^2(k), \tag{2}$$

where $\tilde{C}$ is an order one constant.

The well-known Kolmogorov spectrum can be recovered within this framework for homogeneous, isotropic, statistically steady turbulence. For this case the nonlinear dynamical time scale is

$$\tau_{nl}(k) \equiv \ell/u_k = [k^3 E(k)]^{-1/2}, \tag{3}$$

where $\ell \propto 1/k$ is a length scale in the inertial range and $u_k = [kE(k)]^{1/2}$ is the characteristic velocity of eddies with wave number k. In Kolmogorov analysis, this is the only time scale available, and therefore $\tau_T(k) = \tau_{nl}$. It follows immediately from Eqs. (2) and (3) that
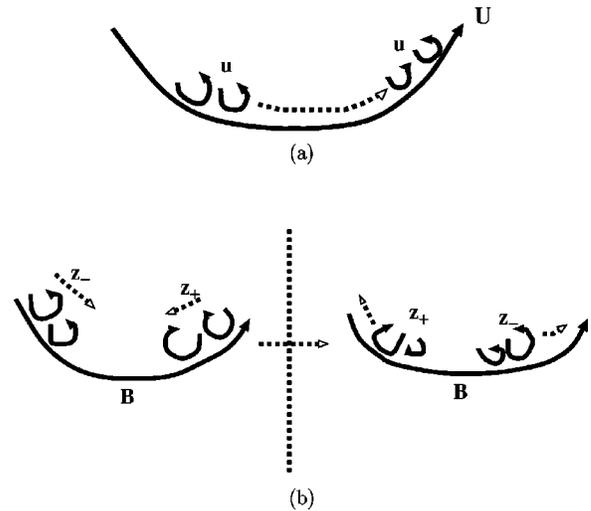
FIG. 1. Comparison of hydrodynamics and magnetohydrodynamics: (a) In hydrodynamics a mean or large-scale flow sweeps the small-scale eddies without affecting the energy transfer between length scales; (b) in magnetohydrodynamics a mean or large-scale magnetic field **B** sweeps oppositely propagating fluctuations $\mathbf{z}_-$ and $\mathbf{z}_+$, which affects the energy transfer (illustrated as distortions after the two types of fluctuation have passed through each other).

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}, \tag{4}$$

which is the Kolmogorov spectrum. Note that $\tau_{nl}(k) \sim \epsilon^{-1/3} k^{-2/3}$. Here C_K is the Kolmogorov constant (for a collection of recent results on the Kolmogorov constant, see Sreenivasan, 1995 and Yeung and Zhou, 1997).

C. Straining and sweeping

The classical Kolmogorov spectrum is based on a "cascade picture" in which the energy transfers between scales much like a series of waterfalls, each one filling a pool that overflows into the next one below (Tennekes and Lumley, 1972). This cascade occurs mainly as a consequence of interactions between eddies of nearly equal size (locality in wave-number space). The interactions consist of *straining* motions in which a vortex produces gradients in the velocity that distort other vortices. On the other hand, a large-scale flow will carry the small-scale vortices but will not induce much distortion on their internal dynamics. The direct interactions of the large and small scales consists then of *sweeping* motions which do not involve significant energy transfer in wave-number space and will not change the form of the hydrodynamic energy spectrum. This is sketched in Fig. 1(a). Later (Sec. III.A) we shall discuss how the influence of wave propagation in MHD may be viewed as a generalization of nonlocal sweeping effects, but with somewhat different consequences.

Instead, the time correlations or equivalently the form of the frequency spectrum (obtained from time series of the velocity at a fixed point) can be strongly influenced by the sweeping effect, since any small-scale fluctuations (which come from the straining motions) will be ad-

vected and passed through the probing point, thus introducing strong fluctuations in the time series.

The Taylor (1938) frozen-turbulence approximation assumes that a large-scale flow, with velocity $\mathbf{U}$, sweeps the turbulence by the point of observation. This approximation, with a large constant speed U ($\gg$ the fluctuation speed u), is used in wind-tunnel studies and in single-spacecraft studies of solar wind turbulence (Jokipii, 1973) to convert time-lagged correlations into spatial correlations. The underlying idea is that the large-scale flow at speed U sweeps the spatial fluctuations past the observation point faster than local nonlinearities can produce distortions. Then the frequency spectrum has the form $E(\omega) \sim \epsilon^{2/3} U^{2/3} \omega^{-5/3}$. Sweeping by random flows (with large scale but random $\mathbf{U}$) gives a similar result (Tennekes, 1975; Chen and Kraichnan, 1989). In contrast, when sweeping is negligible compared to straining motions, the spectrum can be predicted by dimensional analysis, by requiring that the spectral density in frequency depend only upon the energy cascade rate and the frequency (Tennekes, 1975; Nelkin, 1994). This implies that $E(\omega) \sim \epsilon \omega^{-2}$. As can be seen, the presence or absence of a large-scale flow U is relevant for the frequency spectrum if a sweeping hypothesis is assumed.

On the other hand, it seems rather certain that sweeping does not enter directly into the form of the hydrodynamic wave-number energy spectrum in Eq. (4) (see Chapman, 1979; Saddoughi and Veeravalli, 1994; Yeung and Zhou, 1997). However, there is a clearly established relationship (Nelkin and Tabor, 1990) between sweeping and the higher-order moments (e.g., structure functions and kinetic-energy spectra). In particular, the controlling influence that random sweeping exerts on frequency spectra (Chen and Kraichnan, 1989) can directly translate into a similar influence on the (fourth-order) spectrum of kinetic-energy density.[3] Heuristically, the frequency spectrum at a fixed point is a direct consequence of the large-scale flow sweeping a spatially nonuniform (intermittent) distribution of fluctuation energy past the observation point (see Fig. 1). Evidently Kolmogorov scaling, with its single nonlinear time scale $\tau_{nl}(k)$, does not immediately extend to higher-order moments, which may include the influence of the sweeping and other time scales. This view is supported by experiments (Van Atta and Wyngaard, 1975; Zhou *et al.*, 1993; see also Dutton and Deaven, 1972) that show that higher-order inertial-range spectra have the same power-law exponents as the energy spectrum, rather than the values that would be implied by a pure straining argument. These experiments demonstrate the importance of the sweeping effect, the multiplicity of turbulence time scales, and the role of nonlocality.

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[3]The energy spectrum in Eq. (4) is related by a Fourier transform to a correlation of the form $\langle uu' \rangle$, where the prime denotes a spatially lagged variable. The spectrum of kinetic energy is related to $\langle u^2 u'^2 \rangle$ and is therefore a fourth-order moment of the underlying probability distribution function.

D. Time scales, cascade, and closures

How can additional time-scale effects be built into a simple theory of turbulence spectra? One possibility is to look further into the physical underpinnings of Eq. (2). Suppose we write the transfer rate or energy flux in the suggestive form

$$\epsilon = \Pi(k) = \frac{u_k^2}{\tau_{sp}}. \tag{5}$$

Here u_k^2 is (twice) the energy per unit mass associated with the velocity at scales near $1/k$. For compatibility with Eqs. (2) and (3), we can identify the spectral transfer time as $\tau_{sp}(k) = \tau_{nl}^2(k)/\tau_T(k)$. Thus, by developing different approximations for the triple correlation time τ_T, we can arrive at a variety of models of turbulence spectra.

The deeper significance of this rather symbolic procedure can be seen by examining how relationships akin to Eqs. (2) and (5) emerge in more formal mathematical treatments of turbulence and how precisely defined time scales that enter these theories become associated with the terms that appear in the phenomenological treatments.

A particularly revealing form of the evolution equation for the energy spectrum emerges from the closure known as the eddy-damped, quasinormal Markovian (EDQNM) approximation. As a brief background (Orszag, 1970, 1977; Monin and Yaglom, 1975; McComb, 1990; Lesieur, 1997), let us consider the structure (and not the details) of this approach. Consider, in a highly symbolic notation, the wave-number space Navier-Stokes equation, that is, $\partial_t \hat{u}(\mathbf{k}) = \hat{u}\hat{u} - \nu k^2 \hat{u}(\mathbf{k})$. The notation $\hat{u}$ refers to the Fourier representation of the velocity field and reminds us that we are suppressing Cartesian indices, summations over indices, wave-vector arguments, and time arguments (see Lesieur, 1997). The equation for the modal spectrum $E(k)/4\pi k^2 \sim \langle \hat{u}\hat{u} \rangle$ is symbolically of the form

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) \langle \hat{u}\hat{u} \rangle = \langle \hat{u}\hat{u}\hat{u} \rangle. \tag{6}$$

This involves the third-order moments ("triple correlations") $\hat{u}\hat{u}\hat{u}$, which obey an equation of the form

$$\left(\frac{\partial}{\partial t} + \nu(k^2 + p^2 + q^2) \right) \langle \hat{u}(\mathbf{k})\hat{u}(\mathbf{p})\hat{u}(\mathbf{q}) \rangle = \langle \hat{u}\hat{u}\hat{u}\hat{u} \rangle, \tag{7}$$

where $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$ each denote wave vectors. The "closure problem" refers to the occurrence of the fourth-order correlations in the equation for the third-order correlation, and so on. Closure methods adopt approximations for higher-order moments in terms of lower-order moments. The quasinormal approximation (QNA) (Millionschikov, 1941) represents the fourth-order moment in Eq. (7) as a sum over products of second-order moments. This allows a solution for the third-order moments, which are then substituted into Eq. (6), which becomes a closed equation for the second-order moments, thereby also giving the energy spectrum. Orszag

(1970) and others introduced additional refinements and approximations[4] that have become known as the EDQNM approximation. This is in many ways an acceptable approximate model for turbulence, and it leads to an equation for the spectrum that can be written as

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)E(k,t) = \int\int_\Delta dpdq\Theta_{kpq}E(q,t)$$
$$\times [A(k,p,q)E(p,t)$$
$$- B(k,p,q)E(k,t)]. \tag{8}$$

Here the integrals are over values of the wave vectors restricted to $\mathbf{q}=\mathbf{k}-\mathbf{p}$, and A and B are coupling coefficients having the dimension of wave number. The inertial-range energy flux can then be calculated from the integral $\Pi(k)=-\partial/\partial t\int_0^k E(k,t)dk$ by using the EDQNM expression on the right-hand side of Eq. (8). Finally, by estimating that the dominant contributions to the integrals are from $p\approx q\approx k$, we can arrive at the estimate $\epsilon=\Pi(k)=\tau_T(k)k^4 E^2(k)$ previously obtained by dimensional analysis in the inertial range. Notably, the "triple decay time" $\tau_T(k)$ that earlier we assumed to appear on heuristic grounds is now identified with the eddy-damping time Θ_{kkk} that emerges as the key time scale in the evaluation of the EDQNM energy flux. It is very reassuring that this can be done, because now the dimensional result acquires a context within an analytical theory. However, it is very important to keep in mind that the role played by the "eddy-damping rate" of the triple correlations, on physical grounds, is to restore, in an approximate way, the decorrelation effect of the third-order cumulants[5] that were already neglected in the QNA. In effect, the choice of eddy-damping rate "rigs" the EDQNM approximation to yield a particular spectral law. Thus, in spite of its algebraic elegance, the EDQMN closure requires that we correctly understand the physics that determines time decorrelation. Even so, the identification $\tau_T(k)=\Theta_{kkk}$ gives us confidence about how such time scales act in more formal theories, while providing a chain of reasoning that connects the dimensional analysis with the mathematical structure of turbulence. This is a useful background when we extend the use of the cascades phenomenology to include other time-scale effects in MHD.

An additional point of contact with formal theories warrants mention. Another very well known closure theory, Kraichnan's (1957) direct interaction approxima-

tion (DIA), perhaps the archetype for statistical closure theories of turbulence, proceeds through a formal perturbation expansion in which the lowest-order velocity field obeys exactly Gaussian statistics. The DIA seeks a solution to the turbulence closure problem by expanding in a parameter δ, letting the nonlinear terms in the evolution equation be of order δ, a parameter eventually set equal to unity. To facilitate solution, a propagator (Green's function) $\tilde{G}(\mathbf{k},t,t')$ is defined, which is the system response to a delta-correlated Gaussian forcing function. A key step is to arrive at coupled renormalized equations for the averaged propagator $G=\langle\tilde{G}\rangle$ and for the time-lagged spectral function $Q(\mathbf{k},t,t')$, which determines how rapidly correlations decay in time.[6] While the details are unimportant here (see, Leslie, 1973; McComb, 1990), it is clear from the structure of the theory that the simultaneous solution for Q and G establishes the nature of the two time decorrelations at each wave number. Accordingly, through the propagator formalism, the time dependence of third-order correlations is established in terms of the second-order correlations.

In general the EDQNM and DIA models are quite distinct, but McComb (1990; p. 308) argues that there is an interesting *ad hoc* modification to the DIA that shows a structural connection between those models. Suppose that instead of solving the isotropic DIA equations for Q and G in the usual way, one departs from the DIA and makes the very simple approximation, for $t > t'$, that $Q(\mathbf{k},t,t')=S(\mathbf{k},t')e^{-\gamma(k)(t-t')}$ and $G(\mathbf{k},t,t') = e^{-\gamma(k)(t-t')}$ where the decorrelation rate is taken to be the reciprocal of the nonlinear time, $\gamma(k)=1/\tau_{nl}(k) \sim \epsilon^{1/3}k^{2/3}$. Then this "pseudo-DIA" equation for spectral evolution becomes identical to the EDQNM spectral equation, Eq. (8). This identification should not be taken too seriously, as the DIA prescribes the time decorrelation in its own way. However, to the extent that McComb's limit is realizable, the EDQNM approximation, the above-modified DIA procedure, and the phenomenological approach are in agreement with one another: All quantities of interest are determined by the wave-number-dependent correlation damping rate, just as in the successful EDQNM closures (Orszag, 1970; Lesieur, 1997), the spectral power law is determined by the choice of the time scale Θ_{kpq}. In this light, it seems reasonable to pursue a phenomenological treatment of MHD spectra to understand the variety of conclusions that may be applicable in space and astrophysical plasmas.

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[4]The "eddy-damping approximation" consists of inserting a time dependence of the triple correlations (Orszag, 1970) that is associated with nonlinear distortions (strain). In a formal sense this amounts to letting $\nu k^2 \to \nu k^2 + \gamma(k)$ with the rate of decay of triple correlations enhanced by the nonlinear rate of strain, $\gamma(k)=[\tau_{nl}(k)]^{-1}$. The "Markovian" approximation further simplifies the theory by asserting that the spectra are essentially unchanged during a triple decay time (Leith, 1971).

[5]A cumulant is the contribution to a moment of the probability distribution that is due to departure of the distribution from that of a Gaussian distribution.

[6]Two-point correlations that involve both a spatial lag $\mathbf{r}$ and a time lag $\tau=t-t'$ may be defined as $R_{ij}(\mathbf{r},\tau)=\langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t+\tau)\rangle$. The time-lagged spectral tensor $S_{ij}(\mathbf{k},\tau)$ is the Fourier transform of R_{ij} with respect to space, the time lag now denoting the time-decorrelation of each spectral element. The DIA spectral function Q is essentially the trace $S=S_{ii}(\mathbf{k},\tau)$.

III. MAGNETOHYDRODYNAMIC TURBULENCE

A. MHD equations and basic physical concepts

Having established a baseline understanding of how various time scales enter hydrodynamic turbulence, we now turn to the magnetohydrodynamic case, in which things are more complicated, but, as we shall see, some of the same ideas can be applied to understand what spectra are expected.

A plasma, described here as an electrically conducting gas or fluid, evolves in response to both mechanical and electromagnetic forces. For simplicity, as we did in hydrodynamics, we focus upon the constant-density incompressible model, which provides an adequate context for the issues of MHD turbulence that are of primary concern here (see Biskamp, 2003). The incompressible MHD model, in terms of the fluid velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$, includes a momentum equation,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{u}, \tag{9}$$

and a magnetic induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \mu \nabla^2 \mathbf{B}. \tag{10}$$

The plasma density ρ, the kinematic viscosity ν, and the magnetic diffusivity μ, are assumed to be uniform constants. The velocity and magnetic field are solenoidal, $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0$, and the pressure p is determined by taking the divergence of Eq. (9). The dimensionless Reynolds number $R = uL/\nu$ (where u is a typical velocity and L a typical length scale) and magnetic Reynolds number $R_m = uL/\mu$ are measures of the relative strength of the nonlinear terms and linear (dissipative) terms in the dynamical equations. Highly turbulent MHD occurs at high values of R and R_m.

Before continuing, let us recall that MHD is frequently applied to space and astrophysical plasmas for which the derivation of the model is not so clear as one would like. This stands in contrast to the rather more firm conceptual status of hydrodynamics or gas dynamics for which well-established methods, either macroscopic or perturbative (such as the Chapman Enskog expansion), exist and are given in standard texts on statistical mechanics. For low-collisionality plasmas, the basic structure of MHD emerges from conservation of mass, momentum, and energy, along with the Maxwell-Ampère and Faraday laws, upon ignoring displacement current and adopting a suitable form of Ohm's law. However, for most applications there is usually not a clear path to closing the system with a single isotropic pressure field, nor is there usually a convincing calculation of viscosity, resistivity, and other transport coefficients such as thermal conductivity.

A customary approach in numerical work is to adopt scalar dissipation coefficients, choosing values based on numerical limitations of spatial resolution rather than on physical realism. For turbulent MHD this may be justified in part by assuming that the nonlinear cascade is mainly from large to small scales, and the role of the specific dissipation mechanism is simply to absorb whatever energy arrives at small scales through spectral transfer. This is only partially satisfactory, and a more comprehensive theoretical understanding of the nature of dissipation in low-collisionality MHD applications is desirable, though it may be neither simple in nature nor of universal form. On the bright side, when observations are available, as they are in the solar wind, one can see that the inertial range is broadband, and therefore the energy-containing range is well separated in scale from the dissipation range, where the spectrum steepens (Leamon *et al.*, 1998). On this basis one can infer a kind of effective Reynolds number for the solar wind, or by analogy, for any turbulent plasma for which the extent of the inertial range is known. For example, ignoring the difference between viscous and resistive dissipation, one might employ the hydrodynamic estimate of the dissipation wave number $k_d = (\epsilon/\nu^3)^{1/4}$. Using the Taylor–von Karman estimate of the decay rate $\epsilon = u^3/\lambda$, this can be cast in the form $k_d \lambda = R^{3/4}$, or $R = (k_d \lambda)^{4/3}$ where R is the Reynolds number. The quantity $k_d \lambda$ is approximately the bandwidth of the inertial range. Therefore for a three-to-four-decade inertial range (e.g., the solar wind, approximately), one has $R \approx 10^5$. For the lower solar corona, a five-to-six-decade inertial range is estimated, so $R \approx 10^8$.

In general, whenever there is a broadband inertial range extending over decades in scale, one can infer that the effective large-scale Reynolds number is very large, even when a formal theory for the dissipation mechanism is not yet available.

The magnetic field may contain a uniform part $\mathbf{B}_0$ (below, the *dc magnetic field*) or a smoothly varying part (which we identify as a local mean magnetic field) plus small-scale fluctuations $\mathbf{b}$, that is, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$. The large-scale magnetic field supports propagation of hydromagnetic waves; here, for the incompressible case, we call these Alfvén waves. These waves are fluctuations transverse to the mean magnetic field, propagating along the mean magnetic field direction at the Alfvén speed $V_A = B_0/\sqrt{4\pi\rho}$.

Even the simplest MHD case, assuming incompressibility, isotropy, stationarity, and homogeneity, is more complex than hydrodynamics. There are two distinct fields to deal with, the magnetic and velocity field, and additional complexity due to Alfvén wave propagation effects. There are at least two classes of time scales involved, the nonlinear time and the Alfvén time (the time for a fluctuation to propagate a given length scale). Moreover, the large-scale magnetic field introduces a preferred direction, and anisotropic effects on the fluctuations are present.

Here we arrive at a major difference between fluid and MHD turbulence. Unlike fluid turbulence, the non-local effect of large scales upon the small scales, described above as "sweeping," is an important issue for

MHD turbulence. Beginning with the work of Iroshnikov (1963) and Kraichnan (1965), it has been argued that such effects play a significant role in MHD turbulence, even when dc magnetic fields are absent. If there is a strong, large-scale magnetic field, the small-scale fluctuations are subject to a sweep-like effect due to Alfvén wave propagation. To discuss this it is useful to write MHD in a more symmetric form, in terms of the so-called Elsässer fields (Elsässer, 1956), $\mathbf{z}_+ = \mathbf{u} + \mathbf{b}/\sqrt{4\pi\rho}$ and $\mathbf{z}_- = \mathbf{u} - \mathbf{b}/\sqrt{4\pi\rho}$,

$$\frac{\partial \mathbf{z}_\pm}{\partial t} \mp \mathbf{V_A} \cdot \nabla \mathbf{z}_\pm = -\mathbf{z}_\mp \cdot \nabla \mathbf{z}_\pm - \frac{1}{\rho} \nabla P + \mu \nabla^2 \mathbf{z}_\pm, \quad (11)$$

where we have explicitly separated a term involving the large-scale magnetic field (written here in terms of the Alfvén velocity $\mathbf{V_A}$). For simplicity, we have assumed $\nu = \mu$. The total pressure $P = p + B^2/8\pi$ acts to enforce the constraints $\nabla \cdot \mathbf{z}_\pm = 0$.

Setting either $\mathbf{z}_+ = 0$ or $\mathbf{z}_- = 0$ provides exact solutions of the ideal (without dissipation) MHD equations. The nonzero field is often said to correspond to *wave packets* that propagate along the mean field direction. This description can be misleading because the "packets" may not be localized and also may not propagate. Nonpropagating fluctuations with wave vectors strictly perpendicular to the mean magnetic field have zero phase speed. In any case, one sees from the MHD equations that both types of fluctuations $\mathbf{z}_\pm$ are needed for the nonlinear terms to be nonzero and to sustain turbulence. That fact was pointed out by Kraichnan (1965) and discussed in the context of space physics applications (Dobrowolny *et al.*, 1980) and more recently in solar coronal heating models (Dmitruk *et al.*, 2001).

Kraichnan (1965) noted that the mean magnetic field sweeps the small-scale structures which interact, and during that time nonlinear transfer of energy between length scales occurs (in the Kraichnan picture the "wave packets" suffer brief "collisions" during which energy transfer occurs). This is illustrated in Fig. 1(b). One can see then that the mean magnetic field induces an inhibition of the nonlinear energy cascade (Chen and Kraichnan, 1997).

For high-Reynolds-number MHD turbulent flows in astrophysical and space environments, there is scale separation to distinct physical processes at large and small scales. Specifically, one divides the dynamics into a small-scale part that contains "small-small" and "large-small" couplings, and a large-scale part (see Zhou and Matthaeus, 1990a). When the small-scale fields are broadband, one tends to treat the "small-small" coupling as turbulent, involving couplings that are principally local in wave-number space.

Studies of MHD turbulence in space and astrophysical contexts often become tractable at some level when one introduces some form of scale separation. In the simplest approximation a small part of an inhomogeneous MHD system might be treated as "locally homogeneous." Turbulence in the solar wind is a case in point, one in which the wealth of observational data has made
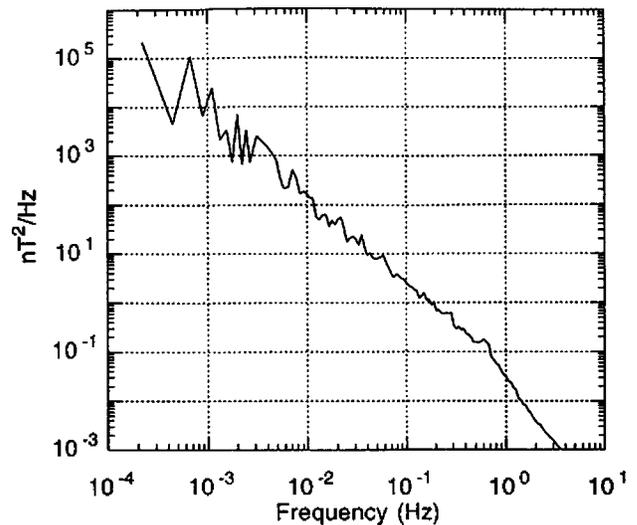

FIG. 2. The trace of the power spectral matrix of the magnetic field fluctuations (nT=nanotesla) from a 74-min sample of data values, each of which is an average value taken over 0.12 sec of data from the Mariner 10 magnetometer on March 20, 1974. The solar wind data indicate a $-5/3$ Kolmogorov scaling for the slope of the spectrum. Adapted from Goldstein *et al.*, 1995.

possible a productive interplay between theory and observation. (See Tu and Marsch, 1995 and Goldstein *et al.*, 1995 for reviews). Early observational studies (e.g., Coleman, 1968) found that spacecraft-frame temporal fluctuations of the plasma fluid velocity admitted a power-law wave-number spectrum (see Fig. 2), reminiscent of the Kolmogorov description of fluid turbulence (Batchelor, 1970; Tennekes and Lumley, 1972; Monin and Yaglom, 1975; Pope, 2000). Observations (Coleman, 1968; Belcher and Davis, 1971; Jokipii, 1973) also revealed a distinctive correlation between velocity and magnetic field that suggests outward-traveling large-amplitude Alfvén waves (Fig. 3).

The solar wind, like most real astrophysical systems in which turbulence is found, is compressible and inhomogeneous at the larger scales. Large-scale inhomogeneities, such as velocity shear or temperature and density gradients, can supply energy to the small-scale turbulence. In the solar wind the observed fluctuations are broadband with correlation scales ($\lambda \sim 0.02$ AU, astronomical unit, at Earth orbit) that are much smaller than the scale of the system (1 AU or more). The MHD turbulence inertial range extends roughly from λ down to scales 1000 times smaller, near the thermal ion gyroscale. Thus turbulence activity of interest is well separated in length scale from the large-scale solar wind inhomogeneities. Moreover, the large-scale properties, such as mean flow and mean magnetic field, are relatively coherent and reproducible. Therefore the solar wind is often described in terms of a canonical average flow and magnetic-field properties, such as low-speed, quiet solar wind at low latitudes, a hotter, less dense, and faster wind at high latitudes, an Archimedean spiral magnetic field, and other idealized large-scale features. Even when dynamic large-scale structures (e.g., interaction re-
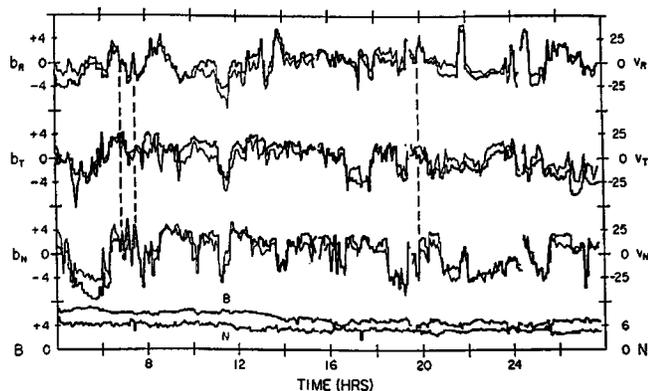
FIG. 3. Twenty-eight hours of magnetic-field and plasma data demonstrating the presence of nearly pure Alfvén waves. The upper six curves are 5.04-min bulk velocity components and magnetic-field components averaged over the plasma probe sampling period. The vertical axes on the right are for the Cartesian velocity components (R=radial, T=tangential, N=normal, relative to ecliptic plane) in km/sec; the vertical axes on the left are for the magnetic-field components in nT. The lower two curves are magnetic-field strength and proton number density (in cm^{-3}). From Belcher and Davis, 1971. Reprinted with permission from Journal of Geophyscal Research, American Geophysical Union.

gions between streams) are present, these features can be viewed to some degree as reproducible.

In contrast, the observed small-scale solar wind fluctuating fields are generally viewed as random and locally homogeneous. These fluctuations were treated originally (Parker, 1965; Hollweg, 1973, 1974, 1986; Barnes and Hollweg, 1974; Jacques, 1977; Barnes, 1979; Heinemann and Olbert, 1980) using linearized weakly inhomogeneous MHD ("WKB" theory), which describes the propagation of short-wavelength Alfvénic fluctuations in an inhomogeneous flow. The present perspective is that the medium is locally incompressible (Matthaeus *et al.*, 1990) and is described to an acceptable degree of approximation as MHD turbulence.

The basic dichotomy between the nonlinear "turbulence" picture and the linear "wave" picture pervades 40 years of studies of the solar wind and mirrors the basic theme of the present Colloquium: the observable features of MHD turbulence emerge as a balance between, on the one hand, wave-propagation or sweeping effects, and on the other hand, nonlinear distortions or straining effects.

B. Phenomenology of MHD decay

Whereas in hydrodynamic turbulence we deal with one energy density u^2 and one associated two-point correlation tensor, the presence of two dynamical fields in MHD introduces four types of correlations or energies (per unit mass), the kinetic energy $E_u = \langle |\mathbf{u}|^2 \rangle / 2$, the magnetic energy $E_b = \langle |\mathbf{b}|^2 \rangle / 2$, the cross helicity $H_c = \langle \mathbf{u} \cdot \mathbf{b} \rangle = \langle |\mathbf{z}_+|^2 - |\mathbf{z}_-|^2 \rangle / 4$ and the energy difference $D = \langle |\mathbf{u}|^2 - |\mathbf{b}|^2 \rangle / 2 = \langle \mathbf{z}_+ \cdot \mathbf{z}_- \rangle / 2$. (Here we are using Alfvén speed

units in which the magnetic field $\mathbf{b} \to \mathbf{b}/\sqrt{4\pi\rho}$ has dimensions of a velocity.) Note that the cross helicity is the difference of the Elsässer energies $Z_+^2 = \langle |\mathbf{z}_+|^2 \rangle / 4$ and $Z_-^2 = \langle |\mathbf{z}_-|^2 \rangle / 4$. The most symmetric situation is when we have equipartition, $D = 0$ and zero cross helicity ("non-Alfvénic") $H_c = 0$ turbulence. For this case neither the velocity and magnetic field, nor the two Elsässer variables are correlated with one another. In this very "plain" type of MHD, $Z_-^2 = Z_+^2 \equiv Z^2$, and a simple extension to the hydrodynamic decay phenomenology works well (Hossain *et al.*, 1995) for MHD decay at moderate Reynolds numbers. In particular, $dZ^2/dt = -\alpha Z^3/\lambda$ and $d\lambda/dt = \beta Z$, with α and β order-one constants and λ a similarity or energy-containing scale. The choice of constants can be given a physical interpretation as in the hydrodynamic case (Matthaeus *et al.*, 1996). This hydrodynamic-like approach to MHD decay has been used in transport modeling of turbulence in the solar wind; such modeling provides a reasonably accurate account of the radial profile of the solar wind turbulence level and temperature (Smith *et al.*, 2001) from earth orbit (1 AU) to beyond 60 AU.

More generally, zero cross helicity cannot be assumed, and a decay phenomenology should take into account asymmetry between Z_+^2 and Z_-^2. This introduces additional time scales. The basis for this is found, for example, in the phenomenological discussions of Iroshnikov (1963), Kraichnan (1965), and Dobrowolny *et al.* (1980), and in the detailed treatment of MHD closures by Pouquet *et al.* (1976) and Grappin *et al.* (1983). Here we adopt the phenomenological approximation (Hossain *et al.*, 1995) that

$$\frac{dZ_\pm^2}{dt} = -\alpha_\pm \frac{Z_\pm^2}{\tau_{sp}^\pm} \tag{12}$$

in terms of constants α_+ and α_-. The simplest estimate for the nonlinear time scales (Pouquet *et al.*, 1976; Grappin *et al.*, 1983) is that the spectral transfer time is identified with the eddy turnover or nonlinear time $\tau_{sp}^\pm = \tau_{nl}^\pm$ while the latter, taking into account the nature of the interactions between $\mathbf{z}_+$ and $\mathbf{z}_-$, can be estimated as $\tau_{nl}^\pm = \lambda_\pm / Z_\mp$ for similarity scales $\lambda_\pm$. For this choice the energy decay equation becomes

$$\frac{dZ_\pm^2}{dt} = -\alpha_\pm \frac{Z_\pm^2 Z_\mp}{\lambda_\pm}. \tag{13}$$

To close the system of decay equations one should choose (and verify if possible) an evolution equation for the similarity scales $\lambda_\pm$. One possibility is (Hossain *et al.*, 1995) $d\lambda_\pm/dt = \beta_\pm (Z_+ Z_-)^{1/2}$, but there remains some difficulty in verifying this behavior of the similarity scales.[7]

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[7]One usually attempts in simulations of turbulence to allow maximum possible spatial resolution at the small scales, in order to capture an accurate direct cascade of energy. However, to simulate the correct growth of the energy-containing scales during decay also requires simulations with adequate large-scale resolution.

Another difficulty with MHD decay phenomenology is the degree of certainty and generality with which one makes the identification $\tau_{sp}^{\pm} \rightarrow \tau_{nl}^{\pm}$ that we used above to arrive at Eq. (13). Indeed one may question whether another time scale enters into global decay, for example, the Alfvén crossing time of the energy-containing eddies, or some other sweeping-like scale that can decorrelate the large-scale nonlinear interactions, thereby modifying the global rate of energy decay. Such issues are discussed at some length below in the context of the inertial range of MHD turbulence, for the cases of both isotropic and anisotropic MHD with a large-scale magnetic field.

However, for the energy-containing range of scales, the issue remains ambiguous at present. For homogeneous periodic turbulence, Hossain *et al.* (1996) argue that the development of anisotropy relative to a large-scale magnetic field acts to saturate and minimize the decorrelation effect on the energy range of scales. However this presupposes that the large-scale Alfvén crossing time $\tau_A = \lambda / V_A$ is not too small. This may depend upon the initial conditions or driving, as well as the boundary conditions. A particularly sensitive issue in applications (see Dmitruk *et al.*, 2001) is whether the boundary conditions permit the persistence of nonpropagating structures, such as 2D turbulence, that are unaffected by the wave-sweeping time τ_A. Also, the interplay between boundary conditions, wave propagation effects, and nonlinear interactions can have an impact on the turbulence level (measured by the energy transfer rate) maintained by the system (see Dmitruk and Matthaeus, 2003 for a particular application in a coronal heating model). For now, however, we note that the multiplicity of MHD time scales may pervade energy range dynamics as well. In such cases the problem-specific details may influence the energy decay time. For certain standard problems, such as periodic or homogeneous MHD with band-limited initial conditions, numerical evidence supports the statement that the global decay time is associated mainly with nonlinear effects and that sweeping or Alfvén time effects are not significant. Clearly this conclusion would need to be readdressed in other problems. For example, the case of initially spatially localized Elsässer fluctuations under the influence of a large-scale dc magnetic field (Parker, 1979) may present an interesting contrast to the case of homogeneous turbulence. With this background we now turn our attention to summarizing the role of time scales in the variety of possible MHD cascades in the inertial range.

C. Isotropic MHD

1. When local straining is dominant: Kolmogorov scaling

Montgomery and co-workers (Fyfe *et al.*, 1977) suggested that the original Kolmogorov reasoning and its associated $k^{-5/3}$ spectrum are applicable to MHD. The implicit assumption here seems to be that the nonlinear

distortion of eddies is faster than the decorrelation effects associated with wave propagation. This implies that the relevant time scale is τ_{nl} and that the straining dominates over random sweeping or propagation. This approach seems most reasonable when the cross helicity is small, the velocity and magnetic fields are close to equipartition, and the large-scale magnetic field is not too large.

One of the strongest sources of support for the $k^{-5/3}$ scaling in MHD comes from *in situ* spacecraft observation of solar wind, in which the $-5/3$ spectral law is often statistically distinguishable from other proposed power laws. An example is shown in Fig. 2. Typically the magnetic energy spectrum $E_b(k)$ displays a near-power-law behavior for some three decades in wave number. Matthaeus and Goldstein (1982) report such power laws between $10^{-11}/\mathrm{cm}^{-1}$ and $3 \times 10^{-9}/\mathrm{cm}^{-1}$, with spectral index -1.73 ± 0.08. The spectral decomposition of the total energy, $E(k) = E_b(k) + E_u(k)$ also typically follows a power law and is nearly equipartitioned between kinetic and magnetic energy in the inertial range. For the total energy, Matthaeus and Goldstein (1982) report a power-law dependence of the form $E(k) \sim k^{-1.69 \pm 0.08}$ at all but the lowest wave numbers. The expectation of equipartition in inertial-range scales is known as the *Alfvén effect* (Kraichnan, 1965). The frequent occurrence of Alfvénic fluctuations in the inner heliosphere (Fig. 3) is indicative not only of near energy equipartition, but also of the presence of cross helicity (Coleman, 1968; Dobrowolny *et al.*, 1980; Grappin *et al.*, 1982, 1983; Pouquet *et al.*, 1986). Generally speaking the solar wind evolves in the outer heliosphere toward less Alfvénic states but remains nearly equipartitioned between kinetic and magnetic energy (Roberts *et al.*, 1987, 1990), and usually $1 < E_u(k) / E_b(k) < 2$.

Simulations have also addressed the issue of MHD spectral indices. The energy spectra obtained from earlier MHD turbulence computations were inconclusive. For example, at a numerical resolution of $180 \times 180 \times 180$ rectangular grid points (Politano *et al.*, 1995) could not generate an extended inertial range. Recent high-resolution direct numerical simulation of MHD turbulence (Biskamp and Müller, 2000) provides strong support for the Kolmogorov $-5/3$ law (see Fig. 4). The spectrum shown has been multiplied by $k^{5/3}$, resulting in a flat region that indicates a clearly discernible, although short, inertial range $\hat{E}(\hat{k})\hat{k}^{5/3} = \hat{k}^{5/3} E_K / (\epsilon \eta^5)^{1/5} = \tilde{C}_K F(\hat{k})$, where $\tilde{C}_K$ is a constant and $\hat{k} = k l_d$ with $l_d = (\mu^3 / \epsilon)^{1/4}$ the Kolmogorov dissipation length (assuming $\mu = \nu$).

Small-scale current sheets are the dominant high-wave-number dissipative feature of MHD turbulence in three dimensions (Biskamp and Müller, 2000) and in two dimensions (Matthaeus and Lamkin, 1986). An example of the formation of small-scale current sheets in MHD turbulence is shown in Fig. 5 for the 2D case. The crucial role of current sheet formation and turbulence-driven reconnection can also be seen (Dmitruk *et al.*, 2002) in wave-driven reduced MHD models that conceptually lie between purely 2D and purely 3D models. In three di-
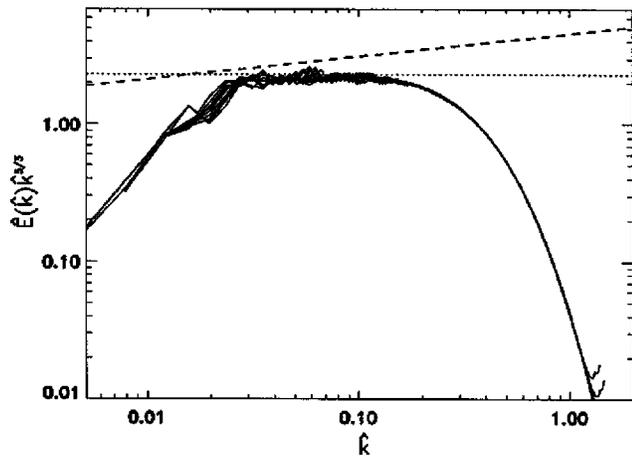
FIG. 4. Plot of the normalized angle-integrated energy spectrum of MHD turbulence vs $\hat{k} \equiv k\ell_d$ (where ℓ_d is the Kolmogorov dissipation length). All curves are multiplied by $k^{5/3}$. The Kolmogorov scaling is strongly suggested by this direct numerical simulation (solid curve), from a run that does not have magnetic helicity. The dashed line indicates the Iroshnikov-Kraichnan spectrum $k^{-3/2}$ (see Sec. III.C.2), and the dotted line the Kolmogorov spectrum [see. Eq. (4)] with $C_K=2.3$. From Biskamp and Müller, 2000.

mensions, current sheets are much more distorted and reconnection sites are harder to identify than in the 2D case (Politano *et al.*, 1995). As an example, Fig. 6 shows a snapshot of the spatial distribution of magnetic-field
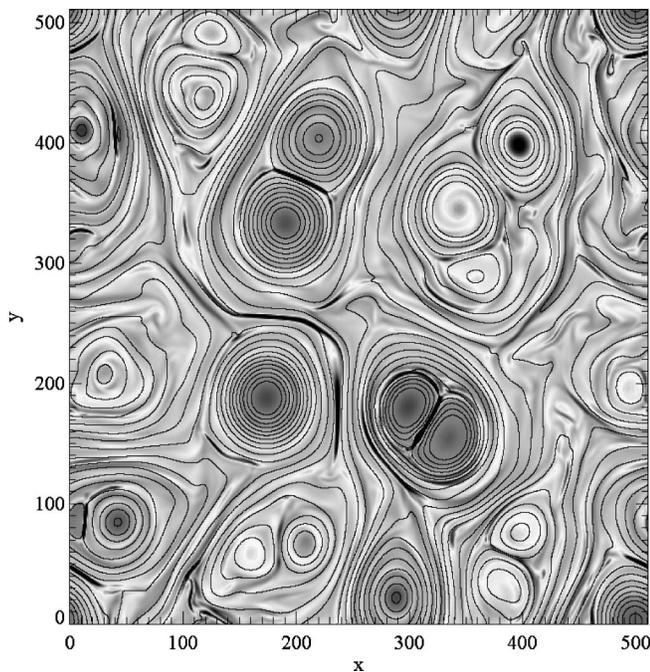

FIG. 5. Magnetic-field lines from a spectral-method simulation of 2D MHD turbulence, and associated randomly occurring electric current sheets and filaments that are oriented so that the current is perpendicular to the magnetic-field activity. The magnitudes of the currents are indicated by gray shading, with the most intense levels being white for out-of-the-plane currents and black for into-the-plane currents.
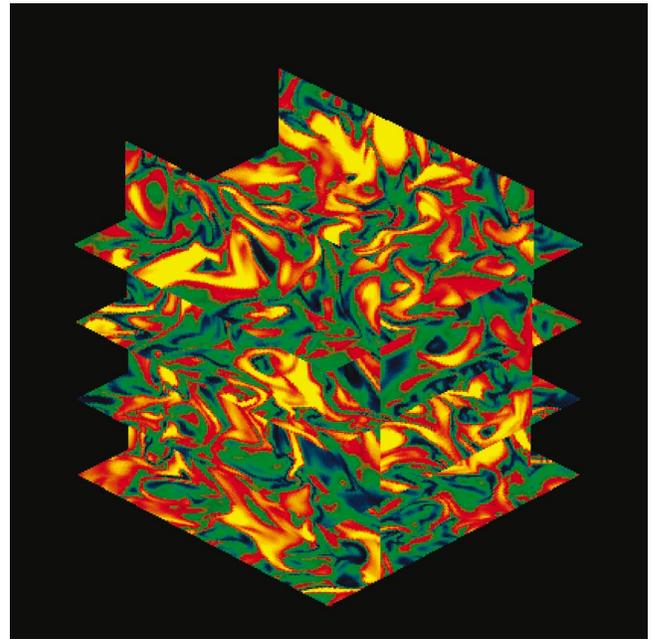
FIG. 6. (Color) Color map of the spatial distribution of magnetic-field intensity in a spectral-method simulation of 3D MHD turbulence. Spatial resolution is $256 \times 256 \times 256$ rectangular grid points. The mechanical and magnetic Reynolds numbers are 1000.

intensity in 3D MHD turbulence, which serves to illustrate the characteristic degree of spatial complexity. The formation of current sheets associated with reconnection of nearby magnetic structures is a fundamental aspect of MHD turbulence that is related to nonlinear strain-type motions.

The parallel dynamics of Alfvén waves do not control the turbulence, which is instead governed by cross-field eddy-type motions, which appear hydrodynamic-like. Biskamp and Müller argue that in three dimensions the swirling motion can easily dominate over Alfvén wave dynamics; therefore random sweeping is weaker than straining in 3D MHD turbulence in the absence of a dc magnetic field. This in turn indicates that the local energy transfer and local interactions are dominant.

2. When random sweeping is dominant: Iroshnikov-Kraichnan scaling

The simplest way to incorporate sweeping effects is to assume isotropic statistics but with the decorrelation time scale controlled by a characteristic Alfvén wave period. In this way, Iroshnikov (1963) and Kraichnan (1965) retain the basic Kolmogorov assumptions of isotropy and locality in the wave numbers of nonlinear interactions. Small-scale fluctuations are viewed as Alfvén wave packets traveling along the large-scale magnetic field and suffering brief collisions with oppositely propagating packets (we have cautioned the reader before about the limits of this pictorial view of the dynamics). Specifically, Iroshnikov and Kraichnan suggested that the triple velocity correlations in MHD turbulence decay in a time of the order of an Alfvén wave period.

Therefore $\tau_T = \tau_A$, $\tau_A = (V_A k)^{-1}$, and $\epsilon = \tilde{C}^2 \tau_T(k) k^4 E^2(k)$. As a result, the well-known Iroshnikov-Kraichnan $k^{-3/2}$ spectrum is obtained.

Grappin *et al.* (1982) examined the properties of the Iroshnikov-Kraichnan cascade using the 3D EDQNM approximation and found, after a few large eddy turn-over times, a quasistationary state that exhibits a −3/2 inertial range with zero correlation between velocity and magnetic fields. Two-dimensional direct numerical simulations (Biskamp and Welter, 1989; Biskamp, 1993; Galtier *et al.*, 1999) offer support for Iroshnikov-Kraichnan scaling. Biskamp and Müller (2000) pointed out that in two dimensions the swirling motions are weak, as manifested by the steep energy spectrum in 2D fluid turbulence. Hence straining is weakened and Alfvén-wave-induced sweeping is the dominant decorrelation effect.

3. Extended phenomenology

Matthaeus and Zhou (1989) and Zhou and Matthaeus (1990b) have developed a framework in which both time scales, τ_{nl} and τ_A, coexist, in a fashion analogous to the composition of triple decay times in the EDQNM closures (Pouquet *et al.*, 1976). The viewpoint is that the lifetime of transfer function correlations $\tau_T(k)$ is more accurately treated by taking into account the influences of both the external agent and turbulent nonlinear interactions. Composing the associated rates leads to

$$\frac{1}{\tau_T(\mathbf{k})} = \frac{1}{\tau_{nl}(\mathbf{k})} + \frac{1}{\tau_A(\mathbf{k})}. \tag{14}$$

Note that in general, but within the approximation of local nonlinear transfer, the nonlinear time may be a function of the vector $\mathbf{k}$. This reduces to the expected limiting cases when the effective magnetic-field strength goes either to zero or to infinity and therefore τ_T approaches either τ_{nl} or τ_A. Accordingly, for the classical case of isotropic turbulence, the energy spectra, $E(k) \sim k^{-m}$, as illustrated in Fig. 7, have a scaling exponent $3/2 \leqslant m \leqslant 5/3$ and reduce to either the Iroshnikov-Kraichnan or the Kolmogorov forms in the appropriate limit. This case may be difficult to realize, because, as we shall discuss in the next section, anisotropy, both global and local, may act to further reduce the Alfvén wave propagation effect.

D. Anisotropic MHD

Kraichnan was aware that the presence of a large-scale magnetic field that supports the propagation of Alfvén waves could result in inducing anisotropy (Galtier *et al.*, 2000). Alfvénic sweeping diminishes non-linear interaction, in the Iroshnikov-Kraichnan view, between the Elsässer fluctuations $\mathbf{z}_\pm$, which appear symmetrically in the MHD Eq. (11). A large-scale magnetic field suppresses the growth of gradients parallel to the magnetic field, but since the perpendicular gradients are not affected in this way, nonlinear (strain) effects con-
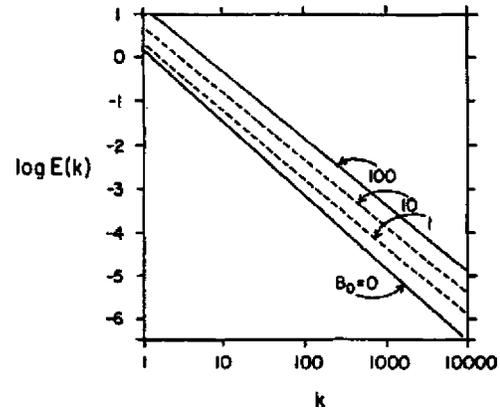

FIG. 7. Omnidirectional energy spectrum vs wave number, computed from a generalized energy spectrum (Matthaeus and Zhou, 1989). Three values of the magnetic-field strength B_0 =0, 1, 10, and 100 are shown. The slope flattens and the spectral level increases as the field strength is increased, indicating a smooth transition from the Kolmogorov −5/3 scaling (B_0 =0) to a spectrum B_0=100 that approaches the Iroshnikov-Kraichnan −3/2 scaling. From Mathaeus and Zhou, 1989.

tinue to pump smaller scales, but anisotropically. Under some circumstances this leads to quasi-two-dimensional states.

When the turbulence is sufficiently two dimensional, the sweep time scale due to propagation parallel to B_0 is no longer short compared with the intrinsic strain interaction times of the fluctuations, and the dynamics of z_+ and z_- become similar to those of purely 2D MHD turbulence and are therefore almost independent of B_0 (Hossain *et al.*, 1995; Chen and Kraichnan, 1997). This explains why all major conclusions of 3D MHD simulations, with an externally imposed dc magnetic field (Oughton *et al.*, 1994), are consistent with the two-dimensional studies of Shebalin *et al.* (1983).

The structure of the spectrum is expected to be highly anisotropic in the presence of a dc field, as was originally suggested on the basis of experimental measurements in the Culham Zeta Device (Robinson and Rusbridge, 1971). Two-dimensional MHD, which is unaffected by the presence of a strong perpendicular dc field, was intensively studied in the 1970s and it was suggested (Fyfe and Montgomery, 1976; Fyfe *et al.*, 1977) that the Kolmogorov analysis, and its associated −5/3 spectral slope, would be applicable to the 2D inertial range associated with a direct energy cascade to small scales. This result stimulated theoretical debate that has continued for more than 20 years.

Unlike the 2D case considered by Fyfe *et al.* (1977), in which the dc field defines a perpendicular plane, Shebalin *et al.* (1983) studied 2D MHD in a plane containing the dc field. This defines an additional preferred direction, and anisotropy can develop within the 2D plane. The 2D incompressible simulations of Shebalin *et al.* (1983) revealed the development of a strong and distinctive anisotropy. The energy preferentially builds up in wave vectors $\mathbf{k}$ perpendicular to $\mathbf{B}_0$ relative to modes with $\mathbf{k}$ parallel to $\mathbf{B}_0$. Oughton *et al.* (1994) confirmed the
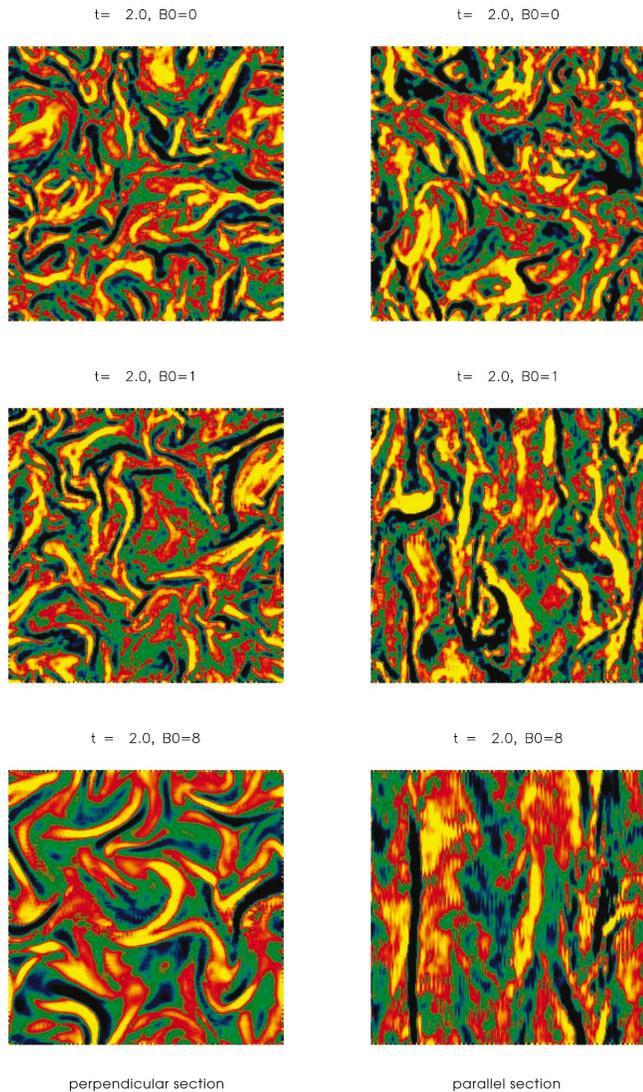
t= 2.0, B0=0   t= 2.0, B0=0

t= 2.0, B0=1   t= 2.0, B0=1

t = 2.0, B0=8   t = 2.0, B0=8

perpendicular section   parallel section

FIG. 8. (Color) Color maps of the current density j_z in cross sections x, y and x, z from 3D simulations of MHD turbulence with an applied dc magnetic field $B_0\hat{\mathbf{z}}$: top, $B_0=0$; middle, $B_0/\delta B=1$; bottom, $B_0/\delta B=8$).

result of Shebalin *et al.* (1983) in three dimensions, finding that, with a dc magnetic field, energy transfer to perpendicular modes is enhanced, relative to a parallel one. Oughton *et al.* (1994) found that the anisotropy tends to increase with (i) the strength of B_0 (with saturation occurring for values of $B_0 \geqslant 3b$); (ii) wave number k; (iii) mechanical and magnetic Reynolds numbers; (iv) time (with a saturation depending on the Reynolds numbers); and (v) decreasing cross correlation.

The manifestation of spectral anisotropy in real space is the emergence of gradients across the mean magnetic-field direction that are stronger than the gradients along the field. As a result correlation lengths are longer along the field, and structures are expected to appear elongated in the mean-field direction. This feature is illustrated using numerical simulation results in Fig. 8.

Anisotropy is also found in solar wind turbulence. Evidence for spectral anisotropy in the solar wind is at

this point indirect but has nonetheless gained considerable weight as consistent indications of anisotropy have come from various types of studies (Matthaeus *et al.*, 1995). Direct observations suggest that solar wind fluctuations are anisotropic (Carbone *et al.*, 1995) and contain a significant admixture of excitations at near-perpendicular wave vectors (Matthaeus *et al.*, 1990; Bieber *et al.*, 1996). In addition (Belcher and Davis, 1971; Roberts *et al.*, 1991) the inertial range of solar wind turbulence admits a distinctive component variance anisotropy, with a suppressed parallel variance. In simulations the appearance of this feature appears to require weak compressibility (Matthaeus *et al.*, 1996).

To offer a simple and physically appealing interpretation of the observed development of anisotropy in the direction perpendicular to the applied dc magnetic field, Shebalin *et al.* (1983) appeal to a three-wave resonance interaction argument. This interpretation is based loosely on a weak-turbulence theory (see, for example, Zakharov *et al.*, 1992), which only computes the first-order corrections to the solutions of the linearized MHD equations. Within this framework, the nonlinear terms in MHD equations exactly cancel for Alfvén wave solutions, so waves propagating in the same direction do not generate additional modes. Two excited Fourier modes can exchange energy efficiently with a third mode only if the triad obeys the standard resonance condition (Montgomery and Matthaeus, 1995): $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$, and $\omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) = \pm \omega(\mathbf{k}_3)$. Here $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wave vectors associated with two excited Fourier modes that are resonantly exciting the third wave vector $\mathbf{k}_3$. In the linear limit, all three modes are assumed to have an associated $\exp(-i\omega t)$ time dependence and satisfy a relation between frequency and wave vector $\omega(\mathbf{k}) = \mathbf{k} \cdot \mathbf{V}_A$, where $\mathbf{V}_A = \mathbf{B}_0/(4\pi\rho)^{1/2}$ is the vector Alfvén velocity associated with the mean dc magnetic field. For nonvanishing coupling strength, interacting waves must propagate in opposing directions, which accounts for the sign difference on the left-hand side of the frequency-matching condition. The triads of waves satisfying $\mathbf{k}_1 \cdot \mathbf{V}_A - \mathbf{k}_2 \cdot \mathbf{V}_A = \pm \mathbf{k}_3 \cdot \mathbf{V}_A$ will have nonzero coupling only if either $\mathbf{k}_1 \cdot \mathbf{V}_A = 0$ or $\mathbf{k}_2 \cdot \mathbf{V}_A = 0$. Therefore either $\mathbf{k}_1$ or $\mathbf{k}_2$ has a zero component along $\mathbf{B}_0$.

The physical consequences of the dominance of the three wave couplings can be summarized as follows: To leading order there is no transfer parallel to an imposed dc magnetic field. Transfer in the perpendicular direction is not impeded by the Alfvénic wave couplings that suppress parallel transfer. Consequently MHD perpendicular transfer proceeds very much as it does in 2D MHD (Fyfe *et al.*, 1977). On this basis one expects a $k_\perp^{-5/3}$ spectrum for perpendicular wave vectors. In the parallel direction, weak transfer occurs, but for each step in $k_\parallel$, much of the energy is diverted into higher $k_\perp$. Thus the parallel spectrum is expected to be exponential $\sim \exp(-k_\parallel)$. The first explicit suggestion of this appears to have been given by Montgomery (1987). See also Goldreich and Sridhar (1995) and Kinney and McWilliams (1998).

Although the physical justifications for the occurrence of spectral anisotropy have been given in wave-number space, one would expect that this phenomenon should have a real-space manifestation that possesses some degree of locality. A sufficiently large-scale magnetic field should induce effects locally that are indistinguishable from a strictly uniform dc field. Consequently one expects to be able to understand the occurrence of anisotropy entirely in the context of a system having localized electric current and magnetic fields that have strictly finite scale. Recent numerical studies (Cho and Vishniac, 2000; Milano *et al.*, 2001) address this issue and conclude that the spectral anisotropy described above indeed does possess a purely local analog. The basic point goes back to the zeta measurements (Robinson and Rusbridge, 1971) that found that the correlation of magnetic fluctuations falls off much faster in directions perpendicular to a large-scale applied field than it does in the parallel direction. Application of these ideas to simulation data indicates that anisotropy does indeed occur locally. Correlations fall off more rapidly in directions transverse to the locally computed mean magnetic field. The anisotropy is found to be greater at small scales (Cho and Vishniac, 2000) and greater where the local mean-field strength is larger.

IV. DISCUSSION AND CONCLUSIONS

On the basis of the physical properties discussed above, it is possible to develop a simple and approximate methodology for estimating energy spectra and correlation functions in various regimes of MHD turbulence. The program is as follows: A *spectral transfer* time scale is estimated, incorporating effects due to both nonlinear straining motions and the sweeping-like influence of wave propagation. The relative influence of these effects will be closely related to the degree and type of anisotropy expected, e.g., whether anisotropy is relative to a strong externally supported dc magnetic field or is relative to the local magnetic field. Accordingly, spectral transfer is either isotropic, when large samples of plasma are considered, or is anisotropic, when there is a strong large-scale mean field. On this basis we can examine the phenomenological steady energy transfer rate through a physical assessment of the formula

$$\epsilon = \Pi(k) = \tau_T(k) \frac{kE(k)}{\tau_{nl}^2}, \tag{15}$$

which is a restatement of Eq. (5). Several important elements of the physics of MHD turbulence are brought together in this relation:

(i)   The energy transfer rate must be proportional to the lifetime of triple correlations, as in Eq. (2).

(ii)   The strength of nonlinear interactions is measured by an eddy turnover, or nonlinear time scale, as in Eq. (3).

(iii) Finally, the spectral flux of energy must be defined in a way [Eq. (5)] that is compatible with (i) and

(ii). This is more than a formal relation, and in fact it can be used to develop estimates of the form of the spectrum in a variety of physically interesting cases.

This procedure allows us easily to understand the physics of, and the physical differences between, the classical Kolmogorov and Iroshnikov-Kraichnan theories. No complex closure theories or perturbation schemes are required. The approach is exploited in further detail in Appendix A, for several additional physical situations. If, in addition, we wish to develop approximations for the time-dependent correlation functions, such as the Eulerian (single spatial point, two-time) correlation functions, or the two-time decorrelations that appear in closure theories, we may proceed in an analogous way: We adopt a reasonable functional form for the time correlation function, depending upon a single time scale, namely, the same spectral transfer time. Appendix B catalogs some of the possibilities for time correlation functions approximated in this way.

In this Colloquium we have attempted to give an overview of the influential role of distinct time scales in establishing energy transfer, cascade, nonlocality, and anisotropy in MHD turbulence. Like hydrodynamics, the "native" time scales of the dynamics can be divided into straining motions that are due to self distortion of eddies, and sweeping motions that displace smaller-scale structures under the influence of the large-scale fields. In MHD, as in hydrodynamics, the straining motions are dominantly local in scale, in that nonlinear distortions are most effective for interactions between eddies of approximately the same size. In MHD, however, the sweeping-type motions are more complex than in hydrodynamics because of the Alfvén propagation effect. Alfvénic sweeping introduces a new level of nonlocality in MHD and a strong tendency for spectral transfer to occur anisotropically relative to the magnetic-field direction.

In the past much has been made of differences between Kolmogorov and Iroshnikov-Kraichnan spectra, and the two possibilities are often compared in simulations or interplanetary observations with the apparent goal of drawing firm distinctions between these possibilities. In the present view, in which the main effect that we vary is the time scale for decay of the transfer correlation functions, we emphasize that in MHD turbulence there is a smooth variation between such limits. Observation of distinct spectral indices in various cases is indicative of the enhanced effect of sweeping effects versus straining effects, or of local effects versus nonlocal effects, in accordance with how the prevailing conditions impact the relevant time scales. Anisotropy is controlled by variation of the directed Alfvénic decorrelation time in comparison to the local nonlinear time scale. This leads to a wide range of possibilities for the spectral anisotropy and for spectral indices that can emerge in MHD.

As discussed above and elaborated upon in Appendix A, a smooth variation from Kolmogorov to Iroshnikov-

Kraichnan is expected in isotropic MHD turbulence as the ratio of the Alfvén time to the nonlinear time is changed. The ratio is also wave-number dependent, and more wavelike decorrelation generally occurs at smaller scales. With a strong uniform magnetic field, the resulting anisotropic energy spectrum can reduce to $k_\perp^{-5/3}$ when resonant interactions and quasi-two-dimensional strain is the dominant decorrelation effect, or to $k_\perp^{-3/2}$ when large-scale quasi-2D Alfvénic decorrelation is strong. When quasi-2D effects are weak (see Appendix A), the spectrum can become either $k_\perp^{-2}$ ("weak turbulence"), when local interactions are dominant, or $k_\perp^{-3}$ when nonlocal interactions are dominant. When both local and nonlocal interactions are present, the spectrum varies smoothly between these limits.

Similar considerations of time scales allows us to outline an approach to modeling the Eulerian correlation functions that appear in MHD. Several possibilities are suggested in Appendix B that may be useful in applications.

It should be noted that observational and simulation evidence has thus far been identified and fully analyzed for only some of the MHD regimes that we have discussed, and considerably more study is needed to examine all MHD parameter regimes, including widely ranging Reynolds numbers, cross helicity, very strong anisotropy, the transition between local and nonlocal spectral transfer, MHD that greatly departs from equipartition of kinetic and magnetic energy, and the influence of various possible kinetic dissipation effects. Many of these parametric investigations are ongoing in view of the current high level of interest in MHD turbulence in astrophysical settings (see, for example, Maron and Goldreich, 2001; Cho and Lazarian, 2002; Cho *et al.*, 2002; Schekochihin *et al.*, 2002). The availability of several time scales makes MHD turbulence more complex and multifaceted than its hydrodynamic counterpart, and it is likely to remain an active area of study into the future.

Needless to say, the present format cannot accommodate an exhaustive review of the entire domain of MHD turbulence theory and its applications, even with our emphasis specifically on time scales. We mention a few topics of interest that could represent interesting extensions of the present discussion.

Perhaps the most glaring exclusion has been our focus on the incompressible model. While incompressibility captures most of the essential physics of classical hydrodynamic turbulence and its extensions to MHD, compressibility can have important effects in various limits. For example, remote sensing of the diffuse interstellar medium indicates that electron density fluctuations exhibit a turbulence-like power-law spectrum (Armstrong *et al.*, 1981). This "Kolmogorov" $-5/3$ density spectrum can be explained (Montgomery *et al.*, 1987) based on MHD turbulence, by assuming near incompressibility and a density that is a linear response to the incompressible pressure variations. This situation gives rise to a statistical balance of thermal and magnetic pressures, up to the order of the turbulent Mach number squared. It is noteworthy that approximate pressure balance is observed frequently in the solar wind (Burlaga and Ogilvie, 1970; Matthaeus *et al.*, 1990). Montgomery *et al.* (1987) showed that the density spectrum at sufficiently high wave number k in the inertial range will have a $k^{-5/3}$ spectrum whenever the inertial-range magnetic energy spectrum also has this law. Using nearly incompressible MHD equations, Zank and Matthaeus (1993) have argued that density fluctuations behave as a passive scalar and follow a Kolmogorov-like spectrum. Clearly, there seems to be an interesting and realizable range of weakly compressible MHD in which features of the incompressible model are seen, with important additional features.

Highly compressible MHD turbulence is also found in nature. Several actively studied examples are found in the interstellar medium. One such case is dense molecular clouds where the important process of star formation occurs and where turbulent Mach numbers may be in excess of ~15. In recent years compressible MHD turbulence has been intensively studied in these regions (see, for example, Mac Low *et al.*, 1998; Mac Low, 1999), in view of the importance that turbulent magnetic fields may have in slowing gravitational collapse and therefore contributing to the regulation of star formation rates. Strongly compressible MHD turbulence remains an important area of study (see Lithwick and Goldreich, 2001; Cho and Lazarian, 2002) that may reveal features that are not simple extensions of the incompressible model.

Finally, we would like to mention that many of these considerations can be pursued in configuration space. Instead of studying the wave-number spectra, one can define the structure functions by various powers of two-point velocity differences. However, this alternative approach is beyond the scope of this Colloquium.

ACKNOWLEDGMENTS

APPENDIX A: ENERGY SPECTRA OF MHD TURBULENCE

To estimate the energy spectrum in the various regimes of MHD turbulence, we examine the phenomenological steady energy transfer rate

$$\epsilon = \Pi(k) = \tau_T(k)\frac{kE(k)}{\tau_{nl}^2} \tag{A1}$$

in several special cases. With this structure and some additional physical reasoning we can construct specific forms for the energy flux and thus deduce the steady MHD inertial-range spectral behavior for a variety of circumstances. We catalog some of these here.

1. Isotropic MHD

a. Isotropic, strain-dominated MHD

Consider the case in which the dc magnetic field is weak or absent and the straining effect is dominant. We further assume that the MHD turbulence is isotropic when viewed over a sample of turbulent plasma of sufficiently large size (i.e., large enough that the magnetic field averages to zero). The turbulence will be strain dominated if the sweeping relative to the magnetic field is not strong enough to displace straining as the dominant source of decorrelation. These circumstances can be realized even when there is moderately strong anisotropy relative to the magnetic-field direction. This form of MHD is then similar to fluid turbulence—energy transfer is mainly local in wave number. This viewpoint is supported by interplanetary and interstellar observations as well as high-resolution MHD simulations (Biskamp and Müller, 2000). In Eq. (15) we take $\tau_T = \tau_{nl} = 1/k u_k = k^{-3/2} E^{-1/2}$, leading to the $-5/3$ scaling law.

b. Isotropic MHD with sweeping

In this case, the energy transfer of MHD turbulence is suppressed by the sweeping effect, which rapidly moves the smaller-scale components of $\mathbf{z}_+$ and $\mathbf{z}_-$ past one another under the influence of a large-scale magnetic field. The turbulence is globally isotropic since the large-scale magnetic field is randomly distributed in direction, but the sweeping effect must be incorporated and the time scale that determines the energy transfer is given by the Alfvénic time scale. Accordingly the decay time of transfer correlations is the Alfvén time $\tau_T = 1/k V_A$ and the nonlinear time remains $\tau_{nl} = k^{-3/2} E^{-1/2}$. Thus the spectrum is given by Iroshnikov-Kraichnan $-3/2$ scaling, $E(k) = (V_A \epsilon)^{1/2} k^{-3/2}$ instead of the classical Kolmogorov spectrum. This is supported by the EDQNM calculations of Grappin *et al.* (1983) and Galtier *et al.* (2000) as well as 2D direct numerical simulation of Biskamp and Müller (2000).

c. Isotropic MHD with cross helicity

When the cross helicity is significantly far from zero, the two Elsässer energies are no longer equal. The global decay rates and the inertial-range energy fluxes must take into account that nonlinearities involve interactions of $\mathbf{z}_+$ with $\mathbf{z}_-$, but not of either Elsässer field with itself. This is an essential feature of the phenomenologies discussed by Kraichnan (1965), Dobrowolny *et al.* (1980), and Hossain *et al.* (1995), and it is also central in the closure and numerical simulation studies of Pouquet *et al.* (1976), Grappin *et al.* (1982, 1983), and Politano *et al.* (1989). We may examine two corresponding energy fluxes,

$$\epsilon^{\pm} = \Pi^{\pm}(k) = \tau_T^{\pm}(k) \frac{k E_{\pm}}{\tau_{nl}^{(\pm)2}}, \qquad (A2)$$

and proceed to estimate the spectral characteristics under various assumptions.

Suppose first that the large-scale magnetic field is strong (but averages to zero to maintain isotropy) so that the triple decay times are identical and Alfvénic. Then $\tau_T^{\pm} = \tau_A(k) = 1/k V_A$. Also assume that $\tau_{nl}^{\pm} = 1/k z_k^{\pm} = (k^{3/2} E_{\mp}^{1/2})^{-1}$ in accordance with the structure of the Elsässer form of the nonlinear terms (Pouquet *et al.*, 1976; Grappin *et al.*, 1982). Then we find that $\epsilon^+ = \epsilon^- = \epsilon/2$ and $2 \epsilon V_A = E_+(k) E_-(k) k^3$. Assuming power laws for both Elsässer spectra with indices μ_+ and μ_-, respectively, leads to

$$\mu_+ + \mu_- = 3. \qquad (A3)$$

This important and useful result for strong, isotropic MHD with nonzero cross helicity was apparently first stated by Grappin *et al.* (1982, 1983) and reduces to the Iroshnikov-Kraichnan $k^{-3/2}$ result when the two fields become identical in the zero cross-helicity limit.

When the large-scale magnetic field is weaker, we can find an analogous result for strain-dominated spectra with cross helicity. Letting $\tau_T^{\pm}(k) \to \tau_{nl}^{(\pm)}(k)$ in Eq. (A2), equating the fluxes of Z_+^2 and Z_-^2, and again assuming power laws μ_+ and μ_- for the two spectra, we find now two equations for the unknown power-law indices. The sole solution is $\mu_+ = \mu_- = 5/3$. The identical result is found if, instead of assuming $\epsilon^+ = \epsilon^-$, we allow for different decay rates for the two Elsässer energies (Grappin *et al.*, 1983), setting $\epsilon^+ = C_\epsilon C^-$ for a k-independent constant C_ϵ. Therefore we conclude that for a strain-dominated phenomenology, $E_+(k)$ and $E_-(k)$ each vary as $\sim k^{-5/3}$, and the dimensionless cross helicity $\sigma_c(k) = [E_+(k) - E_-(k)]/[E_+(k) + E_-(k)]$ is independent of k.

The contrast of the above two results suggests that the behavior of the cross helicity in the inertial range is a direct indicator of the relative dominance of strain- or sweep-type decorrelation effects. In reality, neither simulations nor solar wind observations (Matthaeus and Goldstein, 1982; Tu and Marsch, 1995) typically show pure power laws for the $E_{\pm}(k)$ spectra, nor do they show constant $\sigma_c(k)$ in the inertial range (see Fig. 9). This supports the view (Matthaeus and Zhou, 1989) that for many reasonable parameters, the low-k end of the inertial range will be strain dominated, while the high-k end becomes sweeping/propagation dominated. Pure power laws are not expected in that event.

2. Anisotropic MHD

For anisotropic MHD turbulence the situation is more complicated than the cases discussed above. The dc or large-scale magnetic field imposes a preferred direction, the sweeping effect is very strong, and energy transfer is suppressed in the parallel direction. In order to properly assess the spectral behavior it remains necessary, however, to identify the correct transfer decay time.

a. Anisotropic MHD with dominant resonant interactions

Suppose that MHD turbulence is highly anisotropic relative to a strong large-scale field. Furthermore, sup-
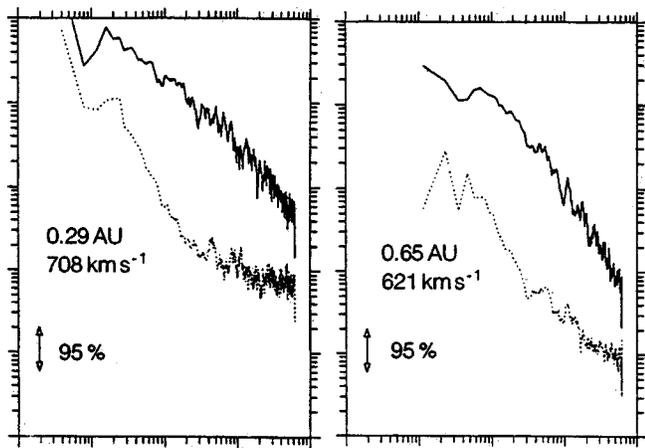
FIG. 9. Observations in the inner heliosphere by the Helios mission in time intervals associated with high-speed solar wind: solid lines, one-dimensional spectra $E_+(k)$; dobted lines, $E_-(k)$. While the larger Elsässer field has approximately a power-law spectrum in both examples, the spectrum of the smaller field does not run parallel to it (as would be expected in the strain-dominated case) nor does it have a pure power-law spectrum that extends into the dissipation range where it intersects the spectrum of the larger field (as would be expected in the sweep-dominated case). Adapted from Marsch, 1991.

pose that there is considerable energy in 2D or quasi-2D modes that satisfy $\tau_A(k_\parallel) > \tau_{nl}(k)$. Such modes are either nonpropagating or have very low frequency.[8] The dominant features are those of 2D MHD turbulence or quasi-2D turbulence equivalent to the "mean mode" of reduced MHD turbulence (Zank and Matthaeus, 1992). However, some excitation may be in modes not lying in the quasi-2D part of wave-number space, namely, excitations having higher $k_\parallel$ and smaller $\tau_A(k_\parallel)$. Such Fourier modes will experience substantial resonant nonlinear coupling that involves quasi-2D modes. The triple correlations associated with these dominant nonlinear couplings will decay at a rate determined by the quasi-2D turbulence, unless the energy in quasi-2D motions is very small. In the latter case nonresonant triad couplings can destroy the resonant triple correlations (see "weak turbulence" below). Assuming the former case, which may be thought of as "strong" anisotropic MHD turbulence, the parallel Alfvén time becomes essentially irrelevant in determining the perpendicular spectrum. However, there still remain two cases to consider depending upon how the triple correlations are destroyed within the set of quasi-2D modes themselves.

First, suppose the quasi-2D turbulence experiences decay of triple correlations mainly due to propagation effects in the 2D plane, due to the large-scale 2D magnetic field. This is a reasonable possibility since 2D turbulence tends to build up large-scale structure in the

magnetic field. This effect is associated with a so-called *inverse cascade* that rearranges the spectrum[9] of the 2D magnetic flux function, leading to a buildup of excitations at the longest allowed scales (see Fyfe and Montgomery, 1976). The net effect is the simultaneous buildup of long-wavelength magnetic field and accompanying transfer of energy to small scales. Indeed, it has been argued on the basis of simulation studies (Politano *et al.*, 1989; Biskamp 1995) that the steady inertial-range spectrum for pure 2D MHD turbulence is determined by these "transverse" propagation effects. For the case at hand, which we call *anisotropic sweep-dominated resonant MHD*, we choose the Alfvén time associated with sweeping in the dynamically active 2D plane. Therefore $\tau_T(k_\perp) = 1/k_\perp \delta v_A$, where δv_A is the Alfvén speed associated with the large-scale quasi-2D magnetic field, and $\tau_{nl}(k_\perp) = k_\perp^{-3/2} E^{-1/2}(k_\perp)$. It follows that $E(k_\perp) = (\epsilon \delta v_A)^{1/2} k_\perp^{-3/2}$ which is simply the quasi-2D Iroshnikov-Kraichnan spectrum.

As a second case, we assume that the quasi-2D motions are governed by strain (the *anisotropic strain-dominated case*). This is expected to be appropriate when the large-scale magnetic field lying in the 2D plane is not very strong, for example, when the 2D inverse cascade (Fyfe and Montgomery, 1976) is not strong. This occurs when the ratio of mean-square magnetic flux function to total energy is small and the average magnetic island size is small compared to the system size. Accordingly we choose $\tau_T(k_\perp) = \tau_{nl}(k_\perp)$. Now we find that $E(k_\perp) = \epsilon^{2/3} k_\perp^{-5/3}$. This is the quasi-2D (or 2D) Kolmogorov spectrum for MHD. Upon inserting an appropriate parallel spectral dependence [such as an exponential in $k_\parallel$ (Montgomery, 1987)] it is equivalent to the spectral form constructed by Goldreich and Sridhar (1995) using an anisotropic EDQNM closure.

b. Anisotropic MHD with dominant parallel sweeping: "weak turbulence"

Another possibility for highly anisotropic MHD turbulence is that the quasi-2D modes are not strong enough to control the decorrelation of the more wave-like modes. In that case neither the quasi-2D cascade itself nor the resonant cascade it induces is the dominant feature that determines the spectrum of non-quasi-2D waves. For such a case, the transfer correlations should decay due to higher-order wave propagation processes. This is the domain of "weak MHD turbulence."

Ng and Bhattacharjee (1997) and Galtier *et al.* (2000) modified the argument by Kraichnan by taking into account the anisotropic feature in the characteristic Alfvén time scale

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[8] We reserve the designation "2D modes" for those Fourier amplitudes having $k_\parallel \equiv 0$ and therefore infinite τ_A. "Quasi-2D modes" refer to $k_\parallel$ small emough that the reduced MHD inequality $\tau_A(k_\parallel) > \tau_{nl}(k)$ is satisfied.

[9] "Inverse cascade" refers to a spectral transfer process that preferentially transfers some types of excitations to low wave number k, while the ordinary "direct cascade" proceeds towards higher k. This "dual cascade" can occur in both 2D and 3D MHD and can lead to formation of long-lived, large-scale structure. See Kraichnan and Montgomery, 1980.

$$\tau_A(k_\parallel) \sim 1/[V_A k_\parallel], \tag{A4}$$

while still maintaining that the propagation effect is primarily responsible for decay of the triple correlations that limit the nonlinear couplings between oppositely propagating Elsässer wave packets. In particular, assuming that parallel transfer has frozen out and the spectral cascade involves only $k_\perp$, one easily sees that the wavenumber dependence of the Alfvén time no longer directly influences the spectral law. That is, assuming $\tau_T(k) = \tau_A(k_\parallel) = 1/V_A k_\parallel$ and inserting this along with $\tau_{nl}^2(k_\perp) = [k_\perp^3 E(k)]^{-1}$ into Eq. (15), one finds that $V_A k_\parallel \epsilon = k_\perp^4 E^2(k_\perp)$. It is now clear that for $k_\parallel \neq 0$, the perpendicular spectral law becomes $E(k_\perp) \sim k_\perp^{-2}$ (see also Dmitruk *et al.*, 2003 for numerical simulations of boundary-driven reduced MHD satisfying this power law). This has become known as "weak turbulence" (Galtier *et al.*, 2000).[10]

c. Anisotropic MHD with varying cross helicity and energy partition

Nonzero cross helicity has an effect on anisotropic spectra analogous to its effects in the $Z_+^2 = Z_-^2 \equiv Z^2$ case. For strong turbulence, in which decorrelation is controlled by quasi-2D dynamics, there are again two cases, associated with regulation either by 2D sweeping motions or by 2D straining motions. In the sweeping case, assuming $E_\pm(k) \sim k^{-\mu_\pm}$, one again finds that the power-law indices sum as $\mu_+ + \mu_- = 3$. When straining in the perpendicular plane is the dominant source of decorrelation, once again the power-law indices are determined to be $\mu_+ = \mu_- = 5/3$ as in the isotropic case.

Things become different in the case of weak turbulence (which is of necessity dominated by sweeping), because now the Alfvénic decorrelation depends upon par-

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[10]Some of the controversy surrounding the anistropic MHD spectrum derives from matters of terminology, but physical issues are also involved. Sridhar and Goldreich (1994) originally claimed that three wave couplings are empty and that the arguments of Kraichnan (1965) and Shebalin *et al.* (1983) were incorrect. It turns out that in certain formal perturbation theories (see Goldreich and Sridhar, 1997) there is some substance to that claim, since canonical weak-turbulence expansions have formal restrictions against including zero-frequency modes. In the present terminology that would require excluding the 2D and quasi-2D modes, which precludes resonant-coupling-dominated MHD. Both Ng and Bhattacharjee (1997) and Montgomery and Matthaeus (1995) provided counterexamples to the claim that three wave couplings are empty, and the presence of the 2D zero-frequency modes plays a central role in this demonstration (see also a recent retraction of the empty-couplings argument in Lithwick and Goldreich, 2003). The work of Galtier *et al.* (2000) shows that the $k_\perp^{-2}$ law supplants both Kolmogorov $k^{-5/3}$ and Iroshnikov-Kraichnan $k^{-3/2}$ for cases in which wavelike modes interact weakly in the absence of strong low-frequency quasi-2D modes. The phenomenological description given above (see also Ng and Bhattacharjee, 1997) provides a less formal derivation of the weak-turbulence spectrum.

allel wave number. In that case one finds that $k_\parallel V_A \epsilon^\pm \propto k_\perp^4 E_+(k) E_-(k)$. The fluxes need not be equal, but provided they are proportional to one another independently of k, the conclusion is that $\mu_+ + \mu_- = 4$. This result was discussed by Ng and Bhattacharjee (1997) using perturbation theory, and by Galtier *et al.* (2000) by examination of Kolmogorov-like solutions to the kinetic equation for weak turbulence.

There are accessible regimes in which the transfer becomes highly nonlocal. One such case occurs when the Elsässer energies are highly asymmetric. Ng and Bhattacharjee (1997) discuss this possibility, leading to a $k_\perp^{-3}$ spectrum, based upon perturbation theory for equipartition of velocity and magnetic field at zeroth order. Here we describe two distinct cases in which a conclusion similar to this emerges.

First, consider the equipartitioned case in which $\langle \mathbf{z}_+ \cdot \mathbf{z}_- \rangle = 0$, which has been the focus of most of the above considerations. Suppose that $Z_+^2 \gg Z_-^2$ and that the smaller of the two energies is very flat, say $E_-(k_\perp) \sim 1/k_\perp$. For this scale-invariant spectrum the notion of local transfer become difficult to realize, since the characteristic amplitude at scale $1/k_\perp$ becomes independent of $k_\perp$, that is, $z_-(k_\perp) \sim [k_\perp E_-(k_\perp)]^{1/2} = \tilde{z}_-$ is independent of $k_\perp$. This is simply the statement that a $1/k_\perp$ spectrum has equal energy per wave-number interval. For finite bandwidth, $\tilde{z}_-$ is of the order of the total amplitude Z_-. As a consequence, using Eq. (A1) with $E_+ \rightarrow E$, $\tau_T = \tau_A(k_\parallel)$, $k \rightarrow k_\perp$, and $\tau_{nl} = [k_\perp^{3/2} E_-(k_\perp)]^{-1/2} \rightarrow 1/(k_\perp \tilde{z}_-)$, we find that $V_A k_\parallel \epsilon = \tilde{z}_-^2 E(k_\perp) k_\perp^3$. This implies a steady spectrum of $E(k_\perp) \sim k_\perp^{-3}$. This is driven by a low level of Z_- fluctuations, corresponding to high cross helicity, and exhibits features of nonlocality (Ng and Bhattacharjee, 1997). Note also that this -3 spectral slope is a special, limiting case of the sum rule that $\mu_+ + \mu_- = 4$, for anisotropic Iroshnikov-Kraichnan phenomenology with cross helicity.

An analogous regime of $k_\perp^{-3}$ behavior, also with a strong nonlocal flavor, has been identified in very different circumstances (Dmitruk *et al.*, 2003), in a model of solar coronal loops stirred by photospheric motions (Parker, 1972; Gómez and Ferro Fontán, 1988). The rectangular box has periodic transverse (x, y) coordinates and field lines anchored on flat boundaries at the top and bottom. The bottom is fixed with zero velocity, but the top models photospheric motion by prescribing a stream function that establishes a stirring pattern. This deflects field lines at the base and low-frequency disturbances propagate into the box, exciting turbulence. For strong driving, strong turbulence emerges, punctuated by intermittent reconnection events. However, for very slow forcing very little kinetic energy is injected. Nonlinear effects persist however, and a power-law regime is established in both magnetic and velocity spectra.

To explain this, we make use of $E_b(k_\perp) \gg E_u(k_\perp)$ and moreover use the property, observed numerically (Dmitruk *et al.*, 2003), that $E_u(k_\perp)$ is very flat and varies approximately as $\sim 1/k_\perp$. The leading-order effect is spectral transfer of magnetic energy to higher wave

number under the influence of the weak velocity field. The nonlinear time should therefore be computed in terms of the velocity alone. Since $E_u(k_\perp) \sim k_\perp^{-1}$, we estimate the nonlinear time as $\tau_{nl} = 1/[k_\perp(k_\perp E)^{1/2}] \rightarrow 1/(k_\perp \tilde{u})$ where $\tilde{u}^2$ is the energy per unit mass in the scale-invariant velocity-field spectrum. Transfer is now due to nonlocal effects of the velocity field upon the magnetic field, and the equation for energy transfer becomes $\epsilon = \tau_T E_b(k) \tau_{nl}^{-2} \rightarrow (V_A k_\parallel)^{-1} \tilde{u}^2 E_b(k_\perp) k_\perp^3$. Therefore a $k_\perp^{-3}$ spectral regime can emerge in the flux-tube problem in the regime of very weak forcing. This has been seen in the computations (Dmitruk *et al.*, 2003) and it will be interesting to see if additional evidence for nonlocal transfer accompanied by k^{-3} or other spectral laws emerges in other problems.

APPENDIX B: TIME-CORRELATIONS OF MHD TURBULENCE

So far we have focused on inertial-range spectra and the influence that various time scales have on them. While the time scale for decay of the energy transfer correlations (also known as the lifetime of triple correlations) has entered this discussion, it is, in fact, an important topic in its own right. The two-point Eulerian velocity autocorrelation $F_u(\tau) = \langle \mathbf{u}(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t+\tau) \rangle$ depends upon the relative importance of sweeping and nonlocal effects (Tennekes, 1975; Chen and Kraichnan, 1989). The structure of F_u (Nelkin and Tabor, 1990) relates to intermittency and the higher-order statistics of the kinetic-energy spectrum. In MHD the Eulerian decorrelation is more complex, due to the multiplicity of possible time scales, and is once again much less well studied than its hydrodynamic counterpart.

For MHD there are several Eulerian correlation functions that may be of interest, for example, $F_b(\tau) = \langle \mathbf{b}(\mathbf{x},t) \cdot \mathbf{b}(\mathbf{x},t+\tau) \rangle$, $F_+(\tau) = \langle \mathbf{z}_+(\mathbf{x},t) \cdot \mathbf{z}_+(\mathbf{x},t+\tau) \rangle$, and $F_-(\tau) = \langle \mathbf{z}_-(\mathbf{x},t) \cdot \mathbf{z}_-(\mathbf{x},t+\tau) \rangle$. For symmetrical special cases, these can become redundant or equivalent. For Alfvénic fluctuations with $2H_c/(E_u + E_b) \rightarrow \pm 1$, we have $\mathbf{u} = \pm \mathbf{b}$ and $F_b \rightarrow F_u$. When $\mathbf{z}_+ \rightarrow \mathbf{z}_-$ then $\mathbf{b} \rightarrow 0$ and $F_+ = F_- = F_u$ while $F_b \rightarrow 0$ and hydrodynamics is recovered. When $\mathbf{z}_+ \rightarrow -\mathbf{z}_-$ then $\mathbf{u} \rightarrow 0$ and $F_+ = F_- = F_b$ while $F_u \rightarrow 0$ and a purely magnetic state is recovered. However, in general, the Eulerian autocorrelations as well as the Eulerian cross correlations such as $F_D = \langle \mathbf{z}_+(\mathbf{x},t) \cdot \mathbf{z}_-(\mathbf{x},t+\tau) \rangle$ and $F_c = \langle \mathbf{b}(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t+\tau) \rangle$, are constrained independent quantities.

Consider one of these Eulerian correlations, say, $F_b(\tau)$. It is easy to see how F_b is related to the spectral decomposition of the random field $\mathbf{b}$, in that $F(\tau) = \int d^3k S(\mathbf{k},\tau)$ is a special case of the two-time two-position correlation $R(\mathbf{r},\tau) = \int d^3k e^{-i\mathbf{k}\cdot\mathbf{r}} S(\mathbf{k},\tau)$. We can in addition let $S(\mathbf{k},\tau) = S(\mathbf{k})\Gamma(\mathbf{k},\tau)$ where $S(\mathbf{k})$ is the trace of the spectral tensor $S_{ij}(\mathbf{k}) = (2\pi)^{-3} \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} R_{ij}(\mathbf{r},0)$. The quantity $\Gamma(\mathbf{k},\tau)$ is the dynamical decorrelation function, representing the decay in time of the spectral information at wave vector $\mathbf{k}$. Γ and therefore the (magnetic) Eulerian correlation play important roles in cosmic-ray

scattering (Bieber *et al.*, 1994). For low-energy particles, Γ regulates the rate at which particles scatter through a 90° pitch angle. By controlling the rate of reversal of particle velocity along the large-scale magnetic field, the dynamical turbulence effect represented by Γ can in some circumstances control the spatial diffusion of energetic particles and cosmic rays in astrophysical plasmas. However, the dynamical decorrelation function is also integrally associated with turbulence properties. Indeed, in the context of either EDQNM or other hydrodynamic closures, the influence of time decorrelation on the triple correlation lifetime becomes, through approximation, embodied in the second-order Eulerian decorrelation. Similar relationships exist in MHD (Yoshizawa *et al.*, 2003), but involving several time scales. For applications in both turbulence and in energetic-particle transport it is useful to model the form and physical content of the MHD time correlations.

There are several features that enter into modeling any of the MHD time correlations. First, a useful ansatz can be given for the time correlation in the similarity form

$$\Gamma(k,\tau) = r[\gamma(k)\tau]. \tag{B1}$$

Here τ is the time difference and γ is the rate of the triple decay, as we have discussed in earlier sections.[11] We further simplify matters by estimating the decay rate γ in the same way as in earlier sections. If isotropic Alfvénic sweeping is the dominant decorrelation, we would let $\gamma \rightarrow \gamma_{sw} = k\delta V_A$ where δV_A is the characteristic magnitude of the large-scale magnetic field. If nonlinear straining is the dominant factor, $\gamma \rightarrow \gamma_{nl} = \epsilon^{1/3} k^{2/3}$, according to the standard estimate. More generally we can write $\gamma = \Sigma_i \gamma_i$ for a set of independent decorrelation rates $\{\gamma_i\}$ of various origins. As in the case of spectral transfer time-scale estimation, we can accommodate anisotropy relative to a large-scale magnetic field of strength V_A by changing the sweeping decay rate to $\gamma_{sw} = k_\parallel V_A$ while allowing for full anisotropy of nonlinear strain by writing $\gamma_{nl} = \epsilon^{1/3} k_\perp^{2/3}$ for components of wave vector parallel and perpendicular to the large-scale magnetic field. When necessary it is also reasonable to draw distinctions between strain rates associated with the two Elsässer fields, or strain due to velocity or magnetic field separately. As a rule of thumb it seems reasonable to us, for most circumstances, to let $\gamma(k) = 1/\tau_T(k)$ where the transfer decay time $\tau_T(k)$ is selected in accordance with the discussion in previous sections.

It remains to adopt a reasonable functional form (or forms) for $r(\gamma\tau)$. One possibility, perhaps most appropriate for strain-dominated decorrelation would be that $dS(\mathbf{k},t)/dt = -\gamma_{nl} S(\mathbf{k},t)$ so that the decorrelation takes on an exponential form. For the isotropic case,

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[11]This is an oversimplification for asymmetric MHD in which there are distinct transfer correlation lifetimes for fields such as $\mathbf{u}$ and $\mathbf{b}$, or $\mathbf{z}_+$ and $\mathbf{z}_-$. For now we assume that one such similarity variable is adequate.

$$\Gamma(\mathbf{k},\tau) = e^{-\tau/\tau_{nl}} = e^{-C_e \epsilon^{1/3} k^{2/3} \tau}, \tag{B2}$$

where C_e is an order one constant. Note that this is easily adapted for spectral anisotropy by the replacement $k \to k_\perp$. This exponential form is compatible with the EDQNM closure and the associated simplified form of the direct interaction approximation described in Sec. II.D. Kraichnan's (1957) analysis of DIA correlations gives rise to the alternative functional form

$$r(\gamma\tau) = \frac{J_1(2\gamma(k)\tau)}{\gamma(k)\tau}, \tag{B3}$$

where J_1 is a Bessel function of order one. The Bessel function, in anisotropic form, also emerges from the anisotropic Eulerian DIA for MHD (Nakayama, 1999).

Analyses of the Lagrangian DIA for hydrodynamics (Kaneda, 1981) and for anisotropic MHD (Nakayama, 2001) suggest a function form related to a Gaussian,

$$r[\gamma(k)\tau] = \exp\{-C_g[\gamma(k)\tau]^2\}, \tag{B4}$$

which was derived for a Lagrangian modification of the DIA. For Kineda's hydrodynamic analysis, $C_g = 0.81\pi/4$. Nakayama's MHD analysis corresponds approximately to a different value of C_g, which seems to us to be poorly determined enough in MHD that it should be treated as a fitting parameter. In particular, Nakayama's arguments suggest that the sweeping effect in MHD is stronger than in hydrodynamics. An alternative way to arrive at a Gaussian form of decorrelation is to employ Chen and Kraichnan's (1989) analysis of sweeping, in which case, for isotropic velocity variance v_0, one finds $r(kv_0\tau) = \exp[-(kv_0\tau)^2/2]$. This suggests $C_g = 1/2$ with the choice that decorrelation is due to sweeping at a rate $\gamma_{sw} = kv_0$. For MHD with decorrelation due to a random large-scale magnetic field of strength $\delta V_A = \delta B/\sqrt{4\pi\rho}$, the analogous form is obtained by letting $v_0 \to \delta V_A$.

The influence of propagation along a large-scale uniform magnetic field of strength V_A can be included by multiplying the appropriate functional form above with a factor $\cos(k_\parallel V_A)$. This emerges in both Lagrangian and Eulerian DIA for anisotropic MHD (Nakayama, 1999, 2001).

At present it is difficult to argue for any specific functional form of the time decorrelation that would be valid in all cases of interest. However, combining the perspectives described above, one can construct a reasonable proposed functional form, such as

$$\Gamma(\mathbf{k},\tau) = \cos(k_\parallel V_A \tau)e^{-\gamma_{nl}(k)\tau}e^{-[\gamma_{sw}(k)\tau]^2/2}. \tag{B5}$$

The basis for this suggestion seems to us to be reasonable, although entirely without rigor. One might view this as emerging from an appropriate generalization of the frequency superposition method described by Chen and Kraichnan (1989). In the above it is envisioned that V_A is the strength of the large-scale dc magnetic field, γ_{nl} is an appropriately selected nonlinear time scale (isotropic or anisotropic), and γ_{sw} is the sweeping time, which may include either large-scale advective effects, large-scale Alfvénic sweeping effects, or both. Here it is as-

sumed that the distribution of the sweep speeds is Gaussian (Chen and Kraichnan, 1989), but in some cases, especially for Alfvénic sweeping, another distribution may be more appropriate and this would modify the functional form of the last term. It should be mentioned that the above ansatz generalizes the previously suggested functional forms of magnetic decorrelation that have been employed in studies of cosmic-ray scattering in dynamical turbulence (Bieber *et al.*, 1994; Matthaeus and Bieber, 1999). Physically motivated improvements to the representation of the MHD Eulerian decorrelation functions may be useful in these applications. Eventually this modeling needs to be examined critically using guidelines from observations and numerical simulations.

REFERENCES

Armstrong, J. W., J. M. Cordes, and B. J. Rickett, 1981, Nature (London) **291**, 561.

Barnes, A., 1979, in *Solar System Plasma Physics*, edited by E. N. Parker, C. F. Kennel, and L. J. Lanzerotti (North-Holland, Amsterdam), Vol. I, p. 251.

Barnes, A., and J. V. Hollweg, 1974, J. Geophys. Res. **79**, 2302.

Batchelor, G. K., 1970, *The Theory of Homogeneous Turbulence* (Cambridge University Press, New York).

Belcher, J. W., and L. Davis, Jr., 1971, J. Geophys. Res. **76**, 3534.

Bieber, J. W., W. H. Matthaeus, C. W. Smith, W. Wanner, M. B. Kallenrode, and G. Wibberenz, 1994, Astrophys. J. **420**, 294.

Bieber, J. W., W. Wanner, and W. H. Matthaeus, 1996, J. Geophys. Res. **101**, 2511.

Biskamp, D., 1993, Europhys. Lett. **21**, 563.

Biskamp, D., 1995, Chaos, Solitons Fractals **5**, 1779.

Biskamp, D., 2003, *Magnetohydrodynamic Turbulence* (Cambridge University Press, New York).

Biskamp, D., and W.-C. Müller, 2000, Phys. Plasmas **7**, 4889.

Biskamp, D., and H. Welter, 1989, Phys. Fluids B **1**, 1964.

Burlaga, L. F., and K. Ogilvie, 1970, Sol. Phys. **15**, 61.

Carbone, V., F. Malara, and P. Veltri, 1995, J. Geophys. Res. **100**, 1763.

Chapman, D., 1979, AIAA J. **17**, 1293.

Chen, S., and R. H. Kraichnan, 1989, Phys. Fluids A **1**, 2019.

Chen, S., and R. H. Kraichnan, 1997, J. Plasma Phys. **57**, 187.

Cho, J., and A. Lazarian, 2002, Phys. Rev. Lett. **88**, 245001.

Cho, J., A. Lazarian, and E. T. Vishniac, 2002, Astrophys. J. **564**, 291.

Cho, J., and E. T. Vishniac, 2000, Astrophys. J. **539**, 273.

Coleman, P. J., 1968, Astrophys. J. **153**, 371.

Dmitruk, P., D. O. Gómez, and W. H. Matthaeus, 2003, Phys. Plasmas **10**, 3584.

Dmitruk, P., and W. H. Matthaeus, 2003, Astrophys. J. **597**, 1097.

Dmitruk, P., W. H. Matthaeus, L. J. Milano, S. Oughton, G. P. Zank, and D. J. Mullan, 2002, Astrophys. J. **575**, 571.

Dmitruk, P., L. J. Milano, W. H. Matthaeus, and S. Oughton, 2001, Phys. Plasmas **8**, 2377.

Dobrowolny, M., A. Mangeney, and P. Veltri, 1980, Phys. Rev. Lett. **45**, 144.

Dutton, J. A., and D. G. Deaven, 1972, in *Lecture Notes in Physics*, edited by M. Rosenblatt and C. Van Atta (Springer, Berlin), Vol. 12, p. 402.

Elsässer, W. M., 1956, Am. J. Phys. **24**, 85.

Fyfe, D., and D. Montgomery, 1976, J. Plasma Phys. **16**, 181.

Fyfe, D., D. Montgomery, and G. Joyce, 1977, J. Plasma Phys. **17**, 369.

Galtier, S., S. V. Nazarenko, A. C. Newell, and A. Pouquet, 2000, J. Plasma Phys. **63**, 447.

Galtier, S., E. Zienicke, H. Politano, and A. Pouquet, 1999, J. Plasma Phys. **61**, 507.

George, W. K., 1992, Phys. Fluids A **4**, 1492.

Goldreich, P., and S. Sridhar, 1995, Astrophys. J. **438**, 763.

Goldreich, P., and S. Sridhar, 1997, Astrophys. J. **485**, 680.

Goldstein, M. L., D. A. Roberts, and W. H. Matthaeus, 1995, Annu. Rev. Astron. Astrophys. **33**, 283.

Gómez, D. O., and C. Ferro Fontán, 1988, Sol. Phys. **116**, 33.

Grappin, R., U. Frisch, J. Leorat, and A. Pouquet, 1982, Astron. Astrophys. **105**, 6.

Grappin, R., A. Pouquet, and J. Leorat, 1983, Astron. Astrophys. **126**, 51.

Heinemann, M., and S. Olbert, 1980, J. Geophys. Res. **85**, 1311.

Hollweg, J. V., 1973, J. Geophys. Res. **78**, 3643.

Hollweg, J. V., 1974, J. Geophys. Res. **79**, 1539.

Hollweg, J. V., 1986, J. Geophys. Res. **91**, 4111.

Hossain, M., P. C. Gray, D. H. Pontius, Jr., W. H. Matthaeus, and S. Oughton, 1995, Phys. Fluids **7**, 2886.

Hossain, M., P. C. Gray, D. H. Pontius, Jr., W. H. Matthaeus, and S. Oughton, 1996, in *Solar Wind Eight*, edited by D. Winterhalter, J. T. Gosling, S. R. Habbal, W. S. Kurth, and M. Neugebauer, AIP Conf. Proc. No. 362 (AIP, Woodbury, NY), p. 358.

Iroshnikov, P., 1963, Sov. Astron. **7**, 566.

Jacques, S. A., 1977, Astrophys. J. **215**, 942.

Jokipii, J. R., 1973, Annu. Rev. Astron. Astrophys. **11**, 1.

Kaneda, Y., 1981, J. Fluid Mech. **107**, 131.

Kinney, R. M., and J. C. McWilliams, 1998, Phys. Rev. E **57**, 7111.

Kolmogorov, A. N., 1941a, C. R. Acad. Sci. URSS **30**, 301.

Kolmogorov, A. N., 1941b, C. R. Acad. Sci. URSS **31**, 538.

Kolmogorov, A. N., 1941c, C. R. Acad. Sci. URSS **32**, 16.

Kraichnan, R. H., 1957, J. Fluid Mech. **5**, 497.

Kraichnan, R H., 1965, Phys. Fluids **8**, 1385.

Kraichnan, R. H., and D. Montgomery, 1980, Rep. Prog. Phys. **43**, 547.

Leamon, R. J., C. W. Smith, N. F. Ness, W. H. Matthaeus, and H. K. Wong, 1998, J. Geophys. Res. **103**, 4775.

Leith, C. E., 1971, J. Atmos. Sci. **28** 145.

Lesieur, M., 1997, *Turbulence in Fluids* (Kluwer, Dordrecht).

Leslie, D. C., 1973, *Developments in the Theory of Turbulence* (Clarendon Press, Oxford).

Lithwick, Y., and P. Goldreich, 2001, Astrophys. J. **562**, 279.

Lithwick, Y., and P. Goldreich, 2003, Astrophys. J. **582**, 1220.

Mac Low, M.-M., 1999, Astrophys. J. **524**, 169.

Mac Low, M.-M., R. S. Klessen, A. Burkert, and M. D. Smith, 1998, Phys. Rev. Lett. **80**, 2754.

Maron, J., and P. Goldreich, 2001, Astrophys. J. **554**, 1175.

Marsch, E., 1991, in *Physics of the Heliosphere II*, edited by R. Schwenn and E. Marsch (Springer, Berlin).

Matthaeus, W. H., and J. W. Bieber, 1999, in *Solar Wind 9*, edited by S. R. Habbal, R. Esser, J. V. Hollweg, and P. A. Isenberg, AIP Conf. Proc. No. 471 (AIP, Woodbury, NY), p. 515.

Matthaeus, W. H., J. W. Bieber, and G. P. Zank, 1995, Rev. Geophys. **33**, 609.

Matthaeus, W. H., S. Ghosh, S. Oughton, and D. A. Roberts, 1996, J. Geophys. Res. **101**, 7619.

Matthaeus, W. H., and M. L. Goldstein, 1982, J. Geophys. Res. **87**, 6011.

Matthaeus, W. H., M. L. Goldstein, and D. A. Roberts, 1990, J. Geophys. Res. **95**, 20.

Matthaeus, W. H., and S. L. Lamkin, 1986, Phys. Fluids **29** 2513.

Matthaeus, W. H., and Y. Zhou, 1989, Phys. Fluids B **1**, 1929.

McComb, W. D., 1990, *The Physics of Fluid Turbulence* (Oxford University Press, New York).

Milano, L. J., W. H. Matthaeus, P. Dmitruk, and D. C. Montgomery, 2001, Phys. Plasmas **8**, 2673.

Millionshchikov, M. D., 1941, C. R. Acad. Sci. URSS **32**, 619.

Monin, A. S., and A. M. Yaglom, 1975, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA), Vol. 2.

Montgomery, D., 1987, in *Lecture Notes on Turbulence*, edited by J. R. Herring and J. C. McWilliams (World Scientific, Singapore), p. 75.

Montgomery, D., M. R. Brown, and W. H. Matthaeus, 1987, J. Geophys. Res. **92**, 282.

Montgomery, D., and W. H. Matthaeus, 1995, Astrophys. J. **447**, 706.

Nakayama, K., 1999, Astrophys. J. **523**, 315.

Nakayama, K., 2001, Astrophys. J. **556**, 1027.

Nelkin, M., 1994, Adv. Phys. **43**, 141.

Nelkin, M., and R. Tabor, 1990, Phys. Fluids A **2**, 81.

Ng, C. S., and A. Bhattacharjee, 1997, Phys. Plasmas **4**, 605.

Orszag, S. A., 1970, J. Fluid Mech. **41**, 363.

Orszag, S. A., 1977, in *Fluid Dynamics: Les Houches 1973*, edited by R. Balian and J. L. Peube (Gordon and Breach, New York), p. 237.

Oughton, S., E. R. Priest, and W. H. Matthaeus, 1994, J. Fluid Mech. **280**, 95.

Parker, E. N., 1965, Space Sci. Rev. **4**, 666.

Parker, E. N., 1972, Astrophys. J. **174**, 499.

Parker, E. N., 1979, *Cosmical Magnetic Fields: Their Origin and Activity* (Oxford University Press, Oxford, UK).

Politano, H., A. Pouquet, and P. L. Sulem, 1989, Phys. Fluids B **1**, 2990.

Politano, H., A. Pouquet, and P. L. Sulem, 1995, Phys. Plasmas **2**, 2931.

Pope, S. B., 2000, *Turbulent Flows* (Cambridge University Press, New York).

Pouquet, A., U. Frisch, and J. Leorat, 1976, J. Fluid Mech. **77**, 321.

Pouquet, A., M. Meneguzzi, and U. Frisch, 1986, Phys. Rev. A **33**, 4266.

Roberts, D. A., S. Ghosh, M. L. Goldstein, and W. H. Matthaeus, 1991, Phys. Rev. Lett. **67**, 3741.

Roberts, D. A., M. L. Goldstein, and L. W. Klein, 1990, J. Geophys. Res. **95**, 4203.

Roberts, D. A., M. L. Goldstein, L. W. Klein, and W. H. Matthaeus, 1987, J. Geophys. Res. **92**, 12023.

Robinson, D., and M. Rusbridge, 1971, Phys. Fluids **14**, 2499.

Rose, H., and P. L. Sulem, 1978, J. Phys. (Paris) **39**, 441.

Saddoughi, S. G., and S. V. Veeravalli, 1994, J. Fluid Mech. **268**, 333.

Shebalin, J. V., W. H. Matthaeus, and D. Montgomery, 1983, J. Plasma Phys. **29**, 525.

Schekochihin, A. A., J. L. Maron, S. C. Cowley, and J. C. McWilliams, 2002, Astrophys. J. **576**, 806.

Smith, C. W., W. H. Matthaeus, G. P. Zank, N. F. Ness, S.

Oughton, and J. D. Richardson, 2001, J. Geophys. Res. **106**, 8253.

Speziale, C. G., and P. S. Bernard, 1992, J. Fluid Mech. **241**, 645.

Sreenivasan, K. R., 1995, Phys. Fluids **7**, 2778.

Sridhar, S., and P. Goldreich, 1994, Astrophys. J. **432**, 612.

Taylor, G. I., 1935, Proc. R. Soc. London, Ser. A **151**, 421.

Taylor, G. I., 1938, Proc. R. Soc. London, Ser. A **164**, 476.

Tennekes, H., 1975, J. Fluid Mech. **67**, 561.

Tennekes, H., and J. L. Lumley, 1972, *A First Course in Turbulence* (MIT, Boston).

Tu, C.-Y., and E. Marsch, 1995, Space Sci. Rev. **73**, 1.

Van Atta, C. W., and J. C. Wyngaard, 1975, J. Fluid Mech. **72**, 673.

von Karman, T., and L. Howarth, 1938, Proc. R. Soc. London, Ser. A **164**, 192.

von Karman, T., and C. C. Lin, 1949, Rev. Mod. Phys. **21**, 516.

Yeung, P. K., and Y. Zhou, 1997, Phys. Rev. E **56**, 1746.

Yoshizawa, A., S. J. Itoh, and K. Itoh, 2003, *Plasma and Fluid Turbulence: Theory and Modeling* (IOP, Bristol).

Zakharov, V. E., V. L'vov, and G. E. Falkovich, 1992, *Kolmogorov Spectra of Turbulence I: Wave Turbulence* (Springer, Berlin).

Zank, G. P., and W. H. Matthaeus, 1992, J. Plasma Phys. **48**, 85.

Zank, G. P., and W. H. Matthaeus, 1993, Phys. Fluids A **5**, 257.

Zhou, Y., 1993a, Phys. Fluids A **5**, 1092.

Zhou, Y., 1993b, Phys. Fluids A **5**, 2511.

Zhou, Y., and W. H. Matthaeus, 1990a, J. Geophys. Res. **95**, 10291.

Zhou, Y., and W. H. Matthaeus, 1990b, J. Geophys. Res. **95**, 14881.

Zhou, Y., A. Praskovsky, and G. Vahala, 1993, Phys. Lett. A **178**, 138.

Zhou, Y., and C. G. Speziale, 1998, Appl. Mech. Rev. **51**, 267.

Zhou, Y., P. K. Yeung, and J. G. Brasseur, 1996, Phys. Rev. E **53**, 1261.