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Hamiltonian theories of the fractional quantum Hall effect

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This paper reviews progress on the fractional quantum Hall effect (FQHE) based on what we term Hamiltonian theories, i.e., theories that proceed from the microscopic electronic Hamiltonian to the final solution via a sequence of transformations and approximations, in either the Hamiltonian or path-integral approach, as compared with theories based on exact diagonalization or trial wave functions. The authors focus on the Chern-Simons approach, in which electrons are converted to Chern-Simons fermions or bosons that carry along flux tubes, and on their own extended Hamiltonian theory, in which electrons are paired with pseudovortices to form composite fermions whose properties are a lot closer to the ultimate low-energy quasiparticles. The article addresses a variety of qualitative and quantitative questions: In what sense do electrons really bind to vortices? What is the internal structure of the composite fermion and what does it mean? What exactly is the dipole picture? What degree of freedom carries the Hall current when the quasiparticles are localized or neutral or both? How exactly is the kinetic energy quenched in the lowest Landau level and resurrected by interactions? How well does the composite-fermion picture work at and near $\nu=1/2$? Is the system compressible at $\nu=1/2$? If so, how can composite fermions be dipolar at $\nu=1/2$ and still be compressible? How is compressibility demonstrated experimentally? How does the charge of the excitation get renormalized from that of the electron to that of the composite fermion in an operator treatment? Why do composite fermions sometimes appear to be free when they are not? How does one compute (approximate) transport gaps, zero-temperature magnetic transitions, the temperature-dependent polarizations of gapped and gapless states, the NMR relaxation rate $1/T_1$ in gapless states, and gaps in inhomogeneous states? It is seen that though the Chern-Simons and extended Hamiltonian approaches agree whenever a comparison is possible, results that are transparent in one approach are typically opaque in the other, making them truly complementary.

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I. INTRODUCTION

Twenty years ago, von Klitzing, Dorda, and Pepper (1980) made the first discovery in what has proved to be a vital and exciting subfield of condensed-matter physics to the present day, that of the quantum Hall effects. Their discovery that the Hall conductance σ_{xy} of a two-dimensional electron gas is quantized at integer multiples of $e^2/2\pi\hbar$, the quantum unit of conductance, is known as the integer quantum Hall effect (IQHE). Soon after, Tsui, Störmer, and Gossard (1982) discovered the even more puzzling fractional quantum Hall effect (FQHE). Since the first observation of the fraction $\frac{1}{3}$, many more have been seen (see, for example, Störmer, Tsui, and Gossard, 1999).

This review focuses on what we term the *Hamiltonian theories of the FQHE*, by which we mean theoretical approaches that begin with the microscopic Hamiltonian for interacting electrons and try to obtain a satisfactory description of the underlying physics through a sequence of transformations and approximations in the operator or path-integral formalism. We bother to give a special name to what appears to be business as usual because an alternate approach, pioneered by Laughlin (1983a, 1983b) and based on writing trial wave functions, has proven so extraordinarily successful. While we shall of course discuss the wave-function approach here because the Hamiltonian approach is inspired by it, the discussion (of this and any other topic) will be aimed at serving our primary goal: Providing a cogent and critical description of the Hamiltonian approach in one place, including all the hindsight and insight that the intervening years have provided. Of necessity, many topics will have to be omitted or treated summarily. Fortunately many excellent reviews exist[1] and the reader is directed to them.

The Hamiltonian theories described here address both qualitative and quantitative issues. They give a concrete operator realization of many heuristic pictures that have been espoused and they make precise under what conditions and in what sense these pictures are valid. They allow one to compute (within reasonable approximations) a large number of quantities at zero and nonzero temperatures at equal and unequal times. They open the door for a treatment of disorder.

The Hamiltonian theories themselves fall into two categories. The first, which we call the Chern-Simons approach, consists of making a singular gauge transformation on the electronic wave function that leads to a *composite particle*, which is the union of an electron and some number of point flux tubes. Composite particles have a nondegenerate ground state at mean-field level. They are coupled to a gauge field that is fully determined by the particle coordinates. This formalism is best suited for computation of a variety of response functions. It is particularly effective at $\nu=1/2$, where adherence to gauge invariance is of paramount importance if certain low-energy phenomena pertaining to the overdamped mode, coupling to surface acoustic waves, or compressibility are to be properly described. However, composite particles do not exhibit in any transparent way the quasiparticle properties (such as charge e^* or effective mass m^*) deduced from trial wave functions, though in principle they would surface after considerable work. It is also very hard to obtain a smooth limit as the electron mass $m\to 0$ in this approach. The *extended Hamiltonian theory* (EHT), which we have developed over the years, addresses some of these issues.[2] By the adjective *extended* we signify both that our work is an extension of older Chern-Simons work and that our Hamiltonian is defined in an extended or enlarged Hilbert space with additional degrees of freedom. The enlarged space allows us to introduce a quasiparticle that is a much better approximation to what we expect, to easily disentangle low-energy (lowest-Landau-level) and high-energy physics, to compute a variety of gaps and finite temperature properties, to describe inhomogeneous states, and so on. However, the proper treatment of the additional degrees of freedom is very difficult to ensure in the computation of certain very-low-energy quantities. Both the nonzero compressibility and an overdamped mode at $\nu=1/2$ (and its experimental consequences) are all but invisible in this approach, though they can be extracted with some effort. No contradictions exist between the predictions of the Chern-Simons and extended Hamiltonian approaches, which by and large tend to make predictions in complementary regions. Their predictions can be shown to agree in overlapping regions, but only with some effort.

[1]The earliest comprehensive introduction to the full range of the quantum Hall effects is that of Prange and Girvin (1990). Huckestein (1995) concentrates on the integer quantum Hall effect. A more recent comprehensive review is that of Das Sarma and Pinczuk (1997), while a more focused treatment of the enormous body of research on composite fermions is presented by Heinonen (1998).

[2]We used to refer to this as THE Hamiltonian formalism, but were persuaded to consider a name change that better reflected the state of affairs. Our compliance could not have been more total—the new acronym is the exact reverse of the old one.

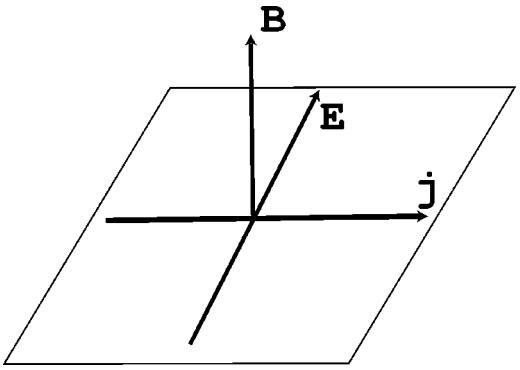

FIG. 1. A schematic of the Hall experiment in a rectangular geometry.

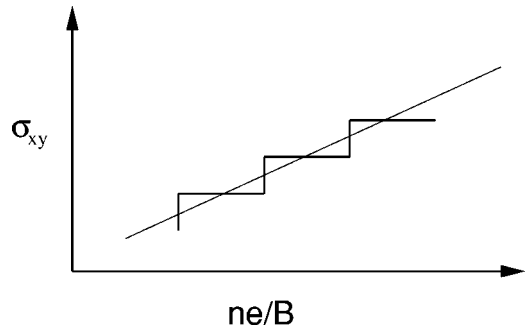

FIG. 2. A schematic of the measured Hall conductance at zero temperature. The straight line is the Galilean invariant result (valid when no disorder is present), while the steps describe the data.

A. The experiment

Figure 1 shows in schematic form the experiment in a rectangular geometry, with the current j_x, magnetic field B_z, and electric field E_y in mutually perpendicular directions. By definition

$$j_x = \sigma_{xy} E_y . \tag{1}$$

If we multiply both sides by the width of the sample we obtain

$$I_x = \sigma_{xy} V_y \tag{2}$$

as the relation between current and voltage, reminding us that in this $d=2$ problem the Hall conductance and conductivity are the same.[3]

Let us ask what one expects for σ_{xy} based on the simplest ideas, so as to place the experimental discoveries in perspective. Let us ignore interactions and disorder. We can then assert that, in the steady state, the electric and magnetic forces balance,

$$eE = evB, \tag{3}$$

where e is the charge of the electrons and v is their velocity. (Vector indices are omitted when obvious.) Therefore

$$j = nev = \frac{neE}{B}, \tag{4}$$

where n is the number density. Thus

$$\sigma_{xy} = \frac{ne}{B}. \tag{5}$$

This result is unaffected by interactions and relies only on Galilean invariance (relativity). To see this, let us perform a boost to a frame in which the electric field (to leading order in v/c)

$$\mathbf{E}' = \mathbf{E} - \mathbf{v} \times \mathbf{B} \tag{6}$$

vanishes, that is, to a frame with $v = E/B$. In this frame $j=0$. Boosting back to the lab frame we obtain $j = neE/B$ and regain Eq. (5).

Figure 2 is a sketch of what is measured at $T=0$ in the thermodynamic limit. The key feature is that σ_{xy}, instead of varying linearly with ne/B, is quantized at steps or plateaus. This article focuses on steps given by

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu, \tag{7}$$

where

$$\nu = \frac{p}{2ps+1}, \quad p = 1,2, \quad s = 0,1,2, \ldots . \tag{8}$$

The IQHE corresponds to $s=0$, $\nu=p$, while the FQHE refers[4] to $s>0$. We explicitly display the $\hbar$, which will soon be set equal to unity.

On each step the longitudinal conductances vanish. Thus the conductivity tensor assumes the form

$$\sigma_{ij} = \frac{e^2}{2\pi\hbar}\begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}. \tag{9}$$

Consequently the diagonal part of the resistivity tensor is also zero and the transport is dissipationless. In the real world, due to nonzero temperature T and finite sample size, the transitions between steps acquire a finite width, with $\sigma_{xx}>0$ therein.

Why is the quantization of conductance so surprising, given that we have seen so many instances of quantization of observables before? The answer is that the observable in this case does not refer to an atom or molecule, but to a *macroscopic* sample, with sample-specific disorder. The role of disorder is truly paradoxical in the quantum Hall effect: on the one hand, without disorder, we cannot escape the Galilean-invariant relation $\sigma_{xy} = ne/B$; on the other hand, despite disorder (which surely varies from sample to sample), the conductance in any step is constant to better than a few parts in 10^{10}. A fairly detailed explanation of this now exists and will be presented below. An integral part of this resolution is also the explanation for why $\sigma_{xx}=0$, for it is basic to an understanding of why σ_{xy} is so well quantized even

[3]The longitudinal conductance and conductivity are related by the (aspect) ratio of length to width.

[4]Fractions with denominator $2ps-1$ require only minor modifications and are left as an exercise for the reader.

though the leads used to measure V_y may not be perfectly aligned—there will be no unwanted contributions from the longitudinal voltage drop.

B. What is special about the steps?

In searching for the physics underlying any one step or plateau, it makes sense to begin with the points where the straight line (the Galilean-invariant result) intercepts the steps. Not only are these points singled out by experiment, but they also have a Hall conductance that we could hope to understand without including disorder.[5]

At these points we have

$$\frac{ne}{B}=\frac{e^2}{2\pi\hbar}\frac{p}{2ps+1}, \tag{10}$$

which we rewrite as

$$\frac{B}{n(2\pi\hbar/e)}=2s+\frac{1}{p}. \tag{11}$$

Thus the ratio of B, the flux density, and n, the particle density, has some special rational values at these points. For example, in the case of the fraction 1/3, there are three quanta of flux per electron.

To understand what is so special about these values, recall the following textbook results (Shankar, 1994) about a single particle of mass m and charge e moving in two dimensions in a perpendicular magnetic field.

- The energy is quantized into Landau levels located at $E=(e\hbar B/m)(n+1/2)\equiv\hbar\omega_0(n+1/2)$, where ω_0 is called the *cyclotron frequency*.
- Each Landau level has a degeneracy equal to Φ/Φ_0, the flux in units of the flux quantum $\Phi_0=2\pi\hbar/e$, or

  $$\text{Degeneracy per unit area of each Landau level}=\frac{B}{(2\pi\hbar/e)}. \tag{12}$$

- The wave functions of the lowest Landau level (LLL) are, in the symmetric gauge,

  $$\psi_{LLL}=z^m e^{-|z|^2/4l^2},\quad z=x+iy,\quad m=0,1,\ldots, \tag{13}$$

  where $l=\sqrt{\hbar/eB}$ is the magnetic length. We shall often drop the Gaussian factor in LLL wave functions. Higher Landau-level wave functions depend on both z and $\overline{z}$.
- Thus Eqs. (11) and (12) imply that at the special points

$$\frac{\text{Number of states in a Landau level}}{\text{Number of particles}}=\frac{\text{Flux quanta}}{\text{particles}}=\frac{B}{n(2\pi\hbar/e)}=2s+\frac{1}{p}=\nu^{-1}. \tag{14}$$

Note that ν, which stood for the dimensionless conductance, is thus the number of occupied Landau levels. It is called the *filling factor*. In the FQHE ν is a fraction, restricted in this article to be $<1/2$ unless otherwise mentioned.

If $\nu<1$, there are more LLL states than particles, and in the noninteracting limit they can all be fit into the lowest Landau level with room to spare. For example, at $\nu=1/3$, there are three LLL states per electron. This macroscopic degeneracy in the noninteracting ground state means that we cannot even get started with a perturbative treatment of disorder and interactions. This is the central problem. While this problem exists for any ν that is not an integer, experiments suggest that $\nu^{-1}=2s+1/p$ is somehow preferred by Nature. Indeed, at such points there is a way out, but it is not simple perturbation theory. This approach is what this article is all about.

An alternative to perturbation theory, the Hartree-Fock approximation, also does not work if applied directly to the electronic Hamiltonian. Let us see why. In Hartree-Fock one takes the interaction (quartic in fermion operators) and obtains an expression quadratic in the fermion operators by replacing various bilinears by their ground-state averages, initially taken as free parameters, and dropping some higher-order fluctuations. The quadratic Hamiltonian is then solved and the ground-state energy is then minimized as a function of the assumed averages. At the minimum, the actual averages will self-consistently come out equal to the assumed averages. Here is a toy model that illustrates the main ideas. The model has just two fermion operators c and d:

$$H=\varepsilon_c c^\dagger c+\varepsilon_d d^\dagger d+u_0 c^\dagger c d^\dagger d. \tag{15}$$

Let us separate out the bilinears into averages $\langle c^\dagger c\rangle=\lambda_c$ and $\langle d^\dagger d\rangle=\lambda_d$ and fluctuating parts $:c^\dagger c:$ and $:d^\dagger d:$,

$$c^\dagger c=:c^\dagger c:+\langle c^\dagger c\rangle\equiv:c^\dagger c:+\lambda_c, \tag{16}$$

$$d^\dagger d=:d^\dagger d:+\langle d^\dagger d\rangle\equiv:d^\dagger d:+\lambda_d. \tag{17}$$

We now rewrite H as

$$H=\varepsilon_c c^\dagger c+\varepsilon_d d^\dagger d+u_0[\lambda_d c^\dagger c+\lambda_c d^\dagger d-\lambda_c\lambda_d +:c^\dagger c::d^\dagger d:] \tag{18}$$

and neglect the last term, quadratic in the fluctuations, to obtain the Hartree-Fock Hamiltonian.

Let $E_0(\lambda_c,\lambda_d)$ be the energy of the ground state $|0\rangle$ of the Hartree-Fock Hamiltonian. From the Feynman-Hellman theorem

$$\frac{\partial E_0}{\partial\lambda}=\left\langle 0\left|\frac{\partial H}{\partial\lambda}\right|0\right\rangle, \tag{19}$$

[5] In the absence of disorder any state would have the desired Hall conductance. We are looking for a nontrivial correlated state that is robust under the inclusion of disorder and capable of dominating the plateau to which it belongs.

which can be proven using the fact that $\langle 0|0\rangle = 1$ has no λ derivative. It follows that, at the minimum, $\lambda_c = \langle c^\dagger c\rangle$ and $\lambda_d = \langle d^\dagger d\rangle$.

Why does this fail in the FQHE problem? For integer filling this procedure gives the expected state of an integer number of Landau levels filled (at least for not-too-strong interactions). However, it turns out that for generic fractional filling translational symmetry is spontaneously broken in Hartree-Fock, and the solution corresponds to a crystalline (usually a Wigner crystal) state. Due to the breaking of translational symmetry, the Landau level and angular momentum indices (n,m) are no longer good quantum numbers. The single-particle states are now bands in the crystal, parametrized by a Bloch quasimomentum $\mathbf{k}$ and a band index n_b. The Hartree-Fock Hamiltonian for the FQHE will be of the form $H = \Sigma_{n_b \mathbf{k}} \varepsilon_{n_b}(\mathbf{k}) c^\dagger_{n_b}(\mathbf{k}) c_{n_b}(\mathbf{k})$ up to c-number terms (the analogs of $\lambda_c \lambda_a$). The self-consistent solutions were worked out by Yoshioka and Fukuyama (1979), Fukuyama and Platzman (1982), and Yoshioka and Lee (1983). The Hartree-Fock solution is compressible, is translationally noninvariant, and has no preference for any particular density. Thus it does not describe the FQHE phenomenology.

C. The integer quantum Hall effect—A warmup

We begin with a brief look at the IQHE case $s=0$, or $\nu = p$, which paves the way for Jain's (1989) view of the FQHE as the IQHE of entities called composite fermions.

In the IQHE, exactly p Landau levels are filled in the noninteracting limit. There is exactly one totally antisymmetric ground state, which we denote by χ_p. The simplest example is $p=1$, with just the lowest Landau level filled up. The corresponding wave function is

$$\chi_1 = \mathrm{Det}\begin{vmatrix} z_1^0 & z_1^1 & z_1^2 & \cdots \\ z_2^0 & z_2^1 & z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} \cdot \mathrm{Gaussian}$$

$$= \prod_{i<j} (z_i - z_j) \exp\left[-\sum_i \frac{|z_i|^2}{4l^2} \right]. \tag{20}$$

This nondegenerate ground state is separated from excited states by a gap equal to the cyclotron energy ω_0. Though now we have a starting point for perturbation theory, we still need to actually carry out the perturbative calculations and in particular understand why the Hall conductance is unaffected by these perturbations and stays at the Galilean-invariant value. We also need to understand why the conductance is unchanged as we make small changes in density, i.e., why there are steps.

The explanation of the IQHE, discovered over the years, can be most easily understood in the noninteracting limit. The single-electron problem with certain special types of disorder can be exactly solved (Aoki and Ando, 1981; Prange, 1981; Prange and Joynt, 1982), and the exact solution shows that the Hall current is independent of disorder. Moving beyond this, Trugman (1983) showed in a seminal paper that for generic disorder in very high magnetic fields the electronic states follow the equipotentials of the disorder potential, and each Landau level gets broadened into a band. The problem has many fruitful analogies with percolation. In particular, there is exactly one energy at the band center at which extended states can exist, while states at all other energies are localized. When the chemical potential μ crosses this energy, the Hall conductance changes by an integer. Contrariwise, when μ lies in an energy range corresponding to localized states, changes of μ (changes of filling) produce no change in σ_{xy}, which explains the steps.

From a somewhat different point of view, Laughlin (1981) and Halperin (1982) showed by a gauge argument that in order to have a quantized Hall conductance in the noninteracting limit, all one needs is $\sigma_{xx}=0$, which is assured if the chemical potential lies in the region of localized states. This argument shows that the Hall conductance (in units of the quantum unit of conductance $e^2/2\pi\hbar$) is a topological integer. There is also a field-theoretic analysis of the noninteracting quantum Hall problem in the presence of disorder that yields similar results (Levine, Libby, and Pruisken, 1983, 1984a, 1984b, 1984c; Pruisken, 1984, 1985a, 1985b).

The IQHE can also be understood in the other limit, where interactions dominate, as follows: In this case if one is not exactly at $\nu =$ integer, the excess (or deficit) of particles form a Wigner crystal due to interactions. One turns off the interactions and imposes an external periodic potential of the same period as the Wigner crystal. In this case Thouless *et al.* (1982) showed that the dimensionless Hall conductance has to be an integer and that this integer is topological. Later work by Thouless (1983), Niu and Thouless (1984), and Niu, Thouless, and Wu (1985) showed that, since this integer is topological, it is robust to the adiabatical introduction of disorder and interactions. As long as the charge gap does not close and there are no ground-state transitions, the Hall conductance will remain unchanged.[6] This allows one to turn off the external periodic potential as the interactions are being turned on. For further details on the IQHE the reader is referred to an excellent review by Huckestein (1995).

D. The fractional Quantum Hall effect

We turn now to fractions, such as 1/3. There is nothing special about disordered single-particle states at such a filling: They are expected to be localized except right near the middle of the Landau band (Trugman, 1983). If we ignore disorder and interactions, we are left with the macroscopically degenerate manifold of many-body states. Presumably interactions will select the ground state. How is one to find it?

[6] By definition, an integer cannot change continuously.

Here the story branches into two trails. The one blazed by Laughlin (1983a, 1983b) consists of writing down inspired trial wave functions; the other is the Hamiltonian approach, which starts with an assault on the degeneracy problem.

1. Laughlin's answer

After some experimentation, Laughlin wrote down the following celebrated trial wave function for $\nu = 1/(2s+1)$:

$$\Psi_{1/(2s+1)} = \prod_{j<i} (z_i - z_j)^{2s+1} \exp\left(-\sum_i |z_i|^2/4l^2 \right), \tag{21}$$

which was shown to have nearly unit overlap with the numerical solution for small systems for generic repulsive interactions (Laughlin, 1983a, 1983b, 1990). We shall refer to $\nu = 1/(2s+1)$ as Laughlin fractions, as per convention.

Of the many remarkable properties of this function, we list those that have the greatest bearing on what follows. First, it is from the lowest Landau level and obeys the Pauli principle under particle exchange, since $2s+1$ is odd and spin is assumed polarized. Halperin (1983, 1984) zeroed in on one of its central features: *it has no wasted zeros*, by which he meant the following: Consider $\psi(z_1)$, which is $\Psi(z_1,\ldots,z_N)$ as a function of any one variable, randomly chosen to be z_1, with all others held fixed.[7] Given that the sample is penetrated by $N/\nu = N(2s+1)$ quanta of flux, the phase of $\psi(z_1)$ has to have an Aharonov-Bohm phase change of $2\pi N(2s+1)$ per particle, or $N(2s+1)$ zeros given the LLL condition of analyticity. Only N of these had to lie on other electrons by the Pauli principle. But they all do lie on other electrons, thereby keeping the electrons away from each other very effectively, producing a low potential energy. (The kinetic energy is of course the same for any function in the lowest Landau level.)

By showing that quantum averages in these ground states are precisely statistical averages in a one-component plasma (Baus and Hansen, 1980; Caillol *et al.*, 1982), Laughlin (1983a, 1983b) showed that the system is an *incompressible fluid*, which means a fluid that abhors density changes. Unlike a Fermi gas, which increases or decreases its density globally when compressed or decompressed, an incompressible fluid is wedded to a certain density and will first show no response to any applied pressure, and then suddenly nucleate a localized region of different density (just the way a type-II superconductor, in which a magnetic field is not welcome, will allow it to enter in quantized units in a region that turns normal).

Laughlin (1983a, 1983b) also provided the wave function for a state with such a localized charge deficit. If one imagines inserting a tiny solenoid at a point z_0 and slowly increasing the flux to one quantum, one must, by gauge invariance, return to an eigenstate of H, and each particle must undergo a 2π phase shift as it goes around z_0.[8] This condition and analyticity point to the ansatz

$$\Psi_{qh} = \prod_i (z_i - z_0)\Psi_{2s+1}. \tag{22}$$

This is a *quasihole*. There is a more complicated state with a quasiparticle.

The prefactor is a *vortex* at z_0. Since vortices play a significant role in the FQHE, let us digress to understand them better. Consider first a real flux tube inserted into the sample at a point z_0. Clearly every electron will "see" this tube at z_0, i.e., $\psi(z_1)$, the wave function, seen as a function of any one coordinate, chosen to be z_1, will see a 2π phase shift as z_1 goes around z_0. The flux tube clearly has a reality and location of its own, independent of the locations of the electrons. The vortex is an analytic LLL version of the flux tube: the 2π phase change is accompanied by a zero, i.e., there is an analytic zero at z_0 for every coordinate. The zeros associated with the vortex have the key feature that their location does not depend on the location of the particles, i.e., it too has an independent reality and location like the flux tube.

There are also vortices in Laughlin's ground-state wave function. The only difference is that, instead of sitting at some point z_0, they are anchored to the particles themselves. To see this, consider $\nu = \frac{1}{3}$ and focus on the part involving just z_1:

$$\prod_{j>1} (z_j - z_1)^3. \tag{23}$$

If we freeze z_1 and view it as a parameter like z_0, we see that there is a triple vortex on particle 1 and, by symmetry, each particle. Apart from the vortex mandated by the Pauli principle, there are two vortices bound to each electron. (The Pauli zero comes with the turf, for being a fermion, and is not included in the count of vortices attached to enforce correlations.) The term *vortex* is again appropriate here, since the location of the vortex is independent of any coordinate except, of course, the electron to which it is attached.

Contrast the zeros that constitute a vortex to the zeros of a generic antisymmetric analytic polynomial of the same degree as the Laughlin wave function. Once again, in $\psi(z_1)$, there is one zero at z_i $(i>1)$ independent of all other z_j by the Pauli principle. This zero is part of a vortex, since all particles will "see" (in the wave functions ψ) the Pauli zeros anchored on the other particles. The non-Pauli zeros of $\psi(z_1)$, by contrast, will be parametric functions of all other particle coordinates $i = 2,\ldots,N$ (what else can they be?). If we now consider $\psi(z_2)$, its non-Pauli zeros will in general bear no rela-

[7]The discussion is independent of this choice.

[8]The flux must be inserted slowly enough to prevent transitions to other excited states, but fast enough to prevent a lapse to the original ground state at the end, a possibility that exists in finite systems with disorder.

tion to those of $\psi(z_1)$. Since the location of the non-Pauli zeros moves parametrically with all the z's, they do not belong to or form vortices. The reader must remember these distinctions in order to follow the discussion on the internal structure of composite fermions that follows shortly.[9]

Returning now to the quasihole, we note another important property, that it represents a charge deficit of $1/(2s+1)$ in electronic units. One way to show this is to employ the plasma analogy (Laughlin 1983a, 1983b, 1990). Here is a more general way (Su and Schrieffer, 1981; Su, 1984) that depends only on the state's being gapped, incompressible, and having a quantized Hall conductance. As the flux quantum Φ_0 is adiabatically inserted to create the quasihole, the charge driven out to infinity is given by integrating the radial current density j produced by the Hall response to the induced azimuthal E field

$$Q=-\int j(r,t)2\pi r dt=\sigma_{xy}\int E2\pi r\ dt$$

$$=-\sigma_{xy}\int \frac{d\Phi}{dt}dt=-\Phi_0\sigma_{xy} \tag{24}$$

$$=-\frac{2\pi\hbar}{e}\frac{e^2}{2\pi\hbar}\nu=-e\nu. \tag{25}$$

For non-Laughlin fractions, the charge driven out by inserting one flux quantum is that of p quasiparticles, each of which has a charge equal to $1/(2ps+1)$ (Su, 1984). Note that the fractional σ_{xy} is the cause behind the fractional charge. The fractional charge of the quasiparticles has been confirmed experimentally (Goldman and Su, 1995; Goldman, 1996). The quasiparticles also have fractional statistics, as was pointed out by Halperin (1984) and explicitly confirmed in a wave-function calculation by Arovas *et al.* (1985).[10]

The insensitivity of the Hall conductance to disorder in the FQHE can be established by an extension of the Laughlin-Halperin gauge arguments for the IQHE. To show this in toroidal geometry, the old arguments have to be supplemented by the fact that at fractions of the form p/q, there are q ground states, and the assumption that upon adding q flux quanta one returns to the starting state. There seems to be a consensus that Hall conductance is a ground-state property that is impervious to a reasonable amount of disorder, very much the way superfluidity is.

Laughlin (1983a, 1983b, 1990) explained the plateaus as we move off the magic fraction as follows. Suppose we move slightly off $\frac{1}{3}$, to, say, $\nu=\frac{1}{3}\pm 0.01$. The system has two choices. It can either go to a FQHE ground state at this fraction or it can be at $\frac{1}{3}$ plus some number of quasiparticles (quasiholes) of charge $\pm\frac{1}{3}$. It turns out that the latter has a lower energy. The reason σ_{xy} is locked at $\frac{1}{3}$ is that the quasiparticles/quasiholes, being sharply defined entities, get localized by the disorder potential and do not contribute to the Hall current. Thus the Hall conductance is expected[11] to be unaffected by disorder. As we move further off $\frac{1}{3}$, the system switches to another ground state with its own quasiparticles, and σ_{xy} jumps to the neighboring plateau. It is clear that if the fate of the quasiparticles/quasiholes is to get localized, their fractional charge cannot explain the fractional Hall conductance. As seen earlier, the connection is the other way around.

Thus a fairly complete description of the FQHE emerged from the Laughlin wave functions for the ground and quasiparticle/quasihole states. However, numerous other issues surfaced and were tackled by subsequent work in the intervening two decades. We now turn our attention to those bearing directly on this paper.

Sticking for a moment to the Laughlin fractions, one important question that was raised was whether the system was really gapped, i.e., could there be lighter excitations than Laughlin's quasiparticles and quasiholes? Perhaps a quasiparticle and quasihole could bind to form a gapless excitation. Haldane and Rezayi (1985) verified by exact diagonalization of small systems that the $\nu=\frac{1}{3}$ system was gapped. Girvin, MacDonald, and Platzmann (1986) explored the question of neutral particle-hole excitations using an analogy with Feynman's work on superfluids and found a way to calculate the dispersion relations of the magnetoexciton within the lowest Landau level and showed that they were gapped in this sector for $\nu=\frac{1}{3}$.

2. Jain's composite fermions

Let us now proceed to the non-Laughlin fractions. Here the central question is this: Given that non-Laughlin fractions like $\frac{2}{5}$ are seen in experiment, what is their wave function? Simply replacing the factors $(z_i-z_j)^{2s+1}$ by $(z_i-z_j)^{5/2}$ in Laughlin's ansatz is not acceptable, since this does not produce the mandatory change of sign under particle exchange. In the hierarchy approach (Haldane, 1983; Halperin, 1983, 1984) the quasiparticles of the Laughlin states condense into their own FQHE states, whose quasiparticles in turn do the same thing, and so on. While this approach gives a natural way to generate additional fractions, it does not give explicit wave functions in terms of electrons.

[9]If the Laughlin function is perturbed slightly, $\psi(z_1)$ will have one Pauli zero on every other electron and two others nearby. As long as the perturbation is small we can say, at length scales bigger than the excursion of the zeros, that electrons are bound to double vortices. For stronger perturbations that take us far from the Laughlin ansatz, there will be no simple description of correlations in terms of vortices.

[10]Fractional statistics was shown to be an emergent property of composite fermions by Goldhaber and Jain (1995).

[11]There is currently no way of rigorously calculating transport coefficients in the FQHE in the presence of disorder, hence the qualifier "expected."

Explicit trial states for fractions $\nu = p/(2ps+1)$ were provided by Jain (1989, 1990, 1994), who also explained why they were natural in terms of objects called *composite fermions*.

Jain's scheme is based on the seminal idea of flux attachment, which plays such a central role in the FQHE that it merits a little digression. It was introduced first by Leinaas and Myrrheim (1977) in terms of wave functions, and was explored in the language of Chern-Simons field theories by Wilczek (1982a, 1982b). The crux of the idea is that in $d=2$ one can have particles (dubbed *anyons* by Wilczek) that suffer a phase change $e^{i\theta}$ upon exchange, with $\theta = 0, \pi \pmod{2\pi}$ corresponding to bosons and fermions, respectively. To obtain one of these particles one takes a fermion and drives a point flux tube through its center, the amount of flux being decided by the desired θ. In particular, if this tube contains an even/odd number of flux quanta, the composite particle one gets is a fermion/boson.

Jain (1989, 1990) exploited flux attachment for $\nu^{-1} = 2s + 1/p$ as follows. Suppose we trade our electrons for composite fermions carrying $2s$ point flux quanta pointing opposite to the external B. On average the composite fermions effectively see $1/p$ flux quanta per particle and fill up exactly p Landau levels. At the mean-field level, this approach gives the following trial wave function:

$$\Psi_{p/2ps+1} = \prod_{i<j} \left[\frac{(z_i - z_j)}{|z_i - z_j|} \right]^{2s} \cdot \chi_p(z, \bar{z}), \tag{26}$$

where χ_p is the composite-fermion wave function with p-filled Landau levels and the prefactor takes the composite-fermion wave function back to the electronic wave function. (This will be made clearer shortly.)

Jain improved this ansatz in two ways and proposed

$$\psi_{p/2ps+1} = \mathcal{P} \prod_{i<j} (z_i - z_j)^{2s} \cdot \chi_p(z, \bar{z}), \tag{27}$$

where

$$\prod_{i<j} (z_i - z_j)^{2s} \tag{28}$$

is called the *Jastrow factor*, and

$$\mathcal{P}: \bar{z} \to 2l^2 \frac{\partial}{\partial z} \tag{29}$$

projects the wave function to the lowest Landau level by eliminating all $\bar{z}$'s in χ_p by turning them to z derivatives on the rest of the wave function, except for the Gaussian. (We shall see[12] that in the lowest Landau level $[z, \bar{z}] = -2l^2$.)

In making the change $\Pi_{i<j}[(z_i - z_j)/|z_i - z_j|]^{2s} \to \Pi_{i<j}(z_i - z_j)^{2s}$, Jain replaces flux tubes by vortices. Although in both cases each particle picks up an extra phase shift of $4\pi s$ on going around another, only the vortex factor has the desirable multiple zero that keeps the particles apart and lies in the lowest Landau level. Jain does not need to justify these modifications, since in writing a trial wave function, he is free to resort to any changes that improve its energy. However, in the Hamiltonian approach, the replacement of flux tubes by vortices and the projection to the lowest Landau level prove difficult but not insurmountable.

At $p=1$, since $\chi(z, \bar{z}) = \Pi_{i<j}(z_i - z_j) \cdot \text{Gaussian}$, we do not need $\mathcal{P}$ to get back Laughlin's answer. For $p>1$ we have concrete expressions for Ψ in terms of electron coordinates, although the action of $\mathcal{P}$ can be quite involved and has a big impact on the wave function, as we shall see.

Thus, while the degeneracy of the noninteracting problem is present for any $\nu<1$, at the Jain fractions one can beat it by thinking in terms of composite fermions. As we move off the Jain fractions, the incremental composite fermions (particles or holes) become localized, giving rise to the plateaus. This is the sense in which the composite-fermion approach allows one to think of the FQHE as the IQHE of the composite fermions. Since the IQHE can be understood without invoking interactions, it is sometimes suggested that composite fermions are free. Later we shall see why this cannot be so.

The reader will have noted that both Laughlin and Jain wave functions *make no reference to the interelectron potential*. This feature, which permits them to work for a whole class of potentials, also renders them insensitive to specific features, including even the range of the interaction.

3. Vortices, zeros, and the dipole picture reexamined

Let us return to the vortices that are attached to electrons in the Jastrow factor. We have already discussed Halperin's observation (Halperin, 1983, 1984) that electrons are bound to vortices in the Laughlin wave function. This concept was taken up and vigorously pursued by Read (1989). In particular, Read made a clear distinction between flux attachment and vortex attachment, which were often loosely interchanged. Attaching electrons to flux tubes is a mathematical trick; flux tubes are unphysical and neutral. Vortices, however, are physical excitations of the Laughlin ground states, and electrons would naturally bind to them since the two are oppositely charged. The reason the composite fermion sees a weaker field is not due to any mysterious capture of flux tubes. Indeed, the external field is uniform and not quantized into point flux tubes. What really happens in the Laughlin case is that each electron pairs with $2s$ vortices, and when the composite object goes around on a loop, it sees a phase change of $-2\pi(2s+1)$ per enclosed particle due to the external B and $2\pi(2s)$ per enclosed composite fermion due to the vortices attached to each one.

Next, since the $2s$-fold vortex has a charge $-2s/(2s+1)$ as per the flux insertion argument (Su, 1984), Eq. (25), the vortices reduce e down to the quasiparticle or composite fermion charge

[12] For a nice review of states, operators, and matrix elements within the lowest-Landau level, see Girvin and Jach (1984).

$$e^* = 1 - \frac{2s}{2s+1} = \frac{1}{2s+1}. \tag{30}$$

In a Fermi liquid an added electron is also screened by a correlation hole; here the difference is that the possible response of the Laughlin liquid to an extra electron is limited to an integer number of vortices of quantized charge.

Read (1994, 1996) next applied the notion of electrons binding to vortices to the case $\nu = \frac{1}{2}$ and derived what is called the *dipole picture*. Let us consider this concept more closely. Since the dipole picture comes from the projection involved in the $\nu = \frac{1}{2}$ wave function, it is instructive to start with an understanding of the impact of $\mathcal{P}$ in a general Jain fraction where higher composite-fermion Landau levels ($p > 1$) are filled.

Consider N electrons at $\nu = \frac{2}{5}$. Prior to projection, each electron in the Jain wave function has a double vortex sitting on it due to the Jastrow factor. As we explained earlier, the term vortex is merited since every other particle sees it at the same place. At this stage it is correct to view the composite fermion as an electron plus a double vortex.[13] All this changes after projection to the lowest Landau level. Now there can be only $5N/2$ zeros of the wave function as a function of, say, z_1; N of them must lie on other electrons by the Pauli principle (forming single vortices anchored to electrons), with $3N/2$ left over. Clearly they cannot all be on electrons or be associated with them uniquely (since we have 3/2 zeros per electron). If the zeros are not on electrons, their location must depend parametrically on the locations of *all* the electrons, and they cannot organize themselves into vortices.

Thus the wave-function analysis leaves us with the following quandary. The composite fermion cannot be viewed as an electron-vortex complex in the projected non-Laughlin states, but the quasiparticle charge, $e^* = 1 - 2ps/(2ps+1)$, is robust under $\mathcal{P}$ since e^* is tied to σ_{xy}, which is presumed robust under projection. What, if any, is the entity that binds to the electron to bring e down to e^*? How does this entity enter the theory? What makes it bind to the electron? We shall see that the extended Hamiltonian theory answers such questions.

Let us now return to $\nu = \frac{1}{2}$. In the unprojected wave function the double-vortex charge fully cancels the electron charge so that $e^* = 0$. Read (1994) has argued that the neutral composite fermion of momentum k has a dipole moment $d^* = kl^2$, based on the wave function at $\nu = \frac{1}{2}$, also called the Rezayi-Read (Rezayi and Read, 1994) wave function, in which χ_p in Eq. (27) is replaced by the filled Fermi sea:

$$\Psi_{1/2} = \mathcal{P} \prod_{i<j} (z_i - z_j)^2 \mathrm{Det}|e^{i\mathbf{k}_i \cdot \mathbf{r}_j}|. \tag{31}$$

[13]There is also another zero, generally nonanalytic, in χ_p, which is antisymmetric. This is not part of the vortex count.

Read considers

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \exp\left(\frac{i}{2}(k\bar{z} + \bar{k}z)\right), \quad k = k_x + ik_y. \tag{32}$$

Since under the action of $\mathcal{P}$, $\bar{z}$ acts on the analytic part of the wave function as $2l^2\partial/\partial z$, $e^{il^2k(\partial/\partial z)}$ causes the shift $z \to z + ikl^2$ in the Jastrow factor. This motion of the vortex off the electron produces the dipole moment $d^* = kl^2$. The energy needed to separate the vortex from the electron (the Coulomb attraction) must begin with a term quadratic in the separation or momentum. This then gives the LLL fermions a kinetic energy or an effective mass m^* resulting from interactions.

Attractive though this picture is, closer scrutiny reveals that the preceding line of reasoning is incorrect in the following ways:

- Since every z_i gets translated, $(z_i - z_j)^{2s} \to (z_i - z_j + ik_i - ik_j)^{2s}$ and particle i sees a multiple zero associated with z_j at $z_j + i(k_j - k_i)$. We cannot call this a vortex since the location of the zero varies with the label i. Different particles see it in different places. This problem is actually moot because of the next, more serious one.
- Even this multiple zero is there for one particular assignment of k's, or one term in the determinant. Upon antisymmetrization, we cannot relate the zeros of the sum over the $N!$ terms to the zeros of individual terms. The situation would have been different if we had been talking about poles, which can survive such a sum. In fact, one of the two zeros that moved off the particles (in each individual term) should return upon antisymmetrization to lie exactly on the particles by Pauli's principle, leaving another zero to depend parametrically on all other coordinates, and all the k_i. Thus there is no reason to expect any simple relation between the location of the electrons and the non-Pauli zeros in $\Psi_{1/2}$ or to conclude that these zeros form vortices. We commend to the reader experimentation with the limited but nonetheless instructive case of just two particles, to see some of these ideas in concrete form.
- If one looks at $\nu = 1/2$ in isolation, the fact that there is one non-Pauli zero per electron (in the thermodynamic limit) may tempt one to suggest that perhaps in this case they organize themselves into vortices. In addition to our analysis of the gapped fractions, which shows that this is extremely unlikely, there is the general argument that zeros that do not lie on particles or external flux tubes (i.e., locations with an independent reality) must vary parametrically with all coordinates and therefore cannot be organized into vortices.

Thus at $\nu = 1/2$ there seem to be two choices: Either use the unprojected wave function in which the Jastrow factor explicitly has two vortices per electron but the dipole moment is zero (since the vortices are on the electrons), or go to the lowest Landau levels, where the vortices disintegrate into a smaller number of ordinary

zeros, not correlated with the electrons in any simple fashion. So what happens to the dipole?

Our extended Hamiltonian theory, which provides an operator realization of the composite fermion, will show that the dipole picture is more robust than the wave-function-based arguments pointing to it. However, one must look for it not in the wave function, but in operators and correlation functions. Even then, the value of d^* will be sensitive to the details of the wave function. In this respect, it is quite unlike σ_{xy}, which is robust under changes in the ground-state wave functions (including projection to the lowest Landau level) and depends only on the filling factor.

E. Hamiltonian approach

Let us begin by reviewing the challenges facing any Hamiltonian theory. In the FQHE there is more than enough room in the lowest Landau level for all the electrons, leading to a macroscopic number of degenerate ground states in the absence of interactions. This preempts many of the standard approximations. It is expected that when interactions are turned on, there will emerge from this degenerate manifold a unique ground state, separated by a gap from other low-lying excited states. While the true ground state and low-lying excitations will of course contain an admixture of higher Landau levels, one expects that in the limit of bare mass $m \to 0$, or $\omega_0 \to \infty$, there will emerge a low-energy sector spanned by LLL states. In other words, in the limit $m \to 0$ a nonsingular low-energy theory must emerge. The m-independent Laughlin wave functions for the ground state, quasiparticles, and quasiholes illustrate this point.

If all memory of m is lost, the only energy scale is set by the interactions. How is one to isolate the LLL physics, starting with the full Hilbert space? How is one to battle the degeneracy of the noninteracting ground state? How is one to get rid of the $1/m$ dependence of the kinetic energy? These are problems for the Hamiltonian approach.

Next, there is one vestige of m dependence even in the lowest Landau level, discovered by Simon, Stern, and Halperin (1996): if the external field B varies slowly in space, the zero-point energy $eB/2m$ can no longer be eliminated by redefinition of the zero. In fact, the particles behave as if they have a magnetic moment $\mu^* = e/2m$ coupled to B. The theory must reproduce this moment.[14]

We also do not want to ban higher Landau levels too soon, since the Hall conductance necessarily involves them.[15] Finally, Kohn's theorem (Kohn, 1961) assures us that despite interactions, the cyclotron mode at $q=0$ must be at $\omega_0 = eB/m$ and it must saturate certain sum rules at small q. The Hamiltonian theory must pass this test: it must succeed in keeping the m dependence where it belongs and exorcise it elsewhere.

The next set of issues concerns the quasiparticles, the composite fermions. A heuristic picture that we saw arising from the unprojected wave functions is that the quasiparticles are composites of electrons and $2s$ vortices. How is this actually realized in the Hamiltonian approach? After all, this is not analogous to the statement that mesons are made of quarks and antiquarks, for unlike antiquarks, which appear in the Hamiltonian, vortices do not. Vortices are not independent of electrons, and to speak of them and electrons at the same time surely paves the way for overcounting. In the non-Laughlin cases, the issue is further complicated by the fact that there are not enough zeros to form $2s$ vortices per electron, and in any case the zeros are not organized into vortices. Yet some object of the same charge as the $2s$-fold vortex seems to bind with the electron since in the end e gets reduced to e^*. Somehow this object has to enter the theory and give rise to the composite fermion.

Other issues surround the composite fermion. How strong are interactions between composite fermions? Why do they sometimes appear to be free when we can give persuasive arguments for why they cannot be? How is disorder to be included?

Besides addressing these questions of principle, we need answers for quantitative questions. For example, what is the gap at $\nu = 1/3$? Very precise answers can be given for such a question in the trial-wave-function approach[16] or by exact diagonalization.[17] While these results are founded on the microscopic Hamiltonian, the intermediate steps are computer intensive. It would be nice to have a theory that displayed transparently the features of the composite fermion deduced from the study of wave functions and that furnished quantitative results to, say, 10% accuracy. The same goes for polarizations in various states and transitions between them.

An especially interesting set of questions arises at and near $\nu = 1/2$. Can the notion of the composite fermion survive in this region without a robust gap? In particular, will they really behave like particles of charge e^* in this region? Since the average effective magnetic field is $B^* = 0$ at $\nu = 1/2$, we have fermions in zero magnetic field, except for fluctuations. Will this Fermi system form a Fermi liquid after including fluctuations? What are its response functions? Will it be compressible? If so, how do particles with $e^* = 0$ manage to be compressible or even have a nonzero Hall conductance? What are the

[14]This magnetic moment is actually a compact way to describe an effect that is orbital in origin.

[15]The current operator goes as $1/m$ while σ_{xy} does not. The $1/m$ in the current is canceled by the energy denominator of order ω_0 that comes from virtual transitions to the higher Landau levels. For a more detailed explanation, see Girvin, MacDonald, and Platzman (1986) and Sondhi and Kivelson (1992).

[16]For a review of the trial-wave-function method with complete references, see Jain and Kamilla (1998).

[17]Due to limitations of space we present only the earliest references: Girvin and Jach (1983), Su (1984), Haldane and Rezayi (1985), Morf and Halperin (1986, 1987), Yoshioka (1986), with one exception, Morf *et al.* (2002).

collective modes and how can they be detected? How is disorder to be handled? Exact diagonalization and the wave-function-based approaches are not very helpful for unequal-time correlations and finite-temperature physics. Hamiltonian theories should fill this void and answer all the questions raised above.

F. Organization of this review

Section II introduces the reader to some notation and the physics of the lowest Landau level, including Kohn's theorem, which is often referred to in our work. Section III describes the Chern-Simons theory of flux attachment, including both composite bosons and fermions. Section IV deals with the region at and near $\nu=1/2$ and is mainly about the work of Halperin, Lee, and Read (1993). Section V explains our extended Hamiltonian theory, EHT, formulated in an enlarged Hilbert space. It is shown that there are two approaches to solving our final equations: the conserving approximation, in which the constraints are respected but the composite fermion physics is hidden, and a shortcut in which the opposite is true. The compressibility paradox at $\nu=1/2$ is discussed. Section VI illustrates the conserving approximation in the EHT via the calculation of the structure factor at small q in the Jain series and the magnetoexciton dispersion relations for 1/3 and 2/3. Section VII is devoted to the computation of gaps and comparison to numerical work and experiment. Section VIII deals with magnetic phenomena at $T=0$ for gapped and gapless systems. The question of whether composite fermions are free or not is discussed and answered in the negative, along with an explanation of why they sometimes appear to be so. Section IX addresses physics at $T>0$, describing the computation of polarization and relaxation at $\nu=1/2$ and polarization at 1/3 and 2/5. The results are then compared to experiment. Section X deals with inhomogeneous states. Section XI gives a critical evaluation of the EHT. A summary and a discussion of open problems follow in Sec. XII.

II. PRELIMINARIES AND NOTATION

This section focuses on a single electron in two dimensions. The Hamiltonian is

$$H_0=\frac{(\mathbf{p}+e\mathbf{A})^2}{2m}=\frac{\mathbf{\Pi}^2}{2m}. \tag{33}$$

Here $\mathbf{A}$ is the vector potential that leads to the external magnetic field $\mathbf{B}=\nabla\times\mathbf{A}=-B\hat{\mathbf{z}}$, $\mathbf{p}$ is the canonical momentum of the electron, $-e$ is its charge, and m is its bare or band mass. Note that $\mathbf{B}$ points along the negative z axis.

Although one needs to pick a gauge for $\mathbf{A}$ in order to find the wave functions, we can obtain the spectrum without making that choice. Let us define a *cyclotron coordinate*

$$\boldsymbol{\eta}=l^2\hat{\mathbf{z}}\times\mathbf{\Pi}, \tag{34}$$

where $l=\sqrt{\hbar/eB}$ is the magnetic length. Despite the name, the two components of $\boldsymbol{\eta}$ are not commuting but canonically conjugate variables:

$$[\eta_x,\eta_y]=il^2. \tag{35}$$

It follows that

$$H_0=\frac{\boldsymbol{\eta}^2}{2ml^4} \tag{36}$$

describes a harmonic oscillator with energies

$$E=\left(n+\frac{1}{2}\right)\omega_0, \tag{37}$$

where n is the Landau-level index.

The Landau levels are highly degenerate because another conjugate pair that commutes with $\boldsymbol{\eta}$ called the *guiding-center coordinate*,

$$\mathbf{R}=\mathbf{r}-\boldsymbol{\eta}, \tag{38}$$

is cyclic. The components of $\mathbf{R}$ obey

$$[R_x\ ,R_y]=-il^2. \tag{39}$$

Thus l^2 plays the role of $\hbar$. Since in the lowest Landau level $\langle\eta^2\rangle=l^2$, $\mathbf{R}$ roams all over the sample, whose area L^2 plays the role of phase space. The degeneracy of the lowest Landau level (or any Landau level) is, from the Bohr-Sommerfeld quantization rule,

$$D=\frac{L^2}{\text{``}h\text{''}}=\frac{L^2}{2\pi l^2}=\frac{eBL^2}{2\pi\hbar}=\frac{\Phi}{\Phi_0}, \tag{40}$$

where Φ_0 is the flux quantum. This leads to the following result worthy of committing to memory:

$$\frac{\text{LLL states}}{\text{particles}}=\frac{D}{N}=\frac{\text{flux quanta density}}{\text{particle density}}=\frac{eB}{2\pi n\hbar}. \tag{41}$$

At points where steps in σ_{xy} cross the straight line dictated by Galilean invariance,

$$\sigma_{xy}=\frac{ne}{B}=\frac{e^2}{2\pi\hbar}\nu=\frac{e^2}{2\pi\hbar}\frac{p}{2ps+1}, \tag{42}$$

and

$$\frac{\text{LLL states}}{\text{particles}}=\frac{\text{flux quanta density}}{\text{particle density}}=\frac{eB}{2\pi\hbar n}=2s+\frac{1}{p}=\nu^{-1}. \tag{43}$$

Hereafter we set $\hbar=1$.

A. Gauge choices

There are two famous choices for gauge. In the *Landau gauge* $\mathbf{A}=-\mathbf{j}Bx$ the Hamiltonian is cyclic in y, and hence has the eigenfunctions e^{iky}. Making the ansatz

$$\psi(x,y)=e^{iky}\phi(x), \tag{44}$$

we find that ϕ obeys the equation for a displaced harmonic oscillator,

$$-\frac{1}{2m}\frac{d^2\phi}{dx^2}+\frac{1}{2}m\omega_0^2(x-kl^2)^2\phi=E\phi. \tag{45}$$

The lowest Landau level corresponds to putting the oscillator in its ground state. The degeneracy of any Landau level can be computed by considering a sample with sides L_x and L_y, with periodic boundary conditions in the y direction. This forces $k=2\pi j/L_y$. Since the wave function is centered on $x=kl^2<L_x$, we must demand $0\leqslant j\leqslant L_xL_y/2\pi l^2$. Thus the degeneracy D of each Landau level is (going back to a square sample) $D=BL^2/\Phi_0$, in agreement with our prior gauge-invariant counting of states. We shall use this gauge in the computation of the magnetoexciton.

In the rest of this article we employ the *symmetric gauge*, in which

$$\mathbf{A}=\frac{eB}{2}(\mathbf{i}y-\mathbf{j}x), \tag{46}$$

$$\boldsymbol{\eta}=\frac{1}{2}\mathbf{r}+l^2\hat{\mathbf{z}}\times\mathbf{p}, \tag{47}$$

$$\mathbf{R}=\frac{1}{2}\mathbf{r}-l^2\hat{\mathbf{z}}\times\mathbf{p}. \tag{48}$$

B. Getting to know the lowest Landau level

As with any simple harmonic oscillator, we can construct ladder operators from the canonically conjugate pair η_x and η_y (with l^2 playing the role of $\hbar$),

$$a_\eta=\frac{1}{l\sqrt{2}}(\eta_x+i\eta_y), \tag{49}$$

$$a_\eta^\dagger=\frac{1}{l\sqrt{2}}(\eta_x-i\eta_y). \tag{50}$$

The LLL condition $a_\eta|LLL\rangle=0$ gives

$$\psi=e^{-|z|^2/4l^2}f(z), \tag{51}$$

where $z=x+iy$. A basis for ψ is

$$\psi_m(z)=z^me^{-|z|^2/4l^2}, \qquad m=0,1,\ldots. \tag{52}$$

The Gaussian is often suppressed. The state has angular momentum $L_z=m$.

If $\nu=1$ (one electron per LLL state), there is a unique noninteracting ground state, which may then be perturbed by standard means,

$$\chi_1=\prod_{i<j}(z_i-z_j)\cdot\text{Gaussian}$$
$$=\text{Det}\begin{vmatrix} z_1^0 & z_1^1 & z_1^2 & \cdots \\ z_2^0 & z_2^1 & z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}\cdot\text{Gaussian}. \tag{53}$$

For $\nu<1$, we often want to focus on the limit $\omega_0\to\infty$ and work entirely within the lowest Landau level. If in $H=T+V$ we set T equal to a constant ($eB/2m$ per particle), all the action is in V. Why is this a problem if V is a function of just coordinates? After all,

$$\rho(\mathbf{r})=\sum_j\delta(\mathbf{r}-\mathbf{r}_j) \tag{54}$$

and

$$V=\frac{1}{2}\int d^2r\int d^2r'\rho(\mathbf{r})v(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \tag{55}$$

$$=\frac{1}{2}\sum_{\mathbf{q}}\rho(\mathbf{q})v(q)\rho(-\mathbf{q}), \tag{56}$$

$$\rho(\mathbf{q})=\int d^2r\rho(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}=\sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}. \tag{57}$$

The point is that if one sets $T=eB/2m$, the LLL value, one must project the operator $\mathbf{r}$ to the lowest Landau level. The coordinates x and y, which commute in the full Hilbert space, no longer commute in the lowest Landau level and V now contains noncommuting operators.

C. Projection to the lowest Landau level

Let $\mathcal{P}$ denote projection to the lowest Landau level.[18] Then

$$\mathcal{P}: \quad \mathbf{r}=\mathbf{R}+\boldsymbol{\eta}\Rightarrow\mathbf{R}. \tag{58}$$

The projected components do not commute:

$$[R_x,R_y]=-il^2 \quad \text{or} \quad [z,\bar{z}]=-2l^2 \quad \text{in the lowest Landau level.} \tag{59}$$

As for the densities,

$$\mathcal{P}: \quad e^{-i\mathbf{q}\cdot\mathbf{r}}\Rightarrow\langle e^{-i\mathbf{q}\cdot\boldsymbol{\eta}}\rangle_{LLL}e^{-i\mathbf{q}\cdot\mathbf{R}}=e^{-q^2l^2/4}e^{-i\mathbf{q}\cdot\mathbf{R}}. \tag{60}$$

Thus the projected problem is defined by

$$\bar{H}=\frac{1}{2}\sum_{\mathbf{q}}e^{-q^2l^2/2}\bar{\rho}(\mathbf{q})v(q)\bar{\rho}(-\mathbf{q}), \tag{61}$$

$$\bar{\rho}(\mathbf{q})=\sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_j}. \tag{62}$$

The projection $\bar{\rho}$ is the *magnetic translation operator*,[19] which differs from the projected density by a factor $e^{-q^2l^2/4}$. We shall often refer to this as the density, but take care to include the Gaussian in Eq. (61). The commutation rules of $\bar{\rho}$ define the magnetic translation algebra:

$$[\bar{\rho}(\mathbf{q}),\bar{\rho}(\mathbf{q}')]=2i\sin\left[\frac{(\mathbf{q}\times\mathbf{q}')l^2}{2}\right]\bar{\rho}(\mathbf{q}+\mathbf{q}'), \tag{63}$$

[18] A very nice introduction to LLL physics appears in Girvin and Jach (1984)

[19] These operators and the projective group they form have a long history and were first used to describe symmetries of the noninteracting electron Hamiltonian in a magnetic field. For references, see Peterson (1960), Brown (1964), and Zak (1964a, 1964b). To the best of our knowledge, Girvin, MacDonald, and Platzman (1986) were the first to concentrate on the algebra of this operator.

which was thoroughly exploited in the work of Girvin, MacDonald, and Platzman (1986).

There is no small parameter in $\bar{\bar{H}}$ and the overall energy scale is set by $v(q)$. This is why the FQHE problem is unique. As mentioned earlier, the Hartree-Fock solution is also not an option at fractional filling.

D. Kohn's theorem

By Kohn's theorem we shall mean the following results. Consider, in a translationally invariant system, the density-density response function $K(\mathbf{q},\omega)$, the Fourier transform of

$$K(\mathbf{q},t)=i\theta(t)\langle 0|[\rho(\mathbf{q},t),\rho(-\mathbf{q},0)]|0\rangle. \tag{64}$$

In the frequency domain we define

$$K(\mathbf{q},\omega)=\int_{-\infty}^{\infty}K(\mathbf{q},t)e^{i\omega t}dt. \tag{65}$$

From just the canonical commutation rules it can be shown that K obeys the sum rule

$$\int_0^{\infty}\mathrm{Im}\,K(\mathbf{q},\omega)\frac{d\omega}{\pi}=\frac{q^2n}{2m}. \tag{66}$$

Kohn showed that K must have a pole, the magnetoplasmon, at the cyclotron frequency $\omega_0=eB/m$ with a residue that saturates the above sum rule as $q\to 0$.

It follows that $\bar{\bar{\rho}}(\mathbf{q})$, the restriction of $\rho(\mathbf{q})$ to the lowest Landau level, cannot have (transition) matrix elements that are linear in q for small q.

The structure factor $S(q,\omega)$,

$$S(q,\omega)=\sum_n|\langle 0|\rho(q,0)|n\rangle|^2\delta(\omega-E_n), \tag{67}$$

is related to $K(q,\omega)$ for $\omega>0$ by

$$\frac{1}{\pi}\mathrm{Im}\,K(q,\omega)=S(q,\omega). \tag{68}$$

Kohn's theorem[20] tells us that if we limit ourselves to the lowest Landau level, $S(q)\simeq q^4$ for small q.

III. HAMILTONIAN THEORY I—THE CHERN-SIMONS APPROACH

The Hamiltonian for electrons in a vector potential $\mathbf{A}$ is

$$H=\sum_j\frac{[\mathbf{p}_j+e\mathbf{A}(\mathbf{r}_j)]^2}{2m}+V, \tag{69}$$

where V is the electron-electron interaction, say, Coulombic. Disorder is not included. As stated earlier, the field $\mathbf{A}$ is such that there are $2s+1/p$ flux quanta or LLL states per electron and the attendant degeneracy frustrates perturbative analysis.

The first step in the Chern-Simons approach is to deal with the degeneracy by resorting to flux attachment. For the Laughlin fractions there are actually two options involving either composite bosons or composite fermions.

A. Composite bosons

Historically, the first treatment came from Zhang, Hansson, and Kivelson (1989), who considered Laughlin fractions $\nu=1/(2s+1)$. They traded electrons for composite bosons carrying $2s+1$ flux quanta in opposition to the applied field so that at mean-field level the bosons saw zero field and had a unique ground state.[21] The trading is done by introducing a Chern-Simons wave function Ψ_{CS} defined as follows:

$$\Psi_e=\prod_{i<j}\frac{(z_i-z_j)^{2s+1}}{|z_i-z_j|^{2s+1}}\Psi_{\mathrm{CS}}$$

$$\equiv\exp\left((2s+1)i\sum_{i<j}\phi_{ij}\right)\Psi_{\mathrm{CS}}, \tag{70}$$

$$H_{CS}=\sum_i\frac{[\mathbf{p}_i+e\mathbf{A}(\mathbf{r}_i)+\mathbf{a}_{cs}(\mathbf{r}_i))^2}{2m}+V, \tag{71}$$

where ϕ_{ij} is the phase of the coordinate difference z_i-z_j. Since Ψ_e changes sign under particle exchange and the prefactor produces $2s+1$ extra minus signs, Ψ_{CS} describes bosons.

The Chern-Simons gauge field, $\mathbf{a}_{cs}$, comes from the action of $\mathbf{p}$ on the prefactor (which is just the phase of the Jastrow factor) that multiplies Ψ_{CS}:

$$\mathbf{a}_{cs}(\mathbf{r}_i)=2s\nabla\sum_{j\neq i}\phi_{ij}, \tag{72}$$

$$\oint\mathbf{a}_{cs}(\mathbf{r}_i)\cdot d\mathbf{r}_i$$

$$=2s\oint\sum_{j\neq i}\nabla\phi_{ij}\cdot d\mathbf{r}_i \tag{73}$$

$$=2\pi(2s+1)(\text{number of particles enclosed}), \tag{74}$$

$$\nabla\times\mathbf{a}_{cs}=2\pi(2s+1)\rho. \tag{75}$$

Equation (75) shows explicitly that the flux quantum density is $2s+1$ times the particle density. This is what is meant by flux attachment. Equations (71) and (75) define a Chern-Simons theory.[22] The possibility that the

[20] To see this, introduce a complete set of exact eigenstates $|n\rangle$ of H between the two factors of ρ in Eq. (64), express the Heisenberg operator $\rho(t)$ in terms of $\rho(0)$ and H, and do the time integral.

[21] This mean-field idea was first applied to anyon superconductivity by Laughlin (1988). Many more works on this topic followed: Chen *et al.* (1989); Fetter, Hanna, and Laughlin (1989); Halperin, March-Russell, and Wilczek (1989); Hanna, Laughlin, and Fetter (1989, 1991); Lee and Fisher (1989); Dai *et al.* (1992).

[22] Note that these manipulations could just as well be done in the path-integral formulation. We have chosen to use the first-quantized operator version for all our discussions, in the interest of uniformity.

FQHE would be described by a Chern-Simons theory was presaged by Girvin (1987).

Since the idea of flux attachment is to cancel the applied field on average, Zhang, Hansson, and Kivelson (1989) separate $\mathbf{a}_{cs}$ and ρ into average and fluctuating parts:

$$\nabla\times\langle\mathbf{a}_{cs}\rangle+\nabla\times:\mathbf{a}_{cs}:=2\pi(2s+1)n+2\pi(2s+1):\rho:. \tag{76}$$

This gives

$$H_{CS}=\sum_i \frac{(\mathbf{p}+e\mathbf{A}+\langle a_{cs}\rangle+:\mathbf{a}_{cs}:)_i^2}{2m}+V$$
$$=\sum_i \frac{(\mathbf{p}+:\mathbf{a}_{cs}:)_i^2}{2m}+V \tag{77}$$

upon using the fact that the flux due to $e\mathbf{A}$ precisely cancels that due to $\langle\mathbf{a}_{cs}\rangle$. Bosons in zero field have no degeneracy problem (assuming they have some repulsive interactions) and allow one to describe much of the FQHE physics in the familiar language of superfluids. For example, it has been shown by computing response functions that the superfluidity of the bosons implies the FQHE for electrons (Zhang, Hansson, and Kivelson, 1989) and that the vortex in the superfluid is Laughlin's quasihole (Lee and Zhang, 1991). The nature of the collective modes has also been explored (Kane *et al.*, 1991).

Neglecting $:\mathbf{a}_{cs}:$ (the mean-field approximation) and the interaction Ψ_{CS}, we obtain the wave function for bosons [in Eq. (70)] as just unity and that of electrons as

$$\Psi_e=\prod_{i<j}\left(\frac{z_i-z_j}{|z_i-z_j|}\right)^{2s+1}\cdot 1, \tag{78}$$

which is the phase of Laughlin's answer. Kane *et al.* (1991) showed that if long-wavelength Gaussian fluctuations are included, the full Laughlin wave function is obtained at long distances. [For an alternative route see Rajaraman and Sondhi (1996).] But the same fluctuations also reduce the long-range order in the boson field down to the power-law order found by Girvin and MacDonald (1987), who analyzed a gauge-transformed version of the Laughlin wave function.[23] Read (1989) then showed that for Laughlin fractions one could form an operator (which was a composite of an electron and $2s+1$ vortices) that was neutral and had true long-range order. The corresponding Landau-Ginzburg theory for the order parameter was, however, very complicated. Constraints of time and space prevent us from describing composite bosons any further. We refer the reader to primary sources and excellent reviews (Zhang, 1992; Karlhede, Kivelson, and Sondhi, 1993).

There are, however, some shortcomings in the composite-boson approach. First, it is restricted to Laughlin fractions. Next, since bosons have to be interacting to be stable, a noninteracting starting point does not exist. Finally, there is singular dependence on m as $m\to 0$ and it is hard to carry out quantitative computations. Nonetheless, this work has served as a paradigm for the Hamiltonian approach.

B. Composite fermions

We turn now to composite fermions. It was seen in the review of Jain's work (Sec. I.D.2) that by trading electrons for composite fermions, which carry $2s$ flux quanta in opposition to the applied field, we could get a fermionic system that on the average sees a field $\mathbf{A}^*$ that is just right to fill p Landau levels. Lopez and Fradkin (1991, 1992, 1993, 1998) were the first to accomplish this in the Hamiltonian approach.[24] They traded the electronic wave function Ψ_e for Ψ_{CS} defined as follows:

$$\Psi_e=\prod_{i<j}\frac{(z_i-z_j)^{2s}}{|z_i-z_j|^{2s}}\Psi_{\rm CS}\equiv\exp\left(2is\sum_{i<j}\phi_{ij}\right)\Psi_{\rm CS}, \tag{79}$$

$$H_{CS}=\sum_i\frac{[\mathbf{p}_i+e\mathbf{A}(\mathbf{r}_i)+\mathbf{a}_{cs}(\mathbf{r}_i)]^2}{2m}+V. \tag{80}$$

Since Ψ_e describes fermions, so does Ψ_{CS}, since the phase factor is even under particle interchange. The Chern-Simons gauge field $\mathbf{a}_{cs}$ now obeys

$$\nabla\times\mathbf{a}_{cs}=4\pi s\rho. \tag{81}$$

Separating $\mathbf{a}_{cs}$ and ρ into average and fluctuating parts,

$$H_{CS}=\sum_i\frac{(\mathbf{p}+e\mathbf{A}+\langle\mathbf{a}_{cs}\rangle+:\mathbf{a}_{cs}:)_i^2}{2m}+V$$
$$=\sum_i\frac{(\mathbf{\Pi}+:\mathbf{a}_{cs}:)_i^2}{2m}+V, \tag{82}$$

$$\mathbf{\Pi}=\mathbf{p}+e\mathbf{A}+\langle\mathbf{a}_{cs}\rangle\equiv\mathbf{p}+e\mathbf{A}^*, \tag{83}$$

Lopez and Fradkin obtained

$$\nabla\times(e\mathbf{A}+\langle\mathbf{a}_{cs}\rangle)=-eB+4\pi ns \tag{84}$$
$$=-\frac{eB}{2ps+1}$$
$$\equiv -eB^*\quad\left(\mathbf{A}^*=\frac{\mathbf{A}}{2ps+1}\right), \tag{85}$$

$$l^*=\frac{1}{\sqrt{eB^*}}=l\sqrt{2ps+1}. \tag{86}$$

The following important results emerged from the work of Lopez and Fradkin (1991, 1992, 1993):

- If we ignore $:\mathbf{a}_{cs}:$ and V, the composite fermions see $1/p$ flux quanta each (since $2s+1/p\to 1/p$ under flux attachment) and have a unique ground state χ_p of p filled Landau levels. Excitations are given by pushing fermions into higher composite-fermion Landau levels.

[23]This is to be expected given that the gauge transformation of Girvin and MacDonald (1987) is the same Chern-Simons transformation of Eq. (70).

[24]Note that, in the interest of uniformity, throughout this paper we do not distinguish between the functional-integral formalism that they employed and the operator approach that we used in our work.

There is, however, a problem: If we excite a fermion from level p to $p+1$, the energy cost (activation gap) of the particle-hole pair is $\Delta = eB^*/m$ plus corrections due to neglected terms. This divergent dependence on m flies in the face of the nonsingular $m \to 0$ limit we have argued must exist.[25] We want $\Delta \simeq e^2/\varepsilon l$ in the Coulomb case.

- At the mean-field level, the composite-fermion wave function χ_p transformed back to electrons is

$$\Psi_e = \prod_{i<j} \left(\frac{z_i - z_j}{|z_i - z_j|} \right)^{2s} \chi_p(z, \bar{z}). \tag{87}$$

Lopez and Fradkin showed that fluctuations at one loop give the square of the wave function (at long distances) for Laughlin fractions. (The factors $|z_i - z_j|^{2s}$ in the denominator of the mean-field wave function are eliminated by fluctuations.)

- They calculated time-dependent density-density response functions in the random-phase approximation (RPA; this will be explained in the next section). The cyclotron mode appears with the right position and residue. However, between the cyclotron mode (eB/m) and the LLL excitations, there are many spurious modes attributable to the ubiquitous presence of m, which prevents a clear separation of LLL and non-LLL energy scales. This is a problem common to all Chern-Simons theories and was well appreciated by the authors.

The Lopez-Fradkin work paved the way for subsequent work to which we now turn.

IV. PHYSICS AT AND NEAR $\nu = 1/2$

In the early days of the composite fermion, there was a widespread belief that its utility was confined to fractions with a robust gap: the all-forgiving gap allowed one to neglect, in a first approximation, interactions, disorder, and gauge-field fluctuations. It was therefore quite a surprise to see that the composite fermion survived even when $p \to \infty$ or $\nu \to 1/2s$ and the composite-fermion cyclotron gap $eB^*/m = eB/(2ps+1)m$ approached zero.

A. Physics at $\nu = 1/2$

Kalmeyer and Zhang (1992) were the first to discuss the case $\nu = 1/2$. Their work emphasized the following important point. One may expect that the effect of disorder will be rather small since the electron donors lie at a respectable distance from the electron gas itself. However, any small charge inhomogeneity induced by disorder is accompanied by a corresponding flux density in a Chern-Simons theory, and this can cause significant scattering.

A landmark study of the region at and near $\nu = 1/2$ was made by Halperin, Lee, and Read (1993), who seriously pursued the remarkable possibility that a Fermi liquid could be hiding deep in the fractional quantum Hall regime. Halperin, Lee, and Read started with the following Hamiltonian *at* $\nu = 1/2$:

$$H_{CS} = \sum_i \frac{(\mathbf{p} + :\mathbf{a}_{cs}:)_i^2}{2m} + V, \tag{88}$$

where $:\mathbf{a}_{CS}:$ is related to charge fluctuations by the Chern-Simons condition

$$\nabla \times :\mathbf{a}_{cs}: = 4\pi :\rho:. \tag{89}$$

Note that Chern-Simons theory dictates that m must be the bare mass, since flux attachment by minimal coupling is a gauge transformation performed on the bare electron. If one wants to take the view that this is an effective low-energy theory with an effective mass m^*, one must also be prepared for the possibility that the coupling of the fermion to the gauge field is more complicated.

The cornerstone of Halperin, Lee, and Read's work is the computation of the electromagnetic response functions. Let us recall some general results on this topic so we may better understand and appreciate their work. In writing this description we were greatly aided by the reviews of Halperin (in Das Sarma and Pinczuk, 1997) and Simon (in Heinonen, 1998).

If an external four-potential $eA_\mu^{ext}(\mathbf{q}, \omega)$ is applied to the system, it will generate a four-current $j_\mu(\mathbf{q}, \omega)$ (whose components are the particle-number current and density) as per

$$j_\mu(\mathbf{q}, \omega) = eK_{\mu\nu}(\mathbf{q}, \omega) A_\nu^{ext}(\mathbf{q}, \omega). \tag{90}$$

Linear-response theory tells us that

$$K_{\mu\nu} = \int_{-\infty}^{\infty} dt e^{i\omega t} i\theta(t) \langle 0 | [j_\mu(\mathbf{q}, t), j_\nu(-\mathbf{q}, 0)] | 0 \rangle, \tag{91}$$

where $|0\rangle$ is the vacuum state.

For pedagogical purposes let us consider just K_{00}, the density-density correlator. In free-field theory, K_{00} is given by K_{00}^0, the particle-hole bubble, in which the particle and hole created by one ρ are absorbed by the other. The full theory, of course, requires us to include interactions. Once again, for pedagogical purposes, let us begin with the case with just the Coulomb interactions. The exact K_{00} is given by an infinite sum of Feynman graphs in which the electrostatic propagator $v(q) = 2\pi e^2/q$ appears in all possible ways. It is possible to organize the sum as in Fig. 3. Each bubble K_{00}^v, called the *Coulomb-irreducible response*, has the property that it cannot be cut into two disjoint pieces by snipping just one Coulomb propagator $v(q)$, denoted by the wiggly line connecting the irreducible bubbles. Performing the geometric sum, one finds

[25] Jain does not have this problem since he does not use H_{CS} or χ_p or its excitations directly. For him the Chern-Simons picture is a step towards getting electronic wave functions for the ground and excited states by attaching the Jastrow factor and projecting. The energy gap is computed as the difference in $\langle V \rangle$ between the ground and excited electronic wave functions.

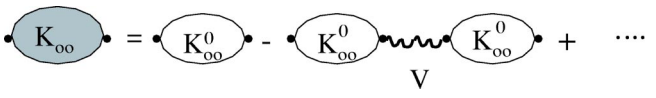

FIG. 3. The structure of the RPA expansion. Here the wiggly line is the interaction and the bubble is the free particle-hole pair. If the bubble is made to include all interaction diagrams except those that have a horizontal wiggle (upon snipping which the diagram will be disjoint), the result is the full exact answer.

$$K_{00}=\frac{K^v_{00}}{1+v(q)K^v_{00}}=\frac{1}{[K^v_{00}]^{-1}+v(q)}. \tag{92}$$

The response K^v_{00} has the following significance. Consider some conducting system with an applied potential $e\phi^{ext}$. An electron inside the conductor feels in addition the potential generated by the charges themselves, i.e., it feels the total potential $e\phi^T=e\phi^{ext}-v(q)\rho(\omega,\mathbf{q})$, and K^v_{00} is the response to this total field. To verify this let us write

$$\rho(\omega,\mathbf{q})=eK^v_{00}[e\phi^{ext}-v(q)\rho(\omega,\mathbf{q})], \tag{93}$$

for which we get the solution

$$\rho(\omega,\mathbf{q})=\frac{eK^v_{00}}{1+v(q)K^v_{00}}\phi^{ext}. \tag{94}$$

Who cares about the total field? Answer: The voltmeter we use to measure the drop in a wire. The voltmeter responds to the total electric field and not to the externally applied one that would exist in the absence of the conductor. The longitudinal conductivity σ_{xx} can be related to K^v_{00} as

$$\sigma_{xx}=\frac{ej}{E^T}=e^2\frac{\omega}{iq^2}K^v_{00}, \tag{95}$$

where we have used the continuity equation $qj=\omega\rho$ and $E=iq\phi$.[26]

In the RPA, K^v_{00} is approximated by K^0_{00}, the free particle-hole bubble. Thus RPA takes into account Coulomb interaction via the internal field the particle themselves generate, but ignores all vertex and self-energy corrections coming from exchanging $v(q)$ inside the particle-hole bubble. Figure 3 shows the connection between the irreducible and total responses.

The work of Halperin, Lee, and Read differs from this illustrative example in that there are now two types of gauge fields, the Coulombic $v(q)$ and the Chern-Simons gauge field, so that the wiggly line in Fig. 3 is described by a 2×2 matrix propagator. The Halperin-Lee-Read version of RPA consists of summing repeated bubbles irreducible with respect to both propagators, and approximating the irreducible part by the free-field response. Thus the Chern-Simons field and $v(q)$ are included only to take into account the internal fields produced by the induced charges and currents. In matrix notation it is still true that

$$[K]^{-1}=[K^0]^{-1}+U, \tag{96}$$

where U is a matrix propagator for the Coulombic $v(q)$ and the Chern-Simons gauge fields. Let us focus on just the 00 element K_{00}, which assumes the form

$$K_{00}=\frac{1}{v(q)+[K^0_{00}]^{-1}+\left(\frac{2\pi\bar{\phi}}{q}\right)^2K^0_{11}}, \tag{97}$$

where K^0_{11} is the free-field transverse-current response and $\bar{\phi}$ is the number of flux quanta attached. Though $\bar{\phi}=2s$ in our notation, we still use Halperin, Lee, and Read's notation here to help readers who may wish to consult that work for more details.

Let us now examine this expression in various regimes.

1. Static compressibility

The behavior of K_{00} at $\omega=0$, $q\to0$, gives the static compressibility of the system. For $\omega=0$ and $q\ll k_F$, Halperin, Lee, and Read find

$$K_{00}=\frac{1}{v(q)+2\pi(1+\bar{\phi}^2/6)/m}, \tag{98}$$

which shows that just as in a Fermi liquid, the compressibility is nonzero if $v(q)$ is short ranged and vanishes as q if Coulombic. This vanishing only means that any applied external force is unable to change the density because the field inside the medium is strongly screened by the medium.

2. Cyclotron mode

The poles of K_{00} define the natural modes of oscillation of the system, for at these poles we get a response with no applied potential [see Eq. (90)]. If at any q there is a pole at some $\omega(q)$, it means that there is a mode of energy $\omega(q)$. For example, in a three-dimensional (3D) electron gas, a plasmon pole appears at the plasma frequency as $q\to0$ and then moves as a function of q. At larger q, when decay into particle-hole pairs is possible, the pole position acquires an imaginary part denoting a finite lifetime.

In the present case, as $q\to0$, at very high ω, K_{00} has a pole at the cyclotron frequency $\omega_0=4\pi n/m$ (where m is the bare or band mass) with a residue in accordance with Kohn's theorem.

3. The overdamped mode

At very low ω and q, one finds

$$[K_{00}]^{-1}=\frac{2\pi}{m}\left(1+\frac{\bar{\phi}^2}{12}\right)+v(q)-i\left(\frac{2\pi\bar{\phi}}{q}\right)^2\frac{2n\omega}{qk_F}. \tag{99}$$

[26]The extra e in front comes in because ρ and j refer to the particle density and not charge density.

The zero occurs at the *overdamped mode* with the dispersion relation

$$\omega \simeq iq^3 v(q) \simeq q^2 \tag{100}$$

for the Coulomb case. If $v(q)$ is short ranged, the mode is subdiffusive. The reason the charge will diffuse rather than move ballistically (even in the clean system) is that the magnetic field makes the charge move perpendicular to the Coulomb force, which tries to even out the density gradient. The overdamped mode and the effects it produces lie at the heart of the work of Halperin, Lee, and Read.

Note that if one first sent $m \to 0$, one would miss the overdamped mode. Of course in a real system we can always find a q such that $1/q$ dominates $1/m$. It is, however, a theoretical problem that we cannot send $m \to 0$ here. A way out will be discussed later.

4. Longitudinal conductivity

Upon examining Eq. (99), we can see that the Coulomb-irreducible part (obtained by dropping v) is dominated by the last term in the limit of small q and ω. Thus

$$\sigma_{xx} = e^2 \frac{\omega}{iq^2} K_{00}^v = \frac{e^2}{8\pi} \frac{q}{k_F}. \tag{101}$$

The above result holds only for $q \gg 1/l_m$, where l_m is the mean free path for the composite fermion. For $q \ll 1/l_m$,

$$\sigma_{xx} = \frac{e^2}{4\pi k_F l_m}. \tag{102}$$

It is interesting to see how one arrives at Eq. (102).

Let us define a composite-fermion conductivity by

$$e\mathbf{j} = \sigma_{CF} \mathbf{E}^T, \tag{103}$$

where $\mathbf{E}^T$ is the total field, which is the sum of the applied field and the internal field $\mathbf{e}$ generated by the fermions themselves. To compute $\mathbf{e}$, let us imagine a particle current $\mathbf{j}$ and a unit length perpendicular to it. In one second j particles cross it, carrying with them $2j$ flux quanta of Chern-Simons flux. However, the composite fermion makes no distinction between real flux and Chern-Simons flux, since the particles enter H_{CS} only via their sum. It will therefore sense a Chern-Simons electric field $\mathbf{e} = (4\pi/e)\hat{\mathbf{z}} \times \mathbf{j}$. Thus

$$e\mathbf{j} = \sigma_{CF} \left[\mathbf{E} + \frac{4\pi}{e} \hat{\mathbf{z}} \times \mathbf{j} \right]. \tag{104}$$

The electron resistivity tensor ρ (with components $\rho_{\alpha\beta}$ and not to be confused with the density operator) defined by

$$E_\alpha = \rho_{\alpha\beta} \ (ej_\beta) \tag{105}$$

is then

$$\rho = \rho_{CS} + \rho_{CF}, \tag{106}$$

where

$$\rho_{CS} = \frac{4\pi}{e^2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{107}$$

At mean-field level, composite fermions do not see a field of any kind. The effect of disorder is to produce

$$\sigma_{CF,xx} = \sigma_{CF,yy} = \frac{ne^2\tau}{m} = \frac{ne^2 l_m}{k_F} = \frac{k_F^2 e^2 l_m}{4\pi k_F} = \frac{e^2 k_F l_m}{4\pi}, \tag{108}$$

where τ is the elastic-scattering time and l_m is the mean free path. If we now use Eq. (106) and assume $k_F l_m \gg 1$ we obtain Eq. (102).

Note that the resistivity of the electron is the sum of the resistivities of the composite fermion and the Chern-Simons term. If we move off to a general Jain fraction and ignore disorder, we will have

$$\rho_{CF} = \frac{2\pi}{e^2} \begin{bmatrix} 0 & \frac{1}{p} \\ -\frac{1}{p} & 0 \end{bmatrix} \tag{109}$$

since the composite fermions fill p Landau levels of their own. When the resistivity matrices are added, one obtains the correct Hall resistivity for electrons at $\nu = p/(2p+1)$.

5. Surface acoustic waves

When a surface acoustic wave is coupled to the electronic system, it is predicted to undergo a velocity shift and an attenuation described by

$$\frac{\delta v_s}{v_s} - \frac{i\kappa}{q} = \frac{\alpha^2/2}{1 + i\sigma_{xx}(q)/\sigma_m}, \tag{110}$$

where α is a piezoelectric constant, v_s is the sound velocity, κ describes the attenuation, $\sigma_{xx}(q) = \sigma_{xx}(q, \omega = qv_s)$ where v_s is the sound velocity, and $\sigma_m = v_s \varepsilon / 2\pi$. Theory fits the experiments of Willett *et al.* (1990) and Willet and Pfeiffer (1996; for a review see Willett, 1997) with a σ_m that is about five times larger than expected. The reader is strongly urged to consider Halperin, Lee, and Read (1993) and the reviews for more details.

6. Mass divergences

Halperin, Lee, and Read (1993) predicted a divergence in the effective mass m^* at the Fermi surface arising from the fermion self-energy diagram involving the emission and absorption of the overdamped mode. For the Coulomb case one has

$$m^*(\omega) \simeq \ln \omega. \tag{111}$$

(Shorter-range interactions lead to more violent divergences.) Assuming that this mass can be used near $\nu = 1/2$, one expects that the gaps will be given by

$$E_\nu = \frac{eB^*}{m^*}, \tag{112}$$

where m^* is self-consistently defined by $m^*(\omega = E_\nu)$. For $\nu = p/(2p+1)$, this implies, as $p \to \infty$, that $E_p \simeq 1/(p \ln p)$. The log divergent effective mass will also imply a specific heat $C(T) \simeq T \ln T$. It has not been possible to confirm these logarithms in numerical or experimental work.

Since the nature of the mass divergence depends on the range of the potential, it cannot be reproduced by trial wave functions, which make no explicit reference to the potential, in particular, its range.

It has been shown that the mass divergences do not affect bosonic (e.g., density-density) correlations.[27]

B. Physics near $\nu = 1/2$

Halperin, Lee, and Read's (1993) predictions transcend $\nu = 1/2$ and describe its immediate vicinity. The key idea is that in this region, it is useful to think of the composite fermion as a particle seeing a weak effective magnetic field $B^* = B/(2p+1)$. This means that it will describe a cyclotron orbit of radius

$$R^* = \frac{\hbar k_F}{eB^*} \tag{113}$$

with $k_F = \sqrt{4\pi n}$ (given that spin is fully polarized). Note that this result is independent of the fermion mass, whose treatment is quite tricky.

The surface acoustic wave results of Willett *et al.* (1990, 1996) found that away from $\nu = 1/2$, there was a resonance in the velocity shift when the wavelength of the surface acoustic wave coincided with $2R^*$.

Confirmation of the R^* concept was also found in the experiments of Kang *et al.* (1993), Goldman, Su, and Jain (1994), and Smet *et al.* (1996). To visualize the Goldman *et al.* experiment, imagine a semi-infinite system in the upper half-plane. If a current is introduced at the origin up the y axis, it should bend and return to the x axis at $x = 2R^*$, after completing one semicircle. It would then bounce off and start the next semicircle. It follows that if a return path is provided on the x axis, the maximum current will flow if the drain is located an integral multiple of $2R^*$ from the source. If a return location is held fixed and B is varied, a maximum is expected, and found, whenever the spacing between source and drain is a multiple of $2R^*$.

Kang *et al.* built an array of antidots, in which each antidot is a region where electrons are absent. It was found that the sheet conductance had peaks when the antidot lattice constant equaled $2R^*$. While a detailed formula for conductance is not known in this context, the correlation is very suggestive and is what was seen with ordinary electrons in a weak field. The fact that composite fermions, which entered the theory as a mathematical device, manifest themselves so clearly in transport is a stunning affirmation of the theoretical framework.

[27]See Kim, Furusaki, *et al.* (1994); Kim, Lee, *et al.* (1994); Kim, Lee, and Wen (1995); Stern and Halperin (1995).

C. The theory of Halperin, Lee, and Read—room for improvement?

Despite its many remarkable and experimentally confirmed predictions, the theory of Halperin, Lee, and Read leaves room for improvement mainly because of the Chern-Simons approach on which it is based.

One such area has to do with the dependence on bare mass m. We have argued that at the starting point H_{CS} must contain the bare mass, since flux attachment by minimal coupling of the Chern-Simons field is an exact transformation done on electrons. This choice certainly helps to get the cyclotron frequency in accord with Kohn's theorem. However, in our extraction of various low-energy long-wavelength quantities, like the overdamped mode from Eq. (99), we had to assume that $1/q$ dominates over $1/m$. While this is certainly valid for realistic and fixed values of m, one would like to be able to extract what is evidently the correct physics even if $m = 0$ is imposed first.

Simon and Halperin (1993) and Simon, Stern, and Halperin (1996) proposed a way out. They assumed, as in Landau theory, that H_{CS} is an effective theory with an effective m^*. Kohn's theorem and the overdamped mode can both be salvaged if a suitable Landau parameter F_1 is introduced. This is reasonable since Kohn's theorem relies on Galilean invariance and Landau's theory uses this principle to relate m, m^*, and F_1. However, this leaves the origin of m^* inside a black box.[28] It is also not clear why in the effective theory the fermions and the Chern-Simons field should be minimally coupled.

In the case of Coulomb interactions, one can also argue that the bare mass is swamped by the renormalized $m^* \simeq \ln \omega$ generated by the exchange of the overdamped mode. Again it would be nice to be able to follow in detail the separation of the low-energy physics, controlled by an m^* generated by interactions, and high-energy physics controlled by m.

Another shortcoming is that there is no evidence of the neutral fermion one expects at $\nu = 1/2$, and more generally a fermion of charge $e^* = e/(2ps+1)$ at other Jain fractions. This was to be expected since the composite fermion in the work of Halperin *et al.* and in earlier Chern-Simons work of Lopez and Fradkin was an electron bound to two flux tubes that carried no charge.

Also missing was the effective magnetic moment $\mu^* = e/2m$ that Simon, Stern, and Halperin (1996) would later argue must go with each composite fermion.

In short, the composite fermion that appears in the Chern-Simons theory does not have, in obvious form, the right charge or energy scale of the ultimate quasiparticle. While the Chern-Simons transformation is exact and can yield all these in principle, they are not manifest in the RPA.

[28]Why do we view this inability to calculate m^* as a weakness, when in Landau theory it is simply accepted as a fact of life? As we shall see, in the FQHE we can go a long way towards computing m^* from the interactions by using the extended Hamiltonian theory.

Finally, Lee, Krotov, *et al.* (1997, 1998) have raised the following issue. Suppose we push all Landau levels up to infinity except for the lowest Landau level and assume particle-hole symmetric disorder at $\nu = 1/2$. Then it can be verified that a nonzero σ_{xx} for electrons implies for the Chern-Simons fermion $\sigma_{xy}^{CS} = -e^2/(4\pi)$. However, both the mean-field approximation and the RPA corrections to it give $\sigma_{xy}^{CS} = 0$. We are not aware of a resolution of this issue, arising from the fact that resistivities add.

V. HAMILTONIAN THEORY II—EXTENDED HAMILTONIAN THEORY

We now turn to our extension of the Chern-Simons formalism.[29] In this extension, a composite fermion with all the known properties can be made manifest and a variety of LLL quantities computed with no $1/m$ singularities. But the computation of certain low-energy long-wavelength quantities (like the compressibility), straightforward in the Chern-Simons approach, becomes extremely delicate. Hopefully we shall impart to the reader a sense of which approach to use for what purpose.

We want to eschew the historical route and furnish an axiomatic presentation of our work with a minimum of preamble, setting the stage for exact or approximate calculations as rapidly as possible. However, to establish the context, we shall begin with a brief sketch of our earlier work so as to take some of the mystery out of the end product and provide some degree of motivation.

Let us recall the work of Bohm and Pines (1953) on the electron gas in three dimensions. At small q the spectrum consists of particle-hole pairs at low energies and the plasmon at high energies. The particle and hole are part of the original Hamiltonian, while the plasmon comes from summing an infinite class of diagrams in the density-density response. It is not an independent entity, even though it is a sharply defined excitation that can be experimentally produced and detected. Bohm and Pines showed that, by introducing extra canonical oscillator degrees of freedom at small q, one could describe plasmons as independent objects in an enlarged Hilbert space. In order to prevent double counting, they imposed constraints on state vectors of the form

$$\bar{\bar{\chi}}(\mathbf{q})|\text{physical state}\rangle = 0, \tag{114}$$

with one $\bar{\bar{\chi}}(\mathbf{q})$ for each $\mathbf{q}$ at which a plasma mode was introduced as an independent canonical oscillator. The plasmons, initially coupled to the fermions, were approximately decoupled, leaving behind fermions with a renormalized mass and constraints that essentially froze out any putative plasmons with small q. Such a simple description of plasmons, or isolation of high-energy physics, would have been impossible within the confines of the electronic Hilbert space, wherein plasmons are complicated collective excitations of the electrons themselves.

What we originally did for the FQHE (Shankar and Murthy, 1997; Murthy and Shankar, 1998a) was similar in many ways. We enlarged the Hilbert space to include at each $\mathbf{q}$ a new set of independent, canonically conjugate variables—a transverse vector field $\mathbf{a}(\mathbf{q})$ and a longitudinal vector field $\mathbf{P}(\mathbf{q})$. Using these it was possible, by a unitary transformation, to get rid of the dependent Chern-Simons vector potential $\mathbf{a}_{CS}$. While all the operators in our Hamiltonian were now independent, the physical sector was defined by the (Chern-Simons) condition

$$(\nabla \times \mathbf{a} - 4\pi s\rho)|\text{physical state}\rangle = 0. \tag{115}$$

The conjugate variables $\mathbf{a}$ and $\mathbf{P}$ formed oscillators near, but not exactly at, the cyclotron frequency. They were coupled to fermions. We found a way to decouple the oscillators in the limit $ql \to 0$ by a second unitary transformation. The decoupled oscillators now complied with Kohn's theorem for pole position and residue. The formula for $\bar{\bar{\rho}}(\mathbf{q})$, the electronic charge density projected to the lowest Landau level, was derived to order ql, as was the constraint $\bar{\bar{\chi}}(\mathbf{q})$, which now acted only within the particle sector. At this point a guess was made (Shankar, 1999) to extend the small ql answer to all orders by exponentiation of leading-order terms. The final theory of the LLL sector took the following form:

$$\bar{\bar{H}} = \frac{1}{2}\sum_{\mathbf{q}} \bar{\bar{\rho}}(\mathbf{q}) v(q) e^{-q^2 l^2/2} \bar{\bar{\rho}}(-\mathbf{q}), \tag{116}$$

$$[\bar{\bar{H}}, \bar{\bar{\chi}}(\mathbf{q})] = 0, \tag{117}$$

$$0 = \bar{\bar{\chi}}(\mathbf{q})|\text{physical state}\rangle. \tag{118}$$

The Hamiltonian $\bar{\bar{H}}$ is just the potential energy V projected to the lowest Landau level, written in terms of the projected electron charge density $\bar{\bar{\rho}}(\mathbf{q})$,

$$\bar{\bar{\rho}}(\mathbf{q}) = \sum_j \exp(-i\mathbf{q}\cdot\mathbf{R}_{ej}), \tag{119}$$

$$\mathbf{R}_e = \mathbf{r} - \frac{l^2}{1+c}\hat{\mathbf{z}} \times \mathbf{\Pi}, \tag{120}$$

where we define a very important and frequently occurring variable:

$$c^2 = \frac{2ps}{2ps+1}. \tag{121}$$

From the commutation relations

$$[\mathbf{R}_{ex}, \mathbf{R}_{ey}] = -il^2 \tag{122}$$

it is clear that $\mathbf{R}_e$ is just the electron guiding-center coordinate but now expressed in terms of composite-fermion variables $\mathbf{r}$ and $\mathbf{\Pi} = \mathbf{p} + e\mathbf{A}^*$.

The physical states are annihilated by the constraint

$$\bar{\bar{\chi}}(\mathbf{q}) = \sum_j \exp(-i\mathbf{q}\cdot\mathbf{R}_{vj}), \tag{123}$$

[29] Shankar and Murthy (1997); Murthy, Park, Shankar, and Jain (1998); Murthy and Shankar (1998a, 1998b, 1999, 2002); Murthy (1999, 2000a, 2000c, 2001b, 2001c); Shankar (1999, 2000, 2001).

$$\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1+c)}\hat{\mathbf{z}}\times\mathbf{\Pi}. \tag{124}$$

The *pseudovortex coordinate* $\mathbf{R}_v$ describes an object of charge $-c^2 = -2ps/(2ps+1)$:

$$[\mathbf{R}_{vx}, \mathbf{R}_{vy}] = \frac{il^2}{c^2} \tag{125}$$

and commutes with $\mathbf{R}_e$:

$$[\mathbf{R}_e, \mathbf{R}_v] = 0. \tag{126}$$

Thus the constraints commute with $\bar{\bar{H}}$ and form an algebra:

$$[\bar{\bar{\chi}}(\mathbf{q}), \bar{\bar{\chi}}(\mathbf{q}')] = -2i \sin\left[\frac{l^2(\mathbf{q}\times\mathbf{q}')}{2c^2}\right]\bar{\bar{\chi}}(\mathbf{q}+\mathbf{q}'). \tag{127}$$

The problem is just like Yang-Mills theory.

The expressions for $\mathbf{R}_e$ and $\mathbf{R}_v$ in terms of $\mathbf{r}$ and $\mathbf{\Pi}$, which were already encoded in the small-q theory, jumped out upon exponentiation. They, together with the constraints, lie at the heart of our approach.

We shall now show how one can get to the final result, Eqs. (116)–(118), simply by making a certain enlargement of the electronic Hilbert space and following it with a change of variables. Any reader who wants to know more about choice of these variables should consult our earlier work. Those who just want to use the results can go ahead.

A. The axiomatic introduction to the extended Hamiltonian theory

Let us begin afresh with the primordial Hamiltonian in terms of electronic variables (which carry the subscript e to distinguish them from other coordinates to be introduced):

$$H = \sum_j \frac{\boldsymbol{\eta}_{ei}^2}{2ml^4} + \frac{1}{2}\sum_{i,j,\mathbf{q}} v(q) e^{i\mathbf{q}\cdot(\mathbf{r}_{ei}-\mathbf{r}_{ej})} \equiv H_0 + V. \tag{128}$$

This Hamiltonian contains complete information about the problem. It can be used to study Landau-level mixing and the computation of Hall current, which requires higher Landau levels in an essential way. These topics will be discussed in due course. We first focus on the main challenge of Hamiltonian theories: extracting the m-independent physics of the lowest Landau level. As discussed in Sec. II, projecting to the lowest Landau level as such is no problem: one drops the first term and makes the replacement $\mathbf{r}\to\mathbf{R}$ in the density operator and $v(q)\to v(q)e^{-q^2l^2/2}$. The catch is that the projected Hamiltonian lives in the highly degenerate lowest Landau level, frustrating both perturbation theory and the Hartree-Fock approximation.

In the Chern-Simons approach one resorts to flux attachment to beat the degeneracy of the kinetic term, but that can be done only in the full electronic Hilbert space. Consequently m gets into everything and the low-energy (LLL) and high-energy sectors get hopelessly entangled. What we really want to do is work within the lowest Landau level and attach flux tubes, which in the lowest Landau level translates into vortices, by analyticity. However, when we say vortices, we do not mean zeros of the wave function, for such a thing does not exist as a degree of freedom within the Hamiltonian and, as we have seen, there are not enough of them in the wave function to go around anyway. Instead we mean by a vortex some object that has the charge of the $2s$-fold vortex and corresponds to an excitation that can be created by inserting $2s$ flux quanta into the Hall system. Since such an object does not exist in the original Hilbert space (as an independent entity), we enlarge it to make room for this entity, which we call the *pseudovortex*, the *vortex* to emphasize its similarity to the vortices in the wave functions and the *pseudo* to emphasize its differences.

The enlargement of Hilbert space can be explained in terms of just one electron, with cyclotron and guiding-center coordinates $\boldsymbol{\eta}_e$ and $\mathbf{R}_e$. Let us temporarily focus on just the LLL physics and ignore $\boldsymbol{\eta}_e$, which does not participate in the change of variables and will be reinstated subsequently.

First we introduce an extra guiding-center coordinate $\mathbf{R}_v$ (the pseudovortex), defining it algebraically by its commutation relations, which represent a charge $-c^2$,

$$[R_{vx}, R_{vy}] = \frac{il^2}{c^2}. \tag{129}$$

Next we combine $\mathbf{R}_e$ and $\mathbf{R}_v$ to form the composite-fermion space. (Note that it takes two canonical pairs to make one regular fermion in two dimensions.) This composite-fermion space can be defined in terms of either the position $\mathbf{r}$ and velocity $\mathbf{\Pi}$ of the composite fermion, or in terms of its cyclotron and guiding-center coordinates, $\boldsymbol{\eta}$ and $\mathbf{R}$. *Note that henceforth variables carrying no identifying subscripts will refer to the composite fermion*. The composite-fermion coordinates $\boldsymbol{\eta}$ and $\mathbf{R}$ obey the commutation rules of the cyclotron and guiding-center coordinates of an object of charge $e^* = 1-c^2 = 1/(2ps+1)$:

$$[\eta_x, \eta_y] = il^{*2} = \frac{il^2}{1-c^2}, \tag{130}$$

$$[R_x, R_y] = -il^{*2}. \tag{131}$$

The rule for forming the composite fermion from $\mathbf{R}_e$ and $\mathbf{R}_v$ is the following:

$$\mathbf{R} = \frac{\mathbf{R}_e - c^2\mathbf{R}_v}{1-c^2}, \tag{132}$$

$$\boldsymbol{\eta} = \frac{c}{1-c^2}(\mathbf{R}_v - \mathbf{R}_e). \tag{133}$$

The first equation says that the composite-fermion guiding center is the weighted sum of its parts. The second can be found by demanding that $\boldsymbol{\eta}$ be linear in $\mathbf{R}_e$ and $\mathbf{R}_v$, commute with $\mathbf{R}$, and have an overall scale that produces the right commutator.

The inverse transformation is

$$\mathbf{R}_e = \mathbf{R} + \boldsymbol{\eta} c, \tag{134}$$

$$\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c. \tag{135}$$

In terms of $\mathbf{r}$ and $\boldsymbol{\Pi}$, the composite-fermion coordinate and velocity operators are

$$\mathbf{R}_e = \mathbf{r} - \frac{l^2}{(1+c)}\hat{\mathbf{z}}\times\boldsymbol{\Pi}, \tag{136}$$

$$\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1+c)}\hat{\mathbf{z}}\times\boldsymbol{\Pi}, \tag{137}$$

which are just the expressions encountered in the brief historical review.

Ignoring the zero-point energy, here is where we stand in the LLL sector:

$$\bar{\bar{H}} = \frac{1}{2}\sum_{i,j,\mathbf{q}} v(q) e^{-q^2 l^2/2} e^{i\mathbf{q}\cdot(\mathbf{R}_{ei} - \mathbf{R}_{ej})} \tag{138}$$

$$= \frac{1}{2}\sum_{i,j,\mathbf{q}} v(q) e^{-q^2 l^2/2} \times \exp\{i\mathbf{q}\cdot[(\mathbf{R}_i - \mathbf{R}_j) + c(\boldsymbol{\eta}_i - \boldsymbol{\eta}_j)]\}. \tag{139}$$

While it is true that we have managed to get rid of m and cleanly isolate the lowest Landau level, the reader may ask what we have gained, since algebraically the problem is the same as in electronic coordinates. The answer is that now there is a natural nondegenerate Hartree-Fock ground state in the extended space. This is because the Hartree-Fock Hamiltonian is now written in terms of composite-fermion operators $\mathbf{R}$ and $\boldsymbol{\eta}$, and the particle density is just right to fill exactly p filled composite-fermion Landau levels, i.e., χ_p is the ground state. The proof of its Hartree-Fock nature is found in Appendix C. This key step opens up all the usual approximation schemes.

Depending on what we want to compute, there are two distinct schemes. Both rely on the nondegenerate Hartree-Fock ground state, and both acknowledge a huge symmetry group of $\bar{\bar{H}}$, which comes from the following fact: $\bar{\bar{H}}$, as embedded in the composite-fermion space, does not depend on $\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c$, the pseudovortex coordinate. Equivalently, it depends on $\mathbf{R}$ and $\boldsymbol{\eta}$ only through the combination $\mathbf{R}_e = \mathbf{R} + c\,\boldsymbol{\eta}$. The other (commuting) combination $\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c$ can be used to define a *pseudovortex density*

$$\bar{\bar{\chi}}(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_v}, \tag{140}$$

which commutes with $\bar{\bar{H}}$:

$$[\bar{\bar{H}}, \bar{\bar{\chi}}(\mathbf{q})] = 0. \tag{141}$$

Let us understand this symmetry. If we view $\bar{\bar{H}}$ as a function of $\boldsymbol{\eta}$ and $\mathbf{R}$, or $\mathbf{r}$ and $\boldsymbol{\Pi}$, its eigenfunctions will depend on two coordinates, which can be chosen to be one component each of $\boldsymbol{\eta}$ and $\mathbf{R}$ or just x and y. If we view H as a function of $\mathbf{R}_e$ and $\mathbf{R}_v$, it depends only on $\mathbf{R}_e$. The energy eigenfunctions will be of the form

$$\psi(z_e, z_v) = \psi_e(z_e)\psi_v(z_v), \tag{142}$$

where $\psi_v(z_v)$ is arbitrary, since nothing in $\bar{\bar{H}}$ determines it. This degeneracy is of the same type as that of the noninteracting electron Hamiltonian, which depends on $\boldsymbol{\eta}_e$ but not $\mathbf{R}_e$. However, since $\mathbf{R}_v$ is unphysical, and all physical observables depend only on $\mathbf{R}_e$, this is a gauge symmetry.

We deal with the gauge symmetry as usual, by selecting a representative from each orbit, which in this case means eliminating the degeneracy due to the arbitrary function $\psi(z_v)$ that tags along for the ride. Let us make a particular choice $\psi_{v0}(z_v)$. All we require is that it be translationally invariant so that $\langle\bar{\bar{\chi}}(\mathbf{q})\rangle = 0$ in this state. The gauge-fixed Hilbert space now consists of functions of the form $\psi_e(z_e)\psi_{v0}(z_v)$. In this sector $\bar{\bar{\chi}}(\mathbf{q}) = 0$ in the weak sense: any Green's function involving a string of $\bar{\bar{\chi}}(\mathbf{q})$'s will vanish since (i) $\bar{\bar{\chi}}(\mathbf{q},t) = \bar{\bar{\chi}}(\mathbf{q},0) \equiv \bar{\bar{\chi}}(\mathbf{q})$, (ii) $\langle\bar{\bar{\chi}}(\mathbf{q})\rangle = 0$,[30] in the one-dimensional space spanned by $\psi_{v0}(z_v)$. [Imagine inserting the projector to this state between any two $\bar{\bar{\chi}}(\mathbf{q})$'s in the Green's function.]

In the path-integral language we can make $\bar{\bar{\chi}}(\mathbf{q})$ *vanish weakly*, that is to say, vanish whenever it appears in a Green's function, by a method similar to what is done in gauge theories. Imagine writing a path integral in the full composite-fermion space and then inserting a delta function imposing $\bar{\bar{\chi}}(\mathbf{q}) = 0$ for all $\mathbf{q}$ at any one time.[31] Since $\bar{\bar{\chi}}(\mathbf{q})$ does not change with time, this restriction will hold automatically for all times and specify the fate of $\bar{\bar{\chi}}(\mathbf{q})$ the way a gauge-fixing term does for the longitudinal degrees of freedom.

The theory is thus defined (in schematic form) by the equations

$$\bar{\bar{H}} = \frac{1}{2}\sum_{\mathbf{q}} v(q) e^{-q^2 l^2/2} \bar{\bar{\rho}}(\mathbf{q})\bar{\bar{\rho}}(-\mathbf{q}), \tag{143}$$

$$[\bar{\bar{H}}, \bar{\bar{\chi}}(\mathbf{q})] = 0, \tag{144}$$

$$\bar{\bar{\chi}}(\mathbf{q}) \simeq 0, \tag{145}$$

where $\simeq 0$ means vanish weakly.

Now we turn to the two approximate ways of dealing with this problem: the conserving approximation and the shortcut.

B. The conserving approximation

Given a Hamiltonian, a reasonable first approximation to try is Hartree-Fock. It is shown in Appendix C that composite-fermion Landau-level states are Hartree-Fock states of $\bar{\bar{H}}$. How good is this Hartree-Fock approximation likely to be? For example, how will we fare if we compute the activation or *transport gap* in a fully polarized sample as

$$\Delta = \langle \mathbf{p} + PH | H | \mathbf{p} + PH\rangle - \langle \mathbf{p} | H | \mathbf{p}\rangle, \tag{146}$$

[30] An exception occurs if the string contains only $\bar{\bar{\chi}}(0) = N$. This does not affect what we plan to do.

[31] Since $\bar{\bar{\chi}}(\mathbf{q})$ involves $\mathbf{r}$ and $\mathbf{p}$, this will have to be done in the phase-space path integral.

where $|\mathbf{p}\rangle$ stands for the Hartree-Fock ground state with p-filled Landau levels and PH stands for a widely separated particle-hole pair?

There are at least two good reasons to expect that this naive Hartree-Fock result will require fairly strong corrections. First, if we compute the matrix element of the projected electron density between any two Hartree-Fock states, the answer will be linear in q, whereas in the exact theory, and within the lowest Landau level, it must go as q^2 as per Kohn's theorem. To see this note that

$$e^{-i\mathbf{q}\cdot\mathbf{R}_e}=1-i\mathbf{q}\cdot(\mathbf{R}+c\boldsymbol{\eta})+\mathcal{O}(q^2). \tag{147}$$

While $\mathbf{R}$ has no transition matrix elements between different composite-fermion Landau levels, $\boldsymbol{\eta}$ does. The second problem is that as $ql\to 0$, the projected electronic density [which reduces to $\Sigma_j \exp(-i\mathbf{q}\cdot\mathbf{r}_j)$] has a unit contribution from each composite fermion while we would like it to be e^*. Evidently the Hartree-Fock result will receive strong corrections that will renormalize these quantities until they are in line with these expectations.

These shortcomings are to be expected since the Hartree-Fock solution does not obey the constraint, or equivalently, does not factorize into the form $\psi=\psi_e(z_e)\psi_v(z_v)$.

The conserving approximation (Anderson, 1958a, 1958b; Rickayzen, 1958; Nambu, 1960; Baym and Kadanoff, 1961) is a sophisticated procedure for improving the Hartree-Fock state with additional diagrammatic corrections so that $\bar{\bar{\chi}}(\mathbf{q})\simeq 0$ in Green's functions. For $\nu=1/2$, Read (1998) showed that this procedure restores Kohn's theorem and reveals a dipolar structure for density-density correlations. We shall say more about this in connection with the compressibility paradox.

Now we shall discuss the other approximation, the shortcut.

C. The shortcut: The preferred charge and Hamiltonian

Consider the exact solution to the gauge-fixed problem. Suppose, in the Hamiltonian and elsewhere, we replace $\bar{\rho}(\mathbf{q})$ by the *preferred combination*

$$\bar{\rho}^p(\mathbf{q})=\bar{\rho}-c^2\bar{\bar{\chi}}. \tag{148}$$

This makes no difference (to the computation of anything physical) in an exact calculation, since $\bar{\bar{\chi}}$ is essentially zero.

However, in the Hartree-Fock approximation it makes a big difference if we start with the Hamiltonian written in terms of $\bar{\rho}^p(\mathbf{q})$. To see why, consider its expansion in powers of ql:

$$\bar{\bar{\rho}}^p=\sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}\left(\frac{1}{2ps+1}-il^2\mathbf{q}\times\boldsymbol{\Pi}_j+0\cdot(\mathbf{q}\times\boldsymbol{\Pi}_j)^2+\cdots\right). \tag{149}$$

- The transition matrix elements are of order q^2 between Hartree-Fock states because the coefficient of $\mathbf{q}$ is proportional to the composite-fermion guiding-center coordinate $\mathbf{r}-l^{*2}\hat{\mathbf{z}}\times\boldsymbol{\Pi}$ with no admixture of the composite-fermion cyclotron coordinate. This is more transparent if we use $\mathbf{R}$ and $\boldsymbol{\eta}$ to write

$$\bar{\rho}^p(\mathbf{q})=[1-i\mathbf{q}\cdot(\mathbf{R}+c\boldsymbol{\eta})+\cdots]-c^2[1-i\mathbf{q}\cdot(\mathbf{R}+\boldsymbol{\eta}/c)+\cdots] \tag{150}$$

$$=(1-c^2)(1-\mathbf{q}\cdot\mathbf{R})+\mathcal{O}(q^2). \tag{151}$$

- With no further fixing, we see that the electronic charge density associated with $\bar{\rho}^p(\mathbf{q})$ is now $1-c^2=e^*$.
- We see from Eq. (149) that when $\nu=1/2$, the preferred density couples to an external electric field like a dipole of size $l^2\hat{\mathbf{z}}\times\mathbf{p}$.

Thus there is no *a priori* need for strong corrections, at least in the long-wavelength limit.

The Hamiltonian we work with,

$$\bar{\bar{H}}^p=\frac{1}{2}\sum_{\mathbf{q}}\bar{\rho}^p(\mathbf{q})v(q)e^{-q^2l^2/2}\bar{\rho}^p(-\mathbf{q}), \tag{152}$$

subsumes a lot of the right low-energy physics and composite-fermion properties. Unlike $\bar{\bar{H}}$, which commutes with $\bar{\bar{\chi}}(\mathbf{q})$, $\bar{\bar{H}}^p$ is weakly gauge invariant, that is,

$$[H(\bar{\rho}^p),\bar{\bar{\chi}}(\mathbf{q})]\simeq 0, \tag{153}$$

where the $\simeq 0$ symbol means that it vanishes in the subspace obeying $\bar{\bar{\chi}}(\mathbf{q})=0$. Thus neither $H(\bar{\rho}^p)$ nor $\bar{\rho}^p$ will mix physical and unphysical states.

The significance of $H(\bar{\rho}^p)$ is the following. If the constraint $\bar{\bar{\chi}}=0$ is imposed exactly, there are many equivalent Hamiltonians depending on how $\bar{\bar{\chi}}$ is insinuated into it. However, in the Hartree-Fock approximation, these are not equivalent and $H(\bar{\rho}^p)$ best approximates, between Hartree-Fock states and at long wavelengths, the true Hamiltonian between true eigenstates. In contrast to a variational calculation in which one searches among states for an optimal one, here the Hartree-Fock states are the same for a class of Hamiltonians (where $\bar{\bar{\chi}}$ is introduced into H in any rotationally invariant form), and we seek the best Hamiltonian: $H(\bar{\rho}^p)$ encodes the fact that every electron is accompanied by a correlation hole of some sort, which leads to a certain e^*, d^* and obeys Kohn's theorem (q^2 matrix element for the LLL projected charge density).

Note that when we use the preferred charge and Hamiltonian we shall make no further reference to constraints and simply carry out the Hartree-Fock approximation. This is based on the expectation that, even if we found some way to include the effect of constraints, it would make no difference in the small ql region. This is because the leading renormalization of e to e^* and suppression of q matrix elements down to q^2 that are achieved by the conserving approximation by summing ladder diagrams following Read (1998) or using the time-dependent Hartree-Fock (Murthy, 2001a, Sec. VI) are built in here.

It is in this approximate, operator sense, where we use $\bar{\bar{\rho}}^p$ in place of $\bar{\bar{\rho}}$, that the binding of electrons and pseudovortices to form composite fermions is realized in the Hamiltonian theory. Since the pseudovortices have coordinates that are independent of the electrons, there is no double counting here. The problems that we encountered with vortices in the wave-function approach for non-Laughlin fractions (with the limiting case being the dipole picture of $\nu=1/2$) are also absent, since we are not talking about those vortices. Since the pseudovortices per electron are independent of electrons, their number does not change when ν changes (though their charge, tied to $c^2=2s\nu$, does). Whereas antisymmetrization fragmented the vortices in the Jastrow factor into ordinary parametric zeros (except for the Pauli zero), it does nothing to the pseudovortices. Antisymmetrization is accomplished by simply writing the operator $\bar{\bar{\rho}}(\mathbf{q})$ in terms of second-quantized (composite) fermion operators.

The reader will recall that any simple picture of quasiparticles, whether it be in Landau's Fermi-liquid theory or in BCS theory, is best captured by approximate and not exact descriptions. The quasiparticles are all caricatures of some exact reality and therein lies their utility. Similarly the composite fermion in our extended formalism appears only in the Hartree-Fock approximation to $\bar{\bar{H}}^p$. Recall that we brought in the coordinate $\mathbf{R}_v$ to become the electron's partner in forming the composite fermion. However, $\mathbf{R}_v$ was cyclic in the exact Hamiltonian $\bar{\bar{H}}$. Thus the exact dynamics never demanded that $\mathbf{R}_v$ be bound to $\mathbf{R}_e$ or even be anywhere near $\mathbf{R}_e$. However, in the Hartree-Fock approximation, since we wanted the right charge and transition matrix elements of the density operator (Kohn's theorem) to be manifest, we needed to replace $\bar{\bar{\rho}}(\mathbf{q})$ by $\bar{\bar{\rho}}^p(\mathbf{q})$, and trade $\bar{\bar{H}}$ for $\bar{\bar{H}}^p$, the preferred Hamiltonian. In $\bar{\bar{H}}^p$, $\mathbf{R}_v$ is coupled to $\mathbf{R}_e$. The Hartree-Fock approximation and this coupling go hand in hand. The exact eigenfunctions of the original $\bar{\bar{H}}$ are factorized in the analytic coordinates z_e and z_v and presumably reproduce the electronic correlations of the FQHE states. On the other hand, in the Hartree-Fock approximation to $\bar{\bar{H}}^p$, the wave functions (e.g., p-filled Landau levels) mix up z_e and z_v, and $\bar{\bar{H}}^p$, the preferred Hamiltonian, dynamically couples $\mathbf{R}_e$ and $\mathbf{R}_v$. The net result is that, at least at long wavelengths, these two wrongs make it right and mimic what happens in the exact solution.

Another advantage of $\bar{\bar{H}}^p$ is that it gives an approximate formula for m^* originating entirely from interactions. This is best seen at $\nu=1/2$. When we square $\bar{\bar{\rho}}^p$ [Eq. (149)], we get a double sum over particles whose diagonal part is the one particle (free-field) term:

$$H^0_{\nu=1/2}=2\sum_j \int \frac{d^2q}{4\pi^2}\sin^2\left[\frac{\mathbf{q}\times\mathbf{k}_j l^2}{2}\right]v(q)e^{-q^2l^2/2}. \tag{154}$$

This is not a Hamiltonian of the form $k^2/2m^*$. However, if the potential is peaked at very small q, we can expand the sine and read off an approximate $1/m^*$,

TABLE I. Basic equations.

CS Theory

$$H_{CS}=\frac{1}{2m}\sum_i (\mathbf{p}+e\mathbf{A}+\mathbf{a}_{cs})_i^2+V$$

$$\nabla\times\mathbf{a}_{cs}=4\pi s\rho$$

Extended Hamiltonian theory (lowest Landau level only)

$$\bar{\bar{H}}=\frac{1}{2}\sum_{\mathbf{q}} v(q)e^{-q^2l^2/2}\bar{\bar{\rho}}(\mathbf{q})\bar{\bar{\rho}}(-\mathbf{q})$$

$$[\bar{\bar{H}},\bar{\bar{\chi}}(\mathbf{q})]=0$$

$$\bar{\bar{\chi}}(\mathbf{q})\simeq 0$$

$$\bar{\bar{\rho}}(\mathbf{q})=\sum_j \exp(-i\mathbf{q}\cdot\mathbf{R}_{ej})$$

$$\mathbf{R}_e=\mathbf{r}-\frac{l^2}{1+c}\hat{\mathbf{z}}\times\mathbf{\Pi}=\mathbf{R}+\boldsymbol{\eta}c$$

$$\bar{\bar{\chi}}(\mathbf{q})=\sum_j \exp(-i\mathbf{q}\cdot\mathbf{R}_{vj})$$

$$\mathbf{R}_v=\mathbf{r}+\frac{l^2}{c(1+c)}\hat{\mathbf{z}}\times\mathbf{\Pi}=\mathbf{R}+\boldsymbol{\eta}/c$$

$$\bar{\bar{H}}^p=\bar{\bar{H}}(\bar{\bar{\rho}}^p)$$

$$\bar{\bar{\rho}}^p(\mathbf{q})=\bar{\bar{\rho}}(\mathbf{q})-c^2\bar{\bar{\chi}}(\mathbf{q})$$

$$\frac{1}{m^*}=\int \frac{q\,dq\,d\theta}{4\pi^2}[(\sin^2\theta)(ql)^2]v(q)e^{-q^2l^2/2}, \tag{155}$$

which has its origin in electron-electron interactions. Indeed, we can do more: we have the full H_0 as well as the interactions. The point to emphasize is that H is not of the traditional form $(p^2/2m+V)$ and there is no reason it has to be.

The reader should verify that if we use $H(\bar{\bar{\rho}})$ instead, the one-particle piece will be a constant with no momentum dependence. The entire energy will be due to the Fock term, as in Read's (1998) conserving calculation.

For the benefit of readers who may be overwhelmed by seeing too many approaches, we present the key equations of the Chern-Simons and extended Hamiltonian approaches in Table I.

To summarize, in any context where $\bar{\bar{\rho}}^p$ can be reliably employed, we can say that the composite fermion appears to be the bound state of an electron and the pseudovortex. We shall see that this includes calculations of the gap and magnetization properties at zero and nonzero temperatures, even at $\nu=1/2$. We shall turn to these later on. But first, we list some cases in which $\bar{\bar{\rho}}(\mathbf{q})^p$ and $\bar{\bar{H}}^p$ fail to capture the right physics.

D. The conserving approximation and compressibility paradox

Halperin, Lee, and Read (1993) predicted that the $\nu=1/2$ system has a static compressibility that is nonzero for short-range forces and vanishes as q for the Coulomb interaction. This result appears paradoxical in the

present approach in which the dipolar nature of the composite fermion has been transparently exposed (using $\bar{\bar{\rho}}^p$). Imagine coupling the system to an external potential Φ. The dipole will couple to the *gradient* of Φ and the resulting response will be a dipolar density whose *divergence* will give the induced charge. These two italicized factors imply a q^2 in the response even for short-range interactions. Indeed, this is what we first obtained (Shankar and Murthy, 1997; Murthy and Shankar, 1998a) upon doing a simple RPA calculation. How is this to be reconciled with the Halperin-Lee-Read (1993) result, assuming that their (compressible) answer is right?

Halperin and Stern (1998) first raised this question and provided the key to its resolution. They first established a matter of principle, namely, that dipolar objects could be compressible, by considering the following Hamiltonian:

$$H=\sum_i \frac{\mathbf{p}_i^2}{2m}-\frac{1}{2mn}\sum_{i,j,\mathbf{q}}^{n,n,Q}\mathbf{p}_i\cdot\mathbf{p}_j e^{i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j)}. \tag{156}$$

This Hamiltonian arose in our earlier work (Murthy and Shankar, 1998a) when we decoupled the magnetoplasmon oscillators from the fermions. There we had chosen the upper cutoff $Q=k_F$ so that the $i=j$ term from the double sum contributes $-\mathbf{p}^2/2m$ to each particle and cancels the first sum, rendering $1/m^*=0$. Halperin and Stern considered the limiting case $\mathbf{Q}\to 0$. In this limit H takes the form

$$H=\sum_{ij}\frac{(\mathbf{p}_i-\mathbf{p}_j)^2}{2m}=\sum_i\frac{\left(\mathbf{p}_i-\frac{1}{n}\sum_j\mathbf{p}_j\right)^2}{2m} \tag{157}$$

and is invariant under the simultaneous shift of all momenta. This symmetry, called *K invariance*, had also been pointed out by Haldane (1995) in unpublished work and arose as part of a gauge symmetry in our work. The symmetry implies that it costs no energy to move the Fermi surface as a whole. Consequently there are some very soft modes that could lead to a singular density response which can offset the q^2 from the dipolar factors, provided these soft modes are not merely gauge artifacts that couple to nothing physical. The detailed calculation of Stern *et al.* (1999), which we now describe, demonstrated that gauge-invariant soft modes do exist and lead to nonzero compressibility.

The first order of business was to start with a Hamiltonian that had K invariance for small Q and not just $Q=0$, since one needed to consider the response functions at small but nonzero q. The Hamiltonian in Eq. (157) had to be augmented by more terms in order to assure this. These terms were derived by Stern *et al.* (1999) as follows. They go a step back and start with the following Hamiltonian and constraints for the coupled oscillators and particles:

$$H=\sum_j\frac{(\mathbf{p}_j-\mathbf{a})^2}{2m}+V \tag{158}$$

$$=\sum_j\frac{\mathbf{p}_j^2}{2m}-\sum_{\mathbf{q}}^{Q}\mathbf{g}(\mathbf{q})\frac{1}{2mn_{CS}}\mathbf{a}(-\mathbf{q})+\int d^2r[n_{CS}(\mathbf{r})|\mathbf{a}(\mathbf{r})|^2]+V, \tag{159}$$

$$\mathbf{g}(\mathbf{q})=\sum_j\mathbf{p}_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}, \tag{160}$$

$$0=(\nabla\times\mathbf{a}-4\pi n_{CS})|\text{physical state}\rangle, \tag{161}$$

where n_{CS} is the fermion number-density operator. The only minor difference from our work is that a_x and a_y (instead of the longitudinal and transverse components a and P) are canonically conjugate. The cutoff on q at Q means that the Chern-Simons fermions now carry flux tubes, smeared over a distance $1/Q$ (sometimes called "fat" flux tubes; see Halperin, 1992).

Next Stern *et al.* (1999) approximate n_{CS} by n, the average density, and obtain

$$H=\sum_j\frac{\mathbf{p}_j^2}{2m}-\frac{1}{2mn}\sum_{\mathbf{q}}\mathbf{g}(\mathbf{q})\mathbf{g}(-\mathbf{q})+\sum_{\mathbf{q}}\frac{n}{2m}\left|\mathbf{a}(\mathbf{q})-\frac{1}{n}\mathbf{g}(\mathbf{q})\right|^2. \tag{162}$$

Now they invoke the unitary transformation we employed to decouple the **a** fields from the fermions in the small-ql limit:

$$U=\exp\left[\frac{i}{4\pi n}\sum_{\mathbf{q}}^{\mathbf{Q}}\mathbf{g}(\mathbf{q})\times\mathbf{a}(-\mathbf{q})\right]. \tag{163}$$

Since a_x and a_y are conjugates, U is just the shift operator $\mathbf{a}-\mathbf{g}/n\to\mathbf{a}'$. Here comes the big difference. While we kept just the H from Eq. (157), they augment H with additional terms to ensure that the constraints

$$\rho(\mathbf{q})=-\frac{il^2}{2}\mathbf{q}\times\mathbf{g}(\mathbf{q}) \tag{164}$$

commute with H_{aug} to the desired order. The reader should verify that this constraint is just

$$\bar{\bar{\chi}}(\mathbf{q})=\sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_{vj}}=\sum_j\exp\left[-i\mathbf{q}\cdot\left(\mathbf{r}_j+\frac{l^2}{2}\hat{\mathbf{z}}\times\mathbf{p}_j\right)\right]=0 \tag{165}$$

expanded to order ql.[32] To this order the electron density is (dropping **a** terms, which do not matter at low energies)

$$\rho^e(\mathbf{q})=\sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_{ej}}=\sum_j\exp\left[-i\mathbf{q}\cdot\left(\mathbf{r}_j-\frac{l^2}{2}\hat{\mathbf{z}}\times\mathbf{p}_j\right)\right] \tag{166}$$

[32]Since the Chern-Simons constraint commutes with H prior to the action of U, it does so after the action of U, but of course only to leading order in ql, since U was not implemented exactly.

$$\simeq \sum_j \exp(-i\mathbf{q}\cdot\mathbf{r}_j)\left(1-\frac{il^2}{2}\mathbf{q}\times\mathbf{p}_j\right) \tag{167}$$

$$=\rho(\mathbf{q})-\frac{il^2}{2}\mathbf{q}\times\mathbf{g}(\mathbf{q}). \tag{168}$$

Recall that the constraint generates gauge transformations. To zeroth order the constraint operator is just $\Sigma_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}=0$ and its action is

$$\mathbf{r}_j\to\mathbf{r}_j, \quad \mathbf{p}_j\to\mathbf{p}_j+\mathbf{q}e^{-iqr_j}. \tag{169}$$

Thus respecting the constraint ensures K invariance.

Stern *et al.* (1999) now perform an RPA calculation of the electronic density-density correlation. The random-phase approximation works because, at a given $\mathbf{q}$, the gauge field (of that $\mathbf{q}$) enters only in the wiggles connecting irreducible bubbles in Fig. 3 and nowhere inside these bubbles: every internal exchange brings with it a sum over q that introduces a small parameter Q. Since in the limit $Q\to 0$, RPA diagrams are all we have, RPA respects the symmetry of H_{aug}. We shall now argue that if the constraint is respected, compressibility follows. For the electron-density operator we have many choices, starting with Eq. (168) and using the constraint Eq. (164). In particular we can write it as

$$\rho^e=2\rho \tag{170}$$

or as

$$\rho^e=-2\frac{il^2}{2}\mathbf{q}\times\mathbf{g}(\mathbf{q}). \tag{171}$$

Written the first way, a nonzero compressibility is not surprising since there are no powers of q in the operator. In the second way, we see the possible paradox, since there is a q up front in each of the two factors of ρ^e. It is here that the overdamped mode appears in the transverse sector, couples to $\mathbf{g}(\mathbf{q})$, and saves the day with a factor $q^2v(q)$ in its static propagator, yielding results that coincide with those of Halperin, Lee, and Read (1993).

The details of this formidable calculation are not shown here in the interest of brevity. Suffice it to say that five different operators (ρ and the two components each of $\mathbf{g}$ and $\mathbf{C}$, where $\mathbf{C}$ is another vector operator) get coupled and $K^{-1}=K_0^{-1}+U$ is a relation among 5×5 matrices. It is remarkable that in the end all the physics of Halperin, Lee, and Read, including the overdamped mode, finite compressibility, and mass divergences at the Fermi surface, emerge from the more physical quasiparticles, though after a lot of work.

This is not quite the end, since we have discussed only the $Q\to 0$ limit, while the actual theory has no such limit on Q. Stern *et al.* (1999) finesse this question with a two-step argument: (i) First they show that K invariance guarantees that the Landau parameter F_1 has to be -1. (ii) Then they show that this condition generally produces very soft modes which restore compressibility to the dipole gas. Let us elaborate a little. Suppose we want to define an effective low-energy theory for the composite fermions. How do we ensure that it has K invariance? We cannot possibly find all the higher-order correction in the Q expansion. This is also the typical situation in Landau theory: although in principle the Landau parameters can be calculated given the microscopic interaction, there is no way to do this reliably in practice. However, there are exceptions where a symmetry is involved. For example, Galilean invariance can be used to relate the bare mass, the physical mass, and the Landau parameter F_1. It turns out that here, too, K invariance can be assured for the actual problem if we choose $F_1=-1$. The reason is that the energy cost of boosting the Fermi sea is measured by $(1+F_1)$. Choosing $F_1=-1$ makes the boost cost-free, i.e., implies K invariance. In other words, the dipolar fermion cannot have an arbitrary Fermi-liquid interaction: its F_1 must equal -1. At the level of diagrams, if $F_1=-1$ is included as an interaction, the correlation function of two dipolar densities will go as q^0 and not q^2.

The compressibility of the $\nu=1/2$ system was also established by Read (1998) using a conserving approximation. He begins with the problem of bosons at unit filling, a problem first studied by Pasquier and Haldane (1998), though not in a conserving approximation. The bosons can be traded for fermions in zero average field by attaching a flux quantum. The role of fluctuations is the same as in the Halperin-Lee-Read problem except for the strength of the gauge field-fermion coupling. The Hamiltonian is just the electronic interaction written in the composite-fermion basis with

$$\mathbf{R}_e=\mathbf{r}-\frac{l^2}{2}\overline{\mathbf{z}}\times\mathbf{p}, \tag{172}$$

and the constraint is the density corresponding to

$$\mathbf{R}_v=\mathbf{r}+\frac{l^2}{2}\overline{\mathbf{z}}\times\mathbf{p}. \tag{173}$$

The advantage of this starting point is that the constraint exactly commutes with $\bar{\bar{H}}$. Thus one can look for approximation schemes in which the constraint is respected at the level of Green's functions, i.e., Green's functions with any number of $\bar{\bar{\chi}}(\mathbf{q})$'s in them vanish. Read's (1998) calculation begins with the filled Fermi sea (which does not respect the constraint) and embellishes it with diagrammatic corrections. An infinite sum of ladders (whose legs contain fermions in zero field) leads to the collective mode. The density-density correlator has the appearance of dipolar objects that exchange a transverse gauge field. The transverse propagator (which is just the overdamped mode) introduces a $v(q)q^2$ in the denominator that offsets the q^2 upstairs. It is found that correlators containing $\bar{\bar{\chi}}(\mathbf{q})$'s vanish and Ward identities are satisfied.

It is instructive to look at Read's (1998) density-density correlator, which takes the form

$$K_{irr}(q,\omega)=\int d^2k\,d^2k'(e^{il^2\mathbf{k}\times\mathbf{q}}-1)M(\mathbf{k},\mathbf{k}',\mathbf{q},\omega) \times(e^{-il^2\mathbf{k}'\times\mathbf{q}}-1), \tag{174}$$

where the factors at each end reduce to dipoles at small ql, and $M=M_0+M_T$ is a sum of two terms, one from the free Fermi sea and one from the exchange of the transverse collective mode, which peaks at $i\omega\simeq q^3\,v(q)$. At high frequencies, M_T can be ignored and the dipoles emerge as free objects, while at low frequencies, they are coupled by the overdamped mode and do not behave like classical dipoles. We begin to see how the dipoles appear in the correlators (because we can tune the frequency and wavelength in K to expose them) but not the wave functions, which describe equal-time correlations and thus involve an integral over all frequencies.

Arguments for the compressibility of the dipolar system were also given by Lee (1998).

The resolution of the compressibility paradox did much to assure the community that various descriptions of the quasiparticle, each tailor-made for a different occasion, were mutually compatible and consistent.

E. Higher Landau levels

The advantage of the extended Hamiltonian is that it keeps track of the electronic cyclotron coordinate and does not go to the lowest Landau level prematurely. This has many benefits. If we want to compute the Hall conductance, we can couple the system to an external potential and find the response. As stated previously, the response involves higher Landau levels so that the presence of $\boldsymbol{\eta}_e$ is crucial. The details are shown in Appendix B. We can also study the effects of Landau level mixing. Instead of using just H_{LLL} we can try to get an effective theory within the lowest Landau level, which subsumes the effects of virtual transitions to higher Landau levels.

To extract the leading correction due to higher Landau levels we write the Schrödinger equation in schematic form as

$$\begin{bmatrix} H_{00} & H_{0n'} \\ H_{n'0} & H_{n'n'} \end{bmatrix}\begin{bmatrix} \phi \\ \xi \end{bmatrix}=E\begin{bmatrix} \phi \\ \xi \end{bmatrix}, \tag{175}$$

where ϕ is restricted to the space spanned by Fock states composed of just the LLL states and ξ stands for everything else. Likewise the subscripts 0 and n' stand for collective labels in and above the lowest Landau level, respectively. The exact equation obeyed by ϕ is

$$\left(H_{00}+H_{0n'}\frac{1}{E-H_{n'n'}}H_{n'0}\right)\phi=E\phi, \tag{176}$$

which is not an eigenvalue problem since E appears on both sides. However, we may approximate as follows:

$$\frac{1}{E-H_{n'n'}}=-\frac{1}{H_{n'n'}}+\mathcal{O}(v/\omega_0), \tag{177}$$

since the eigenvalue E we are interested in is of order v and the eigenvalues of $H_{n'n'}$ are of order ω_0. To the same accuracy in $\kappa=v/\omega_0$ we can also replace

$$H_{n'n'}\simeq H^0_{n'n'}=\sum_\alpha a^\dagger_\alpha a_\alpha n'_\alpha\omega_0, \tag{178}$$

which leads to

$$H^{\text{eff}}_{00}=H_{00}-\sum_{n'}H_{0n'}\frac{1}{n'\omega_0}H_{n'0}. \tag{179}$$

Once we have the effective theory in the lowest Landau level, we can switch to the composite-fermion formalism: We introduce $\mathbf{R}_v$, exchange $\mathbf{R}_e$ and $\mathbf{R}_v$ for $\mathbf{R}$ and $\boldsymbol{\eta}$, and proceed as before by a Hartree-Fock calculation. The results are in accord with earlier works showing that Landau level-mixing reduces the transport gap, but that finite thickness reduces this effect (Yoshioka, 1986; Price, Platzman, and He, 1993; Melik-Alaverdian and Bonesteel, 1995; Price and Das Sarma, 1996; Melik-Alaverdian, Bonesteel, and Ortiz, 1997). Full details can be found in Murthy and Shankar (2002).

VI. CORRELATION FUNCTIONS IN THE CONSERVING APPROXIMATION

This section illustrates how one does the conserving calculation within the Hamiltonian approach by deriving density-density correlation functions (Murthy, 2001a). This calculation differs from that of Read (1998) in two ways. First, it is done for nonzero effective field ($\nu\neq 1/2$), so that the composite fermions are in Landau levels instead of plane wave states. Next, the calculation is done in the operator approach (Anderson, 1958a, 1958b; Rickayzen, 1958) using equations of motion as compared to diagrams (Nambu, 1960). The second difference is only cosmetic and is introduced here to promote harmony with the rest of the paper.[33]

The Hamiltonian and constraint to be solved are

$$H=\frac{1}{2}\sum_{\mathbf{q}}\bar{\rho}(\mathbf{q})v(q)e^{-(ql)^2/2}\bar{\rho}(-\mathbf{q}), \tag{180}$$

$$\bar{\bar{\chi}}(\mathbf{q})\simeq 0. \tag{181}$$

Let us define a time-ordered pseudovortex-electron density-density Green's function $G_{ve}(q,t)$ as follows (with G_{ee} and G_{vv} similarly defined):

$$\begin{aligned} G_{ve}(\mathbf{q},t-t')&=-i\langle T\bar{\bar{\chi}}(\mathbf{q},t)\bar{\rho}(-\mathbf{q},t')\rangle \\ &=-i\Theta(t-t')\langle\bar{\bar{\chi}}(\mathbf{q},t)\bar{\rho}(-\mathbf{q},t')\rangle \\ &\quad -i\Theta(t'-t)\langle\bar{\rho}(-\mathbf{q},t')\bar{\bar{\chi}}(\mathbf{q},t)\rangle, \end{aligned} \tag{182}$$

which evolves as

$$\begin{aligned} -i\frac{\partial}{\partial t}G_{ve}(\mathbf{q},t-t')&=-\delta(t-t')\langle[\bar{\bar{\chi}}(\mathbf{q},t),\bar{\rho}(-\mathbf{q},t')]\rangle \\ &\quad -i\langle T[H,\bar{\bar{\chi}}(\mathbf{q},t)]\bar{\rho}(-\mathbf{q},t')\rangle. \end{aligned} \tag{183}$$

Since $\bar{\bar{\chi}}$ commutes with H, one immediately sees that G_{ve} is a constant. If the constraint is set to zero initially, then it remains zero, and all its correlators also remain zero.

[33] For an illustration of calculations in the gapped fractions using the diagrammatic approach, see Green (2001).

The above is true in an exact treatment of the theory. Of course, we can only do approximate calculations. A calculation that respects $G_{ve}=G_{vv}=0$ respects the symmetries of the theory at the level of correlators and is called conserving. Let us see what a natural approximation scheme might be. Consider

$$i\frac{\partial}{\partial t}G_{ee}(\mathbf{q},t)=\delta(t)\langle[\bar{\rho}(\mathbf{q},t),\bar{\rho}(-\mathbf{q},0)]\rangle - i\langle T[H,\bar{\rho}(\mathbf{q},t)]\bar{\rho}(-\mathbf{q},0)\rangle. \tag{184}$$

Since $[\bar{\rho},\bar{\rho}]\simeq\bar{\rho}$, a Green's function involving three densities will arise. Additional time derivatives will produce higher-order density correlators, leading to a hierarchy of equations for more and more complicated Green's functions. The natural way to truncate this hierarchy is to make a mean-field approximation at some stage that reduces a product of two densities to a single density. One of the simplest of such approximations (Baym and Kadanoff, 1961) reduces $[H,\bar{\rho}]$, which is a product of four Fermi operators, to a product of only two, by using the averages

$$d^\dagger_{\alpha_1}d^\dagger_{\alpha_2}d_{\beta_2}d_{\beta_1}\to\langle d^\dagger_{\alpha_1}d_{\beta_1}\rangle d^\dagger_{\alpha_2}d_{\beta_2}+\langle d^\dagger_{\alpha_2}d_{\beta_2}\rangle d^\dagger_{\alpha_1}d_{\beta_1} - \langle d^\dagger_{\alpha_1}d_{\beta_2}\rangle d^\dagger_{\alpha_2}d_{\beta_1}-\langle d^\dagger_{\alpha_2}d_{\beta_1}\rangle d^\dagger_{\alpha_1}d_{\beta_2}. \tag{185}$$

Here $\langle d^\dagger_\alpha d_\beta\rangle=\delta_{\alpha\beta}N_F(\alpha)$, where $N_F(\alpha)$ is the Fermi occupation of the single-particle state α.

Using the Hartree-Fock states and their occupations in the above truncation is known as the time-dependent Hartree-Fock (TDHF) approximation.[34] We shall use the operator approach to TDHF as expounded by Anderson (1958a, 1958b) and Rickayzen (1958). We shall explicitly see below that it is conserving for all principal fractions.

The physical picture underlying our calculation is the following. When a bosonic operator such as $\bar{\rho}(\mathbf{q})$ or $\bar{\chi}(\mathbf{q})$ acts on the ground state, it creates a linear combination of particle-hole pairs. In the Landau gauge, each pair is labeled by an index $\nu=n_1,n_2$ (not to be confused with the filling factor) that keeps track of the composite-fermion Landau-level indices of the particle and hole, and a total pair momentum $\mathbf{q}$, which is conserved because the particle and hole have opposite charges and do not bend in the magnetic field. It is clear that in order to calculate time-dependent response functions we have to understand the time evolution of these pairs. In the exact theory, the Hamiltonian can scatter a pair into two pairs, two pairs into four pairs, etc. This is what leads to the hierarchy of equations. However, the great simplicity of the TDHF approximation is that in this approximation a particle-hole pair scatters only into another particle-hole pair. At a given $\mathbf{q}$ we thus have a matrix labeled by the indices (ν,ν') of the incident and scattered particle-hole pairs. The magnetoexciton spectrum comes from the eigenvalues of this matrix, which, together with its eigenvectors, will be seen to explicitly determine the Green's function.

Clearly we must begin with an operator that creates an exciton in a state of definite momentum $\mathbf{q}$, starting with states labeled by composite-fermion Landau-level indices $\nu=(n_1\ n_2)$. The following operator does the job:

$$O_{n_1n_2}(\mathbf{q})=\sum_X e^{-iq_xX}d^\dagger_{n_1,X-q_yl^{*2}/2}d_{n_2,X+q_yl^{*2}/2}. \tag{186}$$

Why is this so? First note that in this gauge [Eq. (45)], the wave functions are plane waves in y with momentum k and localized in x at $X=kl^{*2}$. Thus $d^\dagger_{m_1,X-q_yl^{*2}/2}d_{m_2,X+q_yl^{*2}/2}$, which creates a hole at $X-q_yl^{*2}/2$ and a particle at $X+q_yl^{*2}/2$, creates an exciton in a state of momentum q_y in the y direction, centered at X. Multiplying by e^{-iq_xX} and summing over X creates a state of momentum q_x in the x direction.

Now we define

$$G(\nu;\nu';\mathbf{q};t-t')=-i\langle TO_\nu(\mathbf{q},t)O_{\nu'}(-\mathbf{q},t')\rangle. \tag{187}$$

Taking the time derivative we get

$$-i\frac{\partial}{\partial t}G(\nu;\nu';\mathbf{q};t)=-i\langle T[H,O_\nu(\mathbf{q},t)]O_{\nu'}(-\mathbf{q},0)\rangle - \delta(t)\langle[O_\nu(\mathbf{q},t),O_{\nu''}(-\mathbf{q},0)]\rangle. \tag{188}$$

The last piece is the standard inhomogeneous "source" term. The dynamics are controlled by the commutator with the Hamiltonian, to which we now turn:

$$[H,O_\nu(\mathbf{q})]=[\epsilon(n_1)-\epsilon(n_2)]O_\nu(\mathbf{q})+[N_F(n_2) - N_F(n_1)]\sum_{\nu''}\left(\frac{v(q)}{2\pi(l^*)^2}e^{-q^2l^2/2}\rho_{n_1''n_2''}(\mathbf{q}) \times\rho_{n_2n_1}(-\mathbf{q})-\int\frac{d^2s}{(2\pi)^2}v(s) \times e^{-s^2l^2/2}\rho_{n_1''n_1}(\mathbf{s})\rho_{n_2n_2''}(-\mathbf{s}) \times e^{i(l^*)^2\mathbf{s}\times\mathbf{q}}\right)O_{\nu'}(\mathbf{q}), \tag{189}$$

where $\varepsilon(\nu)$ is the Fock energy in a state of composite-fermion Landau-level index ν, and the TDHF approximation has been made.

Note that the action of commuting with H on O_ν in the TDHF approximation can be represented as the right multiplication by a matrix $\mathcal{H}(\nu;\nu'';\mathbf{q})$. It follows that if we form a linear combination of operators $O_\Psi=\Sigma_\nu\Psi(\nu;\mathbf{q})O_\nu$, the column vector Ψ will transform lin-

[34]The TDHF approximation has been the method of choice in computing magnetoexciton dispersions in the IQHE. Here are some of the early references: Chiu and Quinn (1974); Horing and Yildiz (1976); Theis (1980); Bychkov, Iordanskii, and Eliashberg (1981); Bychkov and Rashba (1983); Kallin and Halperin (1984); Hawrylak and Quinn (1985); MacDonald (1985); Marmorkos and Das Sarma (1992); Longo and Kallin (1993).

early under the action of $\mathcal{H}$. Diagonalizing $\mathcal{H}$ will also enable us to solve for the Green's function, for the eigenvectors represent linear combinations of particle-hole states that are normal modes of $\mathcal{H}$. Assume that one has found the right and left eigenvectors and corresponding eigenvalues $E_\alpha(q)$, labeled by α,

$$\mathcal{H}(\nu;\nu'';\mathbf{q})\Psi^R_\alpha(\nu'';\mathbf{q}) = E_\alpha(q)\Psi^R_\alpha(\nu;\mathbf{q}), \tag{190}$$

$$\Psi^L_\alpha(\nu;\mathbf{q})\mathcal{H}(\nu,\nu'';\mathbf{q}) = E_\alpha(q)\Psi^L_\alpha(\nu'';\mathbf{q}), \tag{191}$$

where sums over repeated indices are implicit.

Assuming that the matrix $\mathcal{H}$ has a complete set of eigenvectors, the Green's function can be written as

$$G(\nu;\nu';\mathbf{q};\omega) = \frac{L^2}{2\pi(l^*)^2}\sum_\alpha \Psi^R_\alpha(n_1 n_2;\mathbf{q})\frac{1}{\omega - E_\alpha} \times \Psi^L_\alpha(n_2' n_1';\mathbf{q})[N_F(n_1') - N_F(n_2')], \tag{192}$$

where the factor $[N_F(n_1') - N_F(n_2')]$ comes from the source term [the last term of Eq. (188)].

An important property of $\mathcal{H}$ from the point of view of the conserving approximation is that it always has one left eigenvector with zero eigenvalue for every $\mathbf{q}$, namely,

$$\Psi^L_0(n_1 n_2;\mathbf{q}) = \tilde{\chi}_{n_1 n_2}(\mathbf{q}), \tag{193}$$

where

$$\tilde{\chi}_{n_1 n_2}(\mathbf{q}) = \langle n_1 | e^{-i\mathbf{q}\cdot(\boldsymbol{\eta}/c)} | n_2 \rangle. \tag{194}$$

The existence of this eigenvector with zero eigenvalue [for any $v(q)$] is shown explicitly by Murthy (2001a). This zero eigenvalue is related to the constraint and is one of the conditions for the TDHF approximation to be conserving. To see this let us go back to the exact theory. There the Hamiltonian commutes with $\bar{\bar{\chi}}(\mathbf{q})$. This means that $\bar{\bar{\chi}}(\mathbf{q})$ acting on the ground state should produce a zero-energy state at every $\mathbf{q}$. In the TDHF approximation the action of the Hamiltonian on this state is approximated by the action of the matrix $\mathcal{H}$ on the left vector $\Psi^L_0(n_1 n_2;\mathbf{q})$, corresponding to the operator $O_{\bar{\bar{\chi}}(\mathbf{q})} = \Sigma\tilde{\chi}_{n_1 n_2}(\mathbf{q})O_{n_1 n_2}(\mathbf{q})$. The fact that $\mathcal{H}$ admits Ψ^L_0 as a left eigenvector with zero eigenvalue means that the zero-energy state that had to be present in the exact theory is also present in the TDHF approximation.

To put it slightly differently, the TDHF approximation represents a truncation of the Hilbert space of neutral excitations to states having only a single particle-hole pair above the ground state. *A priori* this truncation need not have respected the constraint, but it does.

The above condition is necessary for TDHF to be conserving, but not sufficient. The other condition is that the physical sector [excitations created by $\bar{\bar{\rho}}(\mathbf{q})$] not couple to the zero-energy sector of $\mathcal{H}$. This can also be verified in a straightforward way (Murthy, 2001a).

A. Small-q structure factor

We shall now put the conserving approximation to another test. Recall that the density-density correlator has two parts: one coming from the cyclotron pole, with a residue of q^2, which is unrenormalized as per Kohn's theorem and saturates the sum rule, and another coming from the dynamics in the lowest Landau level. Hence the LLL-projected structure factor $\bar{\bar{S}}(q)$ has to vanish faster than q^2, and if it is analytic in q^2, it has to go like q^4. We shall see if this is true. Luckily, to obtain the leading behavior of $\bar{\bar{S}}(q)$ in the TDHF approximation one need keep only a finite-dimensional submatrix of the infinite TDHF matrix.

The results are as follows. For $\nu = 1/3$ we have, upon diagonalizing a 4×4 matrix,

$$\bar{\bar{S}}(q) = \frac{(ql)^4}{8} + \cdots. \tag{195}$$

We can extend this result to all the Laughlin fractions $1/(2s+1)$ to obtain

$$\bar{\bar{S}}(q) = \frac{1}{8}(ql)^4 + \cdots. \tag{196}$$

The coefficient 1/8, independent of s, differs from the result $s/4$ obtained from Laughlin's wave function (Girvin, MacDonald, and Platzman, 1986). However, no general principle requires that we regain this coefficient.

One can carry out a very similar, but much more tedious, calculation for all the principal fractions (Murthy, 2001a). The result, upon diagonalizing a 6×6 matrix, is

$$\bar{\bar{S}}(q) = \frac{(ql)^4}{2}\frac{p^4 - 3p^3 + \frac{5}{4}p^2 + 3p + \frac{7}{4}}{p^2 - 1}. \tag{197}$$

This expression, with its divergence as $p\to\infty$ or $\nu \to 1/2$, is consistent with the result (Read, 1998) that for a problem equivalent to the $\nu = 1/2$ problem, $\bar{\bar{S}}(q) \simeq q^3\ln(q)$. As $p\to\infty$, the radius of convergence of the power-series expansion of $\bar{\bar{S}}(q)$ must go to zero [or else the structure factor would diverge for a range of q according to Eq. (97)], and the $p\to\infty$ limit does not commute with the $q\to 0$ limit. The formula also does not hold for $p=1$, the Laughlin fractions, for which we must use the results quoted earlier.

Surprisingly, the results are independent of the form of the potential, which does enter the intermediate stages of the calculation.

B. Magnetoexciton dispersions for 1/3 and 2/5

We saw that the small-q behavior of $\bar{\bar{S}}(q)$ can be satisfactorily addressed using the TDHF approximation. It turns out that this approximation also works well for computing the dispersion of magnetoexcitons, by which we mean the lowest-energy physical eigenstate of $\mathcal{H}$ at each $\mathbf{q}$.

As shown in the previous section, for very small q, the naive magnetoexcitons (bare particle-hole states created by O_ν) do not mix with others and become the true eigenstates of $\mathcal{H}$. As q increases they become increasingly coupled, and both level repulsion (between posi-

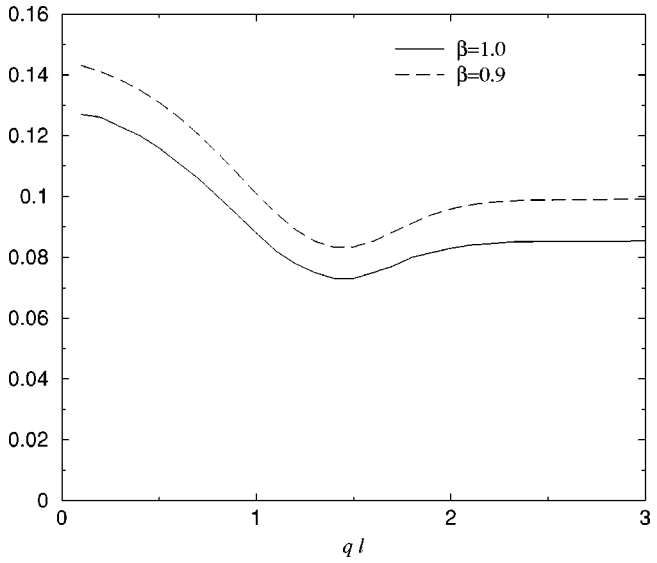

FIG. 4. The lowest-energy spin-polarized magnetoexciton for $\nu = 1/3$ for two values of β. Note the "magnetoroton" at $ql = 1.4$.

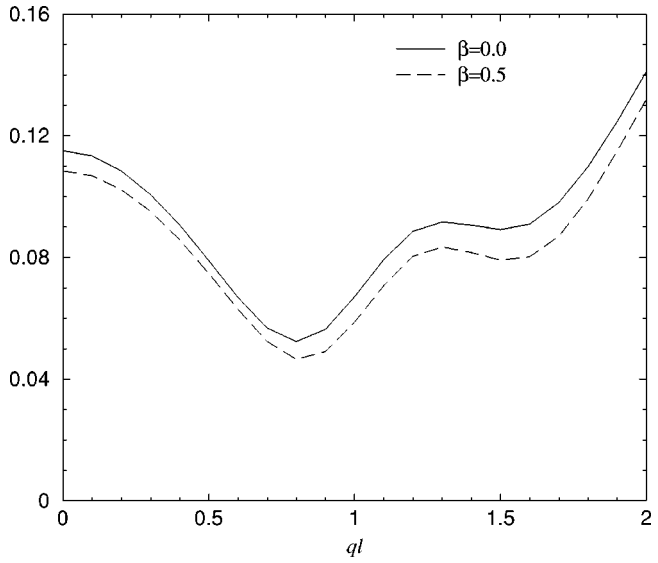

FIG. 5. The lowest-energy spin-polarized magnetoexciton for $\nu = 2/5$ for two values of β. Note the minima at $ql = 0.8$ and $ql = 1.5$.

tive energy states) and level attraction (between positive and negative energy states) manifest themselves, giving rise to complicated magnetoexciton dispersions. The matrix $\mathcal{H}$ is infinite dimensional, but in any numerical calculation only a finite matrix can be diagonalized. We saw that the lowest nontrivial result for $\bar{\bar{S}}(q)$ could be obtained by keeping at most a 6×6 matrix. As q increases, more composite-fermion Landau levels have to be kept to obtain accurate results. The accuracy of the truncation was checked by two different methods. First, the number of composite-fermion Landau levels kept was increased until the energy of the magnetoexciton was stable. Second, we knew that at every q there should be two zero eigenvalues corresponding to the unphysical sector. The number of composite-fermion Landau levels kept in the calculation was increased until these null eigenvalues were at least four orders of magnitude smaller than the smallest physical eigenvalue.

Figures 4 and 5 show the potential $v(q) = (2\pi e^2/\epsilon q)\, e^{-q^2\beta^2/2}$ for various values of β. It is worth noting that though the depths of the minima are β dependent and different from what is found in numerical diagonalizations or from composite-fermion wave functions, the positions are correct.[35] This indicates that the TDHF approximation to the extended Hamiltonian theory does capture the important physics of the electronic problem even at fairly large q.

In the case of 1/3, the magnetoexciton energy has stabilized at large ql, enabling us to read off the *activation gap* Δ_a, defined as the minimum energy needed to produce a widely separated particle-hole pair. However, the situation deteriorates rapidly as p increases. It becomes prohibitively hard, even at $p=2$, to get the large-ql limit of the magnetoexciton spectrum, as is evident from Fig. 5. This is because the decoupling of the naive magnetoexcitons is asymptotic only for large q. However, as p increases, the shortcut gives very good answers (Figs. 6 and 7 below) with hardly any additional work. We now turn to it.

VII. GAPS

Having developed the extended Hamiltonian theory at length and having established several qualitative results and matters of principle, we now turn to mainly quantitative issues. As mentioned at the outset, one can compute just about anything within this approach, some things more accurately than others. In the next few sections we shall discuss a variety of such quantities. Rather than run through an endless list, we shall focus on a few that give the flavor of the method and expose its strengths and shortcomings, and point out references containing more details or examples. An invaluable benchmark will be provided by results from trial wave functions and exact diagonalization. Comparison with experiment will be made with the clear understanding that no systematic attempt is made to incorporate disorder.

In comparing to the results of exact diagonalization and trial wave functions one must bear in mind their limitations. Exact diagonalization (ignoring machine errors) suffers from the fact that the systems are necessarily small. In comparing these results to ours (valid for infinite systems) we must examine the approach to the thermodynamic limit and the scaling of gaps. We refer the reader to Morf, d'Ambrumenil, and Das Sarma (2002) for the most recent study of this kind.

[35]See Haldane and Rezayi (1985); Morf and Halperin (1987); Su and Wu (1987); d'Ambrumenil and Morf (1989); He, Simon, and Halperin (1994); He and Platzman (1996); Kamilla, Wu, and Jain (1996a, 1996b); Jain and Kamilla (1998).

While trial wave functions can be written down for any number of particles, the evaluation of gaps requires once again that the number of particles be small (though not as small as in exact diagonalization). The details may be found in Park, Meskini, and Jain (1999), to whose work we compare our numbers. It must also be borne in mind that the correct wave functions may not be of the form being tried, in which case even a flawless evaluation of gaps is irrelevant. So far there is good reason to believe that for $1/3<\nu\leqslant 1/2$ the Jain-like functions are good. At very low densities the correct state is believed to be a Wigner crystal, and when higher Landau levels are involved, more complicated functions like Pfaffians [the Moore-Read (1991) state] need to be explored. With these caveats in mind we proceed with the comparison of our extended Hamiltonian theory with numerics.

In the remaining sections, we shall use the preferred density and pay no further regard to constraints. As stated earlier, this is the most efficient way to work on problems that do not depend crucially on the deep-infrared region [$\omega \simeq q^3\, v(q)$]. In the cases discussed, this is ensured by either a gap, a nonzero temperature, or both.

Let us begin with the activation or *transport gap* in a fully polarized sample, defined as the minimum energy needed to produce a widely separated particle-hole pair:

$$\Delta=\langle \mathbf{p}+PH|H|\mathbf{p}+PH\rangle-\langle \mathbf{p}|H|\mathbf{p}\rangle, \tag{198}$$

where $|\mathbf{p}\rangle$ stands for the Hartree-Fock ground state with p-filled Landau levels and PH stands for a widely separated particle-hole pair. We shall use a boldface symbol such as $\mathbf{p}$ to label a Slater determinant with p occupied Landau levels. Nonboldface symbols will label single-particle states. Note also that the highest occupied composite-fermion Landau level index n for the state labeled by p is $n=p-1$ since the composite-fermion lowest Landau level has index $n=0$. As shown in Appendix C, the particle-hole excitations of $|\mathbf{p}\rangle$ are Hartree-Fock states of our H.

The expression for the gap written above is formally the same in the wave-function-based approach of Park, Meskini, and Jain (1999), to whose results we shall compare ours. However, the notation hides a big difference. They work in the *electronic basis* where $\rho(\mathbf{q})=\Sigma_j \times \exp(-i\mathbf{q}\cdot\mathbf{r}_j)$, and the states are the simple wave functions χ_p, multiplied by the Jastrow factor and then projected to the lowest Landau level. (Projection leads to a very complicated expression for the wave functions.) In the present approach we have tried to incorporate these effects by going in the reverse direction, from electrons to composite fermions, and obtaining complicated expressions for the charge and other operators, but with simple expressions for the wave functions. While these operator expressions are unusual in form, they are simple to evaluate within the Hartree-Fock calculation.

In all our calculations we shall use the Zhang–Das Sarma potential (Zhang and Das Sarma, 1986)

$$v_{ZDS}=\frac{2\pi e^2 e^{-q l\lambda}}{q}, \tag{199}$$

which is a crude model for the electron-electron interaction in a sample of finite thickness. We simply take it to be a one-parameter family of potentials. In terms of the Haldane pseudopotentials V_m (which give the interaction in a state of relative angular momentum m; see Haldane, 1990), we know that just one (typically V_1) dominates. We can think of λ as controlling the operative pseudopotential.

Rather than work with a widely separated particle-hole pair, we find the energy in a state with just the particle and add to it the energy of a state with just the hole and subtract double the ground-state energy. Relegating the details to Appendix E, we present the central idea.

We begin with the second-quantized expression for the preferred charge operator $\bar{\bar{\rho}}^p(\mathbf{q})$:

$$\bar{\bar{\rho}}^p(\mathbf{q})=\sum_{m_2n_2;m_1n_1} d^\dagger_{m_2n_2} d_{m_1n_1} \rho_{m_2n_2;m_1n_1}(\mathbf{q}), \tag{200}$$

where $d^\dagger_{mn}$ creates a particle in the state $|mn\rangle$ where m is the angular momentum and n is the Landau-level index of the composite fermion in the weakened field $A^*=A/(2ps+1)$ with a magnetic length

$$l^*=l\sqrt{2ps+1}. \tag{201}$$

The key ingredient in the Hartree-Fock calculation is the matrix element $\rho_{m_2n_2;m_1n_1}$, which factorizes (as shown in Appendix A) into

$$\rho_{m_2n_2;m_1n_1}=\rho^m_{m_2m_1}\otimes\rho^n_{n_2n_1}. \tag{202}$$

The gaps depend only on $\rho^n_{n_2n_1}$, the superscript of which will be generally dropped. Often we shall use the *dimensionless activation gap* δ_a defined by

$$\Delta_a=\frac{e^2}{\varepsilon l}\,\delta_a. \tag{203}$$

Figure 6 shows the gaps computed for 1/3, 2/5, 3/7, and 4/9 for the Zhang–Das Sarma potential and compared to the work of Park, Meskini, and Jain (1999) in the region $0\leqslant\lambda\leqslant 3$. The following features are noteworthy.

- At $\lambda=0$, the Coulomb case, the gaps are finite in contrast to the small-q theory (Murthy and Shankar, 1999). This is due to the Gaussian factor $e^{-q^2l^2/2}$ in $H(\bar{\rho})$ [Eq. (61)], which was absent in the small-q theory. The slope of the graphs in the present theory is nonzero at this point. It is readily verified that $d\Delta/d\lambda$ at $\lambda=0$ is the gap due to a delta-function potential and should vanish for spinless fermions. The description of the composite fermion in terms of $\bar{\bar{\rho}}^p$ does not give good answers for potentials as short ranged as the delta function, as anticipated earlier. Indeed, even the Coulomb interaction is too singular, and the theory begins to work well only beyond $\lambda\simeq 1$.
- Beyond $\lambda\simeq 1$ the agreement is quite fair in general

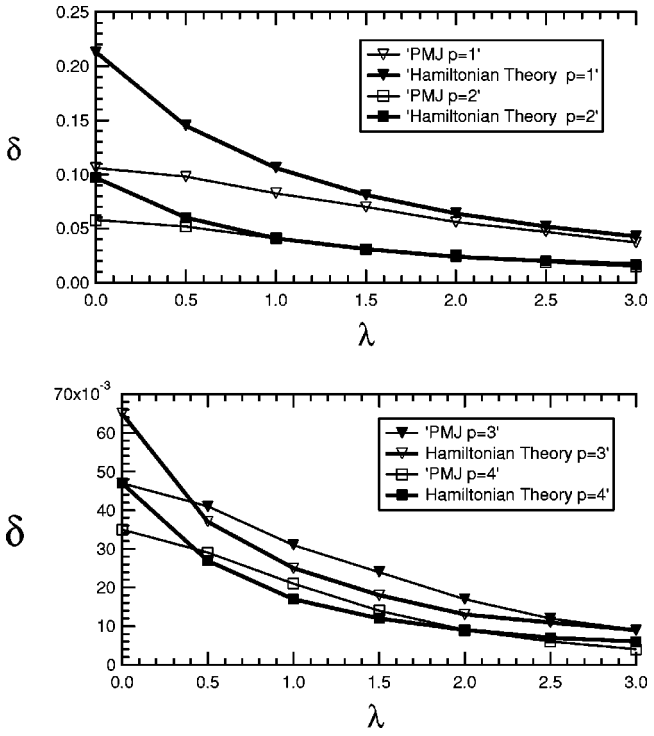

FIG. 6. Comparison of the dimensionless activation gaps δ_a to the work of Park, Meskini, and Jain (1999; PMJ) for the fractions 1/3, 2/5, 3/7, and 4/9 (that is, $p=1,2,3,4$ and $s=1$) as a function of λ, the thickness parameter in the Zhang–Das Sarma potential.

and best for 2/5. However, we cannot go to too large a λ since in this range the system may not be an FQHE state.

- The gaps that do not vanish for any fraction and any finite λ exceed the Park, Meskini, and Jain (1999) values for 1/3 and 2/5 and lie below them for 3/7 and 4/9. This result is at odds with the general belief that Hartree-Fock always overestimates the gaps by neglecting fluctuations.

The situation is different when we compare with the exact diagonalization results of Morf, d'Ambrumenil, and Das Sarma (2002), who used a potential

$$v(q)=\frac{2\pi e^2}{q}e^{(qlb)^2}\mathrm{Erfc}(qlb), \tag{204}$$

where b is the analog of λ.[36] Our numbers are compared in Fig. 7. The calculated gaps always lie above the exact diagonalization results for the two fractions shown (as well as for the 1/3 case, not shown).

The general disagreement with our theory is worse for this potential than for the Zhang–Das Sarma case because at large q this potential goes as $1/q$, while the Zhang–Das Sarma potential falls exponentially.

[36]This potential is not very different in form from the Zhang–Das Sarma potential except at very short distances.

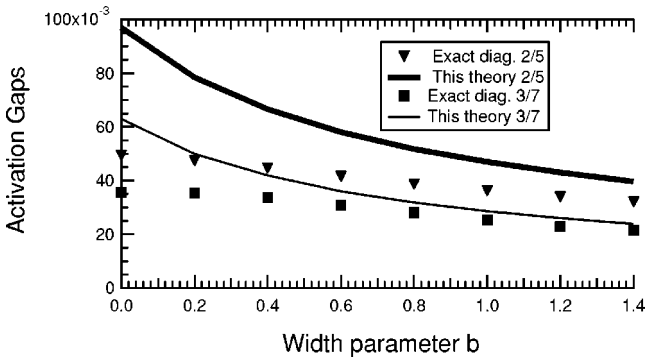

FIG. 7. Comparison of the predictions of the Hamiltonian theory to the exact diagonalization results of Morf, d'Ambrumenil, and Das Sarma (2002) for $p=2$ and 3.

A. Activation gaps

We have computed gaps for many other fractions, including $s=2$, when four vortices are attached to form composite fermions. Rather than show more plots, we shall now analyze the theory in terms of the *activation mass* m_a defined by

$$\Delta_a=\frac{eB^*}{m_a}=\frac{eB}{(2ps+1)m_a}. \tag{205}$$

Comparison to Eq. (203) shows that

$$\frac{1}{m_a}=\frac{e^2l}{\varepsilon}\delta_a(2ps+1)\equiv\frac{e^2l}{\varepsilon}C_a. \tag{206}$$

Thus

$$C_a=\delta_a(2ps+1). \tag{207}$$

Based on the exact diagonalization results of d'Ambrumenil and Morf (1989), Halperin, Lee, and Read (1993) pointed out that C_a approaches a limit as we approach $\nu=1/2$ or $p\to\infty$. The Halperin-Lee-Read theory expects C_a to be modified by logarithms. We too expect these logarithms once we approach $\nu=1/2$ and are forced to include the overdamped mode. However, we find, as did Halperin, Lee, and Read (1993), a good fit to the calculated gaps without invoking the logarithms, which are operative in a very tiny region near $\nu=1/2$ (Morf, d'Ambrumenil, and Das Sarma, 2002). The only difference here is that C_a approaches a limit that depends on λ, a parameter they set equal to zero (Coulomb case).

We next turn to Pan *et al.* (2000), whose experiments detected that the *normalized mass* defined by

$$m_a^{nor}=\frac{m_a}{m_e\sqrt{B(T)}}, \tag{208}$$

where m_e is the electron mass and $B(T)$ is the field in tesla, is nearly the same for $s=1$ and $s=2$, i.e., two and four flux tubes. The theory predicts that m^{nor} are comparable for $s=1$ and $s=2$, but makes it clear that no fundamental significance can be attached to this result since it depends on λ, or more generally, the potential. Further details may be found in Shankar (2001).

TABLE II. Activation gaps as a function of λ for $1 \leqslant \lambda \leqslant 2$ according to the Hamiltonian theory. Note the convergence of $C_a^{(2)}$ as $p \to \infty$. The superscript (2) refers the number of flux tubes attached.

p	$\Delta_a^{(2)}/k_B = 50\sqrt{B(T)}\,\delta_a^{(2)}$	$\delta_a^{(2)}$	$C_a^{(2)}$
1	$5.31\sqrt{B(T)}/\lambda$	$0.106/\lambda$	$0.32/\lambda$
2	$2.08\sqrt{B(T)}/\lambda$	$0.042/\lambda$	$0.21/\lambda$
3	$1.23\sqrt{B(T)}/\lambda$	$0.025/\lambda$	$0.17/\lambda$
4	$0.87\sqrt{B(T)}/\lambda$	$0.017/\lambda$	$0.16/\lambda$

Consider next the experiments of Du *et al.* (1993), who have extensive data on activation gaps. We shall limit ourselves to $\nu \leqslant 1/2$, to which states with $1 \geqslant \nu \geqslant 1/2$ are related by particle-hole symmetry if full polarization is assumed. Given that the experiments, unlike that of Park, Meskini, and Jain (1999), have an unknown contribution from Landau-level mixing and impurities, it is not clear how to apply the theory. There is no *ab initio* calculation that includes these effects. (There is, however, reliable evidence that Landau-level mixing is a very small effect at the values of λ under consideration. Recall also our results from Sec. V.)

We shall compute gaps using the Zhang–Das Sarma potential with λ as a free parameter and ask what λ fits the data, just to get a feel for its size. The results are summarized in Tables II–IV.

Comparing the above values of λ extracted from data to the local-density approximation (LDA) (Price and Das Sarma, 1996, and references therein) and exact diagonalization calculations (Park, Meskini, and Jain, 1999; Morf, d'Ambrumenil, and Das Sarma, 2002), which suggest $\lambda \simeq 1$, we see that disorder has a substantial effect on activation gaps.

It is possible to compute the charge density in a state with a widely separated particle-hole pair in some gapped fractions. The details, and a comparison to the unpublished work of Park and Jain, may be found in Shankar (2001).

B. Other potentials

Figure 8 shows a comparison to the Park, Meskini, and Jain (1999) results for a Gaussian potential

$$v(q) = 2\pi e^2 l e^{-q^2 l^2/2}. \tag{209}$$

Note that except for $\nu = 1/3$ the agreement is exceptional. This is the kind of ultraviolet soft potential for which the present theory works best, though unfortunately we do not presently know of any system in which it is operative. Its study serves to instruct us on the domain of validity of our approach. Not surprisingly, work on the potential

$$v(r) = \frac{e^{-\kappa r}}{r} \tag{210}$$

shows an agreement that is worse than for the Coulomb case, since this potential is just as bad as $r \to 0$ and does not give the large r values a chance. Likewise $1/r^2$ fares worse than $1/r$.

Note that if $\lambda \to \infty$ we may be in trouble, since the potential may not lead to FQHE states.

In summary, it appears as if our approach works well in any problem that is not sensitive to distances smaller than about a magnetic length. Since the extended formalism is mathematically equivalent to the original electronic problem in the lowest Landau level, it is the twin approximations—the use of $\bar{\rho}^p$ to deal with constraints and Hartee-Fock—that are responsible for deviations from our benchmarks.

While our approach can reproduce the numbers of the wave-function and exact diagonalization approaches to within about 10% (if $\lambda > 1$) with a lot less work (a few seconds on a PC), without these benchmarks we would have known neither its range of validity nor its degree of accuracy. Without such feedback, we would not have applied the method with such confidence to other phenomena not treated by wave functions or exact diagonalization, such as relaxation rates and polarization at $T > 0$.

TABLE III. Approximate numbers used in this paper, with k_B the Boltzmann's constant, m_e the electron mass, and $B(T)$ the field in tesla.

$\frac{eB}{m_e k_B} = 1.34 B(T)$ K	$\frac{e^2}{\varepsilon l k_B} = 50\sqrt{B(T)}$ K	$\frac{\varepsilon}{e^2 l} = 0.026\, m_e \sqrt{B(T)}$

TABLE IV. Comparison of activation masses to sample A of Du *et al.* (1993), which has a density $n = 1.12 \times 10^{11}$ cm^{-2}. The last column gives the best fit to λ.

ν	$B(T)$	Δ_a^{exp} (K)	Δ_a^{theo} (K)	λ
$\frac{1}{3}$	13.9	8.2	$5.3\sqrt{B(T)}/\lambda$	2.4
$\frac{2}{5}$	11.6	3	$2.08\sqrt{B(T)}/\lambda$	2.4
$\frac{3}{7}$	10.8	2	$1.23\sqrt{B(T)}/\lambda$	2.0

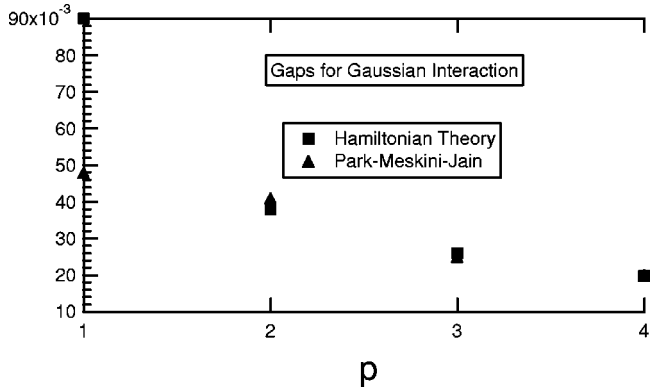

FIG. 8. Comparison of activation gaps predicted by the Hamiltonian theory with the work of Park, Meskini, and Jain (1999) for the Gaussian potential. The agreement is almost perfect for $p > 1$.

VIII. MAGNETIC TRANSITIONS AT $T=0$

Now we turn to the behavior of the spin of the system, which so far has been assumed to be frozen along the applied field. The reader might wonder why we bother, since the magnetic fields are of order at least a few tesla. The reason, originally pointed out by Halperin (1983), is that the Zeeman energy $E_Z=g^*(e/2m)B$ is the smallest energy scale in the problem, owing to a combination of two factors. First, due to band-structure effects, the band mass of the electron in GaAs is $0.068m_e$, with m_e the electron's mass in vaccuum. This makes the cyclotron frequency about 14 times what it would have been in empty space. Next, due to spin-orbit coupling, the effective g factor of the electron in GaAs is $g=0.44$ instead of the empty space value of 2. Thus the Zeeman energy is about 64 times smaller than the cyclotron energy ω_0. Typically the interaction energy is of the same order as ω_0, leading to E_Z's being the smallest energy. Since the original realization by Halperin (1983), spin transitions have been seen in experiment,[37] and explored by exact diagonalization.[38] We shall compare our results with the recent work of Park and Jain (1998, 1999) in the composite-fermion wave-function approach.

The coupling of electron spin to the applied field is given by the Zeeman term

$$H_Z=-g\left(\frac{e}{2m_e}\right)\frac{S}{2}B, \tag{211}$$

where $g=0.44$, m_e is the electron mass in free space, S is given by

$$S=nP, \tag{212}$$

where n is the density, and P is the polarization, to which each electron contributes ± 1.

Since the uniform external field couples to the $q=0$ component of the spin density, which is unaffected by the canonical transformations, H_Z will have the same form in the final composite-fermion representation.

A. Magnetic transitions in gapped fractions

When H_Z is large, we expect the system to be fully polarized ($P=1$). As we lower H_Z, we expect P to drop. If translationally invariant composite-fermion states are considered for the gapped fractions, there is a discrete set of allowed values of P. At $\nu=p/(2ps+1)$, these correspond to states of the form $|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle$ in which $p-r$ composite-fermion Landau levels are occupied by up-spins and r composite-fermion Landau levels by down-spins.[39] Thus the allowed values of polarization are given by

$$P=\frac{p-2r}{p}. \tag{213}$$

For example, when $p=4$, the allowed values are $P=1$, 0.5, and 0 corresponding to $|\mathbf{4},\mathbf{0}\rangle$, $|\mathbf{3},\mathbf{1}\rangle$, and $|\mathbf{2},\mathbf{2}\rangle$.

Our goal is to calculate the critical fields at which the system will jump from one value of r to the next as H_Z is varied. Let

$$E(p-r,r)=\langle \mathbf{p}-\mathbf{r},\mathbf{r}|H|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle, \tag{214}$$

where H does not contain the energy due to H_Z. This will be the case for the single-particle and ground-state energies, with one exception, which will be clearly pointed out. Since H_Z is diagonal in the Hartree-Fock states that have definite spin, its effects can be trivially incorporated.

The Hartree-Fock calculation of $E(p-r,r)$ is detailed in Appendix F.

The critical field B^c for the transition from r to $r+1$ is given by

$$E(p-r,r)-E(p-r-1,r+1)=g\frac{eB^c}{2m_e}\frac{n}{p}, \tag{215}$$

where the right-hand side denotes the Zeeman cost of flipping the n/p spins in the Landau level that switched its spin. This discussion assumes that B is perpendicular to the sample. If there is a tilt θ, we write

$$E(p-r,r)-E(p-r-1,r+1)=g\frac{eB_\perp^c}{2m_e\cos\theta}\frac{n}{p}. \tag{216}$$

When these energy differences are calculated, the same remarkable regularity first noted by Park and Jain (1998, 1999) emerges: the differences can be fit in the following sense by a theory of free fermions of mass m_p, the polarization mass, that occupy Landau levels with a gap $\Delta_p=eB^*/m_p$. In a free theory of gap Δ_p, we would have

$$E(p-r,r)-E(p-r-1,r+1)=\frac{n(p-2r-1)}{p}\Delta_p, \tag{217}$$

since (n/p) spin-up fermions of energy $(p-r-1+1/2)\Delta_p$ drop to the spin-down level with energy $(r+1/2)\Delta_p$. Suppose we evaluate the left-hand side of Eq. (217) in the Hartree-Fock approximation and define

$$\Delta_p(r)^{def}=\frac{p}{n}\frac{E(p-r,r)-E(p-r-1,r+1)}{p-2r-1}. \tag{218}$$

Given that H is not free, there is no reason why $\Delta_p(r)^{def}$ should be r independent. But it is very nearly so. For example, at $p=6,\lambda=1$,

[37] Clark *et al.* (1989), Eisenstein *et al.* (1989), Furneaux *et al.* (1989), Buckthought *et al.* (1991), Du *et al.* (1995, 1997).

[38] Chakraborty and Zhang (1984a, 1984b), Rasolt, Perrot, and MacDonald (1985), Chakraborty, Pietiläinen, and Zhang (1986).

[39] In Jain's approach, the actual wave function will be such a state times the Jastrow factor, followed by projection to the lowest Landau level. In the present approach, $|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle$ is literally the state, but the operators for charge and spin are obtained by transformations to the composite-fermion basis. For the interested reader we mention that these transformations are spin independent.

$$\Delta_p(0,1,2)^{def}=\frac{e^2}{\varepsilon l}(0.00660,0.00649,0.00641), \tag{219}$$

The three terms describe the transitions $|\mathbf{6},\mathbf{0}\rangle\rightarrow|\mathbf{5},\mathbf{1}\rangle$, $|\mathbf{5},\mathbf{1}\rangle\rightarrow|\mathbf{4},\mathbf{2}\rangle$, and $|\mathbf{4},\mathbf{2}\rangle\rightarrow|\mathbf{3},\mathbf{3}\rangle$. This r independence of the gaps was true for every fraction and every value of λ we looked at. We shall soon demystify this apparent free-field behavior, which has a counterpart in the gapless case as well.

The transition $|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle\rightarrow|\mathbf{p}-\mathbf{r}-\mathbf{1},\mathbf{r}+\mathbf{1}\rangle$ occurs when the spin-flip energy equals the energy difference of the two competing ground states:

$$g\frac{e}{2m_e}\frac{B_\perp^c}{\cos\theta}=(p-2r-1)\Delta_p. \tag{220}$$

B. Magnetic properties of gapless fractions

Let us turn now to the gapless fractions $\frac{1}{2}$ and $\frac{1}{4}$. The discrete labels $p-r$ and r of the Hartree-Fock states that count the spin-up and spin-down Landau levels are now replaced by continuous variables $k_{\pm F}$, which label the Fermi momenta of the spin-up and spin-down seas. These momenta are such that the total number of particles equals n:

$$k_{+F}^2+k_{-F}^2=k_F^2=4\pi n, \tag{221}$$

where k_F denotes the Fermi momentum of a fully polarized sea.

In the gapped case there were several critical fields B^c, each corresponding to one more composite-fermion Landau level flipping its spin, each describing one more jump in the allowed values of P. In the gapless case the situation is different. For very large Zeeman energy, the sea will be fully polarized. It will not be worth including even one fermion of the opposite spin, since the Zeeman energy cost alone will exceed the Fermi energy of the polarized sea. As we lower the Zeeman term, we shall reach a critical field at which it will be worth introducing one fermion of the other spin with zero (effective) kinetic energy. At this point the energy of a particle on top of the spin-up sea obeys

$$\mathcal{E}_+(k_{+F})=g\frac{e}{2m_e}\frac{B_\perp^c}{\cos\theta}. \tag{222}$$

If we lower the Zeeman term further, the polarization will fall continuously and be determined by $\mathcal{E}_\pm(k_{\pm F})$, the energies of the particles on top of these two seas, according to

$$\mathcal{E}_+(k_{+F})-\mathcal{E}_-(k_{-F})=g\frac{e}{2m_e}\frac{B_\perp}{\cos\theta}. \tag{223}$$

This equation states that the system is indifferent to the transfer of a particle from one sea to another, i.e., it has minimized its energy with respect to polarization.

Since the effective magnetic field vanishes at the gapless fractions, we deal with a very simple expression for $\bar{\rho}(\mathbf{q})^p$:

$$\bar{\rho}^p(\mathbf{q})=\int\frac{d^2k}{4\pi^2}(-2i)\sin\left(\frac{\mathbf{q}\times\mathbf{k}l^2}{2}\right)d^\dagger_{\mathbf{k}-\mathbf{q}}d_{\mathbf{k}}. \tag{224}$$

It is easy to do a Hartree-Fock calculation and obtain

$$\begin{aligned}\mathcal{E}_\pm(k)=&2\int\frac{d^2q}{4\pi^2}\check{v}(q)\sin^2\left[\frac{\mathbf{k}\times\mathbf{q}l^2}{2}\right]\\&-4\int\frac{d^2k'}{4\pi^2}n_\pm^F(|k'|)\check{v}(|\mathbf{k}-\mathbf{k}'|)\\&\times\sin^2\left[\frac{\mathbf{k}'\times\mathbf{k}l^2}{2}\right]\equiv\mathcal{E}_0+\mathcal{E}_I,\end{aligned}$$

where the Zeeman energy is not included, $n_\pm^F$ is the Fermi (step) function for the two species, $\mathcal{E}_0$ and $\mathcal{E}_I$ represent single-particle energy (due to what was called H_0 earlier) and the energy of interaction of this particle at the Fermi surface with those inside the sea, and

$$\check{v}(k)=v(k)e^{-k^2l^2/2}. \tag{225}$$

When this result is used to compute $\mathcal{E}_+(k_{+F})-\mathcal{E}_-(k_{-F})$, we find once again that the numbers fit a free theory in the following sense. Imagine that composite fermions were free and had a mass m_p. We would then have

$$\mathcal{E}_+(k_{+F})-\mathcal{E}_-(k_{-F})=\frac{k_{+F}^2-k_{-F}^2}{2m_p}. \tag{226}$$

What we find is that the Hartree-Fock number for $\mathcal{E}(k_{+F})-\mathcal{E}(k_{-F})$ may be fit very well to the above form with an m_p that is essentially constant as we vary $k_{\pm F}$; i.e., the relative sizes of the up and down seas (this is analogous to an m_p that does not depend on the index r in the gapped case). This constant m_p defined by

$$\frac{1}{m_p}=2\frac{\mathcal{E}_+(k_{+F})-\mathcal{E}_-(k_{-F})}{k_{+F}^2-k_{-F}^2} \tag{227}$$

matches smoothly with that defined for the nearby gapped fractions.

This free-field behavior is surprising because we shall shortly see that there are many reasons to believe that the composite fermions are not free. For now, note that the Hartree-Fock energies are not even quadratic in momenta: for example, at $\nu=1/2$ and $\lambda=1$ there is a hefty quartic term:

$$\frac{\mathcal{E}(k_{\pm F})}{(e^2/\varepsilon l)}=a\left(\frac{k_{\pm F}}{k_F}\right)^2+b\left(\frac{k_{\pm F}}{k_F}\right)^4, \tag{228}$$

where $a=0.075$ and $b=-0.030$.

There is no reason that the composite-fermion kinetic energy should be quadratic in momentum. These particles owe their kinetic energy to electron-electron interactions, and given this fact, all we can say is that their energy must be an even function of k, starting out as k^2 at small k. What constitutes small k is an open question that is answered unambiguously here: Our expression of the energy has substantial k^4 terms for momenta of interest, but not from higher powers. An analogous result

holds for the gapped fractions, in which the composite-fermion Landau levels are not equally spaced (Murthy, 1999; Mandal and Jain, 2001a).

The proper interpretation of this free-field behavior will be taken up next.

C. Composite fermions: free at last?

The fact that magnetic phenomena at $T=0$ can be described (to excellent accuracy) by free fermions of mass m_p (Park and Jain, 1998, 1999; Shankar, 2000, 2001) needs to be properly understood and interpreted. In particular, one must resist the thought that perhaps by some further change of variables one could take the present Hamiltonian and convert it to a free one. This is because if there were really an underlying free theory, it would have a single mass m_{CF} for both activation and polarization phenomena, with the composite fermions forming Landau levels of spacing eB^*/m_{CF}. But we know from the extended Hamiltonian theory, Jain's approach, or experiment that there are two masses m_a and m_p that differ by at least a factor of 2. Furthermore the shape of the magnetoexciton dispersions (Jain and Kamilla, 1998, and Fig. 5) also points to sizable composite-fermion interactions: as ql^2, the distance between the particle and hole, varies, the (binding) energy varies by an amount comparable to their individual energies, whose sum is given by the value at large ql.

It has been shown (Shankar, 2000, 2001) that a single assumption about the form of the ground-state energy, an assumption that is not equivalent to the free-field assumption or even to a quadratic dispersion relation in the gapless cases, will explain this behavior for gapped and gapless fractions. Consider $E(S)$, the ground-state energy as a function of $S=nP$, where P is the polarization. By rotational invariance it must have only even powers of S in its series. Assume that the series is dominated by the first two terms:

$$E(S)=E(0)+\frac{\alpha}{2}S^2, \tag{229}$$

where α is the inverse linear static susceptibility.

Consider first the gapless case. When dn particles go from spin-down to spin-up,

$$dE=\alpha S dS=\alpha S(2dn) \tag{230}$$

$$=\alpha\frac{k_{+F}^2-k_{-F}^2}{4\pi}(2dn) \tag{231}$$

using the volumes of the Fermi seas. We see that dE has precisely the form of the kinetic-energy difference of particles of mass m_p given by

$$\frac{1}{m_p}=\frac{\alpha}{\pi}. \tag{232}$$

Thus m_p is essentially the static susceptibility, which happens to have dimensions of mass in $d=2$. The statement that m_p has no r dependence in the gapped case or no spin dependence in the gapless case is the same as saying that the full nonlinear susceptibility does not depend on the spin S, which in turn means that $E(S)$ is quadratic in S.

Note that the free-field form of dE comes from $E\simeq S^2$ *and* $d=2$: in $d=3$, we would have $dE/dn\simeq S\simeq(k_{+F}^3-k_{-F}^3)$, which no one would interpret as a difference of kinetic energies.

Let us see how this general argument applies to the specific example we have been working on.

Consider the Hartree-Fock energies quoted earlier:

$$\frac{\mathcal{E}(k_{\pm F})}{(e^2/\varepsilon l)}=a\left(\frac{k_{\pm F}}{k_F}\right)^2+b\left(\frac{k_{\pm F}}{k_F}\right)^4. \tag{233}$$

The quartic terms miraculously drop out in the energy cost of transferring a particle from the top of the spin-down sea to the top of the spin-up sea:

$$\frac{dE}{(e^2/\varepsilon l)}=a\frac{k_{+F}^2-k_{-F}^2}{k_F^2}+b\frac{k_{+F}^4-k_{-F}^4}{k_F^4} \tag{234}$$

$$=a\frac{k_{+F}^2-k_{-F}^2}{k_F^2}+b\frac{(k_{+F}^2-k_{-F}^2)(k_{+F}^2+k_{-F}^2)}{k_F^4} \tag{235}$$

$$=\frac{(a+b)}{k_F^2}(k_{+F}^2-k_{-F}^2) \tag{236}$$

using

$$k_{+F}^2+k_{-F}^2=k_F^2. \tag{237}$$

Note how $d=2$ was essential to this argument: in $d=3$ we would have $k_{+F}^3+k_{-F}^3=k_F^3$.

Thus the k^4 terms in $\mathcal{E}(k_\pm)$ are not the cause of the S^4 term. However, a k^6 term in $\mathcal{E}(k_\pm)$ can be shown to produce an S^4 term in $E(S)$.

Thus the apparent free-field behavior is tied to the smallness of terms of order k^6 and higher. To understand why the k^6 term is so small, we turn to Eq. (154) for H_0. Expanding the $\sin^2$ in a series, we find that the k^6 term is down by a factor of at least 15 (50) relative to the k^2 term, at $\lambda=0$ ($\lambda=1$), all the way up to $k=k_F$. Presumably this feature (and its counterpart in the gapped case) persists in the Hartree-Fock approximation to H and keeps $E(S)$ essentially quadratic, which in turn mimics free-field behavior.

The reader is referred the original work (Shankar, 2001) for a proof that $E(S)=E(0)+(\alpha/2)S^2$ implies that $\Delta(r)$ will be r independent in the gapped case as well.

Composite fermions are not free fermions but are like Landau quasiparticles in a Fermi liquid.[40] These objects, too, are labeled by free-particle quantum numbers and are long-lived. They do have fairly strong interactions: the dimensionless Landau parameters that describe these interactions are not small and produce effects like

[40]This was already suggested by Halperin, Lee, and Read (1993). Also in this context, see the work of Mandal and Jain (2001a).

zero sound. They are adiabatically connected to free fermions in zero field just as the composite fermions are adiabatically connected to free fermions in a reduced field B^*.

D. Effective potentials for experimental systems with disorder

In comparing theory to experimental results one cannot neglect disorder. Our theory ignores disorder and our results are completely determined by the electron-electron interaction. Here we ask if it is possible that a Zhang–Das Sarma potential with some effective λ could describe a dirty system. First of all, we realize that this could not be true with respect to all observables, if it were true for any. For example, if one were considering conductance, one would know the electron in a disordered potential would typically get localized, whereas no Zhang–Das Sarma interaction would predict this. As for transport gaps, the present-day samples, with a disorder broadening of the same order as the gaps, again preclude this possibility. Magnetic transitions, however, are controlled by total energies and one might expect that disorder would have a rather innocuous effect and could be represented in an average way by some translationally invariant interaction. We raise this issue because, in several magnetic phenomena to be described shortly, it appears that a single λ characterizes a sample. Specifically, λ extracted from one data point can be used to explain the rest of the data from that sample. If the other data points differ only in the temperature T, the same λ is used. If they differ in B or n or ν, the following scaling argument applies (Ando, Fowler, and Stern, 1982): In a heterojunction, the donors of density n produce a confining linear potential of slope that goes as n. If one considers a variational wave function of the Fang-Howard (1966) form $\psi(z)=A(w)z\exp(-z/w)$ in the transverse direction, then the optimal $\overline{w}$ (to which Λ, the well width, must be proportional) varies as $\overline{w}\simeq n^{-1/3}$. Consequently the dimensionless width, $\lambda=\Lambda/l$, varies as

$$\lambda\simeq n^{-1/3}B^{1/2}\simeq B^{1/6}\nu^{-1/3}\simeq n^{1/6}\nu^{-1/2}. \tag{238}$$

Arguments can be given (Shankar, 2001) for why in certain limiting conditions, not realized in today's experiments, an effective potential exists. The question of why this works in realistic situations far from this limit remains unanswered.

With this preamble, let us turn to comparison with experimental results. Kukushkin, von Klitzing, and Eberl (1999) vary both n and B and drive the system through various transitions at $T=0$ (by extrapolation). The field B is always perpendicular to the sample. We compare the Hamiltonian theory to these experiments by calculating the critical fields at which the $\nu=\frac{1}{2}$ and $\nu=\frac{1}{4}$ systems saturate ($P=1$) and the gapped fractions undergo transitions from one quantized value of P to the next.

Let us recall that, as far as these transitions go, the systems behave like free fermions of mass m_p, which is independent of the index r that labels how many Landau levels have reversed their spins in the gapped case or the size of the up and down Fermi circles in the gapless cases.

TABLE V. Critical fields based on a fit at $\frac{3}{7}$. The rows are ordered by the last column, which measures density.

ν	Comment	B^c (expt)	B^c (theor)	νB^c (expt)
$\frac{4}{9}$	(3,1)→(2,2)	2.7 T	1.6 T	1.2
$\frac{2}{5}$	(2,0)→(1,1)	3 T	2.65 T	1.2
$\frac{1}{4}$	saturation	5.2 T	4.4 T	1.3
$\frac{3}{7}$	(3,0)→(2,1)	4.5 T	4.5 T	1.93
$\frac{4}{9}$	(4,0)→(3,1)	5.9 T	5.9 T	2.62
$\frac{1}{2}$	saturation	9.3 T	11.8 T	4.65

We consider B^c's at which the systems at $\frac{1}{4}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{1}{2}$ lose full polarization ($r=0$ for gapped cases, saturation for the gapless cases) and, for $\frac{4}{9}$, also the $r=1$ transition, $|\mathbf{3},\mathbf{1}\rangle\rightarrow|\mathbf{2},\mathbf{2}\rangle$.

An experimental complication needs to be addressed first. Each of these transitions seems to take place via a narrow intermediate step (Kukushkin, von Klitzing, and Eberl, 1999) with a polarization halfway between those allowed by composite-fermion theory based on spatially homogeneous states. We use the center of these narrow steps as the transition points for comparison to the present theory. The physics of these intermediate steps will be addressed in Sec. X.

In accordance with our strategy we find λ by fitting the theory to the experiment for a particular transition. We use this value [or for samples with changing field and density, Eq. (238)] to predict B^c for other transitions using Eq. (220). We obtain $\lambda_{3/7}=1.42$ from the transition $|\mathbf{3},\mathbf{0}\rangle\rightarrow|\mathbf{2},\mathbf{1}\rangle$ at $B^c=4.5$ T.

For the gapless cases, there are two equivalent approaches. First, at the critical field the Fermi energy of the up-spins equals the Zeeman energy of the down-spins:

$$g\left[\frac{eB^c}{2m_e}\right]=\frac{k_F^2}{2m_p}=\frac{2\pi n}{m_p}=\frac{eB\nu}{m_p}. \tag{239}$$

Equivalently we can write for the total ground-state energy density $E^Z(S)$ (where the superscript indicates that the Zeeman energy is included)

$$E^Z(S)=\frac{\alpha}{2}S^2-g\frac{e}{2m_e}\frac{B_\perp S}{\cos\theta}, \tag{240}$$

where $\alpha=\pi/m_p$. This expression is minimized (for $P\leqslant 1$) to give P. Setting $P=1$ gives the critical fields.

The comparison to experiment is made in Table V. Note that in rows above $\frac{3}{7}$, where we fit λ, the predicted B^c's are lower than the observed values, i.e., the actual λ's are less than what Eq. (238) gives, and in the rows below $\frac{3}{7}$, the predicted B^c's are higher than the values observed. This is consistent with the expectation that interactions will increase the effective thickness with increased density. If we fit to the $\frac{2}{5}$ point, we obtain similar numbers, with the agreement worsening as we move off

in density from $\frac{2}{5}$. Thus $\frac{3}{7}$ was chosen as the fitting point, since its density was somewhere in the middle of all the densities considered.

IX. PHYSICS AT NONZERO TEMPERATURES $T>0$

So far we have seen that the extended Hamiltonian theory may be used to compute quantities such as gaps, particle-hole profiles, critical fields for magnetic transitions, and so on to 10–20 % accuracy. All such quantities have been readily computed using trial wave functions, giving numbers that are superior to ours. Our main emphasis has been to expose the underlying physics as transparently as possible and to resolve questions such as why composite fermions behave like free particles on some occasions.

We turn now to physics at finite T (Murthy, 2000c; Shankar, 2000, 2001), where the Hamiltonian method has few rivals. Exact diagonalization (Chakraborty and Pietiläinen, 1996; Chakraborty, Niemelä, and Pietiläinen, 1998) is limited to very small systems, and trial wave functions typically cover the ground state and very-low-energy excitations. The Hamiltonian approach is able to yield the temperature dependence of polarization P and the relaxation rate $1/T_1$ for the gapless states in the thermodynamic limit. We shall see that, if λ is treated as before (fit to one data point per sample), it is possible to give a very satisfactory account of experiments in gapless systems up to about 1 K, which is of the order of the Fermi energy (Shankar, 2000, 2001).

We shall then address finite-temperature polarization in gapped states, which is complicated by a nonzero spontaneous polarization and the attendant spin waves. It turns out to be essential to take the finite-T behavior of these spin waves into account. Once this is done, the theoretical predictions are in excellent agreement with experiment up to several degrees Kelvin (Murthy, 2000c).

We start with the gapless case since it is simpler.

A. Polarization and relaxation in gapless states

The polarization P is computed as follows. First we compute the Hartree-Fock energy of a particle including the Zeeman energy, which is the self-consistent solution to

$$\mathcal{E}^Z_\pm(k)=\mp\frac{1}{2}g\left[\frac{eB}{2m}\right]+2\int\frac{d^2q}{4\pi^2}\check{v}(q)\sin^2\left[\frac{\mathbf{k}\times\mathbf{q}l^2}{2}\right]-4\int\frac{d^2k'}{4\pi^2}n^F_\pm(|k'|)\check{v}(|\mathbf{k}-\mathbf{k}'|)\times\sin^2\left[\frac{\mathbf{k}'\times\mathbf{k}l^2}{2}\right],$$

where the superscript on $\mathcal{E}^Z_\pm$ reminds us that it is the total energy including the Zeeman part, and the Fermi functions

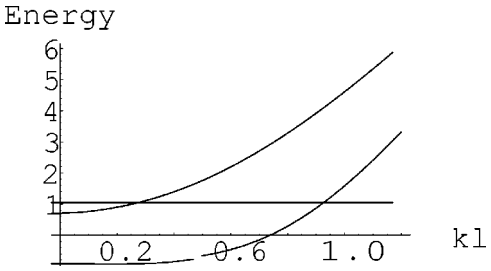

FIG. 9. Hartree-Fock energies at $\nu=1/2$ for up- and down-spins (upper and lower curves) at $T=0.3$ K for $B=5.52$ T and no tilt. Note that they are not simply quadratic in the momenta and that at the chemical potential, indicated by the horizontal line, the two graphs have very different slopes, that is, different densities of states.

$$n^F_\pm(|k|)=\frac{1}{\exp[(\mathcal{E}^Z_\pm(k)-\mu)/kT]+1}\tag{241}$$

depend on the energies $\mathcal{E}^Z_\pm(k)$ and the chemical potential μ. Figure 9 shows some typical results. At each T, one must choose a μ and solve for $\mathcal{E}^Z_\pm(k)$ until a self-consistent answer with the right total particle density n is obtained. From this, one may obtain the polarization by taking the difference of up and down densities. As usual we use the Zhang–Das Sarma potential, for which

$$\check{v}(q)=e^{-q^2l^2/2}\frac{2\pi e^2e^{-ql\lambda}}{q}.\tag{242}$$

The computation of the longitudinal nuclear relaxation rate $1/T_1$ is more involved (Shankar, 2000, 2001). The fermions are in a quantum well, with their density varying across the width, so the nuclear relaxation rate will be a function of position. Consider a nucleus at the center of the quantum well (as well as the $x-y$ plane) where the density is the largest. Let us call this point the origin and let $1/T_1$ be the relaxation rate here. The theory predicts

$$\frac{1}{T_1}=4\pi k_BT\left(\frac{K^{\max}_\nu}{n}\right)^2\times\int_{E_0}^\infty dE\left(\frac{dn^F(E)}{dE}\right)\rho_+(E)\rho_-(E)F(k_+,k_-),\tag{243}$$

$$F=e^{-(k_+^2+k_-^2)l^2/2}I_0(k_+k_-l^2),\tag{244}$$

$$\rho_\pm(E)=\int\frac{kdk}{2\pi}\delta[E-\mathcal{E}^Z_\pm(k)],\tag{245}$$

where $k_\pm$ are solutions to $\mathcal{E}^Z_\pm(k_\pm)=E$, I_0 is the Bessel function, E_0 is the lowest possible energy for up-spin fermions, and $K^{\max}_\nu$ is the measured maximum Knight shift (at the center of the sample) for the fraction $\nu=\frac{1}{2}$ or $\frac{1}{4}$.

Here is a rough description of the derivation, the details of which may be found in Shankar (2000). Suppose for a moment we were dealing with electrons and not composite fermions. The Knight shift at the chosen

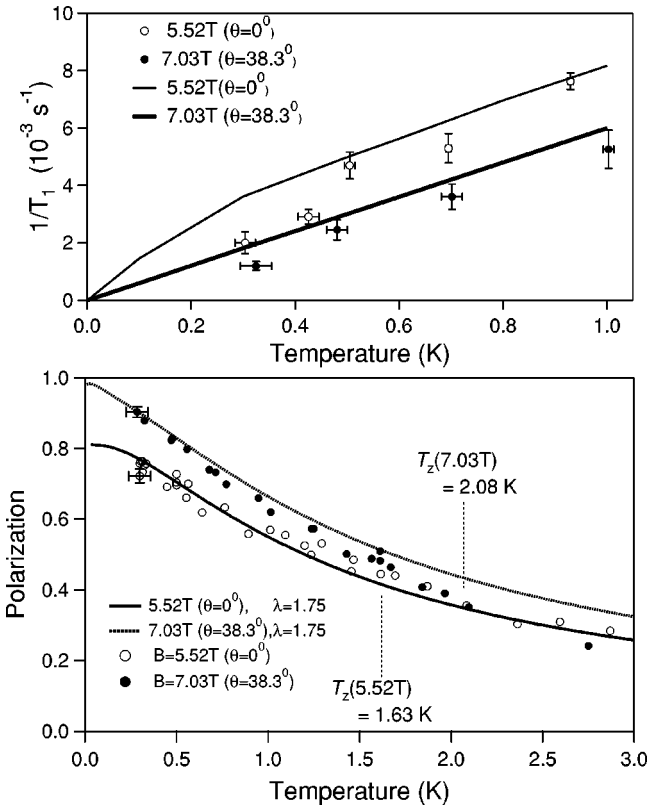

FIG. 10. Comparison of the predictions of the Hamiltonian theory for the NMR relaxation rate and polarization to the work of Dementyev *et al.* (1999). The value of λ used in the theory is fit to P at 300 mK, $B_\perp = 5.52$ T, and used for other T and $B_\perp$. Notice the correlation between the curvature of $1/T_1$ and the limit of P as $T \to 0$.

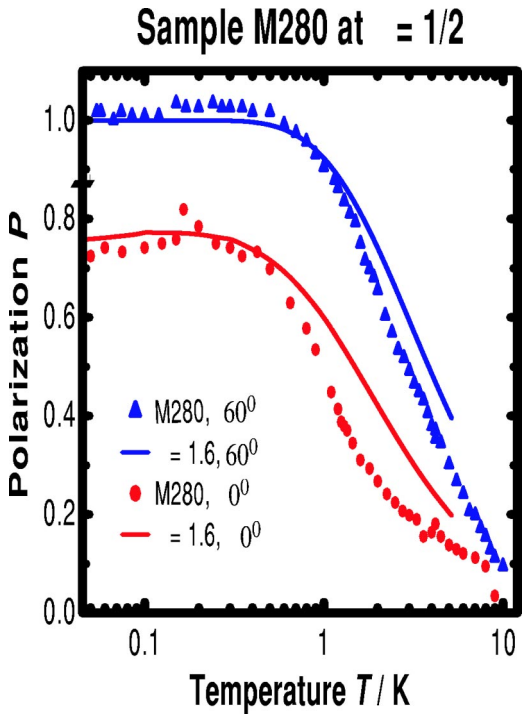

FIG. 11. (Color in online edition) Comparison of the predictions of the Hamiltonian theory with the data of Melinte *et al.* (2000) for two values of the tilt angle $\theta = 0°$ and $\theta = 60°$. The value of λ used in the theory is fit to P at 60 mK and $B_\perp = 7.1$ T.

point, the origin, will be determined by the spin density there. The same parameter enters the $1/T_1$ calculation quadratically. This is why $K_\nu^{\max}$ enters the answer. Thus $K_\nu^{\max}$ is not calculated *ab initio* but is taken from the same experiment. The density of states and Fermi factor are standard. The only new feature here is the presence of $F(k_+, k_-)$, which reflects the fact that the spin density has to be projected into the lowest Landau level when going to the composite-fermion basis. The effect of this factor (which is none other than the $e^{-q^2 l^2/2}$ that appeared on the projected charge density) is to suppress processes with momenta much larger than $1/l$, as these have no place within the lowest Landau level.

We now compare to some experiments at $\nu = \frac{1}{2}$ and $T > 0$. Consider first Dementyev *et al.* (1999). From their data point $P = 0.75$ for $B = B_\perp = 5.52$ T at 300 mK we deduce

$$\lambda = 1.75. \tag{246}$$

We have once again chosen to see to what extent a sole parameter λ can describe P and $1/T_1$ for the given sample at a given $B_\perp$, but at various temperatures and tilts.

Since there does not exist a model, including disorder, that describes how λ should vary with tilt, we include no such variation.

Dementyev *et al.* (1999) find $K_{1/3}^{\max} = 4.856 \times 10^{-7}$ K, which is believed to describe a saturated system at $P = 1$. They estimate that $K_{1/2}^{\max} = 0.953 K_{1/3}^{\max}$, which is what we need here. Given this information, $1/T_1$ follows.

The top and bottom halves of Fig. 10 compare the Hartree-Fock calculation of $1/T_1$ and P, respectively, to the data. The graphs for $1/T_1$ differ slightly from those in Shankar (2000), since the present calculation treats the spin of the composite fermion more carefully. The $1/T_1$ graph at 5.52 T appears a little jagged, since it was computed at just six points, which were then connected. This is not apparent in the tilted case, since the points lie on a straight line.

Dementyev *et al.* (1999) had pointed out that a two-parameter fit (using a mass m and interaction J) led to disjoint sets of values (in the $m - J$ plane) for these four curves. Given that H is neither free nor of the standard form $[p^2/2m + V(x)]$, this is to be expected. By contrast, a single λ is able to describe the data here rather well, since H has the right functional form. Given how the theory fits the polarization data up to the Fermi energy of $\simeq 1$ K, it is clear that changing the data point used to fix λ will be inconsequential.

The present work establishes a phenomenological, nontrivial, and nonobvious fact that a single λ parameter, determined from one data point, can describe both P and $1/T_1$ for the given sample under a variety of conditions. That the fitted λ is larger than the local-density-approximation value makes sense, as both disorder and Landau-level mixing will lower the gap and raise λ.

Consider next sample M280 of Melinte *et al.* (2000), which had $P=0.76$ at 0.06 K and $B=B_\perp=7.1$ T, from which we deduced $\lambda=1.6$. Figure 11 compares the theoretical T dependence of $P(T)$ with data. There is a factor-of-2 difference between theory and experiment for $1/T_1$ (not shown).

The fair agreement for the tilted cases is unexpected in both the Dementyev *et al.* (1999) and Melinte *et al.* (2000) data. First of all, orbital effects have to be considered due to the tilt. The thickness parameter Λ can be affected by it. As pointed out by Jungwirth in a private communication, once there is an in-plane component of B, the problem is no longer rotationally invariant. This means that our states are no longer Hartree-Fock states and get scattered into each other by the potential. No attempt was made here to take into account all the effects of the tilt. Instead we included just the increased Zeeman coupling and hoped for the best.

For the benefit of others who measure $1/T_1$ at $\nu=1/2$ in the future on similar samples, we give some very approximate formulas (to be used for zero or small tilts). From Fig. 10 we note that, in general, the graphs of $1/T_1$ become linear and parallel for temperatures above 0.3 K. In this region we can write

$$\frac{d(1/T_1)}{dT}\simeq 3\left[\frac{\overline{K}}{\overline{n}}\right]^2\times 10^{-3}\ \mathrm{s}^{-1}\,\mathrm{K}^{-1}\quad \text{for } T>0.3\ \mathrm{K}, \tag{247}$$

with $\overline{K}$ the Knight shift in KHz and

$$\overline{n}=\frac{n}{10^{10}/\mathrm{cm}^2}. \tag{248}$$

[In this approximate formula, we ignore the λ dependence of Eqs. (243)–(245) and the distinction between the average and maximum Knight shift.]

The graphs do not generally obey the Korringa-like law because as $T\to 0$ they are sublinear or superlinear for saturated or unsaturated cases, respectively. Only the critical case with $P(0)\to 1$ as $T\to 0$ is linear. For $T>0.3$ K (which in general must be replaced by either the energy gap or energy overlap between the up and down Fermi energies) we have the approximate result

$$\frac{1}{T_1}\simeq[3T\ \mathrm{K}+C]\left[\frac{\overline{K}}{\overline{n}}\right]^2\times 10^{-3}\ \mathrm{s}^{-1}, \tag{249}$$

$$C=0\quad \text{(critical)} \tag{250}$$

$$\geqq 0\quad \text{(unsaturated)} \tag{251}$$

$$\leqq 0\quad \text{(saturated)}. \tag{252}$$

For the critical case (only), we have a Korringa law

$$\frac{1}{T_1 T\ \mathrm{K}}\simeq[3]\left[\frac{\overline{K}}{\overline{n}}\right]^2\times 10^{-3}\ \mathrm{s}^{-1}\,\mathrm{K}^{-1}. \tag{253}$$

For Dementyev *et al.* (1999) $C\simeq 1$. This value may be used as a first approximation. For more accurate results one must solve Eqs. (243)–(245).

B. Polarization in gapped states

Let us now turn to the theoretical description of the finite-temperature polarization of the $\nu=1/3$ state, detailed measurements of which have recently been carried out (Khandelwal *et al.*, 1998; Melinte *et al.*, 2000; Freytag, 2001). Theoretical details can be found in Murthy (2000c).

While the composite-fermion Hartree-Fock approximation provides an adequate description of $\nu=1/2$ it is qualitatively incorrect for a spontaneously polarized state like $\nu=1/3$, because it underestimates the effects of the excitations that destroy order. For example, it predicts a nonzero spontaneous polarization at $E_Z=g\mu B_{\mathrm{tot}}=0$ and $T>0$, in violation of the Hohenberg-Mermin-Wagner theorem (Mermin and Wagner, 1966; Hohenberg, 1967) that forbids spontaneous breaking of a continuous symmetry in two dimensions, except at $T=0$. The $T=0$ spontaneous polarization is driven by the fact that fermions of the same spin will avoid each other due to the Pauli principle and thus have a lower interaction energy, just as in the $\nu=1$ spontaneous quantum ferromagnet,[41] which has been extensively studied theoretically[42] and experimentally.[43]

Thus we must begin by understanding the disordering mechanism and seeking a proper description of it. Consider a fully polarized $\nu=1/3$ state at $T=0$ with all the fermions in the composite-fermion LLL state. As the system is heated, some fermions will go to the $n=1$ composite-fermion Landau level with spin-up (which does not change the polarization) and some will go to spin-down $n=0$ which reduces the polarization and costs an energy per spin-flip of Δ_{SR}, the *spin-reversed gap*.

This description completely misses the spin waves, which are related to the particle-hole excitations as follows. Just as in the case of the magnetoexciton (Fig. 4), where no spin was flipped, a spin-flip particle-hole excitation has an energy that varies with q.[44] As $q\to\infty$, the dispersion settles down at Δ_{SR}, the energy to create a widely separated pair. However, as $q\to 0$ the pair energy vanishes, for $E_Z=0$, by Goldstone's theorem. For nonzero E_Z the long-wavelength limit for the spin-wave energy is E_Z according to Larmor's theorem. These are the modes to reckon with, for they are very low in energy and plentiful at low temperatures.

The simplest way to describe these spin waves (including their self-interaction) is the *continuum quantum ferromagnet* (Read and Sachdev, 1996). This model assumes that all high-energy modes (at the electronic and

[41]The classic study of nontrivial interaction effects (beyond ferromagnetism) at $\nu=1$ is that of Sondhi *et al.* (1993).

[42]For theoretical work, see Kopietz and Chakravarty (1989); Kasner and MacDonald (1996); Read and Sachdev (1996); Haussmann (1997); Timm *et al.* (1998); Kasner, Palacios, and MacDonald (2000).

[43]Experimental studies include those of Barrett *et al.* (1995); Tycko *et al.* (1995); Aifer, Goldberg, and Broido (1996); Manfra *et al.* (1996).

[44]For an example of this calculation, see Murthy (1999).

composite-fermion cyclotronic scales) have been integrated out. The only modes left are slow and long-wavelength fluctuations of the spin polarization, which have the action

$$S=\int d^dx\int_0^{1/T}d\tau\Big(iM_0\mathbf{A}(\mathbf{n})\cdot\nabla_\tau\mathbf{n}+\frac{\rho_s}{2}(\nabla_x\mathbf{n})^2 - M_0\mathbf{H}\cdot\mathbf{n}+\cdots\Big), \tag{254}$$

where M_0 is the magnetization density, $\mathbf{n}$ is a local vector of unit length pointing in the direction of the magnetization, $\mathbf{A}(\mathbf{n})$ is the field that implements the Berry's phase needed to obtain the correct quantum commutation relations between the spin components, ρ_s is the spin stiffness, and $\mathbf{H}=g^*\mu_B\mathbf{B}$ is the Zeeman field ($|\mathbf{H}|=E_Z$).

This model is still nontrivial (because of the condition $|\mathbf{n}|^2=1$). However, it can be solved (Read and Sachdev, 1996) in the limit when $N\to\infty$, where N is the number of components of $\mathbf{n}$. The limit $N\to\infty$ appears to describe the actual case of $N=3$ in the case $\nu=1$.

Our strategy, then, will be to deduce reasonable values for M_0, the magnetization density, and ρ_s, the spin stiffness, and then plug them into the known results from the large-N limit for the magnetization as a function of T, given by

$$P(T)=M_0\Phi_M(r,h), \tag{255}$$

where $r=\rho_s/T$ and $h=E_Z/T$ are scaling variables, and Φ_M is a known scaling function (Read and Sachdev, 1996).[45]

To find the values of the parameters M_0 and ρ_s corresponding to the $\nu=1/3$ state, we shall use some Hartree-Fock results. Since the underlying fermionic theory responds to temperature by self-consistently modifying occupations and energies, we expect to obtain temperature-dependent values[46] $M_0(T)$ and $\rho_s(T)$.

First consider the spin stiffness $\rho_s(T)$. At a given temperature T the self-consistent occupations $N_{F,GS}(\sigma,n)$ and energies $\epsilon(\sigma,n)$ in the ground state are computed using the procedure described in the gapless case. Now one creates a twisted spin state and computes the Hartree-Fock energy of the twisted ground state (Murthy, 2000c), and thence the excess energy to order q^2. Comparing to the energy cost of a twist in the continuum quantum ferromagnet, which is $(\rho_s/2)L^2q^2$, one finds the spin stiffness

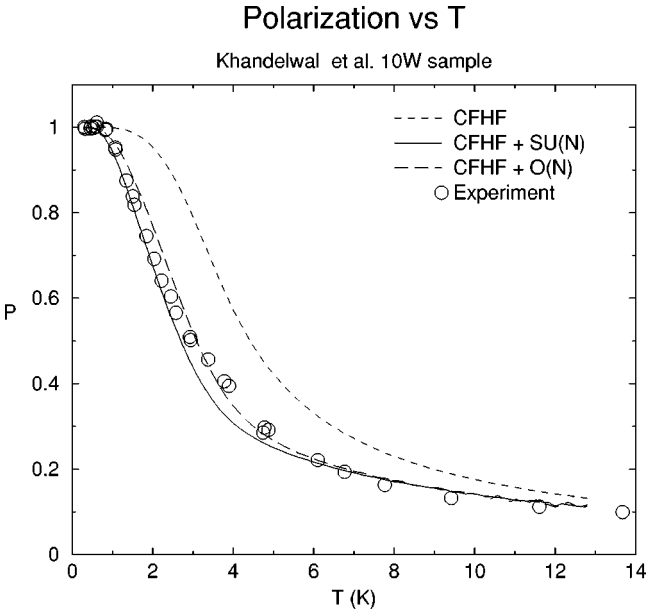

FIG. 12. Comparison of the predictions of the Hamiltonian theory for the temperature-dependent polarization $P(T)$ to the Khandelwal *et al.* (1998) data of 10W sample. The thickness parameter has been set to $\lambda=1.5$ and other parameters appropriate to the 10W sample have been used. Note the important role played by the spin waves in bringing the Hartree-Fock value of P down to excellent agreement with experiment.

$$\rho_s=\frac{1}{16\pi}\int\frac{d^2s}{(2\pi)^2}v(s)\sum_{n_1,n_2}|\rho_{n_1n_2}(s)|^2 \times[N_F(\uparrow,n_1)-N_F(\downarrow,n_1)] \times[N_F(\uparrow,n_2)-N_F(\downarrow,n_2)], \tag{256}$$

where L^2 is the area of the system, and $\tilde{\rho}_{n_1n_2}$ is the matrix element of Eq. (D1). The above should be regarded as an estimate for the twist rather than a rigorous calculation (even in Hartree-Fock), since ideally one should compute the free-energy cost of a twist, rather than just the internal energy cost, as we have done.

To find $M_0(T)$ we need a more devious approach (Murthy, 2000c) based on Eq. (255). We already know how to compute the composite-fermion Hartree-Fock magnetization $P_{HF}(T)$. We set

$$P_{HF}(T)=M_0(T)\Phi_M(r=0,h=\Delta_{SR}/T) \tag{257}$$

and justify it as follows. In the composite-fermion Hartree-Fock theory the particles and holes are treated as independent, or noninteracting, with a gap equal to Δ_{SR} independent of the distance between them. This corresponds to a collective mode dispersion that is completely flat, $\omega(q)=\Delta_{SR}$. The continuum quantum ferromagnet description that corresponds most closely to the composite-fermion Hartree-Fock theory is the one that has the same spin-flip excitation spectrum, namely, one with no spin stiffness ($r=0$) and an effective Zeeman field $E_Z^{\text{eff}}=\Delta_{SR}$.

Armed with $M_0(T)$ and $\rho_s(T)$, we evaluate the large-N scaling functions and hence $P(T)$. The predictions

[45]There are actually two different large-N approximations corresponding to the fact that the symmetry group can be viewed as an example of O(N) with $N=3$, or as an example of SU(N) with $N=2$. Both are considered by Read and Sachdev (1996) and scaling functions are given.

[46]This should be contrasted with the case $\nu=1$ in which, due to the huge exchange gap, particle-hole excitations are frozen at all temperatures of interest and the parameters of the continuum quantum ferromagnet are T independent.

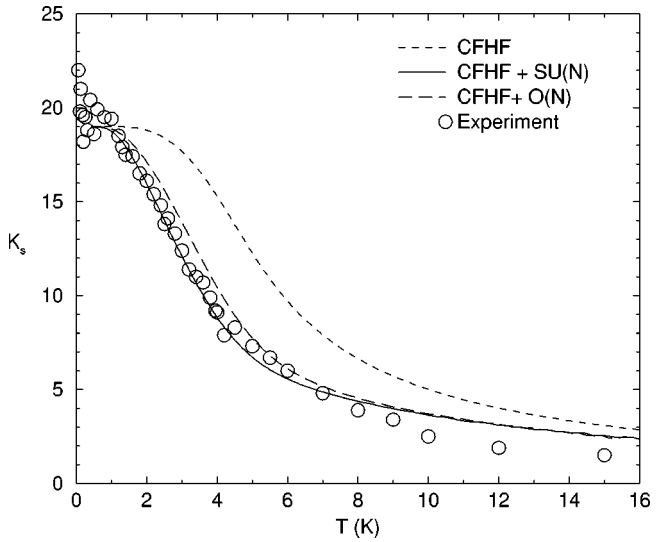

FIG. 13. Comparison of the predictions of the Hamiltonian theory for the temperature-dependent polarization $P(T)$ to the Melinte *et al.* (2000) data of M242 sample. The thickness parameter has been set to $\lambda = 1.5$ and other parameters appropriate to the M242 sample have been used. Once again spin waves are seen to be important. CFHF = composite-fermion Hartree-Fock.

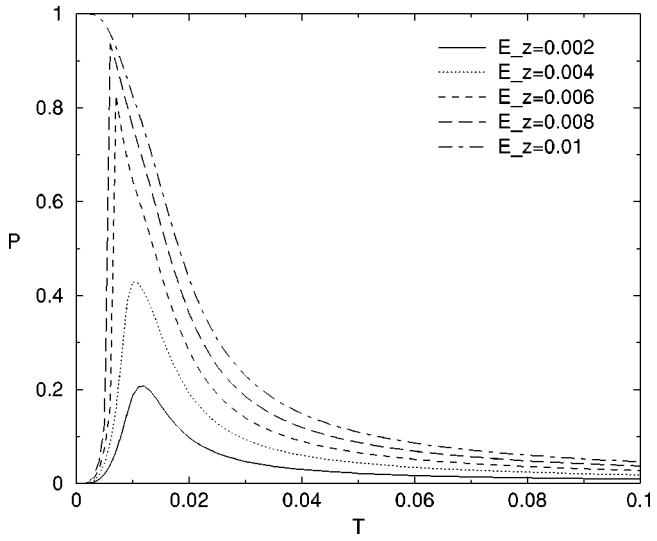

FIG. 14. Theoretical predictions for the temperature-dependent polarization $P(T)$ for $\nu = 2/5$ for various values of E_Z. Both T and E_Z are in units of the interaction energy $e^2/\varepsilon l$. Note the transition to a fully polarized state at around $E_Z = 0.01$.

are shown in Fig. 12 for parameters corresponding to the experiment of Khandelwal *et al.* (1998). As can be seen, composite-fermion Hartree-Fock theory overestimates the spin magnetization considerably at low temperatures, but the inclusion of interacting spin waves gives a prediction in almost perfect agreement with the data. The results are quite insensitive to λ, and a typical value of $\lambda = 1.5$ was used. The results are also insensitive to which of the two large-N approximations is used. Similar good agreement over a wide range of temperatures is found in the comparison to the Melinte *et al.* (2000) data of M242 sample, shown in Fig. 13. Here we have plotted the Knight shift vs the temperature. The Knight shift at very low temperature shows considerable scatter, and an appropriate intermediate value has been used to fit to the theory.

It is somewhat surprising that the theory agrees so well with the data up to temperatures much higher than those for which agreement was found for the gapless fractions. We expect the theory to work only when composite-fermions are well defined, which should be true up to temperatures of the order of the 1/3 gap. In fact, the agreement persists to about 10 K, which is higher than a typical activation gap for 1/3 in such samples. For more details see Murthy (2000c).

The same considerations can be applied to the $\nu = 2/5$ state. Since it is unpolarized for $E_Z = 0$ we can stop at the composite-fermion Hartree-Fock stage. Figure 14 shows the $P(T)$ curves for $\nu = 2/5$ for $\lambda = 1.5l$ for a range of Zeeman couplings. The rise and fall of $P(T)$ at small E_Z can be understood as follows. At $T = 0$, the system is in a singlet state with the up- and down-spins occupying the composite-fermion lowest Landau level. As the system is heated, some particles go to the next Landau level. Here the ones with spin parallel to the external field are preferred due to the Zeeman term and hence a nonzero polarization develops. At higher T, the two spin species start to get occupied with nearly equal probabilities and P starts to decline. There is a transition to the fully polarized state around $E_Z = 0.01E_C$.

One may compute $P(T)$ for arbitrary fractions by exact diagonalization (keeping all the excited states) and subsequent calculation of thermodynamic quantities (Chakraborty and Pietiläinen, 1996). Due to computational limitations, this method is restricted to fairly small systems. For example, the largest system studied by Chakraborty and Pietiläinen (1996) for $\nu = 1/3$ has five electrons and for $\nu = 2/5$ has four electrons. Allowing for this, the results seem fairly consistent with ours. Experimental data are not currently available for comparison.

Finally, one can use the Chem-Simons theory to do finite T calculations, using m^* as a free parameter instead of λ.

X. INHOMOGENEOUS STATES OF COMPOSITE FERMIONS

We have repeatedly emphasized that composite fermions are interacting particles. The uniform liquid states studied so far were not contingent on composite-fermion interactions—such states could exist even for noninteracting composite fermions. We now turn to some interesting inhomogeneous states, which would not exist if the composite fermions were free, such as the high-field Wigner crystal and possibly the partial polarized states seen in the gapped fractions by Kukushkin, von Klitzing, and Eberl (1999).

A. The high-field Wigner crystal

The clearest instance of an inhomogeneous state in the fractional quantum Hall regime is the high-field Wigner crystal (Yoshioka and Fukuyama, 1979). If we start with a free-electron gas and slowly crank up the interactions or lower the density, the emphasis goes from kinetic to potential energy. In the Wigner crystal the electrons seek an arrangement aimed at minimizing the potential energy. In the Hall case, due the quenching of kinetic energy, the Wigner crystal seems even more likely at low densities. It was initially proposed (Fukuyama and Platzman, 1982) as a possible explanation of the fractional quantum Hall effect. However, it was soon realized (Yoshioka and Lee, 1983) that this state is very different from the fractional quantum Hall states, because both its longitudinal and its Hall conductances vanish at zero temperature and when the crystal is pinned by disorder (Yoshioka, 1983; MacDonald, 1984), leading to an insulating state. Further, the Wigner crystal state is not tied to any particular commensurate filling.

There are now experimental observations[47] that support the existence of the Wigner crystal near $\nu = 1/5$. In fact, experiments see a reentrant transition (Jiang *et al.*, 1990, 1991) in which there is a putative Wigner crystal both above and below 1/5. To see how this might happen, recall that the incompressibility of the Laughlin state results in a downward cusp in the ground-state energy as a function of filling factor in the neighborhood of (say) 1/5. Thus one can easily imagine that in a small neighborhood of 1/5 the Laughlin liquid has a lower energy than the Wigner crystal, leading to the reentrant Wigner crystal near 1/5.

Theoretical work on the Wigner crystal,[48] based on a study of trial wave functions and collective excitations, has established that the Laughlin state becomes unstable around $\nu \approx 1/6$ to a density wave even in the absence of disorder. In the wave-function approach, Hartree-Fock and (weakly) correlated wave functions have also been written down and their energy evaluated. By studying the excitonic instabilities of the Laughlin liquid, Jain and Kamilla (1998) showed that it becomes unstable to crystallization around $\nu = 1/9$.

Let us begin with one of the simplest Hartree-Fock wave functions for the crystal, that of Maki and Zotos (1983):

$$\Psi_{HF}(\{\mathbf{r}_i\}) = \mathcal{A}\prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i), \tag{258}$$

where $\mathcal{A}$ is the antisymmetrization operator and $\phi_{\mathbf{R}_i}$ is a single-particle wave function that is localized at $\mathbf{R}_i$ (lattice site) and belongs to the lowest Landau level. It is given by

$$\phi_{\mathbf{R}_i}(\mathbf{r}) = e^{-|\mathbf{r}-\mathbf{R}_i|^2/4l_0^2 - i\mathbf{r}\times\mathbf{R}_i\cdot\hat{z}/2l_0^2}. \tag{259}$$

The wave function (258) can be improved by adding a correlation factor corresponding to including small fluctuations around the Hartree-Fock state. The energy of this state becomes lower than that of the liquid state at about the experimentally right filling fraction ($\nu \approx \frac{1}{7}$; Maki and Zotos, 1983; Lam and Girvin, 1984; Levesque, Weiss, and MacDonald, 1984).

Experiments (Jiang *et al.*, 1990, 1991) show transport gaps two orders of magnitude smaller than the theoretical estimate as calculated using the Hartree-Fock approximation (Fukuyama, Platzman, and Anderson, 1979; Yoshioka and Fukuyama, 1979). The measurements of the Hall resistivity ρ_{xy} are surprising as well (Goldys *et al.*, 1992; Goldman *et al.*, 1993). As mentioned above, the Wigner crystal is expected to have a vanishing Hall conductance $\sigma_{xy} = 0$ (when pinned), which implies a vanishing Hall resistance $\rho_{xy} = 0$. However, the experiments see "Hall insulating" behavior (Zhang, Kivelson, and Lee, 1992), that is, $\rho_{xy} \approx 2\pi\hbar/\nu e^2$. These problems led Yi and Fertig (1998) to consider crystalline states with correlation zeros that keep electrons apart. Each electron is combined with $2s$ vortices to obtain the trial wave function

$$\Psi(\{\mathbf{r}_i\}) = \mathcal{A}\prod_{i\neq j}(z_i - z_j)^{2s}\prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i). \tag{260}$$

They also find that near 1/5 the best energies are obtained by attaching four zeros to each electron. In other words, they take the Laughlin FQHE wave function at $\frac{1}{5}$, which is a product of a Jastrow factor with quartic zeros and a single filled composite-fermion Landau level χ_1, and replace χ_1 with a crystal. The Coulomb energy for this wave function is computed using Monte Carlo methods. Yi and Fertig (1998) have shown that the ground-state energy of the correlated Wigner crystal is lower than that of the usual Wigner crystal at experimentally relevant filling fractions. Moreover, by introducing Laughlin-Jastrow correlations between the interstitials and the lattice electrons, one can explain the experimentally observed ρ_{xy} (Zheng and Fertig, 1994a, 1994b). Unfortunately, the method becomes too computationally demanding to allow one to calculate other quantities of interest, such as the excitation spectrum.

This is a situation tailor-made for the extended Hamiltonian theory. Before we describe the theory and its results, let us note that there are two features of the experiment that hint that a composite-fermion Wigner crystal is involved. The first is the nonmonotonic behavior of the gap near $\frac{1}{5}$, which shows that the $\frac{1}{5}$ Laughlin liquid correlations are being felt in the nearby Wigner crystal state. Second, the threshold electric field beyond which the Wigner crystal becomes depinned and starts sliding increases near $\frac{1}{5}$. Below we shall see its connection with the structure and properties of the composite-fermion Wigner crystal.

Now let us see how to set up the extended Hamiltonian theory. Since attaching the zeros to electrons converts them into composite fermions, we are naturally led

[47] See, for example, Jiang *et al.* (1990, 1991); Engel *et al.* (1992); Goldys *et al.* (1992); Goldman *et al.* (1993). For a recent review of the experimental situation, see Shayegan (1997).

[48] See Maki and Zotos (1983); Lam and Girvin (1984); Levesque, Weiss, and MacDonald (1984); Price, Platzman, and He (1993). For a recent review of the theory, see Fertig (1997).

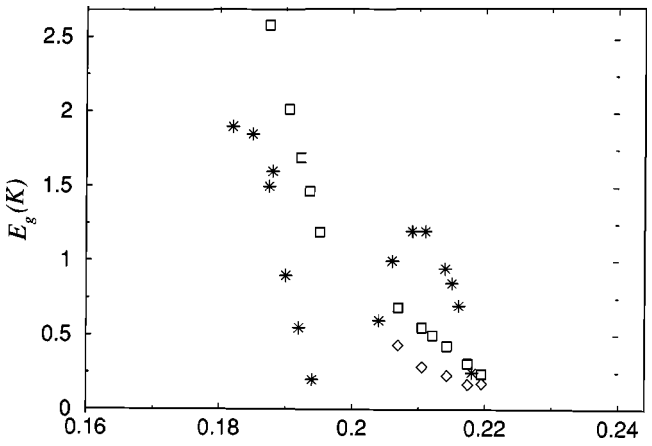

FIG. 15. Experimental (stars, from Jiang *et al.*, 1991) and theoretical (squares and diamonds, from Narevich, Murthy, and Fertig, 2001) gaps vs ν for the Wigner crystal. Theoretical gaps are for a composite-fermion Wigner crystal where the composite fermion has four vortices attached. Squares are the gaps for a triangular lattice, while the diamonds are for an oblique lattice. A disorder-driven change of shape just above 1/5 could explain the nonmonotonicity of the data.

to consider a Wigner crystal of composite fermions. Here is how one proceeds (details can be found in Narevich, Murthy, and Fertig, 2001). The crystal is characterized by density-wave order parameters at the reciprocal-lattice vectors $\mathbf{G}$:

$$\Delta_{nn'}(\mathbf{G}) = \frac{2\pi(l^*)^2}{L^2} \sum_X e^{-iG_x X} \times \langle d^\dagger_{n,X-G_y l^{*2}/2} d_{n',X+G_y l^{*2}/2} \rangle. \quad (261)$$

First, one has to assume a particular lattice structure (shape and size). An important parameter of the lattice is the number of quanta of effective flux that penetrate each unit cell.[49] Things become simple when this number is rational, of the form p/q, where p and q have no common factors. In this case each composite-fermion Landau level breaks up into p subbands with the total number of states in the original composite-fermion Landau level being equally divided among the subbands (Yoshioka and Fukuyama, 1979). While the original composite-fermion Landau level had a sharp energy, the subbands have a nonzero energy dispersion, which can be found from the Hartree-Fock Hamiltonian. One finally closes the circle by demanding self-consistency; the ground state formed by filling up the subbands with the correct number of particles should reproduce the assumed form of Eq. (261). Once one has a self-consistent Hartree-Fock solution, the transport gap is found as the energy difference between the centers of the highest occupied subband and the lowest unoccupied subband. The Wigner crystal is characterized by one particle per unit cell, and, depending on the filling, this translates to different p and q.

Here are the results: The gaps calculated from the Hamiltonian theory are within a factor of 2 of the experimental results (Fig. 15), which is to be contrasted to the more-than-two-orders-of-magnitude disagreement with electronic Hartree-Fock calculations. The theory also predicts that the gap below 1/5 should be substantially more than the gap between 1/5 and 2/9 (the next quantum Hall liquid state). Narevich, Murthy, and Fertig (2001) are not able to go very close to 1/5 because then p and q become large and the problem becomes computationally prohibitive.

Two other results emerge at a qualitative level. First, the shear modulus becomes very small as $\frac{1}{5}$ is approached, which means that the crystal becomes very soft to deformations. The reason is that as $\frac{1}{5}$ approaches the Wigner crystal tends more and more to a Laughlin liquid. The density-wave order parameter decreases, and the shape of the crystal matters less and less. This is connected to the depinning threshold of the Wigner crystal to electric fields and can be understood as follows (Fukuyama and Lee, 1978; Blatter *et al.*, 1994): When the crystal is stiff it is not able to take advantage of all the minima in the disorder potential, and so is not strongly pinned. However, when the crystal becomes soft, it does not cost a lot of energy to deform and take advantage of local minima of the disorder, and the crystal becomes more strongly pinned. Disorder may also have a dominant role in the behavior of the gap near $\frac{1}{5}$. Since the crystal is soft, its actual shape is determined by the local disorder. Different shapes give gaps varying by about a factor of 2.

The second qualitative result is that the density inhomogeneity in the composite-fermion Wigner-crystal state is quite small in absolute terms (about 20% of the background density) for all ν, not just near $\frac{1}{5}$. This is at odds with the conventional view that electrons are localized in a Wigner crystal. The correlation zeros seem to prefer a more homogeneous state.[50] The electronic Wigner crystal has to partially melt and become more homogeneous to accommodate Laughlin-Jastrow correlations. This has interesting similarities with earlier ideas concerning cooperative ring exchanges in a Wigner crystal and the melting of the Wigner crystal (Kivelson *et al.*, 1986a, 1986b).

B. Partially polarized Hall crystal states

Another class of candidates for an inhomogeneous state of composite fermions stems from the observation of Kukushkin, von Klitzing, and Eberl (1999), who measured the spin polarization of various principal fractions as a function of E_Z, the Zeeman energy, and found polarizations not expected in the composite-fermion theory with uniform states. As an example, consider ν

[49]Since the composite fermions see effective flux rather than external flux, this point is crucial.

[50]It is easy to see that correlation zeros become meaningless if electrons are strictly localized; then they do not come close enough for the zeros to be operative.

$=\frac{2}{5}$. There are two filled composite-fermion Landau levels, and at small E_Z we expect that the state is unpolarized with the $n=0,\uparrow$ and $n=0,\downarrow$ Landau levels occupied and the rest empty. At large E_Z we expect a fully polarized state with $n=0,\uparrow$ and $n=1,\uparrow$ occupied. If these are the only states we are allowed to think about, then at some intermediate E_Z there should be a first-order transition. Such transitions between $P=0$ and $P=1$ were extensively discussed earlier.

However, Kukushkin, von Klitzing, and Eberl (1999) observe a plateau at half the maximum polarization in a narrow region of E_Z for $\frac{2}{5}$. Analogous plateaus are observed at other principal fractions.[51] Other puzzling phenomena near this transition are that of hysteresis over a range of fillings near 2/5 (Cho *et al.*, 1998) and very slow dynamics (Eom *et al.*, 2000). One of us (Murthy, 2000a) has proposed that these observations might be explained by considering partially polarized Hall crystals of composite fermions.[52]

Let us first review some terminology. A state that shows the co-existence of density-wave and quantum Hall order (a quantized σ_{xy}) is called a *Hall crystal*. Such states were implicit in the ring-exchange theory (Kivelson *et al.*, 1986a, 1986b) but were first postulated explicitly by Halperin, Tesanovic, and Axel (1986) and analyzed in great detail with explicit examples by Tesanovic, Axel, and Halperin (1989).[53] A Wigner crystal is an example of a Hall crystal, as is the lattice of quasiparticles formed at a filling $\nu=\frac{1}{3}+\delta$, provided the density wave is pinned. The reason the partially polarized state at 2/5 has to be a Hall crystal (if it is a crystal at all) is that no changes in the quantized Hall conductance are observed as E_Z is varied.

Now let us understand the nature of the proposed state (Murthy, 2000a). Imagine that composite fermions were really free. Then as E_Z is raised from 0 the system remains in a singlet state until the $n=0,\downarrow$ Landau level (whose energy is increasing) crosses the $n=1,\uparrow$ Landau level (whose energy is decreasing) at a special Zeeman energy E_Z^*. This is when the system makes the first-order transition. Now let us "turn on" interactions between composite fermions, which we know are present in the full theory. At E_Z^* there are twice the number of degenerate states (from the $n=0,\downarrow$ and $n=1,\uparrow$ composite-fermion Landau levels) as there are composite-fermions at the Fermi level. The true many-body ground state will therefore be picked by the interactions. This state should be stable in a small range of E_Z near E_Z^* determined by the strength of the composite-fermion interactions. One possibility is a charge/spin-density wave state.

As we saw in the previous subsection, when the system self-consistently chooses a particular density-wave order with p/q quanta of effective flux per unit cell, each composite-fermion Landau level splits up into p subbands of equal degeneracy (Yoshioka and Fukuyama, 1979). The partially polarized 2/5 state has two quanta of effective flux penetrating every unit cell, which means $p=2$, $q=1$, and each composite-fermion Landau level is split into two subbands. The proposed state corresponds to occupying three subbands of $\uparrow$ composite fermions and one subband of $\downarrow$ composite fermions. Once again a composite-fermion Hartree-Fock calculation allows us to calculate the gap. The theoretical prediction for the range in E_Z over which the state should be stable is consistent with the experiment (Murthy, 2000a).

This possibility can be investigated in more detail in the integer quantum Hall analog of $\nu=2/5$, which is $\nu=2$. There it can be shown (Murthy, 2000b) that there are indeed charge/spin-density-wave states whose energy (in Hartree-Fock) is less than either the singlet or the fully polarized liquid states. It can also be shown that these states are Hall crystals with a nonzero quantized Hall conductance (Murthy, 2000b).

However, there are other possibilities for the partially polarized state at $\nu=2/5$. For example, Apalkov *et al.* (2001) have proposed that the composite fermions belonging to the $n=0,\downarrow$ and $n=1,\uparrow$ composite-fermion Landau levels form a Halperin (111) state (Halperin, 1983), which is a liquid. Their proposal is based on a finite-size calculation in a spherical geometry where only the composite fermions belonging to the two "active" levels were kept, and their interaction was approximated as being a two-body interaction. While their state remains a contender, Apalkov *et al.* (2001) wrongly conclude, by neglecting some terms in the ground-state energy (see Murthy, 2001c), that the ground-state energy of the charge/spin-density-wave state would be higher than the Halperin (111) analog state.

At the moment it is fair to say that the situation is not resolved. There are some key differences between the two proposals that could be used to distinguish them experimentally. For example, in the liquid state of Apalkov *et al.* (2001), one would expect no gapless excitations whatsoever and no spatial variation in the spin density, and thus the NMR Knight shift. On the other hand, there are generically gapless density excitations in the partially polarized Hall crystal state (Tesanovic, Axel, and Halperin, 1989; Murthy, 2000b), and one would expect (at very low temperatures when the crystal is frozen) periodic spatial variation in the Knight shift, with the maximum Knight shift being the same as in the

[51]Very recently, polarizations not allowed by translationally invariant composite-fermion states have been seen at $\frac{2}{3}$ by NMR techniques as well; see Freytag *et al.* (2001).

[52]An alternate explanation is provided by Mariani *et al.* (2002).

[53]The topological numbers associated with these states had been noticed before by a number of authors: Wannier (1978); Johnson and Moser (1982); Thouless *et al.* (1982); MacDonald (1983); Thouless (1983); Avron and Seiler (1985); Avron and Yaffe (1986); Dana, Avron, and Zak (1985); Kunz (1986). However, Tesanovic, Axel, and Halperin (1989) were the first to give a concrete example in which nontrivial topological integers were realized and to make an important connection between the topological integers and gapless collective excitations. For an extension of the Hall crystal concept to the fractional quantum Hall regime, see Kol and Read (1993).

fully polarized state. A complete understanding of these partially polarized states will hinge on future experiments.

XI. CRITICAL REVIEW OF EHT AND ITS APPROXIMATE SOLUTIONS

The extended Hamiltonian theory (EHT) was arrived at by starting with the electronic Hamiltonian, introducing a new (pseudovortex) coordinate $\mathbf{R}_v$, combining it with the electronic guiding center $\mathbf{R}_e$ to form the composite-fermion variables $\boldsymbol{\eta}$ and $\mathbf{R}$. A (weak) constraint $\bar{\tilde{\chi}}(\mathbf{q})\simeq 0$ completed the picture. For $\nu=1/2$ the formalism coincides with that of Pasquier and Haldane (1998).

As an exact restatement of the original electronic problem, the EHT is as good as any, with no loss of information, including that pertaining to higher Landau levels. Its main claim to superiority over other formulations is its amenability to Hartree-Fock and attendant approximations, because the problem is now expressed in terms of composite-fermion variables ($\boldsymbol{\eta}$ and $\mathbf{R}$), which admit a natural Hartree-Fock ground state that fills an integral number of Landau levels.

We discussed two approximation schemes, both of which rely on a nondegenerate Hartree-Fock state but differ in how the constraints are handled. Since the separation of the LLL physics (and thus the limit $m\to 0$) was straightforward in this formalism, we focused on this limit and the lowest Landau level.

In a *conserving approximation*, one finds approximate Green's functions that respect the constraints. The time-dependent Hartree-Fock (TDHF) approximation is conserving. It was employed (Murthy, 2001a) to show that in the lowest Landau levels, $S(q)\simeq q^4$ for gapped states, in compliance with Kohn's theorem. For $\nu=1/2$ it was used by Read (1998) to show that the state is compressible and the fermion effective mass has a logarithmic divergence at the Fermi surface, because the overdamped mode of Halperin, Lee, and Read (1993) responsible for all this is generated by summing ladder diagrams. Thus there is no conflict between the Halperin-Lee-Read results based on Chern-Simons fermions (which have unit charge) and those based on composite fermions with charge e^* (which vanishes for $\nu=1/2$.) However, results that are easily obtained in the Chern-Simons approach are obtained only following a careful, gauge-invariant calculation in the EHT. It may well be that without the Chern-Simons results, the compressibility at $\nu=1/2$ would have been overlooked in a description that succeeded in displaying dipolar expressions for the electron charge operator. While the primary use of the conserving approximation thus far has been to demonstrate that correct results in the infrared, long-wavelength limit require a proper implementation of constraints, it seems to capture numerically some of the short-distance physics, as well, as evidenced by the magnetoexciton dispersions for 1/3 and 2/5 (Murthy, 2001a, Sec. VI.B).

In the other approximation, *the shortcut*, one uses the preferred charge $\bar{\rho}^p(\mathbf{q})=\bar{\rho}(\mathbf{q})-c^2\bar{\tilde{\chi}}(\mathbf{q})$ in place of $\bar{\rho}(\mathbf{q})$ on the grounds that in an exact calculation such a substitution is permitted. This single substitution manages to accomplish at tree level some of the work of the constraints: $e\to e^*$, $d\to d^*$, and $S(q)\simeq q^2\to S(q)\simeq q^4$. The constraints are then ignored, so that their effects at higher orders in ql are not built in and may or may not be described well by the simple Hartree-Fock calculation that follows. Thus, although the EHT itself is exact, the small parameter ql is needed in the shortcut because of the way we handled the constraints. Note that it is ql and not ql^* that has to be small, as demonstrated by the success of the approach near and at $\nu=1/2$ where $l^*\to\infty$.

Also missing in the shortcut are the overdamped mode and its attendant consequences. However, in problems with a gap or $T>0$ this does not seem to matter. On this basis one expects it to work in problems in which the potential favors small ql and the problem is still described by composite fermions.[54]

These expectations were borne out in the computation of gaps, magnetic transitions, and T dependence of polarization and nuclear-spin relaxation, which were compared to results from exact diagonalization, trial wave functions, and experiment. The very special form of the Hamiltonian that comes out of this theory does what *ad hoc* Hamiltonians parametrized by standard kinetic and interaction terms cannot: It explains a host of phenomena in any given sample (polarization, relaxation) with a single parameter describing electron-electron interactions. We are not aware of any other quantitative, analytical approximation scheme for computing these quantities that is not plagued by singularities as $m\to 0$.

The preferred charge description also resolves many qualitative issues: In what sense does an electron bind to its correlation hole (represented here by pseudovortices) to form the composite fermion? How can composite fermions appear to be free in some magnetic phenomena when they are surely not free? What does the dipole picture really mean? How is an effective mass generated from the interactions alone?

On the down side, the preferred charge as a way to implement constraint is peculiar to the FQHE and is unprecedented. We do not know *a priori* how well this procedure will work and how to systematically improve it. We have no internal signal that the composite-fermion description is failing as the interaction is varied (say, by sending $\lambda\to\infty$ in the Zhang–Das Sarma interaction) or by mapping $\nu>1$ problems into effective Landau-level problems with modified interactions, where Pfaffian states and striped states might be the answer.

XII. SUMMARY AND OPEN QUESTIONS

The goal of the Hamiltonian theory was to start with the Hamiltonian for two-dimensional interacting elec-

[54] If one lets the "thickness" parameter of the Zhang–Das Sarma potential become very large, composite fermions will likely become unstable, and the state will not be describable in the extended Hamiltonian theory.

trons in a magnetic field and arrive at a comprehensive description of fractional quantum Hall phenomena by a sequence of transformations to the final quasiparticles. The final theory was expected to encode and display the wisdom inherited from the study of trial wave functions, make the various pictures precise, and permit the computation of all physical quantities (to some accuracy) by the use of standard approximation techniques such as Hartree-Fock.

Thus we first surveyed the wave-function approach to see what it could teach us and then described the two Hamiltonian approaches: the Chern-Simons approach and our extension of it, the extended Hamiltonian theory.

The wave functions taught us the following. The FQHE ground states are described by incompressible fluids. The wave functions for the ground states and quasiparticle (or quasihole) excitations are independent of the mass m (in the $m\to 0$ limit) and are built entirely of LLL wave functions. In the Laughlin fractions, the elementary excitation is a vortex of charge $-1/(2s+1)$ in electronic units. When an extra electron is introduced into the system, it is screened by $2s$ such vortices, leading to a composite fermion of charge $e^*=1/(2s+1)$. These vortices sit on the electron. In the Jain fractions the wave functions are obtained by projection to the lowest Landau level. Prior to projection one can see the composite fermion as made of an electron and $2s$ vortices sitting on them. After projection, many zeros are annihilated; typically they move off the electrons and are no longer organized into vortices. This holds for $\nu=1/2$ as well, as is clear when $\nu=1/2$ is seen as a limit of nearby Jain fractions. This means that the naive dipole picture based on zeros of the wave functions is not tenable. Remarkably, despite the dissolution of vortices on projection, the composite fermion still has the charge of an electron and $2s$ vortices. The proper interpretation of this requires the Hamiltonian approach, in particular the EHT, which in turn is an offshoot of Chern-Simons theory.

We emphasize that our aim was not simply to get yet another exact reformulation of the problem, but to find one that lends itself to approximations. For example, the problem in terms of electrons,

$$H=\sum_j \frac{\boldsymbol{\eta}_{ei}^2}{2ml^4}+\frac{1}{2}\sum_{i,j,\mathbf{q}} v(q)e^{i\mathbf{q}\cdot(\mathbf{r}_{ei}-\mathbf{r}_{ej})}\equiv H_0+V, \tag{262}$$

contains all the answers in principle, but is flawed in practice because the LLL degeneracy of H_0 precludes the use of Hartree-Fock or perturbation theory.

The degeneracy of the noninteracting problem is overcome by the flux attachment transformation of Chern-Simons theory. Lack of space prevented us from discussing the very first application of this idea to the Laughlin fractions by Zhang, Hansson, and Kivelson (1989). They converted the problem to one of composite bosons in zero average field, which led to a detailed analogy to superfluidity. This review focused on composite fermions, turning to the work of Lopez and Fradkin (1991, 1992, 1993), who implemented Jain's idea in operator form by attaching $2s$ flux quanta to electrons and converting them to Chern-Simons fermions described by

$$H_{CS}=\sum_i \frac{(\Pi+:\mathbf{a}_{cs}:)_i^2}{2m}+V, \tag{263}$$

$$\mathbf{\Pi}=\mathbf{p}+e\mathbf{A}^* \quad \left(\mathbf{A}^*=\frac{\mathbf{A}}{2ps+1}\right), \tag{264}$$

$$\nabla\times:\mathbf{a}_{cs}:=4\pi s:\rho:. \tag{265}$$

At mean field one now has a unique vacuum of p-filled Landau levels of composite fermion upon neglecting V and $:\mathbf{a}_{cs}:$. Their effects could be included perturbatively. In particular, in an RPA formulation one could include both of these and obtain the cyclotron mode at the right frequency, and in the Laughlin fraction obtain the wave function in the long-distance limit (Lopez and Fradkin, 1991, 1992, 1993).

However, two problems remained: A singular limit as m, the bare mass of the electron, vanished, and a quasiparticle charge of unity (at tree level) instead of e^*, owing to the fact that flux tubes and not vortices were attached. These features made quantitative predictions difficult.

We mentioned Kalmeyer and Zhang (1992), who considered $\nu=1/2$ and pointed out that impurity scattering would be stronger than naively expected, since any charge inhomogeneity led to additional gauge flux (by the Chern-Simons condition), which produced strong scattering.

We then moved to the Halperin, Lee, and Read (1993) treatment of $\nu\simeq 1/2$, which was extensive and aimed at confronting theory with experiment at a quantitative level. The success of the composite-fermion theory in the region of small or zero gap was quite unexpected. Halperin, Lee, and Read not only highlighted the fact that at $\nu=1/2$ the system saw zero gauge field on average and therefore had a Fermi surface, they also argued that the best way to think of the region near $\nu=1/2$ was in terms of composite fermions seeing an effective field $B^*=B-B_{1/2}$. This meant the particles would bend with a radius $R^*=\hbar k_F/eB^*$ with $k_F=\sqrt{4\pi n}$, a result that was verified experimentally by Kang *et al.* (1993), Goldman, Su, and Jain (1994), and Smet *et al.* (1996).

Halperin, Lee, and Read (1993) computed the electromagnetic response within the RPA. They showed that the system was as compressible as a traditional Fermi liquid. It had a longitudinal conductivity $\sigma_{xx}=(e^2/8\pi)\times(q/k_F)$. This was tested by the damping and velocity shift of surface acoustic waves by Willett *et al.* (1990, 1996), who also found that away from $\nu=1/2$ there was a resonance in the velocity shift when the wavelength of the surface acoustic wave coincided with $2R^*$.

Halperin, Lee, and Read (1993) identified an overdamped mode in the density-density correlation with dispersion $\omega\simeq iq^3v(q)$. This mode, responsible for the compressibility of the system, also produces a (logarithmic) divergence in the composite-fermion mass by enter-

ing its self-energy. This in turn leads to a gap $E \simeq 1/(p \ln p)$ for $\nu = p/(2p+1)$ as $p \to \infty$. It has been shown that the mass divergences do not affect bosonic (e.g., density-density) correlations.[55]

These successes notwithstanding, the Halperin-Lee-Read approach had some room for improvement. It did not allow a clear separation of the LLL physics, i.e., it did not have a smooth $m \to 0$ limit. The Chern-Simons fermion, obtained by attaching two flux tubes to electrons, had a charge of unity (and not $e^* = 0$). It did not make contact with the dipole picture (Read, 1994, 1996) based on a wave-function analysis.

The extended Hamiltonian theory was devised to address some of these issues. The key idea is that in order to discuss the correlation hole that accompanies the electron to form the composite fermion in tractable form, one must enlarge the Hilbert space to describe collective charge degrees of freedom and to place a suitable number of constraints. This idea, along with expressions for the new coordinates and a certain change of variables, was arrived at via a somewhat tortuous route (Shankar and Murthy, 1997; Murthy and Shankar, 1998a) from electrons to composite fermions. In this review, we have spared the reader details of the historical route and given an axiomatic description (Murthy and Shankar, 2002) of the EHT.

Here we begin with the electronic Hamiltonian Eq. (262), and add for each electron a new pseudovortex guiding-center coordinate $\mathbf{R}_v$ (whose algebra corresponds to charge $-c^2$). Thus the Hilbert space becomes larger and $\mathbf{R}_v$ has no dynamics.

If we are interested in physics at the cyclotron scale, we can focus on the kinetic term. The mass m appears here as it should, in ω_0. The collective coordinate formed out of $\boldsymbol{\eta}_e$ carries the entire Hall current. (This answers the question of what carries the Hall current at $\nu = 1/2$ when $e^* = 0$.) In fact the η_e collectively carry the Hall current even at gapped fractions where $e^* \neq 0$, thus substantiating the general belief that the Hall current is not affected by disorder because it is carried by collective coordinates. Having identified them as our oscillators (for which we have a specific Hamiltonian), we have paved the way for a detailed analysis of the Hall response. We also showed how to extract an effective LLL theory by approximately integrating out the coordinate $\boldsymbol{\eta}_e$ in our discussion of Landau level mixing.

Let us now proceed to drop higher Landau levels and work with

$$\bar{\bar{H}}(\bar{\bar{\rho}}) = V = \frac{1}{2} \sum \bar{\bar{\rho}}(q) v(q) e^{-(ql)^2/2} \bar{\bar{\rho}}(q), \tag{266}$$

$$\bar{\bar{\rho}}(\mathbf{q}) = \sum_j \exp\left[-i\mathbf{q} \cdot \left(\mathbf{r}_j - \frac{\mathbf{z} \times \boldsymbol{\Pi}_j}{1+c} \right) \right] \equiv \sum_j \exp[-i\mathbf{q} \cdot \mathbf{R}_{ej}]. \tag{267}$$

The densities $\bar{\bar{\rho}}(\mathbf{q})$ obey the magnetic translation algebra, since $\mathbf{R}_e$ is the electron guiding-center coordinate in the composite fermion basis. The advantage of this basis is that the velocity operator $\boldsymbol{\Pi}$ sees a weaker field B^*, leading to a nondegenerate Hartree-Fock ground state.

The Hamiltonian commutes with the operators

$$\bar{\bar{\chi}}(\mathbf{q}) = \sum_j \exp\left[-i\mathbf{q} \cdot \left(\mathbf{r}_j + \frac{\mathbf{z} \times \boldsymbol{\Pi}_j}{c(1+c)} \right) \right] \equiv \sum_j \exp[-i\mathbf{q} \cdot \mathbf{R}_{vj}]. \tag{268}$$

The pseudovortex densities $\bar{\bar{\chi}}(\mathbf{q})$ thus form a closed algebra, the symmetry algebra of H.

In the first derivation of the theory (Shankar, 1999), the following constraint naturally emerged:

$$\bar{\bar{\chi}}(\mathbf{q}) |\text{physical state}\rangle = 0. \tag{269}$$

In the extended approach (Murthy and Shankar, 2002), $\mathbf{R}_v$ is a cyclic coordinate with no dynamics. One is free to supplement the theory with the above constraint, since nothing physical depends on the auxiliary coordinate $\mathbf{R}_v$. We made this choice so that there would be just one set of equations to deal with.

We described two ways to proceed, the choice being dictated by what we want to calculate: the conserving approximation and the shortcut using the preferred charge.

For situations in which the symmetries of H are important one uses the conserving approximation, in which the constraint is respected at the level of Green's functions. This amounts to ensuring gauge invariance. We reviewed the compressibility paradox in which gauge invariance made all the difference. The paradox concerned the system at $\nu = 1/2$, which Halperin and Stern (1998) argued must be compressible [as had been predicted by Halperin, Lee, and Read (1993)]. This result seemed to be at odds with our operator description in which the charge was dipolar. Halperin and Stern gave heuristic arguments for how a system of dipoles could still be compressible if their Hamiltonian had K invariance—invariance of the energy under the shift of all momenta, a symmetry first noted by Haldane, and which appeared in our work as part of a gauge symmetry.

The more detailed analysis of Stern *et al.* (1999) drove the point home. They considered first a model in which the flux tubes were spread over a distance $1/Q$ and the Hamiltonian had K invariance to sufficiently high order in Q to examine the question of compressibility. The density-density correlation (now of the dipole-dipole form) was computed in the RPA (which was exact in the limit $Q \to 0$) and the factors of q^2 in the numerators from the dipoles were canceled by the exchange of the overdamped mode. They then showed that in the actual FQHE problem, K invariance could be built in if the Landau parameter $F_1 = -1$. The conserving approximation was also employed by Read (1998) within the Pasquier-Haldane (1998) formalism for $\nu = 1$ bosons (to which our theory reduces if we set $c = 1$ or $\nu = 1/2$).

[55] Kim, Furusaki, *et al.* (1994); Kim, Lee, *et al.* (1994); Kim, Lee, and Wen (1995); Stern and Halperin (1995).

Read (1998) established that the fermions interacted with each other as dipoles by exchanging a transverse collective mode, but that, as $q \to 0$, the propagator for this mode produced enough negative powers of q to overturn the q^2 coming from the dipolar factors from the two ends. These calculations were important in showing that if constraints were taken into account, the correct charge of the composite fermion would emerge and that despite the weaknesses in the wave-function-based arguments leading to the dipole picture (see Sec. I.D.3), the predicted dipole moment is correct in the above sense. Murthy (2001a) was able to show that in the gapped fractions one could obtain structure factors in accord with Kohn's (1961) theorem ($\bar{\bar{S}} \simeq q^4$) in a conserving calculation. He also showed how to compute magnetoexciton dispersions.

In all other situations [when a gap or temperature or both suppress the deep-infrared region $\omega \simeq q^3 v(q)$] the shortcut using $\bar{\bar{\rho}}^p(\mathbf{q})$ was the weapon of choice. Here we make the replacement $\bar{\bar{\rho}} \to \bar{\bar{\rho}}^p = \bar{\bar{\rho}} - c^2 \bar{\bar{\chi}}$, which is allowed in the exact theory. This choice allows us to employ naive Hartree-Fock calculations that respect Kohn's theorem ($\bar{\bar{S}} \simeq q^4$). As a bonus the composite-fermion charge and dipole moment emerge in the power-series expansion of $\bar{\bar{\rho}}^p$. It is only in this sense that the composite fermion can be viewed as the union of an electron and a correlation hole of charge $-c^2$. With these features built in at tree level, the usual approximations such as Hartree-Fock are applicable as long as the large-ql region is avoided by the potential.

The EHT gives a uniform and precise description of the internal structure of the composite fermion. Whereas in the wave-function-based description, the Laughlin fractions allowed for a simple picture of the composite fermion (an electron bound to $2s$ vortices) and the rest of the Jain series (upon projection) did not, in the EHT the composite fermion is viewed as an electron plus a pseudovortex. Especially interesting is the case of $\nu = 1/2$. The expansion for $\bar{\bar{\rho}}^p$ in a power series shows that it begins with a term that couples to an external electric field exactly as a dipole moment of strength $l^2 \mathbf{z} \times \mathbf{p}$ would. This formula, in operator form, makes no reference to zeros of the wave function or vortices, neither of which is robust. It does not have the problems of the wave-function-based dipole picture. These problems arose upon antisymmetrization, which we carry out by expressing $\bar{\bar{\rho}}^p$ in second-quantized form in terms of fermionic operators. We saw that the place to look for dipoles is not in the wave function but in correlation functions at high frequencies at and above the composite-fermion Fermi energy.

The operator approach gives a concrete realization of one of the primary expectations in FQHE: Once the kinetic energy is quenched by restricting electrons to the lowest Landau level, it will be resurrected by interactions, and this low-energy problem will be characterized by one common scale, the electron-electron interaction. Equation (61), which gives H as a quadratic function of $\bar{\bar{\rho}}$ (or $\bar{\bar{\rho}}^p$ if we use the preferred combination), embodies all of these expectations.

The Hartree-Fock approximation to $H(\bar{\bar{\rho}}^p)$ was used to compute transport gaps (Murthy and Shankar, 1999; Shankar, 2001) good to within 10–20 % for potentials that vanish rapidly with ql. It could be used to explore magnetic transitions from one quantized value of magnetization to the next (Shankar, 2000, 2001). In the absence of disorder it is clear that all physical quantities pertaining to the FQHE (restricted to the lowest Landau level) are functionals of the potential $v(q)$. For the Zhang–Das Sarma potential which we use for most of our work, this means a function of λ. We saw that, given a value of λ from experiment, we did not need several masses for several phenomena; they all came from one potential (Shankar, 2000, 2001). This is because the Hamiltonian for the composite fermions is rather unusual[56] and contains the kinetic and potential terms in a monolithic form. Hidden in it are the various mass scales m_a and m_p appropriate to various phenomena (activation, polarization), all functionals of the interaction $v(q)$.

The operator approach also clarified the question of whether or not composite fermions are free. Given the prominent variations in the magnetoexciton dispersions and the fact that it takes two very different masses to describe polarization and activation, it is clear that they are not. But why do they appear to be free for some magnetic phenomena? Our theory (Shankar, 2000, 2001) shows that it is an accident coming from rotational invariance and $d=2$.

The EHT allows us to compute physical quantities at $T>0$, such as polarization and relaxation rates for gapless states (Shankar, 2000, 2001). In the experiments of Dementyev *et al.* (1999), a λ determined from one polarization data point gives the polarization and relaxation curves for two tilts and a range of temperatures. This is to be contrasted with attempts to fit the data with a mass and interaction pair (m,J), where the four curves require four nonoverlapping values of these pairs. In the case of Melinte *et al.* (2000) and Freytag (2001), the $1/T_1$ predictions (which vary over orders of magnitude) are off by a factor of 2, but the polarization data are well described by a single λ. Once again the success can be traced back to the fact that the composite-fermion Hamiltonian is of a nonstandard form, parametrized by a λ, which in turn determines all the mass and energy scales relevant to each given process.

When calculating the polarization of gapped states, it turns out to be essential to take into account spin waves for spontaneously polarized cases. This is done (Murthy, 2000c) by mapping the low-energy dynamics of the problem onto the continuum quantum ferromagnet treated in the large-N approximation (Read and Sachdev, 1996), the parameters of which are extracted from the Hartree-Fock treatment of the problem in the Hamiltonian theory. The results (Murthy, 2000c) agree extremely well with experiment (Khandelwal *et al.*, 1998; Melinte *et al.*, 2000) up to very high temperatures.

[56]Consider, for example, Eq. (154) for $\nu = \frac{1}{2}$.

This calculation, like all the others in the Hamiltonian theory, is in the thermodynamic limit and free from finite-size effects.

The EHT allows us to calculate the gaps of the Wigner crystal in terms of composite fermions (Narevich, Murthy, and Fertig, 2001). These gaps are off by a factor of 2 when compared to experiment (Jiang *et al.*, 1990, 1991). This should be contrasted to previous approaches in which the gaps were off by two orders of magnitude. This approach also allows us to consider inhomogeneous states (Murthy, 2000a) with polarizations not allowed by the composite-fermion theory of homogeneous ground states.

Hopefully we have succeeded in establishing that the Hamiltonian theory of the FQHE is a comprehensive scheme for addressing and answering a variety of qualitative and quantitative questions, for gapped and gapless states, at zero and nonzero temperatures. We have given the reader a taste of what can be done using this formalism. While many things have been clarified, there are many open problems to which we believe this approach may be fruitfully applied.

One outstanding open problem is that of computing transport coefficients in the quantum Hall regime from a microscopic theory. By identifying the collective mode that carries the Hall current, we have set the stage for a study of transport in the presence of disorder.

The fractional quantum Hall edge[57] is another open problem to which the Hamiltonian approach is applicable. Since Wen's (1990a, 1990b, 1992) description of the edge as a chiral Luttinger liquid, other descriptions have appeared based on wave-function and field-theoretic approaches.[58] The field-theoretic descriptions are all effective theories whose connection to the electron problem has not been rigorously established. While the wave-function approaches are microscopic, it is impractical to calculate time-dependent response functions in them. As we have seen in the extended formalism, we have an exact rewriting of the microscopic electron problem, but with the added advantage of a nondegenerate starting point. The calculation of edge reconstructions (MacDonald, Yang, and Johnson, 1993; Chamon and Wen, 1994) in composite-fermion Hartree-Fock theory and the number and dispersions of the edge collective modes in TDHF seem to be very accessible in the Hamiltonian approach. One of the interesting things to consider in the edge problem is tunneling.[59] For this one needs a description of electron creation and destruction operators within the composite-fermion basis, which is an open problem.

Since there is a gap, the effects of disorder on the Jain series ought to be describable in some simple approximation, such as the self-consistent Born approximation for single-particle properties. Such a treatment seems to capture many of the experimental facts at $\nu=1$ (see Murthy, 2001b), such as the reduction of the transport gap due to disorder, the variation of the transport gap as a function of E_Z (see, for example, Schmeller *et al.*, 1995), the polarization at $\nu=1$,[60] etc. Such an approach ought to be applicable to the gapped fractions. The effect of disorder on excitons is also interesting and can be treated in a "self-consistent exciton approximation" (Kallin and Halperin, 1985).

A more ambitious problem that might benefit from the Hamiltonian approach is the question of whether the $\nu=1/2$ state remains metallic at $T=0$, when disorder is included. The Chern-Simons formalism of Halperin, Lee, and Read (with its logarithmic corrections) suggests that metallic behavior disappears. It is known that noninteracting fermions are always localized in two dimensions, regardless of how weak disorder is (Abrahams *et al.*, 1979). By extension, it seems plausible that a Fermi liquid, which is adiabatically connected to noninteracting fermions, should also be an insulator on the longest length scales. The $\nu=1/2$ system is unique in that the state is produced by interactions. The extended Hamiltonian has additional symmetries absent in the zero-field problem.

We have excluded from this review many interesting topics to which the Hamiltonian is applicable, such as double-layer systems,[61] paired states,[62] etc. These omissions reflect the double constraints of limited space and our own lack of expertise.

ACKNOWLEDGMENTS

Over the years, as we have tried to understand the FQHE, we have acquired valuable insights from countless discussions with many of our colleagues and have gratefully acknowledged them in our various papers. To

[57]For an excellent review, see Kane and Fisher (1997).

[58]For wave-function-based approaches, see Zülicke and MacDonald (1999), Goldman and Tsiper (2001), and Mandal and Jain (2001b). For field-theoretic descriptions, see Lee and Wen (1998), Lopez and Fradkin (1999), and Levitov, Shytov, and Halperin (2001).

[59]The theory was described in a series of beautiful papers by Kane and Fisher (1992a, 1992b, 1994). Recently some truly remarkable tunneling experiments have appeared in which the data span several orders of magnitude of voltage; see Grayson *et al.* (1998) and Chang *et al.* (2001).

[60]See Barrett *et al.* (1995); Tycko *et al.* (1995); Aifer *et al.* (1996); Manfra *et al.* (1996).

[61]For some of the early theoretical references, see Fertig (1989); Wen and Zee (1992, 1993); Ezawa and Iwazaki (1993); Yang *et al.* (1994); Moon *et al.* (1995). For a recent review, see Girvin and MacDonald (1997). For recent experiments in double-layer systems see Spielman *et al.* (2001) and Kellogg *et al.* (2002).

[62]The experiment that launched this subfield was the observation of a plateau in the Hall resistance at $\nu=5/2$; see Willett *et al.* (1987). For some of the theoretical references, see Haldane and Rezayi (1988); Moore and Read (1991); Greiter, Wen, and Wilczek (1992); Ho (1995); Morf (1998); Park *et al.* (1998); Bonesteel (1999); Read and Green (2000).

mention them all here is unrealistic. We take this opportunity to thank them all collectively, while apologizing for the anonymity. This work, performed over a period of six years, was made possible by Grant Nos. DMR-0071611 (G.M.) and DMR-0103639 (R.S.) from the National Science Foundation. Without this support, it would not have been possible for us to pursue this fascinating subject to our heart's content.

APPENDIX A: MATRIX ELEMENTS

Many of the calculations performed in this paper deal with the preferred density $\bar{\bar{\rho}}^p$. In second quantization we write it as

$$\bar{\bar{\rho}}^p(\mathbf{q})=\sum_{m_2n_2;m_1n_1} d^\dagger_{m_2n_2}d_{m_1n_1}\rho_{m_2n_2;m_1n_1}, \tag{A1}$$

where $d^\dagger_{m_2n_2}$ creates a particle in the state $|m_2\ n_2\rangle$ where m is the angular momentum and n is the Landau-level index. They are related to the composite-fermion cyclotron and guiding-center coordinates $\mathbf{R}$ and $\boldsymbol{\eta}$ as follows. Let

$$b=\frac{R_x-iR_y}{\sqrt{2l^{*2}}},\quad b^\dagger=\frac{R_x+iR_y}{\sqrt{2l^{*2}}}, \tag{A2}$$

where $l^*=l/\sqrt{1-c^2}$ is the composite-fermion magnetic length. These obey the oscillator algebra

$$[b,b^\dagger]=1 \tag{A3}$$

given

$$[R_x,R_y]=-il^{*2}. \tag{A4}$$

Similarly we define, in terms of the cyclotron coordinates,

$$a=\frac{\eta_x+i\eta_y}{\sqrt{2l^{*2}}},\quad a^\dagger=\frac{\eta_x-i\eta_y}{\sqrt{2l^{*2}}}, \tag{A5}$$

which obey the oscillator algebra

$$[a,a^\dagger]=1 \tag{A6}$$

given

$$[\eta_x,\eta_y]=il^{*2}. \tag{A7}$$

The states $|mn\rangle$ are just the tensor products

$$|mn\rangle=\frac{(b^\dagger)^m}{\sqrt{m!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|00\rangle, \tag{A8}$$

where $|00\rangle$ is annihilated by both a and b.

We shall now show that

$$\langle m_2|e^{-i\mathbf{q}\cdot\mathbf{R}}|m_1\rangle=\sqrt{\frac{m_2!}{m_1!}}e^{-x/2}\left(\frac{-iq_+l^*}{\sqrt{2}}\right)^{m_1-m_2}\times L_{m_2}^{m_1-m_2}(x), \tag{A9}$$

where

$$x=q^2l^{*2}/2,\quad q_\pm=q_x\pm iq_y. \tag{A10}$$

L is the associated Laguerre polynomial and $m_1\geqslant m_2$. If $m_1<m_2$ one may invoke the relation

$$\langle m_2|e^{-i\mathbf{q}\cdot\mathbf{R}}|m_1\rangle=\langle m_1|e^{+i\mathbf{q}\cdot\mathbf{R}}|m_2\rangle^*. \tag{A11}$$

Likewise to establish Eq. (A9), consider the coherent states

$$|z\rangle=e^{b^\dagger z}|0\rangle=\sum_{m=0}^{\infty}\frac{|m\rangle}{\sqrt{m!}}z^m \tag{A12}$$

with the inner product

$$\langle\bar{z}|z\rangle=e^{\bar{z}z}. \tag{A13}$$

First we write from the definitions given above

$$\langle\bar{z}|e^{-i\mathbf{q}\cdot\mathbf{R}}|z\rangle=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{\bar{z}^{m_2}}{\sqrt{m_2!}}\frac{z^{m_1}}{\sqrt{m_1!}}\langle m_2|e^{-i\mathbf{q}\cdot\mathbf{R}}|m_1\rangle \tag{A14}$$

$$\equiv R(\bar{z},z,\mathbf{q}). \tag{A15}$$

On the other hand,

$$\langle\bar{z}|e^{-i\mathbf{q}\cdot\mathbf{R}}|z\rangle=\langle\bar{z}|\exp\left(-\frac{il^*}{\sqrt{2}}(q_+b^\dagger+q_-b)\right)|z\rangle \tag{A16}$$

$$=\left\langle\bar{z}-\frac{il^*}{\sqrt{2}}q_+\middle|z-\frac{il^*}{\sqrt{2}}q_-\right\rangle e^{q^2l^{*2}/4} \tag{A17}$$

$$=\exp\left[\bar{z}z-\frac{il^*}{\sqrt{2}}(\bar{z}q_-+q_+z)\right]e^{-q^2l^{*2}/4} \tag{A18}$$

$$\equiv R(\bar{z},z,\mathbf{q}). \tag{A19}$$

Comparing Eqs. (A14)–(A18) and matching powers of $\bar{z}^az^b$ we obtain Eq. (A9) if we recall

$$L_{m_2}^{m_1-m_2}(x)=\sum_{t=0}^{m_2}\frac{m_1!}{(m_2-t)!(m_1-m_2+t)!}\frac{(-1)^t}{t!}x^t. \tag{A20}$$

Likewise, to establish

$$\langle n_2|e^{-i\mathbf{q}\cdot\boldsymbol{\eta}}|n_1\rangle=\sqrt{\frac{n_2!}{n_1!}}e^{-x/2}\left(\frac{-iq_-l^*}{\sqrt{2}}\right)^{n_1-n_2}L_{n_2}^{n_1-n_2}(x) \tag{A21}$$

(again for $n_1\geqslant n_2$), we just need to remember that the commutation rules of the components of $\boldsymbol{\eta}$ have a minus sign relative to those of $\mathbf{R}$, which exchanges the roles of creation and destruction operators and hence q_+ and q_-.

Now we consider matrix elements of $\bar{\rho},\bar{\bar{\chi}},\bar{\bar{\rho}}^p$. As a first step, let us express the operators $\mathbf{R}_e$ and $\mathbf{R}_v$ in terms of composite-fermion guiding-center and vortex coordinates $\mathbf{R}$ and $\boldsymbol{\eta}$. We have seen that in the composite-fermion representation

$$\mathbf{R}_e=\mathbf{r}-l^2\frac{\hat{\mathbf{z}}\times\boldsymbol{\Pi}}{1+c}=\mathbf{R}+\boldsymbol{\eta}c \tag{A22}$$

if we recall $l^2 = l^{*2}(1-c^2)$.

It can similarly be shown that

$$\mathbf{R}_v = \mathbf{r} + l^2 \frac{\hat{\mathbf{z}} \times \mathbf{\Pi}}{c(1+c)} = \mathbf{R} + \boldsymbol{\eta}/c. \tag{A23}$$

Thus in first quantization

$$\bar{\bar{\rho}}^p = \bar{\bar{\rho}} - c^2 \bar{\bar{\chi}}, \tag{A24}$$

$$\bar{\bar{\rho}} = \sum_i \exp(-i\mathbf{q}\cdot\mathbf{R}_i)\exp(-i\mathbf{q}\cdot\boldsymbol{\eta}_i c), \tag{A25}$$

$$\bar{\bar{\chi}} = \sum_i \exp(-i\mathbf{q}\cdot\mathbf{R}_i)\exp(-i\mathbf{q}\cdot\boldsymbol{\eta}_i/c). \tag{A26}$$

Armed with Eqs. (A9) and (A21) we may finally write for the matrix elements of $\bar{\bar{\rho}}(\mathbf{q})$ defined in Eq. (A1)

$$\bar{\bar{\rho}}_{m_2 n_2; m_1 n_1} = \sqrt{\frac{m_2!}{m_1!}} e^{-x/2} \left(\frac{-iq_+ l^*}{\sqrt{2}} \right)^{m_1 - m_2} L_{m_2}^{m_1 - m_2}(x) \otimes \left[\sqrt{\frac{n_2!}{n_1!}} \left(\frac{-icq_- l^*}{\sqrt{2}} \right)^{n_1 - n_2} e^{-xc^2/2} L_{n_2}^{n_1 - n_2}(xc^2) \right.$$
$$\left. - c^2 \cdot f \cdot \left(\frac{-iq_- l^*}{\sqrt{2}c} \right)^{n_1 - n_2} e^{-x/2c^2} L_{n_2}^{n_1 - n_2}(x/c^2) \right] \equiv \rho^m_{m_2 m_1} \otimes \rho^n_{n_2 n_1}.$$

Superscripts on $\rho^m_{m_2 m_1}$ and $\rho^n_{n_2 n_1}$, which will be apparent from the subscripts, will usually be suppressed.

APPENDIX B: HALL RESPONSE IN THE EXTENDED PICTURE

To compute the dc Hall conductance we just need the zero-momentum component of the current operator

$$\mathbf{J}(0) = \frac{\partial H}{\partial \mathbf{A}_{\mathbf{q}=0}} = \frac{e}{m} \sum_j \Pi_j = -\frac{e}{ml^2} \sum_j \hat{\mathbf{z}} \times \boldsymbol{\eta}_{ej}. \tag{B1}$$

Note that the current at $q=0$ depends only on the cyclotron coordinate, just as it depended only on the oscillator coordinate in the small treatment. Upon coupling the system to an external potential $\Phi(\mathbf{q})$, we obtain H,

$$H = \sum_j \frac{\boldsymbol{\eta}_{ej}^2}{2ml^4} + e\Phi(\mathbf{q}) \sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}. \tag{B2}$$

If we now keep the interaction to $\mathcal{O}(\mathbf{q})$ and recall $\mathbf{r}_e = \mathbf{R}_e + \boldsymbol{\eta}_e$, we obtain

$$H = \sum_j \frac{\boldsymbol{\eta}_{ej}^2}{2ml^4} + e\Phi(\mathbf{q}) \sum_j [-i\mathbf{q}\cdot(\mathbf{R}_{ej} + \boldsymbol{\eta}_{ej})] + \cdots. \tag{B3}$$

If we complete squares on $\boldsymbol{\eta}_{ej}$, we can read off its mean value in the presence of Φ or the corresponding electric field $i\mathbf{q}\Phi$. From this we obtain a mean current $\mathbf{J}(0)$ corresponding to the right Hall conductance of ne/B.

This exercise should make it clear that the oscillator coordinates are just the collective coordinates formed from $\boldsymbol{\eta}_e$ at small q. If we go to higher orders in q we shall find that the current involves both $\boldsymbol{\eta}_e$ and $\mathbf{R}_e$. Our extended formalism allows us to explore corrections due to this mixing in a small-q expansion.

Note that in this approach the magnetic moment of Simon *et al.* need not be put in by hand, since $\boldsymbol{\eta} + e$ is still in the picture and can respond to a slowly varying magnetic field by a changing zero-point energy.

The objections of Lee *et al.* (1998) to the composite-fermion Hall conductance are moot since we do not add resistivities as in Eq. (107) but rather conductivities (of the $\boldsymbol{\eta}_e$ and composite-fermion variables).

APPENDIX C: PROOF OF HARTREE-FOCK NATURE OF TRIAL STATES

Consider

$$\langle f|H|i\rangle = \langle \mathbf{p}|d_f H d_i^\dagger|\mathbf{p}\rangle, \tag{C1}$$

where $|\mathbf{p}\rangle$ stands for the (ground) state with a p-filled Landau-level and i,f label single-particle excitations on top of this ground state. We want to show that this matrix element vanishes if $i \neq f$, i.e., the Hamiltonian does not mix these putative Hartree-Fock particle states. (This result was established for the small-q theory by Murthy, 1999.) The proof, which relies on just the rotational invariance of the potential, applies with trivial modifications to the hole states, i.e., to

$$\langle \mathbf{p}|d_f^\dagger H d_i|\mathbf{p}\rangle. \tag{C2}$$

The matrix element in question takes the schematic form

$$\langle f|H|i\rangle = \int_q \langle \mathbf{p}|d_f d_1^\dagger d_2 d_3^\dagger d_4 d_i^\dagger|\mathbf{p}\rangle \rho_{12}(\mathbf{q})\rho_{34}(-\mathbf{q}), \tag{C3}$$

where 1 stands for $m_1 n_1$ and so on, and $\int_q$ stands for an integral over a rotationally invariant measure:

$$\int_q = \frac{1}{2}\int \frac{d^2q}{4\pi^2} v(q) e^{-q^2 l^2/2}. \tag{C4}$$

Now we use Wick's theorem and perform pairwise contractions on the vacuum expectation value, bearing in mind that

- We cannot contract the indices 1 and 2 or 3 and 4 since this will require that $q=0$, at which point the measure (which contains the potential) vanishes.
- If we contract i and f we already have the desired result.

Here is a representative of the contractions we can get:

$$\rho_{12}\rho_{34}\delta_{f1}(1-n_1^F)\delta_{23}(1-n_2^F)\delta_{4i}(1-n_4^F), \tag{C5}$$

where n_1^F is the Fermi function for the Landau level labeled by n_1,

$$n_1^F = \theta(p-1-n_1) \tag{C6}$$

and so on. Since $f=1$ and $i=4$, the factor $(1-n_2^F)(1-n_4^F)=1$. The integrand assumes the form

$$\sum_{m_2=0}^{\infty}\sum_{n_2=p}^{\infty} \rho_{f2}(\mathbf{q})\rho_{2i}(-\mathbf{q}) = \left[\sum_{m_2=0}^{\infty} \rho_{m_f m_2}(\mathbf{q})\rho_{m_2 m_i}(-\mathbf{q})\right] \times \left[\sum_{n_2=p}^{\infty} \rho_{n_f n_2}(\mathbf{q})\rho_{n_2 n_i}(-\mathbf{q})\right]$$

$$= \delta_{m_f m_i}\sum_{n_2} q_-^{n_i-n_f} F(|q|),$$

where we have also used the fact that $e^{-i\mathbf{q}\cdot\mathbf{R}}\cdot e^{i\mathbf{q}\cdot\mathbf{R}} = I$ (the identity operator) in doing the sum over m_2, and where $F(|q|)$ is some rotationally invariant function. It follows that every term in the sum over n_2 vanishes unless $n_f = n_i$ due to the angular integral in $\mathbf{q}$.

APPENDIX D: ON THE CONSERVING NATURE OF TDHF

We shall now verify that the constraint is a left eigenvector of $\mathcal{H}$, as required of the conserving approximation.

To this end we shall need the matrix elements

$$\tilde{\rho}_{n_1 n_2}(\mathbf{q}) = \langle n_1 | e^{-ic\mathbf{q}\cdot\boldsymbol{\eta}} | n_2\rangle, \tag{D1}$$

$$\tilde{\chi}_{n_1 n_2}(\mathbf{q}) = \langle n_1 | e^{-i/c\,\mathbf{q}\cdot\boldsymbol{\eta}} | n_2\rangle. \tag{D2}$$

Note that only the cyclotron parts of the electron and pseudovortex coordinates appear in these exponentials. An important result we shall need is

$$\sum_n \tilde{\chi}_{n_1 n}(\mathbf{q}_1)\tilde{\rho}_{n n_2}(\mathbf{q}_2) = \langle n_1 | e^{-i/c\,\mathbf{q}_1\cdot\boldsymbol{\eta}} e^{-ic\mathbf{q}_2\cdot\boldsymbol{\eta}} | n_2\rangle \tag{D3}$$

$$= e^{-(i/2)\,\mathbf{Q}_1\times\mathbf{Q}_2}\langle n_1 | e^{-i(\mathbf{q}_1/c + c\mathbf{q}_2)\cdot\boldsymbol{\eta}} | n_2\rangle, \tag{D4}$$

$$\mathbf{Q} = \mathbf{q}l^*, \tag{D5}$$

where we have used the completeness of the states $|n\rangle$ and the commutation rules obeyed by $\boldsymbol{\eta}$. Finally we separate the exponentials in the reverse order to get the second useful identity,

$$\sum_n \tilde{\chi}_{n_1 n}(\mathbf{q}_1)\tilde{\rho}_{n n_2}(\mathbf{q}_2) = e^{-i\mathbf{Q}_1\times\mathbf{Q}_2} \times \sum_n \tilde{\rho}_{n_1 n}(\mathbf{q}_2)\tilde{\chi}_{n n_2}(\mathbf{q}_1). \tag{D6}$$

Note that $\tilde{\rho}$ and $\tilde{\chi}$ do not commute, even though $\bar{\rho}$ and $\bar{\bar{\chi}}$ do.

Let us now right-multiply the putative left eigenvector $\tilde{\chi}$ by $\mathcal{H}$:

$$\sum_{n_1' n_2'} \tilde{\chi}_{n_1' n_2'}(\mathbf{q})\mathcal{H}(n_1' n_2'; n_1 n_2; \mathbf{q}) = [\epsilon(n_1)-\epsilon(n_2)]\tilde{\chi}_{n_1 n_2}(\mathbf{q}) + \sum_{n_1' n_2'} [N_F(n_2')-N_F(n_1')]\frac{v(q)}{2\pi(l^*)^2} e^{-q^2 l^2/2}\tilde{\chi}_{n_1' n_2'}(\mathbf{q}) \times \tilde{\rho}_{n_2' n_1'}(-\mathbf{q})\tilde{\rho}_{n_1 n_2}(\mathbf{q}) - \sum_{n_1' n_2'} [N_F(n_2')-N_F(n_1')] \times \int \frac{d^2 s}{(2\pi)^2} v(s) e^{-s^2 l^2/2}\tilde{\rho}_{n_1 n_1'}(\mathbf{s})\tilde{\chi}_{n_1' n_2'}(\mathbf{q})\tilde{\rho}_{n_2' n_2}(-\mathbf{s}) e^{i(l^*)^2 \mathbf{s}\times\mathbf{q}}. \tag{D7}$$

Let us consider the direct and exchange terms separately. In the direct term, one n' index can always be summed freely, while the other is constrained by the Fermi occupation factor N_F. The sum over the free n' gives, according to Eq. (D4),

$$\begin{aligned}\text{direct term} = &\sum_{n_2'} N_F(n_2')\tilde{\rho}_{n_1 n_2}(\mathbf{q}) \\ &\times\langle n_2'|e^{-i(1/c-c)\mathbf{q}\cdot\boldsymbol{\eta}}|n_2'\rangle \\ &-\sum_{n_1'} N_F(n_1')\tilde{\rho}_{n_1 n_2}(\mathbf{q}) \\ &\times\langle n_1'|e^{-i(1/c-c)\mathbf{q}\cdot\boldsymbol{\eta}}|n_1'\rangle. \end{aligned} \tag{D8}$$

The two terms are immediately seen to cancel. Now let us turn to the exchange terms and consider the one that has the factor $N_F(n_2')$ and a free sum over n_1'. In this term, one can use Eq. (D6) to exchange the $\tilde{\rho}$ and $\tilde{\chi}$ matrix elements, to obtain

$$\begin{aligned}\text{exchange term} = -&\sum_{n_2'} N_F(n_2')\sum_{n_1'} \tilde{\chi}_{n_1 n_1'}(\mathbf{q}) \\ &\times\int \frac{d^2s}{(2\pi)^2} v(s) e^{-s^2 l^2/2}\tilde{\rho}_{n_1' n_2'}(\mathbf{s}) \\ &\times\tilde{\rho}_{n_2' n_2}(-\mathbf{s}). \end{aligned} \tag{D9}$$

Notice that the phase factor $e^{i(l^*)^2\mathbf{s}\times\mathbf{q}}$ has been canceled by an opposite phase factor from Eq. (D6). Now the angular $\mathbf{s}$ integral forces $n_1'=n_2$ for a rotationally invariant potential, and the result contains the Fock energy of the state n_2,

$$\epsilon^F(n_2)\tilde{\chi}_{n_1 n_2}(\mathbf{q}). \tag{D10}$$

Similarly, the other exchange term proportional to $N_F(n_1')$ ends up giving $-\epsilon^F(n_1)\tilde{\chi}_{n_1 n_2}(\mathbf{q})$. Due to the peculiar nature of the Hamiltonian, the Hartree energy is a constant independent of the composite-fermion Landau-level index, and the difference of the Fock energies is the same as the difference of the full Hartree-Fock energies. Thus the exchange contributions cancel the diagonal term $[\epsilon(n_1)-\epsilon(n_2)]\tilde{\chi}_{n_1 n_2}(\mathbf{q})$, and $\tilde{\chi}_{n_1 n_2}(\mathbf{q})$ is indeed a left eigenvector with zero eigenvalue for $\mathcal{H}$. In addition, this property is independent of the form of $v(q)$ as long as it is rotationally invariant.

APPENDIX E: ACTIVATION GAPS

Now we need to find the energy cost of producing a widely separated particle-hole (PH) pair. This will be done by evaluating

$$\Delta_a = \langle \mathbf{p}+P|H|\mathbf{p}+P\rangle + \langle \mathbf{p}+H|H|\mathbf{p}+H\rangle - 2\langle \mathbf{p}|H|\mathbf{p}\rangle \tag{E1}$$

$$= \int_q E(P)+E(H), \tag{E2}$$

$$\int_q = \frac{1}{2}\int \frac{d^2q}{4\pi^2} v(q) e^{-q^2 l^2/2}, \tag{E3}$$

where P denotes a particle added to the state labeled $\mu=(n=p,m=0)$ and H denotes a state in which a hole has been made in the state $\mu=(n=p-1,m=0)$. Let us consider

$$E(P) = \langle \mathbf{p}|d_\mu d_1^\dagger d_2 d_3^\dagger d_4 d_\mu^\dagger|\mathbf{p}\rangle \rho_{12}\rho_{34}. \tag{E4}$$

In performing the contractions we

- Do not make any contractions within H. This gets rid of $E_0=\langle \mathbf{p}|H|\mathbf{p}\rangle$, the ground-state energy.
- Do not contract 1 with 2 or 3 with 4 since $v(0)=0$.

We end up with

$$\begin{aligned}\int_q [&\delta_{\mu 1}\delta_{23}\delta_{4\mu}(1-n_1^F)(1-n_2^F)(1-n_4^F) \\ &-\delta_{\mu 3}\delta_{14}\delta_{2\mu}(1-n_3^F)(n_4^F)(1-n_2^F)]\rho_{12}\rho_{34}. \end{aligned}$$

Since $4=\mu=1$ in the first term, we can drop $(1-n_1^F)(1-n_4^F)$ and for similar reasons $(1-n_3^F)(1-n_2^F)$ in the second, giving us

$$E(P) = \sum_{m_2=0}^{\infty}\sum_{n_2=p}^{\infty} \rho_{\mu 2}(\mathbf{q})\rho_{2\mu}(-\mathbf{q}) \tag{E5}$$

$$-\sum_{m_2=0}^{\infty}\sum_{n_2=0}^{p-1} \rho_{2\mu}(\mathbf{q})\rho_{\mu 2}(-\mathbf{q}). \tag{E6}$$

Since the sum over m_2 is unrestricted, we can use completeness and $e^{-i\mathbf{q}\cdot\mathbf{R}}\cdot e^{-i\mathbf{q}\cdot\mathbf{R}}=I$ to get rid of the m index altogether. Thus we end up with

$$E(P) = \left(\sum_{n=p}^{\infty}|\rho_{pn}|^2 - \sum_{n=0}^{p-1}|\rho_{pn}|^2\right) \tag{E7}$$

$$= \left[\langle p|\rho(q)\rho(-q)|p\rangle - 2\sum_{n=0}^{p-1}|\rho_{pn}|^2\right]. \tag{E8}$$

A similar calculation for the hole state gives (upon dropping the ground-state energy as usual)

$$\begin{aligned}E(H) = \Big[&-\langle p-1|\rho(q)\rho(-q)|p-1\rangle \\ &+2\sum_{n=0}^{p-1}|\rho_{p-1,n}|^2\Big], \end{aligned} \tag{E9}$$

where

$$\langle n|\rho(q)\rho(-q)|n\rangle = \sum_{n'=o}^{\infty}|\rho(q)_{nn'}|^2. \tag{E10}$$

Putting all the pieces together, we obtain the gap.

APPENDIX F: CRITICAL FIELDS FOR MAGNETIC TRANSITIONS

We need to calculate

$$E(p-r,r) = \langle \mathbf{p}-\mathbf{r},\mathbf{r}|H|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle, \tag{F1}$$

the energy in a state with $p-r$ spin-up Landau levels and r spin-down Landau levels. Since the Hartree-Fock calculation for the spinless case is very similar, this treatment will be brief. We write

$$H=\sum_{1234}\int_q d_1^\dagger d_2 d_3^\dagger d_4 \rho_{12}\rho_{34} \tag{F2}$$

with the understanding that a label like 1 stands for the triplet (n_1,m_1,s_1), where s is the spin. The matrix elements ρ_{ij} are defined by

$$\rho_{12}=\langle 1|e^{-i\mathbf{q}\cdot\mathbf{R}}[e^{-i\mathbf{q}\cdot\boldsymbol{\eta}c}-c^2 f e^{-i\mathbf{q}\cdot\boldsymbol{\eta}/c}]|2\rangle$$
$$=\rho_{m_1m_2}\otimes\rho_{n_1n_2}\otimes\delta_{s_1s_2}$$

and as a result

$$E(p-r,r)=\int_q \sum_{n_1n_2s} n_1^F(s)[1-n_2^F(s)]|\rho_{n_1n_2}|^2 \times \underbrace{\sum_m \langle m|I|m\rangle}_{=n/p},$$

where we acknowledge the fact that the occupation factors n_1^F and n_2^F can depend on the spin. We have also used the fact that the sum over all values of m is the degeneracy of each composite-fermion Landau-level, n/p. Carrying out the sums over n_1 and n_2, we obtain

$$E(p-r,r)=\frac{n}{p}\int_q\Bigg[\sum_{n_1=0}^{p-r-1}\langle n_1|\rho(q)\rho(-q)|n_1\rangle - \sum_{n_1,n_2=0}^{p-r-1}|\rho_{n_1n_2}|^2 + \sum_{n_1=0}^{r-1}\langle n_1|\rho(q)\rho(-q)|n_1\rangle - \sum_{n_1,n_2=0}^{r-1}|\rho_{n_1n_2}|^2\Bigg].$$

It is now straightforward to compute the critical field for the transition $|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle\rightarrow|\mathbf{p}-\mathbf{r}-\mathbf{1},\mathbf{r}+\mathbf{1}\rangle$ by invoking

$$E(p-r,r)-E(p-r-1,r+1)=g\left[\frac{e}{2m_e}\right]B^c\frac{n}{p}. \tag{F3}$$

ACRONYMS AND SYMBOLS

This paper invokes many symbols, not all of which have been standardized. For the convenience of the reader, a list is supplied below.

LIST OF SYMBOLS

Widely used quantities

ν	$=p/(2ps+1)=$filling fraction
$2s$	Number of vortices of flux tubes attached
p	Number of composite-fermion Landau levels
$\hat{\mathbf{z}}$	Unit vector along z axis
c^2	$2ps/(2ps+1)$
m_e (m)	Electron mass in free space (in solid)
A^* or B^*	Potential or field seen by composite fermion $A^*=A/(2ps+1)$
$\mathbf{\Pi}$	Velocity operator for composite fermion, $\mathbf{p}+e\mathbf{A}^*$
g	g factor of electron or composite fermion, taken to be 0.44
$l(l^*=l/\sqrt{1-c^2})$	Electron (composite-fermion) magnetic length
$\mathbf{R}_e$ or $\mathbf{R}_v$ or $\mathbf{R}$	Electron, pseudovortex, or composite-fermion guiding-center coordinate
$\boldsymbol{\eta}_e$ or $\boldsymbol{\eta}$	Electron or composite-fermion cyclotron coordinate
$\bar{\bar{H}}$	Hamiltonian in the lowest Landau level
$\bar{\bar{\rho}}(\mathbf{q})$	Electron density in the lowest Landau level
$\bar{\bar{\chi}}(\mathbf{q})$	Constraint
$\bar{\bar{\rho}}^p(\mathbf{q})$	$\bar{\bar{\rho}}(\mathbf{q})-c^2\bar{\bar{\chi}}(\mathbf{q})$ (preferred charge)

Quantities related to gaps

$\Delta_{a,p}$	Activation or polarization gap
$m_{a,p}^{(2s)}$	Defined by $\Delta_{a,p}=eB^*/(m_{a,p}^{(2s)})$
δ	$\Delta/(e^2/\varepsilon l)$
λ	Defined by $v_{ZDS}(q)=2\pi e^2 e^{-ql\lambda}/q$

Magnetic quantities

S	Number of spin-up minus spin-down composite fermions
$E(S)$	Ground-state energy density
$\mathcal{E}_\pm(k)$	Hartree-Fock energy for up/down-spin at momentum k
$\|\mathbf{p}-\mathbf{r},\mathbf{r}\rangle$	Composite-fermion state with $p-r$ Landau levels spin-up and r down.

Matrix elements

$\rho_{n_1n_2}$	Single-particle matrix element of $\bar{\bar{\rho}}^p$ between Landau levels n_1 and n_2
$\tilde{\chi}_{nn'}$	$\langle n\|\exp(-i\mathbf{q}\cdot\boldsymbol{\eta}/c)\|n'\rangle$

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