Vortex states and quantum magnetic oscillations in conventional type-II superconductors

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The theory of pure type-II superconductors at high magnetic fields and low temperatures has recently attracted much attention due to the discovery of de Haas–van Alphen oscillations deep in the vortex state. In this article the authors review the state of the art in this rapidly growing new field of research. The very existence of quantum magnetic oscillations deep in the vortex state poses challenging questions to the theorists working in this field. For a conventional extreme type-II superconductor in magnetic fields just below the upper critical field H_{c2}, the quasiparticle spectrum is gapless and the de Haas–van Alphen effect is suppressed with respect to the corresponding normal-state signal due to superconducting induced currents near the vortex cores, which are of paramagnetic nature. Numerical simulations of the quasiparticle band structure in the Abrikosov vortex lattice show the existence of well-separated Landau bands below H_{c2}. An analytical perturbative approach, which emphasizes the importance of phase coherence in quasiparticle scattering by the pair potential in the Abrikosov lattice, predicts a relatively weak magnetic breakdown of the corresponding cyclotron orbits. In contrast to the situation in the Abrikosov lattice state, a theory based on a random vortex lattice model yields large exponential decay of the de Haas–van Alphen oscillations with the superconducting order parameter below H_{c2}. The disordered nature of the vortex state near H_{c2} in real superconductors, where long-range phase coherence in the superconducting order parameter is destroyed, could explain the success of this model in interpreting experimental data below H_{c2}. In the Abrikosov vortex lattice state, which usually stabilizes well below H_{c2}, the residual damping of the de Haas–van Alphen amplitude is significantly reduced. In quasi-two-dimensional superconductors, phase fluctuations associated with sliding Bragg chains along principal axes in the vortex lattice lead to a weak first-order melting transition far below the mean-field H_{c2}. Superconducting fluctuations dominate the additional damping of the de Haas–van Alphen oscillations in this vortex liquid state. Below the first-order freezing point, this damping is predicted to weaken signifiFantly.

CONTENTS

I. INTRODUCTION

The discovery of high-temperature superconductivity in the cuprates caused a flurry of activity in various subfields of condensed-matter research, stimulating not only studies of the basic mechanisms leading to this phenomenon, but also a widespread search for new technological applications. In particular, the extreme type-II character of these materials, which near optimal doping is associated with huge upper critical magnetic fields, H_{c2}, has posed challenging new problems to the community of researchers in high-magnetic-field laboratories.

It was natural that the standard technique of de Haas–van Alphen (dHvA) oscillations (de Haas and van Alphen, 1930), which has been so successful in mapping the Fermi surfaces of many metals (Shoenberg, 1984a, 1984b), would be proposed as a tool in the search for a Fermi surface in this class of materials. However, since dHvA oscillations are observable only at low temperatures T (Lifshitz and Kosevich, 1956), the huge values of $H_{c2}(T)$ at such temperatures immediately implied that, under the constraints imposed by the stationary high-magnetic-field equipment in the existing laboratories, the experiments would have to be carried out in the superconducting state.

This situation provoked a debate as to whether quantum magnetic oscillations could exist in the superconducting state, raising fundamental questions concerning the nature of the low-temperature states of pure type-II superconductors in strong magnetic fields. Some of these questions still remain open even though significant progress in understanding important aspects of the problem has been achieved during the last decade.

The purpose of this review is therefore to present a coherent account of the major developments occurring in this field during the last decade or so. Since some fundamental aspects of the problem are far from being well understood even within the conventional BCS theory, and since most experimental work in this field was done with conventional superconductors, we shall restrict ourselves in the present review to this class of superconducting materials.

A. Superconductivity in high magnetic fields: General aspects

As indicated by the title of this article, the connection between the observation of quantum magnetic oscillations and the nature of the vortex states in type-II superconductors is at the core of the present discussion. Since both of these topics usually seem highly technical to the nonspecialist reader, it will be helpful in this introduction to place the subject in a broader context by describing the more general aspects of the phenomenon of type-II superconductivity under high magnetic fields.

It is well known that an external magnetic field destroys superconductivity in a spin-singlet superconductor. This effect is usually described as a pair-breaking process, taking place at a sufficiently strong magnetic field that the gain in magnetic energy associated with spin polarization of the normal electrons overcomes the gain in the superconducting condensation energy of the unbroken Cooper pairs. An estimate of the (Pauli) pair-breaking critical field H_p was first obtained by Clogston (1962) and by Chandrasekhar (1962), who simply compared the Zeeman energy with the value of T_c at zero field. As will be explained below, this criterion is too rough to yield the actual critical field. Furthermore, under these highly simplified circumstances the phase transition to the superconducting state as a function of magnetic field would be of the first order (see, for example, Fulde, 1969).

Within this framework it would also be expected that a magnetic field could critically diminish the range of singlet superconductivity. The reason can be seen by attempting to pair two electrons with linear momenta, equal in magnitude but opposite in direction, on the Fermi surface and with opposite spin projections in a magnetic field. Here the Zeeman spin splitting prevents the occurrence of the well-known Cooper singularity in the two-electron correlation function (Schrieffer, 1964). One may recover the Cooper singularity in this case by allowing a superconducting state with a nonzero linear momentum of the Cooper pairs to offset the Zeeman spin splitting (Fulde and Ferrel, 1964; Larkin and Ovchinikov, 1964).

These considerations ignore the far more important ingredient of the superconductor's response to the external magnetic field—the orbital (diamagnetic) response. Unlike the sharp Pauli pair-breaking effect, the magnetic orbital response in type-II superconductors acts smoothly but more effectively: the penetrating magnetic field induces a collective cyclotron motion of Cooper pairs, which coexists with the superconducting order. The price paid for this coexistence is an inhomogeneity in the superconducting order parameter, which follows the penetrating magnetic flux lines by a dual network of vortex lines. A continuous transition from the superconducting to the normal state in this picture occurs when the kinetic energy of the collective cyclotron motion of the Cooper pairs overcomes the superconducting condensation energy. The state with the minimum kinetic energy, which corresponds to the lowest Landau level of the Cooper pairs, is the state that sustains the maximal magnetic field—the upper critical field, $H_{c2}(T)$. The corresponding condition for the transition can be thus obtained by equating the cyclotron energy of a Cooper pair in the ground Landau level, $\frac{1}{2}(\hbar 2eH/2mc) = \frac{1}{2}\hbar\omega_c$, with the condensation energy per Cooper pair, $\hbar^2\xi(T)^{-2}/2(2m)$. Here m is the single-electron mass and $\xi(T)$ is the Cooper-pair coherence length. Alternatively, one may express this condition in terms of the length scales $\xi(T)$ and the magnetic length $a_H = \sqrt{c\hbar/eH}$: the former is a measure of the size of the vortex core while the latter is approximately equal to the distance between neighboring vortices. Clearly at $H \sim H_{c2}(T)$, where the vortices form a close-packed lattice, these two length scales coincide [the exact condition is $\sqrt{2}\xi(T) = a_H$].

A typical phase diagram is shown in Fig. 1: Near the bottom of the superconducting-normal phase boundary, where $H_{c2}(T \to T_c) \to 0$, thermal excitations are the dominant mechanism of pair breaking. The characteristic length scale that controls this type of excitation is the thermal mean free path $\zeta(T) = \hbar v_F/\pi k_B T$, $T \approx T_c$, where v_F is the velocity of an electron on the Fermi surface. At lower temperatures (or higher magnetic fields) along the $H_{c2}(T)$ line, where $\zeta(T) > a_H$, or equivalently $k_B T < \hbar\omega_c\sqrt{E_F/\hbar\omega_c}$, the cyclotron currents provide the dominant pair-breaking mechanism over the thermal excitations. A small magnetic field, i.e., corresponding to $\hbar\omega_c \sim k_B T/\sqrt{E_F/\hbar\omega_c}$, is therefore sufficient to enter this regime, since the Fermi energy E_F is typically much larger than $\hbar\omega_c$.

Under these circumstances the effect of the Pauli pair breaking is usually not very important. A simple estimate in the spirit of the original Clogston-Chandrasekhar criterion can show that the corresponding critical field is very close to H_{c2} and that the magnitude of the discontinuous jump of the order parameter is very small. The corresponding calculation goes as follows: considering the energetic advantage of breaking spin-singlet pairs in a magnetic field, the suppression of the superconducting condensation energy by the magnetic field H leads at a certain critical field $H = H_{cp}$ to the situation in which the gain in energy of the normal-state electrons due to spin polarization, i.e., $-\varepsilon_P(H) = \frac{1}{2}N(0)(\hbar eH/m_0 c)^2$, exceeds the gain in energy in the superconducting state $-\varepsilon_{SC}(H) \sim 1.57N(0)\Delta_0^2(H)$, so that the system will become normal. Here $N(0)$ is the single-electron density of states at the Fermi energy, m_0 is the free-electron mass, and $\Delta_0(H) \sim 1.76k_B T_c\sqrt{1-H/H_{c2}}$. The condition for such a transition to occur is thus $\varepsilon_P(H_{cp}) \sim \varepsilon_{SC}(H_{cp})$. A simple calculation shows that as long as $k_B T_c \ll E_F$,

$$H_{cp} \approx H_{c2}[1-(1.76k_B T_c/E_F)^2],$$

which implies a discontinuous jump of the superconducting order parameter at $H = H_{cp} \lesssim H_{c2}$ of very small magnitude, i.e., $\delta\Delta_0(H_{cp})/1.76k_B T_c \sim (1.76k_B T_c/E_F)$.

In approaching the highest sector of the $H_{c2}(T)$ line shown in Fig. 1, where the thermal smearing of the Fermi distribution function is smaller than the cyclotron energy of an electron $\hbar\omega_c$ (that is, where $2\pi^2 k_B T < \hbar\omega_c$), one enters a new region in which the Landau quantization of the normal-electron spectrum (Landau, 1930; Landau and Lifshitz, 1976) can significantly interfere with the process of Cooper pairing. In the normal metallic state this quantization (across a sufficiently abrupt Fermi surface) leads to dHvA oscillations. The condition for entering the new quantum region can be expressed in terms of the elementary length scale of the problem—the magnetic length a_H: the uncertainty in the electron wavelength due to thermal excitations, $\Lambda_T = \sqrt{\hbar^2/4\pi^2 m_c k_B T}$, should be larger than a_H. It is interesting to note here that, since $H_p/H_{c2}(T) \approx [\xi(T)/\pi\Lambda_T]^2$, the original Clogston-Chandrasekhar limiting field H_p is usually much larger than the upper critical field $H_{c2}(T)$ except for the quantum magnetic oscillation regime, in which they are comparable.

As noted by Gunther and Greunberg (1966), in this quantum regime the transition temperature $T_{c2}(H)$ should be an oscillatory function of the magnetic field. This effect can be easily understood by examining a simple BCS formula for T_c, in which the effect of Landau quantization on the normal-electron density of states $N(0)$ is taken into account. Gunther and Gruenberg (1966) also pointed out that at the maxima of the oscillations a (singlet) superconducting state can exist at arbitrarily large magnetic field, provided the temperature is reduced to the exponentially small value $\sim T_c(H=0)e^{-1.1\sqrt{n_F}}$, where $n_F = E_F/\hbar\omega_c \gg 1$. Their conclusion regarding the possible observation of such a dramatic effect was, however, quite pessimistic, dismissing its realistic value by emphasizing the destructive influence of a minute amount of impurity scattering or spin misalignment. A more optimistic and creative attitude towards this possibility was adopted by Rasolt and Tesanovic (1992), who described the fascinating scenario of reentrant (singlet) superconductivity in three-dimensional (3D) low-density electron systems in very high magnetic fields, where only a few electronic Landau levels are occupied. In 2D, or quasi-2D models, where the effective Zeeman spin splitting can be reduced to zero by tilting the magnetic-field direction with respect to the conducting planes (Wosnitza, 1996), the effect is dramatically enhanced even for metallic densities, due to the possibility of "resonant pairing" (see Maniv *et al.*, 1992, and Sec. II.C.2 below).

In fact, by combining orbital (Landau) quantization with Zeeman spin splitting in a consistent fashion, a large variety of pairing states, as illustrated in Fig. 2, can in principle be constructed. It is evident, however, that as long as the Zeeman spin splitting $\hbar\omega_e = \frac{1}{2}g\mu_B$ (where g is the Lande g factor and μ_B is the Bohr magneton) is not a multiple integer of the cyclotron energy $\hbar\omega_c$, any attempt to pair two electrons on the Fermi surface with opposite spin projections and with linear momenta k_z, $k_z' = -k_z$ along the magnetic-field direction will fail. This failure means that the Zeeman splitting usually prevents

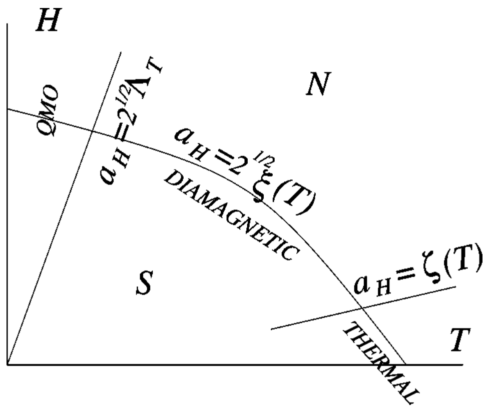


FIG. 1. Schematic phase diagram of a pure type-II superconductor, showing the regions of thermal and diamagnetic pair breaking and the region of strong quantum magnetic oscillations.

the occurrence of the desired Cooper singularity. However, an idea of Fulde and Ferrel (1964) and of Larkin and Ovchinikov (1964) provides a way to recover the Cooper singularity. For any Landau level with index n destructive effect of the Zeeman spin splitting can be compensated by allowing the pair to have a nonzero linear momentum Q_n along the field direction (i.e., by letting $k_z' = Q_n - k_z$; see Fig. 2). The number of possible solutions Q_n to this pairing problem on the Fermi surface is equal to the number of occupied Landau levels (i.e., $\sim n_F \gg 1$).

B. Coexistence of quantum magnetic oscillations and superconductivity

One may conclude from the above discussion that Cooper pairing of electrons in Landau levels can be realized in various, sometimes intriguing, ways and that very unusual effects are theoretically expected. Such effects have not been observed so far in magnetotransport experiments. What is the reason for this failure? To answer this question one should be equipped with a better understanding of the phenomenon of quantum magnetic oscillations in the mixed superconducting state.

Considering a simple BCS type-II superconductor, three questions immediately arise, relevant to the observation of quantum magnetic oscillations below the upper critical field.

(1) Does the superconducting energy gap Δ_0, characterizing the quasiparticle spectrum near the Fermi energy at low magnetic fields, also exist in the mixed state at high fields just below H_{c2}? Such a gap would have drastically damped the oscillations at low temperatures, by a factor of $e^{-\Delta_0/k_B T}$ (mechanism 1).

(2) Does the inhomogeneity of the magnetic induction B within the bulk of the superconductor, associated with the creation of the flux lattice, significantly broaden the Landau levels (mechanism 2)?

(3) Does the intrinsic inhomogeneity in the superconducting order parameter Δ associated with the creation of the vortex lattice lead to significant inhomogeneous broadening of the Landau levels in the quasiparticle spectrum near the Fermi surface (mechanism 3), which would thus make the detection of dHvA oscillations even more difficult?

To summarize the list of questions: Can the dHvA oscillations observable in the normal state of many superconducting metals survive all these potentially destructive effects in the vortex state?

These theoretical obstacles to the observation of dHvA oscillations in the vortex state convinced most people active in the field before 1994 that quantum magnetic oscillations are unlikely to be observable in the superconducting state even under ideal conditions. Graebner and Robbins' pioneering experiment (Graebner and Robbins, 1976), in which magnetothermal oscillations were clearly seen deep in the vortex state of superconducting 2H-$NbSe_2$, thus drew only scant attention, awaiting verification for almost 20 years. Only in 1992 did Onuki *et al.* confirm the observation of Graebner and Robbins in the same material, interpreting, however, the lack of the expected strong attenuation of the dHvA amplitude in the vortex state as an indication that the observed signal was not associated with superconducting quasiparticles but with a normal-electron part of the Fermi surface.

At about the same time, several new papers appeared in the literature reporting the observation of dHvA oscillations in the superconducting state of other materials, namely, in V_3Si (Mueller *et al.*, 1992) and even in the high-T_c compound $YBa_2Cu_3O_{7-\delta}$ (Kido *et al.*, 1991; Fowler *et al.*, 1992; Haanappel *et al.*, 1993). In all of these experiments, however, there was no clear evidence for the superconducting nature of the detected fermionic quasiparticles.

Only the series of papers published during 1994 by the Bristol group, in which they reported on dHvA measurements in the $A15$ compounds V_3S_i (Corcoran, Harrison, *et al.*, 1994) and Nb_3Sn (Harrison *et al.*, 1994), as well as in the classic material 2H-$NbSe_2$ (Corcoran, Meeson, *et al.*, 1994), eventually convinced the skeptics that quantum magnetic oscillations associated with superconducting quasiparticles in the vortex state of type-II superconductors were a reality. This group succeeded where others failed because they adopted the strategy of studying carefully and systematically several conventional superconductors, such as the $A15$ compounds, instead of rushing into the race of searching for quantum magnetic oscillations in high-T_c superconductors. The importance of the results reported in these papers was that for the first time the variation of the superconducting order parameter with magnetic field below H_{c2} was clearly shown to significantly influence the measured oscillations.

To appreciate how large this influence could be, let us compare two characteristic energy scales, the zero-field superconducting energy gap $\Delta_0 \approx 1.7 k_B T_c$ and the cyclotron energy $\hbar\omega_c = \hbar eB/mc$, at magnetic field B close to the zero-temperature upper critical field, $H_{c2}(0) = \phi_0/2\pi\xi(0)^2$. Here $\phi_0 = ch/2e$ is the Cooper-pair flux

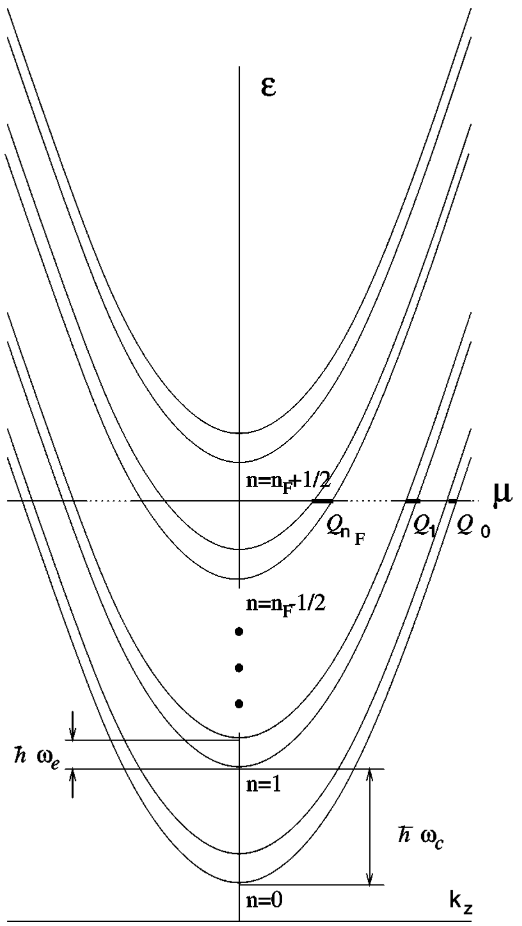


FIG. 2. Illustration of different pairing possibilities in a 3D electron gas, depending on the Fulde-Ferrel-Larkin-Ovchinikov wave number Q_n, selected to offset the Zeeman splitting of the nth Landau branch of the k_z dispesion curve at $\mu \approx E_F$.

quantum, and $\xi(0)=0.18\hbar v_F/k_BT_c$ is the zero-temperature Cooper-pair coherence length.

Using these relations we can show that

$$\frac{\Delta_0}{\hbar\omega_c}\approx 0.61\sqrt{n_F},\quad n_F\equiv\frac{E_F}{\hbar\omega_c}, \tag{1}$$

which is typically a large ratio since $n_F \gg 1$. It is therefore striking that many experiments failed to observe the large effect expected on the basis of such an estimate.

In the next subsections we shall discuss the main qualitative features of the fermionic quasiparticles in the vortex state of an extreme type-II superconductor placed under high magnetic fields. Exploiting a simple quasiparticle picture, we shall explain why no real obstacle to the observation of quantum magnetic oscillations originating from mechanism (1) or (2) is expected in the mixed state near and below H_{c2}. Combining the conclusion of Sec. I.B.3 below with the large characteristic ratio between the nominal strength of the pair potential and the Landau-level spacing, reflected in Eq. (1), it seems plausible that mechanism (3) is potentially the most dangerous killer of the dHvA oscillations in the superconducting state below H_{c2}. An important part of this article is therefore devoted to a detailed investigation of this mechanism. Our model also reveals the existence of an additional mechanism in which the dHvA oscillations in the vortex state are suppressed.

1. Gapless superconductivity

The behavior of a quasiparticle in the vortex lattice state is in some sense similar to that of an electron in a periodic crystal potential under the influence of an external magnetic field. Both systems have the same symmetry, namely, invariance under magnetic translations (see Zak, 1964; Brown, 1968; Bychkov and Rashba, 1983). However, in contrast to the periodic field governing the electronic motion in a crystalline atomic lattice, the pair potential in the vortex lattice is a complex function of space coordinates, which mixes quasiparticle wave functions with quasihole wave functions.

Clearly, in contrast to an atomic (or molecular) lattice in a magnetic field, the very existence of the vortex lattice is a consequence of the magnetic field. Thus the two length parameters controlling the single-particle energy spectrum, the size of the unit cell of the periodic lattice and the magnetic length $a_B=\sqrt{c\hbar/eB}$, have fundamentally different relationships in the two cases. In the vortex lattice the unit cell and the magnetic length are inherently related physical quantities, which differ only by a proportionality constant of order unity, while in the atomic (or molecular) lattice the size of the unit cell a is a field-independent parameter, which is usually much smaller than a_B.

To gain a physical understanding of this complicated problem, let us first work out a highly simplified model for the dynamics of a quasiparticle under high magnetic field. We may invoke the semiclassical nature of the quasiparticle wave functions in a typical superconductor, which arises from the large value of the Fermi energy E_F in comparison with the cyclotron energy $\hbar\omega_c$ for a typical magnetic-field strength. Disregarding the complex particle-hole nature of the actual quasiparticle, we consider it as a simple charged particle moving under the influence of a uniform magnetic field, that is, on a surface of constant energy. These energy surfaces in a uniform superconductor are determined by the well-known BCS-Bogoliubov expression

$$\varepsilon(k)=\pm\sqrt{\xi_k^2+|\Delta|^2}, \tag{2}$$

where $\xi_k=\hbar^2k^2/2m^*-E_F$, and Δ is the superconducting energy-gap parameter.

In the presence of the magnetic field the order parameter is no longer constant in space. In a pure superconductor at thermodynamical equilibrium it has the Abrikosov lattice form, with N zeros of $\Delta(\vec{r})$ determining the normal cores, N being the number of magnetic flux lines that thread the superconductor in a regular triangular lattice (Abrikosov, 1957). Introducing this spatial dependence into Eq. (2) by replacing the constant Δ with $\Delta(\vec{r})$, and then using the relation between the position vector $\vec{r}$ and the vortex crystal momentum $\vec{k}(\vec{r})$, via the Lorenz equation of motion

$$\frac{d\vec{k}}{dt} = -\frac{e}{c\hbar}\left[\frac{d\vec{r}}{dt}\times\vec{B}\right], \tag{3}$$

where $\vec{B}$ is the magnetic induction, we obtain the classical energy surfaces in the reciprocal space from $\varepsilon(\vec{k}) = \pm\sqrt{\xi_k^2+|\Delta(\vec{r})|^2}$.

Using now the classical equation of motion for $\vec{r}$ in terms of this dispersion relation, i.e.,

$$\frac{d\vec{r}}{dt} = \frac{1}{\hbar}\frac{d}{d\vec{k}}\varepsilon(\vec{k}), \tag{4}$$

we may quantize the resulting classical orbits in a way similar to the Lifshitz-Onsager quantization scheme (Lifshitz and Kosevich, 1956) used for a Bloch electron in a magnetic field.

Taking for $|\Delta(\vec{r})|$ in $\varepsilon(\vec{k})$ the modulus of the Abrikosov lattice order parameter, we find that the equations of motion (3) and (4) have extended cyclotron orbit solutions for energies ε higher than $\Delta_0 \equiv \max|\Delta(\vec{r})|$, as well as localized solutions within the vortex cores for energies $\varepsilon \leqslant \Delta_0$. Near H_{c2} the distance between neighboring vortex cores, which is of the order of the magnetic length a_B, and the size of the vortex core region, which is of the order of the zero-temperature coherence length $\xi(0)$, are comparable. Thus quantum-mechanical tunneling of quasiparticles with energies $\varepsilon \leqslant \Delta_0$ between neighboring vortex cores is expected to be significant, allowing extended cyclotron orbitals at all energies near the Fermi energy.

Our simple semiclassical picture shows that near and below H_{c2} the quasiparticle spectrum across the Fermi surface is gapless. The early theoretical works of de Gennes (1966) and Brandt *et al.* (1967) suggested this possibility, whereas experimentally it has been known for many years (Tinkham, 1969) that the superconducting energy gap disappears at high magnetic fields.

2. Inhomogeneous broadening by the flux lattice

In a pure type-II superconductor below H_{c2} the magnetic induction $\vec{B}$ follows the spatial modulation of the order parameter, creating a lattice of magnetic flux lines that is induced by the supercurrents generating the vortex lattice. Consequently the unit cell of the flux lattice is identical to that of the vortex lattice, having dimensions of the order of the magnetic length a_B. For $B \lesssim H_{c2}$ the size of the vortex core region $\xi \sim a_B$ and so the vortices are arranged as impenetrable objects in a close-packed structure. In contrast, the size of a magnetic flux line is of the order of the magnetic penetration depth λ. The reason for selecting extreme type-II superconductors for our study will become apparent in a moment. In such a material $\lambda \gg \xi$, so that the flux lines are arranged in a highly overlapping structure that yields a very weak spatial modulation of the magnetic induction. The maximal amplitude of this variation is known to be (Fetter and Hohenberg, 1969)

$$B_1 \equiv H - B \approx \frac{H_{c2}-H}{2\kappa^2},$$

where $\kappa \equiv \lambda/\xi \gg 1$.

An estimate of the Landau-level broadening due to this inhomogeneity can be made on the basis of a simple calculation in which the quasiparticle is modeled by a charged particle in a 2D space placed in a periodic magnetic field (Rom *et al.*, 1996). For a small modulation B_1, the envelope of the time-dependent Green's function $G(\vec{r},\vec{r};t-t')$ was found, as expected, to be Gaussian at short times, leading to a Gaussian Landau-level broadening, the spectral width at the Fermi energy being

$$\left(\frac{\Delta\omega}{\omega}\right) = \left(\frac{\pi}{\Gamma_l\omega_c}\right) = 0.351 n_F^{1/4}\left(\frac{B_1}{B_0}\right),$$

where B_0 is the mean value of B over the flux lattice.

This result shows that the relative width is of the order of $n_F^{1/4}/\kappa^2$, which is typically very small compared to unity, since $n_F \sim 100$ and $\kappa \gtrsim 10$. It is interesting to note that this broadening effect is relatively weak due to the ordered arrangement of the flux lines. A comparison of this result to the expression for the Landau-level width derived by Aronov *et al.* (1995) for a completely random distribution of flux lines is instructive. The result of this calculation may be written as

$$\left(\frac{\pi}{\Gamma_r\omega_c}\right) = n_F^{1/2}\left(\frac{B_1}{B_0}\right),$$

meaning that inhomogeneous broadening of the Landau level by the random distribution of flux lines is larger by a factor $n_F^{1/4}$ than the broadening by a regular lattice of the same flux-line density.

3. Magnetic breakdown picture

The complex nature of the Landau quantization in the vortex state may be illuminated by further developing the model described in Sec. I.B.1 above. Of special interest is the interplay between the "extended" cyclotron trajectories of a quasiparticle near the Fermi surface, i.e., with Larmor radii $r_F \sim \sqrt{2n_F+1}a_B$, $n_F \gg 1$, and the localized orbits near the vortex cores. Starting with a single cyclotron orbit, one can assure its stability by selecting the initial conditions in such a way that the guiding center of the orbit obtained in the $\Delta=0$ limit coincides with one of the vortex cores [i.e., with zeros of $\Delta(\vec{r})$]. Such a selection corresponds to a self-consistent description of the "extended" quasiparticle cyclotron current and the vortex currents. The corresponding solutions of Eqs. (3) and (4) are closed (periodic) orbits slightly distorted with respect to the normal-state cyclotron orbits, like that shown in Fig. 3.

The distortion is most pronounced near the vortex cores, where the pairing force $\nabla|\Delta(\vec{r})|$ is relatively strong. Note, however, that despite the distortions, the orbital guiding center remains completely stationary, fixed to a vortex core. Thus semiclassical (Lifshitz-Onsager) quantization of such orbits leads to a discrete (Landau-level) energy spectrum. In the semiclassical

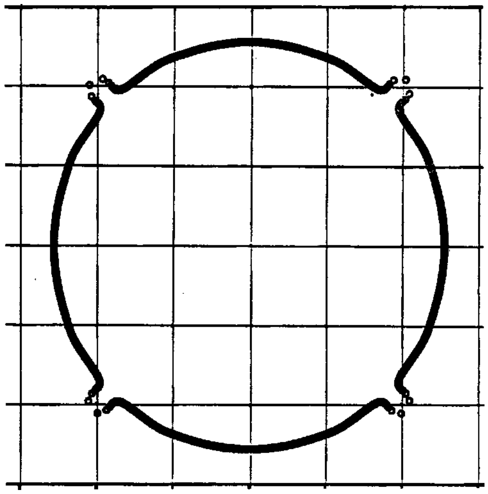


FIG. 3. A classical orbit of a quasiparticle satisfying the dispersion relation, Eq. (2), with an energy slightly above the superconducting gap Δ_0, in a 2D square Abrikosov lattice. The initial conditions were selected such that the orbital center coincides with a vortex core.

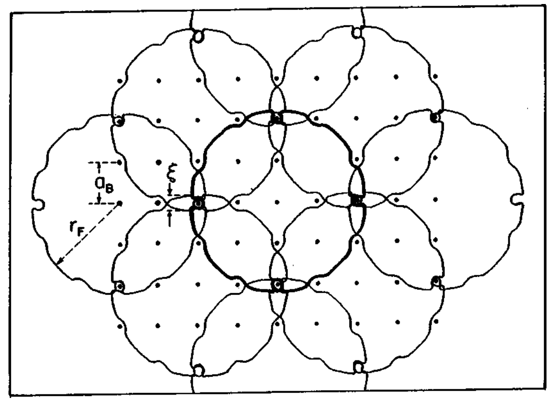


FIG. 4. A magnetic breakdown picture of the occupied orbit shown in Fig. 5, obtained by translating the orbital center from vortex core to vortex core. The size of the unit cell is approximately a_B, the size of the core region is ξ, and r_F is the Larmor radius.

limit $r_F \gg a_B$, each orbit intersects with many other similar orbits (i.e., with different orbital centers).

Quantum-mechanical tunneling between different orbits is expected to be relatively strong near these intersections. The "attractive pairing force" deflecting a quasiparticle towards a nearby vortex core (see Fig. 3) further enhances the tunneling probability by increasing the overlap between the intersecting orbitals. Thus the tunneling in the entire vortex lattice between any such orbits near vortex cores, which may also be described as tunneling of the orbital guiding center from one vortex core to another, transforms the localized cyclotronlike states into extended magnetic Bloch-like states in the entire superconducting sample (see Fig. 4).

This delocalization mechanism is reminiscent of the phenomenon of coherent magnetic breakdown in metals placed in high magnetic fields (Kaganov and Slutskin, 1983), which results in the broadening of the Landau levels into magnetic bands and their splitting into subbands (Gvozdikov, 1986). As a consequence, the amplitude of the dHvA oscillations in the vortex state will be damped in comparison with the normal-state oscillations.

The magnitude of this damping mechanism can be determined only by a detailed quantitative theory, which will be described in Sec. II. The tunneling probability may be estimated, however, within a WKB-like approximation (Kaganov and Slutskin, 1983) by considering only the dominant exponential factor $W \equiv e^{-B^*/B}$ (Gvozdikov, 1986), where the magnetic breakdown field can be written as $B^* = \Phi_0/2\pi\varsigma^2$, with $\Phi_0 = ch/e$ the magnetic-flux quantum and ς a characteristic length. For a typical crystalline lattice, ς is of atomic length scale a, whereas for the vortex lattice it is of the order of the magnetic length a_B. Thus one finds for the former case $W = e^{-(a_B/a)^2}$, which is usually much smaller than unity and strongly field dependent, while for the latter case W is ~ 1 and field independent.

This simple analysis shows that the magnetic breakdown phenomenon in the vortex lattice state can be very significant at any magnetic-field strength, unlike its counterpart in a typical crystalline lattice; this becomes apparent only at very high fields, or under very special circumstances (e.g., for crystals with very large unit cells).

4. Suppression of dHvA oscillations by paramagnetic vortex currents

The model presented above indicates another mechanism for suppressing dHvA oscillations in the vortex state. This can be clearly seen in Fig. 5, where typical empty ($\varepsilon > 0$) and occupied ($\varepsilon < 0$) orbits are sketched. Considering the occupied orbits, which dominate the orbital magnetization at low temperatures, one can see that near a vortex core the effect of the pairing force is attraction toward the core, leading to rotation of the quasiparticle around the core in an opposite sense to that of the large cyclotron orbit. As a result there is an overall reduction in orbital magnetization with respect to normal-state magnetization. Note in contrast that for the empty orbits the vortex currents and the large cyclotron orbit have the same sense of rotation.

This picture is reminiscent of the classic picture of electron diamagnetism (see, for example, Peierls, 1979), where the (paramagnetic) effect of the edge currents cancels that of the (diamagnetic) bulk currents.

It will be instructive to elaborate on this result by the following little calculation. Consider our model quasiparticle near a vortex core located at $\vec{r}_1$, where $|\Delta(\vec{r} \to \vec{r}_1)| \to 0$; since the zeros of $\Delta(\vec{r})$ in the Abrikosov lattice state are of the first order (see Sec. II.B.1), the gradient in this region $\nabla(|\Delta(\vec{r})|^2)$ can be approximated by $2\eta(\vec{r}-\vec{r}_1)|\Delta_0|^2/a_H^2$, where η is a constant of order unity. Thus, with the help of Eq. (3), Eq. (4) can be approximated (with lengths measured in units of magnetic length) by

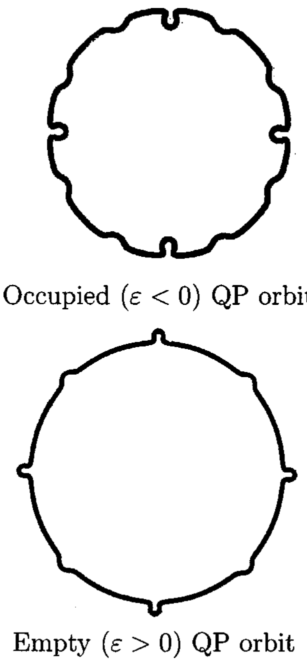


FIG. 5. Typical occupied and empty quasiparticle orbits obtained in a manner similar to that used in Fig. 3.

$$\frac{d\vec{r}}{dt} \approx \frac{\vec{k}}{\varepsilon(\vec{k})}\left(\omega_c \xi_k + \frac{1}{\hbar}|\Delta_0|^2\eta\right), \quad \vec{r} \to \vec{r}_1. \tag{5}$$

For an occupied orbit we replace $\varepsilon(\vec{k})$ in this expression with $-\varepsilon(\vec{k})$, and ξ_k with $-|\xi_k|$, so that

$$\frac{d\vec{r}}{dt} \approx \frac{\vec{k}}{\varepsilon(\vec{k})}\left(\omega_c |\xi_k| - \frac{1}{\hbar}|\Delta_0|^2\eta\right), \quad \vec{r} \to \vec{r}_1. \tag{6}$$

This equation can be rewritten in terms of the local radius vector $\vec{\rho} \equiv \vec{r} - \vec{r}_1$ as

$$\frac{d\vec{\rho}}{dt} \approx \frac{|\xi_k|}{\varepsilon(\vec{k})}\left(\omega_c - \frac{|\Delta_0|^2\eta}{\hbar|\xi_k|}\right)[\vec{\rho}\times\vec{n}]$$
$$\approx \left(\omega_c - \frac{|\Delta_0|^2\eta}{\hbar|\xi_k|}\right)[\vec{\rho}\times\vec{n}],$$

where $\vec{n} \equiv \vec{B}/B$, which looks like an equation for a charged particle rotating about the core with an effective angular velocity,

$$\omega_c^* \approx \left(\omega_c - \frac{|\Delta_0|^2\eta}{\hbar|\xi_k|}\right). \tag{7}$$

The first term in the brackets represents the cyclotron motion of the quasiparticle in the external magnetic field $\vec{B}$, while the second term represents the effect of the pair potential, which leads to the vortex current around the core. The negative sign in front of this term reflects the paramagnetic nature of the vortex current, which tends to cancel the diamagnetic effect of the cyclotron current. The dependence of this suppression effect on the amplitude of the pair potential Δ_0 is approximately quadratic, provided ξ_k is not too close to zero. Very close to the Fermi energy, however, the quadratic dependence crosses over to a linear one, that is,

$$\omega_c^* \to -\frac{|\Delta_0|\eta}{\hbar}, \quad \xi_k \to 0. \tag{8}$$

This result is in agreement with the fully quantum self-consistent calculation reported by Gygi and Schluter (1991), who studied quasiparticle excitations for an isolated vortex line in a type-II superconductor by solving numerically the corresponding Bogoliubov–de Gennes equations. They found that the current density near the vortex core, which is associated mainly with the bound states inside the core, is paramagnetic, whereas the current density far away from the core is mostly due to scattering states and is diamagnetic.

II. MEAN-FIELD THEORIES IN THE ABRIKOSOV VORTEX LATTICE

A. Formulation of the problem

The appearance of dHvA oscillations is a genuinely thermodynamical equilibrium phenomenon, taking place when the external magnetic field H is quasistatically varied and the sample magnetization M is measured as a function of H. The resulting crossing of quantized energy surfaces of a quasiparticle through the Fermi surface leads to oscillatory dependence on magnetic field, which is periodic in $1/H$. To evaluate $M_{osc}(1/H)$, we need to take into account any contribution to M induced by electrical current loops associated with these fermionic quasiparticles near the Fermi surface. This includes both ground and thermally excited states of the interacting many-electron system.

The BCS-Bogoliubov theory places the excited states (i.e., quasiparticle-quasihole pairs) in one-to-one correspondence with those of the free-electron gas, namely, electron-hole pair excitations. In the free-electron gas the same single-electron states are involved in the many-body ground and excited states, the only difference being in the occupation numbers of these states. This simplicity enables one to calculate both the ground-state and excited-state contributions to $M_{osc}(1/H)$ using the same single-particle spectrum. In a superconducting many-electron system, however, the ground state and the quasiparticle excitations are not so simply connected. The ground state is a condensate of a macroscopic number of highly correlated bound electronic states, whereas a quasiparticle excitation is constructed from this state by first breaking a single pair and removing it from the condensate, and then adding an electron to one of the vacant single-particle states. Obviously, this unpaired electron contributes to the magnetization M. However, its contribution alone cannot produce oscillations in M, since the corresponding energy is always positive (i.e., always above the Fermi energy) and so no Fermi-surface crossing can take place. This conclusion remains true even when no superconducting energy gap exists in the quasiparticle spectrum.

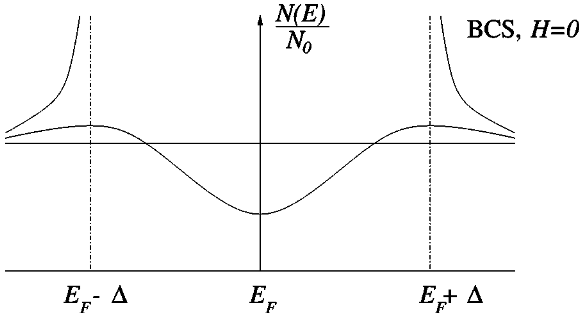


FIG. 6. Bardeen-Cooper-Schrieffer (BCS) density of states at zero magnetic field vs the actual state density obtained from tunneling experiments at strong field. From Tinkham, 1969.

An alternative description of the superconducting state can clarify the situation: In the absence of magnetic field all superconducting electrons participating in the pairing interaction are constructed from single-electron states within some cutoff energy around E_F. These states are in one-to-one correspondence with the fermionic quasiparticle states, whose energies are distributed outside the energy-gap region (see Fig. 6). This means that at $T=0$ all quasiparticle states above $E_F+\Delta_0$ are empty and all states below $E_F-\Delta_0$ are occupied. This filled Fermi sea of quasiparticles constitutes the ground state of the superconductor. At high magnetic field $B \lesssim H_{c2}$ where the energy gap disappears (see Fig. 6 and Tinkham, 1969), the corresponding quantized fermionic quasiparticles contribute to magnetic oscillations.

This heuristic description can be put on a solid formal footing, thanks to an elegant result derived more than 30 years ago by Bardeen *et al.* (1969) on the basis of earlier work due to Eilenberger (1965). To see how this can be done, let us formulate the problem within a more general framework using a more modern approach, which will enable us to treat a broader range of problems.

Starting from the BCS Hamiltonian in a uniform external magnetic field corresponding to vector potential $\vec{A}(\vec{r})$ but neglecting, for the sake of simplicity, the Zeeman energy term associated with the electron spin, we write

$$\mathcal{H}=H_0+\mathcal{H}_{int}=\mathcal{H}_0+V\int \psi_\uparrow^\dagger(\vec{r})\psi_\downarrow^\dagger(\vec{r})\psi_\downarrow(\vec{r})\psi_\uparrow(\vec{r})d^3r, \tag{9}$$

with $\mathcal{H}_0$ representing the single-electron part

$$\mathcal{H}_0=\int \sum_\sigma \left\{\frac{1}{2m}\psi_\sigma^\dagger(\vec{r})\left[\frac{\hbar}{i}\vec{\nabla}-\frac{e}{c}\vec{A}(\vec{r})\right]^2\psi_\sigma(\vec{r})-E_F\right\}d^3r, \tag{10}$$

where $\psi_\sigma^\dagger(\vec{r})$ are the creation field operators for the two spin components $\sigma=\uparrow,\downarrow$, and $V<0$ is the effective electron-electron interaction. The energy is measured relative to the Fermi energy E_F.

Note that magnetic induction associated with supercurrents is neglected here so that $\vec{B}\approx\vec{H}=\vec{\nabla}\times\vec{A}$. All thermodynamic properties of the system are determined by the partition function (Zagoskin, 1998),

$$\mathcal{Z}=\mathrm{tr}e^{-\beta(\mathcal{H}_0+\mathcal{H}_{int})}=\mathrm{tr}\left[e^{-\beta\mathcal{H}_0}T_\tau\exp\left(-\int_0^\beta d\tau\mathcal{H}_{int}(\tau)\right)\right]$$
$$\equiv\left\langle T_\tau\exp\left(-\int_0^\beta d\tau\mathcal{H}_{int}(\tau)\right)\right\rangle_0\mathcal{Z}_0, \tag{11}$$

where $\mathcal{Z}_0=\mathrm{tr}e^{-\beta\mathcal{H}_0}$, $\beta=1/k_BT$, T_τ is the time-ordering operator, and $\mathcal{H}_{int}(\tau)$ is $\mathcal{H}_{int}$ in the (Matsubara) imaginary-time interaction representation $\mathcal{H}_{int}(\tau)=e^{\mathcal{H}_0\tau}\mathcal{H}_{int}e^{-\mathcal{H}_0\tau}$.

Using the Hubbard-Stratonovich transformation [to express the exponential of the quadratic form in $\psi_\downarrow(\vec{r})\psi_\uparrow(\vec{r})$, appearing in $\mathcal{H}_{int}(\tau)$, as an integral over an auxiliary complex field $\Delta(\vec{r},\tau)$ with a Gaussian weight of the exponential of a linear form in $\psi_\downarrow(\vec{r})\psi_\uparrow(\vec{r})$], it is possible to rewrite the partition function as a functional integral,

$$\mathcal{Z}=\mathcal{Z}_0\int \mathcal{D}\Delta(\vec{r},\tau)\mathcal{D}\Delta^*(\vec{r},\tau)$$
$$\times\exp\left(-(1/|V|)\int_0^\beta d\tau\int d^3r|\Delta(\vec{r},\tau)|^2-\beta\Omega_B\right), \tag{12}$$

where $e^{-\beta\Omega_B}\equiv\langle T_\tau\exp[-\int_0^\beta d\tau\mathcal{H}_B(\tau)]\rangle_0$ and

$$\mathcal{H}_B(\tau)=-\int d^3r[\Delta(\vec{r},\tau)\overline{\psi_\uparrow}(\vec{r},\tau)\overline{\psi_\downarrow}(\vec{r},\tau)$$
$$+\Delta^*(\vec{r},\tau)\psi_\downarrow(\vec{r},\tau)\psi_\uparrow(\vec{r},\tau)],$$

with the field operators in the Matsubara representation: $\psi_\sigma(\tau,\vec{r})=e^{\mathcal{H}_0\tau}\psi_\sigma(\vec{r})e^{-\mathcal{H}_0\tau}$ and $\overline{\psi_\sigma}(\tau,\vec{r})=e^{\mathcal{H}_0\tau}\psi_\sigma^\dagger(\vec{r})e^{-\mathcal{H}_0\tau}$.

The stationary-phase approximation (or more precisely, the steepest-descent approximation) for the functional integral over the fields Δ and Δ^* is equivalent to the mean-field approximation. It yields the well-known self-consistency relation

$$\Delta^*(\vec{r},\tau)=|V|\langle T_\tau\overline{\psi_\uparrow}(\vec{r},\tau)\overline{\psi_\downarrow}(\vec{r},\tau)\rangle, \tag{13}$$

where "mean" stands for $\langle\mathcal{A}\rangle=\mathrm{tr}\{e^{-\beta\mathcal{H}_0}T_\tau\mathcal{A}\exp[-\int_0^\beta d\tau\mathcal{H}_B(\tau)]\}/\mathcal{Z}$.

The auxiliary field Δ in the stationary-phase approximation is therefore identical to the superconducting order parameter defined within BCS (mean-field) theory. The physical meaning of $\Delta(\vec{r},\tau)$ in the general case (i.e., as a variable of functional integration) is now also clear: it describes all possible pairing configurations, including fluctuating superconducting droplets as well as the mean-field configuration with long-range superconducting order.

Neglecting fluctuations (see, however, Sec. IV), the thermodynamic potential of the superconductor is then

$$\Omega\equiv-\beta^{-1}\ln\mathcal{Z}$$
$$=-\beta^{-1}\ln\mathrm{tr}e^{-\beta\mathcal{H}_P}+\frac{1}{|V|}\int d^3r|\Delta(\vec{r})|^2, \tag{14}$$

where $\mathcal{H}_P\equiv\mathcal{H}_0+\mathcal{H}_B(0)$. This result is identical to the large-κ limit (i.e., $B\rightarrow H$) of the expression derived by Eilenberger (Eilenberger, 1966). The first term on the right-hand side of Eq. (14) was expressed in terms of the thermal Green's function $G_\omega(\vec{r},\vec{r'})$ obtained from the solution of Gorkov's equations (Gorkov, 1959),

$$-\beta^{-1}\ln \mathrm{tr} e^{-\beta\mathcal{H}_P}=i\beta^{-1}\sum_{\nu=-\infty}^{\infty}\int_{\omega_\nu}^{\infty sgn\,\nu}d\omega \times\int d^3r[G(\vec{r},\vec{r};\omega)-\text{c.c.}], \qquad (15)$$

where $\omega_\nu=(2\nu+1)\pi/\beta$, $\nu=0,\pm1,\dots$ are the Matsubara frequencies.

The mean-field pairing Hamiltonian,

$$\mathcal{H}_P=\mathcal{H}_0-\int d^3r[\Delta(\vec{r})\psi_\uparrow^\dagger(\vec{r})\psi_\downarrow^\dagger(\vec{r}) +\Delta^*(\vec{r})\psi_\downarrow(\vec{r})\psi_\uparrow(\vec{r})], \qquad (16)$$

which was first proposed on rather intuitive grounds by de Gennes (1966), can be diagonalized by a generalized version of the Bogoliubov transformation,

$$\psi_\uparrow(\vec{r})=\sum_\lambda [u_\lambda(\vec{r})\gamma_{\lambda,\uparrow}-v_\lambda^*(\vec{r})\gamma_{\lambda,\downarrow}^\dagger],$$

$$\psi_\downarrow^\dagger(\vec{r})=\sum_\lambda [v_\lambda(\vec{r})\gamma_{\lambda,\uparrow}+u_\lambda^*(\vec{r})\gamma_{\lambda,\downarrow}^\dagger], \qquad (17)$$

where the quasiparticle operators $\gamma_{\lambda,\uparrow}$, $\gamma_{\lambda,\downarrow}$ satisfy fermionic commutation relations. In order to satisfy these relations, the functions $u_\lambda(\vec{r})$ and $v_\lambda(\vec{r})$ must satisfy the orthonormality conditions

$$\int [u_\lambda^*(\vec{r})u_\mu(\vec{r})+v_\lambda^*(\vec{r})v_\mu(\vec{r})]d^3r=\delta_{\lambda,\mu},$$

$$\int [u_\lambda(\vec{r})v_\mu(\vec{r})-v_\lambda(\vec{r})u_\mu(\vec{r})]d^3r=0,$$

and the completeness conditions

$$\sum_\lambda [u_\lambda(\vec{r})u_\lambda^*(\vec{r}')+v_\lambda^*(\vec{r})v_\lambda(\vec{r}')]=\delta(\vec{r}-\vec{r}'),$$

$$\sum_\lambda [u_\lambda(\vec{r})v_\lambda^*(\vec{r}')-v_\lambda^*(\vec{r})u_\lambda(\vec{r}')]=0.$$

The Hamiltonian $\mathcal{H}_P$ is diagonal in the new fermionic quasiparticle representation, i.e.,

$$\mathcal{H}_P=\mathcal{E}_0+\sum_{\lambda(\varepsilon_\lambda>0)}\varepsilon_\lambda(\gamma_{\lambda,\uparrow}^\dagger\gamma_{\lambda,\uparrow}+\gamma_{\lambda,\downarrow}^\dagger\gamma_{\lambda,\downarrow}), \qquad (18)$$

provided that the functions u_λ, v_λ satisfy the following "eigenvalue" equations with respect to the eigenenergies ε_λ:

$$\hat{\xi}u_\lambda(\vec{r})+\Delta(\vec{r})v_\lambda(\vec{r})=\varepsilon_\lambda u_\lambda(\vec{r}),$$

$$-\hat{\xi}^*v_\lambda(\vec{r})+\Delta^*(\vec{r})u_\lambda(\vec{r})=\varepsilon_\lambda v_\lambda(\vec{r}), \qquad (19)$$

where $\hat{\xi}$ is the kinetic-energy operator $\hat{\xi}=(1/2m)[(\hbar/i)\nabla-(e/c)\vec{A}]^2-E_F$.

It is important to note that the resulting quasiparticle spectrum obeys particle-hole symmetry. Indeed, Eqs. (19) have both positive and negative values of ε_λ, for if $\varepsilon_\lambda>0$ corresponds to the spinor solution (u_λ, v_λ), then the spinor $(v_\lambda^*, -u_\lambda^*)$ is also a solution with energy $-\varepsilon_\lambda$. The set (u_λ, v_λ) with $\varepsilon_\lambda>0$ is, however, a complete set of functions.

The first term, $\mathcal{E}_0$, on the right-hand side of Eq. (18) is the ground-state energy of the superconductor. The second is the quasiparticle term, corresponding to the elementary excitations above the ground state. Equation (15) provides the basis for the alternative picture sketched in Fig. 4; indeed, as shown by Bardeen *et al.* (1969), substitution of the quasiparticle spectral representation of the Green's function,

$$G(\vec{r},\vec{r}';\omega)=\sum_\lambda\left(\frac{u_\lambda(\vec{r})u_\lambda^*(\vec{r}')}{i\omega-\varepsilon_\lambda}+\frac{v_\lambda^*(\vec{r})v_\lambda(\vec{r}')}{i\omega+\varepsilon_\lambda}\right),$$

in Eqs. (15) and (14) leads to the formula

$$\Omega\equiv-\beta^{-1}\ln\mathcal{Z} =-2\beta^{-1}\sum_{\lambda(\varepsilon_\lambda<E_F)}\ln\left(2\cosh\frac{1}{2}\beta\varepsilon_\lambda\right) +\frac{1}{|V|}\int d^3r|\Delta(\vec{r})|^2. \qquad (20)$$

The sum over λ is just what one would expect from the free energy of an assembly of independent fermions with energies ε_λ. As in the Hartree-Fock approximation, one should subtract from this expression the interaction energy $-(1/|V|)\int d^3r|\Delta(\vec{r})|^2$, because of double counting. It should be noted that, as far as the magnetization in the mean-field approximation is concerned, the similarity to an assembly of independent fermions is not disturbed by this additional term, since the stationary condition $\partial\Omega/\partial\Delta=0$ guarantees that

$$M=-\frac{\partial\Omega}{\partial B}=2\beta^{-1}\frac{\partial}{\partial B}\sum_\lambda \ln\left(2\cosh\frac{1}{2}\beta\varepsilon_\lambda\right)_\Delta, \qquad (21)$$

where the subscript Δ means that the term in parentheses is assumed constant under the differentiation.

It should be stressed that the particle-hole symmetry inherent to the quasiparticle spectrum in the superconducting state does not exist in the normal-state (Landau-level) spectrum (except for discrete values of the applied magnetic field; see Sec. II.B.2)—a fundamental complication associated with the theory of quantum magnetic oscillations in the superconducting state.

B. Quasiparticles in a quantizing magnetic field

The physical picture emerging from the general analysis presented above leads to some modification of the highly simplified picture outlined in Sec. I.B.3. The Bogoliubov–de Gennes Eqs. (19) describe Andreev reflections (Andreev, 1964a, 1964b, 1964c) of a quasihole with amplitude $u_\lambda(\vec{r})$ by a quasielectron with amplitude $v_\lambda(\vec{r})$ at the "interfaces" between the normal domains [i.e., near the zeros of the order parameter $\Delta(\vec{r})$] and the interstitial regions of significant superconducting order. Thus a quasielectron is transformed near a vortex core into a quasihole that carries the same current in the same direction. In the picture drawn in Sec. I.B.3 the quasiparticle trajectory should therefore be regarded as partly quasielectronlike and partly quasiholelike. Using for u_λ and v_λ a basis set with magnetic translational symmetry automatically takes into account the magnetic breakdown of the cyclotron orbitals, as depicted in Fig. 4. The self-consistency condition [Eq. (13); see also Eq. (22) below] reflects the common origin of the currents in the vortex core and the quasiparticle cyclotron currents.

The problem under consideration now reduces to the calculation of the magnetization from Eq. (21) with the quasiparticle energies ε_λ obtained from solution of the Bogoliubov–de Gennes Eqs. (19). The pair potential $\Delta(\vec{r})$ should be determined self-consistently by Eq. (13). The latter condition can be rewritten in terms of the above solution as

$$\Delta^*(\vec{r},\beta)=|V|\sum_\lambda u_\lambda^*(\vec{r})v_\lambda(\vec{r})\tanh[\beta\varepsilon_\lambda(\Delta^*)/2]. \quad (22)$$

Using for the spatial dependence of Δ a form corresponding to the Abrikosov lattice of vortices (Abrikosov, 1957), the mean-field pairing Hamiltonian (16) is seen to be invariant under the magnetic translations associated with this lattice. A basis set of eigenfunctions of these symmetry operators (Bychkov and Rashba, 1983) can in principle diagonalize the Hamiltonian and satisfy the self-consistency condition (22). In practice, however, the complete solution of this problem is a formidable task.

1. The magnetic Bloch quasiparticle

Several attempts to solve the Bogoliubov–de Gennes Eqs. (19) by exploiting magnetic translational symmetry have been described in the literature. The most comprehensive efforts are those of Dukan, Andreev, and Tesanovic (1991), Norman, Akera, and MacDonald (1992, 1995), Dukan and Tesanovic (1994), and Norman and MacDonald (1996). Let us describe here in some detail the general framework of the theory before discussing its applications.

Assuming an isotropic 3D electron-gas model in a stationary and uniform external magnetic field $\vec{H}$, oriented along the z axis, we can write the corresponding vector potential in the Landau gauge as $\vec{A}=H(-y,0,0)$. The corresponding Abrikosov form (Abrikosov, 1957) of the superconducting order parameter can be written in terms of an arbitrary 2D lattice spanned in the x-y plane by the primitive vectors $\vec{a}=(a_x,0,0)$, $\vec{b}=(b_x,b_y,0)$,

$$\Delta(\vec{r})=\Delta_0\sum_{k=-\sqrt{N}/2}^{\sqrt{N}/2} c_k\exp[iq_kx-(y+q_k/2)^2], \quad (23)$$

with $c_k=e^{i\gamma k^2}, \gamma=\pi b_x/a_x, q_k=2\pi k/a_x$. Here it is assumed that spatial distances are measured in units of the magnetic length, $a_H=\sqrt{\hbar c/eH}$. The relation $a_xb_y=\pi$ states that a single Cooper-pair flux quantum $\phi_0=ch/2e$ (or alternatively 1/2 electronic flux quantum $\Phi_0=ch/e$) threads a unit cell in this lattice.

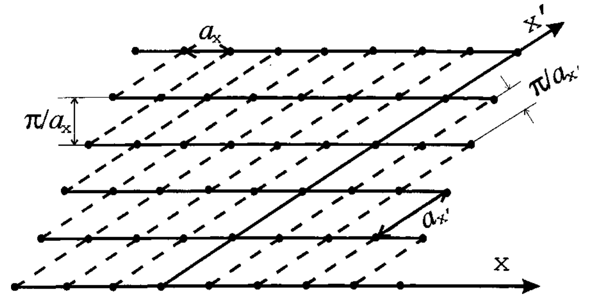


FIG. 7. Two families of Bragg chains in a triangular Abrikosov lattice along the principal axes x and x'.

Let us consider the geometrical meaning of the coefficients $c_k=e^{i\varphi_k}$, with $\varphi_k=\gamma k^2$. Each coefficient describes a set of $\sqrt{N}$ guiding centers, periodically arranged within a certain chain along the principal crystallographic axis x of the vortex lattice (see Fig. 7). The index k labels the vertical (along the y axis) position of the chain, whereas the phase φ_k determines the relative "horizontal" position $x_k=-\varphi_k/q_k=-a_x\gamma k/2\pi$ of the kth chain. The totality of this family of Bragg chains spans the entire Abrikosov lattice. An equivalent family of Bragg chains can be identified along the other principal axis denoted by x' in Fig. 7, with a period $a_{x'}$.

Note that for an arbitrary 2D lattice the lattice constant a_x (in units of magnetic length) and the angle $\Theta/2$ between the principal axes are expressed via γ as $a_x^2=\pi^2/\gamma\tan\Theta$. For a general rhombic lattice $a_x^2=\pi/\sqrt{1-(\gamma/\pi)^2}$, $\cos\Theta=(\gamma/\pi)$, while in the special case of the triangular Abrikosov lattice, $\gamma=\pi/2$.

Equation (23) describes a coherent superposition of $N\gg1$ ground Landau orbitals of Cooper pairs $\varphi_0(x,y+q_k/2)=\exp[iq_kx-(y+q_k/2)^2]$, arranged with their guiding centers in a lattice equivalent to that described in Fig. 7, such that a single flux quantum ϕ_0 is attached to each orbital.

An alternative, more general form of the superconducting order parameter in the lowest Landau-level approximation, illuminating this point, was used in the literature by several authors (Rajagopal and Vasudevan, 1966a, 1966b; Tesanovic *et al.*, 1989, 1991; Rasolt and Tesanovic, 1992). It is written in symmetric gauge,

$$\Delta_{sym}(\vec{r})=e^{ixy}\Delta(\vec{r})=\Delta_0e^{-(1/2)|z|^2}f(z), \quad (24)$$

where $f(z)$, $z=x+iy$, is an arbitrary entire function with $N=\phi/\phi_0$ zeros in the complex plane, where ϕ is the total magnetic flux threading the sample. The function f has the general form

$$f(z)\propto\prod_{j=1}^{N}(z-z_j).$$

The power 1 of all factors $(z-z_j)$ in this expression stems from the physical fact stated above that in the ground state each vortex will carry a single flux quantum ϕ_0. It is more general than Eq. (23), since the positions of the vortices are not necessarily restricted to a regular periodic lattice. This lowest-Landau-level description is a good approximation almost everywhere around $H_{c2}(T)$ [actually down to $\simeq\frac{1}{3}H_{c2}(T)$], except for a small region around $T_c(H\to 0)$ where the Ginzburg critical regions of the lowest Landau level and higher Landau levels start to overlap (see Tesanovic and Andreev, 1994).

The basis set selected by Dukan *et al.* (1991) was chosen as the common eigenfunction $e^{ik_z\zeta}\phi_{\vec{q},n}(x,y)$ of the magnetic translation operators and the kinetic-energy operator $\hat{\xi}$, where ζ and k_z are the spatial coordinate and momentum, respectively, along the magnetic-field direction. The function $\phi_{\vec{q},n}(x,y)$ (a magnetic Bloch function) is constructed as a (tight-binding-like) coherent superposition of localized Landau orbitals $\varphi_n(x,y+y_0)=(1/\sqrt{2^n n!\sqrt{\pi}})\,e^{iy_0x-(1/2)(y+y_0)^2}H_n(y+y_0)$, where $H_n(y)$ is the Hermitian polynomial of degree n, with a given energy $\xi_n=\hbar\omega_c(n+1/2)-E_F$, centered on a lattice spanned by the vectors $2\vec{a},\vec{b}$ (i.e., with projections $y_{0,m}=q_x+\pi m/a_x$ on the y axis):

$$\phi_{\vec{q},n}(x,y)\propto\sum_m e^{(i/2)\,\gamma m^2}e^{-iq_y\pi m/a_x}\varphi_n\left[x,y+\left(q_x+\frac{\pi m}{a_x}\right)\right].$$

The selection of the lattice as a double superlattice of the original Abrikosov lattice assures that a single electronic flux quantum $\Phi_0=ch/e$ threads its unit cell. The quasicontinuous wave vector $\vec{q}$ is restricted to the magnetic Brillouin zone defined by the vectors $\vec{a}^*=(b_y,-b_x),\vec{b}^*=(0,2a_x)$.

The coefficients of the quasiparticle annihilation-creation operators in the Bogoliubov transformation, Eq. (17), are given in this representation by

$$u_{k_z,\vec{q}}(x,y)=e^{ik_z\zeta}\sum_{n=0}^{\infty}u_n(\vec{q})\,\phi_{\vec{q},n}(x,y), \tag{25}$$

$$v^*_{k_z,\vec{q}}(x,y)=e^{-ik_z\zeta}\sum_{n=0}^{\infty}v^*_n(\vec{q})\,\phi_{-\vec{q},n}(x,y),$$

showing clearly the characteristic Andreev reflection mixing mechanism of a quasihole, which carries a momentum $\vec{q}$, with a quasielectron carrying a momentum $-\vec{q}$. This exact relation guarantees that the electrical current carried by the quasiparticle, regarded as a mixture of a quasielectron with a quasihole, is uniquely defined.

The technical advantage of this representation, in addition to its elegant similarity to the paired states at zero magnetic field, is that the corresponding matrix elements of the pairing Hamiltonian are diagonal in momentum space. The price paid for this advantage, however, is quite heavy, since basis functions with different Landau-level indices $n\neq n'$ are strongly mixed by the pairing Hamiltonian. The origin of this complication is, of course, the matrix elements of the pair potential (obviously $\mathcal{H}_0$ is diagonal):

$$\begin{aligned}\Delta_{n,n'}(\vec{q})&\equiv\int d^2r\,\phi^*_{-\vec{q},n'}(\vec{r})\Delta(\vec{r})\phi^*_{\vec{q},n}(\vec{r})\\&=\frac{\Delta_0(-1)^{n'}}{\sqrt{2}2^{n+n'}\sqrt{n!n'!}}\\&\quad\times\sum_k\exp[i\gamma k^2+2ikb_yq_y-(q_x+\pi k/a_x)^2]\\&\quad\times H_{n+n'}[\sqrt{2}(q_x+\pi k/a_x)],\end{aligned} \tag{26}$$

where $\vec{r}$ denotes here a position vector in the x-y plane. Note that $\Delta_{0,0}(\vec{q})$ is just the real-space order parameter $\Delta(\vec{r})$ after the 90° rotation $(x,y)\to(q_y,q_x)$. This result is consistent with the description presented in Sec. I.B.1, where $\Delta(\vec{r})$ was expressed in terms of the classical momentum $\vec{k}(\vec{r})$ through the Lorenz equation of motion, Eq. (3), which implies that $\vec{k}$ is always rotated by 90° with respect to $\vec{r}$.

This immediately implies that the positions of the zeros of $\Delta_{0,0}(\vec{q})$ are just the positions of the zeros of the Abrikosov order parameter rotated by 90°, and that these zeros are all of the first order since, as shown above, the zeros of $\Delta(\vec{r})$ are of the first order.

Clearly, the matrix elements $\Delta_{n,n'}(\vec{q})$ of importance to us are not $\Delta_{0,0}$ but those corresponding to Landau-level indices n,n' of the order of n_F, which is much larger than unity. However, as found in the numerical calculations of Dukan *et al.* (1991), the first-order zeros of $\Delta_{n,n}(\vec{q})$ are at the same positions as those of $\Delta_{0,0}(\vec{q})$, while zeros of higher orders also appear but with relative abundances of $1/n_F$ to the first-order zeros. This property is of crucial importance in the behavior of the quasiparticle under high magnetic fields, as will be discussed later.

2. Magnetic bands and pseudogaps in the quasiparticle spectrum

The analysis presented in the previous subsection enables us now to address the general question of the fate of the Landau-level structure of the quasiparticle spectrum in the vortex state. This problem is closely related to the fundamental particle-hole symmetry requirement imposed by the pair potential in the superconducting state. Indeed, as indicated above, the exact diagonalization of the pairing Hamiltonian in momentum subspace is not followed by a diagonalization in Landau-level subspace. On the contrary, strong Landau-level mixing of the magnetic Bloch basis functions by the pair potential seems to imply that the Landau-level structure should be quickly destroyed by the increasing superconducting order parameter below H_{c2}. Unfortunately this mixing also makes any analytical approach to this problem, except for narrow limiting cases, an extremely complicated task. A very popular limiting case of this type has been advocated by Tesanovic and co-workers in several papers. To explain this type of approximation, let us consider the relatively transparent case of a 2D electron system, and add to the model Hamiltonian analyzed above the Zeeman spin energy term. We make the following change in the kinetic-energy operator: $\hat{\xi}\rightarrow\hat{\xi}_{\uparrow,\downarrow}=\hat{\xi}\mp\frac{1}{2}g\mu_B B$, where g is the Lande g factor and μ_B is the Bohr magneton. We then obtain two sets of separable Bogoliubov–de Gennes equations for $(u_{\lambda,\uparrow},v_{\lambda,\downarrow})$ and for $(u_{\lambda,\downarrow},v_{\lambda,\uparrow})$. Using the expansions (25) in the magnetic translations basis, we find that the first set of equations reads

$$\sum_{n'}[(\xi_{n',\uparrow}-\varepsilon)\delta_{n',n}u_{n'}(\vec{q})+\Delta_{n,n'}(\vec{q})v_{n'}(\vec{q})]=0,$$

$$\sum_{n'}[\Delta^*_{n,n'}(\vec{q})u_{n'}(\vec{q})-(\xi_{n',\downarrow}+\varepsilon)\delta_{n,n'}v_{n'}(\vec{q})]=0,$$

where the spin index was dropped from the functions u,v.

The limit considered by Tesanovic *et al.* corresponds to the special situation in which the g factor vanishes and the chemical potential is very close to one of the Landau levels, say $\xi_n\rightarrow 0$ (or $n_F\rightarrow n+1/2$). If, in addition, the strength of the pair potential Δ_0 is much smaller than the cyclotron energy $\hbar\omega_c$, the off-diagonal elements $\Delta_{n',n}(\vec{q})$, $n'\neq n$ can be neglected and the quasiparticle energies are given approximately by

$$\varepsilon(\vec{q})\approx\pm\sqrt{\xi_n^2+|\Delta_{n,n}(\vec{q})|^2}\rightarrow\pm|\Delta_{n,n}(\vec{q})|. \quad (27)$$

This simple "diagonal approximation" reflects, in a very transparent way, three important features of the quasiparticle spectrum in a strong magnetic field: the broadening of the Landau levels into energy bands, the magnetic Bloch bands; the disappearance of the superconducting energy gap from the Fermi surface (see Sec. I.B.1), and the opening of pseudogaps within the Landau bands. The first feature, which is due to the q dispersion of $\Delta_{n,n}$ in Eq. (27), is of course not surprising in light of the extended nature of the quasiparticle wave function in the entire vortex lattice. The second feature is a direct consequence of the zeros of $\Delta_{n,n}(\vec{q})$ (see an extensive discussion of this aspect by Dukan and Tesanovic, 1994). The third feature is less obvious. It can be readily seen, following the original analysis of Dukan *et al.* (1991), by considering the quasiparticle density of states $\mathcal{D}(\varepsilon)$ near the center of the band. Here $\Delta_{n,n}(\vec{q})\approx 0$, which implies that the dominant contributions to the density of states come from the close vicinities of the zeros of $\Delta_{n,n}(\vec{q})$. Since for $n_F\gg 1$ most of these zeros, $\tilde{q}_j=q_{x,j}+iq_{y,j}$, are of the first order, a leading-order expansion yields

$$\varepsilon(\vec{q})\rightarrow\pm|\Delta_{n,n}(\vec{q})|\propto\pm|\tilde{q}-\tilde{q}_j|, \quad \tilde{q}\rightarrow\tilde{q}_j,$$

which results in the quasiparticle density of states' also linearly vanishing near the band center, i.e.,

$$\mathcal{D}(\varepsilon)\propto|\varepsilon|\rightarrow 0.$$

It is clear that this break at the center of the $n=n_F-1/2$ Landau band is due to the linear dispersion relation $\varepsilon(\vec{q})$ near the first-order zeros of $|\Delta_{n,n}(\vec{q})|$. The existence of this kind of zero is a consequence of the fundamental condition of flux quantization and its realization in the Abrikosov lattice, where each unit cell is threaded by 1/2 electronic flux quantum Φ_0 (or alternatively by a single Cooper-pair flux quantum ϕ_0). Thus it seems that one should expect the appearance of pseudogaps in all Landau levels broadened by the pair potential, as first suggested by Maniv *et al.* (1997).

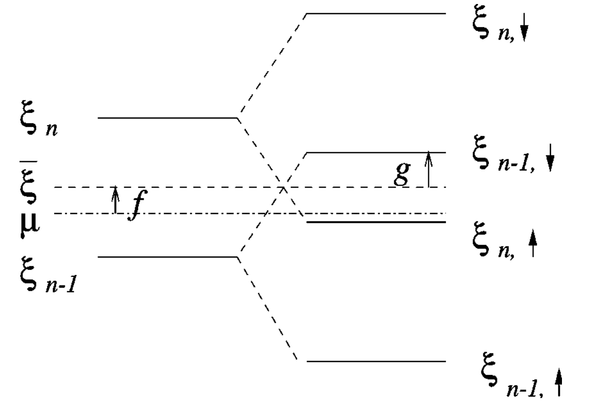


FIG. 8. A quartet of Landau spin-split sublevels around the Fermi energy which are dominantly involved in superconducting pairing in the extreme limit $\Delta_0\ll\hbar\omega_c$.

To convince ourselves that this phenomenon is indeed not a peculiarity of the Landau band containing the chemical potential [the conduction (Landau) band], let us consider the situation when the chemical potential μ is in between two adjacent Landau levels ξ_n and ξ_{n-1} (illustrated in Fig. 8). We may also assume a somewhat more general situation when the Zeeman splitting is not zero. To make the calculation analytically tractable, however, we assume as above that $\Delta_0\ll\hbar\omega_c$.

This situation allows us to neglect all matrix elements $\Delta_{n,n'}(\vec{q})$ except for $n'=n,n-1$, so that after using the symmetry relation $\Delta_{n,n-1}(\vec{q})=-\Delta_{n-1,n}(\vec{q})$, the relevant Bogoliubov–de Gennes Hamiltonian matrix reduces to the 4×4 block (with all energies measured in units of $\hbar\omega_c$),

$$\begin{pmatrix} f-g & d_1 & 0 & d_3 \\ d_1^* & -1-g-f & -d_3^* & 0 \\ 0 & -d_3 & -1-g-f & d_2 \\ d_3^* & 0 & d_2^* & -g-f \end{pmatrix},$$

where $g\equiv\xi_{n-1,\downarrow}-\overline{\xi}$, $f\equiv\overline{\xi}-\mu$, $\overline{\xi}\equiv\frac{1}{2}(\xi_{n-1}+\xi_n)$, $d_1\equiv\Delta_{n,n}(\vec{q})$, $d_2\equiv\Delta_{n-1,n-1}(\vec{q})$, and $d_3\equiv\Delta_{n-1,n}(\vec{q})=-\Delta_{n-1,n}(\vec{q})$.

This matrix can be diagonalized exactly to yield four quasiparticle energies around the chemical potential,

$$\varepsilon_1^{\pm}=-\frac{1}{2}-g+\frac{1}{2}f_{\pm}, \quad \varepsilon_2^{\pm}=-\frac{1}{2}-g-\frac{1}{2}f_{\pm},$$

where $f_{\pm}=(1+4f^2+2|d_1|^2+2|d_2|^2+4|d_3|^2\pm 2\sqrt{Q})^{1/2}$, $Q=4f^2+4|d_3|^2+4f(|d_1|^2-|d_2|^2)+O(d^4)$.

Assuming the chemical potential to be located exactly in the middle of the quartet (electron-hole symmetry), i.e., taking $f=0$ and neglecting second- and higher-order terms in d_i, we find for the occupied quasiparticle energies

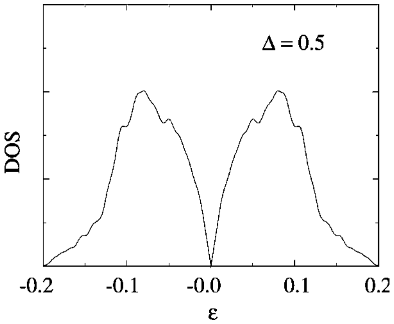


FIG. 9. Quasiparticle density of states calculated numerically by Tesanovic and Sacramento (1998) for a 2D system in which $\tilde{\Delta}_0=\Delta_0/\hbar\omega_c=0.5$ and $n_F=40$. The chemical potential position is at the center of the Landau band.

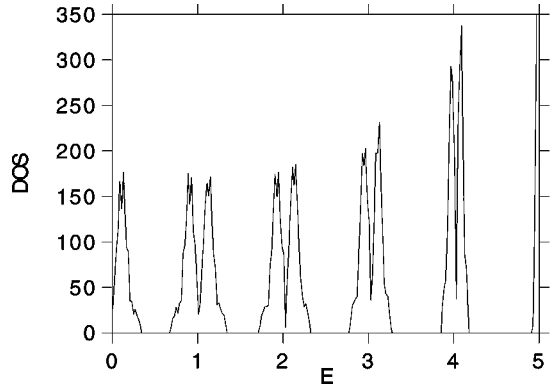


FIG. 10. A result similar to Fig. 9 for $\tilde{\Delta}_0=1$ and $n_F=20$ calculated by Norman and MacDonald (1996), showing the opening of pseudogaps in all Landau bands.

$$\varepsilon_1^{\pm}(\vec{q})\rightarrow -g\pm|\Delta_{n-1,n}(\vec{q})|,$$
$$\varepsilon_2^{\pm}(\vec{q})\rightarrow -1-g\pm|\Delta_{n-1,n}(\vec{q})|.$$

The unoccupied energies can be obtained from the dual set of Bogoliubov–de Gennes equations corresponding to $(u_{\lambda,\downarrow},v_{\lambda,\uparrow})$. Here we find again the same type of behavior as in the previous example, namely, that the two normal-state Landau sublevels $\xi_{n,\uparrow}=-g$ and $\xi_{n-1,\uparrow}=-1-g$ broaden into magnetic bands, which break into subbands by the pair potential. The density of states vanishes linearly with the energy distance from the band centers, i.e., $\mathcal{D}(\varepsilon)\propto|\varepsilon+g|\rightarrow 0$, or $\propto|\varepsilon+1+g|\rightarrow 0$, due to the first-order zeros of $\Delta_{n-1,n}(\vec{q})$. Note, however, that in contrast to the previous case in which the pairing was "diagonal in the Landau levels," in the present case it is "off diagonal." The reason in the present case is, of course, the location of the chemical potential in the cyclotron gap between two Landau sublevels.

The problem becomes much more complicated in cases of lower symmetry, e.g., when the chemical potential is shifted from the Landau level or from the center of the cyclotron gap, or when the strength of the pair potential is not small compared to this gap. In these cases more distant Landau levels should be taken into account and the problem quickly becomes very cumbersome due to the presence of many significant matrix elements far from the diagonal.

However, numerical diagonalizations carried out by Norman and MacDonald (1996), as well as by Tesanovic and Sacramento (1998) for cases when Δ_0 is no longer small compared to $\hbar\omega_c$ (see Figs. 9 and 10), confirmed the general pattern of the quasiparticle density of states obtained above.

Specifically they found that as long as $\tilde{\Delta}_0=\Delta_0/\hbar\omega_c$ does not exceed some critical value $\tilde{\Delta}_{0,c}$, the superconducting quasiparticle spectrum for a 2D electron system, with the chemical potential in a symmetrical electron-hole position, consists of broadened Landau levels, each of which splits into two subbands separated by a pseudogap. In this region the broadened levels do not show any apparent shift with respect to the corresponding free-electron Landau levels, implying the absence of a superconducting energy gap. The critical value $\tilde{\Delta}_{0,c}$ was found to be significantly larger than unity, but its dependence on relevant parameters, such as, for example, n_F, could not be reliably determined from these numerical calculations (see Sec. V for more details).

No result for the quasiparticle density of states for asymmetrical position of the chemical potential has been reported so far. Such a calculation is particularly cumbersome since the quasiparticle spectrum in the superconducting state obeys particle-hole symmetry [see the remark following Eqs. (19)], whereas the normal-state Landau-level representation used above lacks this symmetry at all magnetic fields; discrete values when n_F is integral or half integral are exceptions.

In light of Eq. (21), which is identical (in its general structure) to the well-known formula derived for a gas of normal-state quasiparticles (see, for example, Shoenberg, 1984b), one can readily understand the implications of the main features of the quasiparticle density of states shown above. It is, of course, quite obvious that the broadening of the Landau levels into bands leads to superconducting-induced damping of the dHvA amplitude. The splitting of each Landau band into two subbands is reminiscent of the Zeeman spin splitting. This observation has a deeper origin than may appear at first sight. Indeed, as discussed in Sec. I.B.3, the extremal cyclotron orbit near the Fermi surface, which constitutes the dominant contribution to the dHvA oscillations, consists of many small sectors around the vortex cores where the current density has a paramagnetic nature, in contrast to the diamagnetic character of the intervortex current density (see Fig. 11). These small current loops, which are "attached" to the large diamagnetic cyclotron loop near many vortex cores, can be regarded as spin-1/2 quantum paramagnets.

The quantum analogy to the spin-1/2 case can be described as follows: a paramagnetic current loop is generated by the quasielectron-quasihole (Andreev) mixture rotating in opposite directions around the core in its close vicinity. The corresponding angular momenta along the z axis, equal in magnitude but opposite in direction, may be regarded as the two quantized projections of the "spin" along this axis. Furthermore, the

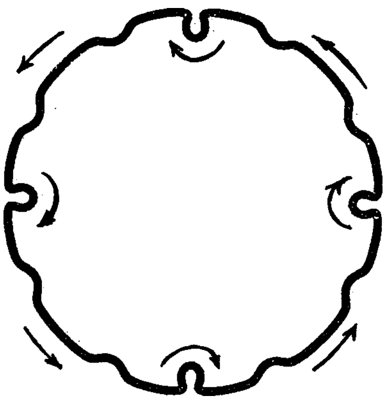


FIG. 11. The vortex core paramagnets "attached" to a diamagnetic cyclotron orbit in the vortex lattice.

charge carried by each branch of the quasiparticle is the electron charge e, so that the phase acquired by each component of its wave function in a full rotation around the core is π [the corresponding phase acquired by the Cooper-pair wave function $\Delta(\vec{r})$ in such a rotation is 2π]. This feature again characterizes a particle of angular momentum 1/2. The first-order vanishing of $\Delta(\vec{r})$ at the vortex cores is responsible for the pseudogap in the quasiparticle density of states (and the breaking of a Landau band into two subbands); it reveals the deep connection between the spin-1/2 character of the vortex core paramagnets and the splitting of the Landau bands into two magnetic subbands.

We thus find that the vortex currents near the cores can be regarded as a coherent ensemble of spin-1/2 quantum paramagnets, in an effective "magnetic field" growing with the pair-potential amplitude $|\Delta_0|$, as suggested by Eqs. (7) and (8). It will be shown in Secs. II.C.3 and II.C.4 that the suppression mechanism of the dHvA oscillations associated with these vortex currents is the leading superconducting effect just below H_{c2}, as proposed in Sec. I.B.3.

C. The Gorkov-Ginzburg-Landau approach

We have already discussed in the previous section the advantages and the disadvantages of approaching the problem of quantum magnetic oscillations in the vortex state through its ultimate microscopic structure—the quasiparticle spectrum. For the somewhat limited physical insight gained from a very detailed analysis of the quasiparticle microscopic features, one pays a high price in the quantitative aspects concerned, e.g., when one needs concrete answers about the magnetic-field dependence of the dHvA amplitude in the vortex state. This "differential" approach produces a very complex band structure, which can be further used in calculating the thermodynamic potential in two possible ways: either by "brute force" numerical computation, or by pursuing limiting cases when simple analytical forms for the quasiparticle dispersion relation exist. Unfortunately, both ways have serious shortcomings. The numerical calculations with their obvious limitations are also restricted to a relatively small number of Landau levels, whereas all attempts made so far to find a simple analytical model of the quasiparticle density of states have failed to produce a valid result at magnetic fields that are not extremely close to H_{c2}.

An alternative, "integral" approach has been taken by several investigators (Maniv, Markiewicz, *et al.*, 1992; Maniv, Rom, *et al.*, 1992; Bruun *et al.*, 1997; Vavilov and Mineev, 1997). Here one deliberately gives up some detailed microscopic information by directly dealing with the thermodynamic potential, in order to be able to make progress in calculating the relevant observables. Basically such an approach is based on the Green's-function formalism developed by Gorkov for spatially inhomogeneous superconductors (Abrikosov, Gorkov, and Dzaloshinski, 1975). Since later in this review we shall be concerned with situations in which fluctuations of the superconducting order parameter are important, we present here a more general formulation than the mean-field approximation used in the classic work of Gorkov by following the functional-integral formalism outlined in Sec. II.A.

1. Perturbative expansions in the superconducting order parameter

Considering the partition function in the functional-integral form, Eq. (12), one may integrate over the electronic fields and then generate a new exponent, which is, however, a complicated functional of the pair field $\Delta(\vec{r},\tau)$. At this point one is forced to make a major approximation: if we assume that the magnitude of Δ is small, we may proceed by expanding the new exponent in powers of Δ and terminate at a desired order. This procedure may be justified sufficiently close to the second-order phase transition at H_{c2} where the superconducting order parameter is small enough.

A careful reader may note here that due to the Pauli paramagnetic pair-breaking effect (Fulde, 1969), the transition should turn first order, in which case the discontinuous jump of the order parameter at the transition point may not allow the proposed expansion. As shown in the Introduction, however, the magnitude of this jump is very small on the full scale of the order parameter.

A more serious problem concerns the use of the normal-state Hamiltonian for a uniform electron gas in an external magnetic field as the zeroth-order Hamiltonian, since the infinite degeneracy of the corresponding unperturbed eigenfunctions invalidates the expansion in the zero-temperature limit. At finite temperatures and/or in the presence of Landau-level smearing by disorder, however, the convergence of this expansion is quickly recovered (for more detailed discussions see Secs. II.C.2 and II.C.5).

Thus, assuming a second-order or nearly second-order phase transition at H_{c2}, neglecting quantum fluctuations (i.e., time dependence) of the pair field, and terminating the expansion in Δ at fourth order, we recover the form (Maniv, Markiewicz, *et al.*, 1992; Maniv, Rom, *et al.*, 1992; Tesanovic and Andreev, 1994)

$$\mathcal{Z}=\mathcal{Z}_0\int \mathcal{D}\Delta(\vec{r})\mathcal{D}\Delta^*(\vec{r})\exp\{-\beta F_G[\Delta(\vec{r}),\Delta^*(\vec{r})]\}, \tag{28}$$

where

$$F_G=\int d^3r_1d^3r_2\left[\frac{1}{V}\delta(\vec{r}_1-\vec{r}_2)-K_2(\vec{r}_1,\vec{r}_2)\right]\Delta(\vec{r}_1)\Delta^\star(\vec{r}_2)$$
$$+\frac{1}{2}\int d^3r_1d^3r_2d^3r_3d^3r_4K_4(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)\Delta(\vec{r}_1)$$
$$\times\Delta^\star(\vec{r}_2)\Delta(\vec{r}_3)\Delta^\star(\vec{r}_4)+O(\Delta^6) \tag{29}$$

and the electronic kernels K_2 and K_4 are expressed in terms of the normal-state electronic Green's function $G^0_\sigma(\vec{r}_1,\vec{r}_2;\omega_\nu)$ with spin components $\sigma=\uparrow,\downarrow$ and Matsubara frequencies $\omega_\nu=(2\nu+1)\pi k_BT$ as

$$K_2(\vec{r}_1,\vec{r}_2)=\frac{k_BT}{\hbar^2}\sum_{\nu=-\infty}^{\infty}G^0_\uparrow(\vec{r}_2,\vec{r}_1;\omega_\nu)G^0_\downarrow(\vec{r}_2,\vec{r}_1;-\omega_\nu) \tag{30}$$

and

$$K_4(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)=\frac{k_BT}{\hbar^4}\sum_{\nu=-\infty}^{\infty}G^0_\uparrow(\vec{r}_2,\vec{r}_1;-\omega_\nu)$$
$$\times G^0_\downarrow(\vec{r}_2,\vec{r}_4;\omega_\nu)G^0_\uparrow(\vec{r}_3,\vec{r}_4;-\omega_\nu)$$
$$\times G^0_\downarrow(\vec{r}_3,\vec{r}_1;\omega_\nu).$$

As discussed in Sec. II.A, the mean-field approximation can be obtained from the stationary condition of F_G with respect to the variation of $\Delta(\vec{r})$ [or $\Delta^*(\vec{r})$]. This yields a Ginzburg-Landau-like equation, which is very difficult to solve exactly because of its nonlinear term. The linear equation

$$\frac{1}{V}\Delta(\vec{r})=\int d^3r_1K_2(\vec{r}_1,\vec{r})\Delta(\vec{r}_1) \tag{31}$$

is useful sufficiently close to the second-order phase transition $H\to H_{c2}$, where $\Delta(\vec{r})\to 0$. It is therefore an equation determining the critical field $H_{c2}(T)$. To solve this equation one may attempt to solve the eigenvalue problem of the integral operator $\int d^3r_1K_2(\vec{r}_1,\vec{r})$, i.e.,

$$\int d^3r_1K_2(\vec{r}_1,\vec{r})\Delta(\vec{r}_1)=\mathcal{A}\Delta(\vec{r}), \tag{32}$$

thus reducing Eq. (31) to an algebraic equation $1/V=\mathcal{A}$.

The schematic phase diagram shown in Fig. 1 corresponds to such a solution. As discussed in the Introduction, near the bottom of the $H_{c2}(T)$ line where $H_{c2}(T\to T_c)\to 0$, thermal excitations provide the dominant mechanism of pair breaking. The spatial decay of the pair-correlation function $K_2(\vec{r}\,',\vec{r})$ is governed by the thermal mean free path $\zeta(T)=\hbar v_F/\pi k_BT$, $T\approx T_c$. The latter, which is of the order of the Cooper-pair coherence length at zero temperature $\xi(T=0)=0.18\hbar v_F/k_BT_c$, is much smaller than the magnetic length a_H [which is approximately equal to the pair-correlation length near the critical temperature $\xi(T\to T_c)\to\infty$]. Since the order parameter $\Delta(\vec{r})$ varies on the scale of $\xi(T)\gg\zeta(T)$, $T\to T_c$, one is allowed to use a gradient expansion of K_2. To second order in this expansion (the first-order term vanishes by symmetry) one recovers the well-known Ginzburg-Landau equation, first derived from the microscopic BCS theory by Gorkov (1959); hence

$$-\left(\vec{\nabla}-\frac{2ie}{c}\vec{A}\right)^2\Delta(\vec{r})\approx\xi(T)^{-2}\Delta(\vec{r}), \tag{33}$$

where $\xi(T)=\xi(0)/\sqrt{\ln[T_c(0)/T]}$ is the temperature-dependent coherence length, $T_c(0)=(2\gamma\hbar\omega_D/\pi k_B)\exp[-1/VN(0)]$ is the well-known BCS (zero-field) transition temperature, ω_D is the Debye cutoff frequency, and $\gamma\approx 0.5772$ is Euler's constant.

This equation has the form of the Schrödinger equation for a free particle with charge $2e$ in a magnetic field $\vec{H}=\vec{\nabla}\times\vec{A}$. For the ground Landau level of this "particle," $(\hbar/2)(2eH/2m^*c)$, where m^* is an effective electronic mass, Eq. (33) reduces to the simple equation $2eH/ch=\xi(T)^{-2}$, or $\sqrt{2}\xi(T)=a_H$, and so determines $H_{c2}(T)=\phi_0/2\pi\xi(T)^2$.

Climbing up the $H_{c2}(T)$ line with the magnetic field by lowering the temperature, such that $k_BT<\hbar\omega_c\sqrt{n_F}$ or $\zeta(T)>a_H$ (see Fig. 1), one enters the broad region of diamagnetic pair breaking. In this region the spatial variation of the electronic kernels is governed by the magnetic length. Since the order parameter varies on the same length scale, the gradient expansion is not valid in this region. It was noted by Helfand and Werthhamer (1964), however, that as in the situation in the thermal regime, a ground-state Landau (gauge) wave function for a particle with charge $2e$ in a magnetic field with an arbitrary guiding-center projection is an exact eigenfunction of the integral operator in Eq. (32). A particular symmetric gauge version of this solution, given by Eq. (24) with $f(z)=\text{const}$, was proposed two years later by Rajagopal and Vasudevan (1966a, 1966b), who provided an explicit expression for the eigenvalue $\mathcal{A}$, that is,

$$\mathcal{A}=\frac{1}{4\pi a_H^2}\sum_{n,n'=0}^{\infty}\frac{(n+n')!}{2^{n+n'}n!n'!}$$
$$\times\int\frac{dk_z}{2\pi}\frac{\tanh(\beta\xi_{n,k_z}/2)+\tanh(\beta\xi_{n',k_z}/2)}{\xi_{n,k_z}+\xi_{n',k_z}}, \tag{34}$$

where $\xi_{n,k_z}=\hbar\omega_c(n+1/2)+\hbar^2k_z^2/2m^*-E_F$.

Three aspects of this solution should be emphasized.

(1) The energy eigenvalue $\mathcal{A}$, Eq. (34), is infinitely degenerate, a condition associated with the complete arbitrariness of the guiding-center projections q_k in Eq. (23). In the symmetric gauge picture the degeneracy is also associated with the arbitrariness of the zeros of the function $f(z)$ in Eq. (24). This degeneracy is just a mathematical reflection of an important physical fact, namely, that the quadratic term in the superconducting free energy (29) does not include any interaction between vortices. The quartic term introduces such an interaction and is therefore crucially important for the determina-

tion of the stable vortex configuration of the superconductor even infinitesimally close to H_{c2}.

(2) For any solution of Eq. (32) the quadratic term of the free energy (29) is a local functional of $\Delta(\vec{r})$,

$$F_{SC}^{(2)}=\left(\frac{1}{V}-\mathcal{A}\right)\int d^3r_1|\Delta(\vec{r}_1)|^2, \tag{35}$$

and so is independent of the phase of the order parameter. This feature is again exclusive to the quadratic term only; the quartic term (and all higher-order terms as well) has a very significant phase dependence, which will be discussed in Sec. II.C.3.

(3) The mixing of different Landau levels involved in pairing is very strong. This aspect of Eq. (34), which is consistent with what was established in Sec. II.B, can be illustrated by considering the asymptotic behavior of the combinatorial factor:

$$\frac{(n+n')!}{2^{n+n'}n!n'!}\rightarrow\left(\frac{1}{\pi n}\right)^{1/2}\exp\left[-\frac{(n-n')^2}{4n}\right] \tag{36}$$

for large Landau-level indices n,n', which reflects a slow decay of the overlap integral between the two Landau orbitals as a function of the relative pair energy. Note that in this limit the Landau-level energy may be replaced by the kinetic energy of a free particle, namely, $\hbar\omega_c(n+1/2)\rightarrow\hbar^2k^2/2m_c$, or $n\rightarrow\frac{1}{2}(ka_H)^2$. In this representation the exponent in Eq. (36) becomes $(k^2-k'^2)^2a_H^2/8k^2$, implying that the relative momentum of the electron pair $\Delta k\sim 1/a_H$, or that the relative spatial distance $\Delta r\sim a_H$.

The highest sector of the $H_{c2}(T)$ line in Fig. 1 is the region of significant quantum magnetic oscillations; here $\hbar\omega_c\gtrsim 2\pi^2k_BT$, or alternatively $\Lambda_T\geqslant a_H$. As we have already indicated, the parameter $n_F=E_F/\hbar\omega_c$ is a large number, of the order of 100 in a typical situation. Consequently, the quantum effects under study here are of an evident semiclassical nature. It was concluded in the previous section, following the detailed study of Dukan *et al.* (1991, 1994), that for large n_F values the relative abundance of the zeros of the matrix element $\Delta_{n,n}(\vec{q})$, which are of order higher than 1, is much smaller than that of first-order zeros (i.e., by a factor of $1/n_F$). The inclusion of off-diagonal matrix elements further complicates the calculations but does not change the nature of the zeros of the quasiparticle spectrum. This feature has important implications concerning the Landau band density of states, as discussed in Sec. II.B.1.

It is therefore desirable to find the explicit dependence of the dHvA amplitude, not only upon the strength Δ_0 of the superconducting order parameter but also on the parameter n_F. With this aim in mind we shall now follow an approach taken by Maniv *et al.* (Maniv, Rom, Vagner, and Wyder, 1992, 1994, 1997; Zhuravlev, Maniv, Vagner, and Wyder, 1997, 1999), in which the semiclassical nature of the electronic propagators is exploited to derive analytical expressions for the relevant physical observables, and thus to reveal their dependence on n_F.

For the sake of simplicity, and without missing the main physical point, we may use here a 2D electron-gas model. This is because for an isotropic 3D electron-gas model the dHvA oscillations in the semiclassical limit are dominated by an effective 2D momentum subspace, which corresponds to the extremal cross section $k_z=0$ of the 3D Fermi sphere. It should be stressed, however, that under some extreme (quantum) conditions the extremal orbit approximation may not be valid; this will be discussed in Secs. II.C.2 and II.C.4.

An important technical advantage of this approach is avoidance of the strong Landau-level mixing encountered in the previous section by using the analytical expression for the electronic Green's function in the symmetric gauge (Bychkov and Gorkov, 1962),

$$G_\sigma^0(\vec{r}_2,\vec{r}_1;\omega_\nu)=\exp[i\chi(\vec{r}_2,\vec{r}_1)]\tilde{G}_\sigma^0(\rho;\omega_\nu), \tag{37}$$

where $\chi(\vec{r}_2,\vec{r}_1)\equiv[\vec{r}_2\times\vec{r}_1]\cdot\vec{H}/H$, $\tilde{G}_\sigma^0(\rho;\omega_\nu)\propto e^{-\rho^2/2}\Sigma_nL_n(\rho^2/2)/(i\omega_\nu-\xi_{n,\sigma})$, $\rho\equiv|\vec{r}_1-\vec{r}_2|$, and L_n is the Laguerre polynomial. Summing over the Landau-level index by means of the Poisson summation formula (Zhuravlev *et al.*, 1997), it is guaranteed that all possible off-diagonal Landau-level pairing will be automatically included. Useful approximations can thus be employed due to the fact that only large values of the Landau-level index (i.e., for $n\sim n_F$) are of importance.

The order parameter used in calculating the free energy (29) has a variational form based on the degenerate lowest-Landau-level solutions of the linear Eq. (31) since, as shown by Eilenberger (1967a, 1967b), the totality of these solutions constitutes a complete basis set of functions in the subspace of the lowest-Landau-level wave functions. The form selected for $\Delta(\vec{r})$ is given by Eq. (23), where Δ_0, as well as γ and a_x, which determine the specific vortex lattice geometry, are variational parameters. The explicit form of the variational coefficients c_k appearing below Eq. (23) can be shown to constitute a set of exact stationary solutions for Gorkov's free-energy functional (29) with respect to $\Delta(\vec{r})$ (see Maniv, Markiewicz, *et al.*, 1992; Maniv, Rom, *et al.*, 1992).

The result of this calculation for the eigenvalue $\mathcal{A}$ in Eq. (35) is instructive since the three regions shown on the phase diagram in Fig. 1 can be readily identified. Thus we find

$$\mathcal{A}=2\mathcal{D}_{2D}\frac{1}{\zeta(T)}\sum_{\nu=0}^{\nu_D-1}\mathrm{Re}(q_\nu)\int_0^\infty d\rho \times\exp\left[-2(2\nu+1)\frac{1}{\zeta(T)}\rho-\frac{1}{2a_H^2}\rho^2\right], \tag{38}$$

where

$$q_\nu=\frac{\exp(2\pi|\tilde{\omega}_\nu|+i\pi\tilde{m}_c)}{\cos 2\pi n_F+\cosh(2\pi\tilde{\omega}_\nu+i\pi\tilde{m}_c)},$$

$\tilde{\omega}_\nu=\omega_\nu/\omega_c$, $\tilde{m}_c=m_c/m_0$, and $\nu_D=T_D/2T$ with T_D denoting the Debye cutoff temperature.

In this expression the integration variable is the common distance [see Eq. (30)] $\rho=|\vec{r}_1-\vec{r}_2|$ propagated by

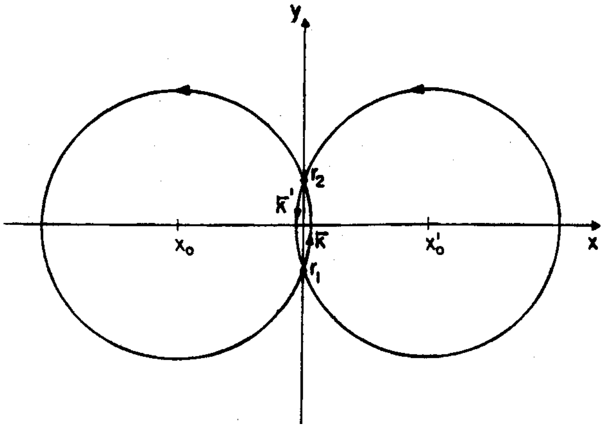


FIG. 12. The semiclassical picture of paired orbits in a 2D electron gas under a perpendicular magnetic field. The paired electrons are in close mutual proximity within a small region of the order of a magnetic length where their local momenta are nearly antiparallel ($\vec{k}'\approx-\vec{k}$).

the two electrons forming a Cooper pair (see Fig. 12). The integral measures the probability amplitude for both electrons to propagate over a common distance in space. It can be seen that in the thermal regime $T \approx T_c$, $a_H \gg \zeta(T)$, so the first term in the exponent in Eq. (38) dominates the integral, which is effectively carried out over a (pair-propagation) distance of the order of $\zeta(T_c)$. In this case the simple integral yields the well-known BCS result

$$\mathcal{A}=\mathcal{D}_{2D}\sum_{\nu=0}^{\nu_D-1}\frac{1}{(\nu+1/2)}=\mathcal{D}_{2D}\int_0^{\pi\nu_D}\frac{d\nu}{\nu}\tanh\nu.$$

In the diamagnetic region $a_H \ll \zeta(T)$ the second exponent dominates the integral as long as the thermal excitation frequency ν does not exceed the value $\zeta(T)/a_H \gg 1$; the effective pair-propagation distance in this region is a_H. Larger thermal frequencies $\nu>\zeta(T)/a_H$ are still important, however, due to the well-known logarithmic divergence, and the final result requires a careful calculation that leads to the superconducting condensation energy in the diamagnetic regime (Helfand and Werthamer, 1966; Werthamer *et al.*, 1966),

$$\mathcal{A}-1/V\approx\mathcal{D}_{2D}\ln\left[\frac{\pi k_B T_D e^{-1/v}e^{1.147}}{(\hbar\omega_c E_F)^{1/2}}\right]$$

$$=\mathcal{D}_{2D}\ln\left[\frac{a_H}{\sqrt{2}\xi(0)}\right], \tag{39}$$

where $\mathcal{D}_{2D}=m_c/2\pi\hbar^2$ is the single-electron density of states (per spin) in a 2D electron gas, and $v=\mathcal{D}_{2D}V$. Note, however, that in this case the effective pair-propagation distance does not exceed the magnetic length a_H.

At this point it is worth emphasizing the short-range character of the pair propagation in space implied from the essentially Gaussian integration over ρ in Eq. (38), and its complete separation from the pair center-of-mass position in the entire vortex lattice, as reflected by the separable integration over $\vec{r}_1$ in Eq. (35). The origin of this feature is described in Fig. 12, where the mechanism of electron pairing near the extremal orbit is illustrated.

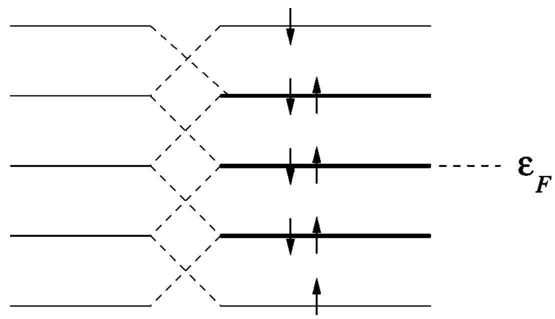


FIG. 13. Single-electron spectrum of a 2D electron gas with a finite Zeeman spin splitting corresponding to $g=4$, where electron pairing under an external magnetic field occurs on degenerate spin-up and spin-down levels (resonant pairing).

The relevant physical meaning of this localization is that the orbital guiding centers of the two electrons cannot move with respect to each other as a result of the pairing interaction over a distance larger than the size of a unit cell. It can therefore be immediately concluded that, to second order in Gorkov's perturbation theory, no magnetic breakdown of the cyclotron orbits by the vortex lattice takes place.

2. Pairing in the presence of Landau quantization

It is now possible, with the aid of the perturbative scheme presented above (Sec. II.C.1), to investigate in some detail the exotic possibilities discussed in the Introduction of electron pairing in Landau levels. Within the framework of this scheme the Landau quantization of the normal-electron states should be reflected (through the condensation energy $\mathcal{A}$) as quantum magnetic oscillations in the transition temperature $T_{c2}(H)$. Let us first consider the ideal 2D model, for which the effect of Landau quantization is complete. We choose the limit of zero g factor, i.e., when $\tilde{m}_c=0$ [see Eq. (38)]. Such a situation can be realized in experiment by tilting the magnetic-field direction with respect to the axis of the cylindrical Fermi surface (Wosnitza, 1996). The effective density-of-states function for a pair of electrons with opposite spins at the Fermi energy

$$q_\nu=\frac{e^{X_\nu}}{\cosh X_\nu+\cos(2\pi n_F)}, \tag{40}$$

where $X_\nu=2\pi|\omega_\nu|/\hbar\omega_c$, is strongly divergent (like $q_\nu \simeq 4/X_\nu^2$) in the very-low-temperature limit $X_\nu \ll 1$, at certain discrete values of the magnetic field where a Landau level coincides with the Fermi energy (i.e., when $n_F=n+1/2$). This divergence is due to the fact that the quantized electronic orbits are crossing the cylindrical Fermi surface in pairs of degenerate spin-up and spin-down states, and so are leading to a resonantly enhanced Cooper-pair density of states. This resonant phenomenon also occurs for nonzero integer values of the g factor, i.e., whenever $\tilde{m}_c$ is an integer (see, for example, Fig. 13).

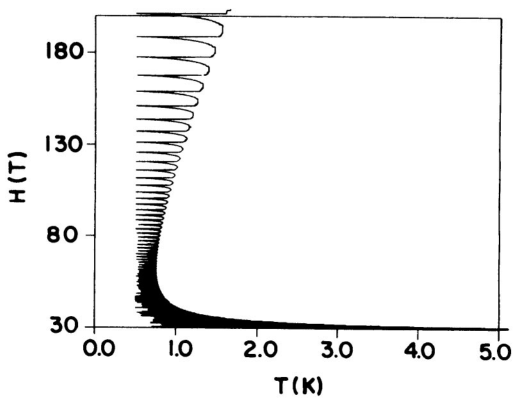


FIG. 14. The $H_{c2}(T)$ line near the critical point to reentrant superconductivity associated with Landau quantization in a 2D electron gas. From Maniv, Rom, *et al.*, 1992.

Under these circumstances the condensation energy term in the linearized self-consistency Eq. (31) diverges at $T\rightarrow 0$ like $1/T$, i.e.,

$$\mathcal{A}\rightarrow \mathcal{D}_{2D}\left(\frac{\hbar\omega_c}{16k_BT}\right)\left(\frac{\hbar\omega_c}{\pi E_F}\right)^{1/2},$$

so that the corresponding transition temperature increases with the strength of the magnetic field as (see Fig. 14)

$$T_c(H)\rightarrow v^*\left(\frac{1}{16\sqrt{\pi}}\right)\left(\frac{\hbar\omega_c}{k_B}\right)n_F^{-1/2},$$

where $1/v^*=1/v-1.15$. The effective enhancement of the electron-electron coupling constant v is due to nonsingular contributions to $\mathcal{A}$ (associated with the off-diagonal terms in the Landau-level representation).

This highly singular behavior is peculiar to the 2D model: introduction of an energy dispersion along the magnetic field (z) direction significantly weakens the singularity. It is also evident that a slight change in the g factor to a fractional value leads, by destroying the resonance condition, to a quick suppression of this singularity. However, as will be shown below, singularities of similar origin, though less dramatic, also exist in 3D electron systems with fractional g factors.

It will therefore be instructive to consider here a model of quasi-2D electron gas in an external magnetic field oriented perpendicularly to the easy-conduction planes, with energy dispersion along the field (z) axis given by the simple tight-binding form $\xi_{k_z}=t_\perp(1-\cos k_z d)$, where d is the corresponding lattice constant, and $t_\perp$ is the miniband width. We also assume a general value (i.e., not necessarily integer) for the g factor. To offset the destructive influence of the Zeeman splitting, one can follow Gruenberg and Gunther (1966), who exploited the idea of Fulde and Ferrel (1964) and Larkin and Ovchinnikov (1964), by allowing for a spatially nonuniform order parameter along the field (z) direction, i.e.,

$$\Delta(x,y,z)=\Delta(x,y)e^{iQz},$$

where Q is a wave number yet to be determined.

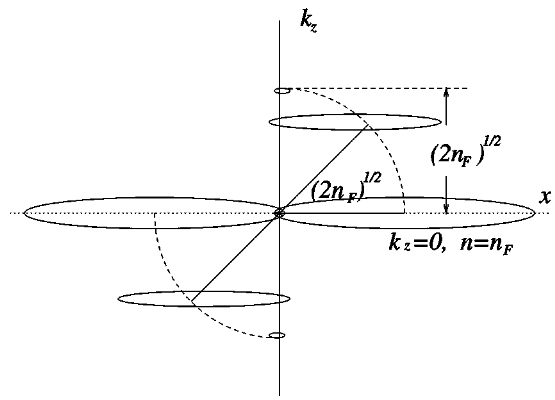


FIG. 15. The semiclassical picture of paired orbits in an isotropic 3D electron gas in a mixed real-space/reciprocal-space representation (in units of magnetic length and inverse magnetic length, respectively). Note the paired extremal orbits with $k_z=0, n=n_F-1/2$, as well as the lowest-Landau-level orbits with $k_z\approx\pm k_F, n=0$.

As discussed in Sec. II.C.1, pair propagation in the plane perpendicular to the external magnetic field is restricted to real-space distances ρ of the order of the magnetic length a_H (see Figs. 12 and 15), which is much smaller than the Larmor radius ρ_F of an electron on the quasicylindrical Fermi surface under study. Thus it is acceptable to linearize the action integral of the semiclassical pair propagator [i.e., the 3D version of Eq. (30)] in the corresponding small parameter $\rho/2\rho_F$, provided that only large-Landau-level indices (i.e., when $n\gg 1$) are considered (see Maniv, Markiewicz, *et al.*, 1992; Maniv, Rom, *et al.*, 1992). Assuming for simplicity that $g<1$, we denote by $n_0(k_z,\sigma)$ the index of the Landau sublevel with spin projection σ crossing the Fermi energy quite close to k_z (see Fig. 2), that is, $n_0(k_z,\sigma)=\mathrm{Int}[(E_F-\xi_{k_z})/\hbar\omega_c-\frac{1}{2}-\frac{1}{4}g\sigma]$. We also denote by $\tilde{\mu}(k_z,\sigma)$ the corresponding fractional residue of the chemical potential, i.e., $\tilde{\mu}(k_z,\sigma)\equiv(n_F-\tilde{\xi}_{k_z}-\frac{1}{4}g\sigma)-[n_0(k_z,\sigma)+1/2]$, where the tilde means energy in units of $\hbar\omega_c$.

In the very anisotropic limit, where $t_\perp\lesssim\hbar\omega_c$, the eigenvalue $\mathcal{A}$ can be written in the same form as Eq. (38), but with q_ν replaced by

$$q_\nu(Q)=(1/2\pi)\int dk_z\,\mathrm{Re}[I_\sigma(k_z,\omega_\nu)I_{-\sigma}(Q-k_z,-\omega_\nu)],$$

$$I_\sigma(k_z,\omega_\nu)=1/\{1-\exp[-X_\nu+2\pi i\tilde{\mu}(k_z,\sigma)\mathrm{sgn}(\omega_\nu)]\}.$$

It should be stressed that the assumption $n_0(k_z,\sigma)\gg 1$ implies that pairings involving small Landau-level indices and large k_z (i.e., $\sim k_F$) are not taken into account here.

Thus in the zero-temperature limit, where $\tilde{\omega}_{\nu=0}\rightarrow 0$, a resonant pairing situation [i.e., when $I_\sigma(k_z,\omega_\nu)I_{-\sigma}(Q-k_z,-\omega_\nu)\simeq 4/X_\nu^2$] can be realized at a certain value of $k_z=k_{z,0}$ if the wave number Q is selected in such a way that

$$\mu(k_{z,0},\sigma)=\mu(Q-k_{z,0},-\sigma)=0 \qquad (41)$$

(see Fig. 16). The wave numbers $k_{z,0}$ and Q depend on the magnetic-field strength H and, as clearly shown in

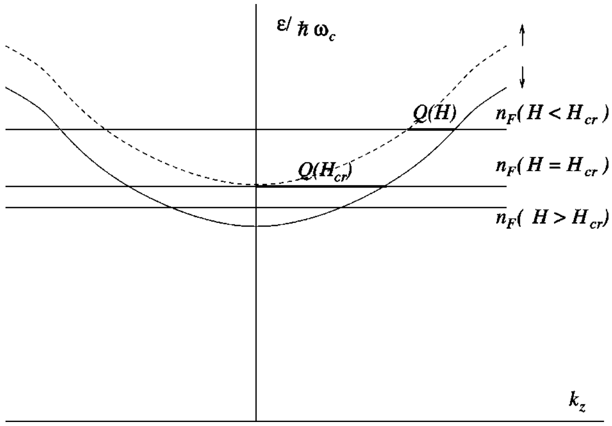


FIG. 16. A figure similar to Fig. 2 for a quasi-2D electron gas. Note the critical behavior of the Fulde-Ferrel-Larkin-Ovchinikov wave number Q as a function of magnetic field.

Fig. 16, such simultaneous solutions exist only within magnetic-field intervals below certain critical values H_{cr}, for which the Fermi energy crosses both of the Zeeman split Landau sublevels. Considering such a situation, it can be shown that

$$\mathcal{A}\rightarrow\frac{1}{8\pi^2 v(k_{z,0},Q)}\mathcal{D}_{q2D}\frac{1}{n_F^{1/2}}\left(\frac{\hbar\omega_c}{t_\perp}\right)\ln\left(\frac{\hbar\omega_c}{2\pi k_B T}\right), \tag{42}$$

where $v(k_{z,0},Q)=\sin(k_{z,0}d)+\sin[(k_{z,0}-Q)d]$, and $\mathcal{D}_{q2D}=(1/d)\,\mathcal{D}_{2D}$.

Note that, as long as $g\neq 0$, $v(k_{z,0},Q)\neq 0$ since the vanishing of the Fermi velocity along the magnetic-field direction, i.e., $\sin(k_{z,0}d)=0$, or $\sin[(k_{z,0}-Q)d]=0$, can take place only separately for each of the paired electrons (see Fig. 16). Only for zero Zeeman splitting are there certain magnetic-field values at which the longitudinal Fermi velocities of both electrons vanish simultaneously, so that $v(k_{z,0},Q)=0$, resulting in a stronger divergence of the pair condensation energy, i.e.,

$$\mathcal{A}\rightarrow\frac{1}{8\pi^2}\mathcal{D}_{q2D}\frac{1}{n_F^{1/2}}\left[\frac{\hbar\omega_c}{(t_\perp k_B T)^{1/2}}\right]. \tag{43}$$

It should be stressed again that in the above analysis the wave number Q was selected following Fulde and Ferrel (1964; see also Larkin and Ovchinnikov, 1965) to offset the Zeeman splitting of the Landau level with the extremal index $n\approx n_F-1/2$. In considering more isotropic 3D systems, however, different selections of the wave number Q could be more favorable, in particular those corresponding to a large momentum along the field direction (i.e., $k_z\approx k_F$) and small Landau-level indices n,n'; see Fig. 2. Indeed, the combinatorial factor $(n+n')!/2^{n+n'}n!n'!$ for $n\approx n'\approx 1$ is considerably larger (by $\pi n_F^{1/2}$) than its value at $n=n'\approx n_F$. Let us look more closely at such a 3D model. The parameter $\mathcal{A}$ can be expressed by the exact quantum-mechanical formula

$$\mathcal{A}=\frac{k_B T}{4\pi a_H^2}\sum_{\nu=-\nu_D}^{\nu=\nu_D}\int dk_z\sum_{n,n'=0}^{\infty}\frac{(n+n')!}{n!n'!2^{n+n'}}\times g_n^{+}\left(k_z-\frac{Q}{2},i\omega_\nu\right)g_{n'}^{-}\left(k_z+\frac{Q}{2},-i\omega_\nu\right), \tag{44}$$

where $[g_n^{\pm}(k_z,i\omega_\nu)]^{-1}=E_F-\hbar\omega_c(n+1/2\pm g/4)-\xi_{k_z}\pm i\omega_\nu$. For $t_\perp\gtrsim E_F$ the corresponding energy dispersion is essentially 3D.

In the singular situation investigated here the dominant contribution to $\mathcal{A}$ originates in the diagonal elements $n=n'$. Selecting the Fulde-Ferrel-Larkin-Ovchinnikov wave number Q to offset the Zeeman splitting of the $n=0$ Landau level [i.e., taking $Q=(1/a_H)(g/\sqrt{8n_F})$; see Fig. 2], we get

$$\mathcal{A}\rightarrow\frac{1}{8\pi^2}\mathcal{D}_{q2D}\left(\frac{\hbar\omega_c}{2t_\perp}\right)^{1/2}\sum_{n=0}^{n_m}\frac{(2n)!}{(n!)^2 2^{2n}}\frac{1}{\sqrt{n_F-n}}\times\ln\left(\frac{\hbar\omega_D}{\sqrt{(\pi k_B T)^2+\Gamma_n^2}}\right), \tag{45}$$

where $\Gamma_n=(gn/8n_F)\hbar\omega_c(g<1)$, with the Matsubara frequency cutoff $\omega_D=(2\nu_D+1)\pi k_B T\ll\mu$, and the Landau-level cutoff $n_m\ll n_F$. If the Zeeman splitting is not too small and $T\rightarrow 0$, the dominant contribution originates from the lowest Landau level $(n=0)$ and

$$\mathcal{A}\rightarrow\frac{1}{8\pi^2}\mathcal{D}_{q2D}\frac{1}{n_F^{1/2}}\left(\frac{\hbar\omega_c}{2t_\perp}\right)^{1/2}\ln\left(\frac{\hbar\omega_D}{\pi k_B T}\right). \tag{46}$$

Note the similarity to Eq. (42) derived with Q corresponding to the extremal Landau level, $n\approx n_F$. The nature of the singularity in the present case, in contrast to the resonant pairing situation described by Eq. (43), does not change in the limit $g\rightarrow 0$; only the prefactor of the (logarithmic) singularity is enhanced. The enhancement [i.e., when the small factor $1/n_F^{1/2}$ in Eq. (46) is replaced by unity] is quite significant due to the large value of n_F. Under these circumstances the global field dependence of the condensation energy $\mathcal{A}$, obtained within the fully isotropic 3D model (i.e., for $t_\perp\sim E_F$) in the semiclassical limit $n_F\gg 1$, is similar to that found by Rasolt and Tesanovic (1992) in the extreme quantum limit $n_F\sim 1$, where a new (reentrant) branch of the $H_{c2}(T)$ line (similar to the upper branch in Fig. 14) was discovered.

In the low-field region (Helfand and Werthamer, 1966), the great number of nonsingular off-diagonal paired Landau-level channels overwhelms the diagonal channels, leading to a condensation energy

$$\mathcal{A}_{QC}\approx\frac{1}{8\pi^2}\mathcal{D}_{q2D}\ln\left(\frac{\hbar\omega_D}{\sqrt{\hbar\omega_c E_F}}\right)$$

for which the $H_{c2}(T)$ curve turns back to the nearly temperature-independent (i.e., the lower) branch shown in Fig. 14.

It is evident that the exotic pairing states described in this section are very sensitive to the scattering of paired electrons by impurities. According to the conventional wisdom, this singularity should be quickly washed out by impurities. A more careful treatment would reveal, however, that the conventional averaging procedure over impurity realizations, which leads to an imaginary self-energy part in the single-electron propagators, is not valid for the extremely low temperatures studied here. At such low temperatures, the presence of long-range coherent paths of electron pairs (Altshuler, 1985; Lee and Stone, 1985), which are not influenced by static disorder, could lead to the creation of mesoscopic superconducting droplets at magnetic fields far above the quasiclassical H_{c2} (see Spivak and Zhou, 1995). A proper theoretical treatment of this problem is, however, far beyond the scope of this article.

Finally, it should be noted that the strength of the divergences described above increases progressively with increasing order in the perturbative expansion of the free energy in the superconducting order parameter, resulting in the breakdown of the entire expansion at some nonzero (though low) temperature. In this case a nonperturbative approach yields in the zero-temperature limit a regular, nonanalytic dependence on Δ_0^2. The explicit form of this dependence is known only in very special limiting cases (see, for example, Sec. II.C.5, or Zhuravlev *et al.*, 1999 and Maniv *et al.*, 1998). Strictly speaking, the nature of the true ground state of the microscopic Hamiltonian in a stationary, uniform external magnetic field is poorly understood. The conventional ladder approximation for the two-particle Green's function, which leads to the Cooper instability at $H=0$ (Schrieffer, 1964), is not rigorously valid at $H\neq 0$ and $T=0$. Divergences in the electron-electron channel (see MacDonald *et al.*, 1992), which are equivalent to the singularities discussed above, as well as in the electron-hole channel (Bychkov, 2000), may reflect competition between the superconducting and charge-density-wave-type instabilities. The electron-hole channel is expected to be particularly dangerous in the more isotropic 3D systems, where Fermi-surface nesting corresponding to the effective 1D energy dispersions becomes significant.

A possible scenario, proposed by Tesanovic (1995), describes the low-temperature phase as a charge-density wave of Cooper pairs, which contains no Abrikosov vortices and exhibits no macroscopic phase coherence at finite T (see Sec. IV for more details). It is determined by a new saddle point of the BCS Hamiltonian, unrelated to the superconducting saddle point.

At present the problem is, however, far from fully settled. Adopting a phenomenological approach, one might argue that the coexistence of superconductivity and quantum magnetic oscillations, verified in many dHvA experiments, justifies the use of the postulated effective BCS model Hamiltonian, whereas the apparent nonanalytic nature of the relevant thermodynamic potentials can be eliminated by introducing a small amount of Landau-level smearing, as explained in Sec. II.C.5.

3. Phase coherence and weak magnetic breakdown in the semiclassical limit

As indicated in the Introduction, potentially the most dangerous killer of dHvA oscillations in the vortex state is the Landau-level broadening effect associated with delocalized orbitals, described heuristically in Sec. I.B.3 and studied systematically in Sec. II.B.2. Here we consider further evidence for the preliminary conclusion of Sec. II.C.1, stating that the pair potential does not lead to significant breakdown of the localized quasiparticle cyclotron orbits. This remarkable phenomenon, which was presented in Sec. II.C.1 by examining only the leading term in the perturbation theory, also seems to hold to higher orders. In contrast to the second-order term, however, in the higher-order terms the existence of phase coherence in the corresponding propagators is of crucial importance. Let us discuss now only the fourth-order term. Integrating the quartic term in Eq. (29) over the four-electron center-of-mass vector $\vec{R}=\frac{1}{4}\Sigma_{l=1}^{4}\vec{r}_l$, one gets an expression in which the local gauge factors of the electronic propagators are grouped together with the order parameters but are separated from the translationally invariant parts of the electronic propagators, i.e. (with all lengths measured now in units of magnetic length),

$$F_{SC}^{(4)}=\int d^2Q e^{-4Q^2}\int d^2S d^2T\mathcal{D}(\vec{S},\vec{T})\frac{1}{\beta}\sum_{\nu}\tilde{K}_{4,\nu}(\{\rho_i\}), \tag{47}$$

where

$$\tilde{K}_{4,\nu}(\{\rho_i\})=\tilde{G}_0(\rho_1,-\omega_\nu)\tilde{G}_0(\rho_2,\omega_\nu)\times\tilde{G}_0(\rho_3,-\omega_\nu)\tilde{G}_0(\rho_4,\omega_\nu) \tag{48}$$

and

$$\mathcal{D}(\vec{S},\vec{T})\propto Ne^{4i(S_yT_x+S_xT_y)}\sum_{m_1,m_2}e^{-2i\gamma m_1m_2+4im_1T_x+4im_2S_x}\times\exp[-(m_1+2S_y)^2-(m_2+2T_y)^2]. \tag{49}$$

In these expressions $\vec{\rho}_l=\vec{r}_{l+1}-\vec{r}_l$, $l=1,\ldots,4$, $\vec{r}_5\equiv\vec{r}_1$, and $\vec{Q}=\frac{1}{8}(\vec{\rho}_2+\vec{\rho}_4-\vec{\rho}_1-\vec{\rho}_3)$, $\vec{S}=\frac{1}{4}(\vec{\rho}_3-\vec{\rho}_1)$, $\vec{T}=\frac{1}{4}(\vec{\rho}_4-\vec{\rho}_2)$ are the various relative coordinates of the four electrons involved. The discrete variables $m_1\equiv\frac{1}{2}q_{k_2-k_1}$ and $m_2\equiv\frac{1}{2}q_{k_3-k_1}$ are distances between the Bragg chains labeled k_1,k_2,k_3 in the Abrikosov lattice (see Fig. 7). The factor N (i.e., the total number of vortices), appearing in the right-hand side of Eq. (49), arises from the translational invariance of the corresponding free-energy density in the Abrikosov lattice with respect to the center-of-mass coordinates.

Equations (47)–(49) provide an exact expression for the quartic term in the Gorkov expansion. Note that it depends only on relative coordinates of electrons (and vortex chains), a result consistent with the nature of the pair potential $\Delta(\vec{r})$ as representing quasiparticle-quasihole interaction.

The Gaussian decaying factors in Eq. (49) allow a considerable simplification by approximating the lattice sum with the dominant contribution at $m_1\approx -2S_y$ and $m_2\approx -2T_y$, resulting in the following expression:

$$\mathcal{D}(\vec{S},\vec{T})\propto Ne^{-4i(2\gamma S_yT_y+S_yT_x+S_xT_y)}.$$

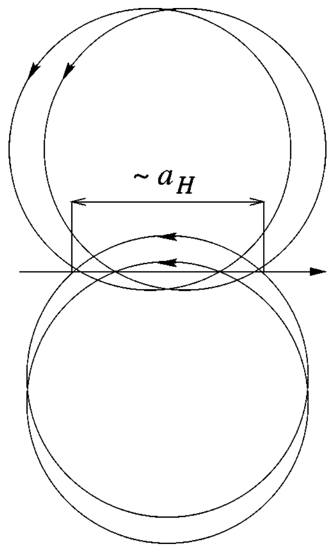


FIG. 17. Four-electron orbits near the Fermi surface with the dominant contributions to the quartic term of the superconducting free energy. Note the collinear configurations of the propagation vectors and their restriction to a region on the order of a magnetic length.

It is very important to stress here that, unlike the situation with the quadratic term, the function $\mathcal{D}(\vec{S},\vec{T})$, which includes all phase-dependent factors of the pair potentials and all gauge factors of the electronic propagators, has a very significant effect. By varying the relative coordinates $\vec{\rho}_l$ over distances much larger than the magnetic length [actually the spatial extent of $\tilde{G}_0(\rho_l,\omega_\nu)$, which is bound only by the cyclotron diameter of an electron at the Fermi surface], the function $\mathcal{D}(\vec{S},\vec{T})$ oscillates violently, interfering strongly with the oscillatory behavior of the electronic kernel $\tilde{K}_{4,\nu}(\{\rho_i\})$. Thus, when one integrates over the independent coordinates $\vec{S}$ and $\vec{T}$, there is a great deal of destructive interference except near stationary points. The stationary points for a fixed $\vec{Q}$ are restricted to $S,T\leq Q$ and correspond to the collinear vectors

$$\vec{\rho}_1=\rho_1\vec{n},\quad \vec{\rho}_2=-\rho_2\vec{n},\quad \vec{\rho}_3=\rho_3\vec{n},\quad \vec{\rho}_4=-\rho_4\vec{n},$$

where $\vec{n}=-\vec{Q}/Q$ and $\Sigma\vec{\rho}_i=0$.

A graphic depiction of the situation is shown in Fig. 17; the circles represent the cyclotron orbits of four electrons near the Fermi surface, with radii $r\sim\sqrt{2n_F}a_H \gg a_H$ corresponding to the reduced Green's functions appearing in Eq. (48). The whole picture is translationally invariant but very sensitive to relative translations of orbits with respect to each other. The contributions to the free energy from most orbital configurations interfere destructively, whereas only the collinear configurations shown in the figure interfere constructively.

Thus, as in the second-order case, no motion of orbital centers relative to each other over a distance larger than the size of a unit cell ($\sim a_H$) is caused by the pair potential to fourth order. Similar behavior is also expected to any higher order. We shall see later how this "coherent" picture is broken down by disorder of the vortex lattice (Sec. III) or by vortex lattice thermal fluctuations (Sec. IV.B). These effects lead to relative motions of orbitals over distances much larger than a_H, which result in a significant incoherent magnetic breakdown of the cyclotron orbits and the destruction of small paramagnetic current loops.

The restriction of all propagation vectors to lie within a region of unit-cell size (which is the smallest length scale of the problem) suggests that the higher-order terms in Gorkov's expansion may be written approximately by a local expression similar to Eq. (35). For the quartic term, Zhuravlev *et al.* (1997) showed that

$$F_{SC}^{(4)}\approx B\int d^2R|\Delta(\vec{R})|^4, \tag{50}$$

where

$$B\propto(1/n_F^{3/2})\int dQf(Q)e^{-4Q^2}\int_{-Q}^{Q}dS\int_{-Q}^{Q}dTe^{-(S^2+T^2)}$$

and $f(Q)=\Sigma_\nu q_\nu^2 e^{-8(2\nu+1)a_H Q/\zeta(T)}$.

This result is of fundamental importance since it shows that the well-known local form of the Ginzburg-Landau free-energy functional, derived originally by Gorkov in the thermal regime near $H_{c2}(T\to T_c)\to 0$, has a much broader range of validity, extending all the way through the diamagnetic regime into the region of quantum oscillations. In addition, we have found that this locality is closely related to the relatively small effect of magnetic breakdown in the semiclassical limit of the Landau levels.

It should be noted here that the coherence effect discussed in this section is a very delicate feature, which can easily be overlooked. For example, Vavilov and Mineev (1997) recently presented an analytical calculation of the quartic term using the diagonal approximation in the Landau-level representation. This approximation eliminates the violently oscillating phase factors that are responsible for gross cancellations of nonlocal terms as discussed in detail above. Another approximation that leads to a similar situation will be discussed in Sec. III.A.

4. Self-consistent harmonic expansion

In a typical dHvA experiment, various mechanisms of Landau-level broadening and phase smearing (Shoenberg, 1984a) lead to strong progressive damping of the Fourier series such that all harmonics may be neglected with respect to the fundamental. In such an experiment the superconducting order parameter always acts in parallel with these mechanisms. The most fundamental type of smearing is, of course, thermal smearing of the Fermi distribution function. Other smearing effects, such as those due to disorder, may be described by an effective temperature, the Dingle temperature (Dingle and Shoenberg, 1950; Springford, 1980). Let us assume here for the sake of simplicity that only thermal smearing is present, which is sufficiently strong [$2\pi^2k_BT/\hbar\omega_c\sim 1$, or $\zeta(T)/a_H\sim 2\pi\sqrt{2n_F}$] to guarantee the quick convergence

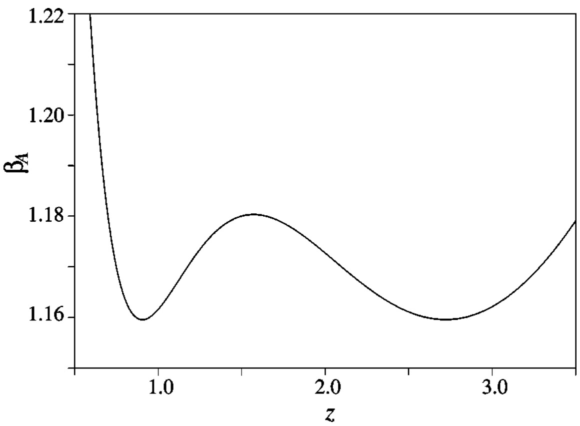


FIG. 18. Dependence of the Abrikosov parameter β_A on $z=(\pi/a_x)^2$. The two minima at $z_1=\pi/2\sqrt{3}$ and $z_2=\sqrt{3}\pi/2$ correspond to a triangular Abrikosov lattice ($\beta_A=1.16$) with differently selected Bragg-chain axes. The maximum at $z_1=\pi/2$, corresponds to the square lattice.

of the harmonic expansion. In the next section we shall discuss specifically the influence of nonmagnetic impurity scattering.

Using the local forms of the quadratic and quartic terms, Eqs. (35) and (50), respectively, in the free-energy functional F_G with the order parameter $\Delta(\vec{r})$ given by the generalized Abrikosov expression Eq. (23), F_G can be written in variational form (Maniv, Markiewicz, *et al.*, 1992; Maniv, Rom, *et al.*, 1992)

$$f_s \equiv \frac{F_G}{N\pi a_H^2} = \mathcal{D}_{2D}\left[-\tilde{\alpha}\Delta_0^2 + \frac{\tilde{B}}{(\pi k_B T_c)^2}\Delta_0^4\right], \qquad (51)$$

where $\tilde{\alpha}=\mathcal{A}/\mathcal{D}_{2D}-1/\lambda$, with $\mathcal{A}$ given by Eq. (38).

In Eq. (51)

$$\tilde{B}=\beta_a \frac{2\pi a_H^3}{\zeta(T)\zeta(T_c)^2}\sum_{\nu=0}^{\nu_D} \mathrm{Re}(q_\nu^2) \times \int_0^\infty d\rho e^{-4(2\nu+1)a_H\rho/\zeta(T)-\rho^2}\,\mathrm{erf}^2(\rho/\sqrt{2}) \qquad (52)$$

and β_a is the well-known Abrikosov parameter (Kleiner, Roth, and Autler, 1964), which depends on the geometry of the lattice through the variational parameters γ and a_x. A particularly useful expression for β_a can be written as

$$\beta_a=\sqrt{z/\pi}\sum_{s,p=-\infty}^{\infty} \exp[-z(s^2+p^2)]\cos(2\gamma sp), \qquad (53)$$

with $z=\pi^2/a_x^2$. The variations over the geometrical parameters γ and a_x lead to a minimum in the free energy for $\gamma=\pi/2$, $a_x=(2\pi/\sqrt{3})^{1/2}$, as well as for $a_{x'}=(2\pi\sqrt{3})^{1/2}$, depending on whether the family of Bragg chains is selected parallel to the principal axis x or to x' (see Fig. 7). Both minima describe a triangular Abrikosov lattice with $\beta_A\simeq 1.159$ (Kleiner *et al.*, 1964; see Fig. 18). Note that each axis can be selected in three equivalent ways in this lattice. All equivalent configurations can be obtained by exploiting the invariance of the Abrikosov parameter, Eq. (53), under the transformations $z'=\pi^2 z/(z^2+\gamma^2)$, $\gamma'=\pi^2\gamma/(z^2+\gamma^2)$, and $z'=z$, $\gamma'=-\gamma$, or $\gamma'=\gamma+n\pi$ with an arbitrary integer n.

As already indicated in the calculation of the condensation energy in the diamagnetic regime $\zeta(T)\gg a_H$, Eq. (39), the uniform (nonoscillatory) part gains a large contribution from many low-lying thermal excitations within the Debye frequency range. Similarly, as can be seen from Eq. (52), the sum over the Matsubara frequency ν for the zeroth harmonic (for which $q_\nu\to a_\nu^{[0]}=2$) is undamped as long as $\nu\lesssim\zeta(T)/a_H\sim\sqrt{n_F}$. The resulting expression for the uniform part of the superconducting free energy (per unit cell in the Abrikosov lattice) is

$$f_s^{[0]}\approx\frac{\hbar\omega_c}{2\pi a_H^2}\left[-\tilde{\Delta}_0^2\ln\left(\frac{a_H}{\sqrt{2}\xi(0)}\right)+\frac{1.38}{n_F}\tilde{\Delta}_0^4\right], \qquad (54)$$

where $\tilde{\Delta}_0\equiv\Delta_0/\hbar\omega_c$.

In contrast, thermal excitations contributing to the fundamental oscillatory part of the free energy are strongly damped by the attenuation factor $a_\nu^{[1]}\sim e^{-2(2\nu+1)\pi^2 k_B T/\hbar\omega_c}$, appearing in the harmonic expansion $q_\nu=a_\nu^{[0]}+a_\nu^{[1]}\cos(2\pi n_F)+\cdots$. This restricts the Matsubara sum to the first few terms only, leading to the following expression for the total magnetization oscillations (Maniv *et al.*, 1994):

$$M^{[1]}\approx M_N^{[1]}\left[1-\frac{\pi^{3/2}\tilde{\Delta}_0^2}{n_F^{1/2}}+\frac{\sqrt{2}\beta_A\pi^{3/2}\tilde{\Delta}_0^4}{n_F^{3/2}}\right]. \qquad (55)$$

Here $M_N^{[1]}$ is the fundamental oscillatory part of the normal-electron magnetization. The additional factor of $1/n_F^{1/2}$ in the superconducting terms of Eq. (55) in comparison with Eq. (54) reflects the remarkable difference between the uniform and the oscillatory parts of the free energy. Since both $M^{[1]}$ and $M_N^{[1]}$ are of first order in the harmonic expansion, the order-parameter amplitude $\tilde{\Delta}_0$ within the brackets should be kept of zeroth order. The self-consistent mean-field value of $\tilde{\Delta}_D$ is determined by minimizing Eq. (54), a condition leading to

$$\tilde{\Delta}_0^2=0.36n_F\ln\left[\frac{a_H}{\sqrt{2}\xi(0)}\right] \approx\frac{(1.7k_BT_c)^2}{(\hbar\omega_c)^2}[1-H/H_{c2}(0)],\quad H\to H_{c2}(0), \qquad (56)$$

which is identical to the well-known high-field limit of the Gorkov-Ginzburg-Landau superconducting order parameter.

The above analysis provides a critical test of consistency for the small Δ expansion presented in this section. Naturally in such a situation one looks for the small dimensionless expansion parameter of the perturbation theory. The Δ expansion appearing in Eq. (55) is controlled by the dimensionless parameter $x=\tilde{\Delta}_0^2/\sqrt{2}\beta_A n_F\sim\tilde{\Delta}_0^2/n_F$. The self-consistency condition, Eq. (56), suggests that this determines a universal critical parameter of the theory, since according to Eq. (56),

$$x \equiv \frac{\tilde{\Delta}_0^2}{\sqrt{2}\beta_A n_F} = 0.22 \ln\left[\frac{a_H}{\sqrt{2}\xi(0)}\right] \approx 0.22[1 - H/H_{c2}(0)], \quad H \rightarrow H_{c2}(0).$$

Finally, one should be aware of the use of a 2D model to describe the quantum magnetic oscillations in the vortex state of 3D superconductors, as is done in this section. Indeed, the analysis presented in Sec. II.C.2 indicates that the possibility of dominant pairing of electrons with small Landau-level indices could seriously call into question the validity of the extremal orbit approximation. It can be shown, however, that as long as the harmonic expansion described here is quickly convergent, the extremal orbit scheme is a very good approximation for the oscillatory part of the free energy. Furthermore, the corresponding calculation shows that within this approximation the ratio $M^{[1]}/M_N^{[1]}$ calculated explicitly for a 2D model [Eq. (55)] is not influenced at all by the k_z dispersion.

5. Scattering by nonmagnetic impurities

In the absence of magnetic field the effect of electron scattering by nonmagnetic impurities on the quasiparticle properties of a conventional superconductor is known to be insignificant (Anderson, 1959). In the presence of strong magnetic field, however, scattering by impurities may have a significant effect due to the breakdown of translational invariance and the consequent removal of the Landau-level degeneracy. The effect is expected to be particularly strong at extremely low temperatures, where thermal smearing of the Fermi distribution function is very small. The problem has been addressed in the literature by several authors (Vavilov and Mineev, 1997; Gvozdikov and Gvozdikova, 1998; Zhuravlev *et al.*, 1999), who considered only the leading (i.e., quadratic) term in Gorkov's expansion and so were restricted to the close vicinity of H_{c2}. Furthermore, the standard technique of averaging over impurity configurations (Abrikosov *et al.*, 1975) was used in these works. This may not be valid at very low temperatures because of the neglect of coherent electron paths (Altshuler, 1985; Lee and Stone, 1985), which may influence electron pairing significantly (Spivak and Zhou, 1995).

Following Vavilov and Mineev (1997), we expand the superconducting free-energy average over impurity configurations to second order in the impurity-dressed pair potential $\Delta(\vec{r},\omega_\nu)$,

$$\overline{F}_{SC}^{(2)} = \frac{1}{V}\int d^2r|\Delta(\vec{r})|^2 - k_B T \sum_\nu \int d^2r_1 d^2r_2 \Phi(\vec{r}_1,\vec{r}_2;\omega_\nu),$$

where

$$\Phi(\vec{r}_1,\vec{r}_2;\omega_\nu) = \Delta(\vec{r}_1,\omega_\nu)\overline{G}^0(\vec{r}_1,\vec{r}_2,\omega_\nu)\Delta^*(\vec{r}_2,\omega_\nu) \times \overline{G}^0(\vec{r}_1,\vec{r}_2,-\omega_\nu)$$

and $\overline{G}^0(\vec{r}_1,\vec{r}_2,\omega_\nu)$ is the normal-electron Green's function averaged over impurity configurations, where Zeeman spin splitting is neglected. It was shown by Vavilov and Mineev (1997) that in the semiclassical limit the correction to the pair potential (i.e., the vertex correction) due to impurities is small ($\sim 1/\sqrt{n_F}$). Neglecting this correction, and using the relaxation time approximation for $\overline{G}^0(\vec{r}_1,\vec{r}_2,\omega_\nu)$ (with an electronic momentum relaxation time $\tau_{sc}=1/2\Gamma$), we recover Eq. (38) obtained for the pure material with the replacement of $|\omega_\nu|$ by $|\omega_\nu+\Gamma|$ in the function q_ν, defined by Eq. (40) with $X_\nu = 2\pi(|\omega_\nu| + \Gamma)/\hbar\omega_c$.

As discussed in detail in Sec. II.C.2, for a clean material in the low-temperature limit, $X_\nu \ll 1$, this effective Cooper-pair density-of-states function diverges as $q_\nu \simeq 4/X_\nu^2$ whenever a Landau level coincides with the Fermi energy (i.e., when $n_F = n+1/2$). The summation over thermal excitation (Matsubara) frequencies significantly weakens this divergency. To see how this happens one may use the Fourier series expansion of q_ν to carry out the Matsubara summation term by term. This procedure determines the damping of each harmonic of the dHvA oscillations. Thus, writing

$$q_\nu = a_\nu^{[0]} + 2\sum_{p=1}^{\infty}(-1)^p a_\nu^{[p]}\cos(2\pi p n_F) \tag{57}$$

with

$$a_\nu^{[p]} = \frac{\exp[-2\pi p(\tilde{\Gamma}+|\tilde{\omega}_\nu|)]}{1-\exp[-4\pi p(\tilde{\Gamma}+|\tilde{\omega}_\nu|)]}$$

and $\tilde{\Gamma} = \Gamma/\hbar\omega_c$, and replacing the Matsubara sum of the fundamental term by an integral, it is found that the suppression factor of the fundamental dHvA oscillation in the vortex state, $\Xi_{\Sigma^z}/\Xi_\Sigma$, is

$$R_s^{[1]} = 1 - \ln\left[\frac{\hbar\omega_c}{2\pi(\Gamma+\pi k_B T)}\right]\frac{\pi^{3/2}\tilde{\Delta}_0^2}{n_F^{1/2}}. \tag{58}$$

It can be shown that the strength of the divergency increases with increasing harmonic. For the entire harmonic series at very strong resonance conditions (i.e., at n_F = half integer) one finds a suppression factor

$$R_s \sim 1 - \frac{\hbar\omega_c}{\Gamma_{ef}}\frac{\tilde{\Delta}_0^2}{n_F^{1/2}}. \tag{59}$$

To complete the discussion let us present here an alternative calculation of the suppression factor in the limit of small $\tilde{\Delta}_0$, when the diagonal approximation described in Sec. II.B.2 is valid. An analytical result derived recently by Zhuravlev *et al.* (1999) in this limiting case sheds light on the relationships between different approaches to the problem. In this calculation the formalism developed by Shoenberg (1984b) for various types of redistribution of the Landau sublevels was modified to include the effect of the pair potential. Assuming an isotropic 3D electron system, the thermodynamic potential is written as

$$\Omega_{QP}^{osc}=\frac{k_BT}{2\pi^2 a_H^3}\sum_{p=1}^{\infty}\frac{\Theta(p)}{p^{3/2}}\cos\left[2\pi p\left(\frac{F}{H}-\frac{1}{2}\right)-\frac{\pi}{4}\right], \tag{60}$$

where

$$\Theta(p)=\int_0^\infty\frac{d\varkappa}{\varkappa}\frac{e^{-(2\pi\Gamma/\hbar\omega_c)\varkappa}}{\sinh\left(\frac{2\pi^2k_BT}{\hbar\omega_c}\varkappa\right)}\Xi_p(\varkappa). \tag{61}$$

The integrand is a product of the Fourier transforms of three distributions: the Fermi distribution function accounting for thermal smearing, the Lorenz distribution arising from scattering by nonmagnetic impurities in the relaxation-time approximation and the distribution associated with "scattering" by the pair potential, $\Xi_p(\varkappa)$. For a small order parameter this integrand is reduced to

$$\Xi_p(\varkappa)=p\,\delta(\varkappa-p)-\left\langle\frac{J_1(x)}{2x}\left|\tilde{\Delta}_{n_F}^{(p)}(\vec{q})\right|^2\right\rangle\left(\frac{\varkappa}{p}\right)\theta(\varkappa-p), \tag{62}$$

where $\tilde{\Delta}_{n_F}^{(p)}(\vec{q})=2\pi p\Delta_{n,n}(\vec{q})/\hbar\omega_c$ at $n=n_F-1/2$, $x\equiv|\tilde{\Delta}_{n_F}^{(p)}(\vec{q})|\sqrt{\varkappa-p}$, $J_1(x)$ is the Bessel function of the first order and $\langle\cdots\rangle$ stands for integration in $\vec{q}$ space over the magnetic Brillouin zone. It is obvious that for a vanishing order parameter, $\Xi_p(\varkappa)\to p\,\delta(\varkappa-p)$ and Eq. (60) leads to the well-known Lifshitz-Kosevich (1956) formula. The superconducting contribution to the thermodynamic potential is represented by the second term on the right-hand side of Eq. (62). The negative sign in front and the proportionality to Δ_0^2 are reminiscent of the paramagnetic effect discussed in Sec. II.B.2. Using Eq. (62) in Eq. (61), we can readily see that the integral over $\varkappa$ diverges logarithmically in the limit when the smearing parameters $\pi k_BT,\Gamma$, as well as $\Delta^{[p]}\sim\sqrt{\langle|\tilde{\Delta}_{n_F}^{(p)}(\vec{q})|^2\rangle}$, all go to zero.

Using the form of the diagonal matrix element presented in Eq. (26), it can be shown that $\langle|\tilde{\Delta}_{n_F}(\vec{q})|^2\rangle=\tilde{\Delta}_0^2/\sqrt{\pi n_F}$. Defining an effective Landau-level width by

$$\Gamma_{ef}\equiv\max(\pi k_BT,\Gamma,\Delta^{[1]}), \tag{63}$$

one then finds that, in the limit when $\Gamma_{ef}\ll\hbar\omega_c/2\pi$, the suppression factor of the fundamental (i.e., $p=1$) dHvA oscillation is

$$R_s^{[1]}=1-\ln\left(\frac{\hbar\omega_c}{2\pi\Gamma_{ef}}\right)\frac{\pi^{3/2}\tilde{\Delta}_0^2}{\sqrt{n_F}}. \tag{64}$$

This result is similar to that derived by Miyake (1993) and by Miller and Gyorffy (1995) in the limiting case when $\Gamma_{ef}\equiv\Delta^{[1]}\ll\hbar\omega_c/2\pi$. In these works the spatial dependence of the order parameter was neglected, so a superconducting gap appeared around the Fermi surface. The similarity to the gapless situation in the small-Δ limit is due to the fact that the opening of a small superconducting gap on both sides of the Fermi energy is similar to the opening of the magnetic pseudogap in the middle of the Landau band. There is, however, a significant difference between the two approaches even in the present small-Δ limit, reflected in the nonlogarithmic prefactor of $\tilde{\Delta}_0^2$; in Miller and Gyorffy's (1995) paper this prefactor is independent of n_F, in contrast to the $1/\sqrt{n_F}$ dependence appearing in Eq. (64). The appearance of such a dependence is due to the spatial dependence of the order parameter (or in other words to the q dispersion of the quasiparticle energy), which is missing in the former models.

It is interesting to note that Eq. (64), derived within the diagonal approximation for $\Delta_0\ll\Gamma$, $\pi k_BT\ll\hbar\omega_c/2\pi$, is almost identical to Eq. (58) obtained from Gorkov's expansion, where the off-diagonal Landau-level pairing is taken into account exactly. The presence of Δ_0 in the logarithmic prefactor in Eq. (64) is obviously due to the nonperturbative nature of the present approach in contrast to the derivation of Eq. (58). However, the former procedure, though nonperturbative in $\tilde{\Delta}_0$, is severely restricted to very small values of $\tilde{\Delta}_0$ since it should satisfy the stringing conditions imposed by the diagonal approximation. These conditions can be summarized by

$$2\pi(\Gamma+\pi k_BT),2\pi^{3/4}\Delta_0/n_F^{1/4}\ll\hbar\omega_c.$$

In the limit when both the thermal and the disorder smearing parameters tend to zero, the effective width Γ_{ef} is determined by the pair potential, $\Gamma_{ef}=\Delta^{[1]}=\Delta_0/(\pi n_F)^{1/4}$. In this case the pair-potential effect renormalizes R_s given by Eq. (59), leading to a linear dependence (nonanalytic in $\tilde{\Delta}_0^2$) on Δ_0:

$$R_s\sim1-\pi^{1/4}\left(\frac{\tilde{\Delta}_0}{n_F^{1/4}}\right). \tag{65}$$

This result is consistent with that reported by Norman and MacDonald (1996), which was obtained from a numerical solution of the Bogoliubov–de Gennes equations for the corresponding quasiparticle spectrum.

In the more realistic situation when the combined impurity and thermal smearing parameter is not small, i.e., for $4\pi(\Gamma+\pi k_BT)/\hbar\omega_c\gtrsim1$, the fundamental term in Eq. (57) is dominant, and R_s is given by Eq. (55), which means that impurity scattering does not significantly influence the suppression mechanisms of the dHvA oscillations in the vortex state.

III. MEAN-FIELD THEORIES FOR DISORDERED VORTEX LATTICES. THE RANDOM VORTEX LATTICE MODEL

Real superconductors always contain some crystal defects and impurities. These usually lead to irregular pinning of vortices, which produces some deviations from the ideal structure of the Abrikosov lattice. The analysis presented in Sec. II.C.3, which revealed the importance of phase coherence in the vortex lattice in suppressing magnetic breakdown, is a clear indication of the great sensitivity of the dHvA amplitude to disorder in the vortex lattice. Thus, without a careful examination of the effects of disorder, the conclusions of the previous sections should be regarded as somewhat academic.

In the present section we describe the relevant models that account for this important effect.

Generally speaking, in evaluating the free energy at magnetic fields H sufficiently close to H_{c2} that Gorkov's expansion can be truncated at the quadratic term in Δ, Eq. (35), the effect of disorder in the Abrikosov lattice is trivially obtained simply by averaging $|\Delta(\vec{r})|^2$ over realizations of the vortex lattice. This extreme simplicity is associated with the fact, discussed in Sec. II.C.1, that any arbitrary distribution of vortices assumed in constructing the lowest-Landau-level wave function, Eq. (24), yields an exact solution of the linear self-consistent Eq. (31). This feature reflects the absence of interactions between vortices in the quadratic term. It can be immediately concluded that to leading order in Gorkov's expansion, disorder in the vortex lattice does not introduce any significantly new effect.

The simplicity is completely lost as H is varied away from H_{c2} and higher-order terms become important. To make some progress, the desire for a self-consistent order parameter should be abandoned, thus opening the way for more phenomenological approaches. The simplest option is to postulate a completely random distribution of vortices. This approach was first taken by Stephen (1992) in his study of the dHvA effect in the mixed state, arguing that the structure of the densely packed vortex lattice does not significantly influence the large cyclotron orbits of quasiparticles near the Fermi surface and only its average is important. This idea has been defended on the grounds that in orbiting through a large number of vortices the quasiparticle is scattered as if the vortices were random scatterers.

The following analysis reveals the flaw of this appealing argument when applied to the ideal Abrikosov lattice. Consider Gorkov's free energy (29) for a completely random distribution of vortices. The corresponding order parameter can be written in the generalized form of Eq. (23),

$$\Delta(\vec{r})=\Delta_0\sum_q c_q \exp[iqx-(y+q/2)^2],$$

where c_q are random variables defined on a quasicontinuous lattice $q=(2\pi/a_x)(k+m/\sqrt{N})$, $k,m=-\sqrt{N}/2+1,\dots,\sqrt{N}/2$ and satisfying the condition

$$\langle c_q c^*_{q'}\rangle \propto \delta_{q,q'}.$$

Using this white-noise condition and exploiting the quasicontinuous nature of the guiding-center projection q by replacing summation with integration, one can readily see that the pair correlation function is

$$\langle\Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\rangle=\Delta_0^2\exp\left[-\frac{1}{2}(\vec{r}_1-\vec{r}_2)^2\right] \times\exp[i(x_1+x_2)(y_1-y_2)]. \quad (66)$$

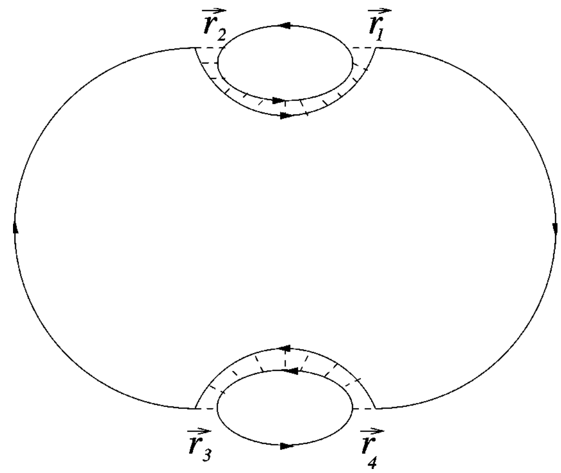


FIG. 19. Feynman diagram corresponding to the quartic term of the superconducting free energy averaged over realizations of the random vortex lattice. Single lines stand for free-electron propagators. Double parallel lines correspond to pair propagators. Note the local character of the pair propagators and the nonlocal nature (on the magnetic length scale) of the free-electron lines.

After averaging the free energy (29) over the realizations of the vortex lattice, one finds that the first nontrivial term is the quartic one. The white-noise condition leads to the factorization

$$\begin{aligned}\langle\Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\Delta(\vec{r}_3)\Delta^*(\vec{r}_4)\rangle &=\langle\Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\rangle\langle\Delta(\vec{r}_3)\Delta^*(\vec{r}_4)\rangle\\ &+\langle\Delta(\vec{r}_1)\Delta^*(\vec{r}_4)\rangle\langle\Delta(\vec{r}_3)\Delta^*(\vec{r}_2)\rangle. \quad (67)\end{aligned}$$

Similar factorization holds to any order in the perturbation expansion.

Examining each term on the right-hand side of Eq. (67) with the aid of the pair-correlation function Eq. (66) and the exact sum rule $\Sigma_{l=1}^4\vec{\rho}_l=0$ characterizing the diagram shown in Fig. 19, one can be readily convinced that, in all regions where the Gaussian factors are not negligibly small, the corresponding total phase is always of the order one or smaller. This means that in the present case there are no violent oscillations of the corresponding phase factor that occur in the ideal Abrikosov lattice case [i.e., the function $\mathcal{D}(\vec{S},\vec{T})$ in Eq. (49)] for any nonlocal configuration of the four electrons involved. Consequently, nonlocal configurations of electron pairs (see Fig. 19) contribute significantly to the superconducting free energy, in contrast to the ideal vortex lattice case. Note that the Gaussian factors localize each pair, i.e., they impose $\vec{r}_1\approx\vec{r}_2$ and $\vec{r}_3\approx\vec{r}_4$ on the first term and $\vec{r}_1\approx\vec{r}_4$ and $\vec{r}_3\approx\vec{r}_2$ on the second term in Eq. (67).

Thus, in contrast to the ideal Abrikosov lattice case, the loss of phase coherence in the random vortex lattice model allows different pairs to propagate independently over distances much larger than the magnetic length, leading to complete breakdown of the local semiclassical picture shown in Fig. 17. In particular, it implies that the corresponding cyclotron orbital centers move relative to each other over distances much larger than the unit cell of the vortex lattice, which is the essence of a strong magnetic breakdown effect associated with the pair potential.

A. The Stephen approach

Assuming the random vortex lattice model discussed above, it is most convenient to follow the procedure used by Stephen (1992) in evaluating the influence of the superconducting order parameter on the dHvA oscillations for a disordered arrangement of vortices caused, for example, by pinning.

Starting from the equation for the ordinary single-electron Green's function [which can be derived from Gorkov's equations (Abrikosov *et al.*, 1975)],

$$G(\vec{r},\vec{r}\,',\omega_\nu)=G^0(\vec{r},\vec{r}\,',\omega_\nu)-\int d^3r_1d^3r_2G^0(\vec{r},\vec{r}_1,\omega_\nu)\times G^0(\vec{r}_2,\vec{r}_1,-\omega_\nu)\Delta(\vec{r}_1)\times\Delta^*(\vec{r}_2)G(\vec{r}_2,\vec{r}\,',\omega_\nu) \quad (68)$$

(where for the sake of simplicity any spin dependence was dropped), and averaging over the realizations of the vortex lattice, the factorization expressed in Eq. (67) can be used to construct a diagram technique in which

$$D(\vec{r}_1,\vec{r}_2)\equiv\langle\Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\rangle \quad (69)$$

plays the role of a pair propagator.

The resulting Dyson-like equation in the self-consistent Born approximation (Ando, Fowler, and Stern, 1982) is

$$\langle G(\vec{r},\vec{r}\,',\omega_\nu)\rangle=G^0(\vec{r},\vec{r}\,',\omega_\nu)-\int d^3r_1d^3r_2G^0(\vec{r},\vec{r}_1,\omega_\nu)\times\Sigma^\star(\vec{r}_1,\vec{r}_2,\omega_\nu)\langle G(\vec{r}_2,\vec{r}\,',\omega_\nu)\rangle, \quad (70)$$

with the corresponding proper self-energy

$$\Sigma^\star(\vec{r}_1,\vec{r}_2,\omega_\nu)=D(\vec{r}_1,\vec{r}_2)\langle G(\vec{r}_2,\vec{r}_1,-\omega_\nu)\rangle.$$

This set of equations has a simple solution in the representation $\varphi_{nqk_z}(x,y,\zeta)=e^{ik_z\zeta}\varphi_n(x,y+q)$, where $\varphi_n(x,y+q)$ is a Landau gauge wave function with a Landau-level index n and guiding-center projection q, and k_z is a momentum along the field direction. It can be written in the form

$$\langle G(\vec{r},\vec{r}\,',\omega_\nu)\rangle=\sum_{n,q,k_z}\frac{\varphi_{nqk_z}(\vec{r})\varphi^*_{nqk_z}(\vec{r}\,')}{i\omega_\nu-\xi_{n,k_z}-\Sigma_{n,k_z}(\omega_\nu)}, \quad (71)$$

where $\xi_{n,k_z}=\hbar\omega_c(n+1/2)+\hbar^2k_z^2/2m^*-E_F$. The simplicity is due to the diagonal form of the self-energy $\Sigma^\star$ in this basis set, namely, to the property

$$\langle\varphi_{nqk}(1)|\Sigma^\star(\vec{r}_1,\vec{r}_2,\omega_\nu)|\varphi_{n'q'k'}(2)\rangle=\delta_{nqk,n'q'k'}\Sigma_{n,k}(\omega_\nu).$$

A self-consistent equation can thus be derived for the reduced self-energy, $\Sigma_{n,k}(\omega_\nu)$, by combining Eq. (71) with Eq. (70), i.e.,

$$\Sigma_{n,k_z}(\omega_\nu)=\Delta_0^2\sum_{n_1}\frac{I_{n,n_1}}{i\omega_\nu+\xi_{n_1,k_z}+\Sigma_{n_1,k_z}(-\omega_\nu)}, \quad (72)$$

where $I_{n,n_1}=(n+n_1)!/n!n_1!2^{n+n_1+1}$.

Two limiting cases are of interest, the high-temperature limit and the limit of very low temperatures.

1. The high-temperature limit, $2\pi^2k_BT/\hbar\omega_c\sim 1$

This case was considered in Sec. II.C.4. In this limit only the first harmonic of the dHvA oscillations is of interest to us. Thus the sum over n_1 in Eq. (72) can be replaced by an integral, since the self-energy should be kept to zeroth order in the harmonic expansion. As in Sec. II.C.3, we also assume small values of Δ_0^2, so that we may neglect the self-energy in the denominator of Eq. (72). Using the asymptotic form, Eq. (36), of I_{n,n_1}, the integral over n_1 yields for the reduced self-energy at the extremal orbit $k_z=0$, $n\approx n_F$,

$$\tilde{\Sigma}_{n_F,0}(\omega_\nu)\approx -i\frac{\sqrt{\pi}\tilde{\Delta}_0^2}{2\sqrt{n_F}}\mathrm{sgn}(\omega_\nu). \quad (73)$$

Such an imaginary self-energy corresponds to a scattering rate $\hbar/\tau_v=\Delta_0^2(\pi/E_F\hbar\omega_c)^{1/2}$ of quasiparticles by the random vortex lattice, which leads to the following superconducting-induced damping of the fundamental dHvA frequency:

$$M^{[1]}\approx M_N^{[1]}\exp\left(-\frac{\pi^{3/2}\tilde{\Delta}_0^2}{n_F^{1/2}}\right). \quad (74)$$

This extremely simple result was first derived by Maki (1991) on the basis of an averaging technique first introduced by Brandt *et al.* (1967). Wasserman and Springford (1996) have provided a detailed analysis of this approach, which will be described in Sec. III.B.

A comparison of this result with the corresponding expression [Eq. (55)] obtained for an ideal Abrikosov lattice is instructive. Expanding the exponential up to second order, we find that the quadratic term in $\tilde{\Delta}_0$ is identical to the corresponding term in Eq. (55), whereas the quartic term is larger by a factor of the order $n_F^{1/2}$ than the corresponding term in Eq. (55). The identity of quadratic terms is in full accord with the well-known property of the quadratic term in the superconducting free energy, discussed at the beginning of Sec. III, namely, that it is independent of the vortex configurations—a reflection of the absence of interaction between vortices. The large difference between the quartic terms can be attributed to the great sensitivity of the quartic term (and all higher-order terms as well) to the presence or absence of phase coherence in the vortex state. The large factor of $n_F^{1/2}$ reflects the dramatic increase in the spatial region traveled by the two electron pairs in Fig. 19 relative to each other after disorder has been introduced into the vortex lattice (see Sec. III.B).

Another interesting difference between the two limiting cases concerns the paramagnetic vortex currents described in Sec. I.B.3 and analyzed in Sec. II.B.2. This effect is completely destroyed in the random vortex state due to the breakdown of the cyclotron orbital structure discussed above. The destruction is reflected in the absence of the magnetic pseudogap from the (Lorenzian) quasiparticle density of states corresponding to Eq. (73). It can be related to the Gaussian form of the pair-correlation function, Eq. (66), which does not contain any remnant of the individual zeros of the vortex lattice order parameter due to averaging over random configurations of vortices.

2. The limit of very low temperatures $2\pi^2 k_B T/\hbar\omega_c \ll 1$

In this case one should be careful not to make any approximation that would destroy the self-consistent nature of Eq. (72), since otherwise the divergences discussed in Secs. II.C.2 and II.C.5 could not be avoided. Thus, considering again small values of Δ_0^2 and assuming the exact half filling condition $n=n_F-1/2$, we find that the dominant contribution to the sum over n_1 corresponds to $n_1=n=n_F-1/2$ and Eq. (72) reduces to

$$\Sigma_{n,0}(\omega_\nu)\approx\Delta_0^2\frac{(1/2\sqrt{\pi n})}{i\omega_\nu+\Sigma_{n,0}(-\omega_\nu)}. \tag{75}$$

An additional equation can be obtained by replacing ω_ν with $-\omega_\nu$ in Eq. (75). Combining the two equations yields $\Sigma_{n,0}(\omega_\nu)=-\Sigma_{n,0}(-\omega_\nu)$ so that the solution of the resulting quadratic equation is

$$\Sigma_{n_F,0}(\omega_\nu)\approx\frac{i\omega_\nu}{2}\pm i\sqrt{\left(\frac{\omega_\nu}{2}\right)^2+\frac{\Delta_0^2}{2\sqrt{\pi n}}}$$

$$\to\pm i\frac{\Delta_0}{2^{1/2}(\pi n_F)^{1/4}},\quad T\to 0. \tag{76}$$

Note that the solution with the negative (positive) sign is of physical meaning for positive (negative) ω_ν.

This result is consistent with the form proposed by Norman *et al.* (1995) on the basis of their numerical solution of the Bogoliubov–de Gennes equations in the low-temperature region $k_BT/\hbar\omega_c\lesssim 0.04$. The corresponding limiting expression for the damping factor of magnetization is

$$R_s\to 1-\frac{2^{1/2}\pi^{3/4}\tilde{\Delta}_0}{n_F^{1/4}}.$$

This is essentially in agreement with the nonanalytic expression obtained in Sec. II.C.5, Eq. (65).

B. Nonlocal behavior

As indicated in the previous section, an alternative averaging procedure based on the original idea of Brandt *et al.* (1967) was first used by Maki (1991) and later by Wasserman and Springford (1996) in analyzing dHvA oscillations in the vortex state. The latter authors utilized a Green's-function technique in momentum space, which neglects the effect of Landau quantization on the self-energy. Furthermore, they used a spatial averaging procedure, which sets severe restrictions on the validity of their approach.

However, the method, which exploits momentum representation, can be useful in illuminating the origin of the dramatic change to the dHvA effect occurring in a random vortex lattice and in analyzing the nonlocal behavior of the corresponding superconducting free energy in real space.

Starting with a slightly modified form of Gorkov's Eq. (68) by redefining $\tilde{G}(\vec{r},\vec{r}\,',\omega_\nu)\equiv e^{-i\chi(\vec{r},\vec{r}\,')}G(\vec{r},\vec{r}\,',\omega_\nu)$, where $e^{i\chi(\vec{r},\vec{r}\,')}$ is the gauge factor of the normal-electron Green's function defined in Eq. (37), we can rewrite this equation in terms of the reduced Green's function $\tilde{G}^0(|\vec{r}-\vec{r}\,'|,\omega_\nu)$ and the effective two-particle interaction

$$V(\vec{r}_1,\vec{r}_2)\equiv\Delta(\vec{r}_1)\Delta^*(\vec{r}_2)e^{-2i\chi(\vec{r}_1,\vec{r}_2)}. \tag{77}$$

Brandt *et al.* (1967), who first introduced such a form of Gorkov's equation in their extensive studies of the quasiparticle density of states at high magnetic fields (see also a review by Tewordt, 1969), have shown that in the Landau gauge, where $\chi(\vec{r}_1,\vec{r}_2)=\frac{1}{2}(x_1+x_2)(y_1-y_2)$, the effective interaction Eq. (77) is a periodic function of the center-of-mass coordinates $\vec{R}=\frac{1}{2}(\vec{r}_1+\vec{r}_2)$, with the periodicity of the Abrikosov lattice. Writing this as a function of the center-of-mass coordinates $\vec{R}$ and the relative coordinates $\vec{\rho}=(\vec{r}_1-\vec{r}_2)$, it is then possible to use the double Fourier transform

$$V(\vec{\rho},\vec{R})=\sum_{\vec{p},\vec{K}}V(\vec{p},\vec{K})e^{i\vec{p}\cdot(\vec{r}-\vec{r}\,')}e^{i\vec{K}\cdot\vec{R}}$$

with $\vec{K}$ running over all discrete points of the 2D lattice reciprocal to the Abrikosov lattice (see Sec. II.B.1). Using a Fourier transform for the full Green's function, the momentum-space representation of Eq. (68) can be written as

$$G_{\omega_\nu}\left(\vec{p}-\frac{1}{2}\vec{k},-\vec{k}\right)$$

$$=\delta_{\vec{k},0}G^0_{\omega_\nu}(\vec{p})-G^0_{\omega_\nu}(\vec{p}-\vec{k})$$

$$\times\sum_{\vec{K}'}G_{\omega_\nu}\left(\vec{p}-\frac{\vec{k}+\vec{K}'}{2},-\vec{k}-\vec{K}'\right)$$

$$\times\sum_{\vec{p}\,'}V(\vec{p}\,',\vec{K}')G^0_{-\omega_\nu}\left(\vec{p}-\vec{p}\,'+\vec{k}+\frac{1}{2}\vec{K}'\right). \tag{78}$$

The key step made by Brandt *et al.* was to average the Green's function $\tilde{G}(\vec{\rho},\vec{R},\omega_\nu)$ over the mean coordinates $\vec{R}$. This was a physically meaningful step in their calculation, which focused on evaluating the quasiparticle density of states for a tunneling experiment, in which the spatial average of the local density of states is what was actually measured. In evaluating magnetization oscillations there is no apparent physical reason for such averaging at the intermediate stage of the self-energy calculation. Furthermore, as we shall see below, this averaging destroys the delicate phase coherence discussed in Sec. II.C.2, exactly as happens after averaging over realizations of the random vortex lattice (Sec. III.A). The latter averaging is, however, fully justified in cases when the vortex lattice is disordered.

Thus, proceeding along the lines drawn above, averaging over the mean coordinates $\vec{R}$ is equivalent to substituting $\vec{k}=0$ in Eq. (78). The resulting equation is still not in a closed form for $G_{\omega_\nu}(\vec{p})$, but can be converted into such a form by neglecting all umklapp terms, i.e., by taking into account only the term with $\vec{K}'=0$ in the reciprocal Abrikosov lattice.

This approximation was justified by Brandt *et al.* (1967) on the basis of the Gaussian form of the interaction V,

$$V(\vec{p},\vec{K})\propto\Delta_0^2 e^{-(1/2)p^2}\exp\left[\frac{1}{2}i\left(K_x p_y-K_y p_x+\frac{1}{2}K_x K_y\right)\right],$$

and of the discrete nature of the vector $\vec{K}$ at high magnetic fields. It is equivalent to the diagonal form of the self-energy obtained by Stephen (1992) in the Landau-level representation after averaging over the random vortex lattice (see Sec. III.A). Note, however, that the latter is an exact result of the ensemble-averaging process while the former is only an approximation of the expression obtained after spatial averaging.

The resulting equation for $G_{\omega_\nu}(\vec{p})$ can be easily solved, yielding

$$G_{\omega_\nu}(\vec{p})=G^0_{\omega_\nu}(p)[1+G^0_{\omega_\nu}(p)\Sigma^0_{\vec{p}}(\omega_\nu)]^{-1}, \tag{79}$$

where $\Sigma^0_{\vec{p}}(\omega_\nu)=\Sigma_{\vec{p}'}V(\vec{p}',0)G^0_{-\omega_\nu}(|\vec{p}-\vec{p}'|)$ is the self-energy part. By using the zero-magnetic-field limit of $G^0_{\omega_\nu}(p)$, one can readily calculate the self-energy at p near the Fermi surface, obtaining

$$\tilde{\Sigma}^0_{\vec{p}}(\omega_\nu)\approx -i\frac{\sqrt{\pi}\tilde{\Delta}_0^2}{2\sqrt{n_F}}\exp\left[-\frac{(p^2-p_F^2)^2}{8p_F^2}\right]\text{sgn}(\omega_\nu), \tag{80}$$

where it is understood that every momentum appearing in an exponent is measured in units of $\hbar/a_H$.

This result can be derived from Eq. (72) in the asymptotic limit of I_{n,n_1}, Eq. (36), after replacing the Landau energy level $\hbar\omega_c(n+1/2)$ with the free-electron energy $p^2/2m_c$ [see the remark following Eq. (36)] and integrating over momentum corresponding to n_1. At $p=p_F$, it is precisely the limiting expression for the Stephen self-energy [Eq. (73)] calculated at the extremal cross section of the Fermi surface. However, in the low-temperature limit discussed in Sec. III.A, this approach fails to properly handle the singularity discussed in Sec. II.C.5, since the electron propagator in the self-energy $\Sigma^0_{\vec{p}}(\omega_\nu)$ is not dressed. Unlike the spatial averaging discussed here, ensemble averaging over random vortex lattice realizations leads to a self-consistent equation for the self-energy that avoids this singularity. The former procedure introduces additional freedom into the electronic Hilbert space, which results in a field-theoretic problem similar to the well-known electron-phonon or electron-photon problem. The Hartree-Fock approximation used in the context of the latter problems (Fetter and Walecka, 1971) is equivalent to the self-consistent Born approximation (Ando, Fowler, and Stern, 1982) outlined in Sec. III.A. In the field of transport properties of disordered media (Ziman, 1979) a similar approximation is known as the coherent potential approximation (CPA). A closely related functional-integral approach will be described in Sec. IV, when thermal fluctuations of the vortex lattice will be taken into account.

The diagonal form of the self-energy in the Landau-level representation obtained by Stephen, and its weak dependence on the thermal frequency ω_ν, allow the use of a simple formula for the thermodynamic potential derived by Wasserman and Springford (1996) for this case. The oscillatory part of this expression can be written in the form

$$\Omega_{osc}=-\frac{k_BT}{\pi a_H^2}\text{Re}\sum_{\nu=-\infty}^{\infty}\sum_{k_z,n}\ln[G^{-1}_{\omega_\nu}(E_{n,k_z})], \tag{81}$$

where $G_{\omega_\nu}(E_{n,k_z})$ is the Green's function obtained from Eq. (79) after approximating $\tilde{\Sigma}^0_{\vec{p}}(\omega_\nu)\approx -i\sigma\,\text{sgn}(\omega_\nu)$, where

$$\sigma\equiv\frac{\sqrt{\pi}\tilde{\Delta}_0^2}{2\sqrt{n_F}}, \tag{82}$$

and replacing $|\vec{p}|^2/2m_c$ with $\hbar\omega_c(n+1/2)+\hbar^2k_z^2/2m^*=\xi_{n,k_z}+E_F$. The "complex energy" argument of the Green's function is defined as $E_{n,k_z}\equiv\xi_{n,k_z}+E_F-i\sigma\,\text{sgn}(\omega_\nu)\hbar\omega_c$.

Using the Dyson Eq. (79) in the form $G^{-1}=(G^0)^{-1}(1+G^0\Sigma)$, the logarithm in Eq. (81) can be expanded in power series of Σ. The first factor yields the thermodynamic potential of the normal-electron gas Ω_n whereas the second factor, $\ln(1+G_0\Sigma)$, is expanded to infinite order in σ. Substituting this expansion into Eq. (81), and using the Poisson summation formula, one finds for the oscillatory part of the thermodynamic potential in the quasiclassical limit $n_F\gg 1$

$$\Omega_{osc}=\Omega_{n,osc}-\frac{k_BT}{\pi a_H^2}\text{Re}\sum_{\nu=-\infty}^{\infty}\sum_{l=1}^{\infty}\frac{1}{l}\int_0^\infty dn \times\sum_{k=1}e^{2\pi ikn}\left[\frac{i\sigma\,\text{sgn}(\omega_\nu)\hbar\omega_c}{i\omega_\nu-E_{n,0}+E_F}\right]^l. \tag{83}$$

It is easy to show that the right-hand side of this equation can be written in the simple form

$$\Omega_{osc}=-\frac{k_BT}{\pi a_H^2}\text{Re}\sum_{\nu=-\infty}^{\infty}\sum_{k=1}e^{-2\pi k(|\omega_\nu|/\hbar\omega_c+\sigma)} \times\frac{\cos 2\pi k(n_F-1/2)}{k}. \tag{84}$$

The term for $k=1$ in this harmonic series reproduces the exponential damping factor Eq. (74). A detailed examination of Eq. (83) can reveal the origin of the strong exponential damping of the dHvA amplitude in the random vortex lattice limit. The sum over l corresponds to the perturbation expansion in the superconducting order parameter. The quartic term ($l=2$) consists of the effective self-energy σ and the energy denominator $(i\omega_\nu - E_{n,0}+E_F)^{-1}$, both to the second power, which correspond to the pair propagators and the free-electron lines, respectively, shown in Fig. 19. Each pair propagator contains an intermediate sum over the Landau-level index [n_1 in Eq. (72)] restricted by the Gaussian in Eq. (80) to a few Landau levels near n_F. This reflects the localized nature of each pair propagator (i.e., within a spatial region of the order of a magnetic length).

The free-electron lines connecting the two self-energy parts in Fig. 19 have no such restriction; the integral over n in Eq. (83) may be transformed into an integral in momentum space, which is dominated by the contribution from the region $\Delta n \sim 1/k \sim \Delta E_{n,0}/\hbar\omega_c \sim v_F \Delta p/\hbar\omega_c$, that is, the region $\Delta p \sim \hbar/a_H\sqrt{n_F}$ for $k \sim 1$. The corresponding real-space propagation distance of this electron is $\Delta\rho \sim \sqrt{n_F}a_H$, which is much larger than the magnetic length a_H. Such extremely nonlocal behavior in real space (or localized behavior in momentum space) is due to the rapidly oscillating exponential $e^{2\pi ikn}$ appearing in Eq. (83) for $k>0$; this does not occur in the nonoscillatory (zero harmonic, $k=0$) component of the thermodynamic potential.

In contrast, the important contributions in the ideal Abrikosov lattice case originate from configurations in which all four electrons are in a close mutual proximity within a spatial region of the order of a magnetic length, resulting in a reduction factor of $1/\sqrt{n_F}$ [see Eq. (55)] relative to the random vortex lattice case.

IV. SUPERCONDUCTING FLUCTUATION EFFECTS AND VORTEX LATTICE MELTING

The discussions presented in the previous sections lead towards one main conclusion, namely, that the amplitude of dHvA oscillations in the mixed superconducting state is very sensitive to the presence or absence of phase coherence in the order parameter. A very important consequence of this sensitivity, discussed in some detail in Sec. III, is the strong additional damping of the dHvA amplitude expected to occur in the mixed state as a result of disorder in the vortex lattice that may be caused by irregular pinning. Theoretical treatments of this situation so far have an important shortcoming: lack of a truly self-consistent framework, since the white-noise condition introduced in Sec. III was not derived from the original free-energy functional, Eq. (29).

A very important, closely related physical situation, which allows a self-consistent treatment, arises when thermal fluctuations of the superconducting order parameter are significant. In conventional superconductors at zero magnetic field, fluctuation effects are primarily very small because the coherence length is much larger than the characteristic interatomic distance. In the presence of a magnetic field the situation changes significantly. The resistive transition is no longer a sharp transition (see Mun *et al.*, 1996), whereas the specific-heat jump at the thermodynamical transition is strongly smeared at high magnetic fields (see, for example, Wanka *et al.*, 1998). It is believed that thermal fluctuations significantly influence the transition to the mixed state at high magnetic fields, though in this region of the phase diagram temperature is relatively low. Fluctuations in the superconducting order parameter are, indeed, expected to be significantly enhance at high magnetic fields when the Landau quantization of the Cooper-pair energy leads to an effective dimensionality reduction of 2 (Brezin *et al.*, 1985).

Naturally, one may consider in this context phase fluctuations, which are known to play a crucial role in controlling the stability of the vortex lattice and its possible melting processes. To begin with, it is most instructive to consider a simple 2D electron gas-model, in particular since 2D systems are much more susceptible to thermal fluctuations than 3D ones, and in view of the unusual melting transitions that characterize 2D solids (Kosterlitz and Thouless, 1973; Halperin and Nelson, 1978). Furthermore, as we shall see in Sec. IV.A.1, the high-field melting transition occurs in 2D systems at a temperature well below the mean-field T_c, which makes this phenomenon relevant to the study of quantum magneto-oscillations. Fortunately, many experimental reports on measurements of the dHvA effect in quasi-2D superconductors are available (Wosnitza, 1993, 1996). Some of these measurements have been carried out successfully in the mixed superconducting state (van der Wel *et al.*, 1995; Sasaki *et al.*, 1998).

A. Phase fluctuations and melting of the vortex lattice in 2D superconductors

In the mixed state of a 2D type-II superconductor for magnetic fields $H<H_{c2}(T)$ at low temperatures $T \ll T_c$, amplitude fluctuations of the superconducting order parameter are suppressed, but at a certain magnetic field $H_m(T)$, $H_{c1}<H_m<H_{c2}$ phase fluctuations can lead to melting of the vortex lattice (Friemel *et al.*, 1996; Sasaki *et al.*, 1998). This remarkable melting phenomenon results from a soft-shear, Goldstone mode, which can be described by long-wavelength phase fluctuations (Moore, 1989). Unfortunately, rigorous analytical approaches to this problem have encountered fundamental difficulties: large-order high-temperature perturbation expansion with Borel-Padé approximants to the low-temperature behavior (Ruggeri and Thouless, 1976; Brezin *et al.*, 1990) gives no indication of an ordered vortex lattice even at zero temperature. The existing nonperturbative approaches have not completely clarified the situation. Indeed, renormalization-group studies (Radzihovsky, 1995; Newman and Moore, 1996), as well as Monte Carlo simulation (O'Neil and Moore, 1992), failed to predict a crystal vortex state in a pure 2D superconductor at finite temperatures, while in the density-functional formalism of Tesanovic and coworkers (Tesanovic and Andreev, 1994; Herbut and Te-

sanovic, 1994, 1995) the low-temperature phase is a charge-density wave of Cooper pairs with no long-range superconducting phase order. Several Monte Carlo simulations have shown (Hikami, Fujita, and Larkin, 1991; Tesanovic and Xing, 1991; Xing and Tesanovic, 1992; Hu and MacDonald, 1993; Kato and Nagaosa, 1993; Sasik and Stroud, 1994) that in a 2D superconductor a true vortex lattice melting phase transition takes place at a finite temperature and that the transition is of first order.

A simple analytical model of vortex lattice melting in 2D extreme type-II superconductors was recently presented by Zhuravlev and Maniv (1999). The model is based on the observation that at low temperature the main correction to the mean-field free energy arises from "Bragg chain fluctuations," that is, fluctuations that preserve long-range periodic order along the principal crystallographic axes in the vortex lattice as depicted in Fig. 7. As a result of these fluctuations the sharp mean-field transition to the Abrikosov lattice state becomes a smooth crossover. Since the strength of the phase-dependent terms in the superconducting free energy is relatively small ($\sim 2\%$ of the condensation energy), the scale of the crossover temperature T_{cm} is well below the mean-field T_c. At temperatures higher than T_{cm} the ordered set of Bragg chains transforms to an ensemble of uncorrelated chains, fluctuating independently around equilibrium lattice positions. Because of a discontinuous (rotational) symmetry change in the average chain structure, there is a weak first-order transition superimposed on the smooth solid-liquid crossover, which is reflected in a small jump of the vortex system entropy at a certain melting temperature $T_m \approx T_{cm}$.

Calculation of the structure factor showed that exact long-range translational order exists only at zero temperature, in agreement with previous results (Newman and Moore, 1996), whereas at finite temperatures below the melting point T_m quasi-long-range (i.e., power-law) translational order (Hu and MacDonald, 1993) was found with only short-range superconducting order (Tesanovic, 1995). Results for various thermodynamic parameters agree well with the numerical calculations of Kato and Nagaosa (1993).

1. The sliding Bragg-chain model of vortex lattice melting

As indicated above, the melting transition occurs at temperature T_m well below the mean-field T_c where amplitude fluctuations are strongly suppressed and phase fluctuations do not completely destroy the order along the principal crystallographic axes, described in Sec. II.B.1. It is thus reasonable to assume that the local form of the Gorkov-Ginzburg-Landau free-energy functional derived in Sec. II.C.2 is valid in this region.

Thus, combining Eq. (35) with Eq. (50), the free-energy functional projected on the subspace of the lowest Landau level can be written in the Ginzburg-Landau form

$$F_{GL}=\int d^2r\left[-\alpha|\Psi(\vec{r})|^2+\frac{1}{2}\beta|\Psi(\vec{r})|^4\right], \quad (85)$$

where $\Psi(\vec{r})$ is the Ginzburg-Landau wave function, which is related to the order parameter via $\Psi(\vec{r}) \equiv (2E_F\mathcal{D}_{2D})^{1/2}\Delta(\vec{r})/\pi k_B T_c$, with the Ginzburq-Landau coefficients expressed in terms of the microscopic parameters [see Eqs. (51) and (52)] $\alpha=\tilde{\alpha}(\pi k_B T_c)^2/2E_F$ and $\beta=\tilde{B}(\pi k_B T_c)^2/2\mathcal{D}_{2D}E_F^2\beta_A$.

As discussed in Secs. II.A and II.C.1, all possible configurations of the order parameter in the lowest-Landau-level subspace can be taken into account by considering the partition function $\mathcal{Z}$ defined in Eq. (28), which reduces in the local approximation to

$$\mathcal{Z}=\int \mathcal{D}\Psi\mathcal{D}\Psi^* \exp[-F_{GL}/k_B T]. \quad (86)$$

As in Sec. III, it is possible to write an arbitrary wave function from the lowest-Landau-level subspace in an infinite superconductor as a one-dimensional integral,

$$\Psi(x,y)=\int dq\, c(q)\phi_q(x,y), \quad (87)$$

where $\phi_q(x,y)=\exp[iqx-(y+q/2)^2]$. Note that all spatial lengths are measured here in units of magnetic length a_H. A system of N vortices with size $L_x=a_x\sqrt{N}$ along the x direction is described by $c(q) =\Sigma_{j=-N/2+1}^{N/2}C_j\delta(q-Q_j)$, where $Q_j=(2\pi/a_x\sqrt{N})j$, $j =-N/2+1,\dots,N/2$. The discussions of Secs. II.B.1 and II.C.3 show that the minimal value of the Ginzburg-Landau free-energy functional is obtained when only $\sqrt{N}$ coefficients from the whole set of N coefficients C_j differ from zero, i.e., $c_k \equiv C_{k\sqrt{N}} \neq 0$, for $k=-\sqrt{N}/2 +1,\dots,\sqrt{N}/2$. At sufficiently low temperatures, when amplitude fluctuations are suppressed, this minimum corresponds to the minimum of the Abrikosov parameter

$$\beta_a=\left(\frac{1}{V}\int d^2r\langle|\Psi|^4\rangle\right)\Bigg/\left(\frac{1}{V}\int d^2r\langle|\Psi|^2\rangle\right)^2, \quad (88)$$

where V is the (2D) volume of the superconductor. From the definition of β_a we see that the absolute minimum $\beta_a=1$ is obtained for a spatially uniform order parameter. Any deviation from $|\Psi|=\text{const}$ leads to an increase in β_a. Under the constraint of the lowest-Landau-level subspace, however, $|\Psi|\neq 0$ cannot be a constant (since $|\Psi|=0$ at the vortex cores), and the minimum $\beta_a=\beta_A\simeq 1.159$ is obtained for a triangular Abrikosov lattice, which is the closest configuration to the homogeneous one. Other periodic lattices yield small positive deviations from β_A, while any departure of Ψ from the quasiuniform distribution of the vortex lattice towards a localized structure leads to a drastic increase of the free energy (Zhuravlev *et al.*, 1999).

Thus at low temperatures the main correction to the mean-field order parameter arises from fluctuations of c_k and a_x where

$$c(q)=\sum_{k=-\sqrt{N}/2+1}^{\sqrt{N}/2} c_k\delta\left(q-\frac{2\pi k}{a_x}\right). \quad (89)$$

Note that regardless of the choice of c_k, $\Psi(x,y)$ is a periodic function of x with a period a_x. Selecting the x

direction to be along the principal crystallographic axis shown in Fig. 7, the corresponding Bragg chains are allowed to slide independently along this axis, where the relative horizontal position, x_k, of the kth chain is determined by the phase $\varphi_k=-i\ln(c_k/|c_k|)$ and the chain's vertical position $q_k=2\pi k/a_x$, namely, $x_k=-\varphi_k/q_k$ (see Sec. II.B.1).

The partition function (86) can therefore be approximated by the multiple functional integral

$$Z\approx\int\prod_k dc_k dc_k^* \exp(-\pi f_{GL}), \tag{90}$$

where

$$f_{GL}\equiv\frac{F_{GL}}{\pi k_B T\sqrt{N}}$$
$$=-\overline{\alpha}\sum_k |c_k|^2+\frac{\overline{\beta}}{2}\sum_{k,s,t}\Lambda^{s^2+t^2}c_k^* c_{k+s+t}^* c_{k+s}c_{k+t} \tag{91}$$

with $\overline{\alpha}=\alpha a_x/\sqrt{2\pi}k_B T$, $\overline{\beta}=\beta a_x/\sqrt{4\pi}k_B T$, and

$$\Lambda=\exp\left(-\frac{\pi^2}{a_x^2}\right). \tag{92}$$

Note that the functional Eq. (91) is invariant under an arbitrary linear shift of the phases φ_k,

$$\varphi_k'=\varphi_k+ak+b, \tag{93}$$

where a and b are arbitrary constants. This symmetry follows from the invariance of the Ginzburg-Landau free-energy [Eq. (85)] under the magnetic translation group (Zak, 1964; Brown, 1968).

The key approximation at this point is based on the small value of the parameter $\Lambda\simeq e^{-\pi}$, which allows us to neglect in Eq. (91) all terms of order higher than Λ^2. This implies that, in addition to the first-order terms in Λ, only the leading-order terms in the phase (φ_k)-dependent part of the free energy are retained. Thus, up to this order in Λ, we have

$$f_{GL}=-\overline{\alpha}\sum_k |c_k|^2+\frac{\overline{\beta}}{2}\sum_k [|c_k|^4+4\Lambda|c_k|^2|c_{k+1}|^2 +4\Lambda^2|c_{k-1}||c_{k+1}||c_k|^2\cos\chi_k]. \tag{94}$$

The angles χ_k are linear combinations of the phases φ_k,

$$\chi_k=-2\varphi_k+\varphi_{k-1}+\varphi_{k+1}, \tag{95}$$

which are clearly invariant under the transformation (93).

It is therefore concluded that the low-lying excitations of the Abrikosov vortex lattice are associated with the sliding motions of the lattice Bragg chains along the principal crystallographic axes. These excitations are closely related to the soft shear modes discussed by Moore (1989) in connection with vortex lattice melting. It should be noted that the soft mode described above is associated with the long-wavelength component of the phase fluctuations in Eq. (94); this becomes clear if we neglect amplitude fluctuations, define the fluctuating phases

$$\varphi_f(k)\equiv\varphi_k-\varphi_k^{(L)}=\varphi_k-\frac{1}{2}\pi k^2,$$

and take the continuous limit [see Eq. (95)], i.e., $\chi_k\to\pi+\partial^2\varphi_f/\partial y^2$, so that the relevant part in the free-energy functional (94) can be written as

$$\delta f_{ph}=K_A\int\cos(\pi+\partial^2\varphi_f/\partial y^2)dy$$
$$\approx\frac{1}{2}K_A\int(\partial^2\varphi_f/\partial y^2)^2 dy, \tag{96}$$

where $K_A\approx\Lambda^2(\overline{\alpha}^2/2\overline{\beta})$. It is instructive to compare this expression with that derived by Moore (1989) for the effective Hamiltonian associated with a smoothly varying phase $\theta(x,y)$, namely, $H_{ph}=\frac{1}{2}c_{66}\int d^2r(\nabla^2\theta)^2$, where c_{66} is an isotropic shear modulus of the vortex lattice, which is given approximately by $\frac{1}{2}(\overline{\alpha}^2/\overline{\beta})$.

The two approaches do not agree completely, however, not only because of the one-dimensional nature of the present model, but also because of the significant difference in the "stiffness" parameters K_A and c_{66}, namely, $K_A/c_{66}\approx\Lambda^2\sim10^{-2}$. The reason for the disagreement can be understood within the present approach by considering shear motion along families of Bragg chains with Miller indices higher than those of the principal chains. For these families the values of a_x are relatively large, and the corresponding values of Λ^2 are not small compared to unity. In the limit of very large Miller indices, $a_x\to\infty$ and $\Lambda^2\to1$, so that the corresponding stiffness parameter approaches c_{66} and becomes independent of the chain orientation, as it is in Moore's theory.

Thus, in contrast to the isotropic shear model used by Moore (1989), the appearance of the small parameter Λ^2 in front of the leading phase-dependent terms of the free-energy functional appearing in Eq. (94) implies that shear motion along the two principal crystallographic axes costs a small fraction of the condensation energy, and so leads to significant distortions of the vortex lattice along these particular directions at temperatures very low relative to the mean-field T_c.

In this low-temperature regime, the calculation of the partition function $\mathcal{Z}$ can be simplified considerably since amplitude fluctuations can be neglected. The functional integrals in Eq. (86) over the order parameter $\{\psi,\psi^*\}$ should be replaced by integrals over the new variables $\{c_k,c_k^*\}\equiv\{|c_k|,\varphi_k\}$, so that after integration over angle variables the partition function can be written as

$$\mathcal{Z}=\mathcal{Z}_v^{\sqrt{N}}\propto\int_0^\infty\prod_k |c_k|d|c_k|e^{-\pi f_s}, \tag{97}$$

where

$$f_s=\sum_k\left\{-\overline{\alpha}|c_k|^2+\frac{\overline{\beta}}{2}(|c_k|^4+4\Lambda|c_k|^2|c_{k+1}|^2)\right.$$
$$\left.-\frac{1}{\pi}\ln I_0(2\overline{\beta}\Lambda^2\pi|c_{k-1}||c_{k+1}||c_k|^2)\right\} \tag{98}$$

and $I_n(x)$ is the modified Bessel function of order n. Neglecting amplitude fluctuations, the integrals in Eq. (97) can be performed by the stationary phase approximation. The solution is similar to the mean-field solution $|c_k|^2=\overline{\alpha}/\overline{\beta}\beta_{fl}$ with the generalized Abrikosov parameter

$$\overline{\beta}_{fl}\simeq 1+4\Lambda-4\Lambda^2\frac{I_1(\tau)}{I_0(\tau)}, \tag{99}$$

where $\tau=[4\Lambda^2/(1+4\Lambda)^2](\pi\overline{\alpha}^2/2\overline{\beta})\equiv T_{cm}/T$. The temperature $T_{cm}(a_x)$ determines a smooth crossover from the mean-field lattice state with $\gamma=\pi/2$, $\overline{\beta}_{fl}=\overline{\beta}_l\equiv 1+4\Lambda-4\Lambda^2$ to a new state corresponding to $\overline{\beta}_{fl}=\overline{\beta}_m\equiv 1+4\Lambda$, where the phase-dependent terms in the free energy are completely destroyed by fluctuations. Note that the energy difference between these states is of the order of the small parameter Λ^2.

In the zero-temperature limit, $T\ll T_{cm}(a_x)$, the parameter $\beta_{fl}\simeq(\sqrt{\pi}/a_x)(1+4\Lambda-4\Lambda^2)$ has a minimum at $a_x^2=2\pi/\sqrt{3}$, and at $a_{x'}^2=2\sqrt{3}\pi$ (see Fig. 18), depending on the choice of Bragg chain family (i.e., along the x or x' axis in Fig. 7). Both minima describe a triangular Abrikosov lattice with $\beta_{fl}=\beta_A\simeq 1.1596$. Both directions can be selected in three equivalent ways in the Abrikosov lattice.

The above-described equilibrium state at $T=0$ is stabilized by the competition between two types of interactions among parallel chains: the repulsive interaction between any two neighboring chains, which is linear in the coupling parameter Λ, and the attractive three-chain phase-dependent interaction (i.e., involving any three neighboring chains), which is quadratic in Λ [see Eq. (94)]. At finite low temperatures, i.e., when $T\sim T_{cm}$, shear fluctuations destroy the phase coherence among parallel Bragg chains, thus diminishing the small attractive interaction and increasing the total free energy. The relatively large repulsive interaction is affected only at higher temperatures.

The interchain coupling parameter Λ depends on the lattice parameter a_x through Eq. (92). Since $a_{x'}>a_x$ (Fig. 7), the chains along x' are closer to each other than those along x; consequently $\Lambda(a_{x'})>\Lambda(a_x)$. Thus, at low temperatures $T\lesssim T_{cm}(a_x)$, when the attractive three-chain interaction decreases with increasing temperature, the first state $(a_{x'})$ is more stable than the second one (a_x), since its corresponding free energy increases more slowly with increasing temperature (see Fig. 20).

At higher temperatures $T\gtrsim T_{cm}(a_{x'})$, the tendency is reversed and the free energy of the first state $(a_{x'})$ increases faster with increasing temperature than that of the second one (a_x). There is therefore an intersection point $T_{cm}(a_x)\lesssim T_m\lesssim T_{cm}(a_{x'})$ at which the free energies of these states are equal but the corresponding entropies are a little different.

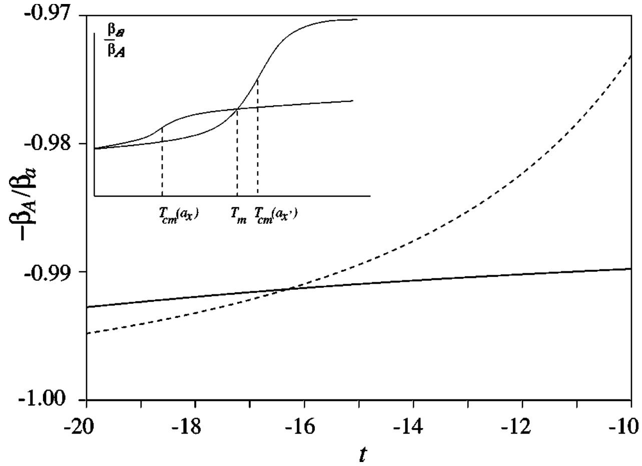


FIG. 20. Free energy of fluctuating Bragg chains (normalized by the corresponding mean-field value), $-\beta_A/\beta_{fl}$, as a function of the parameter t (see text): solid line, Bragg chains along the x direction; dashed line, Bragg chains along the x' direction. The intersection point determines the first-order phase transition. Inset: a schematic illustration of the solid-liquid crossovers in the two principal Bragg families.

Thus at $T=T_m$ there is a weak first-order transition characterized by a small jump of the lattice entropy. With the parameter $t\equiv-\alpha\sqrt{2\pi/\beta k_BT}$, defined after Kato and Nagaosa (1993), the position of the crossing point corresponds to $t=t_m\simeq-16$ and the jump in entropy $[S\equiv-T(\partial F/\partial T)]$ is $\Delta S\simeq 7.5\,10^{-3}F_{MF}/T$. The values of t_m and ΔS agree pretty well with the Monte Carlo simulations carried out by these authors (Kato and Nagaosa, 1993).

The physical nature of this transition can be illuminated by considering the shear modulus μ. The vanishing of the shear modulus in atomic crystals is usually regarded as a definition of the crystal melting point. In the present case μ can be calculated by evaluating the limit

$$\mu=\left(\frac{\partial^2F_{GL}}{\partial\eta^2}\right)_{\eta\to 0}$$

for the transformation $c_n'=e^{i\eta n^2}c_n$ (Sasik and Stroud, 1994). The shear modulus is proportional to the phase factor in the free energy, $\mu\propto\langle\cos\chi_n\rangle$. Normalizing by the mean-field value μ_{MF} with $\cos\chi_n=-1$, it is reduced to

$$\frac{\mu}{\mu_{MF}}=\frac{I_1(\tau)}{I_0(\tau)}. \tag{100}$$

The dependence of the shear modulus on the parameter t is plotted in Fig. 21. At the transition point $t=t_m$, the value of the parameter a_x corresponding to the minimum free energy changes abruptly and the shear modulus jumps from $\mu_1\approx 1$ to $\mu_2\approx 0.3$. It is remarkable that, in contrast to the very small entropy rise, the drop in the shear modulus at $t=t_m$ is very significant. However, its value on the "liquid" side of the transition point

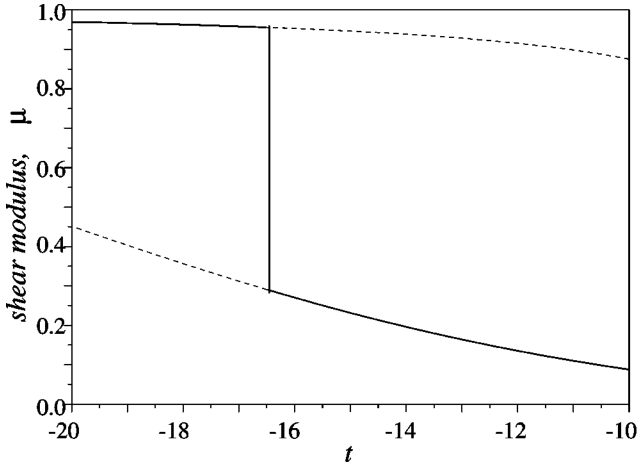


FIG. 21. Dependence of the normalized shear modulus μ on the parameter t (see text). The jump at $t=t_m$ reflects the melting transition.

μ_2 is not 0. The residual shear energy on the high-temperature side reflects an incomplete melting at $t=t_m$. The "liquid" state on this side of the transition point retains some degree of phase correlation between different chains, which continues to decrease gradually to zero with increasing temperature, reaching the complete liquid state only asymptotically. This behavior seems to be due to neglect of intrachain fluctuations in the sliding Bragg-chain model, which is invalid at high temperatures (see Sec. IV.A.2).

The nature of the "quasiliquid" states described above can be revealed by examining the mean values of the first and second moments of the angles χ_k,

$$\langle \chi_k^j \rangle = \frac{1}{\pi I_0(\tau)} \int_0^\pi d\chi_k \chi_k^j \exp(-\tau \cos\chi_k),$$

with $j=1,2$.

In the low-($\tau \gg 1$) and high-($\tau \ll 1$) temperature limits one finds

$$\langle \chi_k \rangle = \pi - \sqrt{\frac{2}{\pi\tau}}, \quad \langle \chi_k^2 \rangle = \pi^2 - \left(\frac{8\pi}{\tau}\right)^{1/2} + \frac{1}{\tau} \quad \text{for } \tau \gg 1, \tag{101}$$

$$\langle \chi_k \rangle = \pi/2 + 2\tau/\pi, \quad \langle \chi_k^2 \rangle = \pi^2/3 + 2\tau \quad \text{for } \tau \ll 1.$$

The square root of the relative variance $\sigma = \sqrt{\langle \chi_n^2 \rangle - \langle \chi_n \rangle^2} / \langle \chi_n \rangle$ is found to be $\sigma \simeq (\pi-2)/\pi\tau \ll 1$ for $\tau \gg 1$, while for $\tau \ll 1$, $\sigma \simeq 1/\sqrt{3}$, implying significant fluctuations in the high-temperature regime. However, since the mean values in Eq. (101) are the same for all chains (i.e., they are independent of the chain index k), on average the fluctuations do not destroy the periodic structure of the Bragg chains, but only change the point symmetry of the corresponding lattice by changing the effective geometrical parameter γ from the Abrikosov lattice value $\pi/2$ to $\langle \chi_k \rangle/2$. The first-order "melting" point at $t=t_m$ thus corresponds to a discontinuous (rotational) symmetry breaking in the lattice of average Bragg chains. One should be careful, however, not to push the picture emerging from the above analysis too far, since it relies on an approximation in which intrachain fluctuations are completely neglected. This approximation can be well justified near and below the melting point, but certainly not in the high-temperature region far above T_m. It is thus conceivable that intrachain fluctuations would completely break the periodic order along the chain axis at sufficiently high temperatures leading to an isotropic liquid phase at high T (see Sec. IV.A.2).

2. Bragg-chain pinning and phase coherence

An important aspect of vortex lattice melting concerns the spatial range over which superconducting phase coherence exists in the mixed state. A related phenomenon is the pinning of vortex lines to crystal defects. The fluctuating phases φ_k are not uniquely determined from the angles χ_k; they depend on the choice of boundary conditions for Eq. (95). Since the general solution of the homogeneous equation ($\chi_k=0$) is a linear function of n [Eq. (93)], two constants, a and b, are required to determine φ_k uniquely. A possible choice is to take $a=b=0$, which corresponds to the selection $\varphi_0=0$ and $\varphi_{-1}=\varphi_1$. The physical meaning of the first condition is that the chain, labeled $k=0$ (i.e., located vertically at $y=0$), is pinned to a fixed horizontal position (i.e., along the x axis). The second condition has a clear physical meaning in the long-wavelength limit, namely, that the horizontal displacement $u_x=(\partial\theta/\partial y)$ of the vortex lines vanishes at the pinning site $y=0$.

The solution of Eq. (95) that satisfies this particular pinning condition is (for $n>0$)

$$\varphi_{\pm n} = \sum_{l=1}^{n} (n-l)\chi_{\pm l} + \frac{n}{2}\chi_0. \tag{102}$$

This transformation enables one to calculate any correlation function of phase factors; in particular, the pair-correlation function

$$\langle e^{i(\varphi_{k'}-\varphi_k)} \rangle = \frac{\prod_\nu \int_0^\pi d\chi_\nu e^{-\tau\cos\chi_\nu} e^{i(\varphi_{k'}-\varphi_k)}}{\prod_\nu \int_0^\pi d\chi_\nu e^{-\tau\cos\chi_\nu}}$$

can be readily evaluated by using Eq. (102) (see Zhuravlev and Maniv, 1999). In the high-temperature limit $\tau \ll 1$ far above the melting point one finds

$$\langle e^{i(\varphi_{k'}-\varphi_k)} \rangle \propto \tau^{(1/2)\,|k^2-k'^2|} \to \delta_{k,k'}, \tag{103}$$

meaning no phase correlation at all. In the low-temperature limit $\tau \gg 1$ one finds

$$\langle e^{i(\varphi_{k'}-\varphi_k)} \rangle \approx \exp\left[i\langle \chi_k \rangle (k'^2-k^2) - \frac{\overline{k}}{2\tau}(\Delta k)^2 \right], \tag{104}$$

where $\Delta k = k'-k$ and $\overline{k} = k'/3 + 2k/3 - 1/2$. Equation (104) is identical to a second-order cumulant expansion with $\langle \varphi_k \rangle = \frac{1}{2}\langle \chi_k \rangle k^2 \approx (\frac{1}{2}\pi - \sqrt{1/2\pi\tau})k^2$.

This result shows that a genuine long-range phase correlation exists only at zero temperature; it also shows that a cluster of highly correlated chains can grow only around a pinned chain, since the phase fluctuations diverge with distance from the pinning chain [see Eq. (102)]. Note that the position of such a chain is arbitrary, since in the Ginzburg-Landau theory used there is no energy cost to pinning. In real samples the translational symmetry is broken by impurities, crystal defects, and the termination of the lattice at the sample surface, which can pin chains of orbital guiding centers to fixed positions. A single pinning center located near a given chain may pin the entire chain due to its rigidity. To maximize the pinning strength, however, additional pinning centers should be distributed uniaxially along the same chain, rather than randomly.

The range of translational order existing in the vortex state at finite temperature can be evaluated by considering the size dependence of the structure factor (Kato and Nagaosa, 1993),

$$S(\vec{G})=\frac{1}{N}\langle|I(\vec{G})|^2\rangle,$$

where

$$I(\vec{G})=\int d^2r|\psi(\vec{r})|^2e^{i(\vec{G}\cdot\vec{r})},$$

$b_y=\pi/a_x$, $b_x=\langle\chi_k\rangle a_x/2\pi$, ν and m are integers, and $\vec{G}$ is a reciprocal lattice vector of the Abrikosov lattice with $G_x=2\pi\nu/a_x$, $G_y=2\pi m/b_y-2\nu b_x$. At zero temperature the long-range order is manifested by the Bragg peaks with $S(\vec{G})\propto N$. At a finite temperature,

$$S(\vec{q})=\frac{\pi a_x^2}{2}e^{-q^2/4}\sum_{k,k',\nu}\delta_{q_x,(2\pi/a_x)\nu}\times C_4(k'+\nu,k,k',k+\nu)e^{-i(\pi/a_x)(k-k')q_y}, \tag{105}$$

where at $\tau\gg 1$

$$C_4(k_1,k_2,k_3,k_4)\equiv\langle e^{i(\varphi_f(k_1)+\varphi_f(k_2)-\varphi_f(k_3)-\varphi_f(k_4))}\rangle =\exp\left[-\frac{s^2(p-s/3+1/3)}{2\tau}+i(\langle\chi_k\rangle-\pi)\nu(k-k')\right],\quad \nu\geqslant 0, \tag{106}$$

with $s=\min(\nu,|k'-k|)$, $p=\max(\nu,|k'-k|)$, $k_1=k'+\nu$, $k_2=k$, $k_3=k'$, and $k_4=k+\nu$. Note that C_4 is the four-chain phase correlation function appearing in the quartic term of the Ginzburg-Landau free energy.

Now, since $\langle\chi_k\rangle\equiv\overline{\chi}$ is independent of k, the sum over k yields a factor $\sqrt{N}$ and Eq. (105) can be rewritten as

$$S(\vec{G})=\sqrt{N}\frac{\pi a_x^2}{2}e^{-G^2/4}\left\{\sum_{|l|\leqslant\nu}\exp\left[il(\nu\overline{\chi}-b_yG_y)-|l|^2\left(\nu-\frac{1}{3}|l|\right)\Big/2\tau\right]+\sum_{|l|>\nu}\exp\left[il(\nu\overline{\chi}-b_yG_y)-\nu^2\left(|l|-\frac{1}{3}\nu\right)\Big/2\tau\right]\right\}. \tag{107}$$

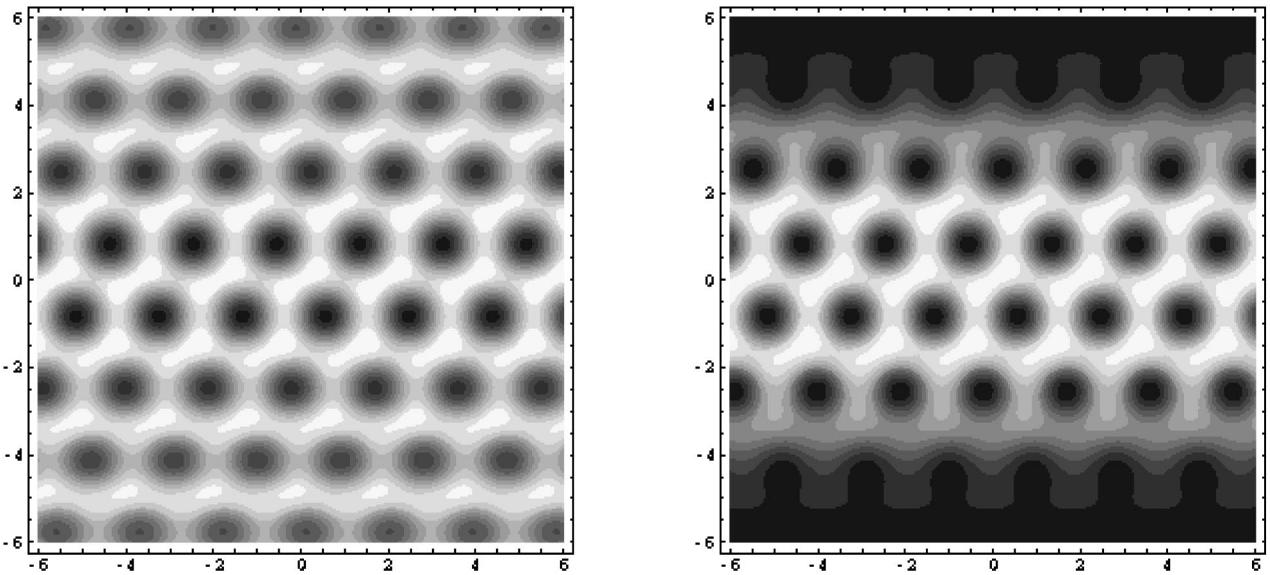
This expression reflects the extreme anisotropy characterizing the Bragg-chain model. Indeed, along the reciprocal-lattice axis $G_x=0$ [i.e., for $\nu=0$ in Eq. (107)], one finds perfect long-range order, since $S(G_x=0,G_y)\sim N$. For any $G_x\neq 0$, however, the corresponding Bragg peaks reflect the 1D long-range order within the real lattice chains, i.e., $S(G_x\neq 0,G_y)\sim N^{1/2}$. This power-law dependence on the macroscopic size of the system is due to the remarkable property that the average $\langle\chi_k\rangle\equiv\overline{\chi}$ is independent of the chain index k.

In an ideal triangular Abrikosov lattice there are three equivalent ways to select the principal axes. In the presence of pinning interactions due to random defects there are, however, many different orientations, determined by the local pinning interaction in the macroscopic sample, along which clusters of Bragg chains prefer to grow. Since the reciprocal-lattice points with $G_x=0$ depend on the specific choice of these axes, it is expected that by averaging over all three equivalent orientations, one will obtain anisotropic size dependence of the structure factor, satisfying $S\sim N^{\sigma}$ with $1/2<\sigma<1$. This result is similar to the quasi-long-range positional order predicted by the Kosterlitz-Thouless-Halperin-Nelson-Young theory of 2D melting (Kosterlitz and Thouless, 1973; Young, 1979; Halperin and Nelson, 1998), according to which $S\sim N^{\sigma}$ with $\sigma\lesssim 5/6$ (see also Hu and MacDonald, 1993; Kato and Nagaosa, 1993).

3. The picture of 2D vortex lattice melting

The simple analytical approach described above enables one to draw a clear picture of the vortex lattice melting process in a 2D superconductor at high magnetic fields. The skeleton of this picture consists of the principal crystallographic axes in a triangular Abrikosov lattice; owing to thermal fluctuations, families of almost rigid Bragg chains slide along these axes nearly freely at low temperatures (see Fig. 7). Similar motion along crystal axes with higher Miller indices costs significantly more energy and is therefore frozen at low temperatures.

Melting of the lattice occurs, essentially, when these fluctuations overcome the weak attractive phase-dependent interactions between chains. This interaction is not the same for the two principal axes; it is stronger for a closely packed Bragg family of chains. Thus the fluctuations at low temperature suppress the phase correlation within a loosely packed Bragg family more effectively than within a closely packed one, and so the stable thermodynamic equilibrium state at low temperatures (i.e., below T_m) corresponds to sliding chains in a



(a) (b)

FIG. 22. Contours of (a) $\langle|\Psi(\vec{r})|^2\rangle$ and (b) $|\langle\Psi(\vec{r})\rangle|^2$, calculated within the framework of the Bragg-chain model at $\tau\equiv T_m/T=2$ (i.e., below the melting point). Note the perfect periodic lattice created by the minima of these contours, which is, however, distorted with respect to the triangular Abrikosov lattice.

closely packed Bragg family. However, at higher temperatures (i.e., above T_m) when the phase correlation within this family is also effectively suppressed, the equilibrium state corresponds to sliding chains in a loosely packed Bragg family, since the dominant interactions between chains at these temperatures are repulsive (and are weaker for a loosely packed Bragg family).

The first-order transition at T_m is therefore a discontinuous transformation between two different configurations of Bragg chains. The low-temperature configuration is characterized by small fluctuations about the mean positions of chains that form an ideal periodic lattice very close to an Abrikosov triangular lattice (i.e., with angle $\Theta/2\approx 30°$ between the principal axes). In the high-temperature configuration the phase fluctuations are significantly larger than in the low-temperature one, whereas the mean positions of the Bragg chains still correspond to an exactly regular (but no longer rhombic) lattice, with a somewhat larger Θ, $\Theta\approx\arctan[\sqrt{3}/(1-\pi^{-3/2}\sqrt{1/2\tau})]$.

This picture is reminiscent of the Cooper-pair charge-density-wave state described by Sasik, Stroud, and Tesanovic (1995), which shows, at a given configuring of $\Psi(\vec{r})$, considerable positional disorder, while the average $\langle|\Psi(\vec{r})|^2\rangle$, taken over the full ensemble of such configurations, exhibits a perfectly periodic triangular modulation. It should be stressed, however, that in the Bragg-chain model the average superfluid density $\langle|\Psi(\vec{r})|^2\rangle$ is not a perfect periodic function of $\vec{r}$; only the positions of the local minima (or maxima) of this average density form a perfectly periodic lattice, as can be inferred from Eq. (104) (see Fig. 22). Note also that, in contrast to the average superfluid density, which exhibits—asymptotically far from the pinned chain $k=0$—periodic order perpendicular to the chain axis, the superconducting order parameter decays to zero over a distance of a few chains only [see the plot of $|\langle\Psi(\vec{r})\rangle|^2$ in Fig. 22]. This picture is similar to the 2D flux-line-lattice phase described by Kato and Nagaosa (1993), which has a strong positional correlation and nonvanishing shear modulus without long-range superconducting order.

The discontinuous nature of the melting transition is peculiar to the Abrikosov vortex lattice. In 2D solids melting (as described by the Kosterlitz-Thouless-Halperin-Nelson-Young theory) is associated with the thermally activated proliferation of dislocations, which reduce the elastic shear modulus to zero in a continuous fashion. In a 2D Abrikosov lattice the discontinuous transition is due to the localized (Gaussian) nature of the ground-state Landau orbitals, which considerably reduces shear stiffness along the principal Bragg families with respect to all the other families. It also allows us to distinguish clearly between the shear moduli of the two principal axes, as explained in Sec. IV.A.1.

Plainly, complete 1D periodic order within a single chain, which characterizes the Bragg-chain model, does not exist at arbitrarily high temperatures. It can be shown that a certain type of amplitude fluctuation, which can be described as a deviation of guiding-center projections q_k from their perfect periodic positions, becomes important at intermediate temperatures $T_m<T\ll T_c$. The energy cost of these fluctuations arises from the two-chain interaction term in the free energy [Eq. (91)], so that at $k_BT\gtrsim 10\Lambda(\pi\alpha^2/2\beta)\gg k_BT_m$ (Zhuravlev and Maniv, 2000) they destroy any remnant of positional order in the entire vortex system.

Finally the Bragg-chain model sheds new light on the mechanism of vortex line pinning: a correlated cluster of

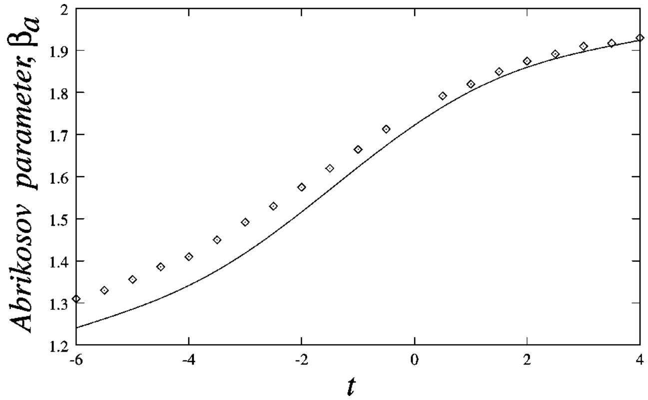


FIG. 23. The Abrikosov ratio $\beta_a(t)$: solid curve, calculated within the sliding Bragg-chain model; dotted curve, obtained by the Monte Carlo simulation of Kato and Nagaosa (1993).

chains nucleates only around a pinned chain. According to this model the pinning force of a whole chain can be strengthened dramatically by distributing pinning centers uniaxially along the chain. This feature suggests a very efficient way of generating pinning defects in quasi-2D superconductors. Such a pinning mechanism may be tested experimentally by producing columnar defects (Klein *et al.*, 1993) along the conducting planes in quasi-2D superconductors.

B. The damping of dHvA oscillations in the vortex liquid

The results presented in Sec. IV.A show very clearly that the effect of thermal fluctuations in the superconducting order parameter on the damping of dHvA oscillations around the mean field H_{c2} can be quite important, particularly in materials in which the electron dispersion near the Fermi surface is quasi-two-dimensional. In what follows the sliding Bragg-chain model, presented in Sec. IV.A.1, will be further developed to study this fluctuation effect.

1. The generalized Stephen approach

Our starting point here is the partition function, Eq. (28), for the Gorkov-Ginzburg-Landau superconducting free-energy functional F_G, given by Eq. (29). The fluctuating superconducting order parameter is represented by the sliding Bragg-chain model, Eqs. (87) and (89). In the vortex liquid state studied here this selection requires justification, since it was shown to be strictly valid only near or below the melting point where intrachain fluctuations are frozen. It turns out, however, that by allowing for amplitude fluctuations in the sliding Bragg-chain model, one preserves many important physical features of the full Gorkov-Ginzburg-Landau free-energy functional.

For example, the Abrikosov parameter $\beta_a(T)$, Eq. (88), calculated within the framework of this model around the mean-field transition temperature $T_{c2}(H)$ (see Fig. 23), is in good quantitative agreement with the results of the Borel-Padé analysis of the high-temperature large-order perturbation expansion (Hikami *et al.*, 1991), as well as with numerical simulations (Kato and Nagaosa, 1993). The small systematic deviations in the intermediate region may be attributed to intrachain fluctuations neglected in the model. This contribution is small (in comparison with the strong amplitude fluctuations) in the high-temperature region around $T_{c2}(H)$, as well as in the low-temperature region near the melting point, where sliding Bragg chains dominate the superconducting fluctuations.

Thus, writing the Gorkov-Ginzburg-Landau free-energy functional [see Eq. (29)] in the Bragg-chain representation, its most general (nonlocal) form is

$$\frac{F_G}{k_B T\sqrt{N}} = -\overline{\alpha}\sum_k |c_k|^2 + \frac{1}{2}\sum_{k,s,p} \mathcal{B}(k,s,p)|c_k||c_{k+s+p}||c_{k+s}||c_{k+p}| \times e^{i(\varphi_{k+s}+\varphi_{k+p}-\varphi_k-\varphi_{k+s+p})}. \tag{108}$$

Note that in the local (Ginzburg-Landau) approximation $\mathcal{B}(k,s,p)\rightarrow\mathcal{B}_{loc}(s,p)\propto\Lambda^{s^2+p^2}$ [compare to Eq. (91)]. Considering the high-magnetic-field region well above the mean field $H_{c2}(T)$ ($\alpha<0$), where the quartic terms are much smaller than the quadratic ones, the integral over phase fluctuations φ_k in the partition function

$$\mathcal{Z}=\prod_k \int |c_k|d|c_k| \int d\varphi_k e^{-F_G/k_BT}$$

can be approximated by the first cumulant, so that

$$-\frac{\ln \mathcal{Z}}{\sqrt{N}} \approx |\overline{\alpha}|\sum_k \langle |c_k|^2\rangle + \frac{1}{2}\sum_{k,p} \mathcal{B}(k,0,p)\langle |c_k|^2\rangle\langle |c_{k+p}|^2\rangle + \frac{1}{2}\sum_{k,s} \mathcal{B}(k,s,0)\langle |c_k|^2\rangle\langle |c_{k+s}|^2\rangle. \tag{109}$$

The factorized form of the quartic term in this expression is equivalent to the form obtained in Sec. III.A [Eq. (67)] in the random vortex lattice model.

The pair-correlation function is not identical to the Gaussian of Eq. (66) obtained in the random vortex lattice model. It is Gaussian in all directions except for the direction of the chains, along which it is a periodic function,

$$\langle \Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\rangle = \langle |\Delta|^2\rangle \exp\left[-\frac{1}{2}(\vec{r}_1-\vec{r}_2)^2\right] \times \exp[i(x_1+x_2)(y_1-y_2)] \times \sum_n \exp\left[-\frac{1}{2}\left(\frac{n\pi}{a_x}-\varsigma\right)^2\right], \tag{110}$$

where $\varsigma=(y_1+y_2)-i(x_1-x_2)$, and $\langle |\Delta|^2\rangle$ is the spatial average of the mean-square fluctuation of the superconducting order parameter.

In the common experimental situation, when the effective smearing parameter is not small (see Sec. II.C.4), the problem is similar to the limiting case discussed in

Sec. III.A in the context of the random vortex lattice model. A detailed calculation shows that the narrow channel of long-range (periodic) pair correlation has little influence on the self-energy, which can be approximated by an expression similar to the Maki-Stephen formula, Eq. (73), with Δ_0^2 replaced by $\langle|\Delta|^2\rangle$. It should be noted that with intrachain fluctuations taken into account, the similarity to the random vortex lattice model will be even stronger.

2. The dHvA amplitude near the vortex liquid freezing point

Near the vortex liquid freezing point, where the shear modulus rises abruptly to nearly its mean-field value, the integral over phase fluctuations, which lead to the factorized form, Eq. (67), in the high-field region, is not trivial as in the latter limit, since the quartic terms are not small. On the crystal side of the melting point, however, the four-chain correlation function has the limiting ($T_m \gg T$) crystal form (where $p \geq s \geq s0$)

$$\langle e^{i(\varphi_{k+s}+\varphi_{k+p}-\varphi_k-\varphi_{k+s+p})}\rangle \to e^{i\pi ps}\exp\left\{-\frac{s}{2\tau}\left[ps-\frac{1}{3}(s^2-1)\right]\right\} \to e^{i\pi ps} \qquad (111)$$

expressing the strong phase correlation in the vortex lattice state.

On the liquid side just above the melting point, calculation of the damping of the dHvA amplitude is a subtle problem. In particular, as indicated at the end of Sec. III.B, the oscillatory part of the free-energy functional, Eq. (108), is strongly affected by phase fluctuations. However, since at $T<T_{c2}(H)$ the leading non-oscillatory (zero harmonic) part of the free energy is not influenced significantly by fluctuations, one may use the local Ginzburg-Landau-like form of the free energy in the Boltzmann weighting factor to calculate the phase-correlation function. Under these circumstances [i.e., in the limit $T_m \ll T < T_{c2}(H)$] it is easy to show that

$$\langle e^{i(\varphi_{k+s}+\varphi_{k+p}-\varphi_k-\varphi_{k+s+p})}\rangle \to \left(\frac{T_m}{T}\right)^{sp} \to (\delta_{s,0}+\delta_{p,0}), \qquad (112)$$

which is equivalent to the factorized form of the quartic term of Eq. (109), found well above H_{c2}. Similarly the pair propagator is given by Eq. (110), and the self-energy by the corresponding Maki-Stephen-like expression with

$$\langle|\Delta|^2\rangle = k_B T \frac{\partial \ln \mathcal{Z}}{\partial \alpha}. \qquad (113)$$

The amplitude of a fundamental dHvA oscillation in the liquid state well above the melting point can therefore be approximated by the Maki-Stephen-like formula

$$M^{[1]} \approx M_N^{[1]} \exp\left(-\frac{\pi^{3/2}\langle|\tilde{\Delta}|^2\rangle}{n_F^{1/2}}\right).$$

A simple analytic expression for the damping parameter can be derived in this region by integrating over amplitude fluctuations (after neglecting a small linear correction in Λ),

$$\langle|\Delta|^2\rangle \simeq \frac{\alpha^{[0]}}{\beta^{[0]}}\left[1+\frac{\exp(-x^2)}{2x\int_{-\infty}^{x}\exp(-y^2)dy}\right], \qquad (114)$$

where $\alpha^{[0]}=(1/2\hbar\omega_c)\ln\sqrt{H_{c2}/H}$, $\beta^{[0]}=1.38/n_F(\hbar\omega_c)^3$, and $x=\alpha^{[0]}/\sqrt{2\beta^{[0]}k_B T}$.

This expression has no singularity at the mean-field transition, where both $\alpha^{[0]}$ and x vanish, and smoothly interpolates between the high-field ($x\to-\infty$) value $k_B T/|\alpha^{[0]}|$ and the mean-field value $\alpha^{[0]}/\beta^{[0]}$ in the case of low field ($x\to\infty$).

It should be stressed here that the limiting expression (112) of the four-chain correlation function is a generalized form of the condition for the vanishing of the shear modulus μ in the liquid state, whereas the limiting expression (111) is a generalized form of the Bragg-chain crystallization. The apparent correlation between the vanishing shear modulus of the Abrikosov lattice and the decoupling of the average quartic term into a product of two pair-correlation functions, which reflects the onset of incoherent damping of the dHvA oscillations, may indicate that the transition from incoherent vortex liquid damping to coherent Abrikosov lattice damping is a sharp transition.

V. DISCUSSION AND COMPARISON WITH EXPERIMENTAL DATA

The theory reviewed in this article provides a reasonable framework for a qualitative description of all important aspects of the dHvA effect in the mixed states of type-II superconductors. It is still insufficiently developed, however, to provide a complete quantitative account of the available experimental data. Furthermore, some fundamental questions concerning the pure limit of a superconductor under high magnetic fields and ultralow temperatures still remain unanswered.

A key conclusion reached here concerns the connections between all the different mean-field approaches to the problem: they are found to be consistent in the region just below the upper critical field H_{c2}, splitting markedly, however, in the lower-magnetic-field region well below H_{c2}.

In this discussion an attempt will be made to find a common framework for all the different approaches in which one could draw a consistent conclusion regarding the behavior of the dHvA oscillations, both away from H_{c2} and in its immediate vicinity. The available experimental data will be systematically examined with respect to this scheme.

In all dHvA experiments performed so far in the mixed superconducting states, the fundamental oscillatory component was typically found to be much larger than the higher harmonics. Consequently, in discussing the existing experiments, the limit of a relatively large smearing effect ($2\pi\Gamma_{ef}\gtrsim\hbar\omega_c$; see Secs. II.C.4, II.C.5, and III.A) applies.

The current experimental situation could change, however, in the future, enabling dHvA measurements on very pure single crystals at ultralow temperatures, when the exotic features discussed in Sec. II.C.2 could be observed. At present the theoretical understanding of such superconducting states is quite limited. Two fundamental difficulties are encountered in attempting to construct such a theory.

(1) The infinite degeneracy of the bare (normal-state) Landau levels, which leads to the breakdown of Gorkov's perturbation expansion in the superconducting order parameter Δ and to the emergence of nonanalytic dependence on Δ. At present it is not at all clear whether a standard nonperturbative approach, like the Borel summation method employed by Maniv *et al.* (1998), is sufficient for this problem; the similarity to the situation encountered in the quantum Hall effect may indicate that a completely new nonperturbative approach is needed.

(2) The lack of particle-hole symmetry in the Landau-level basis sets, on the one hand, and the presence of this symmetry in the quasiparticle spectrum, on the other hand, which turns the crossing of the Fermi energy by the quasiparticle "Landau bands" into an extremely complex process. In interpreting current experimental data, however, one finds that the significant smearing of the Landau-level spectrum by various extrinsic factors dramatically simplifies the picture of this crossing process and allows the use of an expansion in the superconducting order parameter for the oscillatory part of the thermodynamic potential.

Under these circumstances two important parameters can be identified:

$$x_1=\pi^{3/2}\frac{\tilde{\Delta}_0^2}{n_F^{1/2}},\quad x_2=\sqrt{2}\beta_A\frac{\tilde{\Delta}_0^2}{n_F}. \tag{115}$$

For an ideal Abrikosov lattice the reduction factor of the fundamental dHvA oscillation in the vortex state can be written as

$$R_s^{[1]}=1-x_1\Theta(x_2), \tag{116}$$

where $\Theta(x_2)$ has an analytic expansion around $x_2=0$, $\Theta(x_2)=1-x_2+\cdots$ [see Eq. (55)]. In the semiclassical limit $n_F^{1/2}\gg1$ the superconducting-induced suppression of oscillations in the region

$$1\gtrsim x_1\sim n_F^{1/2}x_2\gg x_2 \tag{117}$$

is dominated by the factor x_1 in Eq. (116), which increases linearly with $\tilde{\Delta}_0^2$. This behavior reflects the opposing effect of the paramagnetic vortex currents with respect to the cyclotron diamagnetic currents, as discussed in Sec. I.B.3. In this range of parameters the quasiparticle density of states consists of well-separated Landau bands, as shown in Fig. 10. The Landau-level broadening due to coherent magnetic breakdown, discussed in Secs. I.B.3 and II.B, is controlled by the asymptotically small parameter x_2 through the function $\Theta(x_2)$, and so does not play an important role in this region (as long as $x_2\ll1$).

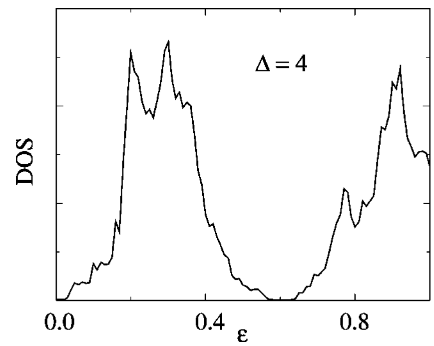


FIG. 24. The quasiparticle density of states calculated numerically by Tesanovic and Sacramento, as in Fig. 9, but for $\tilde{\Delta}_0=4$. Note the shift of the mean positions of the bands relative to the normal-electron Landau levels, which indicates the opening of the superconducting energy gap.

In their extensive numerical study of magnetic oscillations in the mixed state, Norman, MacDonald, and Akera (1995) have shown that the kinetic energy and the pairing contributions to the magnetization oscillations oppose each other. They added that this should be expected, since the superconducting state is always formed at a cost in kinetic energy in order to take advantage of the attractive electron-electron interactions. Since the orbital kinetic energy of the quasiparticles corresponds to cyclotron motion, which is diamagnetic in nature, the pairing contribution in their calculation is evidently of paramagnetic nature, in agreement with the picture described above.

The breakdown of the simple Landau band picture occurs when $\tilde{\Delta}_0$ becomes sufficiently large that the quasiparticle band edges start to overlap and their mean positions significantly shift relative to the normal-electron Landau levels, as shown in Fig. 24. This situation corresponds to strong coherent magnetic breakdown, as well as to the opening of a significant superconducting energy gap in the quasiparticle spectrum. It is conceivable that these phenomena will be reflected in Eq. (116) by the breakdown of the Taylor expansion of $\Theta(x_2)$. Since the form of the full expansion is not known, one may estimate the breakdown value of $\tilde{\Delta}_0^2$ from the criterion $x_2\sim1$, which leads to

$$\tilde{\Delta}_{0,Br}^2\sim0.61n_F. \tag{118}$$

It is interesting to compare this criterion with the available numerical calculations of the quasiparticle spectrum. Norman and MacDonald (1996) examined the mean eigenvalue of the lowest quasiparticle band in their numerical solution of the Bogoliubov–de Gennes equations for a 2D superconductor, as a function of $\tilde{\Delta}_0$ for three fixed values of the filling factor n_F (see Fig. 25), and determined the point at which the corresponding three curves crossed. They interpreted this point as a crossover from a well-separated quasiparticle Landau

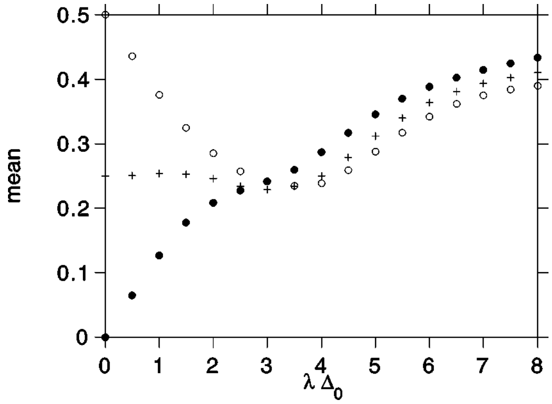


FIG. 25. Mean eigenvalue (in units of $\hbar\omega_c$) of the lowest quasiparticle band vs $\tilde{\Delta}_0$, calculated by Norman and MacDonald (1996) at three values of the filling factor: ●, $n_F=20$; +, $n_F=20.25$; ○, $n_F=20.5$. Note the crossing at about $\tilde{\Delta}_0=3$.

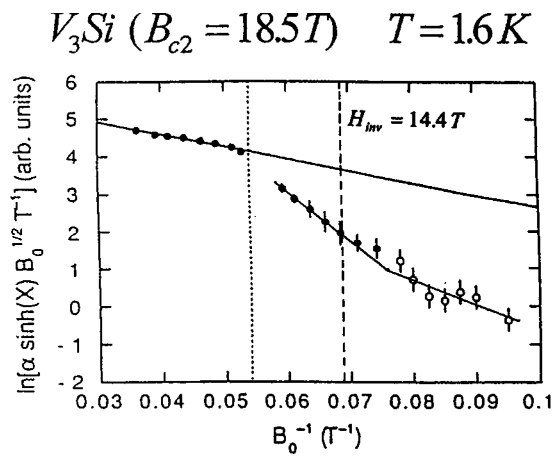


FIG. 26. Dingle plot reproduced from Corcoran, Harrison, *et al.* (1994) for dHvA oscillations in the vortex state of V_3Si. The solid straight lines, drawn as close as possible to the data points, show clearly two different characteristic slopes in the vortex state. The value of H_{inv} indicated in the figure was determined from the value of $\Delta_0(0)$ (see text) yielding the best fit to the data points just below H_{c2}.

band structure, as shown, for example, in Fig. 10, into a region where the emergence of well-separated vortex cores leads to a strong mixture of Landau levels.

Using their numerical value of n_F in Eq. (118), it is found that $\tilde{\Delta}_{0,Br}\approx 3.5$, which should be compared with the crossing point shown in Fig. 25 (i.e., $\tilde{\Delta}_{0,cross}\approx 3$). The agreement is reasonably good, given the relatively small values of n_F (~ 20) used in the numerical calculations and the large values assumed in the analytical expression, Eq. (118).

It should be noted that the quasiparticle density of states shown in Fig. 10 consists of well-separated Landau bands in the parameter region far away from the crossover region. For this calculation, though, $\tilde{\Delta}_0=1$ (which corresponds to $x_1\approx 0.9$), and this is very close to the point at which, according to Eq. (116) the superconducting contribution to the oscillatory free energy becomes comparable to the normal-electron contribution. Naively, one could think that at $x_1\approx 1$ the superconducting contribution to the free energy could not be considered as a small perturbation. The general structure of Eq. (116) in the semiclassical limit indicates, however, that near this point Gorkov's expansion is well within its range of validity.

In real dHvA experiment various inhomogeneities, such as dislocations, vacancies, and grain boundaries, lead to irregular flux-line pinning and consequently to the destruction of the translational long-range order of the Abrikosov lattice (Blatter *et al.*, 1994). This pinning effect is usually reflected in the sample magnetization as a hysteresis loop at magnetic fields near H_{c2} (Janssen *et al.*, 1998) and often also in a large peak in the nonoscillatory component of the magnetization just below H_{c2} (the peak effect; Terashima *et al.*, 1997).

Furthermore, thermal fluctuations in the superconducting order parameter can significantly influence the transition to the superconducting state in the high-field region of the superconducting-normal phase boundary due to an effective dimensionality reduction by the Landau quantization of the energy of Cooper pairs (Tesanovic and Andreev, 1994).

Thus it seems that in the high magnetic-field region near H_{c2}, where dHvA oscillations are observable, the vortex state is usually disordered due to both static (flux pinning) and dynamic effects (thermal and/or quantum fluctuations). Consequently, one usually expects to find in this region the behavior predicted by the Maki-Stephen random vortex lattice model, which is controlled by the single parameter x_1 [Eq. (115)]. The corresponding effective order parameter $\tilde{\Delta}_0$ is not simply related, however, to the amplitude of the Abrikosov solution, nor is it directly connected to the superconducting energy gap, since it is determined by some average of $|\Delta(\vec{r})|^2$ over all possible vortex lattice configurations. This may explain the general trend found in the interpretation of experimental dHvA data (Haworth *et al.*, 1998; Janssen *et al.*, 1998) when the Maki-Stephen simple exponential behavior fits very well the data near H_{c2}, but requires values of $\tilde{\Delta}_0$ that usually do not correlate well with the superconducting energy gap.

The reader should note that strong damping of dHvA oscillations can take place in the close vicinity of H_{c2} because of a completely different mechanism associated with the redistribution of flux lines. This happens in measurements using the field modulation technique, when the nonequilibrium currents induced by the modulation field lead to nonuniform flux distribution inside the superconductor over a macroscopic length scale (which is typically much larger than the Larmor radius r_F) and to a strong phase smearing of dHvA oscillations (Terashima *et al.*, 1997).

The situation changes significantly as the magnetic field is reduced further below H_{c2}. Here both flux-line pinning and thermal fluctuations are suppressed. The regular Abrikosov lattice is recovered, so that according to the general argument presented in Sec. II, the established phase coherence in the vortex lattice should lead

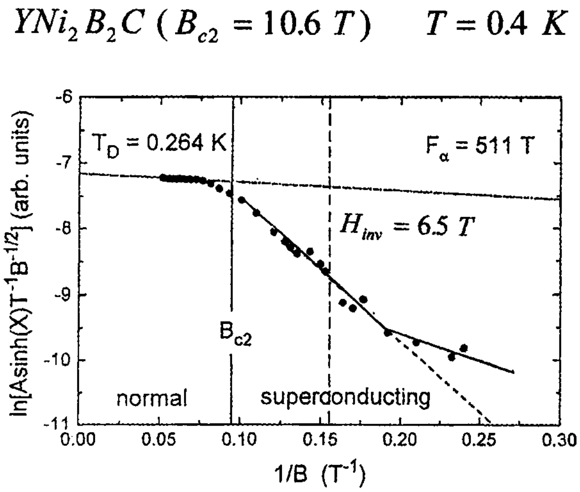


FIG. 27. The same as Fig. 26 for a dHvA measurement in the vortex state of YNi_2B_2C carried out by Goll *et al.* (1996).

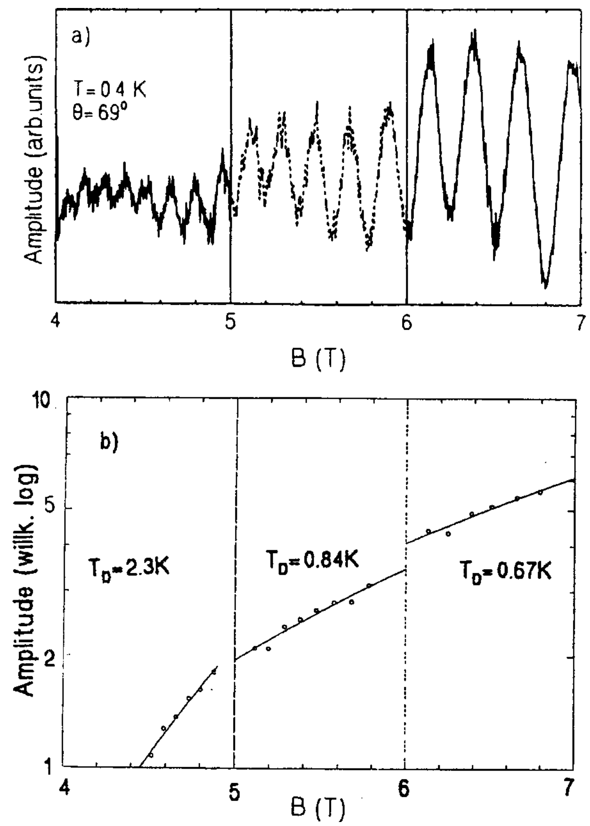


FIG. 28. Influence of temporary vortex line configuration on damping: (a) dHvA oscillations in the mixed state of 2H-$NbSe_2$ measured by Steep *et al.* (1995) for different field sweeps. From left to right: sweep from 3.9 to 5.1 T, stop, sweep back to 4.9 T, sweep from 4.9 to 6.1 T, stop, sweep back to 5.9 T, sweep from 5.9 to 7.1 T, stop of sweep. (b) Dingle plot for the data presented in (a) (Steep *et al.*, 1995, private communication). Note the jump at 6 T and the significant change of slope at 5 T.

to much smaller Landau-level broadening by the pair potential and to weaker damping of the dHvA oscillations.

This consideration seems to explain a rather systematic phenomenon observed in dHvA measurements on several type-II superconductors. Figures 26 and 27 clearly illustrate this phenomenon. The Dingle plots exhibit an initial slope just below H_{c2} that agrees very well with the Maki-Stephen exponential damping factor, Eq. (74), provided the field dependence of $\hbar\omega_c\tilde{\Delta}_0=\Delta_0(H)$ is assumed to be the same as that of the mean-field self-consistent order parameter near H_{c2}, $\Delta_0(H)=\Delta_0(0)\sqrt{1-H/H_{c2}}$ [see Eq. (56)]; $\Delta_0(0)$ is, however, an adjustable parameter. At some field below H_{c2} the straight lines with initially large (negative) slopes break into new lines with significantly smaller slopes.

It should be stressed here that the good agreement of Eq. (74) with the initial straight lines occurs well beyond the region where the linear approximation of this exponential holds [in which it cannot be distinguished from the ideal Abrikosov lattice result, Eq. (116)]. This can be seen in both figures with the help of the location of an auxiliary field H_{inv}, which is defined at the point where R_s in Eq. (116) vanishes (i.e., where $x_1\approx 1$). Since at this field x_2 is still much smaller than one, the Dingle plots would have shown a downward logarithmic divergence at H_{inv} if the ideal behavior, Eq. (116), were valid there. In contrast, it can be seen that for both materials (and for other ones as well; see, for example, Maniv *et al.*, 1998) the Dingle plot is approximated very well by a linear curve all the way down to H_{inv} and slightly below where the data sharply turn upward.

Thus, according to the above interpretation, the smaller slope of the Dingle plots in the low-field region reflects the establishment of long-range translational order in the vortex state, i.e., over a length scale much larger than the magnetic length. The quantitative analysis of the data in this region is, however, complicated by the need to take into account the presence of incomplete (correlated) disorder of an unknown nature in the vortex lattice, as well as by the small signal-to-noise ratio characterizing the experimental data in this field range.

Direct experimental evidence for the important role played by the pinning of vortices in the damping of dHvA signals in superconducting mixed states was provided by Steep *et al.* (1995), who studied the damping of dHvA oscillations along different magnetic-field sweeps in the hysteretic region below H_{c2} of 2H-$NbSe_2$. Discontinuous jumps in the dHvA amplitude, as well as significant changes in the slopes of the Dingle plots, were observed at points where the sweep directions were changed, showing clearly the influence of the temporary vortex line configurations on the damping (see Fig. 28). It is, of course, not clear from this work on what length scale the vortex distribution influencing the dHvA amplitude was most effective.

As mentioned above, the effect of thermal fluctuations in the superconducting order parameter on the damping of dHvA oscillations can be quite important, particularly in materials where the electron dispersion near the Fermi surface is very anisotropic (i.e., quasi-two- or one dimensional). In what follows, we discuss this effect in some detail since, unlike the effect of vortex line pinning, it can be treated readily within a self-consistent approach, which in some simple limiting cases can provide a scheme for a quantitative analysis of the experiment.

Before turning to such a detailed discussion, however, it will be instructive to further examine the available experimental data. Some of the materials studied in these experiments, i.e., V_3Si (see Fig. 26) and Nb_3Sn, which was investigated by Harrison *et al.* (1994), belong to a family of intermetallic compounds possessing the $A15$ (or β-tungsten) crystal structure, known for their relatively high superconducting transition temperatures as well as other anomalous electronic and elastic properties (Weger and Goldberg, 1973). Of special interest here is the linear-chain structure of the β-W lattice, which was proposed as a possible origin of these peculiarities (Weger, 1964). The presence of large thermal fluctuations in the superconducting order parameter, associated with this quasi-1-D electronic structure, was proposed (Maniv, 1978) as the origin of the large enhancement of the nuclear spin-lattice relaxation rate, $1/T_1$, observed in Nb_3Al and Nb_3Sn (Ehrenfreund *et al.*, 1971; Fradin and Cinader, 1977) at temperatures just above $T_c(H=0)$.

It is therefore reasonable to expect that under the high-magnetic-field conditions of the dHvA experiment, superconducting fluctuation effects in these materials play a significant role. Unfortunately, the data taken from both Nb_3Sn and V_3Si in the close vicinity of H_{c2} are strongly disturbed by the large peak effect. In contrast, the borocarbide data shown in Fig. 27 do not suffer from this disturbance, and so the close vicinity of H_{c2} can be carefully examined. The significant rounding of the Dingle plot seen in this region is clear evidence for the effect of superconducting fluctuations. It is thus very interesting to note here that recent studies of YNi_2B_2C single crystals by means of electrical transport (Mun *et al.*, 1996) and nuclear magnetic resonance (Lee *et al.*, 2000) measurements indicate the existence of large thermal fluctuations of vortices in this material, which could lead to a vortex glass transition at high fields (Mun *et al.*, 1996) and to a vortex lattice melting transition at lower fields.

As discussed in detail in Sec. IV, thermal fluctuations in the order parameter of 2D superconductors smear the sharp mean-field second-order phase transition into a broad crossover around the mean field H_{c2}, on which a very weak first-order transition (i.e., a small discontinuous entropy jump) is superimposed well below H_{c2}. The small entropy change is associated, however, with a large drop in the vortex lattice shear modulus (see Fig. 21), indicating the melting of the vortex lattice. Since the damping of dHvA oscillations is sensitive to the local phase of the superconducting order parameter (Sec. II.C.3), the phase fluctuations that drive this melting transition are expected to strongly influence the damping.

Experimentally suitable materials for studying this fluctuation effect can be found among the organic charge-transfer salts $(\text{BEDT-TTF})_2X$ (Saito and Kagoshima, 1990), which are highly anisotropic compounds with nearly two-dimensional electronic structure. Some of these compounds, such as $X=\text{Cu(NCS)}_2$, are extreme type-II superconductors with very small in-plane coherence length. Consequently, the Gorkov critical region is relatively large and one expects drastic deviations from the predictions of mean-field theory for these materials due to strong thermal fluctuations in the superconducting order parameter (Friemel *et al.*, 1996).

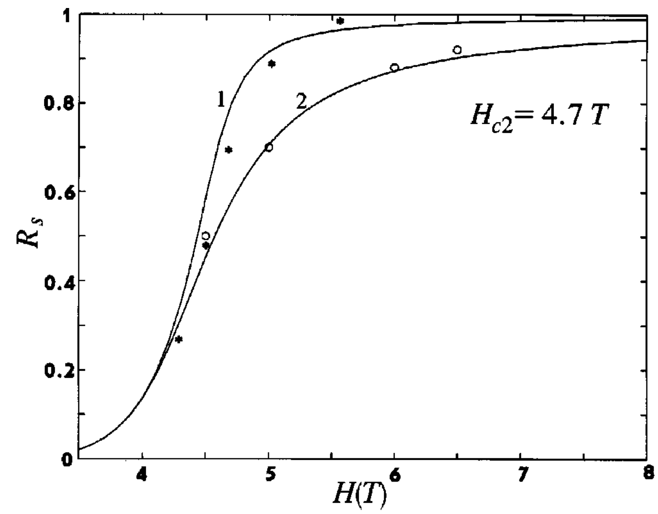


FIG. 29. The magnetization oscillations damping factor R_s as a function of magnetic field H in the quasi-2D superconductor κ-$(ET)_2\text{Cu(SCN)}_2$. Solid lines: theoretical curves for $T=20$ mK, $F=Hn_F=690$ T (1), and $T=120$ mK, $F=600$ T (2); *, ○, corresponding experimental data reproduced from van der Wel *et al.* (1995) and from Sasaki *et al.* (1998), respectively. The other parameters used in the calculations are $T_c(H=0)=10.4$ (with the gap parameter calculated using the BCS weak-coupling formula) and $m^*=3.5m_e$, with m_e denoting the free-electron mass.

Sasaki *et al.* (1998) recently observed dHvA oscillations in the vortex liquid state of this material. Earlier measurements on the same material were reported by van der Wel *et al.* (1995). A comparison of the theory presented in Sec. IV.B with both sets of data is shown in Fig. 29. Assuming that the nucleation of superconducting droplets is associated mainly with the closed Fermi-surface sheet (Wosnitza, 1993, 1996), and using the well-documented normal and superconducting parameters, a remarkably good quantitative agreement with the experimental data is achieved with the mean field H_{c2} as the only adjustable parameter. It is found that a single value of about 4.7 T fits well the two different sets of data, which were taken at quite different temperatures. It should be stressed that in the fluctuation theory the magnetic oscillations are smoothly damped well above H_{c2} and disappear below this value in remarkable agreement with the experiment. This is a sharp contrast with the results of the mean-field Maki-Stephen/Wasserman-Springford theory, where the additional damping begins abruptly at H_{c2} with a significantly stronger damping rate than what is observed experimentally. The theoretically expected sharp change of damping rate at the freezing point, discussed in Sec. IV.B.2, will be difficult to observe in the quasi-2D organic compounds investigated due to the very small value of the melting temperature T_m and due to the strength of the 2D fluctuations. In 3D superconductors, where the role of fluctuations is less important and the melting transition is shifted significantly closer to H_{c2}, one could expect such a transition to be observable. The borocarbide data shown in Fig. 27 could indicate such a freezing transition, as was mentioned earlier.

One should note that the quasi-2D organic superconductors $(\text{BEDT-TTF})_2X$ have been treated here as conventional superconductors. Numerous experiments have indicated, however, that the pairing mechanism in these materials could be unconventional (see, for example, Wosnitza, 1999 or Behnia *et al.*, 1999). The reader may therefore ask what happens to such an anisotropic superconductor when a strong magnetic field is applied perpendicular to its easy conduction planes. Generally speaking, the nature of the low-lying quasiparticles in a superconductor with anisotropic gap, or in a "d-wave" superconductor, at nonzero magnetic fields, is quite different from that found in a conventional superconductor. In this last, nodes in the superconducting gap arise only as a result of center-of-mass motion of pairs in strong magnetic fields near H_{c2}, whereas in d-wave superconductors there are nodes in the gap as a function of the relative momentum of the pairs along symmetry lines, or at some symmetry points of the Fermi surface even at zero magnetic field. If the extremal electron orbit coincides with a zero-gap line, one may expect an enhanced dHvA signal for the corresponding magnetic-field direction (Miyake, 1993). In a d-wave superconductor, the vanishing of the gap at symmetry points is known to yield a finite density of states at the chemical potential (Volovik, 1993). Gorkov and Schrieffer (1998a) have shown that for a $d_{x^2-y^2}$ superconductor at intermediate magnetic fields $H_{c1} \ll B \ll H_{c2}$ the quasiparticle exhibits a Dirac-like energy spectrum $E_n = \pm \hbar \omega_H \sqrt{n}$, $n=0,1,\ldots$ where $\omega_H = 2\sqrt{\omega_c \Delta_0/\hbar}$. The nature of this spectrum is quite different from the equispaced Landau levels characterizing the conventional case. These pioneering studies have recently stimulated intense theoretical activity. Mel'nikov (1999) showed that individual levels are strongly mixed by the effect of spatially varying supercurrent in the vortex lattice, while Franz and Tesanovic (2000) introduced a singular gauge transformation to study the corresponding quasiparticle band structure. Their calculations exhibited strongly dispersed bands rather than the discrete levels discussed above. This conclusion was basically confirmed by the very recent study of Marinelli, Halperin, and Simon (2000). The strong broadening of these "Dirac levels" in the vortex lattice of a d-wave superconductor implies that despite the gapless nature of its quasiparticle spectrum, quantum magnetic oscillations could not be observable deep in the vortex state of such a superconductor.

Finally it should be stressed that in some dHvA experiments on conventional superconductors (see, for example, Terashima *et al.*, 1997) the observed oscillations persist at magnetic fields B surprisingly smaller than H_{c2}, a result that seems difficult to reconcile with the Fermi-surface crossing mechanism discussed in this article. Recently, Gorkov and Schrieffer (1998b) and Gorkov (1998) proposed a new mechanism for quantum magnetic oscillations in isotropic superconductors, which enhances oscillations at $H_{c1} \ll B \ll H_{c2}$ where the superconducting energy gap is almost fully restored. They demonstrated that oscillations in this state are caused by a level crossing at an energy threshold separating "localized" and "extended" states. This energy boundary is introduced to the "gapped" quasiparticle spectrum $\varepsilon(p) = \sqrt{\xi_p^2 + \Delta^2}$ by the Doppler shift $\vec{p} \cdot \vec{v}_s(\vec{r})$, where $\vec{v}_s(\vec{r})$ is the superfluid velocity distribution. The smaller Dingle temperature arising from this threshold mechanism may account for the dHvA oscillations observed deep in the mixed state.

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