Granular matter: a tentative view

P. G. de Gennes

Collège de France, Physique de la Matière Condensée, 75231 Paris Cedex 05, France

Granular matter describes large collections of small grains, under conditions in which the Brownian motion of the grains is negligible (sizes $d>1$ micrometer). The grains can exhibit solidlike behavior and fluidlike behavior, but the description of these states is still controversial. The present discussion is restricted to static problems, for which the main approach is to describe properly the initial state of each volume element, when it was deposited from a fluid flow. [S0034-6861(99)02202-3]

CONTENTS

I. POWDERS IN NATURE AND IN INDUSTRY

Granular matter refers to particle systems in which the size d is larger than one micron. Below one micron, thermal agitation is important, and Brownian motion can be seen. Above one micron, thermal agitation is negligible. We are interested here in many-particle systems, at zero temperature, occupying a large variety of metastable states: if we pour sand on a table, it would like to go to a ground state, with a monolayer of grains giving the lowest gravitational energy. But in reality the sand remains as a heap; the shape of the heap and the stress distribution inside depend critically on how the heap was made. Hence come many difficulties.

We cited sand as an example: a desert like the Sahara provides us with a gigantic laboratory model. The grains are silica (rounded by collisions) of ~100 microns in size. They form ripples and dunes. These deserts have fascinated a number of great men—Lawrence and Thesinger in Arabia, Monod in the western Sahara, and R. Bagnold in the Libyan desert. Bagnold knew physics and fluid mechanics: he had very much the style of G. I. Taylor. He made precise observations in the desert, then returned to England, built a cheap but efficient wind tunnel (with plywood, etc.), and determined with it the basic laws for the transport of sand. His book *Physics of Blown Sand and Sand Dunes*, published in 1941, remains a basic reference sixty years later (Bagnold, 1941). We shall give an ''idealized summary'' of his views in Sec. III.

Of course, there are many other important granular systems in nature: snow is an obvious example; but snow is frightfully complex, because water can show up in all its natural states, and the resulting phase transitions imply deep macroscopic consequences. In the present text, we shall try to concentrate on dry systems. This may be sand, but it may also be mustard seed (the latter being very convenient for certain nuclear resonance studies).

Many industrial products are powders:

- ''clinkers'' (the starting point of cement) are complex mixtures of silicoaluminates, calcium silicates, etc.
- ''builders'' are an important part of a commercial detergent: they are based on inorganic particles such as calcium carbonate.
- most pharmaceutical products are derived from powders, obtained by precipitation, crystallization, or prilling (prilling is based on a molten thread of material, which breaks into droplets via the Rayleigh instability; the droplets then reach a cool region where they freeze, giving grains with a very-well-defined size).

If we measure it by tons, the material most manipulated by man is water; the second-most-manipulated is granular matter. But in our supposedly sophisticated 20th century, the manipulation of powders still involves some very clumsy and/or dangerous operations.

(1) Milling is slow, inefficient, and generates a very broad distribution of final sizes.

(2) The smaller-size component of these distributions is often toxic.

(3) Many powders, when dispersed in air, achieve a composition that is ideal for strong detonations. Certain workshops or silos explode unexpectedly. One of the main reasons for this is electrostatic: many grains, when manipulated, hit each other or hit a wall, generating triboelectric charges, which ultimately end up in sparks. To understand this, a new type of mass spectrometry is now set up, in which the particles are grains rather than molecules. The grains are studied after a sequence of wall collisions; here, the interest is more in the charge than in the weight.

(4) When feeding, for instance, a glass furnace with a mixture of oxides, one finds that the corresponding flow of oxide in the hoppers can lead to *segregation*—thus creating dangerous inhomogeneities in the final glass: the manipulation of mixtures is delicate.

Certain other operations are quite successful, although their basic principles are only partly understood: for instance, by injecting a gas at the bottom of a large column filled with catalytic particles, one can transform them into a *fluidized bed*. This is crucial for many processes, such as the production of polyethylene. But the dynamics of these beds is still not fully understood.

We see, at this level, the importance of fundamental research in granular matter. This was appreciated very early in mechanical and chemical engineering; physicists have joined in more recently. For them, granular matter is a new type of condensed matter, as fundamental as a liquid or a solid and showing in fact two states: one fluidlike, one solidlike. But there is as yet no consensus on the description of these two states! Granular matter, in 1998, is at the level of solid-state physics in 1930.

There are some excellent reviews (e.g., Jaeger *et al.*, 1996) but very few textbooks—apart from Bagnold (1941) and Brown and Richards (1970). The most recent one is (at the moment) published only in French (Duran, 1997).

In the present short survey, we shall talk only about the *statics* of heaps and silos. The dynamics will be presented elsewhere.

II. PREPARING A GRANULAR SAMPLE

"We fill a glass column with sand." This innocent statement hides many subtleties. Did we fill it from a jet of sand near the axis, or did we sprinkle the sand over the whole section? Did we shake the object after filling?

A first, obvious problem is *compaction*. Bernal (1964) and Scott (1962) measured the average density of containers filled with ball bearings. They were in fact concerned with models for amorphous systems at the atomic level, but their results are of wider utility. Computer simulations (Finney, 1970) indicate that the maximum volume fraction achieved in a random packing of spheres is $\phi_{\rm rp}=0.64$—significantly smaller than the face-centered-cubic (or hexagonal) compact packing $\phi_{\max}=0.74$. Compaction is favored by the weight of the grains themselves. Immersing the grains in a fluid of matched density (Onoda *et al.*, 1990), one can study weaker compactions and more or less reach the connectivity limit or, as it is called, the random loose-stacked limit, which for spheres is around $\phi_{\min}=0.56$.

When a powder is gently shaken, it densifies. In fact, a useful method for characterization of a new granular material is based on tapping a vertical column (see, for instance, Selig and Ladd, 1973). Powders that compact fast are expected to flow easily, while powders that compact slowly, more or less refuse to flow. Fundamental studies on the compaction of noncohesive grains have been performed by the Chicago group (Knight *et al.*, 1995; Nowak *et al.*, 1998). The density plots ϕ_n (after n taps) depend on the amplitude of the taps. At small amplitudes, they follow a logarithmic law:

$$\phi_n=\frac{a}{\ln(n)+b}. \tag{1}$$

Many frustrated, frozen systems are expected to show similar forms of creep (Coniglio and Hermann, 1996; Nicodemi *et al.*, 1997). The simplest interpretation of Eq. (1) is based on free-volume models (Knight *et al.*, 1995; Boutreux *et al.*, 1997), which are familiar from the physics of glasses. The case of strong tapping is more complex (Nowak *et al.*, 1998), but some relevant simulations and modelizations have been performed (Barker and Mehta, 1991, 1992, 1993).

Even if we do not perform any tapping, we must specify how the grains were brought in: there is a critical moment, where the grains stop and adopt a *frozen conformation*. For instance, if we build a heap of sand from an axial jet falling on the center, we create avalanches from the center towards the edges; the freezing process takes place via grains that roll and stop.

The distinction between rolling and frozen grains is crucial. It is reminiscent of a phase transition. If we accept it, we may describe the later evolution of the frozen phrase by a *displacement field* $\sim u(x,y,z,t)$. This is defined by the following gedanken experiment. We focus our attention on one rolling grain and watch when it stops, at a certain point *x,y,z*. This will define the origin of its displacements. Later, with other grains added and loading the system, our grain will move by an amount $\sim u(x,y,z,t)$. Its position will thus depend on the whole history of loading. The resulting displacement field is continuous. Inside the frozen phase, we may define deformations ∇u. We may also define a (coarse-grained average) stress field $\sigma_{\alpha\beta}$ and relate it by some empirical relation to the deformations.

This procedure is essentially what has been used in mechanics departments: see, for instance, the review by Biarez and Gourves (1989). But the precise definition of $\sim u$ is not always stated, and thus the very notion of a displacement field has been questioned by a number of physicists (for a recent summary, see Cates *et al.*, 1998a, 1998b).

The present author's belief is that $\sim u$ is well defined, provided that there is a sharp distinction between fluid particles and frozen particles.[1] We shall come back to this discussion later in Sec. III.

Another important point is the role of *boundary conditions*, on the frozen piece:

(a) At the free surface: a heap, for instance, shrinks under its own weight, and this renormalizes the relation between deformations and displacements.

(b) At the interface between the grains and a solid wall, the normal displacements must, of course, be continuous. The delicate part is the description of friction, i.e., of tangential stresses σ_t at the surface. The natural scheme is as follows:

(i) If the tangential component of $\sim u(\sim u_t)$ has grown monotonically and is large enough, the reaction σ_t from the wall is opposed to $\sim u_t$. For a cohesionless interface, we may write the classical

[1]This may exclude certain complex problems such as tapping.

relation (Amontons' law; see, for instance, Bowden and Tabor, 1973)

$\sigma_t = \mu_f \sigma_n$,

where σ_n is the normal stress and μ_f is a friction coefficient. We call this regime "fully mobilized friction."

(ii) If the tangential displacement $|\sim u_t|$ is smaller than a certain microscopic length Δ, the friction is only partly mobilized. We call Δ the "anchoring length" (de Gennes, 1997). It is usually related to the size of microscopic roughness. For macroscopic solids in contact, Δ is of order one micron.

(iii) If we reverse the displacements (as may happen in experiments where weight and thermal expansions are in conflict) the friction force will reverse fully, only if we move backwards by more than 2Δ.

Thus the state of friction may be influenced by minute displacements of the grains (of order Δ) with respect to the container walls. In a recent experiment on columns (Vanel *et al.*, 1998), the apparent weight at the bottom was found to vary cyclically between day and night: as pointed out by the authors, this is probably due to thermal expansion, inducing some (very small) relative displacements between the grains and the lateral walls, and changing drastically the mobilization of friction.

To summarize: the definition of an initial state, in an experiment on granular matter, requires great care. Many theories and some experiments suffer from a lack of precise definitions.

III. MACROSCOPIC STRESS FIELDS

A. The general problem

For more than a hundred years, departments of applied mechanics, geotechnical engineering, and chemical engineering have analyzed the static distribution of stresses in granular samples. What is usually done is to determine the relations between stress and strain on model samples, using the so-called triaxial tests. Then, these data are integrated into the problem at hand, with the material divided into finite elements (see, for instance, Schofield and Wroth, 1968).

In a number of cases, the problem can be simplified, assuming that the sample has not experienced any dangerous stress since the moment when the grains "froze" together: this leads to a *quasielastic description*, which is simple. I shall try to make these statements more concrete by choosing one example: a silo filled with grain.

B. The Janssen picture for a silo

The filled silo is shown in Fig. 1. The central observation is that stresses, measured with gauges at the bottom, are generally much smaller than the hydrostatic pressure $\rho g H$ which we would have in a liquid (here ρ is the density, g is the gravitational acceleration, and H is the column height). A first modelization for this was given long ago by Janssen and Vereins (1895) and Lord Rayleigh (1906a, 1906b, 1906c, 1906d).

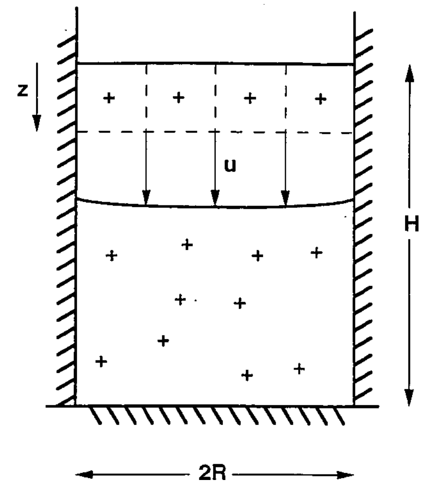

FIG. 1. A silo filled with granular material: the material falls slightly under its own weight, by an amount u. The width of the silo has been exaggerated to display the expected profile of u in a quasielastic model.

(a) Janssen assumes that the horizontal stresses in the granular medium (σ_{xx}, σ_{yy}) are proportional to the vertical stresses:

$$\sigma_{xx} = \sigma_{yy} = k_j \sigma_{zz} = -k_j p(z), \qquad (2)$$

where k_j is a phenomenological coefficient and $p = -\sigma_{zz}$ is a pressure.

(b) An important item is the friction between the grains and the vertical walls. The walls endure a stress σ_{rz}. The equilibrium condition for a horizontal slice of grain (area πR^2, height dz) gives

$$-\rho g + \frac{\partial p}{\partial z} = \frac{2}{R} \sigma_{rz}\big|_{r=R}, \qquad (3)$$

where r is a radical coordinate and z is measured positive towards the bottom.

Janssen assumes that, everywhere on the walls, the friction force has reached its maximum allowed value—given by the celebrated law of L. da Vinci and Amontons (Bowden and Tabor, 1973):

$$\sigma_{rz} = -\mu_f \sigma_{rr} = -\mu_f k_j p, \qquad (4)$$

where μ_f is the coefficient of friction between grains and wall.

Accepting Eqs. (2) and (4), and incorporating them into Eq. (3), Janssen arrives at

$$\frac{\partial p}{\partial z} + \frac{2\mu_f}{R} k_j p = \rho g. \qquad (5)$$

This introduces a characteristic length

$$\lambda = \frac{R}{2\mu_f k_j} \qquad (6)$$

and leads to pressure profiles of the form

$$p(z) = p_\infty [1 - \exp k(-z/\lambda)], \qquad (7)$$

with $p_\infty = \rho g \lambda$. Near the free surface ($z < \lambda$) the pressure is hydrostatic ($p \sim \rho g z$). But at larger depths ($z > \lambda$) $p \to p_\infty$: all the weight is carried by the walls.

C. Critique of the Janssen model

This picture is simple and does give the gross features of stress distributions in silos. But the two assumptions are open to some doubt.

(a) If we take an (excellent) book describing the problem as seen from the point of view of the mechanics department (Nedermann, 1992), we find that Eq. (1) is criticized: a constitutive relation of this sort might be acceptable if *x,y,z* were the principal axes of the stress tensor, but in fact, in the Janssen model, we also need nonvanishing off-diagonal components σ_{xz}, σ_{yz}.

(b) For the contact with the wall, it is entirely arbitrary to assume full mobilization of the friction, as in Eq. (4). In fact, any value σ_{rz}/σ_{rr} below threshold would be acceptable. Some tutorial examples of this condition and of its mechanical consequences are presented in Duran's book (1997). I discussed some related ambiguities in a recent note (de Gennes, 1997) emphasizing the role of the anchoring length.

D. Quasielastic model

When a granular sample is prepared, we start from grains in motion, and each grain freezes at a certain moment. This defines our reference state: (i) the origin of the grain displacements is the freezing point; (ii) the reference density (for defining deformations) is the density achieved immediately upon freezing.

If we fill a silo from the center, we have continuous avalanches running towards the walls, which stop and leave us with a certain slope.

Recent theoretical studies on avalanches (Boutreux *et al.*, 1998) suggest that this final slope, in a "closed-cell" geometry like the silo, should always be below critical: we do not expect to be close to an instability in shear, and the material is under compression everywhere. In situations like this, we may try to describe the granular medium as a *quasielastic medium*. The use of "quasi" must be explained at this point.

When we have a granular system in a certain state of compaction, it will show a resistance to compression, measured by a macroscopic bulk modulus K. But the forces are mediated by small contact regions between two adjacent grains, and the contact areas increase with pressure. The result is that $K(p)$ increases with p. For spheroidal objects and purely Hertzian contacts, one would expect $K \sim p^{1/3}$, while most experiments are closer to $K \sim p^{1/2}$ (Duffy and Mindlin, 1957). Various interpretations of the $p^{1/2}$ law have been proposed (Goddard *et al.*, 1990; de Gennes, 1996).

Evesque and the present author (1998) recently used the quasielastic picture to describe displacements and stresses in a silo. The displacements are vertical and correspond to a slight collapse of the column under its own weight. They increase during filling: their description involves the whole sample history. (The displacements are also slightly smaller near the walls than in the center. This creates the shear stresses that worried Nedermann.)

The result is a Janssen relation of the form of Eq. (2), with a value of k_j that depends only on the Poisson ratio σ_p of the material:

$$k_j = \frac{\sigma_p}{1-\sigma_p}. \tag{8}$$

Although the elastic moduli do depend on pressure, it may be that σ_p and k_j are pressure independent. Then the Janssen pressure profile should hold, provided that mobilization of the wall friction is complete. For long columns ($H \gg \lambda$) the maximum displacement is achieved at mid-height and is

$$|u|_{\max} = \frac{\lambda^2}{\lambda_c}, \tag{9}$$

where $\lambda_c = E/\rho g$ is what we call the compaction length (E = the Young modulus; ρ = the density). Mobilization is indeed complete if $|u|_{\max} \gg \Delta$ (the anchoring length), or equivalently $\lambda > H^*$, where

$$H^* = (\Delta \lambda_c)^{1/2}. \tag{10}$$

In this formula, Δ is very small, but λ_c is very large. Typical values of H^* depend on E, but may be centimetric. Thus, if the quasielastic model makes sense, the Janssen picture should hold for silos ($\lambda \cong$ meters, $\lambda > H^*$) but not necessarily for laboratory columns ($\lambda \cong 1$ cm).

E. Stress distribution in a heap

Below a heap of sand, the distribution of normal pressures on the floor is not easy to guess. In some cases, the pressure is not a maximum at the center point! This has led to a vast number of physical conjectures, describing "arches" in the structure (Bouchaud *et al.*, 1995; Edwards and Mounfield, 1996). In their most recent form (Wittmer *et al.*, 1997), what is assumed is that, in a heap, the principal axes of the stress are fixed by the deposition procedure. Near the free surface, following the pioneering work of Coulomb, it is usually assumed that (for a material of zero cohesion) the shear and normal components of the stress (τ and σ_n) are related by the condition

$$\tau = \sigma_n \mu_i = \sigma_n \tan \theta_{\max}, \tag{11}$$

where μ_i is an interval friction coefficient and $\tan \theta_{\max}$ is the resulting slope. Equation (11) should hold for a dry system with no cohesion between grains. In a two-dimensional geometry, this corresponds to a principal axis that is at an angle $2\theta_{\max}$ from the horizontal (Nedermann, 1992). The assumption of Wittmer *et al.* is that this orientation is retained in the left-hand side of the heap (plus a mirror symmetry for the right-hand side). Once this is accepted, the equilibrium conditions incorporating gravity naturally lead to a "channeling of forces'' along the principal axis, and to a distribution of loads on the bottom that has two peaks. More generally, in the description of Bouchaud *et al.*, the transmission of stresses is described by *hyperbolic* equations, leading to certain preferred directions. In the classical approach from continuum mechanics, the transmission is ruled by *elliptic* equations. In the first picture, the entire heap is pictured as being in some sort of critical state. In the second picture, we are far from criticality, and the heap is not dramatically different from a conventional solid—although the sample history is important for a clear definition of deformations.

The ''critical'' view has been challenged by S. Savage (1997a, 1997b) and by J. D. Goddard (1998). Savage gives a detailed review of the experimental and theoretical literature. He makes the following claims:

(a) for two-dimensional heaps (''wedges'') with a rigid support plane, there is no dip in the experiments.

(b) if the support is (very slightly) deformable, the stress field changes deeply, and a dip occurs. This is another example of the role of minute displacements, which was already emphasized in Sec. III.E.

(c) for the 3D case (''cones''), the results are extremely sensitive to the details of the deposition procedure.

The most recent data on cones are by Brockbank *et al.* (1997). They use an accurate optical measurement of the local load under a conical heap of steel balls. The balls in the bottom layer deform the support, which is made of a transparent rubber film (~2 mm in thickness) lying over a glass surface. They do find a dip with steel, and also with glass heads of diameter 0.18 mm. But, when going to larger glass beads (~0.6 mm), the dip disappears!

Savage also describes finite element calculations, where one imposes the Mohr-Coulomb conditions (to which we come back in Sec. III) at the free surface of a wedge. If we had assumed a quasielastic description inside, we would have found an inconsistency: there is a region, just below the surface, which becomes unstable towards shear and slippage. Thus Savage uses Mohr-Coulomb conditions in a finite sheet near the surface, plus elastic laws in the inner part. With a rigid support he finds no dip, but with a deformable support he gets a dip.

The Savage methodology is similar in spirit to the quasielastic method although the details of the boundary conditions could possibly be altered. For instance, there may exist an extra simplification—which I already announced in connection with the silos. If we look at the formation of the heap, we find that the slope angle upon deposition should be slightly lower than the critical angle $\theta_{\max}$. Thus our system is prepared under noncritical conditions: all of the sample may then be described as quasielastic. This, in fact, should not produce very different results from those of Savage.

But there is a certain doubt, formulated by M. Cates and others: if the grains were glued together by microscopic glue patches at the contact point, indeed we might define displacements and deformations and use the Savage picture. But there is no glue! Certain grains might then be under tension (even if we are under a global compressive load): mechanical integrity is not granted!

In reply to this, the present author proposes three observations, which tend to support the classical view from mechanics.

(i) *Shear tests:* under compressive load (in conditions without fracture) the stress strain relations are clearly history dependent, but do not display (as far as we can tell) any singular power laws.

(ii) *Lack of criticality:* if we examine the local density in a horizontal bed of sand, or the volume fraction ϕ as a function of depth, we find that ϕ is nearly constant and significantly larger than the critical value $\phi_{\min}$ mentioned in Sec. II.[2] For these practical ϕ values (as we shall see in Sec. IV) the few indications available on correlation lengths ξ suggest that ξ is not large (at most of order 5 to 10 grain diameters). The singularities linked with arches, with tensile microcracks, should thus be confined to very small scales $\Delta x<\xi$.

(iii) *Texture:* One of the features that the physicists really wanted to incorporate is the possible importance of an internal *texture*. If we look at the contacts $(1,2,\dots i,\dots p)$ of a grain in the structure, we can form two characteristic tensors: one is purely geometrical and defines preferred directions of contact. It is

$$Q_{\alpha\beta}=\overline{i\sum\, x_\alpha^{(i)}x_\beta^{(i)}}, \tag{12}$$

where x_α are the distances measured from the center of gravity of the grain. $Q_{\alpha\beta}$ is also called the ''fabric tensor'' (Oda, 1972, 1993; Oda and Sudoo, 1989). It is related to the ''ellipsoid of contacts'' introduced by Biarez and Wiendick (1963). The other tensor is the static stress:

$$\sigma_{\alpha\beta}=\frac{1}{2}\, i\sum\, (x_\alpha^{(i)}F_\beta^{(i)}+x_\beta F_\alpha^{(i)}), \tag{13}$$

where $\sim F^i$ is the force transmitted at contact (i). There is no reason for the axes of these two tensors to coincide. For instance, in an ideal hexagonal crystal, one major axis of the Q tensor is the hexagonal axis, while the stresses can have any set of principal axes. In the heap problem, I am personally inclined to believe that the deposition process freezes a certain structure for the Q tensor, but not for the stress tensor. However, this is still open to discussion! Recent arguments defending the opposite viewpoint have been given by Cates *et al.* (1998b).

The presence of a nontrivial Q tensor (or ''texture'') can modify the quasielastic model: instead of using an isotropic medium, we may need an anisotropic medium. In its simplest version, we would assume that the coarse-grained average $Q_{\alpha\beta}$ had two degenerate eigenvalues and a third eigenvalue, along a certain unit vector (the

[2]Note that although ϕ is nearly constant, in a bed of sand elastic moduli increase dramatically with depth. This is the basis of the ''quasielastic'' model.

director) $\sim n(\sim r)$. Thus a complete discussion of static problems (in the absence of strong shear bands) would involve an extra field $\sim n$ defined by the construction of the sample. This refinement may modify the load distribution under a heap. But, conceptually, it is, in my opinion, minor. Texture effects should not alter deeply the quasielastic picture.

F. Strong deformations

Sophisticated tools have been designed for measuring the yield stress τ_y of granular materials in simple shear (Jenike, 1961; for a review, see for instance Brown and Richards, 1970). There is an elastic response at low shears, followed by yield at a certain value of the stress τ_y:

$$\tau_y = C + \mu p_n , \tag{14}$$

where p_n is the normal pressure. The constant C represents adhesive interactions between grains, and μ is a friction coefficient. An important feature of these strongly sheared systems—emphasized long ago by Reynolds (1885) is *dilatancy*: when the material was originally rather compact and is forced to yield, it increases in volume. This can be qualitatively understood by thinking of two compact layers of spheres sliding over each other.

In some cases, these strong deformations, with dilatancy, are present over large volumes. In other cases, they may be concentrated on *slip bands* (see, for instance, Desrues, 1991; Tillemans and Herrmann, 1995). For instance, if we remove sand with a bulldozer, slip bands will start from the bottom edge of the moving plate. Sometimes, the size of these slip bands is large and depends on the imposed boundary conditions (on the sharpness of the plate edge). But there seems to be a minimal thickness for a slip band: for spheroidal grains, without cohesion, it may be of order 5 to 10 grain diameters. We shall come back to this thickness when discussing microscopic properties.

IV. MICROSCOPIC FEATURES

A. Correlation lengths

We have talked about macroscopic stresses σ_{ij}: they must represent some coarse-grained averages over a certain volume. The implicit assumption here is that, indeed, a granular medium can be considered as homogeneous at large scales. This is not obvious: if we were talking about noncompacted material, with a density close to the lower limit $\phi_{\min}$=0.56, we might have a structure of weakly connected clusters (similar to percolation clusters). Exactly at threshold ($\phi = \phi_{\min}$) a structure like this would probably be self-similar and not homogeneous at all. However, in real life, we always operate on systems with $\phi > \phi_{\min}$, and we can expect that, at scales larger than a certain correlation length $\xi(\phi)$, our system may be treated as homogeneous.

Various experiments (Liu *et al.*, 1995) and simulations (Moreau, 1994; Ouageni and Roux, 1995; Zhuang *et al.*, 1995; Radjai *et al.*, 1996) have investigated the local distribution of forces between grains. The central conclusion is that there are *force channels*, which build up a certain mesh with a characteristic size ξ. For spherical objects and ϕ values in the usual range, this ξ is somewhat larger than the grain diameter d ($\xi/d \sim 5$ to 10).

The network is obviously sensitive to variations in size among the grains. This "polydispersity" is always present and plays an important role in the actual value of ξ.

It may well be that the minimum thickness of a slip band (as introduced in Sec. III.F) is equal (within coefficients) to the correlation length ξ. Thus we have at least two empirical ways of estimating ξ for a given system.

B. Fluctuations of the local load

It is also of interest to probe the local distribution of forces on all grains in contact with a supporting (horizontal) plate. This has been done in experiments by the Chicago group (Liu *et al.*, 1995; Mueth *et al.*, 1998), together with some simulations. Their trick is to lay the granular sample on a sequence carbon paper/white paper/solid plate. There is an empirical relation between the size of the dots printed by each grain on the white paper, and the force (w) with which it presses the ground. What Liu *et al.* found was a distribution of w, of the form

$$p(w) = \frac{w^2}{2\overline{w}^3} e^{-2w/\overline{w}}. \tag{15}$$

Liu *et al.* (1995) constructed a simple model for this statistical behavior, ignoring the vector character of the forces. They stipulated that each grain receive a load (w) from three neighbors above it:

$$w = q_1 w_1 + q_2 w_2 + q_3 w_3 , \tag{16}$$

where w_1, w_2, w_3 are the loads on the "parents," and q_1, q_2, q_3 are three coupling factors statistically distributed between 0 and 1, and independent. Conversely, each parent sends some of its weight on to three "children" with fractions q_1', q_2', q_3', and these fractions satisfy the sum rule $\Sigma q_1' = 1$. But apart from this constraint, all the q_s' are independent.

The law (15) can be understood as follows:

(a) for $w \gg \overline{w}$, we must have $q_1, q_2, q_3 \sim 1$, and we can then factorize $p(w) \sim p(w_1) p(w_2) p(w_3)$, with $w = w_1 + w_2 + w_3$. As pointed out by T. Witten, this condition is similar to the problem of a Boltzmann distribution of energies in thermal physics, and the solution is exponential $p(w) \sim \exp(-\alpha w)$.

(b) for $w \ll \overline{w}$, the weights carried by the three parents are much larger than w, and the probability $p(w)$ is essentially proportional to the phase space available in (q_1, q_2, q_3) where the q_s are linked by Eq. (16). This corresponds to a triangle of edges, $(w_1/w, 0, 0)$,

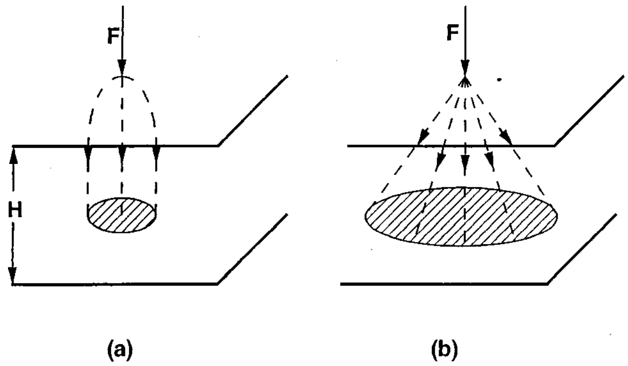

FIG. 2. A crucial experiment, which to the author's knowledge has not yet been performed in a completely conclusive way. A bed of sand is deposited uniformly on a large flat surface and fills a height H. A small local force F is applied vertically at one point of the top surface. What are the resulting extra loads on the bottom plate? (a) In the "elliptic" models, used in soil mechanics, the load is spread over a region of size $\sim H$. (b) In the "hyperbolic" models of Bouchaud *et al.* the load is distributed over an annulus.

$(0,w/w_2,0)$, $(0,0,w/w_3)$ in the (q_1,q_2,q_3) space, with an area $\sim w^2$. (However, on the experimental side, the more recent data of Mueth *et al.* (1998) give a different law!)

To summarize: (i) the fluctuations of w are comparable to the average $(\overline{w})$; (ii) the tail of the distribution at large w is exponential. The probabilities q_1, for very small loads $(w \rightarrow 0)$, are still open to discussion.

A subsidiary question is: what are the correlations $\langle w(\sim x) w(\sim y) \rangle$ between grains at different locations (x,y) on the ground? The natural guess is that the range of these correlations is the correlation length ξ.

Of course, the model should be refined by introducing the vector character of the forces. The vectorial features are crucial when the *average* load is variable from point to point on the bottom plate. Consider, for instance, a horizontal slab of grains, with a thickness H and a very large aspect ratio. Impose a weak localized force F downwards, at the center of the upper surface $(x=y=0$; see Fig. 2).

(a) The scalar model of (Liu *et al.*, 1995) would give an average load profile on the bottom plate with a peak at the center and a width $\Delta x \sim \Delta y \sim \sqrt{dH}$ (where d is the grain diameter).

(b) With a tensorial stress field and a quasielastic model, we expect $\Delta x \sim \Delta y \sim H$.

(c) With "singular" models that predict transmission of the weight only in special directions (e.g., Bouchaud *et al.*, 1995), the load would be concentrated in a ring, and disorder would make this ring slightly diffuse.

V. CONCLUDING REMARKS

The science of granular materials started with outstanding pioneers: Coulomb, Reynolds, Bagnold . . . . In recent years, it has benefited from the impact of very novel techniques—e.g., nuclear imaging of grains at rest or in motion (Nakagawa *et al.*, 1993). A strong stimulus has also come from computer simulations—which have not been adequately described in the present text, because of the author's inexperience. It is clear that virtual experiments with controlled, simplified interactions between grains can have a major impact. A review of the tools, and of certain difficulties, can be found in Duran (1997). Recent advances are described in the proceedings of the Cargèse Workshop (Hermann, 1997).

However, in spite of these powerful tools, and even for the simplest "dry" systems, the statistical physics of grains is still in its infancy. Some basic notions may emerge: (a) the sharp distinction between a fluid phase and a frozen phase, with the resulting possibility of defining a displacement field to describe the evolution of the frozen phase; (b) a displacement field containing a memory of all the sample history; (c) the possibility of describing surface flows with equations coupling the two phases and reduced to a simplicity reminiscent of the Landau-Ginsburg picture of phase transitions.

But we are still left with strong disputes, and large sectors of unraveled complexity.

Two fundamentally different pictures of the static behavior of heaps are facing each other: one represents the material as a deformable solid, the other assumes a completely singular state of matter, with stress fields transmitted along special directions and with microscopic instabilities (earthquakes) occurring all the time (see, for instance, Miller *et al.*, 1996).

We have to know more! Here are some examples:

(a) The problem raised in Fig. 2: if we press *gently* at the free surface of a large, flat bed of sand, are the stresses below widely spread (as expected from a quasielastic solid) or are they localized on a cone (as expected in "singular" models)? The word "gently" is important here: if we go to strong, local loads, we shall of course, generate shear bands.

(b) Acoustic propagation in a granular bed: it is mainly controlled by the (nonlinear) quasielastic features plus mild effects of disorder. Or is it qualitatively different, because a sound wave, even at small amplitudes, starts some sort of earthquake?

(c) Decompaction: if we open the bottom of a vertical column, we see pieces of solidlike matter which separate from each other. Can we think of this as propagation of fractures in a quasielastic solid, or is it completely different?

(d) Similarly, when we perform a sequence of taps on a column, as mentioned in Sec. II, should we visualize the grains during the tap as a solid with microcracks or as a liquid (if the amplitudes are high enough)?

We mentioned some current uncertainties for the solid phase. There are uncertainties of comparable magnitude for the fluid phases. Think, for instance, of fluidized beds: an intelligent literature (describing both transport and macroscopic instabilities) has been built up, but we are still looking for a unified vision. The link between mechanics, tribology, statistical physics, surface chemistry, . . . remains to be built.

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