Brane dynamics and gauge theory

Amit Giveon

Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

David Kutasov

EFI and Department of Physics, University of Chicago, Chicago, Illinois 60637
and Department of Particle Physics, Weizmann Institute of Science,
Rehovot 76100, Israel

The authors review some aspects of the interplay between the dynamics of branes in string theory and the classical and quantum physics of gauge theories with different numbers of supersymmetries in various dimensions. [S0034-6861(99)01004-1]

CONTENTS

 0034-6861/99/71(4)/983(102)/$35.40

I. INTRODUCTION

Non-Abelian gauge theories are a cornerstone of the standard model of elementary particle physics. Such theories (for example, QCD) are often strongly coupled at long distances, and therefore cannot be studied by the standard perturbative methods of weakly coupled field theory. In the last few years important progress has been made in the study of strongly coupled dynamics in a class of gauge theories—supersymmetric Yang-Mills (SYM) theories. New understanding of the constraints due to supersymmetry, the importance of solitonic objects, and electric-magnetic, strong-weak coupling duality has led to many exact results on the vacuum structure of various supersymmetric field theories.

Despite the fact that supersymmetry (a symmetry relating bosons and fermions) is not present in the standard model, there are at least three reasons to study supersymmetric gauge theories:

- It is widely believed that an $N=1$ supersymmetric extension of the standard model describes physics at energies not far above those of current accelerators and is directly relevant to the hierarchy problem and unification of couplings.
- Supersymmetric gauge theories provide examples of many phenomena believed to occur in nonsupersymmetric theories in a more tractable setting. Therefore they serve as useful toy models for the study of these phenomena.
- The study of supersymmetric field theories has many mathematical applications.

Non-Abelian gauge theories also appear in low-energy approximations to string theory, where supersymmetry plays an important role. String theory is a theory of quantum gravity which, moreover, unifies gravity and gauge fields in a consistent quantum theory. Traditionally, the theory has been formulated in an expansion in a (string) coupling, but many of the outstanding problems in the subject have to do with physics outside the weak-coupling domain. String theory has also been undergoing rapid progress in the last few years, driven by similar ideas to those mentioned in the gauge-theory context above.

Some highlights of the progress in gauge and string theory that are relevant for this review are the following:

(1) **Strong-weak coupling duality.** The physics of asymptotically free gauge theory depends on the energy scale at which the theory is studied. At high energies the theory becomes weakly coupled and is well described in terms of the fundamental fields in the Lagrangian (such as quarks and gluons). At low energies the theory is often strongly coupled and can exhibit several different behaviors (or phases): confining, Higgs, Coulomb, free electric, and free magnetic phases.

In the confining phase, the energy of a pair of test charges separated by a large distance R grows linearly with R. Thus such charges cannot be infinitely separated. In the Higgs phase, the gauge bosons are massive and the energy of a pair of test charges goes to a constant at large R. The Coulomb phase is characterized by potentials that go as $1/R$, while the free electric and magnetic phases have logarithmic corrections to this behavior. The standard model of elementary particle physics realizes the confining, Higgs, and free electric phases; other models that go beyond the standard model use the other phases as well.

The determination of the phase structure of non-Abelian gauge theories is an important problem that is in general complicated because it involves understanding the physics of strongly coupled gauge theory. In the last few years, this problem has been solved for many supersymmetric gauge theories. One of the main advances that led to this progress was the realization that electric-magnetic, strong-weak coupling duality is quite generic in field theory.

In a typical realization of such a duality, one studies an asymptotically free gauge theory that becomes more and more strongly coupled as one goes to lower and lower energies. The extreme low-energy behavior is then found to be governed by a different theory, which may be *weakly* coupled, e.g., because it is not asymptotically free.

In other interesting situations, the original theory depends on continuous parameters (exactly marginal deformations), and the duality relates the theory at different values of these parameters. An example of this is the maximally supersymmetric four-dimensional gauge theory known as $N=4$ SYM theory. This theory depends on a complex parameter τ, whose imaginary part is proportional to the square of the inverse gauge coupling; the real part of τ is a certain θ angle. The theory becomes weakly coupled when Im $\tau\to\infty$. It has been proposed that it is invariant under a strong-weak coupling duality $\tau \to -1/\tau$ in addition to the semiclassically manifest symmetry $\tau \to \tau + 1$. This symmetry is a generalization of the well-known symmetry of electrodynamics, which takes $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ and at the same time exchanges electric and magnetic charges. In the last few years convincing evidence has been found for the validity of this duality symmetry of $N=4$ SYM theory.

Many interesting generalizations to theories with less supersymmetry have been found. For example, certain "finite" supersymmetric gauge theories [e.g., $N=2$ SYM theory with gauge group $SU(N_c)$ and $N_f = 2N_c$ "flavors" of fundamental hypermultiplets] also appear to have such symmetries. Furthermore, it has been discovered that different $N=1$ supersymmetric gauge theories may flow to the same infrared fixed point and thus exhibit the same long-distance behavior. As we change the parameters defining the different theories, one of the descriptions might become more weakly coupled in the infrared while another might become more strongly coupled. In some cases, this equivalence relates a strongly coupled interacting gauge theory to an infrared free one. Interesting phenomena have also been shown to occur in other dimensions; in particular, a large class of previously unsuspected nontrivial fixed points in five- and six-dimensional field theory has been found.

String theory has been known for a long time to be invariant under a large discrete symmetry group known as T duality. This duality relates weakly coupled string theories and is valid order by order in the string coupling expansion. It relates different spacetime backgrounds in which the string propagates. A simple example of T duality is the equivalence of string propagation on a circle of radii R and $1/R$. A perturbative fundamental string state that carries momentum n/R around the circle is mapped by T duality to a perturbative fundamental string state corresponding to a string winding n times around the dual circle of radius $1/R$.

In the last few years it has been convincingly argued that the perturbative T-duality group is enhanced in the full string theory to a larger symmetry group, known as U duality, which relates perturbative string states to solitons, and connects different string vacua that were previously thought of as distinct theories. In certain strong-coupling limits string theory becomes 11 dimensional and is replaced by an inherently quantum "M theory." At low energies M theory reduces to 11-dimensional supergravity; the full structure of the quantum theory is not well understood as of this writing.

(2) **Solitonic objects.** Gauge theories in the Higgs phase often have solitonic solutions that carry magnetic charge. Such monopoles and their dyonic generalizations (which carry both electric and magnetic charge) play an important role in establishing duality in gauge theory. In supersymmetric gauge theories their importance is partly due to the fact that they preserve some supersymmetries and therefore belong to special representations of the supersymmetry algebra known as "short" multiplets, which contain fewer states than standard "long" multiplets of the superalgebra. Particles that preserve part of the supersymmetry are conventionally referred to as being "BPS saturated" (for Bogomolny, Prasad, and Sommerfield). Because of the symmetries, some of the properties of these solitons can be shown to be independent of the coupling constants, and thus certain properties can be computed exactly by weak-coupling methods. Often, at strong coupling, they become the light degrees of freedom in terms of which the long-distance physics should be formulated.

In string theory analogous objects have been found. These are BPS-saturated p-branes, p-dimensional objects (with $p+1$-dimensional worldvolumes) that play an important role in establishing U duality. In various strong-coupling regions different branes can become light and/or weakly coupled, and serve as the degrees of freedom in terms of which the dynamics should be formulated. The study of branes preserving part of the supersymmetry in string theory led to fascinating connections, some of which will be reviewed below, between string (or brane) theory and gauge theory.

(3) **Quantum moduli spaces of vacua.** Supersymmetric Yang-Mills (SYM) theories and string theories often have massless scalar fields with vanishing classical potential and therefore a manifold of inequivalent classical vacua $\mathcal{M}_{cl}$, which is parametrized by constant expectation values of these scalar fields. In the nonsupersymmetric case quantum effects generically lift the moduli space $\mathcal{M}_{cl}$, leaving behind a finite number of quantum vacua. In supersymmetric theories the quantum lifting of the classical moduli space is severely constrained by certain nonrenormalization theorems. The quantum corrections to the scalar potential can often be described by a dynamically generated nonperturbative superpotential,[1] which is severely restricted by holomorphicity, global symmetries, and large-field behavior. One often finds an unlifted quantum moduli space $\mathcal{M}_q$. In many gauge theories the quantum superpotentials have been analyzed and the moduli spaces $\mathcal{M}_q$ have been determined. Partial success has also been achieved in the analogous problem in string theory.

Branes have proven useful in relating string dynamics to low-energy phenomena. In certain limits brane configurations in string theory are well described as solitonic solutions of low-energy supergravity, in

[1]There are cases in which the lifting of a classical moduli space cannot be described by an effective superpotential for the moduli (Affleck, Dine, and Seiberg, 1984). We thank N. Seiberg for reminding us of this.

particular black holes. Interactions between branes are then mainly due to ''bulk'' gravity. In other limits gravity decouples and brane dynamics is well described by the light modes living on the worldvolume of the branes. Often, these light modes describe gauge theories in various dimensions with different kinds of matter. Studying the brane description in different limits sheds new light on the quantum mechanics of black holes, as well as on quantum gauge-theory dynamics. Most strikingly, both subjects are seen to be different aspects of a single problem: the dynamics of branes in string theory.

The fact that embedding gauge theories in string theory can help analyze strongly coupled low-energy gauge dynamics is *a priori* surprising. Standard renormalization-group (RG) arguments would suggest that at low energies one can integrate out all fluctuations of the string except the gauge-theory degrees of freedom, which are governed by SYM dynamics (gravity also decouples in the low-energy limit). This would seem to imply that string theory cannot in principle teach us anything about low-energy gauge dynamics.

Recent work suggests that, while most of the degrees of freedom of string theory are indeed irrelevant for understanding low-energy physics, there is a sector of the theory that is significantly larger than the gauge theory in question and that should be kept to understand the low-energy structure. This sector involves degrees of freedom living on branes and their internal fluctuations and embedding in space-time.

We shall see that the reasons for the ''failure'' of naive intuition here are rather standard in the general theory of the renormalization group:

(a) In situations where the long-distance theory exhibits symmetries, it is advantageous to study RG trajectories along which the symmetries are manifest (if such trajectories exist). The string embedding of SYM theory often provides such a trajectory. Other RG trajectories (e.g., the standard quantum-field-theory definition of SYM theory in our case) that describe the same long-distance physics may be less useful for studying the consequences of these symmetries, since they are either absent throughout the RG flow, arising as accidental symmetries in the extreme infrared limit, or are hidden in the variables that are being used.

(b) Embedding apparently unrelated low-energy theories in a larger high-energy theory can reveal continuous deformations of one into the other that proceed through regions in parameter space where both low-energy descriptions fail.

(c) The embedding in string theory allows one to study a much wider class of long-distance behaviors than is possible in asymptotically free gauge theory.

In brane theory, gauge theory arises as an effective low-energy description that is useful in some region in the moduli space of vacua. Different descriptions are useful in different regions of moduli space, and in some regions the extreme IR behavior cannot be given a field-theory interpretation. The underlying dynamics is always the same—brane worldvolume dynamics in string theory. Via the magic of string theory, brane dynamics provides a uniform and powerful geometrical picture of a diverse set of gauge-theory phenomena and points to hidden relations between them.

The purpose of this review is to provide an overview of some aspects of the rich interplay between brane dynamics and supersymmetric gauge theory in different dimensions. We have tried to make the presentation relatively self-contained, but the reader should definitely consult reviews (some of which are listed below) on string theory, D-branes, string duality, and the recent progress in supersymmetric gauge theory, for general background and more detailed discussions of aspects that are only mentioned in passing below.

A. General references

In the last few years there has been a great deal of work on subjects relevant to this review. Below we list a few of the recent original papers and reviews that can serve as a guide to the literature.

For introductions to supersymmetry (SUSY) field theory see, for example, Gates *et al.*, 1983 and Wess and Bagger, 1992. Electric-magnetic strong-weak coupling duality in four-dimensional gauge theory dates back to the work of Montonen and Olive (1977). Reviews of the exact duality in $N=4$ SYM theory and additional references to the literature can be found in the work of Olive (1995), Harvey (1996), and Di Vecchia (1997). Harvey (1996) also includes a pedagogical introduction to magnetic monopoles and other BPS states.

The recent progress in $N=2$ SYM theory started with the work of Seiberg and Witten (1994a, 1994b). Reviews include those of Bilal (1996), Di Vecchia (1996), Lerche (1997), and Alvarez-Gaume and Hasan (1997). The recent progress in $N=1$ SUSY gauge theory was led by Seiberg's, who published two of the important original papers (Seiberg, 1994, 1995a). Some reviews of the work on $N=1$ supersymmetric theories are those of Amati *et al.* (1988), Seiberg, (1995b), Intriligator and Seiberg (1996a), Giveon (1996), Peskin (1997), and Shifman (1997).

The standard reference on string theory is Green, Schwarz, and Witten (1987); for a recent review see Kiritsis (1997). Dirichlet branes are described by Polchinski (1995, 1996) and Polchinski, Chaudhuri, and Johnson (1996). Solitonic branes are discussed by Callan, Harvey, and Strominger (1991a, 1991b, 1991c). A comprehensive review on solitons in string theory is that of Duff, Khuri, and Lu (1995).

T duality is reviewed by Giveon, Porrati, and Rabinovici (1994). The nonperturbative dualities and M theory are discussed by Hull and Townsend (1995), Witten (1995a), Schwarz (1995, 1997a), Townsend (1996, 1997), Vafa (1997), and many additional papers. A recent summary for nonexperts is that of Schwarz (1997b). Finally, reviews on applications of branes to black-hole physics include, for example, those of Maldacena (1996), Peet (1998), and Youm (1997).

B. Plan

The plan of this review is as follows. In Sec. II we introduce the cast of characters—the different 1/2 BPS-saturated branes in string theory.

We start, in Sec. II.A, by describing the field content of 10- and 11-dimensional supergravity and, in particular, the p-form gauge fields to which different branes couple. In Sec. II.B we describe different branes at weak string coupling, where they appear as heavy nonperturbative solitons charged under various p-form gauge fields. This includes Dirichlet branes (D-branes), which are charged under Ramond-sector gauge fields, and solitonic branes charged under Neveu-Schwarz-sector gauge fields. We also describe orientifolds, which are nondynamical objects (at least at weak string coupling) that are very useful for applications to gauge theory.

In Sec. II.C we discuss the interpretation of the different branes in M theory, the 11-dimensional theory that is believed to underlie all string vacua as well as 11-dimensional supergravity. We show how different branes in string theory descend from the membrane and five-brane of M theory and discuss the corresponding superalgebras.

In Sec. II.D we describe the transformation of the various branes under U duality, the nonperturbative discrete symmetry of compactified string (or M) theory. In Sec. II.E we initiate the discussion of branes preserving less than one-half of the supersymmetry, with particular emphasis on their worldvolume dynamics. We introduce configurations of branes ending on branes that are central to the gauge-theory applications and discuss some of their properties.

Section III focuses on configurations of parallel Dirichlet three-branes that realize four-dimensional $N=4$ SYM theory on their worldvolume. We describe the limit in which the worldvolume gauge theory decouples from all the complications of string physics and explain two known features of $N=4$ SYM theory using branes. The Montonen-Olive electric-magnetic duality symmetry is seen to be a low-energy manifestation of the $SL(2,Z)$ self-duality of ten-dimensional type-IIB string theory; Nahm's description of multimonopole moduli space is shown to follow from the realization of monopoles as D strings stretched between $D3$-branes preserving one-half of the supersymmetry. We also describe the form of the metric on monopole moduli space and some properties of the generalization to symplectic and orthogonal groups obtained by studying three-branes near an orientifold three-plane.

In Sec. IV we move on to brane configurations, describing four-dimensional $N=2$ SYM theory. In particular, in Sec. IV.C we explain, using a construction of branes suspended between branes, the observation by Seiberg and Witten that the metric on the Coulomb branch of such theories is given by the period matrix of an auxiliary Riemann surface Σ. In the brane picture this Riemann surface becomes physical and is interpreted as part of the worldvolume of a five-brane. $N=2$ SYM theory is obtained in brane theory by studying the worldvolume theory of the five-brane wrapped around $R^{3,1}\times\Sigma$. We also discuss the geometrical realization of the Higgs branch and various deformations of the theory.

Section V is devoted to four-dimensional theories with $N=1$ supersymmetry. We describe the classical and quantum phase structures of such theories as a function of the parameters in the Lagrangian and explain Seiberg's duality between different theories using branes. In the brane construction, the quantum moduli spaces of members of a dual pair provide different parametrizations of a single space—the moduli space of the corresponding brane configuration. Each description is natural in a different region in parameter space. Seiberg's duality in brane theory is thus reminiscent of the well-known correspondence between two-dimensional sigma models on Calabi-Yau hypersurfaces in weighted projective spaces and Landau-Ginzburg models with $N=(2,2)$ supersymmetry (Greene, Vafa, and Warner, 1989; Kastor, Martinec, and Shenker, 1989; Martinec, 1989), where the relation between the two descriptions can be established by embedding both in the larger framework of the (nonconformal) gauged linear sigma model (Witten, 1993).

In Sec. VI we study three-dimensional theories. In Sec. VI.A we establish, using brane theory, two results in $N=4$ SYM theory. One is that the moduli space of many such theories is identical as a hyper-Kähler manifold to the moduli space of monopoles in a *different* gauge theory. The other is "mirror symmetry," i.e., the statement that many $N=4$ SUSY gauge theories have mirror partners such that the Higgs branch of one theory is the Coulomb branch of its mirror partner and vice versa. In Sec. VI.B we study $N=2$ SUSY theories. We describe the quantum moduli space of $N=2$ supersymmetric QCD using branes and show that the two dualities mentioned above, Seiberg's duality and mirror symmetry, can be extended to this case and teach us new things both about branes and about gauge theories. We also discuss the phase structure of four-dimensional $N=1$ SUSY gauge theory compactified to three dimensions on a circle of radius R.

In Sec. VII we consider two-dimensional theories. We study $2d$ $N=(4,4)$ supersymmetric theories and compactifications of $N=4$ supersymmetric models from three to two dimensions on a circle. We also discuss $N=(2,2)$ SUSY theories in two dimensions. In Sec. VIII we study some aspects of five- and six-dimensional theories, as well as compactifications from five to four dimensions on a circle. Finally, in Sec. IX we summarize the discussion and mention some open problems.

C. Omissions

In the following we briefly discuss issues that will not be reviewed extensively:[2]

- *Gauge theories in Calabi-Yau compactifications.* An alternative (but related) way to study low-energy gauge theory is to compactify string theory to D dimensions on a manifold preserving the required amount of supersymmetry, and take $M_{Planck}\to\infty$ to decouple gravity and massive string modes. This leads to a low-energy gauge theory, some of whose properties can be related to the geometry of the internal space.
  In particular, compactifications of the type-II string on singular Calabi-Yau threefolds—fibrations of asymptotically locally Euclidean (ALE) spaces over CP^1—are useful in the study of $N=2$ SYM theories (Kachru *et al.*, 1996; Klemm *et al.*, 1996); for reviews see (Klemm, 1997; Lerche, 1997). BPS states are related to type-IIB three-branes wrapped around 3-cycles which are fibrations of vanishing 2-cycles in the ALE space. On the base the three-brane is projected to a self-dual string on a Riemann surface Σ, which is the Seiberg-Witten curve. The string tension is related to the Seiberg-Witten differential λ. The existence of stable BPS states is reduced to a geodesic problem on Σ with metric $|\lambda|^2$.
  Similarly, to study $N=1$ SYM theories in four dimensions one compactifies F theory (Vafa, 1996) on Calabi-Yau fourfolds. This "geometric engineering" was initiated by Katz, Klemm, and Vafa (1997; Katz and Vafa, 1997) and is reviewed by Klemm (1997).
- *Probing the geometry of branes with branes.* We shall briefly describe a few (related) examples in which the geometry near branes can be probed by lighter objects. In particular, we shall describe the metric felt by a fundamental string propagating in the background of solitonic five-branes, and by three-branes near parallel seven-branes and orientifold seven-planes. In the latter case, the geometrical data are translated into properties of the four-dimensional $N=2$ supersymmetric gauge theory on the three-branes. The interplay between the gauge dynamics on branes and the geometry corresponding to the presence of other branes was studied by Douglas (1996), Banks, Douglas, and Seiberg (1996), and Sen (1996) and was generalized in many directions.
  For instance, four-branes can be used to probe the geometry of parallel eight-branes and orientifold eight-planes, leading to an interesting connection between five-dimensional gauge theory and geometry (Seiberg, 1996b; Douglas, Katz, and Vafa, 1997; Morrison and Seiberg, 1997). Similarly, p-branes (with $p<3$) can be used to probe the geometry near parallel $(p+4)$-branes and orientifold $(p+4)$-planes, leading to relations between low-dimensional $(D<4)$ gauge theories and geometry (Seiberg, 1996a; Seiberg and Witten, 1996; Banks, Seiberg, and Silverstein, 1997; Diaconescu and Seiberg, 1997), some of which will be discussed in this review. Other brane configurations that were used to study the interplay between geometry and gauge theory appear in articles by Aharony, Kachru, and Silverstein (1997), Aharony, Sonnenschein, *et al.* (1997); Douglas, Lowe, and Schwarz (1997), Sen (1997b, 1997c).
- *Branes in Calabi-Yau backgrounds.* As should be clear from the last two items, there is a close connection between brane configurations and nontrivial string backgrounds. In general one may consider branes propagating in nontrivial backgrounds, such as Calabi-Yau compactifications. The branes may live at points in the internal space or wrap nontrivial cycles of the manifold.
  Such systems have been widely studied (for example, Bershadsky, Sadov, and Vafa, 1996a; Douglas and Li, 1996; Douglas and Moore, 1996; Ahn, 1998a; Ahn and Oh, 1997; Ahn, Oh, and Tatar, 1997; Ahn and Tatar, 1997; Bershadsky, Johanson *et al.*, 1997; Blum and Intriligator, 1997a, 1997b; Hori and Oz, 1997; Intriligator, 1997; Ooguri and Vafa, 1997; Vafa and Zwiebach, 1997, and references therein). In some limits, they are related by duality transformations to the webs of branes in flat space that are extensively discussed below (Ooguri and Vafa, 1996; Kutasov, 1996; Elitzur, Giveon, and Kutasov, 1997; Ooguri and Vafa, 1997). For example, a useful duality, which we shall review below, is the one relating the A-type singularity on $K3$ to a configuration of parallel solitonic five-branes.
- *Quantum mechanics of systems of D0-branes, D instantons, matrix theory.* The quantum mechanics of *D0*-branes in type-IIA string theory (in general in the presence of other branes and orientifolds) has led to fascinating developments which are outside the scope of this review (Bachas, Green, and Schwimmer, 1998; Banks, Seiberg, and Silverstein, 1997; Barbon and Pasquinucci, 1998a; Douglas *et al.*, 1997; Porrati and Rozenberg, 1997). Matrix theory was introduced by Banks *et al.* (1998); for reviews and additional references see Banks (1997); Bigatti and Susskind (1998). D instantons were studied, for example, by Green and Gutperle (1997) and Green and Vanhove (1997) and references therein.
- *Nonsupersymmetric theories.* It is easy to construct brane configurations in string theory that do not preserve any supersymmetry. So far, not much has been learned about nonsupersymmetric gauge theories by studying such configurations (for reasons that we shall explain). Recent discussions include those of Barbon and Pasquinucci, 1998b; Brandhuber, Sonnenschein, *et al.*, 1997a; Evans and Schwetz, 1998; Gomez, 1997; Witten, 1997b. Dynamical supersymmetry breaking in the brane picture has been considered recently by de Boer *et al.* (1998).

II. BRANES IN STRING THEORY

In addition to fundamental strings, in terms of which string theory is usually formulated, the theory contains other extended p-dimensional objects, known as p-branes, that play an important role in the dynamics.

[2]This subsection may be skipped on a first reading.

These objects can be divided into two broad classes according to their properties for weak fundamental string coupling g_s: (1) "solitonic" or Neveu-Schwarz (NS) branes, whose tension (energy per unit p volume) behaves like $1/g_s^2$, and (2) Dirichlet or D-branes, whose tension is proportional to $1/g_s$ (and which are hence much lighter than NS-branes in the $g_s \to 0$ limit).

In this section we describe some properties of the various branes. In supergravity, these p-branes are charged under certain massless $(p+1)$-form gauge fields. We start with a description of the low-energy effective theory corresponding to type-II strings in 10 dimensions as well as 11-dimensional supergravity, the low-energy limit of M theory. We then describe branes preserving half of the supersymmetry in weakly coupled string theory: D-branes, orientifold planes, and solitonic and Kaluza-Klein five-branes. We present the interpretation of the different branes from the point of view of the full quantum eleven-dimensional M theory, and their transformation properties under U duality. We finish the section with a discussion of webs of branes preserving less supersymmetry.

Our notations are as follows: the $(1+9)$-dimensional spacetime of string theory is labeled by $(x^0, x^1, \ldots, x^9)$. The tenth spatial dimension of M theory is x^{10}. The corresponding Dirac matrices are Γ^M, $M=0,1,2,\ldots,10$. Type-IIA string theory has (1,1) spacetime supersymmetry (SUSY); the spacetime supercharges generated by left- and right-moving worldsheet degrees of freedom Q_L, Q_R have opposite chirality:

$$\Gamma^0 \cdots \Gamma^9 Q_L = +Q_L,$$
$$\Gamma^0 \cdots \Gamma^9 Q_R = -Q_R. \tag{1}$$

Type-IIB string theory has (2,0) spacetime SUSY, with both left- and right-moving supercharges having the same chirality:

$$\Gamma^0 \cdots \Gamma^9 Q_L = Q_L,$$
$$\Gamma^0 \cdots \Gamma^9 Q_R = Q_R. \tag{2}$$

Thus type-IIA string theory is nonchiral, while the type-IIB theory is chiral. We shall focus mainly on type-II string theories, but $(1+9)$-dimensional theories with (1,0) SUSY can be similarly discussed. Type-I string theory can be thought of as type-II string theory with orientifolds and D-branes and is, therefore, a special case of the discussion below. Heterotic strings do not have D-branes, but do have NS-branes similar to those described below.

A. Low-energy supergravity

The spectrum of string theory contains a finite number of light particles and an infinite tower of massive excitations with string scale or higher masses. To make contact with low-energy phenomenology it is convenient to focus on the dynamics of the light modes. This can be achieved by integrating out the infinite tower of massive fluctuations of the string and defining a low-energy effective action for the light fields. While it has proven very difficult to formulate a string field theory, it is often helpful to think of string theory as an infinite collection of quantum fields ϕ. Some of these fields are light ϕ_l, and the rest are heavy ϕ_h. Assuming that they are governed by the classical action $S(\phi_l, \phi_h)$, the effective action for the light fields $S_{\rm eff}(\phi_l)$ can in principle be obtained by integrating out the heavy fields:

$$e^{iS_{\rm eff}(\phi_l)} = \int D\phi_h e^{iS(\phi_l, \phi_h)}. \tag{3}$$

In principle Eq. (3) is exact, but in practice it is far from clear how to find the action S and how to integrate out the massive modes of the string. At the same time, the effective action is mainly of interest at energies much lower than the masses of the fields ϕ_h. To find $S_{\rm eff}$ at low energies one can study the S matrix of the string in the low-energy approximation and construct a classical action that reproduces it. The leading terms in such an action are typically determined by the symmetries, such as gauge and diffeomorphism invariance, and supersymmetry.

Following the above discussion for type-II string theory leads to the two $(9+1)$-dimensional type-II supergravity theories, type-IIA and type-IIB. Ten-dimensional type-IIA supergravity can be obtained by dimensional reduction of the unique 11-dimensional supergravity theory, which is of interest in its own right as the low-energy limit of M theory; thus we start with this case.

Eleven-dimensional supergravity includes the bosonic (i.e., commuting) fields G_{MN}, the 11-dimensional metric, and A_{MNP}, a three-index antisymmetric gauge field $(M,N,P=0,1,\ldots,10)$. The only fermionic field is the gravitino, ψ_α^M $(\alpha=1,\ldots,32)$. The Lagrangian describing these fields can be found in the article of Green, Schwarz, and Witten (1987). One can check that there are 128 on-shell bosonic and fermionic degrees of freedom.

The presence of the three-index gauge field A_{MNP} implies that 11-dimensional supergravity couples naturally to membranes and to five-branes. For a membrane with world-volume $X^M(\sigma_a)$, $(a=1,2,3)$, the coupling is (see Bergshoeff, Sezgin, and Townsend, 1988 for a discussion of the full supermembrane worldvolume action)

$$\int d^3\sigma \epsilon^{abc} A_{MNP}(X) \partial_a X^M \partial_b X^N \partial_c X^P. \tag{4}$$

This coupling is the analog of the coupling $\int dx^M A_M$ of a gauge field to a particle, or of a second-rank antisymmetric tensor B_{MN} to a string. Just as these couplings imply that the particle or string carries electric charge or the charge connected with B_{MN}, Eq. (4) implies that the membrane of 11-dimensional supergravity is charged under the three-form gauge field A_{MNP}. The coupling of 11-dimensional supergravity to five-branes is similar, with A_{MNP} replaced by its dual $\tilde{A}_{MNPQRS}$ defined by $*dA = d\tilde{A}$.

Type-IIA supergravity is obtained by dimensional reduction of 11-dimensional supergravity on a circle. Denoting the $(1+9)$-dimensional indices by $\mu,\nu,\lambda=0,1,\ldots,9$, we find that the fields of 11-dimensional supergravity reduce as follows in this limit. The metric G_{MN} gives rise to the metric $G_{\mu\nu}$, a gauge field $A_\mu=G_{\mu,10}$, and a scalar $\Phi=G_{10,10}$. The antisymmetric tensor A_{MNP} similarly gives rise to the antisymmetric tensors $A_{\mu\nu\lambda}$ and $B_{\mu\nu}=A_{\mu\nu,10}$. In the standard Neveu-Schwarz-Ramond quantization of superstrings (Green, Schwarz, and Witten, 1987), the fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ originate in the same sector of the string Hilbert space, the Neveu-Schwarz (or NS) sector, while the gauge fields A_μ and $A_{\mu\nu\lambda}$ are Ramond-Ramond (RR) sector fields. The scalar field Φ is the dilaton; its expectation value determines the coupling constant of the string theory. Since the potential for Φ in type-II string theory vanishes, the theory can be made arbitrarily weakly coupled.

Just as in Eq. (4), the existence of the gauge fields implies that type-II string theory naturally couples to various p-branes. The existence of $B_{\mu\nu}$ means that the theory naturally couples to strings [electrically, as in Eq. (4)] and five-branes (magnetically, via the six-form gauge field dual to $B_{\mu\nu}$). Since the gauge field to which these branes couple is an NS sector field we refer to these branes as NS-branes. The string charged under $B_{\mu\nu}$ is simply the fundamental string that is used to define type-II string theory, while the five-brane is the NS5-brane studied by Callan, Harvey, and Strominger (1991a, 1991b, 1991c).

The gauge fields A_μ and $A_{\mu\nu\lambda}$ couple electrically to zero-branes (particles) and membranes and magnetically to six-branes and four-branes, respectively. Since the corresponding gauge fields originate in the RR sector, these branes are sometimes referred to as Ramond-branes (or D-branes; see below).

Type-IIB supergravity has (2,0) chiral supersymmetry. The massless spectrum contains again the NS sector fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ and the associated NS string and five-brane. The spectrum of RR p-form gauge fields is different from the type-IIA case. There is an additional scalar χ, which combines with Φ into a complex coupling of type-IIB string theory. The antisymmetric tensors one finds have two and four indices, $\tilde{B}_{\mu\nu}$, $A_{\mu\nu\lambda\rho}$. The existence of the former implies that the theory can couple to another set of strings and five-branes, the D string and $D5$-brane. The four-form A is self dual $*dA=dA$; it couples to a three-brane.

In what follows we discuss some properties of the various branes mentioned above. We begin with a description of their construction and properties in weakly coupled string theory.

B. Branes in weakly coupled string theory

1. D-branes

In weakly coupled type-II string theory, D-branes are defined by the property that fundamental strings can end on them (Polchinski, 1996; Polchinski, Chaudhuri, and Johnson, 1996). A Dirichlet p-brane (Dp-brane) stretched in the $(x^1,\ldots,x^p)$ hyperplane, located at a point in $(x^{p+1},\ldots,x^9)$, is defined by including in the theory open strings with Neumann boundary conditions for $(x^0,x^1,\ldots,x^p)$ and Dirichlet boundary conditions for $(x^{p+1},\ldots,x^9)$ (see Fig. 1).

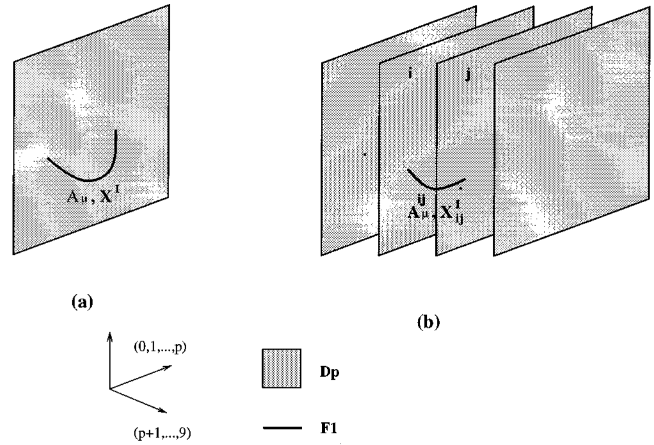

FIG. 1. Low-lying states of fundamental strings with both ends on D-branes: (a) a $U(1)$ gauge field and $9-p$ scalars living on a single D-brane, or (b) a $U(N_c)$ gauge field and $9-p$ adjoint scalars on a stack of N_c D-branes.

The Dp-brane is charged under a Ramond-Ramond (RR) $(p+1)$-form potential of the type-II string. As we saw, in type-IIA string theory there are such potentials with even p and, therefore, Dp-branes with $p=0,2,4,6,8$. Similarly, in type-IIB string theory there are potentials with odd p and Dp-branes with $p=-1,1,3,5,7,9$. The $p=-1$ brane is the D instanton, while the $p=9$ brane is the $D9$-brane that fills the whole $(9+1)$-dimensional spacetime and together with the orientifold (to be described below) turns a type-IIB string into a type-I one. The $D7$-brane is the "magnetic dual" of the D instanton; the $D8$-brane together with the orientifold turns a type-IIA string into a type-I′ one.

The tension of a Dp-brane is

$$T_p=\frac{1}{g_s l_s^{p+1}}, \tag{5}$$

where l_s is the fundamental string scale (the tension of the fundamental string is $T=1/l_s^2$). The Dp-brane tension (5) is equal to its RR charge; D-branes are BPS-saturated objects preserving half of the thirty-two supercharges of type-II string theory. More precisely, a Dp-brane stretched along the $(x^1,\ldots,x^p)$ hyperplane preserves supercharges of the form $\epsilon_L Q_L+\epsilon_R Q_R$ with

$$\epsilon_L=\Gamma^0\Gamma^1\cdots\Gamma^p\epsilon_R. \tag{6}$$

An anti-Dp-brane carries the opposite RR charge and preserves the other half of the supercharges. Equation (6) can be thought of as arising from the presence in the theory of open strings that end on the branes. In the presence of such open strings the left- and right-moving supercharges Q_L, Q_R(1,2) are not independent; Eq. (6) describes the reflection of right-to-left movers at the boundary of the worldsheet, which is confined to the brane.

The low-energy worldvolume theory on an infinite Dp-brane is a $(p+1)$-dimensional field theory invariant under sixteen supercharges. It describes the dynamics of the ground states of open strings both of whose end points lie on the brane [Fig. 1(a)]. The massless spectrum includes a $(p+1)$-dimensional $U(1)$ gauge field $A_\mu(x^\nu)$, $(9-p)$ scalars $X^I(x^\mu)$ $(I=p+1,\ldots,9,\ \mu=0,\ldots,p)$ parametrizing fluctuations of the Dp-brane in the transverse directions, and fermions required for supersymmetry.[3] The low-energy dynamics can be obtained by dimensional reduction of $N=1$ SYM theory with gauge group $G=U(1)$ from $(9+1)$ to $(p+1)$ dimensions. The bosonic part of the low-energy worldvolume action is

$$S=\frac{1}{g_{SYM}^2}\int d^{p+1}x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{l_s^4}\partial_\mu X^I\partial^\mu X_I\right). \tag{7}$$

The $U(1)$ gauge coupling on the brane g_{SYM} is given by

$$g_{SYM}^2=g_s l_s^{p-3}. \tag{8}$$

The g_s dependence in Eqs. (5) and (8) follows from the fact that the kinetic term (7) arises from open-string tree level (the disk), while the power of the string length l_s is fixed by dimensional analysis.

At high energies, the massless degrees of freedom (7) interact with an infinite tower of "open-string" states localized on the brane, and with closed strings in the $(9+1)$-dimensional bulk of spacetime. To study SYM theory on the brane one needs to decouple the gauge-theory degrees of freedom from gravity and massive string modes. To achieve that one can send $l_s\to 0$ holding g_{SYM} fixed. This means (8) $g_s\to 0$ for $p<3$, $g_s\to\infty$ for $p>3$. For $p=3$ g_{SYM} is independent of l_s and the $l_s\to 0$ limit describes $N=4$ SYM theory in $(3+1)$ dimensions. Note that for $p\leqslant 3$ the above limit leads to a consistent theory on the brane, whose UV behavior is just that of $(p+1)$-dimensional SYM theory. For $p>3$, SYM theory provides a good description in the infrared, but it must break down at high energies.

Since D-branes are BPS-saturated objects, parallel branes do not exert forces on each other. The low-energy worldvolume dynamics on a stack of N_c nearby parallel Dp-branes [Fig. 1(b)] is a SYM theory with gauge group $U(N_c)$ and sixteen supercharges, arising from ground states of open strings whose end points lie on the branes (Polchinski, 1994; Witten, 1996a). The scalars X^I (7) turn into $N_c\times N_c$ matrices transforming in the adjoint representation of the $U(N_c)$ gauge group. The N_c photons in the Cartan subalgebra of $U(N_c)$ and the diagonal components of the matrices X^I correspond to open strings both of whose end points lie on the same brane. The charged gauge bosons and off-diagonal components of X^I correspond to strings whose end points lie on different branes. Specifically, the (i,j), (j,i) elements of X^I, A_μ arise from the two orientations of a fundamental string connecting the ith and jth branes [see Fig. 1(b)].

The generalization of Eq. (7) to the case of N_c parallel Dp-branes is described by dimensional reduction of $N=1$ SYM theory with gauge group $G=U(N_c)$ from $(9+1)$ to $(p+1)$ dimensions. The bosonic part of the $(9+1)$-dimensional low-energy Lagrangian,

$$\mathcal{L}=\frac{1}{4g_{SYM}^2}\mathrm{Tr}\,F_{mn}F^{mn};\quad m,n=0,1,\ldots,9,$$
$$F_{mn}=\partial_m A_n-\partial_n A_m-i[A_m,A_n], \tag{9}$$

gives rise upon dimensional reduction to kinetic terms for the $(p+1)$-dimensional gauge field A_μ and adjoint scalars X^I,

$$\mathcal{L}_{\text{kin}}=\frac{1}{g_{SYM}^2}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{l_s^4}\mathcal{D}_\mu X^I\mathcal{D}^\mu X_I\right) \tag{10}$$

(here $\mathcal{D}_\mu X^I=\partial_\mu X^I-i[A_\mu,X^I]$; $F_{\mu\nu}=\partial_{[\mu}A_{\nu]}-i[A_\mu,A_\nu]$), and to a potential for the adjoint scalars X^I,

$$V\sim\frac{1}{l_s^8 g_{SYM}^2}\sum_{I,J}\mathrm{Tr}[X^I,X^J]^2. \tag{11}$$

Flat directions of the potential (11) corresponding to diagonal X^I (up to gauge transformations) parametrize the Coulomb branch of the $U(N_c)$ gauge theory. The moduli space of vacua is $(R^{9-p})^{N_c}/S_{N_c}$; it is parametrized by the eigenvalues of $\vec{X}$,

$$\vec{x}_i=\langle\vec{X}_{ii}\rangle;\quad i=1,\ldots,N_c, \tag{12}$$

which label the transverse locations of the N_c branes. The permutation group S_{N_c} acts on $\vec{x}_i$ as the Weyl group of $SU(N_c)$. For generic positions of the N_c-branes, the off-diagonal components of X^I as well as the charged gauge bosons are massive (and the gauge symmetry is broken, $U(N_c)\to U(1)^{N_c}$). Their masses are read off from Eqs. (10)–(12):

$$m_{ij}=\frac{1}{l_s^2}|\vec{x}_i-\vec{x}_j|. \tag{13}$$

Geometrically Eq. (13) can be thought of as the minimal energy of a fundamental string stretched between the ith and jth branes [Fig. 1(b)]. When n of the N_c-branes coincide, some of the charged particles become massless (13) and the gauge group is enhanced from $U(1)^{N_c}$ to $U(n)\times U(1)^{N_c-n}$.

2. Orientifolds

An orientifold p-plane (Op-plane) is a generalization of a Z_2 orbifold fixed plane to nonoriented string theories (Polchinski, 1996; Polchinski, Chaudhuri, and Johnson, 1996). It can be thought of as the fixed plane under a Z_2 symmetry which acts on the spacetime coor-

[3] We shall usually ignore the fermions below. Their properties can be deduced by imposing supersymmetry.

dinates and reverses the orientation of the string. The fixed plane of the Z_2 transformation[4]

$$x^I(z,\bar{z}) \leftrightarrow -x^I(\bar{z},z); \quad I=p+1,\ldots,9 \tag{14}$$

is an Op-plane extending in the $(x^1,\ldots,x^p)$ directions and time.

Like the usual orbifold fixed planes, the orientifold is not dynamical (at least at weak string coupling). It carries charge under the same RR $(p+1)$-form gauge potential and breaks the same half of the supersymmetry as a parallel Dp-brane. In the presence of an Op-plane, the transverse space R^{9-p} is replaced by R^{9-p}/Z_2. It is convenient to continue to describe the geometry as R^{9-p}, add a Z_2 image for each object lying outside the fixed plane, and implement an appropriate (anti-) symmetrization on the states. Thus D-branes that are outside the orientifold plane acquire mirror images (see Fig. 2). At the fixed plane one can sometimes have a single D-brane that does not have a Z_2 partner and hence cannot leave the singularity. The RR charge of an Op-plane Q_{Op} is equal (up to a sign) to that of 2^{p-4} Dp-branes (or 2^{p-5} pairs of a Dp-brane and its mirror). Denoting the RR charge of a Dp-plane by Q_{Dp}, the orientifold charge is

$$Q_{Op} = \pm 2 \cdot 2^{p-5} Q_{Dp} \tag{15}$$

(this will be further discussed later). The (anti-) symmetric projection imposed on D-branes by the presence of an orientifold plane leads to changes in their low-energy dynamics. On a stack of N_c Dp-branes parallel to an Op-plane one finds a gauge theory with 16 supercharges and the following rank $[N_c/2]$ gauge group[5] G:

- $Q_{Op} = +2 \cdot 2^{p-5} Q_{Dp}$, $\quad N_c$ even: $G = Sp(N_c/2)$.
- $Q_{Op} = -2 \cdot 2^{p-5} Q_{Dp}$: $G = SO(N_c)$.

The light matter consists of the ground states of open strings stretched between different D-branes, giving rise to a gauge field for the group G and $9-p$ scalars X^I in the adjoint of G. Positive orientifold charge gives rise to a symmetric projection on the $N_c \times N_c$ matrices A_μ, X^I and therefore a symplectic gauge group (N_c must be even in that case; as is clear from Eq. (14), for the case of a symmetric projection it is impossible to have a D-brane without an image stuck at the orientifold), while negative orientifold charge leads to an antisymmetric projection and to orthogonal gauge groups.

Geometrically, $(N_c^2 \pm N_c)/2$ of the N_c^2 oriented strings stretched between the N_c Dp-branes survive the (anti-) symmetric projection due to the orientifold. The difference of N_c between the symmetric and antisymmetric cases corresponds to strings stretching between a Dp-brane and its mirror image. These strings transform to themselves under the combined worldsheet and spacetime reflection (14); thus they are projected out in the antisymmetric case and give $2 \times N_c/2$ massless modes in the symmetric case.

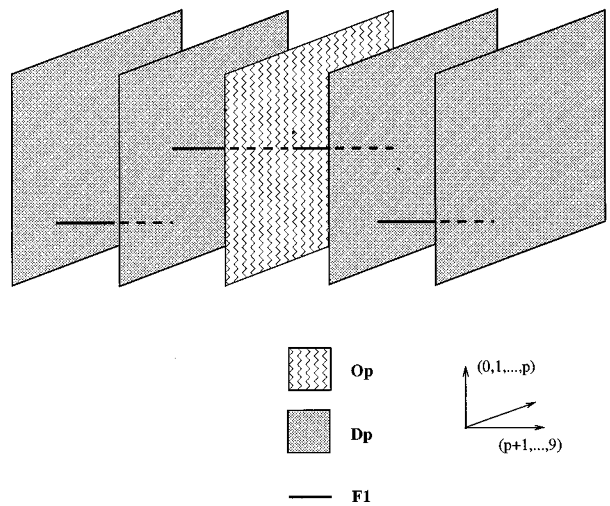

FIG. 2. An orientifold p-plane with two adjacent parallel Dp-branes and their mirror images. Fundamental strings stretched between a D-brane and its image are projected out for negative orientifold charge. Others come in mirror pairs.

Since branes can only leave the orientifold plane in pairs, there are $[N_c/2]$ "dynamical branes" that are free to move. Their locations in the transverse space R^{9-p} parametrize the Coulomb branch of the theory. The $[N_c/2]$ photons in the Cartan subalgebra of G and the scalars parametrizing the Coulomb branch correspond to open strings both of whose end points lie on the same brane. When n of the $[N_c/2]$ Dp-branes coincide outside the orientifold plane the gauge symmetry is enhanced from $U(1)^{[N_c/2]}$ to $U(n) \times U(1)^{[N_c/2]-n}$. When m of the N_c branes coincide with the orientifold plane the gauge group is enhanced to $[SO(m)$ or $Sp(m/2)]$ $\times U(1)^{[(N_c-m)/2]}$.

For high-dimensional orientifolds and D-branes the discussion above has to be slightly modified. In particular, for $p \geqslant 7$ the rank of the gauge group G is bounded since RR flux does not have enough noncompact transverse directions to escape, and therefore the total RR charge must vanish. The case $p=9$ is further special, since there are no transverse directions at all and the reflection (14) acts only on the worldsheet. The requirements that the total RR charge vanish and the orientifold charge $Q_{O9} = -32$ [see Eq. (15)] are in this case directly related to the fact that the gauge group of ten-dimensional type-I string theory is $SO(32)$. The p dependence in Eq. (15) will be discussed in Sec. II.D.

3. The solitonic five-brane

The solitonic five-brane (Callan, Harvey, and Strominger, 1991a, 1991b, 1991c) that exists in weakly coupled type-II and heterotic string theory, is a BPS-saturated object which, like the Dirichlet brane, preserves half of the supersymmetry of the theory and has tension

$$T_{NS} = \frac{1}{g_s^2 l_s^6}. \tag{16}$$

[4] z, $\bar{z}$ parametrize the string worldsheet; $z = \exp(\tau + i\sigma)$.

[5] Our conventions are $Sp(1) \simeq SU(2)$.

It couples magnetically to the NS-NS sector $B_{\mu\nu}$ field and can thus be thought of as a magnetic dual of the fundamental type-II or heterotic string. Since its tension is proportional to $1/g_s^2$ it provides a nontrivial background for a fundamental string in leading order in g_s (i.e., on the sphere). A fundamental string propagating in the background of k parallel NS five-branes located at transverse positions $\vec{x}_i$ is described by a conformal field theory with nontrivial G_{IJ}, B_{IJ}, Φ (metric, antisymmetric tensor, and dilaton) given by

$$e^{2(\Phi-\Phi_0)}=1+\sum_{j=1}^{k}\frac{l_s^2}{|\vec{x}-\vec{x}_j|^2},$$
$$G_{IJ}=e^{2(\Phi-\Phi_0)}\delta_{IJ};\quad G_{\mu\nu}=\eta_{\mu\nu},$$
$$H_{IJK}=-\epsilon_{IJKM}\partial^M\Phi. \tag{17}$$

I,J,K,M label the four directions transverse to the five-brane; μ,ν label the $(5+1)$ longitudinal directions. H is the field strength of B; Φ_0 is the value of the dilaton at infinity, related to the string coupling at infinity $g_s=\exp\Phi_0$. As is clear from Eq. (17), the effective string coupling $\exp(\Phi)$ depends on the distance from the five-brane, diverging at the core.

An NS five-brane stretched in the $(x^1,\dots,x^5)$ hyperplane preserves supercharges of the form $\epsilon_L Q_L+\epsilon_R Q_R$, where for the type-IIA five-brane

$$\epsilon_L=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_L,$$
$$\epsilon_R=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_R, \tag{18}$$

while for the type-IIB five-brane

$$\epsilon_L=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_L,$$
$$\epsilon_R=-\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_R. \tag{19}$$

Thus the nonchiral type-IIA string theory gives rise to a chiral five-brane worldvolume theory with (2,0) supersymmetry in six dimensions, while the chiral type-IIB theory gives rise to a nonchiral five-brane with (1,1) worldvolume supersymmetry. Equations (18) and (19) can be established by a direct analysis of the supercharges preserved by the background (17). As we shall see later, string duality relates them to the supercharges preserved by D-branes (6), and both have a natural origin in 11 dimensions.

The light fields on the worldvolume of a single type-IIA NS five-brane correspond to a tensor multiplet of six-dimensional (2,0) supersymmetry, consisting of a self-dual $B_{\mu\nu}$ field and five scalars (and the fermions needed for supersymmetry). On a single type-IIB five-brane one finds a vector multiplet, i.e., a six-dimensional gauge field and four scalars (+ fermions).

The four scalars in the vector multiplet on the type-IIB five-brane, as well as four of the five scalars in the tensor multiplet on the type-IIA five-brane, describe fluctuations of the NS-brane in the transverse directions. The fifth scalar on a type-IIA five-brane lives on a circle of radius l_s and provides a hint of a hidden 11th dimension of quantum type-IIA string theory (more on this below).

The gauge coupling of the vector field on the type-IIB five-brane is[6]

$$g_{SYM}^2=l_s^2. \tag{20}$$

Since NS-branes are BPS-saturated objects, parallel branes do not exert forces on each other. The low-energy worldvolume dynamics on a stack of k parallel type-IIB $NS5$-branes is a $(5+1)$-dimensional (1,1) $U(k)$ SYM theory (with 16 supercharges), arising from the ground states of D strings stretched between different NS-branes. It is described by Eqs. (10), (11), and (20) with $p=5$. As for D-branes, the four scalars in the vector multiplet are promoted to $k\times k$ matrices, whose diagonal components parametrize the Coulomb branch of the theory, R^{4k}/S_k.

The low-energy theory describing a stack of k parallel type-IIA NS5-branes is more exotic. It can be thought of as a non-Abelian generalization of the free theory of a tensor multiplet on a single NS5-brane and gives rise to a nontrivial field theory with (2,0) supersymmetry in $(5+1)$ dimensions (Strominger, 1996; Witten, 1995b; Seiberg, 1997b). It contains stringlike low-energy excitations corresponding to Dirichlet membranes stretched between the different NS5-branes. These strings are charged under the self-dual $B_{\mu\nu}$ fields on the corresponding five-branes and are light when the five-branes are close to each other. The Coulomb branch of the (2,0) theory, $(R^4\times S^1)^k/S_k$, is parametrized by the expectation values of the diagonal components of the five scalars in the tensor multiplet. At the origin of the Coulomb branch, the (2,0) field theory corresponds to a nontrivial superconformal field theory.

In the limit $g_s\to 0$ the dynamics of the full type-II string theory simplifies and, in particular, all the modes in the bulk of spacetime (including gravity) decouple. The dynamics of a type-II string vacuum with k NS five-branes remains nontrivial in the limit; in the type-IIA case it is described at low energies by the (2,0) field theory described above. The theory of k type-IIB five-branes has (1,1) supersymmetry and reduces at low energies to the (infrared-free) $U(k)$ SYM theory; at finite energies it is interacting. Providing a useful description of the five-brane theory and, in particular, of the low-energy (2,0) field theory of the type-IIA five-branes remains a major challenge as of this writing.

4. The Kaluza-Klein monopole

Compactified type-II string theory has additional solitonic objects. One that will be particularly useful later is the Kaluza-Klein monopole, which is a five-brane in ten dimensions (Duff, Khuri, and Lu, 1995; Townsend, 1997). It is obtained when one of the ten directions, call

[6]Since the NS five-brane is described by a conformal field theory on the sphere, one might have expected the gauge coupling to go like $g_{SYM}^2\simeq g_s^2 l_s^2$ [in analogy to Eq. (8)]. The form (20) is obtained by taking into account the fact that the worldvolume gauge field is a Ramond-Ramond field in the five-brane conformal field theory.

it ρ, is compactified on a circle of radius R. The ten-dimensional graviton gives rise in nine dimensions to a gauge field $A_a = G_{a,\rho}$ $(a=0,\cdots,8)$. The Kaluza-Klein monopole carries magnetic charge R/l_s under this gauge field. Like the monopole of $(3+1)$-dimensional gauge theory it is localized in three additional directions $\vec{r}$ and is extended in the remaining five.

The tension of the Kaluza-Klein five-brane is

$$T_{KK} = \frac{R^2}{g_s^2 l_s^8}. \tag{21}$$

The factor of $1/g_s^2$ is due to the fact that, like the Neveu-Schwarz five-brane, the Kaluza-Klein five-brane "gets its tension" from the sphere (i.e., it is a "conventional soliton"). The other factors in Eq. (21) are the square of the magnetic charge and a $1/l_s^6$ due to the fact that this is a five-brane.

A fundamental string in the background of k parallel Kaluza-Klein monopoles located at transverse positions $\vec{r}_i$ is described by a conformal field theory with the multi-Taub-NUT metric $(B=\Phi=\text{const})$:

$$ds^2 = dx^\mu dx_\mu + ds_\perp^2,$$

$$ds_\perp^2 = U d\vec{r}^2 + U^{-1}(d\rho + \vec{\omega}\cdot d\vec{r})^2 \tag{22}$$

where x^μ label the $(1+5)$-longitudinal directions,

$$U = 1 + \sum_{j=1}^{k} \frac{R}{2|\vec{r}-\vec{r}_j|}, \tag{23}$$

and $\vec{w}$ is the multi-Dirac-monopole vector potential which satisfies

$$\vec{\nabla}\times\vec{\omega} = \vec{\nabla} U. \tag{24}$$

In the limit $R\to\infty$ this background becomes an ALE space with A_{k-1} singularity. On the other hand, in the $R\to 0$ limit the multi-Taub-NUT background (22)–(24) is T dual [in the ρ direction and in an appropriate sense (Gregory, Harvey, and Moore, 1997)] to the multi-NS five-brane solution [Eq. (17); more on T duality later].

C. *M*-theory interpretation

All the different ten-dimensional string theories can be thought of as asymptotic expansions around different vacua of a single quantum theory. This theory, known as "M theory," is in fact $(1+10)$ dimensional at almost all points in its moduli space of vacua [for a review see, for example, Schwarz (1997a), Townsend (1997), and references therein].

In the flat $(1+10)$-dimensional Minkowski vacuum the theory reduces at low energies to 11-dimensional supergravity. There is no adjustable dimensionless coupling; the only parameter in the theory is the 11-dimensional Planck scale l_p. Physics is weakly coupled and well approximated by semiclassical supergravity for length scales much larger than l_p. It is strongly coupled at scales smaller than l_p. The spectrum includes a three-form potential A_{MNP} $(M,N,P=0,1,\ldots,10)$ whose electric and magnetic charges appear as central extensions in the 11-dimensional superalgebra,

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C)_{\alpha\beta} P_M + \frac{1}{2}(\Gamma_{MN} C)_{\alpha\beta} Z^{MN} + \frac{1}{5!}(\Gamma_{MNPQR} C)_{\alpha\beta} Y^{MNPQR}, \tag{25}$$

where $\Gamma_{MN\cdots}$ are antisymmetrized products of the 32×32 Dirac matrices in 11 dimensions, C is the (real, antisymmetric) charge-conjugation matrix, Z^{MN} is the electric charge corresponding to A_{MNP}, and Y^{MNPQR} is the corresponding magnetic charge.[7]

A solitonic M-theory membrane/five-brane $(M2/M5)$ carries electric/magnetic charge Z/Y and breaks half of the 32 supercharges Q (25). An Mp-brane $(p=2,5)$ stretched in the $(x^1,\ldots,x^p)$ directions preserves the supercharges ϵQ with

$$\Gamma^0\Gamma^1\cdots\Gamma^p\epsilon = \epsilon. \tag{26}$$

Its tension is fixed by supersymmetry to be $T_p = 1/l_p^{p+1}$. Large charge branes can be reliably described by 11-dimensional supergravity. The metric around a collection of k Mp-branes located at $\vec{r}=\vec{r}_j$ [where $j=1,\ldots,k$; $\vec{r},\vec{r}_j$ are $(10-p)$-dimensional vectors] is given by

$$ds^2 = U^{-1/3} dx^\mu dx_\mu + U^{2/3} d\vec{r}\cdot d\vec{r}, \tag{27}$$

where x^μ are the $p+1$ directions along the brane, and

$$U = 1 + \sum_{j=1}^{k} \frac{l_p^{8-p}}{|\vec{r}-\vec{r}_j|^{8-p}} \tag{28}$$

and there is also a three-index tensor field that we do not specify.

The ten-dimensional type-IIA vacuum with string coupling g_s can be thought of as a compactification of M theory on $R^{1,9}\times S^1$. Denoting the $(1+9)$-dimensional Minkowski space of type-IIA string theory by $(x^0,x^1,\ldots,x^9)$, and the compact direction by x^{10}, the compactification radius R_{10} and l_p are related to the type-IIA parameters g_s, l_s by

$$\frac{R_{10}}{l_p^3} = \frac{1}{l_s^2}, \tag{29}$$

$$R_{10} = g_s l_s. \tag{30}$$

Thus the strong-coupling limit of type-IIA string theory $g_s\to\infty$ (or equivalently $R_{10}/l_p\to\infty$) is described by the $(1+10)$-dimensional Minkowski vacuum of M theory.

Type-IIA branes have a natural interpretation in M theory:

- A fundamental type-IIA string stretched (say) along x^1 can be thought of as an $M2$-brane wrapped around x^{10} and x^1. It is charged under the gauge field $B_{\mu 1}$

[7] In noncompact space, only the charge per unit volume is finite. Thus Z, Y are best thought of as providing charge densities.

$=A_{10\mu 1}$. Equation (29) is the relation between the wrapped membrane and string tensions.

- The $D0$-brane corresponds to a Kaluza-Klein mode of the graviton carrying momentum $1/R_{10}$ along the compact direction. It is electrically charged under $G_{\mu,10}$. Equation (30) relates the masses of the Kaluza-Klein mode of the graviton and $D0$-brane.
- The $D2$-brane corresponds to a "transverse" $M2$-brane, unwrapped around x^{10}. It is charged under $A_{\mu\nu\lambda}$. The tension of the $M2$-brane $1/l_p^3$ reduces to Eq. (5) using the relation

$$l_p^3=l_s^3 g_s, \tag{31}$$

which follows from Eqs. (29) and (30).

- The $D4$-brane corresponds to an $M5$-brane wrapped around x^{10}. It is charged under the five-form gauge field $\tilde{A}_{10\mu_1\mu_2\cdots\mu_5}$ dual to A $(d\tilde{A}=*dA)$. Its tension (5) is equal to R_{10}/l_p^6 (31).
- The NS5-brane corresponds to a transverse $M5$-brane and is thus charged under $\tilde{A}_{\mu_1\cdots\mu_6}$. Its tension (16) is equal to $1/l_p^6$.
- The $D6$-brane is a Kaluza-Klein monopole. It is magnetically charged under the gauge field $A_\mu=G_{\mu 10}$.
- The $D8$-brane is a mysterious object in M theory whose tension is known to be R_{10}^3/l_p^{12} (Elitzur, Giveon, *et al.*, 1998a).

All this can be summarized by decomposing the representations of $SO(10,1)$ appearing in Eq. (25) into representations of $SO(9,1)$ and rewriting the supersymmetry algebra (25) as

$$\begin{aligned}\{Q_\alpha,Q_\beta\}=&(C\Gamma^\mu)_{\alpha\beta}P_\mu+(C\Gamma^{10})_{\alpha\beta}P_{10}\\&+(C\Gamma^\mu\Gamma^{10})_{\alpha\beta}Z_\mu+\frac{1}{2}(C\Gamma^{\mu\nu})_{\alpha\beta}Z_{\mu\nu}\\&+\frac{1}{4!}(C\Gamma^{\mu\nu\rho\sigma}\Gamma^{10})_{\alpha\beta}Y_{\mu\nu\rho\sigma}\\&+\frac{1}{5!}(C\Gamma^{\mu\nu\rho\sigma\lambda})_{\alpha\beta}Y_{\mu\nu\rho\sigma\lambda},\end{aligned} \tag{32}$$

where $(9+1)$-dimensional vector indices are denoted by $\mu,\nu,\rho,\sigma,\dots$. The momentum in the 11th direction P_{10} is reinterpreted in ten dimensions as zero-brane charge; the spatial components of Z_μ are carried by "fundamental" type-IIA strings. Similarly, $Z_{\mu\nu}$ is the $D2$-brane charge, $Y_{\mu\nu\rho\sigma}$ is the $D4$-brane charge, and $Y_{\mu\nu\rho\sigma\lambda}$ is carried by NS5-branes. The different preserved supersymmetries Eqs. (6) and (18) combine in 11 dimensions into the single relation (26). Note that Eq. (32) includes central charges for p-branes with $p\leqslant 5$. Higher branes (e.g., the $D6$-brane) are inherently tied to compactification; therefore the corresponding central charges have to be added to Eq. (32) by hand.

We mentioned above that the scalar X^{10} describing fluctuations of the type-IIA five-brane in x^{10} lives on a circle of radius l_s. From the point of view of compactified M theory it is clear that the scalar field X^{10} lives on a circle of radius proportional to R_{10}; the proportionality constant is determined for a canonically normalized X^{10} by dimensional analysis to be $1/l_p^3$ as scalars in 5 $+1$ dimensions have scaling dimension two. Using Eq. (29) we arrive at the conclusion that the radius of (canonically normalized) X^{10} is $R_{10}/l_p^3=1/l_s^2$. In the normalization used in Eq. (7), with $g_{SYM}=l_s$ (20), X has dimensions of length and lives on a circle of radius l_s.

The metric around an $M5$-brane transverse to x^{10} [Eqs. (27) and (28)] goes over to that around the NS5-brane (17) as $R_{10}\to 0$. To see this, describe an $M5$-brane at $x^{10}=0$ on the circle as an infinite stack of parallel five-branes located at $x^{10}=nR_{10}$ $(n=0,\pm 1,\pm 2,\dots)$. The harmonic function U (28) is

$$U=1+\sum_n\left[\frac{l_p^2}{|\vec{x}|^2+(nR_{10})^2}\right]^{3/2}. \tag{33}$$

As $R_{10}\to 0$ one can replace the sum by an integral and Eq. (33) approaches [using Eq. (29)]

$$U\simeq 1+l_s^2/|\vec{x}|^2. \tag{34}$$

The component of the metric $G_{10,10}=U^{2/3}$ (27) is related to the ten-dimensional dilaton via $G_{10,10}\equiv\exp(2\gamma)$ $=\exp(4\phi/3)$. The string metric $\mathcal{G}$ is related to the 11-dimensional metric G by a rescaling $\mathcal{G}=G\exp\gamma$. Performing the rescaling leads to the ten-dimensional form (17).

Ten-dimensional type-IIB string theory has a complex coupling,

$$\tau=a+\frac{i}{g_s}, \tag{35}$$

where a is the expectation value of the massless Ramond-Ramond scalar. In the 11-dimensional interpretation, the ten-dimensional type-IIB vacuum corresponds to M theory compactified on a two-torus of complex structure τ and vanishing area. Naively, the theory appears to be $(1+8)$ dimensional in this limit, but in fact as the size of the torus goes to zero, the wrapping modes of the $M2$-brane become light and give rise to another noncompact direction which we shall label by x^B.

M theory on a finite two-torus corresponds to compactifying x^B on a circle of radius R_B. In the special case $a=0$, the M-theory two-torus is rectangular with sides R_9,R_{10}. The mapping of the M-theory parameters (R_9,R_{10},l_p) to the type-IIB ones (R_B,g_s,l_s) is

$$\frac{R_{10}}{l_p^3}=\frac{1}{l_s^2}, \tag{36}$$

$$\frac{R_9}{l_p^3}=\frac{1}{g_s l_s^2}, \tag{37}$$

$$\frac{R_9R_{10}}{l_p^3}=\frac{1}{R_B}. \tag{38}$$

One way to establish Eqs. (36)–(38) is to reinterpret the different type-IIB branes in M theory:

- A fundamental type-IIB string can be thought of as an $M2$-brane wrapped around x^{10}. Equation (36) is the relation between the membrane and string tensions.
- A $D1$-brane (D string) that is not wrapped around x^B corresponds to an $M2$-brane wrapped around x^9. Equation (37) is the relation between the membrane and D-string tensions. A D string wrapped around x^B corresponds to a Kaluza-Klein mode of the 11-dimensional supergraviton carrying momentum in the x^{10} direction. For example, using Eq. (36) and the relation

$$\frac{1}{l_p^3}=\frac{R_B}{g_s l_s^4}, \tag{39}$$

which follows from Eqs. (36) and (38), we find that the masses agree: $1/R_{10}=R_B/g_s l_s^2$.

- A Kaluza-Klein mode of the supergraviton carrying momentum in the x^B direction in type-IIB string theory corresponds to an $M2$-brane wrapped around (x^9,x^{10}); Eq. (38) relates the masses of the two.
- A $D3$-brane unwrapped around x^B corresponds to an $M5$-brane wrapped on (x^9,x^{10}). The tension of the wrapped $M5$-brane R_9R_{10}/l_p^6 reduces to Eq. (5) using Eqs. (36) and (37). A $D3$-brane wrapped around x^B corresponds to an $M2$-brane.
- A $D5$-brane wrapped around x^B corresponds to an $M5$-brane wrapped around x^{10}. The tension of the wrapped $M5$-brane R_{10}/l_p^6 reduces to $R_B/g_s l_s^6$ using Eq. (39). A $D5$-brane unwrapped around x^B corresponds to a Kaluza-Klein monopole charged under the gauge field $G_{\mu,10}$ and wrapped around x^9.
- The NS5-brane wrapped around x^B corresponds to an $M5$-brane wrapped on x^9. Its tension $R_B/g_s^2 l_s^6$ is equal to that of the wrapped $M5$-brane R_9/l_p^6. An NS five-brane unwrapped around x^B corresponds to a Kaluza-Klein monopole charged under the gauge field $G_{\mu,9}$ and wrapped around x^{10}.
- The $D7$-brane wrapped around x^B corresponds to a Kaluza-Klein monopole charged under $G_{\mu,10}$. A $D7$-brane unwrapped around x^B is related to the M-theory eight-brane which reduces to the $D8$-brane of type-IIA string theory.

Orientifolds correspond in M theory to fixed points of Z_2 transformations acting both on space and on the supergravity fields.

D. Duality properties

String (or M) theory has a large moduli space of vacua $\mathcal{M}$ parametrized by the size and shape of the compact manifold and the string coupling (as well as the values of other background fields). At generic points in $\mathcal{M}$ the theory is 11 dimensional and inherently quantum mechanical, while at certain degenerations it has different weakly coupled string expansions.

The space of vacua $\mathcal{M}$ is a nontrivial manifold; in particular, it has an interesting global structure. Some apparently distinct vacua are identified by the action of a discrete group known as "U duality" (Hull and Townsend, 1995). Under this identification different states of the theory are often mapped into each other; an example is the BPS-branes discussed above. What looks like a D-brane in one description may appear to be an NS-brane in another and may even correspond to an object of different dimension.

An important subgroup of U duality is T duality, which takes a weakly coupled vacuum to another weakly coupled vacuum and is therefore manifest in string perturbation theory [for a review see Giveon, Porrati, and Rabinovici (1994) and references therein]. Consider type-IIA string theory in $(1+8)$ noncompact dimensions with the ith coordinate x^i living on a circle of radius R_i. At large R_i the theory becomes $(1+9)$-dimensional type-IIA string theory, while at small R_i it naively becomes $(1+8)$ dimensional. However, winding type-IIA strings with energy nR_i/l_s^2 become light in the limit, producing a continuous Kaluza-Klein spectrum and thus the theory becomes ten dimensional again.

From the discussion of the previous section it is clear what the new $(1+9)$-dimensional theory is. Weakly coupled type-IIA string theory on a small circle $R_i\to 0$ corresponds to M theory on a vanishing two-torus, which we saw before is just type-IIB string theory. How do different states in type-IIA string theory map to their type-IIB counterparts?

The wrapped type-IIA string is a wrapped $M2$-brane [see Eq. (29) and subsequent discussion]; the modes becoming light in the $R_i\to 0$ limit correspond to membranes wrapped n times around the shrinking two-torus labeled by (x^i,x^{10}). Comparing their energy nR_iR_{10}/l_p^3 to Eq. (38) and using Eqs. (29)–(37) we see that the type-IIB string one finds lives on a circle of radius

$$R_i^{(B)}=\frac{l_s^2}{R_i^{(A)}} \tag{40}$$

and has string coupling

$$g_s^{(B)}=g_s^{(A)} l_s/R_i^{(A)}. \tag{41}$$

We shall refer to the transformation (40) and (41) as T_i (T duality in the ith direction).

The different branes of type-IIA string theory transform as follows under T_i:

- As we just saw, a fundamental type-IIA string wound n times around x^i transforms into a fundamental type-IIB string carrying momentum $n/R_i^{(B)}$. An unwound fundamental type-IIA string carrying momentum $m/R_i^{(A)}$ transforms under T_i to a fundamental type-IIB string wound m times around the ith direction.
- A $D0$-brane corresponds in M theory to a Kaluza-Klein graviton carrying momentum $1/R_{10}$. As we saw

earlier, in type-IIB language this is a D string wrapped around the ith direction.

- A $D2$-brane wrapped around x^i corresponds in M theory to a transverse $M2$-brane wrapped around x^i. We saw earlier that in type-IIB language this is a D string unwrapped around x^i. Similarly, a $D2$-brane unwrapped around x^i was seen to correspond to an unwrapped $M2$-brane and was interpreted in type-IIB language as a $D3$-brane wrapped around x^i.
- At this point the pattern for Dirichlet branes should be clear. A type-IIA Dirichlet p-brane wrapped around x^i is transformed under T_i to an unwrapped type-IIB Dirichlet $(p-1)$-brane, while an unwrapped type-IIA Dirichlet p-brane is transformed to a Dirichlet $(p+1)$-brane wrapped around x^i:

$$T_i:\ Dp \text{ wrapped on } x^i \leftrightarrow D(p-1) \text{ at a point on } x^i. \tag{42}$$

- Orientifold planes transform under T_i in the same way as D-branes [Eq. (42)].
- A wrapped type-IIA Neveu-Schwarz five-brane transforms under T_i to a wrapped type-IIB Neveu-Schwarz five-brane. An unwrapped type-IIA Neveu-Schwarz five-brane transforms into the Kaluza-Klein monopole carrying magnetic charge under $G_{\mu,i}$:

$$T_i:\begin{cases}\text{type-IIA NS5 wrapped on } x^i \leftrightarrow \text{type-IIB NS5 wrapped on } x^i\\ \text{NS5 at a point on } x^i \leftrightarrow \text{Kaluza-Klein monopole charged under } G_{\mu,i}.\end{cases} \tag{43}$$

As a check, the tensions of the various (wrapped and unwrapped) Dirichlet and solitonic branes (5), (16), and (21) transform under Eqs. (40) and (41) consistently with the above discussion.

The generalization to T duality in more than one direction $T_{i_1,i_2,\ldots,i_n}\equiv T_{i_1}T_{i_2}\cdots T_{i_n}$ is straightforward:

$$T_{i_1,i_2,\ldots,i_n}:\ (R_{i_1},R_{i_2},\ldots,R_{i_n})\leftrightarrow\left(\frac{l_s^2}{R_{i_1}},\frac{l_s^2}{R_{i_2}},\ldots,\frac{l_s^2}{R_{i_n}}\right),\quad g_s\leftrightarrow g_s\prod_{\alpha=1}^{n}\frac{l_s}{R_{i_\alpha}};\ \ l_s\leftrightarrow l_s. \tag{44}$$

For even n it takes type IIA(B) to itself, while for odd n it exchanges the two.

The discussion above can be used to determine the charge of the Op-plane given in Eq. (15). Starting with the type-I theory on T^n, which contains a single $O9$-plane and 32 $D9$-branes wrapped around the T^n, and performing T duality, $T_{i_1,i_2,\ldots,i_n}$, we find a vacuum with 2^n orientifold p-planes, $p=9-n$, one at each fixed point on T^n/Z_2, as well as 32 Dp-branes. The total Ramond-Ramond $(p+1)$-form charge of the configuration is zero, which leads to Eq. (15).

Another interesting subgroup of U duality is S duality of type-IIB string theory in $(9+1)$ dimensions (Schwarz, 1995), an $SL(2,Z)$ symmetry that acts by fractional linear transformations with integer coefficients on τ [Eq. (35)]. In the M-theory interpretation of type-IIB string theory, this $SL(2,Z)$ is the modular group acting on the complex structure of the two-torus (whose size goes to zero in the ten-dimensional limit). (For a review see Schwarz, 1997a and references therein.) We shall focus on a Z_2 transformation $S\in SL(2,Z)$, which acts as $\tau\to-1/\tau$; we shall furthermore restrict our discussion to the case of vanishing Ramond-Ramond scalar a (namely, a rectangular M-theory two-torus), in which case it acts on the coupling (35) as strong-weak coupling duality: $g_s\to 1/g_s$. In the M-theory interpretation of type-IIB string theory discussed in Eqs. (36)–(38) S acts geometrically by interchanging $R_9\leftrightarrow R_{10}$. Equations (36) and (37) imply that the type-IIB parameters g_s,l_s transform as

$$S:\ g_s\leftrightarrow\frac{1}{g_s};\quad l_s^2\leftrightarrow l_s^2 g_s. \tag{45}$$

Another way to arrive at Eq. (45) is to require that as the string coupling is inverted, the ten-dimensional Planck length $l_{10}^4=g_s l_s^4$ remain fixed. From the discussion following Eq. (38) it is clear that the different type-IIB branes transform under S as follows:

- The fundamental string is interchanged with the D string.
- The $D3$-brane is invariant.
- The NS5-brane is interchanged with the $D5$-brane.
- The $D7$-brane transforms into a different seven-brane.

As a check, the tensions of the various branes [Eqs. (5) and (16)] transform under Eq. (45) consistently with the above discussion. The transformations of orientifold planes under S are more intricate and will be discussed in the context of particular applications below.

The worldsheet dynamics on both the fundamental string and the D string is that of a critical type-IIB string. At weak string coupling the tension of the fundamental string is much smaller than that of the D string, and we can think of the former as "fundamental" and of the latter as a heavy soliton. At strong coupling, the D string is the lighter object and *it* should be used as the basis for string perturbation theory. Since a type-IIB string in its ground state preserves half of the supersymmetry, it can be followed from weak to strong coupling, and the above picture is indeed reliable.

Under the full $SL(2,Z)$ S-duality group, the two different kinds of strings are members of a multiplet of (p,q) strings, with the fundamental string corresponding to $(p,q)=(1,0)$ and the D string corresponding to $(p,q)=(0,1)$. Here p measures the charge carried by the string under the NS-NS $B_{\mu\nu}$ field, while q measures the charge under the Ramond-Ramond $B_{\mu\nu}$ field. In M theory the (p,q) string corresponds to a membrane wrapped p times around x^{10} and q times around x^9; it is stable when p,q are relatively prime. A similar discussion applies to five-branes that carry magnetic charges under the two $B_{\mu\nu}$ fields and thus form a multiplet of (p,q) five-branes. There are also (p,q) seven-branes that carry magnetic charge under the complex dilaton τ.

In M theory compactified on T^d, the $SL(2,Z)$ S dualities corresponding to different $T^2\subset T^d$ are subgroups of the geometrical $SL(d,Z)$ symmetry group of T^d. Together with T duality (44) they generate the U-duality group $E_{d(d)}(Z)$ of type-II strings on T^{d-1} (Elitzur, Giveon *et al.*, 1998a).

E. Webs of branes

So far we have discussed brane configurations that preserve sixteen supercharges. In this section we shall describe some configurations with lower supersymmetry.

We saw before that a stack of parallel D- or NS-branes preserves half of the supersymmetry given by Eq. (6) or Eqs. (18) and (19), respectively. To find the SUSY preserved by a web of differently oriented D- and/or NS-branes one needs to impose all the corresponding conditions[8] on the spinors ϵ. The worldvolume dynamics on such a web of branes is typically rather rich. We shall next consider it in a few examples.

1. The $Dp-D(p+4)$ system

Consider a stack of N_c Dp-branes stretched in the $(x^1,\ldots,x^p)$ hyperplane "parallel" to a stack of N_f $D(p+4)$-branes stretched in $(x^1,\ldots,x^{p+4})$ depicted in Fig. 3. Each stack preserves half of the supersymmetry, and together they preserve $1/2\times 1/2=1/4$ of the 32 supercharges of type-II string theory. The preserved supercharges are those that satisfy Eq. (6):

$$\epsilon_L=\Gamma^0\Gamma^1\cdots\Gamma^p\epsilon_R=\Gamma^0\Gamma^1\cdots\Gamma^{p+4}\epsilon_R. \quad (46)$$

The second equality in Eq. (46) is a constraint on ϵ_R, $\epsilon_R=\Gamma\epsilon_R$ with $\Gamma=\Gamma^{p+1}\Gamma^{p+2}\Gamma^{p+3}\Gamma^{p+4}$. The matrix Γ squares to the identity matrix and is traceless. Thus half of its 16 eigenvalues are $+1$ and half are -1. The constraint on ϵ_R, $\Gamma=1$, preserves 8 of the 16 components of ϵ_R. Given ϵ_R, the first equality in Eq. (46) fixes ϵ_L. Thus the total number of independent supercharges preserved by the configuration is eight.

The light degrees of freedom on each stack of branes were discussed before. On the N_c Dp-branes there is a $(p+1)$-dimensional $U(N_c)$ gauge theory coupled to $(9-p)$ adjoint scalars and some fermions. The adjoint scalars naturally split into $(5-p)$ fields corresponding to fluctuations of the Dp-branes transverse to the $(p+4)$-branes, which together with the gauge field form the vector multiplet of a theory with eight supercharges, and the remaining four fields, which form an adjoint hypermultiplet.

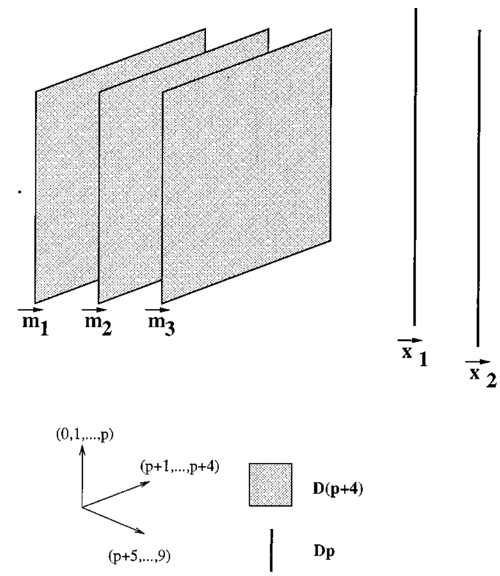

FIG. 3. The $Dp-D(p+4)$ system, consisting of a stack of N_c Dp-branes parallel to N_f $D(p+4)$-branes. Locations in the transverse space $(x^{p+5},\ldots,x^9)$ are labeled by $\vec{x}_a$, $\vec{m}_i$, respectively.

A similar theory with $N_c\to N_f$ and $p\to p+4$ lives on the $D(p+4)$-branes. Each of the two theories has 16 supercharges. The supersymmetry of the full theory is broken down to eight supercharges by additional matter corresponding to strings stretched between the two stacks of branes. From the point of view of the Dp-brane this matter corresponds to N_f flavors in the fundamental representation of $U(N_c)$. From the point of view of the $D(p+4)$-brane, they are N_c pointlike (in the transverse directions) defects in the fundamental of $U(N_f)$. When the Dp-branes are inside the $D(p+4)$-branes, they can be thought of as small instantons (Douglas, 1995).

It is important to emphasize that for an observer who lives on the Dp-brane, the degrees of freedom on the $D(p+4)$-brane are nondynamical background fields (at least in infinite volume). For example, the effective gauge coupling in $(p+1)$ dimensions g_{p+1} of the $U(N_f)$ gauge field on the $D(p+4)$-brane is given by

$$\frac{1}{g_{p+1}^2}=\frac{V_{p+1,\ldots,p+4}}{g_{p+5}^2}, \quad (47)$$

where g_{p+5} is the $U(N_f)$ gauge coupling in $p+5$ dimensions and $V_{p+1,\ldots,p+4}$ is the volume of the $D(p+4)$-brane worldvolume transverse to the Dp-brane. When this volume is infinite, the kinetic energy of $U(N_f)$ excitations is infinite as well and they are frozen at their classical values. The same is true for other excitations on the $D(p+4)$-brane. Thus from the point of view of the Dp-brane, the $U(N_f)$ gauge symmetry of the $D(p+4)$-brane is a global symmetry and

[8]This analysis is valid for widely separated branes and may miss bound states.

the only dynamical fields that appear due to the presence of the $D(p+4)$-brane are the N_f flavors corresponding to strings stretched between the Dp-branes and $D(p+4)$-branes; these modes are localized at the Dp-brane.

The relative locations in space of the various branes correspond to moduli and couplings in the Dp-brane worldvolume theory. Locations of the "heavy" $D(p+4)$-branes correspond to couplings, while locations of the "light" Dp-branes are moduli:

- The locations of the $D(p+4)$-branes in the transverse space $(x^{p+5},\ldots,x^9)$ $\vec{m}_i$ $(i=1,\ldots,N_f)$ correspond to masses for the N_f fundamentals.
- The locations of the Dp-branes in $(x^{p+5},\ldots,x^9)$ $\vec{x}_a$ $(a=1,\ldots,N_c)$ correspond to expectation values of fields $\vec{X}$ in the adjoint of $U(N_c)$ and parametrize the Coulomb branch of the $U(N_c)$ gauge theory, as in Eq. (12).
- The locations of the Dp-branes parallel to the $D(p+4)$-branes [in the $(x^{p+1},\ldots,x^{p+4})$ directions] correspond to expectation values of an adjoint hypermultiplet of $U(N_c)$.

One can think of the Dp-branes as probing the geometry near the $D(p+4)$-brane. For example, the metric on the Coulomb branch of the $U(1)$ gauge theory with N_f flavors on a single Dp-brane adjacent to N_f $D(p+4)$-branes is the background metric of the $D(p+4)$-branes. This is analogous (and in some cases U dual) to the situation described in Sec. II.B.3 where we described the metric felt by a fundamental string propagating in the background of solitonic five-branes.

In general, some of the parameters that one can turn on in the low-energy field theory may be absent in the brane configuration. As an example, in the low-energy $U(N_c)$ gauge theory with eight supercharges one can add a mass term to the adjoint hypermultiplet and a Fayet-Iliopoulos coupling, both of which are absent in the brane configuration. One way to understand this is to note that theories with 16 supercharges do not have such couplings. The theory on a stack of isolated Dp-branes has sixteen supercharges and, while it is broken down to eight by the presence of the $(p+4)$-branes, it inherits this property from the theory with more supersymmetry.

Similarly, some of the moduli of the low-energy gauge theory may not correspond to geometrical deformations in the brane description. In the example above, the Higgs branch of the $U(N_c)$ gauge theory, corresponding to nonzero expectation values of the fundamentals, can be thought of as the moduli space of instantons. Each Dp-brane embedded in the stack of N_f $(p+4)$-branes can be thought of as a small (four-dimensional) $U(N_f)$ instanton which can grow and become a finite-size instanton. The moduli space of N_c instantons in $U(N_f)$ is the full Higgs branch of the theory; it is not realized geometrically. For a more detailed discussion see Douglas (1996).

Clearly, the more of the couplings and moduli of the gauge theory are represented geometrically, the more useful the brane configuration is for studying the gauge theory.

2. More general webs of branes

The system described in the previous subsection can be generalized in several directions: applying U-duality transformations, rotating some of the branes relative to others, adding branes and/or orientifold planes, and considering configurations of branes ending on branes. In this and the next subsections we shall describe some of these possibilities:

- *Orientifolds.* Starting with the $Dp-D(p+4)$ system we can add an Op-plane, an $O(p+4)$-plane, or both, without breaking any further supersymmetry. Adding an Op-plane leads to an $SO(N_c)$ or $Sp(N_c/2)$ gauge theory[9] on the Dp-branes. In gauge theory with eight supercharges and N_f fundamentals the resulting global symmetry is $Sp(N_f/2)$ or $SO(N_f)$, respectively. Therefore it is clear that an orthogonal orientifold projection on the p-branes is correlated with a symplectic projection on the $(p+4)$-branes, and vice versa.
  A similar analysis can be performed for the case of an $O(p+4)$-plane. An example is type-I theory, where an orthogonal projection on nine-branes due to an orientifold nine-plane is correlated with a symplectic projection on five-branes (Gimon and Polchinski, 1996; Witten, 1996b).
- *The $Dp-D(p+2)$ system.* Compactifying the $Dp-D(p+4)$ system of Section II.E.1 and considering different limits gives rise to configurations with the same amount of supersymmetry in different dimensions. These can be studied by using T duality. As an example, compactify x^{p+1} on a circle, T dualize and then decompactify the resulting dual circle. One finds a $D(p+1)-D(p+3)$ system; a stack of N_c $D(p+1)$-branes whose worldvolume stretches in $(x^0,x^1,\ldots,x^{p+1})$ and a stack of N_f $D(p+3)$-branes whose worldvolume lies in $(x^0,x^1,\ldots,x^p,x^{p+2},x^{p+3},x^{p+4})$. The two stacks of branes are now partially orthogonal, with $(p+1)$ of their $(p+2)$ and $(p+4)$-dimensional worldvolumes in common.
  Formally, the degrees of freedom in the common dimensions (which we shall refer to as "the intersection") are the same as before; however, one can no longer talk about a $U(N_c)$ gauge theory on the intersection. All matter in the adjoint of $U(N_c)$ is now classical, as it lives on a "heavy" brane that has one infinite direction (x^{p+1}) transverse to the intersection. The only dynamical degrees of freedom on the $(p+1)$-dimensional intersection region are the N_f fundamentals of $U(N_c)$ that arise from $(p+1)-(p+3)$

[9] N_f and N_c are even here.

strings. Of course, recompactifying x^{p+1} restores the previous physics, and we shall usually implicitly consider this case below.

- *The $Dp-D(p+2)-D(p+2)'$ system:* To reduce the number of supersymmetries from eight to four we can add to the previous system another stack of differently oriented D-branes. A typical configuration consists of a stack of N_c Dp-branes with worldvolume $(x^0,x^1,\ldots,x^p)$, N_f $D(p+2)$-branes $(x^0,x^1,\ldots,x^{p-1},x^{p+1},x^{p+2},x^{p+3})$, and N_f' $D(p+2)'$-branes $(x^0,x^1,\ldots,x^{p-1},x^{p+1},x^{p+4},x^{p+5})$. The gauge group on the Dp-branes is $U(N_c)$, with the following matter:

  (1) N_f fundamental hypermultiplets Q, $\tilde{Q}$ corresponding to strings stretched between the Dp and $D(p+2)$-branes, and N_f' fundamentals Q', $\tilde{Q}'$ corresponding to strings stretched between the Dp and $D(p+2)'$-branes.

  (2) $10-p$ adjoint fields whose expectation values (12) parametrize the locations of the p-branes and the Wilson line of the worldvolume gauge field along the compact x^p direction. These can be split into a complex adjoint field X describing fluctuations of the Dp-branes in the (x^{p+4},x^{p+5}) directions; a complex adjoint field X' corresponding to fluctuations in the (x^{p+2},x^{p+3}) directions; a complex adjoint X'' corresponding to fluctuations in the x^{p+1} direction as well as the gauge field A_p. $4-p$ adjoints parametrize the Coulomb branch of the gauge theory.

X couples to the N_f flavors Q, and X' couples to the N_f' flavors Q' via the superpotential

$$W=\tilde{Q}XQ+\tilde{Q}'X'Q'. \tag{48}$$

Geometrically, the couplings (48) are due to the fact that displacing the Dp-branes in the (x^{p+4},x^{p+5}) directions stretches the $p-(p+2)$ strings, thus giving a mass to the quarks Q, $\tilde{Q}$, etc.

More generally, the coupling matrix of (X,X') and (Q,Q') is governed by the relative angles between the $D(p+2)$ and $D(p+2)'$-branes. Indeed, defining $v=x^{p+2}+ix^{p+3}$ and $w=x^{p+4}+ix^{p+5}$, one can check (Berkooz, Douglas, and Leigh, 1996) that arbitrary relative complex rotations of the different $(p+2)$-branes in v,w,

$$\begin{pmatrix} v \\ w \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{49}$$

preserve four supercharges like the original $Dp-D(p+2)-D(p+2)'$ system. When the relative angle between the $D(p+2)$ and $D(p+2)'$ branes goes to zero, the supersymmetry is enhanced to eight supercharges and one recovers the $Dp-D(p+2)$ system described above.

- *The NS-Dp system:* Starting with the $D3$-$D5$ system and performing an S-duality transformation we find a system consisting of N_c $D3$-branes (x^0,x^1,x^2,x^3), and N_f NS5-branes $(x^0,x^1,x^2,x^4,x^5,x^6)$ preserving eight supercharges. T duality [Eqs. (42) and (43)]—acting on any number of longitudinal directions of the NS-brane—may be used to turn this configuration into other configurations of Dp-branes and NS5-branes. Other T dualities (which act on one direction transverse to the NS-brane) map the system to configurations of Dp-branes wrapped around nontrivial cycles of ALE spaces. Similarly to the D-brane case described above, different NS-branes can be rotated with respect to each other, by complex rotations of the form (49), which preserve four of the eight supercharges.

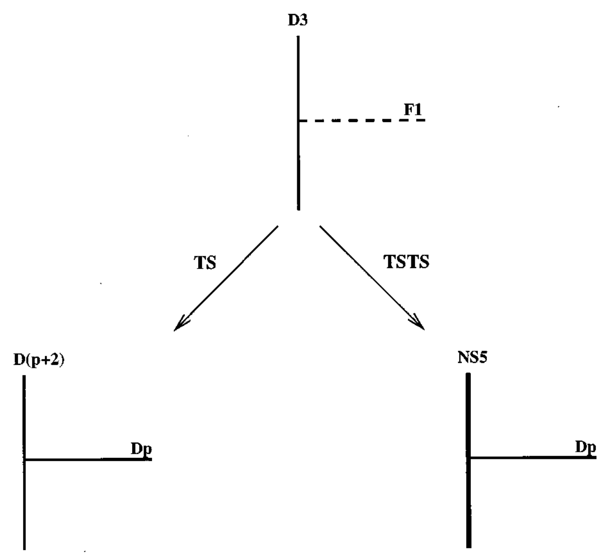

FIG. 4. U duality relating a fundamental string that ends on a $D3$-brane to other supersymmetric configurations: a Dp-brane that ends on a $D(p+2)$-brane and a Dp-brane that ends on a Neveu-Schwarz five-brane.

3. Branes ending on branes

One of the important things branes can do is end on other branes. D-branes are *defined* by the property that fundamental strings can end on them, and by a chain of dualities this can be related to many other possibilities.

Consider a fundamental string ending on a $D3$-brane (Fig. 4). The $D3$-brane itself preserves 16 supercharges, and if we put the open string ending on it in its ground state it preserves 1/2 of these, namely eight. Performing S duality we reach a configuration of a D string ending on the $D3$-brane. By T duality in $(p-1)$ directions transverse to both branes we are led to a configuration of a Dp-brane ending on a $D(p+2)$-brane with a $[(p-1)+1]$-dimensional intersection.

For $p=3$, the configuration of a $D3$-brane ending on a $D5$-brane can be mapped by applying S duality to a $D3$-brane ending on an NS5-brane. Further T duality along the five-brane worldvolume maps this to a configuration of a Dp-brane (with any $p\leq 6$) ending on the NS5-brane.

In M theory, many of the above configurations are related to membranes ending on five-branes. This is most apparent for a $D2$-brane ending on an NS5-brane in type-IIA string theory as well as fundamental and D strings ending on the appropriate five-branes. Others (e.g., a $D4$-brane ending on an NS5-brane) can be thought of as corresponding to a single $M5$-brane with a convoluted worldvolume.

The worldvolume theory on a brane that ends on another brane is a truncated version with eight supercharges of the dynamics on an infinite brane. The light fields are conveniently described in terms of representations of $d=4$, $N=2$ supersymmetry with spin $\leqslant 1$, hypermultiplets, and vector multiplets:

- For a Dp-brane stretched in $(x^0,x^1,\ldots,x^p)$ and ending (in the x^p direction) on a $D(p+2)$-brane stretched in $(x^0,x^1,\ldots,x^{p-1},x^{p+1},x^{p+2},x^{p+3})$ and located at $x^p=0$, the $(p+1)$-dimensional dynamics now takes place on $R^{1,p-1}\times R^+$, where the half line R^+ corresponds to $x^p\geqslant 0$. The three scalars corresponding to fluctuations of the Dp-brane along the $D(p+2)$-brane $(X^{p+1},X^{p+2},X^{p+3})$ combine with the pth component of the Dp-worldvolume gauge field A_p into a massless hypermultiplet with free boundary conditions[10] at $x^p=0$. The scalars describing fluctuations of the Dp-brane perpendicular to the $D(p+2)$-brane $(X^{p+4},\ldots,X^9)$ satisfy Dirichlet boundary conditions $X^I(x^p=0)=0$ $(I=p+4,\ldots,9)$. These $(6-p)$ scalars are paired by supersymmetry with the gauge field A_μ, $\mu=0,1,\ldots,p-1$, into a vector multiplet. Thus the gauge field satisfies Dirichlet boundary conditions as well.
- For a Dp-brane stretched in $(x^0,x^1,\ldots,x^{p-1},x^6)$ and ending (in the x^6 direction) on an NS5-brane stretched in $(x^0,x^1,\ldots,x^5)$, the hypermultiplet contains the scalars (X^7,X^8,X^9) and the sixth component of the Dp-worldvolume gauge field A_6 and satisfies Dirichlet boundary conditions at $x^6=0$. The vector multiplet consisting of the $(6-p)$ scalars $(X^p,\ldots,X^5)$ and the components of the gauge field along $R^{1,p-1}$ is (again, classically) free at the boundary.

Quantum mechanically, we have to take into account that the end of a brane ending on another brane looks like a charged object in the worldvolume theory of the latter. Consider, for example, the case of a fundamental string ending on a Dp-brane. It can be thought of as providing a pointlike source for the p-brane worldvolume gauge field, leading to a Coulomb potential (Callan and Maldacena, 1998; Gibbons, 1998)

$$A_0=\frac{Q}{r^{p-2}}, \tag{50}$$

where Q is the worldvolume charge of the fundamental string and r the distance from the charge on the p-brane. To preserve supersymmetry it is clear from the form of the action (7) that in addition to Eq. (50) one of the p-brane worldvolume scalar fields must be excited, say,

$$X^{p+1}=\frac{Ql_s^2}{r^{p-2}}. \tag{51}$$

The solution (50) and (51) preserves half of the 16 worldvolume supersymmetries and corresponds to a fundamental string stretched along x^{p+1} and ending on the D-brane. We see that the string bends the D-brane: the location of the brane becomes r dependent [Eq. (51)], approaching the "classical" value $x^{p+1}=0$ at large r (for $p>2$). Standard charge quantization implies that the quantum of charge in the normalization (7) is $Q=g_{SYM}^2$. As $r\to 0$, $x^{p+1}\to\infty$; this corresponds to a fundamental string ending on the Dp-brane. Of course, *a priori* we only trust the solution (50) and (51) for large r where the fields and their variations are small. As $r\to 0$ higher-order terms in the Lagrangian, that were dropped in Eq. (7), become important, e.g., one has to replace the Maxwell action by the Born-Infeld action. A detailed discussion of this and related issues appears in the articles of Hashimoto (1997); Callan and Maldacena (1998); Gibbons (1998), , Lee, Peet, and Thorlacius (1998), and Thorlacius (1998).

A similar analysis can be performed in the other cases mentioned above. The conclusion is that when a brane ends on another brane, the end of the first brane looks like a charged object in the worldvolume theory of the second brane. The latter is bent according to Eq. (51) with p the codimension of the intersection in the second brane, and r the p-dimensional distance to the end of the first brane on the worldvolume of the second.[11]

The intersecting brane configurations discussed earlier in this section are intimately related to the configurations of branes ending on branes discussed here. As an example, when the Dp and $D(p+2)$-branes of the previous subsection[12] meet in the transverse space $(x^{p+4},\ldots,x^9)$, the p-brane can split into two parts, $x^p<0$ and $x^p>0$, which can then separate along the $(p+2)$-brane in the $(x^{p+1},x^{p+2},x^{p+3})$ directions. Locally, one has then a configuration of a p-brane ending on a $(p+2)$-brane from the right in x^p and another one ending on it from the left at a different place, as shown in Fig. 5.

In the gauge theory on the intersection of the Dp and NS5-branes this realizes geometrically the Higgs branch of the theory on the D-brane. This will be discussed in detail in the applications below.

III. FOUR-DIMENSIONAL THEORIES WITH $N=4$ SUPERSYMMETRY

At low energies the dynamics on the worldvolume of N_c parallel $D3$-branes in type-IIB string theory is de-

[10]We shall soon see that the boundary conditions are modified quantum mechanically.

[11]This can be shown by U dualizing to a fundamental string ending on a Dp-brane.

[12]There we actually considered $(p+1)$- and $(p+3)$-branes; replace $p\to p-1$ there to get the system discussed here.

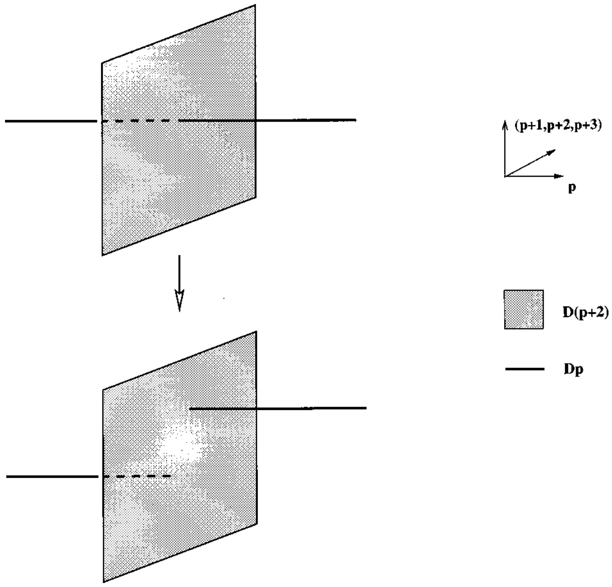

FIG. 5. A Dp-brane intersecting a $D(p+2)$-brane and splitting into two disconnected parts which separate along the $D(p+2)$-brane.

scribed by four-dimensional $N=4$ SYM theory with gauge group $U(N_c)$. Symplectic and orthogonal groups can be studied by considering $D3$-branes near a parallel $O3$-plane. The brane description provides a natural interpretation of the strong-weak coupling duality of $N=4$ SYM theories and leads to a simple geometrical description of BPS-saturated dyons. In this section we describe this circle of ideas, starting with the unitary case.

A. Montonen-Olive duality and type-IIB S duality

Four-dimensional $N=4$ supersymmetric gauge theory with gauge group G can be obtained by dimensionally reducing $N=1$ SYM theory from $(9+1)$ to $(3+1)$ dimensions. Supersymmetry (with 16 supercharges) places strong constraints on the structure. The moduli space of vacua is $6r$ dimensional, where r is the rank of G. It is parametrized by expectation values in the Cartan subalgebra of the six adjoint scalars in the $N=4$ multiplet. Generically in moduli space the gauge symmetry is broken to $U(1)^r$, but at certain singular subspaces some of the non-Abelian structure is restored. The classical and quantum moduli spaces are identical in $N=4$ SYM theory (in contrast with $N=2$ SYM theory, where the metric on the Coulomb branch is generally corrected by quantum effects, and $N=1$ SYM theory, where some or all of the classical moduli space can be lifted; these cases will be discussed later). The leading quantum corrections modify certain nonrenormalizable terms with four derivatives.

The most singular point in the moduli space is the origin, where the full gauge symmetry is unbroken. The theory at that point is conformal and the gauge coupling g_{SYM} is an exactly marginal deformation parametrizing a line of fixed points. The theory also depends on a parameter θ, which together with g_{SYM} forms a complex coupling

$$\tau=\frac{\theta}{2\pi}+\frac{i}{g_{SYM}^2}. \tag{52}$$

The theory at the origin of moduli space is conformal for all τ.

Not all values of τ correspond to distinct theories. Since θ is periodic, taking $\tau\to\tau+1$ leads to the same theory. In addition, $N=4$ SYM theory has a less obvious symmetry, Montonen and Olive's strong-weak coupling duality, which takes $\tau\to-1/\tau$ and exchanges the gauge algebra $\mathcal{G}$ with the dual algebra[13] $\hat{\mathcal{G}}$ (Montonen and Olive, 1977; see Olive, 1995; Harvey, 1997; Di Vecchia, 1997; and references therein). It also acts as electric-magnetic duality on the gauge field and thus interchanges electric and magnetic charges. Together, the two symmetries generate an $SL(2,Z)$ duality group,[14] which acts on τ by fractional linear transformations with integer coefficients:

$$\tau\to\frac{a\tau+b}{c\tau+d};\quad a,b,c,d\in Z,\ ad-bc=1. \tag{53}$$

We shall consider mainly the case of an $SU(2)$ gauge group here, in which states carry electric and magnetic charge under the single Cartan generator and assemble into multiplets of $SL(2,Z)$ that contain states with electric and magnetic charges (e,m), transforming under $SL(2,Z)$ as

$$\begin{pmatrix} e\\ m\end{pmatrix}\to\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} e\\ m\end{pmatrix}. \tag{54}$$

For example, the charged gauge bosons $W^{\pm}$ with charge $(\pm1,0)$ belong to the same multiplet as the magnetic monopole with charge $(0,\pm1)$ and various dyons.

To study $N=4$ SYM theory with gauge group $U(N_c)$ using branes, consider N_c parallel $D3$-branes, whose worldvolumes stretch in (x^0,x^1,x^2,x^3). The $U(N_c)$ gauge bosons $A_\mu^{a\bar{b}}(x^\nu)$, $\mu,\nu=0,1,2,3$, $a,\bar{b}=1,\dots,N_c$, correspond to the ground states of oriented $3-3$ strings connecting the ath and bth three-branes [Fig. 1(b)]. The six scalars $X^I_{a\bar{b}}(x^\mu)$ $(I=4,\dots,9)$ in the adjoint representation of $U(N_c)$ also correspond to $3-3$ strings describing fluctuations of the three-branes in the transverse directions $(x^4,x^5,x^6,x^7,x^8,x^9)$. Together with the ground-state fermionic fields they form an $N=4$ gauge supermultiplet.

The bosonic part of the low-energy Lagrangian is given by Eqs. (10) and (11), with the $U(N_c)$ gauge coupling given by $g_{SYM}^2=g_s$ [Eq. (8)]. The conventional SYM scalar fields Φ^I which have dimensions of energy are related to the scalars X^I which appear naturally in the brane construction via

$$\Phi^I=X^I/l_s^2. \tag{55}$$

[13] $\widehat{su}(N_c)=su(N_c)$, $\widehat{so}(2r)=so(2r)$, $\widehat{so}(2r+1)=sp(r)$.

[14] This was first recognized in lattice models (Cardy and Rabinovici, 1982) and in string theory (Font *et al.*, 1990).

The limit in which the theory on the three-brane decouples from gravity and the four-dimensional dynamics becomes exactly that of $N=4$ SYM theory at all energy scales is $l_s \to 0$ with g_s, Φ^I held fixed. By the latter it is meant that the energy scale studied, E, and the scale set by the expectation values, Φ^I, which typically are comparable, must be much smaller than the string scale $1/l_s$ and the Planck scale $1/l_p$ (which for $g_s \sim 1$ is comparable to the string scale). In particular, the transverse separations of the three-branes parametrizing the Coulomb branch must satisfy $\delta x^i \ll l_s, l_p$.

In the brane picture, the $SL(2,Z)$ Montonen-Olive duality can be thought of as a remnant of the $SL(2,Z)$ S-duality group of type-IIB string theory in the limit $l_s \to 0$. The three-brane is self-dual under S duality. The complex worldvolume gauge coupling (52) is the expectation value of the complex type-IIB dilaton τ [Eq. (35)] on which S duality acts by fractional linear transformations [Eq. (53)], and the type-IIB charges (p,q) that transform under S duality in an analogous way to Eq. (54) are related to the SYM charges (e,m). In what follows we shall study this correspondence in more detail in the case $N_c=2$.

An $N=4$ SYM gauge theory with gauge group $G=SU(2)$ is obtained in the brane description by studying the dynamics on two parallel $D3$-branes (Green and Gutperle, 1996; Tseytlin, 1996). Actually, the gauge group in this case is $U(2)$ but the diagonal $U(1) \subset U(2)$ will play no role in the discussion, as all the fields we shall discuss are neutral under it; therefore it can be ignored. The six-dimensional Coulomb branch of the $SU(2)$ SYM theory is parametrized by the transverse separation of the two branes $\vec{x}_2 - \vec{x}_1$, where $\vec{x} \equiv (x^4, \ldots, x^9)$. Displacing the two three-branes from the origin by $\pm \vec{x}$ (keeping their center of mass corresponding to the decoupled $U(1)$ fixed at the origin) is equivalent to turning on a diagonal expectation value for the adjoint scalar $\vec{X}$:

$$\langle \vec{X} \rangle = \begin{pmatrix} \vec{x} & 0 \\ 0 & -\vec{x} \end{pmatrix}, \tag{56}$$

which breaks $SU(2) \to U(1)$.

The resulting configuration is depicted in Fig. 6. A fundamental string stretched between the two $D3$-branes corresponds to a charged gauge boson in the broken $SU(2)$ with mass given by Eq. (13). In the $N=4$ SYM theory it transforms under electric-magnetic duality [Eqs. (53) and (54)] into a dyon. In the brane description S duality takes a fundamental string to a (p,q) string; thus we learn that a dyon with electric-magnetic charge (p,q) corresponds in the string language to a (p,q) string stretched between the two $D3$-branes.

Note that this is consistent with our discussion of branes ending on branes in Sec. II.E, where we saw that a fundamental string ending on a $D3$-brane can be thought of as an electric charge in the worldvolume theory on the three-brane [Eq. (50)]. Since S duality acts on the three-brane as electric-magnetic duality, this implies that a D string ending on a $D3$-brane provides a magnetic source for the three-brane worldvolume gauge field. The energy of a D string stretched between the two $D3$-branes is $E=2|\vec{x}|/g_s l_s^2$ or in SYM variables $E=2|\vec{\phi}|/g_{SYM}^2$, as expected from gauge theory (the mass of the monopole is $M_{\rm mon}=M_W/g_{SYM}^2$ where M_W is the mass of the charged W boson).

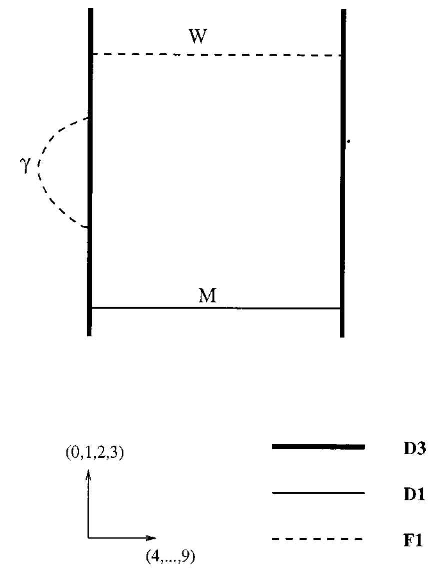

FIG. 6. $U(2)$ $N=4$ supersymmetric Yang-Mills theory on a pair of $D3$-branes broken to $U(1)\times U(1)$ by the separation of the branes. Dyons in supersymmetric Yang-Mills theory, such as the photon γ, the charged gauge boson W, and the magnetic monopole M, are described by (p,q) strings ending on the three-branes.

B. Nahm's construction of monopoles from branes

One application of this construction is to the study of the moduli space of monopoles in gauge theory. To describe the moduli space of k monopoles $\mathcal{M}_k$ one is instructed to study a configuration of k parallel $D1$-branes stretched between the two parallel $D3$-branes (Fig. 7), say in the x^6 direction (Diaconescu, 1997). It is easy to check that the configuration preserves 8 of the 16 supercharges of the three-brane theory, in agreement with the fact that the monopoles are half BPS-saturated objects. The monopole moduli space $\mathcal{M}_k$ is the $4k$ real dimensional space labeled by the locations in (x^1,x^2,x^3) of the k D strings and the Wilson lines of the k $U(1)$ gauge fields along the D strings, A_6.

The brane configuration suggests an alternative point of view on the space $\mathcal{M}_k$. In the $D3$-brane picture it describes a moduli space of k monopoles; from the point of view of the D strings it can be thought of as the moduli space of vacua of the non-Abelian gauge theory on the k D strings stretched between the $D3$-branes! That theory lives in the $1+1$ dimensions (x^0,x^6) and, since the spatial direction x^6 is confined to a finite line segment, it reduces at low energies to supersymmertic quantum mechanics. Of course, supersymmetric quan-

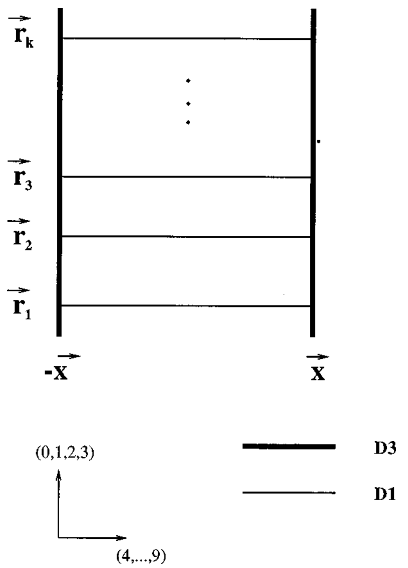

FIG. 7. A point in the moduli space of k $SU(2)$ monopoles represented by D strings stretched between $D3$-branes.

tum mechanics does not have a moduli space of vacua, but there is an approximate Born-Oppenheimer notion of a space of vacua, which arises after integrating out all the fast modes of the system. The low-energy dynamics is described by a sigma model on the moduli space $\mathcal{M}_k$.

The theory on the D strings has eight supercharges and the following matter content. The $U(k)$ gauge field A_0 and five adjoint scalars $(\Phi^4,\Phi^5,\Phi^7,\Phi^8,\Phi^9)$ have Dirichlet boundary conditions at $x^6=\pm x$ (the locations of the two three-branes). The remaining component of the D-string worldvolume gauge field A_6 and the three adjoint scalars (Φ^1,Φ^2,Φ^3) have (formally) Neumann boundary conditions.[15]

To study the dynamics on the worldvolume of the D string we can set to zero all the fields that satisfy Dirichlet boundary conditions, and the gauge field A_6 (by a gauge choice). From the Lagrangian for Φ^1,Φ^2,Φ^3 [Eqs. (10) and (11)],

$$\mathcal{L}\sim \mathrm{Tr}\left(\sum_{I=1}^{3} \partial_s\Phi^I\partial_s\Phi^I-\sum_{I,J}[\Phi^I,\Phi^J]^2\right), \tag{57}$$

where we have denoted x^6 by s, it is clear that ground states satisfy

$$\partial_s\Phi^I+\frac{1}{2}\epsilon^{IJK}[\Phi^J,\Phi^K]=0. \tag{58}$$

The boundary conditions of the fields Φ^I at the edges of the interval $s=\pm x$ are interesting. Naively, one would expect that, at least as long as the k D strings are widely separated in $\vec{r}=(x^1,x^2,x^3)$, we should be able to think of their locations $\vec{r}_i$ as the expectation values of the diagonal components of the matrix fields $\vec{\Phi}_{ii}=\vec{\phi}_i=\vec{r}_i/l_s^2$ [see Eq. (12)]. The off-diagonal components of $\vec{\Phi}$ are massive and could be integrated out in the Born-Oppenheimer approximation. This would lead us to deduce that the boundary conditions for the matrices $\vec{\Phi}$ are

$$\vec{\Phi}(s=\pm x)=\mathrm{diag}(\vec{\phi}_1,\dots,\vec{\phi}_k). \tag{59}$$

However, this picture does not make sense for finite separations of the D strings. We saw [after Eq. (51)] that the "classical" picture of D strings attached to the three-branes at k points $\vec{r}=\vec{r}_1,\dots,\vec{r}_k$ $[\vec{r}=(x^1,x^2,x^3)]$ has to be replaced by a curved three-brane with $s=s(\vec{\phi})$, which approaches the classical location $s=x$ at $|\vec{\phi}|\to\infty$, but is actually described asymptotically by

$$s\simeq\sum_{i=1}^{k}\frac{1}{|\vec{\phi}-\vec{\phi}_i|}+x. \tag{60}$$

Each D string creates a disturbance in the shape of the three-brane of size

$$|\vec{\phi}-\vec{\phi}_i|\simeq\frac{1}{s-x}, \tag{61}$$

which *diverges*[16] as $s\to x$.

Therefore for any finite $|\vec{\phi}_i-\vec{\phi}_j|$ (as measured in the middle of the s interval), the different D strings in fact overlap close to the edges of the s interval. Hence the off-diagonal components of the matrices Φ^I $(I=1,2,3)$ are light and cannot be integrated out, and one expects the matrices $\Phi^I(s\to x)$ not to commute. The only boundary conditions for Φ^I that are consistent with Eqs. (58) and (60) are (for notational simplicity we have set the center of mass of the k monopoles $\vec{r}_0$ to zero)

$$\Phi^I\simeq\frac{T^I}{s-x}, \tag{62}$$

where the $k\times k$ matrices T^I must satisfy Eq. (58):

$$[T^I,T^J]=\epsilon_{IJK}T^K, \tag{63}$$

and, therefore, define a k-dimensional representation of $SU(2)$. The representation T^I must furthermore be irreducible; reducible representations correspond to splitting the k monopoles into smaller groups that are infinitely far apart.

As a check, we can compute the size of the bound state:[17]

$$R^2=\Phi^I\Phi^I\simeq\frac{T^IT^I}{(s-x)^2}=\frac{(k-1)(k+1)}{4(s-x)^2}, \tag{64}$$

i.e., $R\simeq k/2(s-x)$, roughly the size of the k D-string system, as given by Eq. (60), $|\vec{\phi}|\simeq k/(s-x)$. Clearly a similar analysis holds at the other boundary of the s interval, $s=-x$.

[15]As before, $\Phi^I=X^I/l_s^2$ [Eq. (55)].

[16]Note that the asymptotic expression (61) becomes more and more reliable in this regime.

[17]The k-dimensional representation of $SU(2)$ corresponds to $j=(k-1)/2$ and has quadratic Casimir $T^IT^I=j(j+1)=(k-1)(k+1)/4$.

Interestingly, we have arrived [Eqs. (58) and (62)] at Nahm's description of the moduli space of k $SU(2)$ monopoles (Nahm, 1980)! The brane realization provides a new perspective and, in particular, a physical rationale for the construction. It also makes it easy to describe generalizations, e.g., to the case of the moduli space of monopoles in higher-rank groups.

Monopoles in (broken) $SU(N_c)$ gauge theory can be discussed by considering a configuration of N_c $D3$-branes separated in the x^6 direction, and k_a D strings stretched in x^6 between the ath and the $(a+1)$st three-brane, $a=1,\ldots,N_c-1$. Such configurations preserve eight supercharges and describe BPS magnetic monopoles of $SU(N_c)$. The magnetic charge under the natural Cartan subalgebra is $(k_1,k_2-k_1,\ldots,-k_{N_c-1})$. The moduli space of such monopoles can be described by using a generalization of the discussion above.

C. Symplectic and orthogonal groups from orientifolds

To study symplectic and orthogonal groups we add an orientifold three-plane parallel to the N_c three-branes. As described in Sec. II.B the low-energy worldvolume dynamics of the $O3-D3$ system is

$$Sp(N_c/2) \ \ (N_c \text{ even}),$$

$$N=4 \text{ SYM in } 4d \text{ if } Q_{O3}=+\frac{1}{2}Q_{D3},$$

$$SO(N_c), \ N=4 \text{ SYM in } 4d \text{ if } Q_{O3}=-\frac{1}{2}Q_{D3}.$$

In this case we can use the correspondence between gauge theory and brane theory to learn about strong-coupling properties of orientifold planes, by using the correspondence between Montonen-Olive duality in gauge theory and S duality in string theory. From gauge theory we expect $SO(2r)$ to be self-dual under $SL(2,Z)$ while $SO(2r+1)$ and $Sp(r)$ should be dual to each other. The $SO(2r)$ case works in the obvious way: the $D3$-branes and $O3$-plane are self-dual under $SL(2,Z)$. In the non-simply-laced case there is a new element. Consider a weakly coupled $SO(2r+1)$ gauge theory. The orientifold charge is $-Q_{D3}/2$; the $6r$-dimensional Coulomb branch corresponds to removing r pairs of three-branes from the orientifold plane. A single three-brane that does not have a mirror remains stuck at the orientifold.

When the gauge coupling becomes large there are two ways of thinking about the system. We can either continue thinking about it as a (strongly coupled) $SO(2r+1)$ gauge theory or relate it to a weakly coupled theory by performing a strong-weak coupling S-duality transformation. From gauge theory we know that the result should be a weakly coupled $Sp(r)$ theory, which is described by an orientifold with charge $+Q_{D3}/2$.

Thus Montonen-Olive duality of a gauge theory teaches us that a "bound state" of an $O3$-plane with negative Ramond charge and a single $D3$-brane embedded in it [a configuration with Ramond charge $(-1/2+1)Q_{D3}$] transforms under S duality of type-IIB string theory into an $O3$-plane with Ramond charge $+Q_{D3}/2$ (Elitzur, Giveon, *et al.*, 1998b).

Monopoles in broken SO/Sp gauge theory are described as before by D strings stretched between different $D3$-branes. Consider, for example, the rank-one case $N_c=2$. For positive orientifold charge the gauge group is $Sp(1)\simeq SU(2)$ and the moduli space of k $SU(2)$ monopoles that we have discussed previously can be studied by analyzing the worldvolume theory of k D strings connecting the single "physical" $D3$-brane to its mirror image. The gauge group $U(k)$ is replaced by $SO(k)$, and the matrices Φ^I (55) and A_6 now become symmetric $k\times k$ matrices. The discussion of Eqs. (57)–(63) can presumably be repeated, although this has not been done in the literature.

For negative orientifold charge the gauge group is $SO(2)\simeq U(1)$ and one does not expect nonsingular monopoles to exist. This means that D strings cannot connect the single physical $D3$-brane to its mirror image. This is related by S duality to the fact, discussed in Sec. II.B.2, that for negative orientifold charge the ground states of fundamental strings stretched between the $D3$-brane and its image are projected out.

D. The metric on the moduli space of well-separated monopoles

The explicit form of the moduli space metric for k well-separated monopoles in $SU(2)$ gauge theory is known. Setting $g_s=1$, $2|\vec{x}|=1$ [Eq. (56)], and denoting the locations of the monopoles in (x^1,x^2,x^3) by $\vec{r}^i$ and the Wilson lines A_6 by θ^i, so that the $4k$-dimensional monopole moduli space is labeled by $(\vec{r}^i,\theta^i)$, it is (Gibbons and Manton, 1995)

$$ds^2=g_{ij}d\vec{r}^i\cdot d\vec{r}^j+(g^{-1})_{ij}d\tilde{\theta}^i d\tilde{\theta}^j, \tag{65}$$

where

$$g_{jj}=1-\sum_{i\neq j}\frac{1}{r_{ij}}; \quad (\text{no sum over } j)$$

$$g_{ij}=\frac{1}{r_{ij}}; \quad i\neq j,$$

$$d\tilde{\theta}^i=d\theta^i+\vec{W}_{ik}\cdot d\vec{r}^k,$$

$$\vec{W}_{jj}=-\sum_{i\neq j}\vec{w}_{ij}; \quad (\text{no sum over } j)$$

$$\vec{W}_{ij}=\vec{w}_{ij}; \quad (i\neq j). \tag{66}$$

Here $r_{ij}=|\vec{r}_i-\vec{r}_j|$ and $\vec{w}_{ij}$ is the vector potential of a Dirac monopole located at the point $\vec{r}_i$ evaluated at the point $\vec{r}_j$ [Eq. (24)].

In the brane language, one can think of the metric of Eqs. (65) and (66) as the perturbative metric on the "Coulomb branch" of the $U(k)$ gauge theory on the D strings. Classically, $g_{jj}=1$, $g_{ij}=0$ (for $i\neq j$). The corrections proportional to $1/r_{ij}$ in Eq. (66) arise at one loop and can be naturally interpreted as due to the

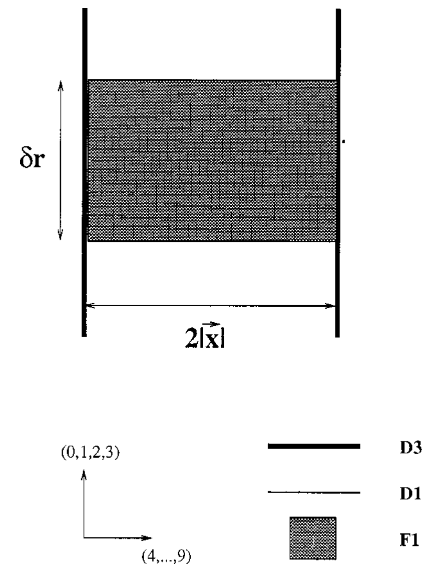

FIG. 8. Nonperturbative corrections to the metric on moduli space, due to Euclidean fundamental strings stretched between the three-branes and adjacent D strings.

asymptotic curving of the three-branes by the D strings [Eq. (51)]. For example, the diagonal components g_{jj} can be thought of as describing the motion of the jth D string in the background of the other $k-1$ strings that curve the two three-branes such that the $\vec{r}$-dependent distance between them (for large $|\vec{r}|$) is

$$\delta x^6 = 1 - \sum_{i\neq j} \frac{1}{|\vec{r}-\vec{r}_i|}. \tag{67}$$

From the point of view of the D-string theory one can interpret Eq. (67) as an $\vec{r}$-dependent gauge coupling. As we shall see in Sec. IV, for systems with eight supercharges the metric g is related to the gauge coupling by supersymmetry. This explains the relation between Eq. (67) and the first line of Eq. (66).

Due to (4,4) supersymmetry, there are no further perturbative corrections to the metric beyond one loop. Nonperturbatively, Eq. (66) cannot be exact, since the diagonal components of the metric are not positive definite. In the brane language, the formula for the curving of the branes, Eq. (67), is only valid asymptotically for large $|\vec{r}|$ while, for $|\vec{r}-\vec{r}_i|\to 0$, x^6 is clearly modified. Instead of diverging, the two three-branes effectively "meet in the middle" of the x^6 interval. Thus Eq. (67) must be modified.

One can think of the nonperturbative corrections to the metric (66) as due to Euclidean fundamental strings stretched between the two $D3$-branes and two adjacent D strings (see Fig. 8). The action of such an instanton is proportional to its area,

$$S = 2|\vec{x}|\delta r/l_s^2 = 2|\vec{\phi}|\delta r = M_w \delta r, \tag{68}$$

where $2\vec{\phi}$ is the Higgs expectation value in the broken $SU(2)$ gauge theory and δr is the separation between adjacent monopoles. The corresponding nonperturbative corrections go like $\exp(-S)\sim\exp(-M_W\delta r)$ where M_W is the mass of the charged W boson. This is consistent with the fact that the size of the magnetic monopole in broken $SU(2)$ gauge theory is M_W^{-1}, much larger than its Compton wavelength $M_{\rm mon}^{-1} = g_{SYM}^2 M_W^{-1}$ for weak coupling g_{SYM}.

Note that the instanton effects (68) are nonperturbative in $l_s^2=\alpha'$, but they survive in the classical string limit $g_s\to 0$. Thus they can be thought of as worldsheet instanton corrections to the metric (66).

IV. FOUR-DIMENSIONAL THEORIES WITH $N=2$ SUPERSYMMETRY

A. Field-theory results

The $N=2$ supersymmetry algebra contains eight supercharges transforming as two copies of the $\mathbf{2}+\overline{\mathbf{2}}$ of spin(1,3). All $N=2$ theories have a global $SU(2)_R$ symmetry which acts on the two supercharges. Scale-invariant theories have in addition a $U(1)_R$ symmetry under which the chiral supercharges have charges ± 1.

To study $N=2$ supersymmetric gauge theory with gauge group G one is interested in two kinds of multiplets. The vector multiplet contains a gauge field A_μ, two Weyl fermions λ_α, ψ_α, and a complex scalar ϕ, all in the adjoint representation of G. The fermions λ, ψ transform in the $\mathbf{2}$ of $SU(2)_R$; A_μ and ϕ are singlets. Under $N=1$ supersymmetry the vector multiplet decomposes into a vector superfield[18]

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu - i\bar{\theta}^2(\theta\lambda) + i\theta^2(\overline{\theta\lambda}) + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{69}$$

with the gauge covariant field strength

$$W_\alpha = \bar{\mathcal{D}}^2(e^{2V}\mathcal{D}_\alpha e^{-2V}) \tag{70}$$

and a chiral superfield

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F. \tag{71}$$

In $N=1$ superspace, the low-energy Lagrangian describing the vector multiplet is

$$\mathcal{L}_{\rm vec} = \mathrm{Im\,Tr}\left[\tau\left(\int d^4\theta \Phi^\dagger e^{-2V}\Phi + \int d^2\theta W_\alpha W^\alpha\right)\right], \tag{72}$$

where the trace runs over the group G and τ is the complex coupling (52). The Lagrangian (72) is invariant under the $U(1)_R$ symmetry $\Phi\to e^{2i\beta}\Phi(e^{-i\beta}\theta)$, which is a consequence of its (classical) conformal invariance. Thus Φ has R charge two.

In components, the bosonic part of the Lagrangian $\mathcal{L}_{\rm vec}$ includes kinetic terms for the fields (10) and a potential for the adjoint scalars ϕ, $\phi^\dagger$ analogous to Eq. (11),

$$V \sim \mathrm{Tr}[\phi^\dagger,\phi]^2. \tag{73}$$

[18]We use the notations of (Wess and Bagger, 1992), except for replacing v_m by A_μ.

$N=2$ SUSY gauge theories in four dimensions can be obtained from $N=1$ SUSY theories in six dimensions by dimensional reduction, i.e., dropping the dependence of all fields on two of the coordinates, say (x^4,x^5). The adjoint chiral superfield in the vector multiplet Φ [Eq. (71)] corresponds to A_4, A_5; the potential (73) arises from the commutator terms in the action (9).

The second multiplet of interest is the hypermultiplet, which in $N=1$ notation consists of two chiral superfields Q, $\tilde{Q}$ in a representation R of the gauge group (and thus contains $2\dim R$ complex scalars and Weyl fermions). The scalar components of Q, $\tilde{Q}$ transform as a doublet under $SU(2)_R$ and carry no charge under $U(1)_R$; the fermions are $SU(2)_R$ singlets and carry $U(1)_R$ charge one. The low-energy Lagrangian describing the hypermultiplet is (in $N=1$ superspace)

$$\mathcal{L}_{\text{hyper}}=\int d^4\theta(Q^\dagger e^{-2V}Q+\tilde{Q}^\dagger e^{-2V}\tilde{Q}) + \int d^2\theta\tilde{Q}\Phi Q+\text{c.c.}, \qquad (74)$$

where $V=V_aT^a$, $a=1,\ldots,\dim G$, and T^a are generators of G in the representation R.

The theory described by Eqs. (72) and (74) has a Coulomb branch corresponding to matrices ϕ satisfying Eq. (73) $[\phi,\phi^\dagger]=0$. It is parametrized by $r=\text{rank}G$ complex moduli corresponding to ϕ in the Cartan subalgebra of G, $\phi=\Sigma_{i=1}^r\phi_iT^i$. The gauge group is generically broken to $U(1)^r$ and the low-energy dynamics is that of r $U(1)$ vector multiplets. $N=2$ supersymmetry ensures that the moduli space of vacua is not lifted by quantum effects but the metric on it is in general modified. The general form of the action consistent with $N=2$ supersymmetry is (Sierra and Townsend, 1983; de Witt *et al.*, 1984; Gates, 1984; Seiberg, 1988)

$$\mathcal{L}_{\text{vec}}=\text{Im Tr}\left[\int d^4\theta\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi_i}\overline{\Phi}_i + \frac{1}{2}\int d^2\theta\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi_i\partial\Phi_j}W^i_\alpha W^\alpha_j\right]. \qquad (75)$$

$\mathcal{F}$ is a holomorphic function of the moduli known as the prepotential. It determines the low-energy $U(1)^r$ gauge coupling matrix τ_{ij}:

$$\tau_{ij}=\frac{\partial^2\mathcal{F}}{\partial\phi_i\partial\phi_j} \qquad (76)$$

and the metric on the moduli space

$$ds^2=\text{Im }\tau_{ij}d\phi_i d\overline{\phi_j}. \qquad (77)$$

Comparing with Eq. (72) we see that, classically, the prepotential is quadratic,

$$\mathcal{F}_0=\frac{1}{2}\tau_{cl}\Phi_i\Phi^i. \qquad (78)$$

After adding the one-loop corrections,[19] it takes the form

$$\mathcal{F}_1=\frac{i}{4\pi}\sum_{\vec{\alpha}>0}(\vec{\alpha}\cdot\vec{\Phi})^2\log\frac{(\vec{\alpha}\cdot\vec{\Phi})^2}{\Lambda^2}, \qquad (79)$$

where the sum runs over positive roots of the Lie algebra of G. The logarithm breaks $U(1)_R$ and is related through the multiplet of anomalies to the one-loop beta function. Higher-order perturbative corrections are absent due to a nonrenormalization theorem, but nonperturbatively Eq. (79) receives a series of instanton corrections that fall off algebraically at large Φ but are crucial for the structure at small Φ.

Seiberg and Witten showed that the prepotential $\mathcal{F}$ can be computed exactly and, in fact, its second derivative τ_{ij} (76) is the period matrix of a Riemann surface (Seiberg and Witten, 1994a). The moduli space of vacua of the $N=2$ SYM theory is thus parametrized by the complex structure of an auxiliary two-dimensional Riemann surface whose physical role seems mysterious. One of our main goals in this section will be to elucidate the meaning of this surface by embedding the problem in string theory.

The prepotential is also important for determining the mass of BPS-saturated states in the theory. From the supersymmetry algebra, one can deduce that the mass of BPS-saturated states with electric charge $(e_1,\ldots,e_r)$ and magnetic charge $(m^1,\ldots,m^r)$ under the r unbroken $U(1)$ gauge fields is

$$M=\sqrt{2}|Z|; \quad Z=\phi^ie_i+\phi_i^Dm^i, \qquad (80)$$

where Z is the central charge and

$$\phi_i^D=\frac{\partial\mathcal{F}}{\partial\phi^i}. \qquad (81)$$

In general, $N=2$ SYM theories also have Higgs branches in which the rank of the unbroken gauge group is decreased. The full phase structure is in general rather rich (see Argyres, Plesser, and Seiberg, 1996; Argyres, Plesser, and Shapere, 1997 for a more detailed discussion). In the remainder of this section we shall describe it in a few examples using branes.

B. Three-branes near seven-branes

As a first example of four-dimensional $N=2$ SYM theory on branes we consider the low-energy worldvolume theory on three-branes in the presence of seven-branes and an orientifold seven-plane in type-IIB string theory (Banks, Douglas, and Seiberg, 1996). This can be thought of as a special case of the $Dp-D(p+4)$ system of Sec. II with a few special features due to the fact that $p+4=7$ is sufficiently large. In particular, the Ramond-Ramond flux of $(1+7)$-dimensional objects does not have enough noncompact transverse directions to es-

[19]For simplicity, we give only the result for gauge theory without matter.

cape. Therefore we should consider configurations with vanishing total Ramond-Ramond charge. In this section we study a particular configuration of this sort.

Consider an $O7$-plane with worldvolume in the $(x^0,x^1,\dots,x^7)$ directions, at a point in the (x^8,x^9) plane. Its Ramond-Ramond charge Q_{O7} is from Eq. (15), $Q_{O7}=-8Q_{D7}$. To cancel this charge we can add $N_f=4$ $D7$-branes and their four mirror images (a total of 8 $D7$-branes with charge Q_{D7} each) parallel to the orientifold seven-plane. When the $D7$-branes coincide with the orientifold plane there is an $SO(8)$ gauge symmetry on their $(1+7)$-dimensional worldvolume. When they are separated from the orientifold this symmetry is generically broken to $U(1)^4$. In a complex parametrization of the (x^8,x^9) plane,

$$w\equiv x^8+ix^9, \tag{82}$$

we can choose the location of the orientifold plane to be

$$w(O7)=0. \tag{83}$$

The locations of the four $D7$-branes and their mirror partners in the (x^8,x^9) plane will be denoted by m_i and $-m_i$, respectively.

In addition, we place a $D3$-brane and its mirror image at[20] w and $-w$, respectively. As explained in Sec. II, the low-energy $(1+3)$-dimensional worldvolume dynamics on the three-branes is an $Sp(1)\simeq SU(2)$ gauge theory with eight supercharges, namely, an $N=2$ supersymmetric gauge theory in four dimensions. The neutral gauge boson W^3_μ corresponds to the ground state of an open string both of whose ends terminate on the three-brane. The charged gauge bosons $W^\pm_\mu$ correspond to the ground states of strings stretched between the $D3$-brane and its mirror image. The $D7$-branes are heavy objects; thus from the point of view of $(1+3)$-dimensional physics, their $SO(8)$ gauge symmetry is "frozen," i.e., the corresponding gauge coupling vanishes. Moduli in the seven-brane theory give rise to parameters in the $(1+3)$-dimensional Lagrangian.

The location of the three-brane in the (x^8,x^9) plane corresponds to the expectation value of the complex chiral field in the adjoint of $SU(2)$:

$$\Phi_{ab}(x^\mu)\equiv X^8_{ab}+iX^9_{ab}\,;\quad a,b=1,2;\quad \mathrm{Tr}\Phi=0, \tag{84}$$

which belongs to the $SU(2)$ vector multiplet (71). It can be diagonalized to

$$\langle\Phi\rangle=\begin{pmatrix} w & 0\\ 0 & -w\end{pmatrix}. \tag{85}$$

When $w=0$, the minimal length of strings stretched between the $D3$-branes vanishes and the charged gauge bosons are massless. When $w\neq 0$, Eq. (85) breaks $SU(2)$ to $U(1)$. Correspondingly, the strings stretched from the three-brane to its mirror image have minimal length $2|w|$—the mass of $W^\pm$ (in string units).

[20]The location of the three-brane in the directions along the $O7/D7$ is not important for what follows.

The $D7$-branes give rise to $N_f=4$ fundamental hypermultiplets $(Q^i,\tilde{Q}_i)$. Their locations m_i are the bare complex masses of quarks. Analyzing configurations of strings stretched between the $D3$- and $D7$-branes we see that at tree level the superpotential is just that expected on the basis of $N=2$ supersymmetry:

$$W=\sum_{i=1}^{4}\,(Q^i\Phi\tilde{Q}_i-m_iQ^i\tilde{Q}_i). \tag{86}$$

The effective masses of the quarks m_i-w and m_i+w are the locations of the four $D7$-branes and their mirror images, respectively, relative to the $D3$-brane. The $SO(8)$ gauge symmetry on the worldvolume of the $D7$-branes turns into a global symmetry of the four-dimensional gauge theory on the three-branes. It is broken when $m_i\neq 0$.

As in the $N=4$ SYM case discussed in the previous section, the complex gauge coupling of the $N=2$ SYM theory on the three-brane corresponds to the complex dilaton (35) of type-IIB string theory. The $D7$-branes and $O7$-plane carry charge under the complex dilaton field. Thus it is nontrivial in their presence and, in particular, when we go once around a $D7$-brane, τ has a monodromy: $\tau\to\tau+1$. Far from the $D7$-branes located at $w=m_i$ and from the $O7$-plane located at $w=0$ we expect τ to behave as

$$\tau(w)=\tau_0+\frac{1}{2\pi i}\left[\sum_{i=1}^{4}\,(\log(w-m_i)+\log(w+m_i))\right.$$
$$\left.-8\log w\right] \tag{87}$$

since there is charge $+1$ at each $w=\pm m_i$ and charge -8 at $w=0$.

Note that one can use the above analysis to understand the identification of the complex dilaton of type-IIB string theory [Eq. (87)] with the gauge coupling of the theory on the $D3$-brane. The metric on the (x^8,x^9) plane implied by Eq. (87) can be interpreted either as the metric induced by the $O7$-plane and $D7$-branes or as the metric on the Coulomb branch of the $N=2$ SYM theory on the worldvolume of the three-brane. In the first interpretation this metric is determined by the complex coupling of type-IIB string theory; in the second, it is related by Eqs. (76) and (77) to the complex gauge coupling τ. This establishes the relation between the two τ's.

The complex coupling τ is gauge invariant; this is made manifest by rewriting Eq. (87) as

$$\tau(u)=\tau_0+\frac{1}{2\pi i}\left[\sum_{i=1}^{4}\,\log(u-m_i^2)-4\log u\right], \tag{88}$$

where u is the gauge-invariant modulus:

$$u=\frac{1}{2}\mathrm{Tr}\Phi^2=w^2. \tag{89}$$

The semiclassical result (88) corresponds in gauge theory to the one-loop corrected prepotential [Eq. (79)].

As in the brane picture, semiclassically, the $SU(2)$ gauge symmetry is restored at the origin $u=0$ where $W^{\pm}$ become massless, while quarks $(Q^i,\tilde{Q}_i)$ become massless when $u=m_i^2$. The appearance of new massless states is the reason for the singularities at $u=0,m_i^2$ in Eq. (88). The coefficient 4 in front of $\log u$ is due to the fact that $W^{\pm}$ carry twice the electric charge of a quark, and the relative sign between the two logs in Eq. (88) is due to the fact that the $W^{\pm}$ belong to a vector multiplet whose contribution to the beta function has an opposite sign to that of a hypermultiplet.

The one-loop result (88) is not corrected perturbatively, but it cannot be exact, since it does not satisfy $\mathrm{Im}\,\tau \geqslant 0$ everywhere in the u plane; for small u, $\mathrm{Im}\,\tau$ becomes large and negative. Therefore we expect that strong-coupling effects will modify the solution for finite u (Seiberg and Witten, 1994a, 1994b; Sen, 1996). Indeed, as discussed in Sec. IV.A, in the $N=2$ SYM analysis one finds that instanton corrections modify Eq. (88). The exact effective coupling is a modular parameter $\tau(u)$ of a torus described by the elliptic curve

$$y^2=x^3+f(u,\tau_0)x+g(u,\tau_0), \tag{90}$$

where $x,y,u\in CP^1$, $f(u)$ is a polynomial of degree 2, $g(u)$ is a polynomial of degree 3 in the gauge-invariant modulus u, and $\exp(i\pi\tau_0)\equiv\Lambda$ is the "QCD scale" of the theory. In the semiclassical limit, namely, for large $\mathrm{Im}\,\tau_0$ and $|u|\gg|\Lambda|^2$, the exact $\tau(u)$ can be rewritten as Eq. (88). Strong-coupling dynamics splits the singularity at the origin into two singularities at $u=\pm\Lambda^2$ corresponding to a monopole or dyon's becoming massless.

The full nonperturbative description of the type-IIB vacuum discussed above involves three-branes in F theory on $K3$ (Vafa, 1996)—a compactification of the type-IIB string on CP^1 labeled by the coordinate u, with a nontrivial complex dilaton describing a two-torus with modular parameter $\tau(u)$ for each point on CP^1. This elliptically fibered surface is given by the algebraic Eq. (90).

In the F-theory description the three-brane moves in the background of six seven-branes located at the singularities of the curve (90). In the weak string-coupling limit four of these branes can be regarded as (1,0) seven-branes, namely, conventional D-branes, while the other two are a (0,1) seven-brane and a (2,1) seven-brane related to a $D7$-brane by $SL(2,Z)$ S-duality transformations. The (1,0) seven-branes are free to move in the u plane, while the (0,1) and (2,1) seven-branes are stuck at $u=\pm\Lambda^2$. As $g_s\to 0$ these branes approach each other and are described at weak coupling by an $O7$-plane. BPS-saturated states with electric and magnetic charges $(e,m)=(p,q)$ in the four-dimensional $N=2$ SYM theory on the three-brane can be described by (p,q) strings stretched between the (p,q) seven-branes and the $D3$-brane (Sen, 1997a).

C. Branes suspended between branes

The fact that branes can end on other branes was deduced in Sec. II.E.3 by starting from a fundamental string ending on a D-brane and applying U duality. Following the same logic we can deduce that branes can be suspended between other branes by starting with a configuration of fundamental strings stretched between two D-branes and applying a chain of duality transformations. As in Sec. II.E.3, one can get in this way Dp-branes stretched between two $D(p+2)$ or NS5-branes. The special case of D strings stretched between two $D3$-branes was used in Sec. III to describe BPS-saturated 't Hooft–Polyakov monopoles of a broken $SU(2)$ gauge theory.

In this section we shall study similar configurations with eight supercharges describing $(3+1)$-dimensional physics and use them to learn about $N=2$ SYM theory. The starting point of our discussion will be brane configurations in type-IIA string theory consisting of solitonic (Neveu-Schwarz) five-branes, $D4$-branes, and $D6$-branes as well as orientifold planes $O4$ and $O6$ parallel to the D-branes. Using Eqs. (6), (18), and (19) it is not difficult to check that any combination of two or more of the following objects:

$$\begin{aligned}&\mathrm{NS5}:\ (x^0,x^1,x^2,x^3,x^4,x^5),\\ &D4/O4:\ (x^0,x^1,x^2,x^3,x^6),\\ &D6/O6:\ (x^0,x^1,x^2,x^3,x^7,x^8,x^9),\end{aligned} \tag{91}$$

preserves eight of the 32 supercharges of type-IIA string theory [Eq. (2)]. In Eq. (91) we specified the directions in which each of the branes is stretched.

The Lorentz group $SO(1,9)$ is broken by the presence of the branes to

$$SO(1,9)\to SO(1,3)\times SO(2)\times SO(3), \tag{92}$$

where the $SO(1,3)$ factor acts on (x^0,x^1,x^2,x^3), the $SO(2)$ on (x^4,x^5), and the $SO(3)$ on (x^7,x^8,x^9). We shall be interested in physics in the $(1+3)$-dimensional spacetime common to all the branes labeled by (x^0,x^1,x^2,x^3); thus we interpret the $SO(1,3)$ factor in Eq. (92) as Lorentz symmetry and the $SO(2)$, $SO(3)$ factors as global symmetries. Due to the ten-dimensional origin of these global symmetries, the supercharges transform as doublets under $SO(3)$ and are charged under $SO(2)$. Thus these are R symmetries. In fact, the $SO(3)$ can be identified with the global $SU(2)_R$ of $N=2$ SYM theory described in Sec. IV.A, while the $SO(2)$ can be identified with the $U(1)_R$ symmetry.

To study situations with interesting $(1+3)$-dimensional physics some of the branes must be made finite. We next turn to a discussion of some specific configurations and their physics (Elitzur, Giveon, and Kutasov, 1997; Hanany and Witten, 1997). We start with a description of the "classical" type-IIA string picture in a few cases involving unitary, symplectic, and orthogonal groups with matter in the fundamental representation of the gauge group, as well as a few more complicated examples, and then study quantum effects.

1. Unitary gauge groups

Consider two infinite NS5-branes oriented as in Eq. (91), separated by a distance L_6 in the x^6 direction and

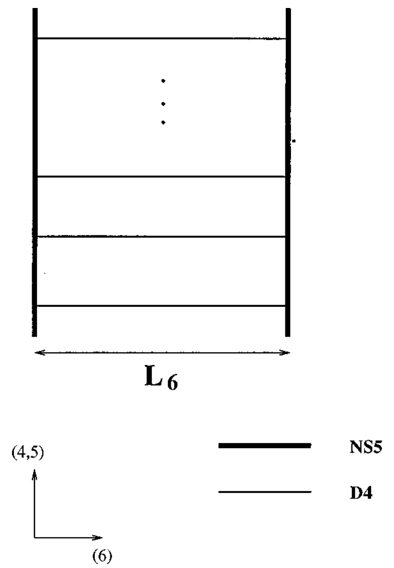

FIG. 9. N_c $D4$-branes stretched between NS5-branes describing $N=2$ supersymmetric Yang-Mills theory with $G=U(N_c)$.

located at the same point in (x^7,x^8,x^9). We can stretch between them in the x^6 direction N_c $D4$-branes oriented as in Eq. (91); see Fig. 9.

To analyze the low-energy physics corresponding to this configuration it is important to distinguish between three kinds of excitations of the vacuum: (1) modes that live in the ten-dimensional bulk of spacetime; (2) modes that live on the two NS5-branes; (3) modes that live on the four-branes. For an observer living in the $1+3$ dimensions (x^0,x^1,x^2,x^3) the first two sets of excitations appear at low energies and in the weak string-coupling limit to be frozen at their classical values by an argument similar to that given around Eq. (47). Essentially, since they correspond to higher-dimensional excitations, any long-wavelength fluctuations away from the classical values of these fields are suppressed by infinite-volume factors.

Excitations attached to the N_c four-branes do not have this property. Despite living in one higher dimension (x^6), they are dynamical in $1+3$ dimensions since the four-branes are finite in x^6. Thus excitations of the four-branes can be thought of as fields living in the $(1+4)$-dimensional space $R^{1,3}\times I$ where I is a finite interval (of length L_6). As in Kaluza-Klein theory, for distance scales much larger than L_6 their physics looks $(1+3)$ dimensional. Depending on the boundary conditions at the ends of the interval, the different fields do or do not give rise to light fields in $1+3$ dimensions.

The analysis of the boundary conditions can be done using the results of Sec. II.E.3. The light excitations on a stack of N_c infinite $D4$-branes [Eq. (91)] are a five-dimensional $U(N_c)$ gauge field A_μ, $\mu=0,1,2,3,6$ and five scalars in the adjoint of $U(N_c)$ corresponding to transverse fluctuations of the four-branes, (X^4,X^5,X^7,X^8,X^9). As we saw in Sec. II.E.3, when the N_c four-branes end on five-branes, (X^7,X^8,X^9) as well as A_6 satisfy Dirichlet boundary conditions on both ends of the interval I. Therefore they give rise in $1+3$ dimensions to states with masses of order $1/L_6$, which are invisible in the low-energy limit of interest, $E\ll 1/L_6$. The remaining light degrees of freedom, the $U(N_c)$ gauge field A_μ, $\mu=0,1,2,3$ and the adjoint scalars (X^4,X^5) satisfy free boundary conditions on I and, therefore, contribute a $U(N_c)$ vector multiplet.

Thus we conclude that the light excitations of the brane configuration above describe an $N=2$ SUSY gauge theory with gauge group $U(N_c)$ and no matter.[21] The gauge coupling of the $(4+1)$-dimensional gauge theory on N_c four-branes is given by standard D-brane techniques [Eq. (8)] to be $g_{D4}^2=g_s l_s$. Upon Kaluza-Klein reduction on a line segment of length L_6 we find a $(3+1)$-dimensional gauge theory with coupling

$$\frac{1}{g^2}=\frac{L_6}{g_s l_s}. \tag{93}$$

It is interesting to ask in what regime the brane configuration is well approximated by an $N=2$ SUSY $(3+1)$-dimensional gauge theory. There are several issues that need to be addressed in this regard. First, the physics on the four-branes looks $(3+1)$-dimensional only at distances much larger than L_6. At shorter distances Kaluza-Klein excitations on the four-branes, whose typical energy is $1/L_6$, begin to play a role and the dynamics becomes $(4+1)$ dimensional. Furthermore, in general there are couplings of the light fields on the four-branes to light fields living on the NS5-branes, to massive excited states of the $4-4$ strings living on the four-branes, and to fields living in the bulk of spacetime, such as gravitons. These couplings are small in the limit $g_s\to 0$ and at distances much larger than l_s.

Thus to study gauge-theory dynamics we are led to consider the brane configuration above in the limit

$$g_s\to 0, \quad L_6/l_s\to 0 \tag{94}$$

with the ratio corresponding to g (93) held fixed. If the gauge coupling at some scale L satisfying $L\gg l_s\gg L_6$ is small but finite, at larger distances the dynamics of the brane configuration will be governed by gauge theory, with the other effects mentioned above providing small corrections.

[21]This theory can be thought of as a reduction of $N=1$ SYM theory in $1+5$ dimensions down to $1+3$ dimensions. In the brane description this process of dimensional reduction is described by compactification, followed by T duality and subsequent decompactification of (x^4,x^5). The six-dimensional version of the theory is obtained by replacing the four-branes in Eq. (91) by six-branes stretched along (x^4,x^5). This enhances the unbroken Lorentz symmetry [Eq. (92)] to $SO(1,5)\times SO(3)$, which is indeed the global symmetry of $N=1$ SYM theory in $1+5$ dimensions. The $SO(3)$ corresponds to the $SU(2)_R$ symmetry of $(1+5)$-dimensional $N=1$ SYM theory, under which the two supercharges in the **4** of spin(1,5) transform as a doublet. Upon reduction to $1+3$ dimensions another global $SO(2)$ symmetry appears [Eq. (92)]. As we shall see later, the consistency constraints in six dimensions are more restrictive than in $4d$; thus only some of the consistent models in $4d$ can be lifted to $6d$.

The classical gauge-theory limit is obtained when, in addition to sending L_6 and g_s to zero, we also send the combination (93) to zero. The theory simplifies in this limit, since when $g=0$ we can ignore the effects of the ends of the four-branes on the NS5-branes. In this section we discuss this classical limit; later we shall describe the modifications that take place when quantum effects are turned on.

Classical $U(N_c)$ $N=2$ SYM theory in 3+1 dimensions has an N_c-(complex)-dimensional moduli space of vacua parametrized by diagonal expectation values of the complex adjoint scalar Φ that belongs to the vector multiplet. At a generic point in the moduli space, $U(N_c)$ is broken to $U(1)^{N_c}$ and the charged gauge bosons are massive.

In the brane description, the complex adjoint field Φ [Eq. (71)] can be thought of as describing fluctuations of the four-branes along the five-branes, $X\equiv\Phi l_s^2=X^4+iX^5$. Turning on an expectation value for Φ corresponds to translations of the N_c four-branes in x^4, x^5. For generic positions of the four-branes along the five-branes, the $4-4$ strings connecting different four-branes [corresponding to vector multiplets charged under a pair of $U(1)$'s] have finite length and, therefore, describe massive states. Note also that Φ has the correct global charges. Turning on an expectation value for Φ breaks the $SO(2)$ symmetry [Eq. (92)]. Thus Φ carries charge (which is two if we normalize the charge of the supercharges to one) under $U(1)_R$. Similarly, it is clear that it transforms as a singlet under $SO(3)\simeq SU(2)_R$; both facts are in agreement with the field-theory discussion of Sec. IV.A.

To have a consistent four-dimensional interpretation of the Coulomb branch we have to require that in the limit (94) the Higgs expectation value $\langle\Phi\rangle$ remain well below the Kaluza-Klein scale $1/L_6$. This means that the typical separation between the four-branes in the (x^4,x^5) plane δx must satisfy $\delta x \ll l_s^2/L_6$. One should also require that δx be less than l_s, L_6 to decouple massive string modes on the four-branes. The resulting hierarchy of scales in the gauge theory limit (94) is

$$\delta x \ll L_6 \ll l_s \ll \frac{l_s^2}{L_6}. \tag{95}$$

To add matter in the fundamental representation of the gauge group we can proceed in one of a number of related ways. One is to add semi-infinite four-branes attached to one of the NS5-branes. For example, one can add N_f four-branes ending on the left NS5-brane from the left, extending to $x^6\to-\infty$, as shown in Fig. 10. Adding the semi-infinite four-branes gives rise to N_f hypermultiplets in the fundamental representation of $U(N_c)$ corresponding to strings stretched between the N_c suspended four-branes and the N_f semi-infinite ones. The locations at which these semi-infinite four-branes attach to the five-brane in the (x^4,x^5) plane are N_f complex numbers $m_1,\ldots,m_{N_f}$, which can be thought of as the masses of the quarks.[22] Note that these locations correspond to expectation values of scalar fields living on the worldvolume of the semi-infinite four-branes. As before, since these four-branes are "heavy" objects, they are frozen at their classical values and give rise to couplings rather than moduli in the four-dimensional low-energy theory.

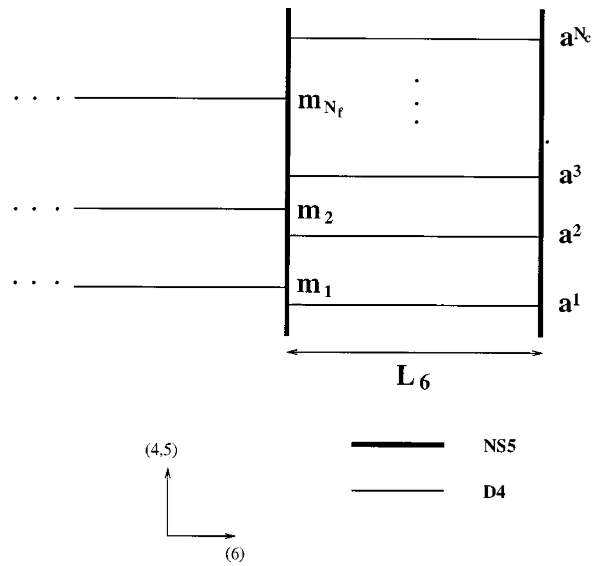

FIG. 10. N_c "color four-branes" stretched between NS5-branes in the presence of N_f semi-infinite "flavor four-branes," describing $N=2$ supersymmetric Yang-Mills theory with $G=U(N_c)$ and N_f fundamental hypermultiplets.

A generic point in the Coulomb branch is parametrized by the N_c complex numbers $a^1,\ldots,a^{N_c}$ corresponding to the locations in the (x^4,x^5) plane of the suspended $D4$-branes. From gauge theory we know that due to the superpotential (86) (with the sum running over all N_f flavors), which is required by $N=2$ supersymmetry, the mass of the quark Q_α^i corresponding to the ith flavor and the αth color is $m_i^\alpha=|m_i-a^\alpha|$. In the brane picture, the mass m_i^α is given by the minimal energy of a fundamental string stretched between the αth suspended brane and the ith semi-infinite one. Just like the adjoint field Φ, the mass parameters m_i (86) carry $U(1)_R$ charge two. Turning on masses breaks $U(1)_R$ (as well as conformal invariance).

While the above way of introducing fundamental matter is appropriate for describing the Coulomb branch of $U(N_c)$ gauge theory with N_f flavors, it does not provide a geometric description of the Higgs branches. Recall that the gauge theory in question has a number of branches of the moduli space of vacua along which some of the quarks Q, $\tilde{Q}$ get expectation values and the rank of the unbroken gauge group decreases. For $N_f\geq 2N_c$ the gauge group can be completely Higgsed and the complex dimension of the corresponding branch of moduli space is $2N_cN_f-2N_c^2$.

[22]Up to a factor of l_s^2, which is needed to fix the dimensions; we shall usually set $l_s=1$ from now on.

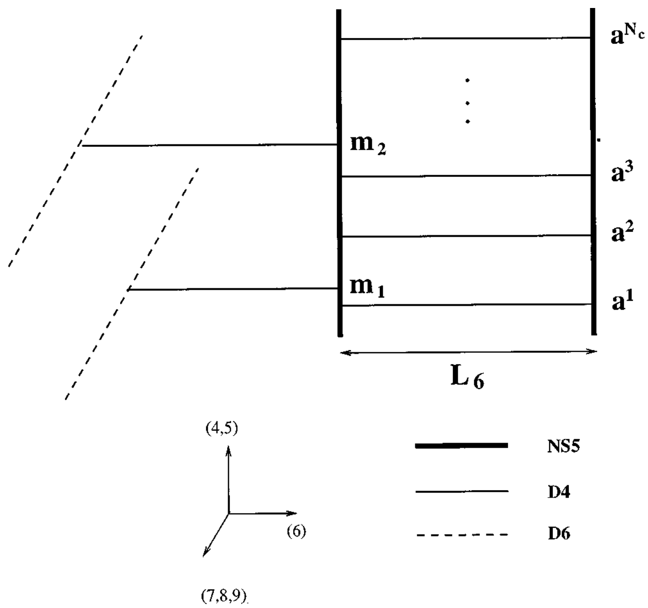

FIG. 11. Replacement of the semi-infinite flavor $D4$-branes of Fig. 10 by $D4$-branes ending on $D6$-branes, which allows one to explore the full phase structure of the theory.

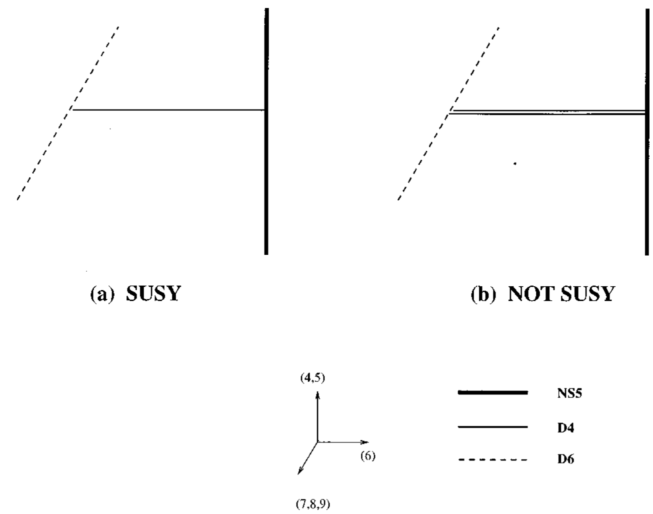

FIG. 12. Supersymmetric and nonsupersymmetric configurations: (a) A supersymmetric configuration containing an NS5-brane connected to a $D6$-brane by a single $D4$-brane. (b) A nonsupersymmetric configuration in which the two are connected by two $D4$-branes.

To recover the Higgs branches it is convenient to generalize the above construction of matter in the manner shown in Fig. 11. Replace the semi-infinite $D4$-branes to the left of the NS5-branes by finite $D4$-branes, each ending on a different $D6$-brane, oriented as in Eq. (91). This opens up the possibility of having additional dynamical degrees of freedom living on these four-branes.

For generic masses $\{m_i\}$, which can now be thought of as positions of the N_f $D6$-branes in the (x^4,x^5) plane and points in the Coulomb branch $\{a^\alpha\}$, there are no new massless states of this kind. Indeed, all potentially light fields living on a four-brane stretched between an NS5-brane and a $D6$-brane have Dirichlet boundary conditions on one or both boundaries and hence do not lead to massless degrees of freedom. That is consistent with the absence of Higgs branches of $N=2$ SYM theory when all the masses m_i are distinct.

When two or more masses coincide, say $m_1=m_2$, we expect from gauge theory to be able to enter a Higgs branch by turning on an expectation value to quarks Q, $\tilde{Q}$. Furthermore, to enter the Higgs branch one needs to go to a particular point in the Coulomb branch for which for some $1\leqslant\alpha\leqslant N_c$, $m_1^\alpha=m_2^\alpha=0$. To reproduce this in the brane description one notes that when two masses m_i coincide, two $D6$-branes are at the same position in the (x^4,x^5) plane. In general they are still separate in the x^6 direction and each is connected to the same NS5-brane by a $D4$-brane. Thus the four-brane connecting the "far" $D6$-brane to the NS5-brane meets in space and intersects the "near" $D6$-brane. From our discussion of brane intersections in Sec. II.E.3 one might conclude at this point that this four-brane can break into two pieces, one stretched between the NS5-brane and the near $D6$-brane and the other between the two $D6$-branes. While the first piece would as before give rise to no massless degrees of freedom in $3+1$ dimensions, the second one would give rise to a neutral (under the gauge group) massless hypermultiplet. The scalars in the hypermultiplet would correspond to displacements of the $D4$-brane along the $D6$-branes in the (x^7,x^8,x^9) directions, and the compact component of the gauge field A_6. This would provide a candidate for a brane realization of the phase transition from the Coulomb to the Higgs phase.

However, the picture we got so far is inconsistent with gauge theory. The process described above appears to be possible for any values of the Coulomb moduli; in gauge theory we have to tune to a particular point in the Coulomb branch in order to be able to enter the Higgs branch. This and many related puzzles are resolved by noting (Hanany and Witten, 1997) that the following "*s* rule" holds in brane dynamics: *A configuration in which an NS five-brane and a D six-brane are connected by more than one D4-brane is not supersymmetric.* The s rule, which is illustrated in Fig. 12, is a phenomenological rule of brane dynamics that has been recently discussed from various points of view for example, by Ooguri and Vafa, 1997; Bachas and Green, 1998; Bachas, Green, and Schwimmer, 1998; Hori, Ooguri, and Oz, 1998. It seems to have to do with the fact that two or more four-branes connecting a given NS five-brane to a given D six-brane are necessarily on top of each other—a rather singular situation. In particular, Bachas and Green (1997); see also Bachas, Green, and Schiwmmer (1998) related it by U duality to Pauli's exclusion principle.

The *s* rule explains why the process described above is forbidden, but the comparison to the gauge-theory picture suggests a way out. If in addition to having two $D6$-branes coincide in the (x^4,x^5) plane we also go to a point in the Coulomb branch where one of the N_c

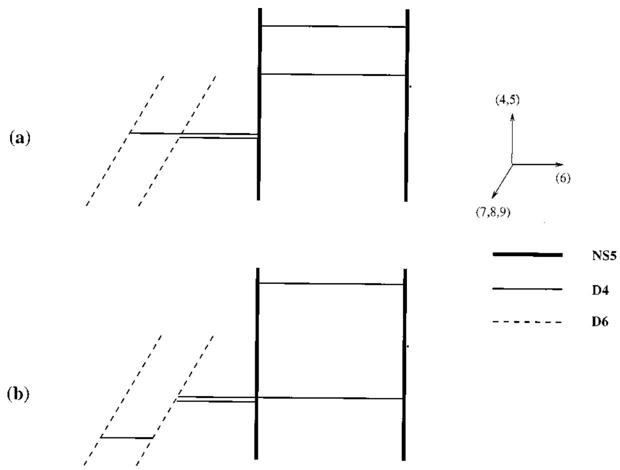

FIG. 13. The transition from Coulomb (a) to Higgs (b) branch.

$D4$-branes suspended between the NS-branes is at the same value of (x^4,x^5) as well, the above phase transition becomes possible (Fig. 13).

All we have to do is first reconnect the $D4$-brane stretched between the left NS5-brane and the far $D6$-brane, combining it with the coincident $D4$-brane suspended between the NS-branes such that it now connects the *right* NS5-brane to the far $D6$. Now there is no conflict with the s rule in breaking the resulting stretched $D4$ into two pieces as described above. As expected in the Higgs mechanism, in the process we replace a massless $U(1)$ vector multiplet corresponding to a $D4$-brane stretched between two NS5-branes by a massless neutral hypermultiplet $\tilde{Q}Q$ corresponding to a $D4$-brane stretched between adjacent (in x^6) $D6$-branes. The expectation value $\langle\tilde{Q}Q\rangle$ is parametrized by the location of this $D4$-brane along the $D6$-branes in (x^7,x^8,x^9) and the Wilson line of A_6. This is consistent with the expected transformation of Q, $\tilde{Q}$ under $SU(2)_R\times U(1)_R$: they are not charged under $U(1)_R$ and transform as a doublet under $SU(2)_R$. Thus the "mesons" $\tilde{Q}Q$ transform as $\mathbf{2}\times\mathbf{2}=\mathbf{3}+\mathbf{1}$.

Thus we learn that the Higgs mechanism is described in brane theory as the reconnection (or breaking) of $D4$-branes stretched between NS5-branes which give rise to vector multiplets that are replaced by hypermultiplets stretched between $D6$-branes and/or NS5-branes consistently with the s rule. Performing all such breakings in the general case of N_c colors and N_f flavors gives rise to the correct (classical) phase structure of $N=2$ SYM theory.

As an example, to compute the dimension of the maximally Higgsed branch (where the gauge symmetry is completely broken) for $N_f\geqslant 2N_c$ we proceed as follows [see Fig. 14(a)]:

- Set all the masses equal to each other, i.e., bring all N_f $D6$-branes to the same point in (x^4,x^5), say the origin. They are still at different positions in x^6.
- Reconnect the $D4$-brane stretching between the left NS5-brane and the leftmost $D6$-brane to stretch between the right NS5 and the leftmost $D6$, by going to the point in the Coulomb branch where one of the N_c "color" $D4$'s is at the origin in the (x^4,x^5) plane as well.
- Break the resulting $D4$ into N_f pieces stretching between the right NS5 and the rightmost $D6$ and the different adjacent $D6$'s. This leads to N_f-1 massless hypermultiplets corresponding to fluctuations of the N_f-1 segments of the $D4$ stretched between the $D6$'s. The expectation values of these are $2(N_f-1)$ complex moduli.
- By bringing in another color $D4$, reconnect the "second longest" $D4$ stretching between the left NS5 and the next-to-leftmost $D6$ to the right NS5. Repeat the breaking procedure. The s rule applied to the *right* NS5 implies that there are now N_f-3 massless hypermultiplets and hence $2(N_f-3)$ complex moduli.
- Continuing this process gives rise to

$$\sum_{i=1}^{N_c}[N_f-(2i-1)]=N_fN_c-N_c^2 \tag{96}$$

massless hypermultiplets and to a $2(N_fN_c-N_c^2)$ complex-dimensional Higgs moduli space, in agreement with gauge-theory expectations.

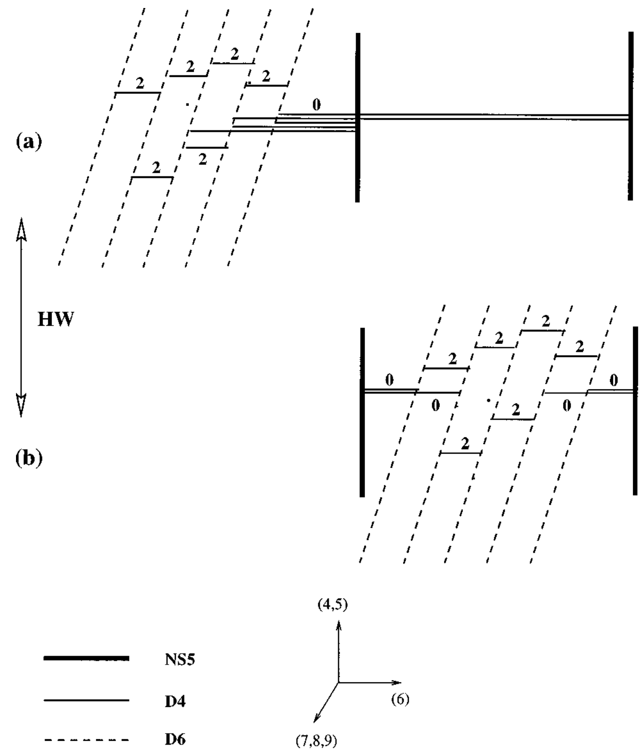

FIG. 14. The fully Higgsed branch of moduli space for $N_f=5$, $N_c=2$; the two equivalent descriptions are related by a series of Hanany-Witten transitions.

A peculiar feature of the above analysis is the (lack of) role of the parameters describing the positions along the x^6 axis of the N_f $D6$-branes providing the flavors. There are no parameters in the low-energy $N=2$ supersymmetric QCD corresponding to changing these pa-

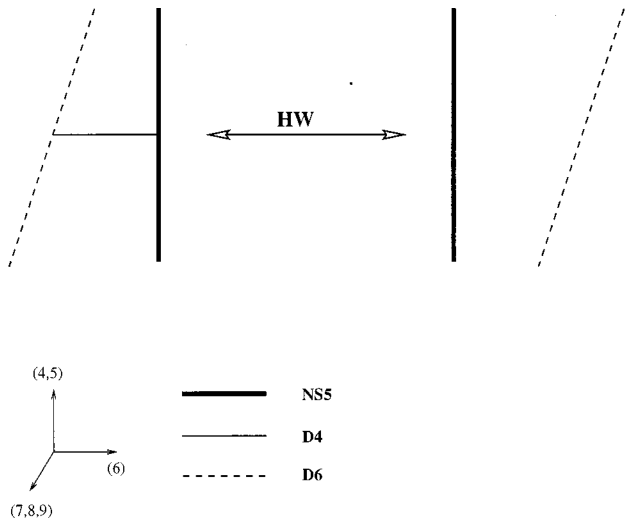

FIG. 15. The Hanany-Witten transition, in which a four-brane is created when an NS5-brane and a $D6$-brane cross in x^6 and exchange positions.

rameters and thus they are irrelevant (in the RG sense). Indeed, the physics of the brane configuration is independent of the locations of the $D6$-branes, at least when they are all to the left (or equivalently all to the right) of the two NS5-branes.

It is interesting to ask what happens if we try to vary the x^6 positions of the $D6$-branes, bringing some or all of them inside the interval between the two NS-branes. Because of the way the different branes are oriented [Eq. (91)], the $D6$ and NS5-branes cannot avoid each other, and when their x^6 values coincide they actually meet in space. What takes place when they switch positions is known as the Hanany-Witten (HW) transition (Hanany and Witten, 1997). The $D4$-brane connecting the $D6$ and NS5 becomes very short as they approach each other and *disappears* when they cross. Conversely, if the $D6$ and NS5 that approach each other do not have a $D4$-brane connecting them, one is created when they exchange positions (Fig. 15).

The Hanany-Witten transition is an interesting effect of brane dynamics which is related by U duality to similar transitions for other branes; it has been investigated from several perspectives (for example, by Bachas, Douglas, and Green, 1997; Danielsson, Ferretti, and Klebanov, 1997; de Alwis, 1997; Ooguri and Vafa, 1997; Bergman, Gaberdiel, and Lifschytz, 1998a, 1998b; Ho and Wu, 1998; Nakatsu *et al.*, 1998b; Ohta, Shimizu, and Zhou, 1998; Yoshida, 1998). Heuristically, it is related to conservation of a certain magnetic charge that can be measured on each brane known as the "linking number." The total charge measured on each brane is

$$L_B=\frac{1}{2}(r-l)+L-R. \tag{97}$$

For an NS5-brane, r and l are the numbers of $D6$-branes to its right and left, respectively, while R and L are the numbers of $D4$-branes ending on the NS5-brane from the right and left, respectively. Similarly, for a $D6$-brane r and l are the numbers of NS5-branes to its right and left, and R and L are as above. Right and left here refer to locations along the x^6 axis.

As an example, for a $D6$-brane connected to an NS5-brane on its right (in x^6) by a $D4$-brane the linking number is $L_{D6}=-1/2$. For the NS5-brane the linking number is $L_{\rm NS5}=+1/2$. If we try to move the $D6$-brane past the NS5-brane, the $D4$ connecting them disappears. The new configuration includes a $D6$ with a disconnected NS5 on its left; the linking numbers are seen from Eq. (97) to be unchanged.

Taking the Hanany-Witten transition into account we can analyze what happens when the $D6$-branes are translated in the x^6 direction and placed in the interval between the two NS-branes. All the $D4$-branes that initially connected the $D6$-branes to an NS5-brane disappear, and we end up with a configuration of two NS-branes at different values of x^6 connected to each other by N_c four-branes, with N_f six-branes located between them in x^6 [see Fig. 14(b)].

Remarkably, the resulting brane configuration describes the same low-energy physics! This is *a priori* surprising since one would in general expect a phase transition to occur as the two branes cross; indeed, we shall see that such transitions occur when the crossing branes are parallel. It is not well understood why there is no phase transition when nonparallel branes cross.

In any case, in the present setup the quarks Q, $\tilde{Q}$ that corresponded to $4-4$ strings before are now due to $4-6$ strings stretched between the N_c suspended four-branes and the N_f six-branes. The locations of the six-branes in the (x^4,x^5) plane still correspond to their masses, and the Higgs branch of the moduli space is described by breaking $D4$-branes stretched between the two NS5-branes on the N_f $D6$-branes. Taking into account the s rule, it is easy to see that the dimension of the Higgs branch is as described above in Eq. (96).

The $N=2$ SYM theory under consideration has gauge group $U(N_c)\simeq SU(N_c)\times U(1)$. Thus one can turn on Fayet-Iliopoulos couplings for the $U(1)$. In $N=1$ superspace they are an $N=1$ Fayet-Iliopoulos D term, and a linear superpotential for the adjoint chiral superfield in the vector multiplet:

$$\mathcal{L}_{FI}={\rm Tr}\left(r_3\int d^4\theta V+r_+\int d^2\theta\Phi+r_-\int d^2\overline{\theta\Phi}\right). \tag{98}$$

Here r_3 is real, while $r_+^*=r_-$. The three Fayet-Iliopoulos couplings $r_3,r_\pm$ transform as the **3** of $SU(2)_R$ and are neutral under $U(1)_R$. For $N_f\geqslant N_c$ the D terms break the gauge group completely and force the system into the Higgs branch. For smaller N_f there is no supersymmetric vacuum.

In the brane language, the Fayet-Iliopoulos couplings correspond to the relative position of the two NS5-branes in the (x^7,x^8,x^9) directions (Fig. 16); note that these parameters have the correct transformation properties under $SO(3)\simeq SU(2)_R$. From the geometry and Eq. (91) it is clear that only when the two NS5-branes are at the same value of (x^7,x^8,x^9) can the $D4$-branes

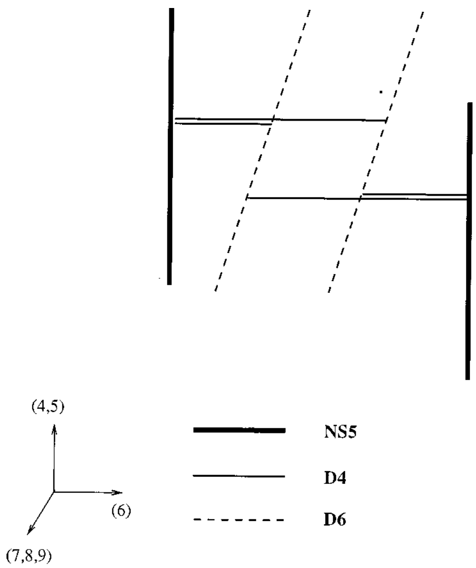

FIG. 16. Fayet-Iliopoulos D terms in the $N=2$ supersymmetric Yang-Mills theory, corresponding to relative displacements of the two NS5-branes in (x^7,x^8,x^9). For nonzero D terms, all color $D4$-branes must break, and the theory is forced into the Higgs phase.

stretch between them without breaking supersymmetry. For nonzero D terms all the $D4$-branes must break on $D6$-branes. This corresponds to complete Higgsing of the gauge group; it is consistent with the s rule for $N_f \geqslant N_c$; for smaller N_f there is no supersymmetric vacuum.

2. Orthogonal and symplectic groups

In this section we discuss configurations of branes near orientifold planes. As we saw before, adding orientifold four- and six- ($O4$ and $O6$) planes as in Eq. (91) does not lead to the breaking of any further supersymmetry or global symmetry [Eq. (92)]. In the simplest cases one finds $N=2$ SYM theories with orthogonal and symplectic gauge groups and matter in the fundamental representation. We next discuss in turn the two cases of an $O6$-plane parallel to the $D6$-branes and of an $O4$-plane parallel to the $D4$-branes.

a. Orientifold six-planes

Consider a configuration in which an NS5-brane is placed at a distance L_6 from an orientifold six-plane [all objects here and below are oriented as in Eq. (91)]. We would like to stretch $D4$-branes between the NS5-brane and the orientifold plane (a useful way to think about these is as $D4$-branes stretched between the NS5-brane and its mirror image with respect to the orientifold). As discussed in Sec. II.B, there are actually two different kinds of $O6$-planes, with positive and negative charge.

The first question we have to address is whether we *can* stretch $D4$-branes between the NS5-brane and its image in this situation without breaking supersymmetry. One might worry that such four-branes are projected out by the orientifold projection for one of the two possible choices of the orientifold charge. For example, we shall see later that it is impossible to stretch a BPS-saturated $D4$-brane between a $D6$-brane parallel to an $O6_-$ plane and its image, and this will have important consequences for the low-energy gauge theory. We next show that for the case of NS5-branes there is no such obstruction.

U duality relates the configuration we have to a more familiar one.[23] Specifically, we perform a T-duality transformation (44) T_{123} which takes the type-IIA string theory to a type-IIB one, followed by an S-duality transformation S [Eq. (45)] on the resulting type-IIB string. T_{123} maps the NS5-brane to itself, the $O6$-plane to an $O3$-plane (x^0,x^7,x^8,x^9), and the N_c $D4$-branes to N_c D strings stretched in x^6 between the two five-branes. The subsequent S-duality transformation acts differently for positive and negative orientifold charge.

For negative orientifold charge, we saw in Sec. III that the $O3$-plane transforms under S to itself. Thus after performing the transformation we end up with a $D5$-brane and its image near an $O3_-$ plane. The original $D4$-branes connecting the NS5-brane to its image turn now into fundamental type-IIB strings connecting the $D5$-brane to its image with respect to the $O3$-plane. The low-energy theory on the $D5$-brane is in this case an $SO(2)$ gauge theory; the configuration is T dual to well-studied systems such as D strings inside an $O9$-plane or $D0$-branes near an $O8$-plane. Translations of the D-brane and its image away from the orientifold plane (in the x^6 direction) are described by an antisymmetric tensor of $SO(2)$, i.e., a singlet.

The original question, whether one can stretch a $D4$-brane between the NS5-brane and its mirror, is translated in the U-dual configuration into the question whether one can stretch a BPS-saturated string between the $D5$-brane and its image. Such a string would correspond to a state charged under the $SO(2)$ gauge symmetry on the $D5$-brane, which goes to zero mass as the $D5$-brane approaches its image. It is well known that such states exist; they correspond to fields describing fluctuations of the D-brane along the orientifold plane, in this case in the (x^7,x^8,x^9) directions. Such fluctuations are described by a symmetric tensor of $SO(2)$ which includes a pair of states charged under $SO(2)$; these states are described by BPS fundamental strings stretched between the $D5$-branes. They are U-dual to $D4$-branes connecting the NS5-brane to its image in the original configuration. In particular, the latter is clearly possible.

For positive orientifold charge we use the fact—explained in Sec. III—that S duality takes an $O3_+$ plane to an $O3_-$ plane with a single $D3$-brane (without a mirror partner) embedded in it. The resulting system of a $D5$-brane near an $O3_-$ plane with a $D3$-brane is similar to that studied in the previous paragraphs. The

[23]Below we freely compactify and decompactify different dimensions. This should not affect the conclusions as to whether various brane configurations are allowed.

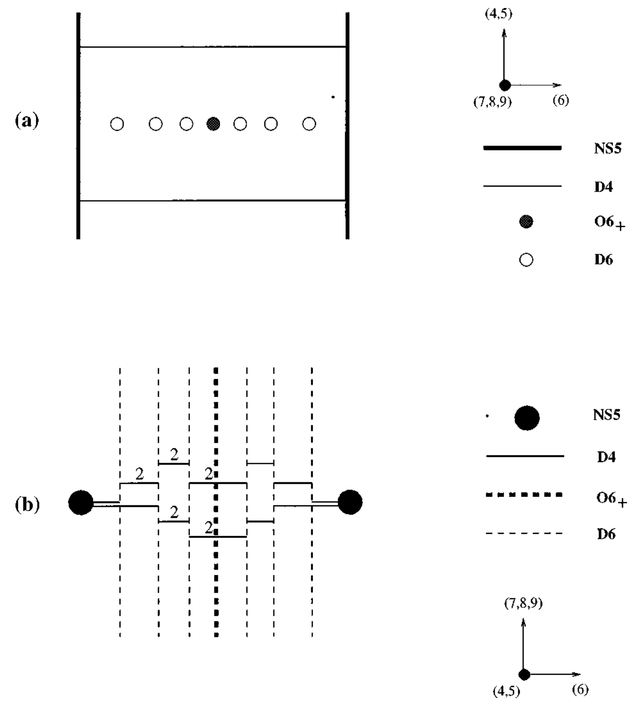

FIG. 17. Branches of moduli space of $SO(2)$ gauge theory with $N_f=3$ charged hypermultiplets: (a) the Coulomb branch; (b) the fully Higgsed branch.

$D3$-brane gives rise to additional matter which plays no role in the discussion. Clearly the rest of the analysis can be repeated as above, with the same conclusions—it is possible to stretch $D4$-branes between an NS5-brane and its image with respect to an $O6$-plane of either sign.

We shall next describe the classical gauge theories corresponding to the two choices of the sign of the orientifold charge, starting with the case of positive $O6$ charge (Fig. 17), which leads to an orthogonal projection on the $D4$-branes. The case of $O6_-$, which leads to a symplectic gauge group, will be considered later.

The gauge group on N_c $D4$-branes connecting an NS5-brane to its mirror image with respect to an $O6_+$ plane is $SO(N_c)$. N_f $D6$-branes parallel to the $O6$-plane located between the NS5-brane and the orientifold give N_f hypermultiplets in the fundamental ($\mathbf{N_c}$) representation of $SO(N_c)$, arising as usual from 4–6 strings. In $N=1$ SUSY notation there are $2N_f$ chiral multiplets Q^i, $i=1,\dots,2N_f$, which are paired to make N_f hypermultiplets. The global flavor symmetry of this gauge theory is $Sp(N_f)$, in agreement with the projection imposed by the positive charge $O6$-plane on the $D6$-branes.

The Coulomb branch of the $N=2$ SUSY gauge theory is parametrized by the locations of the $D4$-branes along the five-brane in the (x^4,x^5) plane. Entering the Coulomb branch involves removing the ends of the four-branes from the orientifold plane [which is located at a particular point in the (x^4,x^5) plane]. Since the four-branes can only leave the orientifold plane in pairs, the dimension of the Coulomb branch is $[N_c/2]$, in agreement with the gauge-theory description.

Higgs branches of the gauge theory are parametrized by all possible breakings of four-branes on six-branes. As for the unitary case there are many different branches; as a check that we get the right structure, consider the fully Higgsed branch which exists when the number of flavors is sufficiently large. From gauge theory we expect its dimension to be $2N_cN_f-N_c(N_c-1)$. The brane analysis gives

$$\dim \mathcal{M}_H=\sum_{i=1}^{N_c} 2(N_f+1-i)=2N_fN_c-N_c(N_c-1)$$
$$=[2N_fN_c-N_c(N_c+1)]+2N_c. \qquad (99)$$

The number in the square brackets is the number of moduli corresponding to segments that do not touch the orientifold, and the additional $2N_c$ is the number of moduli coming from the segments of the four-branes connecting the $D6$-brane closest to the orientifold to its mirror image. These segments transform to themselves under the orientifold projection and thus are dynamical for positive orientifold charge. An example is given in Fig. 17(b).

The $2N_c$ moduli coming from four-branes connecting a $D6$-brane to its image have a natural interpretation in the theory on the $D6$-branes. At low energies this is an $Sp(1)$ gauge theory with sixteen supercharges, and the $D4$-branes stretched between the $D6$ and its mirror can be thought of as $Sp(1)$ monopoles, as in Sec. III. From this point of view the above $2N_c$ moduli parametrize the space of N_c $Sp(1)$ monopoles.

Thus the total dimension of moduli space agrees with the gauge-theory result. It is easy to similarly check the agreement with gauge theory of the maximally Higgsed branch for small N_f, as well as the structure of the mixed Higgs-Coulomb branches.

For negative charge of the $O6$-plane, the configuration of Fig. 18 describes an $Sp(N_c/2)$ gauge theory with N_f hypermultiplets in the fundamental ($\mathbf{N_c}$) representation. Qualitatively, most of the analysis is the same as above, but the results are clearly somewhat different. For example, the dimension of the fully Higgsed branch is in this case $2N_fN_c-N_c(N_c+1)$, smaller by $2N_c$ than the SO case discussed above.

From the point of view of the brane construction, the Higgs branch is different because it is no longer possible to connect a $D6$-brane to its mirror image by a $D4$-brane. Such four-branes are projected out when the $O6$-plane has negative charge. As in Sec. III, this is also clear if we interpret these four-branes as magnetic monopoles in the six-brane theory. In this case the theory on the $D6$-brane adjacent to the orientifold and its image has gauge group $SO(2)$, and there are no nonsingular monopoles.

Therefore the pattern of breaking of the $D4$-branes on $D6$-branes near the orientifold is modified. The result is depicted in Fig. 18(b). We have to stop the usual pattern (99) when we get to the last *two* $D6$-branes before the orientifold, and there we must perform the breaking as indicated in Fig. 18(b). Thus compared with Eq. (99) we lose $2N_c$ moduli. Overall, the brane Higgs

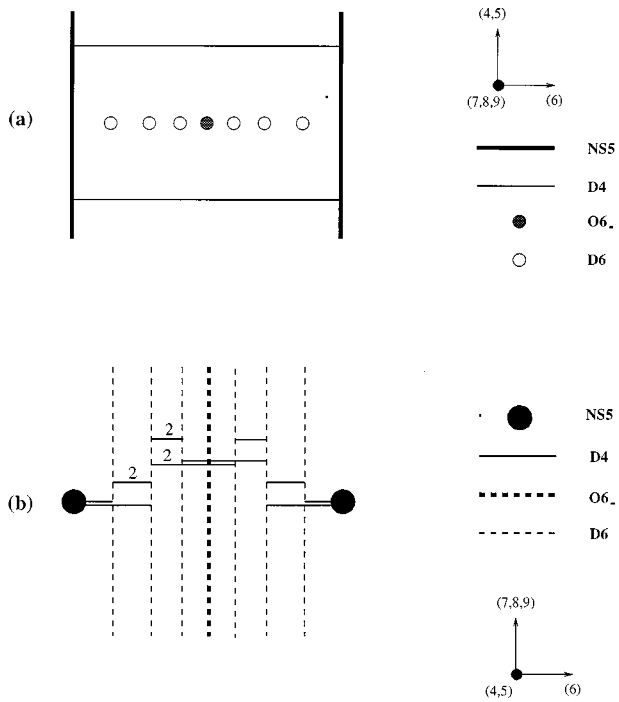

FIG. 18. Branches of moduli space of $Sp(1)$ gauge theory with $N_f=3$ charged hypermultiplets: (a) the Coulomb branch; (b) the fully Higgsed branch.

branch is $2N_fN_c-N_c(N_c+1)$ dimensional, in agreement with the gauge-theory analysis. One can again check that the full classical phase structure of the $Sp(N_c/2)$ gauge theory is similarly reproduced.

Further discussion of gauge theories on brane configurations in the presence of orientifold six-planes appears in the work of Elitzur, Giveon, *et al.* (1997, 1998b), Landsteiner and Lopez (1998).

b. Orientifold four-planes

In this case we suspend N_c four-branes between a pair of NS5-branes stuck on an orientifold four-plane at different locations in x^6 (Fig. 19). $2N_f$ $D6$-branes placed between the NS-branes (in x^6) provide N_f fundamental hypermultiplets. All the branes are as usual oriented as in Eq. (91). Despite much recent work (Ahn and Oh, 1997, Brandhuber, Sonnenschein, *et al.*, 1997b; Elitzur, Giveon, *et al.*, 1997; Evans, Johnson, and Shaper, 1997; Forste, Ghoshal, and Panda, 1997; Johnson, 1997; Ahn, Oh, and Tatar, 1998a, 1998b; Lykken, Poppitz, and Trivedi, 1998b; Tatar, 1998; Terashima, 1998; de Boer *et al.*, 1998), such configurations are not well understood in brane theory, and the discussion below should be viewed as conjectural. The new element in this case, and presumably the origin of the difficulties, is the fact that when an NS5- or $D6$-brane intersects an $O4$-plane, say at $x^6=0$, it splits it into disconnected components corresponding to $x^6>0$ and $x^6<0$. This leads to rather exotic behavior, some aspects of which will be described below.

One way to study what happens when an NS5-brane intersects an $O4$-plane is to start with a pair of such five-branes (i.e., a five-brane and its mirror image) near

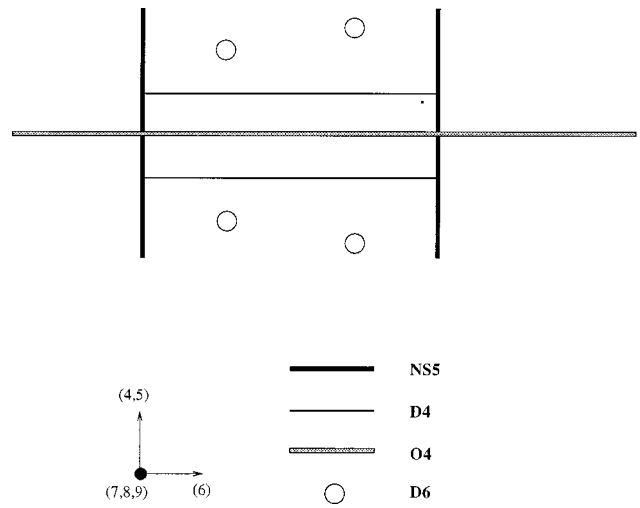

FIG. 19. Four-branes stretched between NS5-branes that are stuck on an $O4$-plane in the presence of $D6$-branes. This configuration provides an alternative description of the $N=2$ supersymmetric Yang-Mills theory with orthogonal and symplectic gauge groups.

the orientifold in (x^7,x^8,x^9), and study the transition in which the pair approaches each other and the orientifold and then splits along the orientifold (in x^6). This process is described in Fig. 20.

A closer look reveals that when the charge of the orientifold is negative, it is in fact impossible to separate

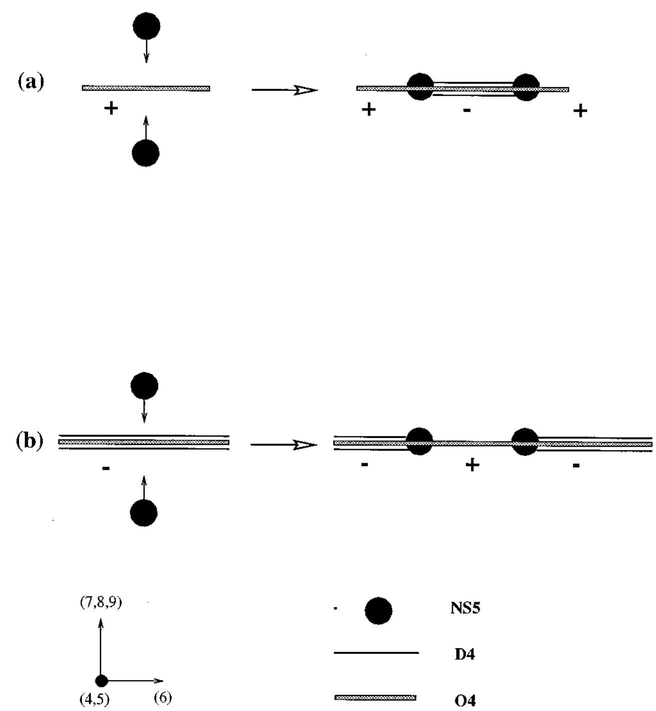

FIG. 20. Sign flipping of orientifold change. (a) An NS5-brane and its mirror image approach an $O4_+$-plane and separate along it (in x^6). The portion of the orientifold between the five-branes flips sign in the process, and a pair of $D4$-branes stretch between the five-branes. (b) When an NS5-brane and its image approach an $O4_-$-plane with two adjacent $D4$-branes, the reverse of (a) happens.

the two NS5-branes along the orientifold. The low-energy worldvolume theory on a pair of NS5-branes near an $O4_-$ plane [more precisely on the $(1+3)$-dimensional spacetime (x^0,x^1,x^2,x^3)] has gauge group $Sp(1)$, and the displacement of the five-branes in the x^6 direction is described by an antisymmetric tensor of $Sp(1)$, i.e., a singlet. Since it is impossible to Higgs $Sp(1)$ using a singlet, it is impossible to separate the two NS5-branes on the $O4_-$ plane.

For positive orientifold charge the separation of Fig. 20(a) is attainable. NS5-branes near an $O4_+$ plane are described by an $SO(2)$ gauge theory. The motions in the x^6 direction are described by a symmetric tensor of $SO(2)$, which includes a pair of charged scalars. Giving an expectation value to the symmetric tensor completely breaks the $SO(2)$ symmetry and corresponds to displacing the two five-branes relative to each other on the $O4_+$ plane.

What happens when an NS5-brane and its image approach an $O4_+$ plane and, after getting stuck on it, separate in the x^6 direction? Each of the five-branes divides the orientifold into two disconnected parts. One can show that the parts of the orientifold on different sides of the five-brane must carry opposite Ramond-Ramond charge. This was first shown by Evans, Johnson, and Shapere (1997) by comparing to gauge theory (see below); a worldsheet explanation of this effect was given in Elitzur, Giveon, *et al.* (1998b). Since far from the five-branes the orientifold charge must (by locality) be positive, between the five-branes it is negative. Furthermore, the total Ramond-Ramond charge must be continuous across each five-brane, since otherwise the net charge would curve the five-brane according to Eq. (51) and, in particular, change its shape at infinity, again violating locality. Therefore one expects to find two $D4$-branes stretched between the five-branes.

Similarly, when a pair of NS5-branes approaches a negatively charged $O4$-plane with two $D4$-branes embedded in it, it *can* split into two five-branes at different values of x^6 and gives rise to the configuration depicted in Fig. 20(b). Both possibilities are useful for describing gauge theories using branes.

Once we understand the behavior of NS5-branes near $O4$-planes, that of $D6$-branes is in principle determined by U duality. In particular, it appears that bringing pairs of $D6$-branes close to an $O4_+$ plane and separating them in x^6 splits the orientifold into components with alternating positive and negative charges. This might at first sight seem surprising, but it is related by U duality to the behavior of NS5-branes intersecting $O4$-planes. Compactifying x^3 one can use T duality to map a $D6$-brane intersecting an $O4$-plane to a $D5$-brane stretched in $(x^0,x^1,x^2,x^7,x^8,x^9)$ intersecting an $O3$-plane stretched in (x^0,x^1,x^2,x^6) and again cutting it into two disconnected pieces. This system can be analyzed by using S duality, and properties of NS5-branes.

Indeed, if we replace the $D5$-brane by an NS5-brane, we arrive at a system similar to that of Fig. 20, with the $O4$-plane replaced by an $O3$-plane. The three-dimensional analog of the transition described in that figure is the following: a pair of NS5-branes approach an $O3_+$ plane and separate in x^6 on it. The segment of the $O3$-plane between the five-branes flips sign and there is a single $D3$-brane embedded in it to make the total Ramond-Ramond charge continuous.

S duality applied to this configuration gives rise (using the results of Sec. III) to a configuration with two $D5$-branes intersecting an $O3$-plane and dividing it into three segments. The leftmost and rightmost parts of the orientifold have negative charge and a $D3$-brane embedded in them, while the segment between the $D5$-branes has positive Ramond-Ramond charge. Thus we conclude that the Ramond-Ramond charge of the $O3$-plane jumps as we cross a $D5$-brane. Since the statement is true for any finite radius of x^3, R, it is also true as $R\to\infty$. Therefore we conclude that the RR charge of the $O4$-plane jumps as it crosses a $D6$-brane.

We shall also need to understand the generalization of the s rule to $D4$-branes stretched between an NS5-brane and a $D6$-brane both of which are stuck on an $O4$-plane. A natural guess is the following. The usual s rule forbids configurations where two or more $D4$-branes are forced to be right on top of each other. In the presence of an $O4$-plane, it is natural to expect that if a part of the $O4$-plane between an NS5-brane and a $D6$-brane has negative charge and no $D4$-branes, one can connect the two branes by a pair of $D4$-branes. If the part of the orientifold between the two branes has positive charge, or negative charge with two $D4$-branes embedded in it (or any combination of these), one cannot stretch any further four-branes between them.

We are now finally ready to turn to applications. When the charge of the segment of the orientifold between the five-branes is negative, the brane configuration of Fig. 19 describes an $SO(N_c)$ gauge theory (we assume that N_c is even for now). To describe matter we add $D6$-branes. Note that when all the $D4$-branes are stretched between the NS5-branes (in the Coulomb branch), the $D6$-branes sit in pairs that cannot be separated further, as discussed above. The number of $D6$-branes must be even; we took it to be $2N_f$, which corresponds to N_f hypermultiplets in the $(\mathbf{N_c})$ of $SO(N_c)$. The $(N_c/2)$-dimensional Coulomb branch is described as usual by displacing the $D4$-branes along the NS5-branes in pairs away from the orientifold plane. The Higgs branch is obtained by studying all possible breakings of the $D4$-branes on $D6$-branes. Taking into account the s rule in the presence of an $O4$-plane explained above leads to the splitting pattern of Fig. 21(a).

The resulting dimension of the fully Higgsed branch (for $N_f>N_c$) is

$$\dim \mathcal{M}_H=2\sum_{i=1}^{N_c/2}[2N_f-(4i-3)]=2N_fN_c-N_c(N_c-1) \tag{100}$$

in agreement with the gauge-theory analysis. It is instructive to verify that one also gets the correct pattern of breaking and vacuum structure for low numbers of flavors where the gauge group cannot be completely

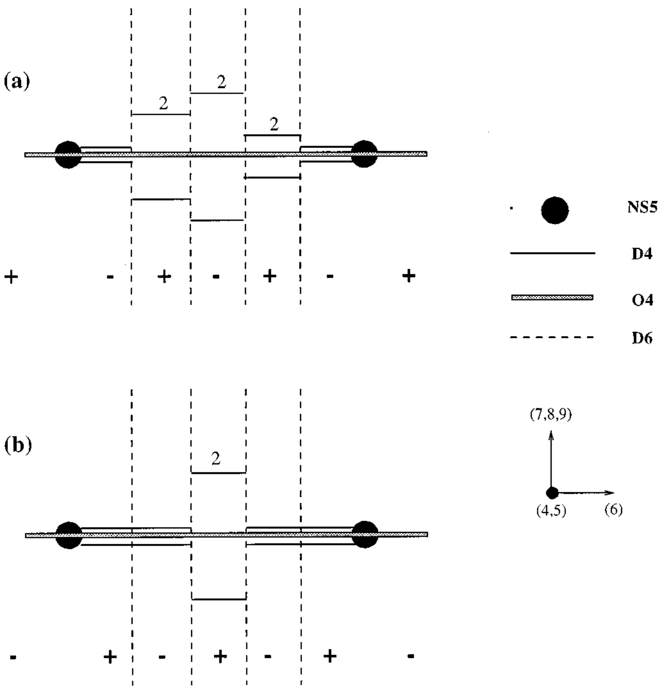

FIG. 21. The fully Higgsed branches of the $N=2$ supersymmetric Yang-Mills theory with (a) $G=SO(2)$ and $N_f=2$ charged hypermultiplets; (b) $G=Sp(1)$ and $N_f=2$ fundamental hypermultiplets. The orientifold charge flips sign, as indicated at the bottom, whenever one crosses a $D6$-brane or NS5-brane.

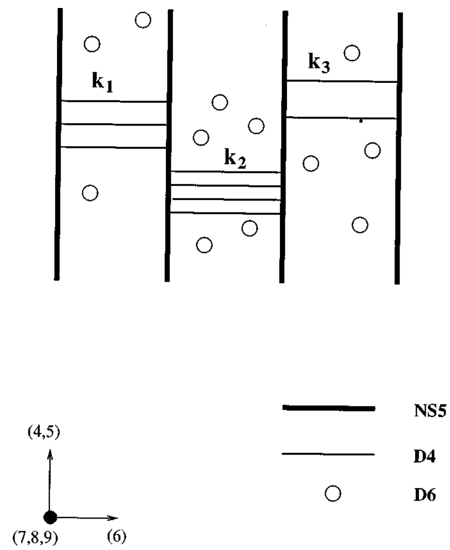

FIG. 22. A theory with a product gauge group and matter in the bifundamental of adjacent factors of the gauge group (as well as fundamental matter of individual factors).

Higgsed. We leave this as an exercise for the reader. One outcome of this exercise is a description of the case of odd N_c, which can be obtained from even N_c by Higgsing one hypermultiplet and breaking $SO(N_c) \to SO(N_c-1)$.

$Sp(N_c/2)$ gauge theory with N_f fundamental hypermultiplets is described by the configuration of Fig. 19 with positive Ramond-Ramond charge between the fivebranes and negative outside. The charge reversal changes the counting (100) in precisely the right way to reproduce the appropriate gauge-theory results. We illustrate the structure of the fully Higgsed branch in Fig. 21(b).

3. Some generalizations

Once the physics of the basic brane constructions has been understood one can generalize them in many different directions. One obvious example is to increase the number of NS5-branes. Consider, for example, a chain of $n+1$ five-branes labeled from 0 to n, with the $(\alpha-1)$st and αth five-brane connected by k_α $D4$-branes. In addition, let d_α $D6$-branes be localized at points between the $(\alpha-1)$st and αth NS5-branes (see Fig. 22 for an example).

The gauge group is in this case $G=\Pi_{\alpha=1}^{n} U(k_\alpha)$. The matter hypermultiplets are the following: 4–4 strings connecting the k_α four-branes in the αth interval to the $k_{\alpha+1}$ four-branes in the $(\alpha+1)$st interval $(\alpha=1,\dots,n-1)$ give rise to (bifundamental) hypermultiplets transforming in the $(k_\alpha,\overline{k}_{\alpha+1})$ of $U(k_\alpha)\times U(k_{\alpha+1})$. In addition we have d_β hypermultiplets in the fundamental representation of $U(k_\beta)$ $(\beta=1,\dots,n)$. We leave the analysis of the moduli space of vacua and the space of deformations to the reader.

If we add to the previous configuration an orientifold four-plane the gauge group becomes an alternating SO/Sp one. For example, for even n (an odd number of NS5-branes) and negative Ramond-Ramond charge of the segment of the $O4$-plane between the first and second NS5-brane, the gauge group is $G=SO(k_1) \times Sp(k_2/2)\times SO(k_3)\times\cdots\times Sp(k_n/2)$ with bifundamental matter charged under adjacent factors of the gauge group.

Brane configurations corresponding to theories with such product gauge groups have been considered in the literature (Brandhuber, Sonnenschein *et al.*, 1997a; Landsteiner, Lopez, and Lowe, 1997; Giveon and Pelc, 1998; Erlich, Naqvi, and Randall, 1998; Tatar, 1998).

Another example is a generalization of the configuration involving an orientifold six-plane that we discussed previously in the context of orthogonal and symplectic gauge groups. Consider a configuration in which one NS5-brane is placed at a distance L_6 from the $O6$-plane as before, and another NS5-brane is placed so that it intersects the orientifold plane. N_c $D4$-branes are stretched between the two five-branes and N_f $D6$-branes parallel to the $O6$-plane are placed between the two NS5-branes (Fig. 23).

The gauge theory describing this configuration has $G=U(N_c)$ and a matter hypermultiplet Z in the symmetric or antisymmetric representation of G (depending on the sign of the orientifold), as well as the usual N_f fundamental hypermultiplets (Landsteiner and Lopez, 1997). The two-index tensor hypermultiplet corresponds to 4–4 strings stretched from one side of the orientifold to the other side, across the stuck NS5-brane.

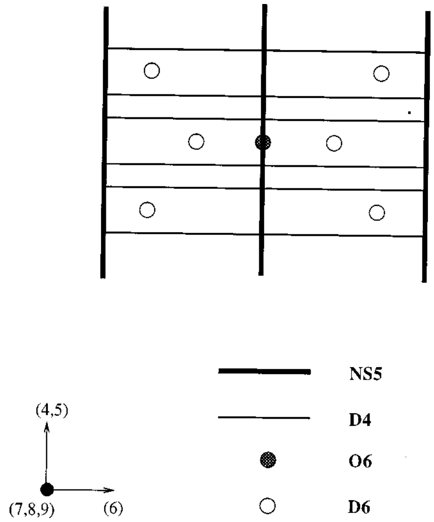

FIG. 23. A theory with $G=U(N_c)$ ($N_c=6$ in this case), one hypermultiplet in the (anti-) symmetric representation, and $N_f=3$ fundamentals.

4. Quantum effects: I

So far in this section we have described the *classical* brane configurations and the corresponding classical gauge-theory dynamics. For finite g [Eq. (93)] there are important qualitative new effects, which are the subject of this section.

We shall first discuss these effects in the context of $U(N_c)$ gauge theory following Witten (1997a). The gauge theory was shown earlier to be described by a system of two NS5-branes in type-IIA string theory with N_c $D4$-branes stretched between them (see Fig. 9). In the next section we shall comment on the generalization to some of the other cases mentioned above.

For finite g_s, the type-IIA string theory becomes eleven dimensional at short distances. The radius of the eleventh dimension x^{10} is proportional to g_s, $R_{10}=l_s g_s$ [Eq. (30)]. Furthermore, as we saw in Sec. II.C, the $D4$-brane can be thought of as an M theory five-brane wrapped around x^{10}. Thus $D4$-branes stretched between NS5-branes are reinterpreted in M theory on $R^{10}\times S^1$ as describing a single five-brane with a curved worldvolume. Since all type-IIA branes are extended in the $1+3$ dimensions (x^0,x^1,x^2,x^3) and are located at a point in (x^7,x^8,x^9), the worldvolume of the M-theory five-brane is $R^{1,3}\times\Sigma$ where Σ is a two-dimensional surface embedded in the four-dimensional space $Q=R^3\times S^1$ labeled by (x^4,x^5,x^6,x^{10}).

It is convenient to parametrize the space Q using the natural complex coordinates

$$s=x^6+ix^{10},$$
$$v=x^4+ix^5. \qquad (101)$$

In the classical type-IIA string limit, the NS5-branes of Eq. (91) are described by $s=$constant, while the $D4$-branes correspond to $v=$constant. If we place the two NS5-branes at $s=s_1,s_2$ and the N_c four-branes stretched between them at $v=v_1,v_2,\ldots,v_{N_c}$, the "classical" surface Σ_{cl} is described by $(s-s_1)(s-s_2)\Pi_{a=1}^{N_c}(v-v_a)=0$, with $\mathrm{Re}(s_1)\leqslant\mathrm{Re}(s)\leqslant\mathrm{Re}(s_2)$. Σ_{cl} is a singular surface with different components which meet at the singular points $s=s_i$, $v=v_a$ ($i=1,2$; $a=1,\ldots,N_c$).

As we shall see shortly, for finite R_{10} and at generic points in moduli space the singularities in Σ are eliminated. To determine the shape of the smooth surface Σ we next consider its large-v asymptotics. Classically, we see at large v the two NS5-branes at fixed $s=s_1,s_2$. However, we know from the discussion of Sec. II.E.3 that the ends of the four-branes on the five-brane look like charges.[24] More precisely, q_L four-branes ending on the five-brane from the left at $v=a_1,\ldots,a_{q_L}$, and q_R four-branes ending on it from the right at $v=b_1,\ldots,b_{q_R}$ curve it asymptotically according to Eq. (51), which in this case is

$$x^6=l_s g_s\sum_{i=1}^{q_L}\log|v-a_i|-l_s g_s\sum_{i=1}^{q_R}\log|v-b_i|. \qquad (102)$$

The fact that the coefficient of the log is proportional to g_s can be understood at weak coupling as a consequence of properties of the NS5-brane. Equations (16) and (20) imply that the supersymmetric Yang-Mills coupling of the theory on the five-brane (reduced to $2+1$ dimensions) has no g_s dependence, while the kinetic term of X^6 is proportional to $1/g_s^2$. Thus the BPS-saturated solution describing a four-brane ending on the five-brane has gauge field $A\simeq Q\log|v|$ with the charge quantum Q independent of g_s, and $X^6\simeq Qg_s\log|v|$. The factor of l_s in Eq. (102) is required by dimensional analysis.

At strong coupling Eq. (102) can be alternatively derived by identifying $l_s g_s$ with R_{10}, the only scale in the problem. The weak-coupling and strong-coupling arguments must agree because the BPS property of the state in question allows one to freely interpolate holomorphic properties between the weak- and strong-coupling regimes (and, as we shall see, Eq. (102) is closely related to a holomorphic quantity).

Note that the fact that the end of a four-brane on a five-brane looks like a codimension-two charged object implies that unlike the case $p>2$ in Eq. (51) one cannot, in general, define quantum mechanically "the location of the NS5-brane" by measuring x^6 at $|v|\to\infty$, since the effects of the four-branes in Eq. (102) are not small for large v. x^6 approaches a well-defined value, which can then be interpreted as the location of the five-brane only when the total charge on the five-brane vanishes, $q_L=q_R$.

The scalar field x^6 is related by $N=2$ supersymmetry to x^{10}; together the two form a complex scalar field s

[24]The theory on the type-IIA five-brane is not a gauge theory, but rather a mysterious non-Abelian theory of self-dual $B_{\mu\nu}$ fields. However, the ends of four-branes on the five-branes are codimension-two objects and therefore the relevant theory is the five-brane theory compactified down to $2+1$ dimensions, which *is* a gauge theory, to which the discussion of Sec. II.E.3 can be applied.

[Eq. (101)] that belongs to the vector multiplet of $N=2$ supersymmetry in the $(3+1)$-dimensional spacetime. For consistency with supersymmetry, Eq. (102) must be generalized to a holomorphic equation for s in Eq. (101),

$$s=R_{10}\sum_{i=1}^{q_L}\log(v-a_i)-R_{10}\sum_{i=1}^{q_R}\log(v-b_i). \tag{103}$$

The real part of this equation is Eq. (102); the imaginary part implies that x^{10} jumps by $\pm 2\pi R_{10}$ when we circle a_i or b_i in the complex v plane. Thus the ends of four-branes on the five-branes look like vortices. Since x^{10} is compact, it will be convenient for later purposes to define

$$t=\exp\left(-\frac{s}{R_{10}}\right) \tag{104}$$

in terms of which Eq. (103) takes the form

$$t=\frac{\Pi_{i=1}^{q_R}(v-b_i)}{\Pi_{j=1}^{q_L}(v-a_j)}. \tag{105}$$

We are now ready to determine the full shape of the surface Σ and thus the embedding of the five-brane corresponding to the classical brane configuration realizing pure $N=2$ SYM theory (Fig. 9) in the eleven-dimensional spacetime. Supersymmetry requires Σ to be given by a holomorphic curve in the two-complex-dimensional space labeled by t,v. It can be described by a holomorphic equation $F(t,v)=0$ for some function F. When we view this as a function of t for large v we expect to see two branches corresponding to the two "NS5-branes" at Eq. (105): $t_1\simeq v^{N_c}$ and $t_2\simeq v^{-N_c}$. Therefore the curve should be described by setting to zero a second-order polynomial in t,

$$F(t,v)=A(v)t^2+B(v)t+C(v)=0, \tag{106}$$

where A, B, and C are polynomials of degree N_c in v, so that for fixed t there will be N_c solutions for v corresponding to the "$D4$-branes" stretched between the five-branes.

As we approach a zero of $C(v)$, a solution of the quadratic Eq. (106) goes to $t=0$, i.e., in Eq. (104) $x^6=\infty$. Thus zeros of $C(v)$ correspond to locations of semi-infinite four-branes stretching to the right of the rightmost NS5-brane. Similarly, $A(v)$ describes semi-infinite four-branes stretching to $x^6=-\infty$ from the left NS5-brane. In the $N=2$ gauge-theory application semi-infinite four-branes give rise to fundamental matter and as a first step we are not interested in them. Thus we set[25]

$$A(v)=C(v)=1. \tag{107}$$

$B(v)$ is taken to be the most general polynomial of degree N_c which can, by rescaling and shifting v, be brought to the form

$$B(v)=v^{N_c}+u_2v^{N_c-2}+u_3v^{N_c-3}\cdots+u_{N_c}, \tag{108}$$

where $u_2,\ldots,u_{N_c}$ are complex constants parametrizing the polynomial B. The curve (106) with the choice of Eqs. (107) and (108) of A, B, and C has the right structure: for fixed t it has N_c roots v_i corresponding to the N_c "four-branes." Note that while classically there should only be such solutions for t between the NS5-branes, because of the bending Eq. (105) there are in fact N_c solutions for v for any $t\neq 0$. Similarly, for all v there are two solutions for t, which for large v behave like

$$t_\pm\simeq v^{\pm N_c}, \tag{109}$$

in agreement with the general structure expected from Eq. (105).

To recapitulate, as in the classical case where there is a one-to-one correspondence between configurations of $D4$-branes stretched between NS5-branes and vacua of classical $N=2$ SYM theory, quantum mechanically there is a one-to-one correspondence between vacua of quantum $N=2$ SYM theory and supersymmetric configurations of an $M5$-brane with worldvolume $R^{3,1}\times\Sigma$, with Σ described by Eqs. (106)–(108). Roots of the polynomial B in Eq. (108) correspond to "the locations of the $D4$-branes" and label different points in the quantum Coulomb branch of the $N=2$ SYM theory.

It is interesting that there are only N_c-1 independent roots, labeled by the moduli $\{u_i\}$. This appears to be in contradiction with the fact that there are N_c massless vector multiplets living on the four-branes for generic values of the moduli $\{u_i\}$ corresponding to the unbroken $U(1)^{N_c}\subset U(N_c)$. In fact, the number of vector multiplets is N_c-1, in agreement with Eq. (108). The $U(1)\subset U(N_c)$ has vanishing coupling and is "frozen."

This can be understood semiclassically by evaluating the kinetic term of the $U(1)$. The N_c "four-branes" ending on an NS5-brane from the left (say) bend it according to Eq. (103). Here a_i are the moduli, and to probe their dynamics one should allow them to slowly vary as a function of (x^0,x^1,x^2,x^3). The kinetic energy of the five-brane behaves for such slowly varying configurations as

$$\begin{aligned}S&\simeq\int d^4x\int d^2v|\partial_\mu s|^2\\&\simeq R_{10}^2\int d^4x\int d^2v\left|\sum_i\partial_\mu a_i\frac{1}{v-a_i}\right|^2.\end{aligned} \tag{110}$$

At large v, where Eq. (110) is expected to be accurate, we find

$$S\simeq R_{10}^2\int\frac{d^2v}{|v|^2}\int d^4x\left|\sum_i\partial_\mu a_i\right|^2. \tag{111}$$

The logarithmically divergent v integral (111) leads to a vanishing coupling for the $U(1)\subset U(N_c)$:

$$\frac{1}{g_1^2}\simeq R_{10}^2\int\frac{d^2v}{|v|^2}\to\infty. \tag{112}$$

[25]We set the QCD scale $\Lambda=1$ here. Restoring dimensions, since v and t^{1/N_c} scale like energy, if we set $A(v)=1$ then $C(v)=\Lambda^{2N_c}$.

Interestingly, Eqs. (106)–(108) describe the Seiberg-Witten curve for $SU(N_c)$ gauge theory with no matter! In gauge theory, the low-energy coupling matrix τ_{ij} (76) is the period matrix of the corresponding Riemann surface. This is also the case in the five-brane construction. The worldvolume theory of a flat five-brane includes a self-dual $B_{\mu\nu}$ field ($H=dB=*dB$). Upon compactification on Σ, B gives rise to g Abelian vector multiplets in $3+1$ dimensions, with g the genus of the Riemann surface Σ. In our case, the surface Σ can be thought of as describing two sheets (the "five-branes") connected by N_c tubes (the "four-branes"), and hence it has genus $g=N_c-1$. The coupling matrix of the resulting $U(1)^{N_c-1}$ gauge theory is the period matrix of Σ (Verlinde, 1995).

Howe, Lambert, and West (1998) have offered another derivation of the relation between the period matrix of the Riemann surface around which the five-brane is wrapped and the coupling matrix of the Abelian gauge theory on the brane that emphasizes the role of the scalar fields living on the brane.

To summarize, the brane analysis agrees with Seiberg-Witten theory. It offers a rationale as to why the low-energy gauge-coupling matrix and metric on moduli space of $N=2$ SYM theory are described by a period matrix of a Riemann surface. The natural context for studying SW theory appears to be as a compactification of the (2,0) field theory of an $M5$-brane (the low-energy limit of the theory of $M5$- or type-IIA NS5-branes) on the Riemann surface Σ.

At this point we should like to comment briefly on the foregoing discussion.

a. Global symmetry, conformal invariance, and the shape of the five-brane

As we discussed in Sec. IV.A, classical $N=2$ SYM theory has at the origin of moduli space a global symmetry $SU(2)_R\times U(1)_R$. The $SU(2)_R$ symmetry is part of $N=2$ supersymmetry; the $U(1)_R$ reflects the classical conformal invariance of the theory and is broken at one loop by the chiral anomaly to Z_{2N_c}.

In the brane description, the $U(1)_R$ symmetry is realized as the $SO(2)$ rotation group of the v plane, which acts as $v\to v\exp(i\alpha)$. The classical configuration of N_c $D4$-branes—all at $v=0$—stretched between the two NS5-branes (say, at $s=0$) is invariant under this $SO(2)$ symmetry. The brane analog of one-loop effects is the leading quantum correction, which is the asymptotic bending [Eq. (103)]. It breaks the $U(1)_R$ symmetry by curving the left and right five-branes[26] to

$$s_L=-N_cR_{10}\log v,$$
$$s_R=+N_cR_{10}\log v. \tag{113}$$

This configuration is no longer invariant under

$$v\to v\exp(i\alpha). \tag{114}$$

Under Eq. (114)

$$s_L\to s_L-N_cR_{10}i\alpha,$$
$$s_R\to s_R+N_cR_{10}i\alpha. \tag{115}$$

For generic α Eq. (115) is clearly not a symmetry; however, there are residual discrete symmetry transformations corresponding to $\alpha=2\pi n/2N_c$ due to the fact that $\mathrm{Im}\, s=x^{10}$ lives on a circle of radius R_{10}. Thus a Z_{2N_c} subgroup of $U(1)_R$ remains unbroken, in agreement with the gauge-theory analysis.

b. Adding flavors

It is easy to add fundamental hypermultiplets to the discussion. As we have noted above, to describe the Coulomb branch of a model with N_f fundamentals of $SU(N_c)$ we can add N_f semi-infinite four-branes, say to the right of the NS5-branes. These are described by turning on $C(v)$ in Eq. (106):

$$C(v)=\prod_{i=1}^{N_f}(v-m_i). \tag{116}$$

Here m_i are the locations of the semi-infinite four-branes in the v plane that, as we have seen, correspond to the masses of the fundamental "quarks." Thus $N=2$ supersymmetric QCD with $G=SU(N_c)$ and N_f fundamentals is described by the Riemann surface

$$t^2+B(v)t+C(v)=0 \tag{117}$$

with $B(v)$ given by Eq. (108) and $C(v)$ by Eq. (116). This agrees with the gauge-theory results of Argyres, Plesser, and Shapere (1995) and Hanany and Oz (1995).

c. $SU(N_c)$ versus $U(N_c)$

We have argued that the brane configuration of Fig. 9, which classically describes a $U(N_c)$ gauge theory, in fact corresponds quantum mechanically to a $SU(N_c)$ one; the coupling of the extra $U(1)$ factor vanishes. This observation appears to be in contradiction with the fact that the moduli and deformations of the brane configuration discussed above seem to be those of a $U(N_c)$ theory. This issue remains unresolved as of this writing; below we explain the specific puzzles.

We saw previously that the moduli space of brane configurations seems to match the classical Higgs branch of $U(N_c)$ gauge theory with N_f hypermultiplets. If the gauge group is $SU(N_c)$, the complex dimension of the Higgs branch is $2N_fN_c-2(N_c^2-1)$ and the brane count is missing two complex moduli. By itself, this need not be a serious difficulty; we saw previously examples where some or all of the field-theory moduli were not seen in the brane analysis. Unfortunately, the mismatch in the structure of the Higgs branch is related to a more serious difficulty having to do with the field-theory interpretation of certain deformations of the brane configuration.

In the classical discussion we interpreted the relative location of the two NS5-branes in (x^7,x^8,x^9) as a Fayet-Iliopoulos D term for the $U(1)\subset U(N_c)$ [Eq. (98)]. If

---

[26]For large v it makes sense to talk about the left and right five-branes although they are connected at small v.

the gauge group is $SU(N_c)$, we have to modify that interpretation, as the theory no longer has a Fayet-Iliopoulos coupling. The question is whether in the quantum theory the parameters corresponding to a relative displacement of the two asymptotic parts of the $M5$-brane in (x^7,x^8,x^9) are moduli in the $(3+1)$-dimensional field theory on the brane, or whether—like the $U(1)$ vector multiplet—they are decoupled. There seem to be two logical possibilities, each of which has its own difficulties (Giveon and Pelc, 1998).

An argument similar to that outlined in Eqs. (110)–(112) would suggest that these parameters are frozen and correspond to couplings in the $(3+1)$-dimensional gauge theory. The kinetic energy of the scalar fields X^I, $I=7,8,9$, seems to diverge [assuming an asymptotically flat metric on the five-brane, as we have for $X^6(x^\mu)$, Eq. (110)] as

$$\mathcal{L}_{\rm kin}\simeq\int d^2v(\partial_\mu X^I)^2. \tag{118}$$

Thus the kinetic energy of the fields X^I is infinite and we must find a coupling in the Lagrangian of $SU(N_c)$ gauge theory that has the same effect on the vacuum structure as a Fayet-Iliopoulos D term [Eq. (98)]. It is not known (to us) how to write such a coupling. In order for such a coupling to exist, the $U(1)$ factor would have to be unfrozen, and the estimate of the kinetic energy (110)–(112) would have to be invalid.

Alternatively, one might imagine that the parameters corresponding to (x^7,x^8,x^9) are in fact moduli in the gauge theory. This would apparently be consistent with gauge theory; these moduli would provide three of the four missing moduli parametrizing the baryonic branch of the theory. However, for this interpretation to be valid we have to come up with a mechanism for rendering the naively divergent kinetic energy [Eq. (118)] finite [without spoiling Eq. (110)]. This sounds even more implausible than the first scenario, as one has to cancel a more divergent kinetic energy. It is not clear to us what is the resolution of this problem.

d. $N_f\geqslant 2N_c$

For $N_f=2N_c$, at the origin of the Coulomb branch and for vanishing quark masses, the curve (117) is

$$t^2+av^{N_c}t+bv^{2N_c}=0 \tag{119}$$

or equivalently

$$t_\pm=\lambda_\pm v^{N_c} \tag{120}$$

with $\lambda_\pm$ the two solutions of

$$\lambda^2+a\lambda+b=0. \tag{121}$$

The $U(1)_R$ symmetry

$$v\to e^{i\alpha}v,$$
$$t\to e^{iN_c\alpha}t \tag{122}$$

is unbroken. Thus the theory at the origin is an interacting non-trivial $N=2$ superconformal field theory. This is consistent with the fact that the curve (120) is singular at $t=v=0$—a hallmark of a nontrivial superconformal field theory. The ratio $w=\lambda_+\lambda_-/(\lambda_+-\lambda_-)^2$ is invariant under rescaling of t, v and can be thought of as parametrizing the coupling constant of the theory. For weak coupling, $w\simeq 0$, one has $w=\exp(2\pi i\tau)$ (τ is the complex gauge coupling), but more generally, due to duality, τ is a many-valued function of w.

For $N_f>2N_c$ the description (117) seems to break down. Both solutions for t behave at large v as $t\sim v^{N_f/2}$, while Eq. (103) (in the presence of N_f semi-infinite "four-branes" stretching to $x^6\to\infty$) predicts $t_1\sim v^{N_c}$, $t_2\sim v^{N_f-N_c}$. Not coincidentally, in this case the gauge theory is not asymptotically free and must be embedded in a bigger theory to make it well defined in the ultraviolet. And, in any case, it is free in the infrared. It is in fact possible to modify Eq. (117) to accommodate these cases (see Witten, 1997a, for details).

e. BPS-saturated states

The five-brane description of $N=2$ SYM theory can also be used to study massive BPS-saturated states. Examples of such states in supersymmetric Yang-Mills theory include charged gauge-boson vector multiplets and magnetic monopole hypermultiplets. In the classical type-IIA limit, massive gauge bosons are described by fundamental strings stretched between different four-branes. For finite R_{10} these fundamental strings are reinterpreted as membranes wrapped around x^{10}. Thus charged W bosons are described in M theory by minimal-area membranes ending on the five-brane. Clearly, the topology of the resulting membrane is cylindrical.

Monopoles are described in the type-IIA limit by $D2$-branes stretched between the two NS5-branes and two adjacent $D4$-branes, as in Fig. 8. In M theory they correspond to membranes with the topology of a disk ending on the five-brane.

There are other BPS-saturated states such as quarks and various dyons, all of which are described in M theory by membranes ending on the five-brane. Membranes with the topology of a cylinder always seem to give rise to vector multiplets, while those with the topology of a disk give hypermultiplets. We shall not describe the corresponding membranes in detail here, referring the reader instead to Mikhailov (1998) and Henningson and Yi (1998).

f. Compact Coulomb branches and finite $N=2$ models

The fact that the gauge coupling of the $U(1)\subset U(N_c)$ vanishes is related to the infinite area of the v-plane [Eq. (112)]. One might wonder what would happen if we compactified (x^4,x^5) on a two-torus.

Already classically we see that in this situation the Coulomb branch of the theory, labeled by locations of $D4$-branes stretched between five-branes, becomes compact. Quantum mechanically we see that since Eq. (103) is a solution of a two-dimensional Laplace equation

$$\partial_v \partial_{\bar{v}} s = R_{10} \sum_{i=1}^{q_L} \delta^2(v-a_i) - R_{10} \sum_{i=1}^{q_R} \delta^2(v-b_i) \qquad (123)$$

on a compact surface, the total charge on each five-brane must vanish: $q_L = q_R$. This means that there must be N_c semi-infinite four-branes attached to each five-brane and the total number of flavors must thus be $N_f = 2N_c$. The solution of Eq. (123) that generalizes Eq. (103) to the case of a two-torus is $(q = q_L = q_R)$

$$s = R_{10} \sum_{i=1}^{q} [\log \chi(v-a_i|\rho) - \log \chi(v-b_i|\rho)], \qquad (124)$$

where ρ is the modular parameter (complex structure) of the v-plane torus ($v \sim v+1$, $v \sim v+\rho$), and

$$\chi(z|\rho) = \frac{\theta_1(z|\rho)}{\theta_1'(0|\rho)}, \qquad (125)$$

where $\log \chi$ is related to the propagator of a two-dimensional scalar field on a torus with modulus ρ (see Green, Schwarz, and Witten, 1987, for notation and references). Note that χ itself is not well defined on the torus; its periodicity properties are

$$\chi(z+1|\rho) = -\chi(z|\rho),$$
$$\chi(z+\rho|\rho) = -e^{-i\pi\rho - 2i\pi z} \chi(z|\rho). \qquad (126)$$

However, we only require that the curve built using χ should exhibit periodicity. To construct this curve, we start with the infinite-volume curve (106) describing this situation,[27]

$$t^2 \prod_{i=1}^{N_c} (v - m_i^{(1)}) + t \prod_{i=1}^{N_c} (v - v_i) + \prod_{i=1}^{N_c} (v - m_i^{(2)}) = 0, \qquad (127)$$

and replace each $(v-a_i)$ by $\chi(v-a_i|\rho)$. This gives

$$t^2 \prod_{i=1}^{N_c} \chi(v - m_i^{(1)}|\rho) + t \prod_{i=1}^{N_c} \chi(v - v_i|\rho) + \prod_{i=1}^{N_c} \chi(v - m_i^{(2)}|\rho) = 0. \qquad (128)$$

Using Eq. (126) and the fact that the moduli v_i and masses m_i satisfy the relations

$$\Sigma_{i=1}^{N_c} (v_i - m_i^{(1)}) = \text{const},$$
$$\Sigma_{i=1}^{N_c} (v_i - m_i^{(2)}) = \text{const} \qquad (129)$$

we find that the curve (128) indeed has the right periodicity properties.

At first sight the generalization of $N=2$ SYM theory with the compact Coulomb branch seems mysterious, but in fact it can be thought of as the moduli space of vacua of a six-dimensional "gauge theory" compactified on a two-torus. Indeed, if v parametrizes a two-torus T^2, we can T-dualize our classical configuration of $D4$-branes ending on NS5-branes and, using the results of Sec. II, reach a configuration of $D6$-branes wrapped around the dual torus $\tilde{T}^2$ and ending on the NS5-branes in the x^6 direction.

From the six-dimensional point of view it is clear that we must require $N_f = 2N_c$, since the only configuration consistent with Ramond-Ramond charge conservation involves in this case N_c infinite $D6$-branes extending to infinity in x^6 and intersecting the two NS5-branes. Wilson lines around the $\tilde{T}^2$ give rise to the parameters m, v [Eq. (128)]. From the gauge-theory point of view, $N_f = 2N_c$ is necessary due to the requirement of cancellation of six-dimensional chiral anomalies.

The curve (128) exhibits an $SL(2,Z)$ duality symmetry corresponding to modular transformations $\rho \to (a\rho + b)/(c\rho + d)$ under which

$$\chi\left(\frac{z}{c\rho+d} \middle| \frac{a\rho+b}{c\rho+d}\right) = \frac{\eta \exp[i\pi c z^2/c\rho+d]}{c\rho+d} \chi(z|\rho), \qquad (130)$$

where η is an eight-root of unity. This $SL(2,Z)$ symmetry provides a geometric realization of the duality symmetry of finite $N=2$ SYM models (which are anomaly free in $6d$ and thus can be lifted to $6d$). Note that the area of the v-plane torus does not appear in Eq. (128). This is essentially because v has been rescaled to absorb a factor of the area. The four-dimensional limit of Eq. (128) is obtained by taking $v \ll 1, \rho$ where $\chi(v|\rho)$ reduces to v.

g. SQCD versus MQCD

As we discussed before, the limit that one needs to take to study decoupled gauge dynamics on the five-brane is $R_{10}, L_6 \to 0$ holding $R_{10}/L_6 = g_{SYM}^2$ fixed. In this limit the five-brane becomes singular although its complex structure (117) is regular. To fully understand gauge dynamics in this limit one needs to study the five-brane theory in the type-IIA limit.

Witten has suggested studying the theory in the opposite limit, $R_{10}, L_6 \to \infty$, R_{10}/L_6 fixed, observing that in that limit Eq. (117) describes a large smooth five-brane and thus can be accurately studied by using low-energy M theory, i.e., eleven-dimensional supergravity (the five-brane dynamics in this limit is sometimes referred to in the literature as "MQCD").

For holomorphic properties of the vacuum, such as the low-energy gauge couplings and metric on moduli space Eqs. (75)–(77), the two limits must agree due to supersymmetry. However, nonholomorphic low-energy features are quite different in the two limits. In particular, in the MQCD limit the five-brane dynamics is no longer effectively four dimensional, and there is large mixing between gauge degrees of freedom and other excitations. Thus it is misleading to refer to the M theory limit as QCD (M or otherwise).

The situation is similar to the well-known worldsheet duality in open-plus-closed string theory. The physics can be viewed either in the open string channel (where light states are typically gauge fields) or as due to closed string exchange (gravitons, dilatons, etc.). Worldsheet

[27]Recall that v_i are moduli parametrizing the Coulomb branch of the theory, while $m_i^{(1)}$, $m_i^{(2)}$ are masses of flavors corresponding to semi-infinite four-branes extending to the left and right, respectively.

duality implies that the two representations must agree, but one may be simpler than the other. In some situations the open string representation is dominated by the massless sector, but then in the closed string channel one needs to sum over exchanges of arbitrarily heavy string states. In such cases, the relevant physics is that of gauge theory.

In other cases, the closed string channel is dominated by exchange of massless modes such as gravitons, but then the open string calculation receives contributions from arbitrarily heavy states and there is no simple gauge-theory interpretation of the physics.

The only known cases (with the possible exception of discrete light-cone quantization matrix theory—reviewed by Banks, 1998, and Bigatti and Susskind, 1998—whose status is unclear as of this writing) where there is a simple interpretation in both the open and closed string channels involve quantities protected by supersymmetry.

In our case, the analog of the closed string channel is the eleven-dimensional "MQCD" limit $R_{10},L_6\to\infty$ where physics is dominated by gravity, while the analog of the open string channel is the type-IIA limit $R_{10},L_6\to 0$. Low-energy features that are not protected by supersymmetry need not agree in the two limits (except perhaps in certain large-N limits). Supersymmetric QCD (SQCD) corresponds to the latter.

h. Nontrivial infrared fixed points

At generic points in the Coulomb branch, the infrared dynamics of $N=2$ SYM theory is described by r massless photons whose coupling matrix is the period matrix of the Riemann surface Σ. At points where additional matter goes to zero mass, the infrared dynamics changes, and in many cases describes a nontrivial superconrformal field theory (Argyres and Douglas, 1995; Argyres *et al.*, 1996). These situations correspond to degenerate Riemann surfaces Σ.

Whenever that happens, the supergravity description breaks down, even if R_{10} and L_6 are large. Thus eleven-dimensional supergravity provides a useful description of the five-brane wrapped on Σ only sufficiently far from any points in moduli space where the infrared behavior changes; in particular, it cannot be used to study (beyond the BPS sector) the nontrivial superconformal field theories discussed by Argyres and Douglas (1995) and Argyres *et al.* (1996).

5. Quantum effects: II

The analysis of the previous section can be easily generalized to the chain of five-branes connected by four-branes mentioned above (Witten, 1997a). Specifically, consider the type-IIA configuration of $n+1$ NS5-branes labeled by $\alpha=0,1,\ldots,n$, with k_α $D4$-branes connecting the $(\alpha-1)$st and αth five-branes (Fig. 22). For simplicity, we assume that there are no semi-infinite four-branes at the edges.

Classically we saw that the gauge group was $\Pi_{\alpha=1}^{n}U(k_\alpha)$, but the n $U(1)$ factors are frozen as before. Thus the gauge group is

$$G=\prod_{\alpha=1}^{n} SU(k_\alpha) \tag{131}$$

with matter in the bifundamental representation $(k_\alpha,\overline{k}_{\alpha+1})$ of adjacent factors of the gauge group. We shall further assume that all factors in Eq. (131) are asymptotically free:

$$2k_\alpha-(k_{\alpha+1}+k_{\alpha-1})\geqslant 0,\quad \forall\alpha. \tag{132}$$

Following the logic of our previous discussion we expect the Riemann surface Σ to be described in this case by the holomorphic equation

$$F(t,v)=t^{n+1}+P_{k_1}(v)t^n+P_{k_2}(v)t^{n-1}+\cdots$$
$$+P_{k_n}(v)t+1=0. \tag{133}$$

The fact that Eq. (133) is a polynomial of degree $n+1$ in t ensures that there are $n+1$ solutions for t corresponding to the $n+1$ NS5-branes. The v independence of the coefficients of t^{n+1} and 1 implies the absence of semi-infinite four-branes. The degrees of the polynomials in v $P_{k_\alpha}=c_\alpha v^{k_\alpha}+\cdots$ are determined by the fact that when one rewrites

$$F(t,v)=\prod_{\alpha=0}^{n}(t-t_\alpha(v)) \tag{134}$$

the locations of the $n+1$ five-branes $t_\alpha(v)$ must behave for large v as Eqs. (103)–(105):

$$t_\alpha(v)\sim v^{k_{\alpha+1}-k_\alpha} \tag{135}$$

(where $k_0=k_{n+1}=0$); to check that this leads to Eq. (133) one has to use Eq. (132). The roots of $P_{k_\alpha}(v)$ correspond to the positions of the k_α four-branes connecting the $(\alpha-1)$st and αth five-branes.

As we have seen in the classical brane construction, semi-infinite four-branes provide a convenient tool for describing the Coulomb branch of SQCD with fundamental matter, but to study the full moduli space of vacua (in particular, to see the Higgs branches) it is necessary to introduce $D6$-branes. Our next task is to understand models with six-branes at finite R_{10}/L_6.

Recall that the $D6$-brane corresponds in M theory to a Kaluza-Klein monopole magnetically charged under $G_{\mu 10}$. The (hyper-Kähler) metric around a collection of Kaluza-Klein monopoles is the multi-Taub-NUT metric (22)–(24). We do not actually need the metric around a Kaluza-Klein monopole, but only its complex structure. The hyper-Kähler manifold (22)–(24) in fact admits three independent complex structures, any of which is suitable for our purposes.

The typical situation we shall be interested in is one where there are N_f Kaluza-Klein monopoles at $v=m_1,\ldots,m_{N_f}$. One of the complex structures of the corresponding multi-Taub-NUT space can be described by embedding it in a three-complex-dimensional space with coordinates y,z,v. It is given by

$$yz=\prod_{i=1}^{N_f}(v-m_i). \tag{136}$$

When all the Kaluza-Klein monopoles coincide, Eq. (136) approaches an A_{N_f-1} singularity $yz=v^{N_f}$. The symmetry $y\to\lambda y$, $z\to\lambda^{-1}z$ of Eq. (136) corresponds to $t\to\lambda t$. Thus one can think of y as corresponding to t (with z fixed) or of z as corresponding to t^{-1} (with y fixed).

Note that the complex structure (136) is insensitive to the x^6 location of the N_f Kaluza-Klein monopoles. That information resides in the Kähler class of the metric (22)–(24), which does depend on x^6. Thus even when different m_i in Eq. (136) coincide, the corresponding A_{N_f-1} singularity may still be resolved by separating the centers of the monopoles in x^6.

Consider as an example the $N=2$ SQCD with gauge group $G=SU(N_c)$ and N_f flavors, realized classically as N_f $D6$-branes situated between the two NS5-branes in the x^6 direction (Fig. 14). At finite L_6/R_{10} we need again to find the shape of an $M5$-brane, except now it lives not in $Q=R^3\times S^1$, but rather in $\tilde{Q}=$resolved A_{N_f-1} multi-Taub-NUT space [Eq. (136)].

We can again describe the five-brane by a curve of the form

$$A(v)y^2+B(v)y+C(v)=0 \tag{137}$$

with some polynomials A,B,C. As before, $A(v)=1$, since otherwise y (and therefore t) diverges at roots of $A(v)$. Rewriting Eq. (137) in terms of $z=\Pi(v-m_i)/y$ and requiring that there be no singularities of z for finite v (these too would correspond to semi-infinite four-branes) one finds that $C=a\Pi(v-m_i)$ (see Witten, 1997a for a more detailed analysis). Finally, $B(v)$ in Eq. (137) is a polynomial of degree N_c as before in Eq. (108).

Thus we recover the solution found before using semi-infinite four-branes. The fact that the result (137) is independent of the x^6 position of the $D6$-branes is consistent with our discussion in Sec. IV.C.1, where this was deduced as a consequence of the Hanany-Witten transition.

The description of $N=2$ SQCD with six-branes (Kaluza-Klein monopoles) can be used to describe the Higgs branch of the theory as well. We refer the reader to Witten (1997a) for a detailed discussion of this.

Finally, $N=2$ gauge theories on N_c four-branes in the presence of six-branes and orientifold planes can be lifted to M theory and used to derive the curves and describe the Higgs branches of $SO(N_c)$ and $Sp(N_c/2)$ theories as well as many product groups (Brandhuber, Sonnenschein, *et al.* (1997b); Fayyazuddin and Spalinski, 1997a; Landsteiner, Lopez, and Lowe, 1997; Nakatsu *et al.*, 1997; Landsteiner and Lopez, 1998 Nakatsu, Ohta, and Yokono, 1998a; Erlich, Naqvi, and Randall, 1998).

V. FOUR-DIMENSIONAL THEORIES WITH $N=1$ SUPERSYMMETRY

In this section we turn our attention to four-dimensional $N=1$ supersymmetric gauge theories which typically have the richest dynamics among the different supersymmetric Yang-Mills theories and are the closest to phenomenology.

As we saw in the previous sections, $N=4$ supersymmetric gauge theory has the simplest dynamics and phase structure. The theory is specified by the choice of a gauge group; all matter is in the adjoint representation. The vacuum structure consists of a Coulomb branch with singularities corresponding to points of enhanced unbroken gauge symmetry. The most singular point is the origin of moduli space, which corresponds to a nontrivial conformal field theory parametrized by the exactly marginal gauge coupling g. The form of the effective action up to two derivatives is completely determined by the symmetry structure; in particular, the metric on the Coulomb branch is flat. The leading terms that receive quantum corrections are certain nonrenormalizable (=irrelevant) four-derivative terms, and these corrections can be controlled since they receive contributions only from BPS-saturated states. The most interesting feature of the dynamics of $N=4$ SYM theory is the discrete identification of theories on the line of fixed points labeled by g provided by Montonen-Olive duality (which acts as $g\leftrightarrow 1/g$). Another interesting phenomenon is the appearance of nontrivial infrared fixed points of the renormalization group at which electrically and magnetically charged particles become massless at the same time.

In the $N=2$ SYM case there are some new features. Theories are now labeled by the choice of a gauge group and a set of matter representations. Nontrivial quantum corrections to the two derivative terms in the vector multiplet action lead to a modification of the metric on the Coulomb branch, described by Seiberg and Witten. In addition, Higgs branches appear in which the rank of the gauge group is reduced; as we saw before, $N=2$ theories typically have a rather rich phase structure.

$N=1$ dynamics generally leads to yet another host of new phenomena (see Amati *et al.*, 1988; Seiberg, 1995b; Giveon, 1996; Intriligator and Seiberg, 1996a; Peskin, 1997; Shifman, 1997, and references therein). It is now possible to write a classical (tree-level) superpotential. Furthermore, the superpotential can in general receive quantum corrections that modify the potential of the light fields. At the same time these corrections are often under control since they are holomorphic, taking the form of a *superpotential* on the classical moduli space. The effect of the quantum superpotential may be to lift a part of the classical moduli space, change its topology, or in some cases break supersymmetry completely, a possibility with obvious phenomenological appeal. $N=1$ SYM theories may also have a chiral matter content and exhibit confinement, possibilities that do not exist in $N\geqslant 2$ theories and that are clearly desirable in a realistic theory. Another interesting phenomenon is the infrared equivalence between different $N=1$ SUSY gauge theories discovered by Seiberg. It provides a generalization of Montonen-Olive duality to theories with a nontrivial beta function. As we discuss below, despite the running of the coupling there is a sense in which Seiberg's duality can be sometimes thought of as a strong-weak coupling duality, and in many cases it allows one to analyze the strongly coupled dynamics of $N=1$ SYM theories.

In this section we shall describe $N=1$ SYM theories using branes (Elitzur, Giveon, and Kutasov, 1997). We shall see that just as in theories with more supersymmetry, embedding $N=1$ SYM theory in brane theory provides a useful qualitative and quantitative guide for studying the classical and quantum vacuum structure of these theories. In particular, brane dynamics can be used to understand Seiberg's infrared duality and a host of other interesting strong-coupling effects. We start with a brief summary of some field-theory results (for more details see the reviews cited above and references therein) and then move on to the brane description.

A. Field-theory results

Pure $N=1$ SYM theory with a simple gauge group G describes a vector multiplet V (69) transforming in the adjoint representation of G. The classical theory has a single vacuum and a $U(1)_R$ symmetry, discussed in Sec. IV.A. As in the $N=2$ case, the existence of the classical R symmetry is related to the classical superconformal invariance of pure $N=1$ SYM theory.

Quantum mechanically, the theory develops a β function that breaks conformal invariance. Accordingly, the $U(1)_R$ symmetry is broken by the gaugino condensate:

$$\langle (\mathrm{Tr}\lambda\lambda)^{C_2}\rangle \sim (N_c\Lambda^3)^{C_2} \tag{138}$$

to a discrete subgroup Z_{2C_2}. Λ is the dynamically generated QCD scale and C_2 is the second Casimir in the adjoint representation; e.g., $C_2=N_c$ for $G=SU(N_c)$, $C_2=N_c-2$ for $G=SO(N_c)$, $C_2=N_c+2$ for $G=Sp(N_c/2)$. The theory has C_2 vacua corresponding to different values of the condensate consistent with Eq. (138):

$$\langle \mathrm{Tr}\lambda\lambda\rangle = \mathrm{const}\times N_c\Lambda^3 e^{2\pi i k/C_2};\quad k=0,1,2,\ldots,C_2-1, \tag{139}$$

which spontaneously breaks the discrete chiral symmetry $Z_{2C_2}\to Z_2$. Each of the C_2 vacua contributes 1 to the Witten index, $\mathrm{Tr}(-)^F=C_2$. It is useful to note for future use that Eqs. (138) and (139) are equivalent to a constant nonperturbative superpotential

$$W_{\mathrm{eff}} = \mathrm{const}\times N_c^2\Lambda^3. \tag{140}$$

Matter is described by chiral multiplets Q_f in representations R_f of G. The classical potential for the scalars in the multiplets (which will be denoted by Q_f as well) includes a "D term" contribution [the analog of Eq. (73)]:

$$V_D(Q)=\sum_a \left(\sum_f Q_f^\dagger T_f^a Q_f\right)^2, \tag{141}$$

where $a=1,\ldots,\dim G$ runs over the generators of the gauge group; f labels different "flavors" or representations; and T_f^a are the generators of G in the representation R_f. The only other contribution to the scalar potential comes from the superpotential,

$$\int d^2\theta W(Q) + \int d^2\bar{\theta} W^*(Q^\dagger), \tag{142}$$

which leads after performing the θ integrals to a potential

$$V_W(Q)\sim \sum_f \left|\frac{\partial W}{\partial Q_f}\right|^2. \tag{143}$$

Classically there are often flat directions in field space along which the potential vanishes. They correspond through Eqs. (141) and (143) to solutions of $V_D=V_W=0$, i.e.,

$$\sum_f Q_f^\dagger T_f^a Q_f = \frac{\partial W}{\partial Q_f} = 0. \tag{144}$$

When the superpotential vanishes, one can show that the space of solutions of $V_D=0$ [Eq. (144)] is parametrized by holomorphic gauge-invariant combinations of the matter fields Q_f. When $W\neq 0$ one has to mod out[28] that space by the second constraint in Eq. (144).

Quantum corrections in general modify the superpotential (142) and consequently lift some or all of the classical moduli space. Because W is a holomorphic function of Q, in many cases the form of the quantum superpotential can be determined exactly. The quantum corrections to the Kähler potential are in general more complicated and are not under control. Fortunately, to study the vacuum structure it is not important what the Kähler potential is precisely, as long as it is nonsingular (and supersymmetry is not broken). Thus below we shall usually ignore the Kähler potential, assuming it is nonsingular in the variables we shall be using. Usually, there is some circumstantial evidence that this is the case (which we shall not review).

In the following we shall discuss a few examples, starting with $N=1$ SQCD—an $SU(N_c)$ SYM theory with N_f flavors $Q^i,\tilde{Q}_i$, $i=1,\ldots,N_f$, in the fundamental and antifundamental representations, respectively. In the absence of a superpotential, the classical global symmetry of the theory is

$$SU(N_f)_L\times SU(N_f)_R\times U(1)_B\times U(1)_a\times U(1)_x. \tag{145}$$

The two $SU(N_f)$ factors rotate the quarks $Q^i,\tilde{Q}_i$, respectively; $U(1)_B$ is a vectorlike symmetry, which assigns charge $+1(-1)$ to $Q(\tilde{Q})$. $U(1)_a$ and $U(1)_x$ are R symmetries under which the gaugino is assigned charge one, and the quarks $Q,\tilde{Q}$ have charge 0 or 1. Only one

[28]In string theory, the operation of modding out by a symmetry consists of removing all states that are not invariant under the symmetry and, in some cases, adding so-called "twisted states," required for consistency (Green, Schwarz, and Witten, 1987).

combination of the two R symmetries is anomaly free—we shall refer to it as $U(1)_R$. The anomaly-free global symmetry of $N=1$ SQCD (with vanishing superpotential) is

$$SU(N_f)_L\times SU(N_f)_R\times U(1)_B\times U(1)_R. \tag{146}$$

The $U(1)_R$ charge of the quarks is

$$R(Q)=R(\tilde{Q})=1-N_c/N_f. \tag{147}$$

The $U(1)_R$ symmetry (147) plays an important role in analyzing the strongly coupled quantum dynamics of SQCD. At long distances the theory flows to a fixed point in which $N=1$ supersymmetry is enhanced to $N=1$ superconformal symmetry. The $U(1)_R$ symmetry (147) becomes part of the superconformal algebra in the infrared. This is important because the superconformal algebra implies that for chiral operators the scaling dimension at the infrared fixed point D is related to their R charge Q via the relation $D=3Q/2$. The fact that the symmetry (147) is a good symmetry throughout the RG trajectory allows one to compute "critical exponents" at a nontrivial IR fixed point by calculating charges of operators at the free UV fixed point.

1. Classical $N=1$ supersymmetric QCD

For $N_f<N_c$ massless flavors of quarks the moduli space of vacua is N_f^2 dimensional. It is labeled by the gauge-invariant meson fields

$$M^i_j\equiv Q^i\tilde{Q}_j, \quad i,j=1,\ldots,N_f. \tag{148}$$

The gauge group can be maximally broken to $SU(N_c-N_f)$. As a check, the quarks have $2N_cN_f$ complex components, out of which $N_c^2-(N_c-N_f)^2$ are eaten by the Higgs mechanism, leaving N_f^2 massless degrees of freedom M^i_j. In various subspaces of the classical moduli space, part or all of the broken gauge symmetry is restored, and the classical moduli space is singular—one has to add additional degrees of freedom corresponding to massless quarks and gluons to describe the low-energy dynamics.

For $N_f\geq N_c$ new gauge-invariant fields appear in addition to Eq. (148), the baryon fields:

$$B^{i_1i_2\cdots i_{N_c}}=\epsilon^{\alpha_1\alpha_2\cdots\alpha_{N_c}}Q^{i_1}_{\alpha_1}Q^{i_2}_{\alpha_2}\cdots Q^{i_{N_c}}_{\alpha_{N_c}}. \tag{149}$$

There are $\binom{N_f}{N_c}$ baryon fields. In particular, for $N_f=N_c$ there is a unique baryon field B,

$$B=\epsilon_{i_1\cdots i_{N_c}}Q^{i_1}\cdots Q^{i_{N_c}}. \tag{150}$$

This structure is doubled since there are also fields $\tilde{B}$ constructed out of the antifundamentals $\tilde{Q}$ in an analogous way to Eqs. (149) and (150).

Since for $N_f\geq N_c$ the gauge group can be completely broken by the Higgs mechanism, the complex dimension of the classical moduli space is $2N_cN_f-(N_c^2-1)$. That means that there are many constraints relating the baryon and meson fields. For example, for $N_f=N_c$ the constraint is

$$\det M-B\tilde{B}=0, \tag{151}$$

which gives the correct dimension of moduli space $N_c^2+2-1=N_c^2+1$. As before, the manifold (151) describing the classical moduli space is singular, with additional massless fields (gluons and quarks) coming down to zero mass when B, $\tilde{B}$ and/or $\det M$ go to zero.

For general $N_f>N_c$ the classical moduli space of vacua is rather complicated. The full set of classical constraints among the mesons and baryons for general N_f,N_c has not been written down.

2. Quantum $N=1$ supersymmetric QCD

For $N_f<N_c$ the classical picture of an N_f^2-dimensional moduli space labeled by the mesons M^i_j, with singularities corresponding to enhanced unbroken gauge symmetry, is drastically modified due to the fact that the theory generates a nonperturbative superpotential for M. The unique superpotential (up to an overall scheme-dependent constant) that is compatible with the symmetries is

$$W_{\rm eff}=(N_c-N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} \tag{152}$$

where Λ is the dynamically generated QCD scale. It has been shown that the superpotential (152) is indeed generated by gaugino condensation in the unbroken gauge group $SU(N_c-N_f)$ for $N_f\leq N_c-2$ and by instantons for $N_f=N_c-1$.

Using Eq. (143) we see that $W_{\rm eff}$ gives rise to a potential with no minimum at a finite value of the fields. Thus the quantum theory exhibits runaway behavior to $M\to\infty$. Adding masses to all the quarks,

$$W=W_{\rm eff}-m^j_iQ^i\tilde{Q}_j, \tag{153}$$

where the rank of the mass matrix is N_f, stabilizes the runaway behavior and gives rise to the N_c vacua of pure $N=1$, $SU(N_c)$ supersymmetric Yang-Mills theory mentioned above. To see this, one integrates out the massive fields M^i_j, which leads to a superpotential

$$W_{\rm eff}={\rm const}\times(\Lambda^{3N_c-N_f}\det m)^{1/N_c}. \tag{154}$$

Using the scale-matching relation between the high-energy theory with N_f flavors, Λ_{N_c,N_f}, and the low-energy theory with no flavors, $\Lambda_{N_c,0}$,

$$\Lambda^{3N_c}_{N_c,0}=\Lambda^{3N_c-N_f}_{N_c,N_f}\det m \tag{155}$$

leads to the pure SYM superpotential (140).

For $N_f=N_c$ the superpotential (152) is singular. One finds that $W_{\rm eff}=0$, but there are still important quantum effects. In particular, the classical constraint (151) is modified to

$$\det M-B\tilde{B}=\Lambda^{2N_c}. \tag{156}$$

Thus in this case quantum effects smooth the singularities in the classical moduli space; in particular, there is no point in moduli space where quarks and gluons go to zero mass and the physics is well described by the mesons and baryons subject to the constraint (156). This means that color is confined. Note also that since the point $M=B=\tilde{B}=0$ is not part of the quantum moduli space, there is no point where the full chiral anomaly-free global symmetry (146) is unbroken. Thus in this case SQCD is confining and breaks chiral symmetry. The moduli space (156) can be thought of as the moduli space of vacua of a sigma model for a set of fields M^i_j, B, $\tilde{B}$ and λ with the superpotential

$$W_{\rm eff}=\lambda(\det M-B\tilde{B}-\Lambda^{2N_c}). \qquad (157)$$

Here λ is a Lagrange multiplier field, which is massive and hence does not appear in the low-energy dynamics. Integrating it out leads to Eq. (156).

For $N_f=N_c+1$ the baryons (149) can be dualized to fields with one flavor index, $B_i=\epsilon_{ii_1\cdots i_{N_c}}B^{i_1\cdots i_{N_c}}$. Classically, the low-energy degrees of freedom $M^i_j, B_i, \tilde{B}_i$ satisfy the constraints

$$\det M(M^{-1})^j_i-B_i\tilde{B}^j=M^i_jB_i=M^i_j\tilde{B}^j=0. \qquad (158)$$

Quantum mechanically, the classical constraint is lifted and the mesons and baryons can be thought of as independent fields, governed by the superpotential

$$W_{\rm eff}=-\frac{\det M-M^i_jB_i\tilde{B}^j}{\Lambda^{2N_c-1}}. \qquad (159)$$

The vacuum equations $\partial_M W_{\rm eff}=\partial_B W_{\rm eff}=\partial_{\tilde{B}}W_{\rm eff}=0$ give the classical constraints (158).

It is at first sight surprising that the quantum meson and baryon fields satisfy the classical constraints (158) only as equations of motion, when in the classical limit they are "Bianchi identities." Two comments are useful to clarify the situation. First, the classical limit corresponds here to $\Lambda\to 0$; in that case the path integral is dominated by configurations satisfying the constraints (158). Second, the situation is analogous to what happens under electric-magnetic duality. In the electric variables, $\partial_\mu F^{\mu\nu}=0$ is an equation of motion while $\partial_\mu\tilde{F}^{\mu\nu}=0$ is a Bianchi identity, while in the magnetic variables the roles are reversed. In fact, as we shall discuss later, the situation here is not only analogous but identical to this example. The relation between $N=1$ SQCD with $N_f=N_c+1$ and the σ model (159) is a special case of a non-Abelian generalization of electric-magnetic duality, which indeed exchanges Bianchi identities and equations of motion.

The resulting quantum moduli space for $N_f=N_c+1$ is identical to the classical one. In particular, it has the same singularity structure, but the interpretation of the singularities is different. While in the classical theory the singularities are due to massless quarks and gluons, in the quantum one they are due to massless mesons and baryons. The theory again confines, but this time the point $M=B=\tilde{B}=0$ *is* in the moduli space and chiral symmetry is unbroken there. It is not difficult to see that adding a mass to one or more of the flavors (153) gives rise to the results (156) and (152), respectively.

For $N_f>N_c+1$ there is no known description of the quantum moduli space in terms of a σ model for the gauge-invariant degrees of freedom M,B. Attempts to write superpotentials consistent with the symmetries typically lead to singularities, suggesting the presence of additional light degrees of freedom. For $N_f\geqslant 3N_c$ it is clear what the relevant degrees of freedom are. In that case the theory is not asymptotically free and at low energies the quarks and gluons are free (up to logarithmic corrections), much as they are in QED. We refer to the theory as being in a free electric phase, since electrically charged sources have a QED-like potential $V(R)\sim 1/R\log R$ in this case.

For $N_f<3N_c$ the theory is asymptotically free. If N_f is very close to $3N_c$ (a possibility that exists, for example, if N_c, N_f are large) there is a weakly coupled infrared fixed point that can be studied perturbatively and describes interacting quarks and gluons. Electrically charged sources have a potential $V(R)\sim\alpha^*/R$, and we say that the theory is in a non-Abelian Coulomb phase. As N_f is decreased, the infrared coupling α^* increases, and perturbation theory breaks down. For most values of N_f in the region $N_c+1<N_f<3N_c$ the theory is strongly coupled and it is not clear what is the infrared dynamics.

The degrees of freedom needed to describe low-energy $N=1$ SQCD in this range were uncovered by Seiberg, who has shown that there is another gauge theory—with a different high-energy behavior—that flows to the same infrared fixed point as SQCD. Specifically, he discovered Seiberg's duality.

The following two theories flow at long distances to the same fixed point:

(1) "Electric" SQCD, with gauge group $G_e=SU(N_c)$ and N_f flavors of quarks $Q^i,\tilde{Q}_i$.
(2) "Magnetic" SQCD, with gauge group $G_m=SU(N_f-N_c)$, N_f magnetic quarks q_i, $\tilde{q}^i$ and a gauge singlet "magnetic meson" chiral superfield M^i_j, which couples to the magnetic quarks via the superpotential

$$W_{\rm mag}=M^i_jq_i\tilde{q}^j. \qquad (160)$$

The singlet mesons M^i_j are the magnetic analogs of the composite mesons $Q^i\tilde{Q}_j$ of the electric theory. Other operators can be mapped as well, but it is not understood in the context of gauge theory how to perform directly a transformation from the electric to the magnetic path integral. In particular, the magnetic quarks and gluons must be rather nonlocal functions of their electric counterparts. For example, the mapping of the baryons (149) implies that (suppressing flavor indices) $q^{N_f-N_c}\sim Q^{N_c}$.

Seiberg's duality allows one to study the low-energy dynamics of the electric theory in the regime $N_c+1<N_f<3N_c$ by passing to the magnetic variables. The magnetic $SU(N_f-N_c)$ gauge theory is not asymptotically free when $N_f<3N_c/2$; in this regime, Seiberg's duality predicts that the strongly interacting electric $SU(N_c)$ gauge theory is in fact free in the appropriate variables. Since the weakly coupled variables in this case are the dual, magnetic variables, we refer to the electric theory as being in a free magnetic phase.

For $N_f > 3N_c/2$ the magnetic theory is asymptotically free, but as in the electric case, when N_f is sufficiently close to $3N_c/2$ it describes weakly interacting magnetic quarks and gluons (as well as the fields M) in the IR. As we increase N_f, the coupling in the IR increases. We see that the electric and magnetic descriptions provide complementary pictures of the non-Abelian Coulomb phase. As N_f increases, the electric description becomes more weakly coupled (and thus more useful) while the magnetic one becomes more strongly coupled and vice versa.

The original SQCD examples constructed by Seiberg were generalized in a few different directions, and many additional examples of the basic phenomenon have been found. There is in general no proof of Seiberg's duality in the context of gauge theory but there is a great deal of evidence supporting it. There are three kinds of independent tests:

- Members of a dual pair have the same global symmetries and the 't Hooft anomaly matching conditions for these symmetries are satisfied.
- The two theories have the same moduli spaces of vacua, obtained by giving expectation values to the first components of chiral superfields.
- The equivalence is preserved under deformations of the theories by the F components of chiral operators. In particular, the moduli spaces and chiral rings agree as a function of these deformations.

It is important to stress that in every one of these tests the classical theories are different and only the quantum theories become equivalent. For example, in SQCD the electric theory does not develop a quantum superpotential (for $N_f > N_c + 1$), while in the magnetic theory the classical superpotential (160) is corrected quantum mechanically to

$$W_{\text{quantum}} \sim \frac{1}{\mu} M q \tilde{q} + \Lambda^{(3N_c - N_f)/(N_c - N_f)} \times (\det M)^{1/(N_f - N_c)}, \tag{161}$$

where μ is some fixed scale.

There is also a crucial difference in the interpretation of the deformations of the two theories. Often, when one theory is Higgsed and becomes weaker, its dual is confining and becomes stronger. This is one reason for interpreting the relation between these theories as electric-magnetic duality.

3. Supersymmetric QCD with an adjoint superfield

An interesting generalization of $N=1$ SQCD is obtained by adding to the theory a chiral superfield Φ in the adjoint representation. The theory without a classical superpotential is very interesting (Kutasov, Schwimmer, and Seiberg, 1996). Unfortunately, not much is known about its long-distance behavior. It is known that the quantum moduli space is identical to the classical one. The only singularities are at points where classically the unbroken gauge symmetry is enhanced. The most singular point in moduli space is the origin. It is expected that the theory at the origin is in a non-Abelian Coulomb phase for all $N_f \geq 1$ (for $N_f = 0$ it actually has $N=2$ supersymmetry and is equivalent to pure $N=2$ SYM theory, described in Sec. IV). As we saw before, the physical interpretation of the singularities in the quantum theory may be different from the classical one.

While the infrared physics at the origin of moduli space is mysterious, some perturbations of the theory by tree-level superpotentials lead to theories whose low-energy behavior is understood. If we add the superpotential

$$W = \lambda \sum_{i=1}^{N_f} \tilde{Q}_i \Phi Q^i \tag{162}$$

we get a theory that can be analyzed easily. When the Yukawa coupling λ is one we recover the $N=2$ SUSY theory discussed in Sec. IV. The moduli space of the theory has a Coulomb branch which has only massless photons at generic points. At special singular points on the moduli space there are more massless particles: massless monopoles, dyons, massless gluons and quarks, and even points with interacting $N=2$ superconformal field theories. More quantitatively, this branch of the moduli space is described by the hyperelliptic curves discussed in Sec. IV.

It is easy to extend the curve away from the $N=2$ limit (Elitzur *et al.*, 1995; Hanany and Oz, 1995). Using the symmetries of the theory this is achieved by replacing factors of $\Lambda^{2N_c - N_f}$ in the curve by $\lambda^{N_f}\Lambda^{2N_c - N_f}$. Therefore, as $\lambda \to 0$, all the features of the Coulomb branch approach the origin; this is clearly a singular limit which is not easy to describe from this point of view.

Another deformation that simplifies the dynamics involves turning on a polynomial superpotential for Φ. When Φ is massive,

$$W = \mu \operatorname{Tr} \Phi^2 + \lambda \sum_{i=1}^{N_f} \tilde{Q}_i \Phi Q^i, \tag{163}$$

we can integrate it out and obtain a superpotential for the quarks of the form

$$W \sim \frac{\lambda^2}{\mu} \tilde{Q}_i Q^j \tilde{Q}_j Q^i. \tag{164}$$

In the limit $\mu \to \infty$ the quartic superpotential (164) disappears and we recover the $SU(N_c)$ theory with N_f flavors considered above.

An interesting deformation corresponds to the pure polynomial superpotential

$$W_{\text{el}} = \sum_{i=0}^{k} \frac{s_i}{k+1-i} \operatorname{Tr} \Phi^{k+1-i}. \tag{165}$$

At first sight the fact that the high-order polynomial appearing in Eq. (165) can have any effect on the physics is surprising. Indeed, the presence of these nonrenormalizable interactions seems irrelevant for the long-distance behavior of the theory, which is our main interest. Nevertheless, these operators have in general strong effects on the infrared dynamics. They are examples of operators that in the general theory of the renormalization group are known as *dangerously irrelevant*.

It was shown by Kutasov (1995), Kutasov and Schwimmer (1995), and Kutasov, Schwimmer, and Seiberg (1996) that in the presence of the superpotential (165) there is a simple dual description. The magnetic theory has gauge group $SU(kN_f - N_c)$ with N_f magnetic quarks $q, \tilde{q}$, an adjoint field φ, and k gauge singlet magnetic meson fields M_j, $j=1,\ldots,k$, which correspond to the composite operators

$$(M_j)^i_l = \tilde{Q}_l \Phi^{j-1} Q^i. \tag{166}$$

The magnetic theory has a superpotential

$$W_{\rm mag} = -\sum_l \frac{t_l}{k+1-l} {\rm Tr}\varphi^{k+1-l} + \sum_{l=0}^{k-1} t_l \sum_{j=1}^{k-l} M_j \tilde{q} \varphi^{k-j-l} q, \tag{167}$$

where $\{t_i\}$ are coordinates on the space of magnetic theories, related to the $\{s_i\}$ by a known coordinate transformation on theory space.

When all the $\{s_i\}$ except for s_0 vanish, the same is true for the magnetic couplings $\{t_i\}$, and the duality relates in general nontrivial strongly coupled gauge theories with $W_{\rm el} \sim {\rm Tr}\Phi^{k+1}$, and $W_{\rm mag} \sim {\rm Tr}\varphi^{k+1}$. When the $\{s_i\}$ are generic, the k solutions of $W'(x)=0$ for both the electric and magnetic theories are distinct and both theories have a rather rich vacuum structure. If we place r_i eigenvalues of Φ in the ith minimum of the bosonic potential corresponding to Eq. (165), the theory describes at low energies k decoupled SQCD systems with gauge group $SU(r_i)$, N_f flavors of quarks, and gauged baryon number. The total gauge group is broken as

$$SU(N_c) \to SU(r_1) \times SU(r_2) \cdots \times SU(r_k) \times U(1)^{k-1}. \tag{168}$$

A similar story holds for the magnetic theory; the electric-magnetic duality between Eqs. (165) and (167) reduces in such vacua to k decoupled versions of the original SQCD duality due to Seiberg. More generally, a matrix version of singularity theory is useful in the analysis of the theory (Kutasov, Schwimmer, and Seiberg, 1996).

B. Branes suspended between nonparallel branes

In Sec. IV.C we discussed configurations of NS5, $D4$, and $D6$-branes [Eq. (91)] that preserve eight supercharges and are useful for describing four-dimensional $N=2$ SUSY gauge theories. To describe $N=1$ SYM theory, we should like to break four supercharges by changing the orientation of some of the branes in the configuration. This problem was encountered and discussed in Sec. II.E.2. We saw there that performing complex rotations such as that given by Eq. (49) leads to configurations depending on continuous parameters, which preserve the same four supercharges for all values of the parameters. In this section we shall use this basic idea to study $N=1$ SYM theory using branes (Elitzur, Giveon, and Kutasov, 1997; Elitzur *et al.*, 1997).

Starting with the brane configuration describing $N=2$ SQCD with $G=SU(N_c)$ and N_f fundamental hypermultiplets (Fig. 11), we can apply complex rotations of the general form (49) to one or both of the NS5-branes, or one or more of the $D6$-branes, such that $N=1$ supersymmetry is preserved. Of course, only the relative orientation in the (v,w) plane of all these objects is meaningful. Recall that NS5-branes are located at some particular value of w and are stretched in the (x^μ, v) directions, where $\mu=0,1,2,3$ and

$$v = x^4 + ix^5, \qquad w = x^8 + ix^9, \tag{169}$$

while $D6$-branes are located at a particular value of v and are stretched in (x^μ, w) (as well as x^7).

If we rotate (say) the rightmost NS5-brane in Fig. 11 by the angle θ (Barbon, 1997) $(v,w) \to (v_\theta, w_\theta)$ [Eq. (49)], where

$$v_\theta = v\cos\theta + w\sin\theta, \qquad w_\theta = -v\sin\theta + w\cos\theta, \tag{170}$$

then the resulting five-brane, which we may refer to as the "NS5$_\theta$-brane," is located at $w_\theta=0$, or

$$w_\theta = 0 \Rightarrow \ w = v\tan\theta \equiv \mu(\theta) v. \tag{171}$$

Obviously, one can also apply rotations of the (x^8, x^9) plane, $w \to e^{i\varphi} w$ [or rotations of the (x^4, x^5) plane, $v \to e^{-i\varphi} v$]. Therefore, generically, μ is complex,

$$\mu(\theta, \varphi) = e^{i\varphi} \tan\theta \tag{172}$$

(we shall usually ignore this possible φ dependence).

$\theta=0$ corresponds to the original NS-brane: NS5$_0 \equiv$NS5. For $\theta=\pi/2$, the rotated brane is stretched in w and it is located at $v=0$. Since this object will be particularly useful below we give it a name, the "NS5$'$-brane": NS5$_{\pi/2} \equiv$NS5$'$. Its worldvolume is

$$\text{NS5}': \ (x^0, x^1, x^2, x^3, x^8, x^9). \tag{173}$$

Note that to be able to rotate one of the NS5-branes relative to the other we need to locate all the $D4$-branes stretched between them in Fig. 11 at $v=w=0$, i.e., approach the origin of the Coulomb branch. The field describing fluctuations of the four-branes along the five-branes, the chiral multiplet in the adjoint representation of $SU(N_c)$ that belongs to the vector multiplet of $N=2$ supersymmetry, gets a θ-dependent mass due to the rotation. Thus the effect of the rotation on the low-energy field theory on the $D4$-branes can be parametrized by the superpotential

$$W \sim \mu(\theta)\Phi^2 + \sum_{i=1}^{N_f} \tilde{Q}_i \Phi Q^i, \tag{174}$$

which is a special case of the theory discussed in Eq. (163).

The mass of the adjoint chiral superfield $\mu(\theta)$ in Eq. (174) clearly breaks $N=2$ supersymmetry to $N=1$. The resulting low-energy theory is $N=1$ SQCD with a superpotential for the quarks (164) obtained by integrating out the massive adjoint field Φ. At least on a qualitative level, the mass μ in Eq. (174) is related to the geometrical complex rotation parameter given in Eq. (171). Indeed, both vanish for $\theta=0$ (the $N=2$ SUSY configuration), while when $\theta\to\pi/2$ we shall see later that the mass μ must go to infinity and we recover SQCD with vanishing superpotential (164).

What happens when we rotate both NS5-branes of the $N=2$ configuration of Fig. 11 by the same angle θ? In the absence of $D6$-branes (i.e., for $N_f=0$) the answer is nothing, since there is a symmetry between v and w, so the low-energy theory is pure $N=2$ SYM theory for all θ. In the presence of $D6$-branes, the relative orientation between the NS5 and $D6$-branes changes, and it is natural to expect that the Yukawa coupling necessary for $N=2$ supersymmetry will change with θ,

$$W=\lambda(\theta)\sum_{i=1}^{N_f}\tilde{Q}_i\Phi Q^i. \tag{175}$$

This is the model discussed after Eq. (162). The massless adjoint chiral superfield Φ is now associated with fluctuations along the v_θ (170) directions. The locations of $D4$-branes along the NS5$_\theta$-branes correspond to the expectation values $\langle\Phi\rangle$ and parametrize the Coulomb branch. The quarks are massive on the Coulomb branch; their mass $\lambda(\theta)\langle\Phi\rangle$ is due in the brane description to open 4–6 strings whose minimal length is $\langle\Phi\rangle\cos\theta$. We thus learn that the Yukawa coupling λ depends on the angle θ via

$$\lambda(\theta)=\cos\theta. \tag{176}$$

Here $\theta=0$ corresponds to $\lambda=1$, the $N=2$ configuration, while for $\theta=\pi/2$ (i.e., after rotating the NS5-branes to NS5′-branes) the superpotential vanishes.

To recapitulate, the dictionary between the deformations of the $N=2$ SUSY brane configuration that preserve $N=1$ supersymmetry and their manifestations in the low-energy theory on the four-branes is as follows. Keeping one of the NS5-branes and all the $D6$-branes fixed and rotating the remaining NS5-brane corresponds to changing the mass of the adjoint chiral superfield Φ (174). Rotating both NS5-branes relative to the $D6$-branes, keeping the two five-branes and all the six-branes parallel among themselves, corresponds to changing the value of the Yukawa coupling between Φ and the quarks (175).

The most general configuration of this sort corresponds to rotating all N_f+2 objects (the N_f $D6$-branes and the two NS5-branes) by arbitrary angles θ_i all of which are different. Since this configuration breaks the $SU(N_f)$ symmetry between the $D6$-branes, to describe it one needs to vary individually the different Yukawa interaction terms of the different flavors. We shall next study a few examples that will hopefully make the general case clear.

Our first example is $N=1$ SQCD. The main goals are to describe the classical and quantum moduli space of vacua and explain Seiberg's $N=1$ duality using branes (Elitzur, Giveon, and Kutasov, 1997; Elitzur *et al.*, 1997). To this end, we explain in the next two sections the brane realization of the classical electric and magnetic SQCD theories. The study of quantum corrections is postponed to the next section.

1. Classical supersymmetric QCD: The electric theory

Consider a configuration of N_c $D4$-branes stretched between an NS5-brane and an NS5′-brane along the x^6 direction. The NS5- and NS5′-branes are separated by a distance L_6 in the x^6 direction, with $x^6(\mathrm{NS5})<x^6(\mathrm{NS5}')$. In addition, there are N_f $D6$-branes to the left of the NS5-brane, each of which is connected to the NS5-brane by a $D4$-brane [see Fig. 24(a)]. The branes involved are extended in the directions given in Eqs. (91) and (173). We call this brane configuration the "electric theory."

An equivalent configuration, which is related to the previous one by a series of Hanany-Witten transitions (see Sec. IV.C.1), consists of N_c $D4$-branes stretched between an NS5-brane and an NS5′-brane along the x^6 direction, with N_f $D6$-branes at values of x^6 that are between those corresponding to the positions of the NS5- and NS5′-branes [Fig. 24(b)].

This brane configuration describes classically $N=1$ SQCD with gauge group $G=U(N_c)$, N_f flavors of chiral superfields in the fundamental and antifundamental representations, and vanishing superpotential. Quantum mechanically, the $U(1)$ factor in $U(N_c)$ will have vanishing gauge coupling and decouple; we shall discuss the quantum case in the next section. The gauge-theory limit corresponds again to $L_6, l_s, g_s\to 0$ with fixed gauge coupling (93)–(95).

It is instructive to relate the supersymmetric deformations of the gauge theory to parameters defining the brane configuration, using the dictionary established in the previous sections:

- *Moduli space of vacua.* The structure of the moduli space of the gauge theory was discussed in Sec. V.A. For $N_f<N_c$, the $U(N_c)$ gauge symmetry can be broken to $U(N_c-N_f)$. The complex dimension of the moduli space of vacua is

$$N_f<N_c:\ \dim\mathcal{M}_H=2N_cN_f-[N_c^2-(N_c-N_f)^2]=N_f^2. \tag{177}$$

For $N_f\geqslant N_c$ the gauge symmetry can be completely broken, and the complex dimension of the moduli space is

$$N_f\geqslant N_c:\ \dim\mathcal{M}_H=2N_cN_f-N_c^2. \tag{178}$$

In the brane description, Higgsing corresponds to splitting four-branes on six-branes. Consider, for example, the case $N_f\geqslant N_c$ (the case $N_f<N_c$ is similar). A generic point in moduli space is described as follows (Fig. 25).

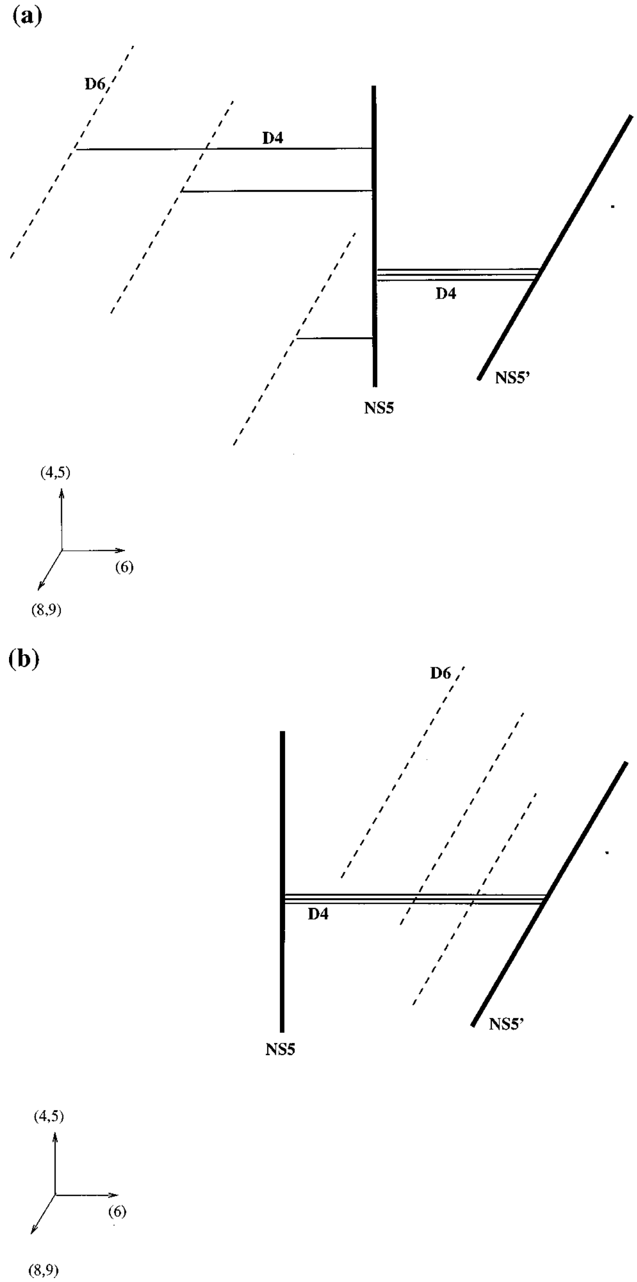

FIG. 24. Two descriptions of $N=1$ supersymmetric QCD with $G=U(N_c)$ and N_f fundamentals ($N_f=N_c=3$), related by a series of Hanany-Witten transitions.

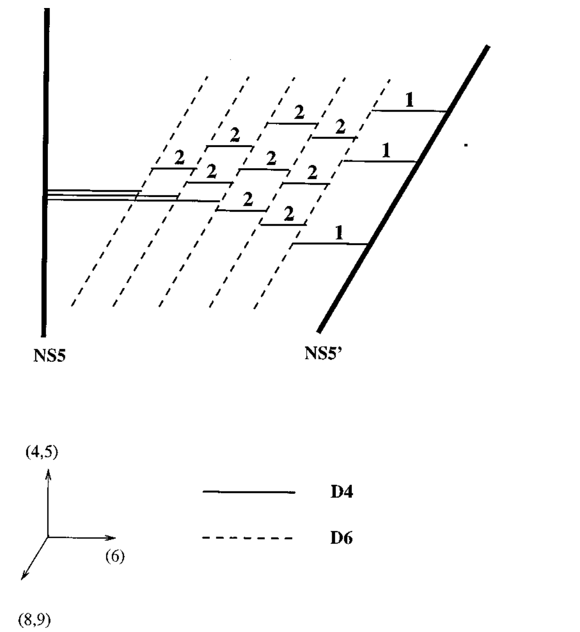

FIG. 25. The fully Higgsed branch of $N=1$ supersymmetric QCD with $G=U(3)$ and $N_f=5$ fundamentals.

The first $D4$-brane is broken into N_f+1 segments connecting the NS5-brane to the first (i.e., leftmost) $D6$-brane, the first $D6$-brane to the second, etc., with the last segment connecting the rightmost $D6$-brane to the NS5′-brane. The second $D4$-brane can now only be broken into N_f segments, because of the s rule (see Sec. IV.C.1): the first segment must stretch between the NS5-brane and the *second* $D6$-brane, with the rest of the breaking pattern as before. We saw in Sec. IV.C.1 that a $D4$-brane stretched between two $D6$-branes has two complex massless degrees of freedom. Similarly, it is geometrically obvious that a $D4$-brane stretched between a $D6$-brane and an NS5′-brane has one complex massless degree of freedom, corresponding to motions in w. Moreover, one does not expect an analog of the s rule (see Sec. IV.C.1) for $D4$-branes stretched between an NS5′-brane and a $D6$-brane, for example, because two such four-branes can be separated in the (x^8,x^9) directions, which are common to both kinds of branes.

Therefore the dimension of moduli space is

$$N_f \geq N_c:\ \dim \mathcal{M}_H = \sum_{l=1}^{N_c} [2(N_f-l)+1] = 2N_fN_c - N_c^2 \tag{179}$$

in agreement with the gauge-theory result (178).

- *Mass deformations.* In gauge theory we can turn on a mass matrix for the (s)quarks, by adding a superpotential

$$W = -m_i^j Q^i \tilde{Q}_j \tag{180}$$

with m an arbitrary $N_f \times N_f$ matrix of complex numbers. In the brane description, masses correspond to relative displacement of the $D6$- and $D4$-branes (or equivalently the $D6$- and NS5′-branes) in the (x^4,x^5) directions. The configuration can be thought of as realizing a superpotential of the form (180), with the mass matrix m satisfying the constraint

$$[m, m^\dagger] = 0. \tag{181}$$

Thus we can diagonalize $m, m^\dagger$ simultaneously; the locations of the $D6$-branes in the v-plane are the eigenvalues of m.

The brane configuration describes only a subset of the possible deformations of the gauge theory. We have already encountered such situations before; they are rather standard in string theory. In this context the constraint (181) can be "explained" by noting that it appears as a consistency condition in $N=2$ supersymmetric gauge theories. Our theory is clearly not $N=2$ supersymmetric; nevertheless, it is not surprising that the condition (181) arises, since one can think of m as the expectation value of a superfield in the adjoint of the $U(N_f)$ gauge group on the $D6$-branes. The theory on the infinite six-branes is invariant under sixteen supercharges in the bulk of the worldvolume, and while it is broken by the presence of the other branes, it inherits Eq. (181) from the theory with more supersymmetry.

• $x^6(D6)$—*A phase transition.* One important difference between the $N=2$ configurations considered in Sec. IV and the present discussion is that it is no longer true that the low-energy physics is completely independent of the positions of the $D6$-branes in x^6. If we move one or more of the $D6$-branes of Fig. 24(a) towards the NS5-brane, as the two branes cross there is no change in the low-energy physics; this is guaranteed by the Hanany-Witten transition. If all the $D6$-branes move to the other side of the NS5-brane we arrive at the configuration of Fig. 24(b), which describes the same low-energy physics as Fig. 24(a) (as for the $N=2$ case).

When the $D6$-branes are displaced towards the NS5′-brane and eventually pass it, the physics changes. No branes can be created in the transition, because the $D6$- and NS5′-branes can avoid each other in space by going around each other in the (x^4,x^5,x^6) directions. Therefore every time a $D6$-brane moves out of the interval between the two NS-branes by passing the NS5′-brane, the theory loses one light flavor of $U(N_c)$.

There is an interesting lesson here. Brane dynamics apparently have the property that when D- and NS-branes that are *not parallel* cross each other, there is no change in the low-energy physics, while crossing of parallel branes leads in general to phase transitions.

• *Fayet-Iliopoulos D term.* In the gauge theory it is possible to turn on a D term for $U(1)\subset U(N_c)$:

$$\mathcal{L}_{FI}=r\int d^4\theta\,\mathrm{Tr}\,V. \tag{182}$$

Note that—unlike the $N=2$ SUSY case (98)—here the D term is a single real number r. For $0<N_f<N_c$ adding Eq. (182) breaks supersymmetry. For $N_f\geqslant N_c$ there are supersymmetric vacua in which the gauge symmetry is completely broken and the system is forced into a Higgs phase. In the brane description, the role of the Fayet-Iliopoulos D term is played by the relative displacement of the NS5- and NS5′-branes in the x^7 direction (Fig. 26). Clearly, when the two are at different values of x^7, a four-brane stretched between them breaks supersymmetry. To preserve supersymmetry, all such four-branes must break on $D6$-branes, which as we saw above is only possible for $N_f\geqslant N_c$ because of the s rule. Once all four-branes have been split, there is no obstruction to moving the NS5- and NS5′-branes to different locations in x^7. At generic points in the Higgs phase, nothing special happens when the D term is turned off. In the brane construction the reason is that once all N_c $D4$-branes have been broken on $D6$-branes in a generic way, nothing special happens when the relative displacement of the two five-branes in x^7 vanishes.

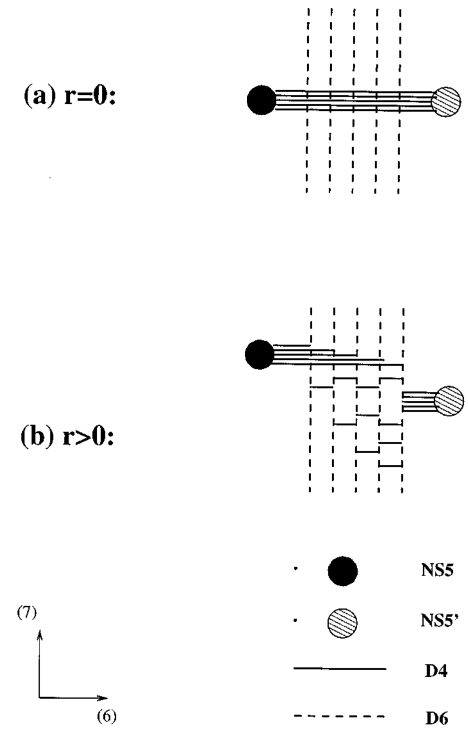

FIG. 26. Displacement of the five-branes in x^7 corresponding to a Fayet-Iliopoulos D term in the worldvolume gauge theory.

• *Global symmetries.* Classical supersymmetric QCD with gauge group $SU(N_c)$ and N_f quarks has the global symmetry (145); quantum effects break it to Eq. (146). The anomaly is a quantum effect that is not expected to be visible in the classical brane construction (we shall exhibit it in the brane description in Sec. V.C).

In our case, the (classical) gauge symmetry is $U(N_c)\simeq SU(N_c)\times U(1)$, with the extra $U(1)$ factor in the gauge group corresponding to gauging baryon number $U(1)_B$. The brane configuration has a manifest (vector) $SU(N_f)$ symmetry, which is a gauge symmetry on the $D6$-branes and a global symmetry on the $D4$-branes. The other (axial) $SU(N_f)$ symmetry is generically not an exact symmetry of the brane configuration of Fig. 24(b) and arises as an effective symmetry when we take the infrared limit. In the general spirit of brane theory—trying to realize as much as possible of the symmetry structure of the low-energy theory throughout the RG flow—one might wonder whether it is possible to realize it too as an exact symmetry of the brane vacuum.

This is indeed the case, as shown by Aharony and Hanany (1997), Brodie and Hanany (1997), and Hanany and Zaffaroni (1998a). The main idea is the following. We saw before that the positions of the $D6$-branes in x^6 are not visible in the low-energy theory, but of course their values influence the high-energy structure. One may thus hope that the full chiral symmetry may be restored for some particular value of these parameters. When the $D6$-branes are placed at the same value of x^6 as the NS5′-brane (Fig. 27) the full chiral symmetry is restored (Brodie and Hanany, 1997).

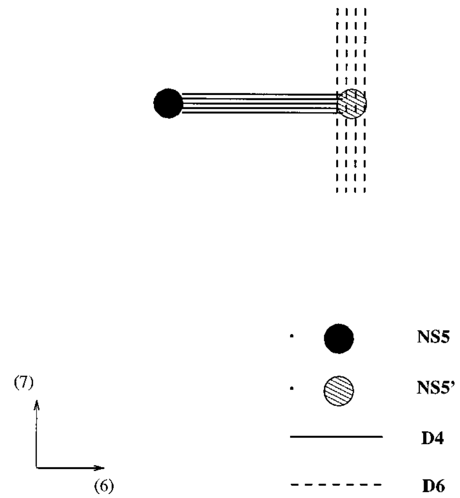

FIG. 27. Enhancement of the global symmetry at finite energies from $SU(N_f)$ to $SU(N_f)\times SU(N_f)$, as a result of placing the $D6$-branes on the NS5′-brane.

To see that this is geometrically plausible, note that when that happens, the NS5′-brane located at (say) $x^7=0$ cuts each $D6$-brane into two disconnected halves, the $x^7>0$ and $x^7<0$ parts.[29] The situation is very similar to that encountered in Sec. IV.C.4 when we discussed compact Coulomb branches. Using our analysis there, it is clear that there are now two separate $SU(N_f)$ symmetries acting on the two disconnected groups of N_f six-branes. Just as in Sec. IV.C.4, despite the fact that the two groups of six-branes are independent, we cannot remove one of them from the configuration. From the brane theory point of view this is due to the fact that, as discussed in Sec. IV.C.4, this would lead to nonconservation of charge. From the point of view of the gauge theory on the four-branes the reason is that the resulting four-dimensional gauge theory, with only fundamentals and no antifundamentals, would be anomalous.

The symmetries $U(1)_x$, $U(1)_a$ [Eq. (145)] are also realized in the brane picture. They correspond to rotations in the (x^4,x^5) and (x^8,x^9) planes, $U(1)_{45}$, $U(1)_{89}$. These rotations are R symmetries because the four preserved supercharges of the brane configuration of Fig. 24 are spinors of the Spin(9, 1) Lorentz group in ten dimensions and therefore are charged under both $U(1)_{45}$ and $U(1)_{89}$. From the discussion of the mass deformations and Higgs moduli space above, it is clear that the mass parameters (180) are charged under $U(1)_{45}$, while the quarks Q, $\tilde{Q}$ are charged under $U(1)_{89}$. If we assign $U(1)_{45}\times U(1)_{89}$ charges (1, 1) to the superspace coordinates θ_α, the quarks Q and $\tilde{Q}$ have charges (0, 1), while the mass parameters m in (180) have charges (2, 0).

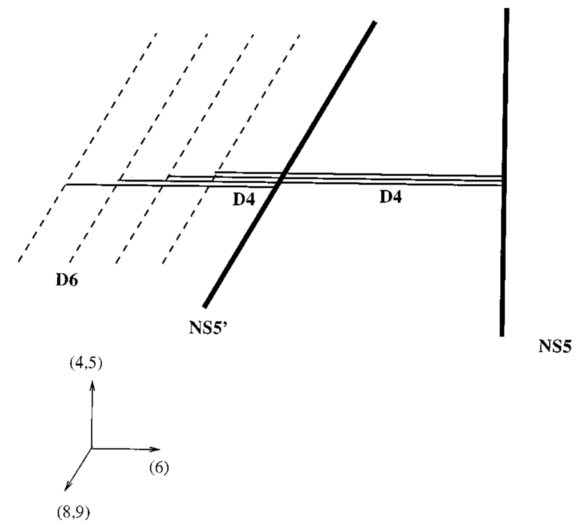

FIG. 28. The magnetic brane configuration.

With these assignments, the mass term (180) is invariant under both global symmetries.[30]

2. Classical supersymmetric QCD: The magnetic theory

The "magnetic" brane configuration—the reason for the name will become clear soon—contains N_c $D4$-branes connecting the NS5′-brane to an NS5-brane on its right (we shall refer to these as "color four-branes") and N_f $D4$-branes connecting the NS5′-brane to N_f $D6$-branes on its left, which we shall refer to as "flavor four-branes." The configuration is depicted in Fig. 28. As usual, all the branes involved are stretched in the directions given in Eqs. (91) and (173). We shall consider the case $N_f\geqslant N_c$ in what follows.

This configuration describes SQCD with "magnetic gauge group" $G_m=U(N_c)$ (with the gauge bosons coming as before from 4–4 strings connecting different color four-branes), N_f flavors of "magnetic quarks" q_i, $\tilde{q}^i$ (4–4 strings connecting the N_c color four-branes with the N_f flavor four-branes). In addition to the $N=1$ SQCD matter content there are now N_f^2 chiral superfields that are singlets under the gauge group G_m, arising from 4–4 strings connecting different-flavor four-branes. Denoting these "magnetic meson" fields by M_j^i $(i,j=1,\ldots,N_f)$, we see that the standard coupling of three open strings gives rise to a superpotential connecting the magnetic mesons and the magnetic quarks,

$$W_{\rm mag}=M_j^i q_i \tilde{q}^j. \tag{183}$$

This is precisely the "magnetic theory" discussed in Sec. V.A.

The analysis of moduli space and deformations of this model are similar to the electric theory, with a few differences due to the existence of the superpotential (183).

[29]Note that this does not happen when the $D6$-branes intersect an NS5-brane. This is consistent with the fact that in the $N=2$ SUSY configurations we do not expect a chiral enhancement of the global symmetry.

[30]The discussion of global charges is somewhat oversimplified. A more precise description of the transformation properties of gauge-invariant observables requires a detailed mapping of the brane and gauge-theory degrees of freedom.

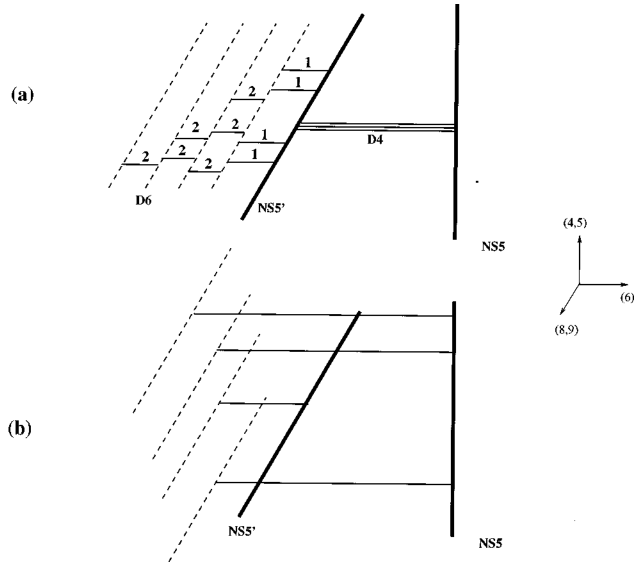

FIG. 29. Magnetic moduli space: (a) The N_f^2-dimensional classical magnetic moduli space corresponding to unbroken gauge symmetry and arbitrary expectation values for the singlet meson M. (b) The brane description of adding a linear superpotential $W=-mM$. The eigenvalues of m correspond to locations of $D6$-branes in the v plane.

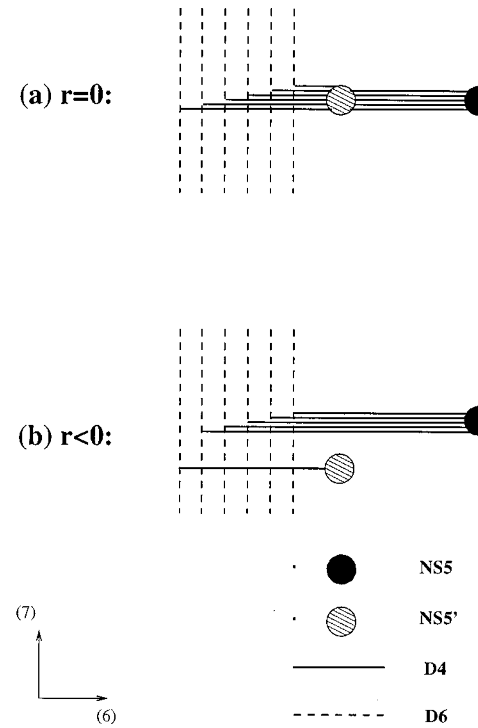

FIG. 30. The brane description of the Fayet-Iliopoulos D term in the magnetic theory: (a) $r=0$; (b) $r<0$.

Consider first mass deformations. In gauge theory we can add a mass term to the magnetic quarks by modifying the superpotential to

$$W_{\rm mag}=M_j^i q_i \tilde{q}^j+\delta M_j^i q_i \tilde{q}^j. \tag{184}$$

The mass parameters δM can be absorbed in the expectation value of the magnetic meson M_j^i and can be thought of as parametrizing a moduli space of vacua. The N_f^2 resulting parameters are described in the brane language by splitting the N_f flavor four-branes on the $D6$-branes in the most general way consistent with the geometry [Fig. 29(a)]. This results in a total of N_f^2 massless modes corresponding to the N_f^2 components of M: N_f of them describe fluctuations in the (x^8,x^9) plane of four-branes stretched between the NS5$'$-brane and the rightmost $D6$-brane, and the remaining $\Sigma_{l=1}^{N_f-1}2l=N_f(N_f-1)$ parametrize fluctuations in (x^6,x^7,x^8,x^9) of the four-branes connecting different six-branes.

Another interesting deformation of the magnetic gauge theory corresponds to adding a linear term in M to the magnetic superpotential:

$$W_{\rm mag}=M_j^i(q_i\tilde{q}^j-m_i^j). \tag{185}$$

Integrating out the massive field M we find that in the presence of the "mass parameters" m_i^j the gauge group is broken; thus the parameters m play the role of Higgs expectation values. In the brane description, these deformations correspond to a process in which color four-branes are aligned with flavor four-branes and reconnected to stretch between the NS5-brane and a $D6$-brane [see Fig. 29(b)]. If m has rank $n(\leq N_c)$, n such four-branes are reconnected. The $D6$-branes on which the reconnected four-branes end can then be moved in the (x^4,x^5) directions, taking the four-branes with them and breaking the $U(N_c)$ gauge group to $U(N_c-n)$. The brane description realizes only a subset of the possible "mass matrices" m, namely, those which satisfy Eq. (181) (the reason is similar to the one described there). We shall soon see that this analogy is not coincidental.

Another deformation of the magnetic gauge theory and of the corresponding brane configuration, which will play a role in the sequel, is switching on a Fayet-Iliopoulos D term for the $U(1)$ subgroup of $U(N_c)$. Again, in the brane construction this corresponds to a relative displacement of the NS5- and NS5$'$-branes in the x^7 direction (Fig. 30). To preserve supersymmetry, all N_c color four-branes have to be reconnected to N_c of the N_f flavor four-branes, leading to a situation in which N_c four-branes stretch between the NS5-brane and N_c different six-branes and N_f-N_c four-branes stretch between the NS5$'$-brane and the remaining six-branes [Fig. 29(b)]. Once this occurs, the two five-branes can be separated in x^7.

Unlike the electric theory, here there is a jump in the dimension of the classical moduli space of the theory as we vary the D term. For nonvanishing D term there are only N_f-N_c four-branes that give rise to moduli (the other N_c are frozen because of the s rule), and the moduli space is easily checked to be $N_f^2-N_c^2$ dimensional. When the D term vanishes, the previously frozen four-branes can be reconnected to yield the original configuration, with unbroken $U(N_c)$, and we gain access to the full (N_f^2)-dimensional moduli space of Fig. 29(a). We shall see in Sec. V.C that quantum mechanically this classical jump in the structure of the moduli space disappears.

The magnetic brane configuration is invariant under the same global symmetries as the electric theory described above by Eq. (145). The charge assignments under the $U(1)_{45}\times U(1)_{89}$ symmetry are as follows: the magnetic quarks q, $\tilde{q}$ have charges (1, 0), the mass parameters m have charges (2, 0), the magnetic meson M has charges (0, 2), and the superspace coordinates θ_α have charges (1, 1).

3. Seiberg's duality in the classical brane picture

We have now constructed using branes two $N=1$ supersymmetric gauge theories, the electric and magnetic theories discussed in the previous two sections. Seiberg has shown that the electric gauge theory with gauge group $U(N_c)$ and the magnetic theory with gauge group $U(N_f-N_c)$ are equivalent in the extreme infrared[31] (i.e., they flow to the same infrared fixed point) (Seiberg, 1995a). Seiberg's duality is a quantum symmetry, but it has classical consequences in situations where the gauge symmetry is completely broken and there is no strong infrared dynamics. In such situations Seiberg's duality reduces to a classical equivalence of moduli spaces and their deformations.

In this section we show using brane theory that the moduli spaces of vacua of the electric and magnetic theories with gauge groups $U(N_c)$ and $U(N_f-N_c)$ coincide. They provide different parametrizations of the moduli space of vacua of the appropriate brane configuration. This explains the classical part of Seiberg's duality. As one approaches the root of the Higgs branch, nontrivial quantum dynamics appears, and we have to face the resulting strong-coupling problem. This will be addressed in Sec. V.C.

Start, for example, with the electric theory with gauge group $U(N_c)$ [the configuration of Fig. 24(a)]. Now enter the Higgs phase by connecting the N_c original four-branes stretched between the NS5- and NS5′-branes to N_c of the N_f four-branes stretched between the NS5-brane and the six-branes; we then further reconnect the resulting four-branes in the most general way consistent with the rules described in Secs. IV.C.1 and V.B.1. The resulting moduli space is $2N_fN_c-N_c^2$ dimensional, as described in Sec. V.B.1. Note that, generically, there are now N_f-N_c $D4$-branes attached to the NS5-branes, and N_c $D4$-branes connected to the NS5′-brane (the other ends of all these four-branes lie on different $D6$-branes).

Once we are in the Higgs phase, we can freely move the NS5-brane relative to the NS5′-brane and, in particular, the two branes can pass each other in the x^6 direction without ever meeting in space. This can be achieved by taking the NS5-brane around the NS5′-brane in the x^7 direction, i.e., turning on a Fayet-Iliopoulos D term in the worldvolume gauge theory (the process is described in Fig. 31). At a generic point in the Higgs branch of the electric theory, turning on such a D term is a completely smooth procedure; this is particularly clear from the brane description, where in the absence of $D4$-branes connecting the NS5-brane to the NS5′-brane, the relative displacement of the two in the

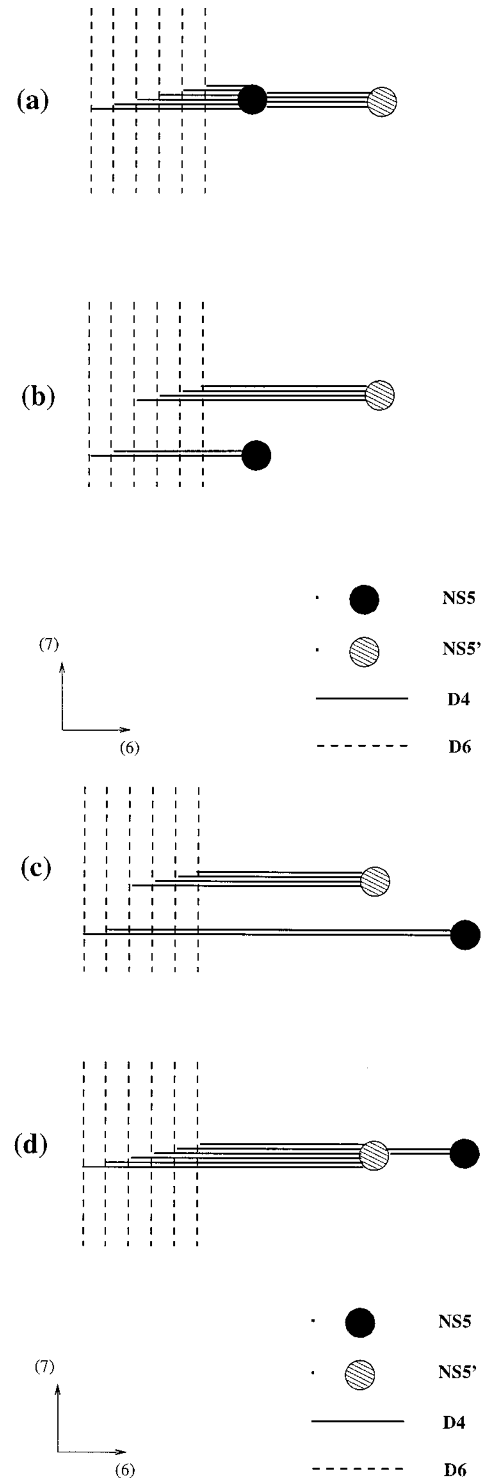

FIG. 31. Continuous connection of electric and magnetic brane configurations. Starting with the electric configuration (a), one can turn on a Fayet-Iliopoulos D term (b), exchange the five-branes in x^6 (c), and switch off the D term, arriving at the magnetic configuration (d).

[31]Seiberg actually considered the $SU(N_c)$ and $SU(N_f-N_c)$ theories (see Sec. V.A), but the statement for $U(N_c)$ and $U(N_f-N_c)$ follows from his results by gauging baryon number.

x^7 direction can be varied freely.

After exchanging the NS5- and NS5$'$-branes, we can interpret the brane configuration we find as describing the Higgs phase of *another* gauge theory. To find out what that theory is, we approach the root of the Higgs branch by aligning the N_f-N_c $D4$-branes emanating from the NS5-brane with the NS5$'$-brane, and the N_c $D4$-branes emanating from the NS5$'$-brane with $D4$-branes stretched between $D6$-branes.

We then reconnect the $D4$-branes to obtain a configuration consisting of N_f-N_c $D4$-branes connecting the NS5$'$-brane to an NS5-brane that is to the right of it; the NS5$'$-brane is further connected by N_f $D4$-branes to the N_f $D6$-branes that are to the left of it [see Fig. 31(d)]. This is the magnetic SQCD of Sec. V.B.2, with gauge group $U(N_f-N_c)$.

To summarize, we have shown that the moduli space of vacua of the electric SQCD theory with (completely broken) gauge group $U(N_c)$ and N_f flavors of quarks, and the moduli space of vacua of the magnetic SQCD model with (broken) gauge group $U(N_f-N_c)$, can be thought of as providing different descriptions of a single moduli space of supersymmetric brane configurations. One can smoothly interpolate between them by varying the scale Λ (related to the displacement of the NS5- and NS5$'$-branes in x^6), keeping the Fayet-Iliopoulos D term fixed but nonzero. Since the only role of Λ in the low-energy theory is to normalize the operators (Kutasov, Schwimmer, and Seiberg, 1996), theories with different values of Λ are equivalent. The electric and magnetic theories will thus share all features, such as the structure of the chiral ring (which can be thought of as the ring of functions on moduli space), that are independent of the interpolation parameter Λ.

The above smooth interpolation relies on the fact that the gauge symmetry is completely broken due to the presence of the Fayet-Iliopoulos D term. As mentioned above, it is not surprising that duality appears classically in this situation since there is no strong infrared gauge dynamics.

The next step is to analyze what happens as the gauge symmetry is restored when the D term goes to zero and we approach the origin of moduli space. Classically, we find a disagreement. In the electric theory, we saw in Sec. V.B.1 that nothing special happens when the gauge symmetry is restored. New massless degrees of freedom appear, but there are no new branches of the moduli space that we gain access to.

In the magnetic theory the situation is different. When we set the Fayet-Iliopoulos D term to zero, we saw in Sec. V.B.2 that a large moduli space of previously inaccessible vacua became available. While the electric theory has a $(2N_fN_c-N_c^2)$-dimensional smooth moduli space, the classical magnetic theory experiences a jump in the dimension of its moduli space from $2N_fN_c-N_c^2$ for nonvanishing Fayet-Iliopoulos D term to N_f^2 when the D term is zero. However, in the magnetic theory when the D term vanishes the $U(N_f-N_c)$ gauge symmetry is restored, and to understand what really happens we must study the quantum dynamics. We shall discuss this in Sec. V.C, where we shall see that quantum mechanically the jump in the magnetic moduli space disappears, and the quantum moduli spaces of the electric and magnetic theories agree.

It is instructive to map the deformations of the classical electric theory onto those of the classical magnetic one. Turning on masses (180) in the electric theory corresponds to moving the $D6$-branes away from the $D4$-branes (or equivalently from the NS5$'$-brane) in the (x^4,x^5) directions. As discussed in Sec. V.B.2, in the magnetic description, the electric mass parameters correspond to Higgs expectation values (185).

Turning on expectation values to the electric quarks, which was described in the brane language in Sec. V.B.1, corresponds on the magnetic side to varying the expectation value of the magnetic meson M (184). This gives masses to the magnetic quarks.

The transmutation of masses into Higgs expectation values and vice versa observed in the brane construction is one of the hallmarks of Seiberg's duality.

4. Other rotated $N=2$ configurations

The brane configurations corresponding to electric and magnetic SQCD were obtained above by rotating branes in the $N=2$ SUSY configuration studied in Sec. IV. Before moving on to the study of quantum dynamics of these theories we should like to discuss a few additional theories that can be realized using such rotations.

a. *$U(N_c)$ with adjoint, N_f flavors, and $W=0$*

Starting with the $N=2$ SUSY brane configuration of Fig. 14, rotate the two NS5-branes as in Eqs. (170) and (172), keeping them parallel to each other. As discussed above [Eq. (175)], the angle of rotation determines the Yukawa coupling; in particular, when the two NS5-branes are rotated into NS5$'$-branes [Fig. 32(a)] the Yukawa coupling disappears. The resulting theory has gauge group $U(N_c)$, the matter content necessary for $N=2$ supersymmetry, i.e., an adjoint chiral multiplet Φ and N_f fundamentals Q^i, $\tilde{Q}_i$, but the superpotential $W=\tilde{Q}\Phi Q$ required by $N=2$ supersymmetry in $4d$ is absent here; instead, $W=0$. This is a model discussed in Sec. V.A.3.

Fluctuations of the N_c four-branes along the NS5$'$-branes parametrize the Coulomb branch of the model. Displacements of the N_f $D6$-branes relative to the NS5$'$-branes in the (x^4,x^5) directions give masses m to the fundamental multiplets Q, $\tilde{Q}$:

$$W=-\sum_{i=1}^{N_f} m_i\tilde{Q}_iQ^i. \tag{186}$$

The relative position of the two NS5$'$-branes in the (x^4,x^5) directions is not an independent parameter; it can be compensated for by a change in the positions of the $D6$-branes in the (x^4,x^5) plane (i.e., the masses of the fundamentals) and an overall rotation of the configuration. The relative displacement of the two NS5$'$-branes in the x^7 direction plays the role of a Fayet-

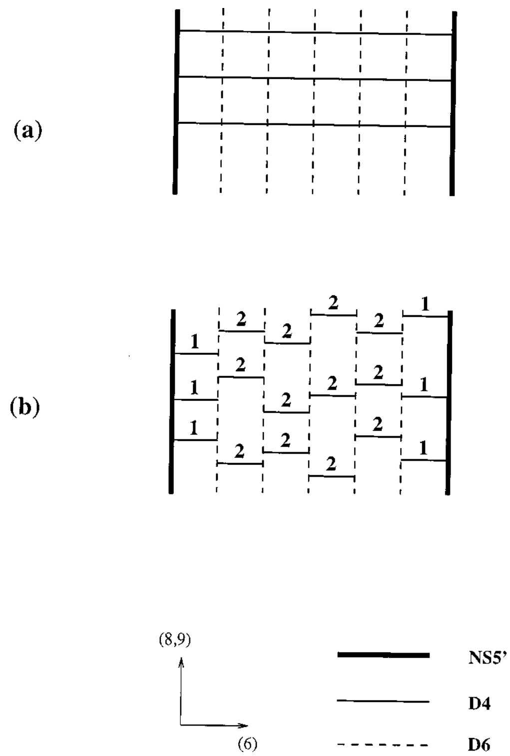

FIG. 32. $U(N_c)$ with adjoint: (a) $N=1$ supersymmetric Yang-Mills theory with $G=U(N_c)$, N_f fundamentals, and an adjoint superfield with vanishing superpotential; (b) the fully Higgsed branch of moduli space.

Iliopoulos D term (182). Complete Higgsing is possible for all $N_f \geqslant 1$; the (complex) dimension of the Higgs branch is

$$\dim \mathcal{M}_H = 2N_f N_c + N_c^2 - N_c^2 = 2N_f N_c . \tag{187}$$

The first two terms on the left-hand side are the numbers of components in the fundamental and adjoint chiral multiplets, and the negative term accounts for degrees of freedom eaten up by the Higgs mechanism.

The brane configuration provides a simple picture of the moduli space of vacua. As usual, complete Higgsing corresponds to breaking all N_c four-branes on various $D6$-branes, as indicated in Fig. 32(b). We find that the dimension of moduli space of brane configurations with completely broken $U(N_c)$ gauge symmetry is

$$\dim \mathcal{M}_H = N_c[2(N_f-1)+1+1] = 2N_c N_f \tag{188}$$

in agreement with the gauge-theory analysis (187).

b. Mixed electric-magnetic theories

A straightforward generalization of the electric and magnetic SQCD brane configurations is a configuration that includes both "electric" and "magnetic" quarks. Consider the configuration of Fig. 33; an NS5-brane connected by N_c $D4$-branes to an NS5$'$-brane that is to its right (in x^6). To the left of the NS5-brane we put N_f $D6$-branes, each of which is connected by a four-brane to the NS5-brane. As before, these represent N_f quarks $Q,\tilde{Q}$. To the right of the NS5$'$-brane we put N_f' $D6$-branes, each of which is connected to the NS5$'$-brane by a four-brane. These represent N_f' quarks $Q',\tilde{Q}'$ and $N_f'^2$ complex scalars M' with a tree-level superpotential $W=M'Q'\tilde{Q}'$. The supersymmetric Yang-Mills theory thus obtained is a "mixed electric-magnetic" $SU(N_c)$ gauge theory with N_f "electric" quarks and N_f' "magnetic" quarks coupled to "magnetic mesons."

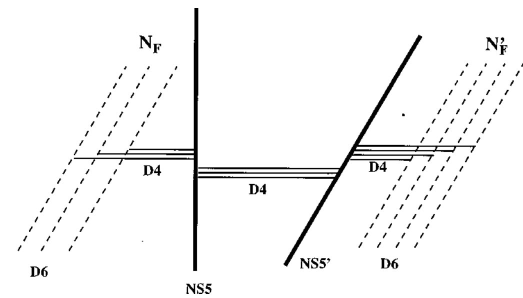

FIG. 33. A "mixed electric-magnetic" theory, in which some of the fundamentals couple to singlet mesons and some do not.

The discussion of Seiberg's duality can be repeated for such theories. Interchanging the two NS-branes in x^6 gives rise to an $SU(N_f+N_f'-N_c)$ theory with N_f magnetic quarks $q,\tilde{q}$ coupled to N_f^2 complex scalars M via $Mq\tilde{q}$, and N_f' electric quarks $q',\tilde{q}'$. Of course, the dual theory is also a mixed electric-magnetic theory in which the roles of electric and magnetic quarks are interchanged. In the particular case $N_f'=0$ the original theory is the electric theory studied in Sec. V.B.1, while its dual is the magnetic theory as considered in Sec. V.B.3.

c. $D6'$-branes

Another interesting deformation of the $N=2$ SUSY configuration involves rotating some of the $D6$-branes as well. Restricting our attention to ninety-degree rotations, for simplicity, we should like to consider, in addition to the objects studied above, rotated six-branes that are located at $w=0$ and stretched in v. We shall refer to these as $D6'$-branes:

$$D6': \quad (x^0,x^1,x^2,x^3,x^4,x^5,x^7). \tag{189}$$

To study brane configurations including both $D6$- and $D6'$-branes one has to keep in mind the following interesting feature of brane dynamics.

Consider a configuration in which a pair of $D4$-branes connect a $D6$-brane to a $D6'$-brane. Naively the configuration preserves four supercharges and there are two complex moduli describing the locations of the two $D4$-branes along the six-branes (in x^7) together with the compact component of the gauge field A_6.

However, there is a superpotential due to Euclidean fundamental strings stretched between the $D4$- and $D6$-branes, as indicated in Fig. 34. If the distance between the six-branes is δl_6 and the separation between the four-branes is δl_7, the superpotential due to these Euclidean strings is of order $\exp(-\delta l_6 \delta l_7 / l_s^2)$. This effect is nonperturbative in l_s but does not go to zero in the

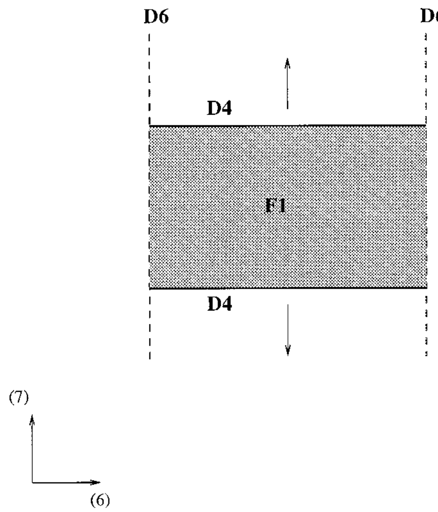

FIG. 34. $D4$-branes stretched between $D6$- and $D6'$-branes: The repulsive interaction between them is due to Euclidean fundamental strings stretched between the various branes.

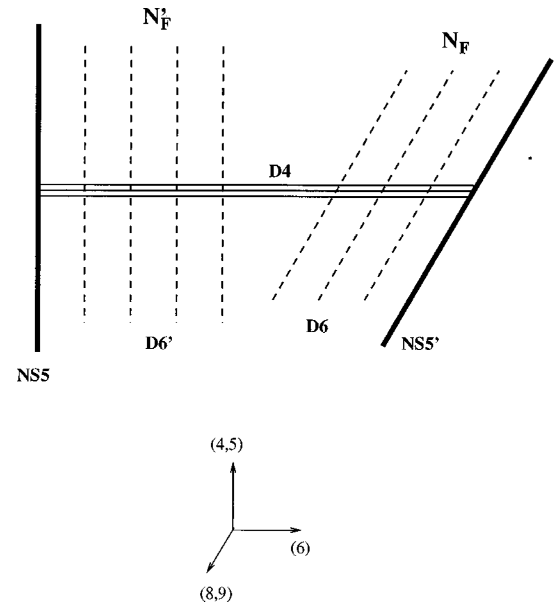

FIG. 35. $N=1$ supersymmetric QCD with vanishing superpotential described by a configuration with both $D6$- and $D6'$-branes.

limit $g_s \to 0$—it is a worldsheet instanton effect. In particular, it does not vanish in the classical gauge-theory limit discussed above and leads to long-range repulsive interactions between the four-branes. It is closely related to the nonperturbative effects discussed in Sec. III in systems with twice as much supersymmetry, where they contribute to the metric on moduli space [see Eq. (68)].

We thus arrive at the following classical rule of brane dynamics: *There is a long-range repulsive interaction between $D4$-branes stretched between a $D6$-brane and a $D6'$-brane. This repulsion does not go to zero in the classical limit $g_s \to 0$.*

Taking this rule into account allows one to understand configurations including both $D6$- and $D6'$-branes. The resulting physics depends on the ordering of the six-branes along the x^6 axis. When a $D6$-brane passes a $D6'$-brane there is a phase transition; this can be seen by U duality, which can be used to map this system to an NS5′-brane and a $D6$-brane; as we saw before, the physics certainly changes when we exchange those. We shall next consider the physics for a particular ordering of the branes; the generalization to other cases is straightforward.

Consider the configuration of Fig. 35. In addition to the usual N_c $D4$-branes stretched between NS5 and NS5′-branes, which give rise to a $U(N_c)$ gauge group, we have N_f $D6$-branes located next to the NS5′-brane and N_f' $D6'$-branes located next to the NS5-brane, which give rise to N_f+N_f' flavors. Clearly the theory does not have a massless adjoint field as there is no Coulomb branch, and by placing the $D6$-branes on top of the NS5′-brane and the $D6'$-branes on top of the NS5-brane we deduce that the symmetry of the theory is at least $SU(N_f)\times SU(N_f)\times SU(N_f')\times SU(N_f')$, which does not allow a superpotential.

The theory is therefore $N=1$ SQCD with gauge group $U(N_c)$, N_f+N_f' flavors of quarks, and $W=0$, which we have analyzed before. The analysis of the moduli space gives the right structure; we leave the details to the reader. To get the correct structure it is important to use the rule stated above, which implies here that configurations are unstable in which multiple $D4$-branes connect a given $D6$- and $D6'$-brane.

C. Quantum effects

In this section we study quantum effects in $N=1$ SYM theory using brane theory. We describe the quantum moduli space of vacua and complete the demonstration of Seiberg's duality. We start by following a similar route to that taken in Sec. IV and studying the form of the M-theory five-brane describing the brane configuration at finite R_{10}/L_6, first semiclassically and then exactly. Then we present a qualitative picture of the moduli space as resulting from certain quantum interactions between branes analogous to the classical interactions encountered in the last section.

1. Semiclassical description

At finite gauge coupling g [Eq. (93)] we should interpret our brane configurations as describing five-branes and six-branes in M theory with finite $R_{10}/L_6 = l_s g_s / L_6$. Recall the definitions

$$s = x^6 + ix^{10},$$

$$v = x^4 + ix^5,$$

$$w = x^8 + ix^9. \tag{190}$$

Classically, i.e., ignoring the size of the x^{10} circle, the $D4$-brane is located at $v=w=0$ and is extended in s, the NS5-brane is at $s=w=0$ and is extended in v, while the NS5′-brane is at $v=0$, $s=L_6$ and is extended in w.

Quantum mechanically, the four-branes and five-branes merge into a single five-brane in M theory, as described in the $N=2$ case in Sec. IV. The vacuum configuration of the five-brane is described by a curve Σ embedded in the space $R^5\times S^1$ (190). As before, for large v and w we can think of the shape of the resulting $M5$-brane in terms of the original NS5- and NS5′-branes, appropriately deformed by the four-branes ending on them [Eq. (103)]. The structure for large v and w is the brane analog of one-loop corrections to classical physics in gauge theory. In this section we shall describe these effects (Elitzur, Giveon, *et al.*, 1997).

Consider the classical electric configuration of Fig. 24(a). According to Eq. (103), far from the origin this configuration is deformed as follows. For large v (and small w), the shape of the $M5$-brane is that of the deformed NS5-brane,

$$s_5=(N_f-N_c)R_{10}\ln v, \tag{191}$$

while for large w (and small v) it looks like the deformed NS5′-brane,

$$s_{5'}=N_cR_{10}\ln w. \tag{192}$$

The two asymptotic regions join in a way that will be discussed later at small v and w.

As explained in Sec. IV.C.4, this bending causes, among other things, the freezing of the $U(1)\subset U(N_c)$. Therefore quantum mechanically we are dealing with an $SU(N_c)$ gauge theory.

In Sec. V.A we saw that the classical electric SQCD is invariant under two $U(1)$ R symmetries (145). In gauge theory only one combination of the two is preserved quantum mechanically; the other is broken by the chiral anomaly or, equivalently, instantons (see Sec. V.A). We shall next examine this effect in brane theory.

The two classical R symmetries correspond in the brane construction to rotations of the (x^4,x^5) plane, $v\to e^{i\alpha}v$, and the (x^8,x^9) plane, $w\to e^{i\beta}w$. The semiclassical configuration, Eqs. (191) and (192), breaks both symmetries. $\mathrm{Re}\, s_5$ and $\mathrm{Re}\, s_{5'}$ are invariant under $U(1)_{45}$ and $U(1)_{89}$, but $\mathrm{Im}\, s_5$ and $\mathrm{Im}\, s_{5'}$ are not invariant. Overall shifts of $\mathrm{Im}\, s$ can be compensated for by a translation in x^{10}. Hence any combination of $U(1)_{45}$ and $U(1)_{89}$ which preserves the relative location of the NS5- and NS5′-branes in x^{10} is a symmetry. The (semiclassically) unbroken R symmetry is, therefore, the one which preserves

$$s_5-s_{5'}=R_{10}\log(w^{-N_c}v^{N_f-N_c}). \tag{193}$$

It is not difficult to check that if (by definition) the R charge of θ under this symmetry is one, that of Q, $\tilde{Q}$ is $R(Q)=R(\tilde{Q})=1-N_c/N_f$, in agreement with the gauge theory answer (147). Of course, so far all we have checked is that this symmetry is conserved semiclassically. In field theory there is no contribution to the anomaly beyond one loop; brane dynamics reflects this, and one can check that the exact form of the five-brane preserves the symmetry as well.

One can tell the same story for the brane construction describing magnetic SQCD. The classical R symmetry corresponds again to $U(1)_{45}\times U(1)_{89}$. The charge assignments of the various fields (q, $\tilde{q}$, M) agree with those found in gauge theory (see Sec. V.A) and with the electric configuration. Quantum mechanically the five-branes are deformed due to the presence of the four-branes; Eqs. (191) and (192), which were found for the electric configuration, are valid for the magnetic one as well. In brane theory this is a consequence of the fact that the two configurations are related by the smooth transition discussed earlier. In gauge theory it is one of the checks of Seiberg's duality.

The foregoing discussion may be used to provide a heuristic explanation of a certain scale-matching relation between the electric and magnetic theories used in gauge-theory studies of Seiberg's duality. We can think of the electric coupling $(s_5-s_{5'})/R_{10}$ [Eq. (193)] as describing the electric QCD scale Λ_e:

$$\Lambda_e^{3N_c-N_f}=\mu^{3N_c-N_f}e^{-(s_5-s_{5'})/R_{10}}$$
$$=\mu^{3N_c-N_f}w^{N_c}v^{N_c-N_f}, \tag{194}$$

where μ is some fixed scale. If we start with a large and negative $\mathrm{Re}(s_5-s_5')$ the QCD scale Λ_e is large. Exchanging the branes as discussed above leads to a theory with $\mathrm{Re}(s_5-s_5')>0$ and, therefore, small Λ_e. In this situation we can continue thinking about the theory as the electric theory with a small Λ_e; alternatively, we can switch to the magnetic point of view and define the magnetic QCD scale Λ_m:

$$\Lambda_m^{3\bar{N}_c-N_f}=\mu^{3\bar{N}_c-N_f}e^{+(s_5-s_{5'})/R_{10}}, \tag{195}$$

where $\bar{N}_c\equiv N_f-N_c$. Equations (194) and (195) lead to the scale-matching relation

$$\Lambda_e^{3N_c-N_f}\Lambda_m^{3\bar{N}_c-N_f}=\mu^{N_f}, \tag{196}$$

which has been argued to hold in gauge theory, with μ a constant related to the coefficient of the magnetic superpotential (161) (Kutasov, Schwimmer, and Seiberg, 1996). Equation (196) emphasizes the strong-weak coupling aspect of Seiberg's duality, since if Λ_e becomes small (thus making the electric theory weakly coupled) Λ_m is large, and vice versa.

2. Exact results

So far we have focused on the large v and w forms of the $M5$-brane into which the type-IIA five-branes and four-branes merge for finite R_{10}/L_6. Following the logic of Sec. IV we next derive its exact form.

We start with the case of pure supersymmetric Yang-Mills theory with $G=SU(N_c)$ and no matter, described by the brane configuration of Fig. 24 (without six-branes). We can proceed as in the $N=2$ case studied in Sec. IV. The worldvolume of the $M5$-brane is $R^{3,1}\times\Sigma$, where the complex curve Σ is now embedded in the three-complex-dimensional space $Q\simeq R^5\times S^1$ parametrized by (v,w,s). The shape of the curve Σ can be determined by using the symmetries and singularity structure (Hori, Ooguri, and Oz, 1997; Witten, 1997b).

Defining the variable t as in Eq. (104), we know that as $v\to\infty$ on Σ (the region corresponding to the NS5-brane), t diverges [Eq. (191)] as $t\simeq v^{N_c}$, while w goes to zero. Similarly, as $w\to\infty$ [the NS5′-brane (192)], $t \simeq w^{-N_c}$ while $v\to 0$. More generally, t should be a function of v that does not have poles or zeros except at $v=0$ (which is $w=\infty$) and $v=\infty$. The unique solution to all the constraints, up to an undetermined constant ζ, is

$$v^{N_c}=t,$$
$$w^{N_c}=\zeta^{N_c}t^{-1},$$
$$vw=\zeta. \tag{197}$$

One way of arriving at the curve (197), which also helps us to understand the role of the parameter ζ, is to start with the $N=2$ SUSY configuration described in Sec. IV and rotate one of the NS5-branes as described in Sec. V.B (Hori, Ooguri, and Oz, 1997; Witten, 1997b).

The $N=2$, $SU(N_c)$ brane configuration is given by the curve (106)–(108)

$$t^2+B(v,u_k)t+\Lambda_{N=2}^{2N_c}=0, \tag{198}$$

where we have restored the dependence on the QCD scale $\Lambda_{N=2}$. We should like to find the curve corresponding to a configuration in which the right NS5-brane has been rotated as in Eqs. (170)–(172), which corresponds to turning on a (complex) mass μ to the adjoint field (174), breaking $N=2$ supersymmetry to $N=1$. In order to ''rotate the NS5-brane'' we must consider configurations in which the genus-(N_c-1) curve (198) degenerates to a genus-zero one. In gauge theory this is the statement that the adjoint mass lifts the Coulomb branch, except for isolated points. In the classical type-IIA limit there is one such point, where all the $D4$-branes are placed together, corresponding to the origin of the Coulomb branch. For finite R_{10}/L_6 there are N_c points where the curve (198) is completely degenerate. These points are related by the discrete unbroken Z_{2N_c} subgroup of $U(1)_{45}$ whose action on v, t was described in Sec. IV.C.4 [after Eq. (115)]. It acts on the QCD scale as

$$\Lambda_{N=2}^2\to e^{2\pi i/N_c}\Lambda_{N=2}^2. \tag{199}$$

At one of these degenerate points the curve takes the form

$$v=t^{1/N_c}+\Lambda_{N=2}^2 t^{-1/N_c}. \tag{200}$$

Rotating the right NS5-brane from $w=0$ to $w=\mu v$ implies that at large t we should like for the curve to approach $v^{N_c}=t$ and for w to be small, while for $t\to 0$ we want it to approach $w=\mu v$ with large v, w (corresponding to the NS_θ-brane). This is achieved by supplementing Eq. (200) by

$$w=\mu\Lambda_{N=2}^2 t^{-1/N_c}. \tag{201}$$

To make contact with Eq. (197) we should like to take the adjoint mass $\mu\to\infty$. Scale matching between the high-energy theory with the adjoint field and the low-energy theory obtained by integrating it out,

$$\Lambda_{N=1}^3=\frac{\mu}{N_c}\Lambda_{N=2}^2, \tag{202}$$

implies that at the same time we have to take $\Lambda_{N=2}\to 0$ holding the $N=1$ SYM scale (202) fixed. Rewriting Eqs. (200) and (201) in terms of $\Lambda_{N=1}$,

$$v=t^{1/N_c}+\frac{N_c}{\mu}\Lambda_{N=1}^3 t^{1/N_c},$$
$$w=N_c\Lambda_{N=1}^3 t^{-1/N_c} \tag{203}$$

and dropping the term proportional to μ^{-1} in the first equation of Eq. (203) leads to the curve (197) with[32]

$$\zeta=N_c\Lambda^3. \tag{204}$$

We saw earlier that pure $N=1$ SYM theory with $G=SU(N_c)$ has a $U(1)_R$ symmetry that is broken at one loop to Z_{2N_c} by the chiral anomaly and is further spontaneously broken nonperturbatively to Z_2, giving rise to N_c vacua with different values of the gaugino condensate, Eqs. (138) and (139). This pattern of breaking of the chiral $U(1)_R$ symmetry has a direct analog in the brane language. In the previous section we saw that the brane analog of the one-loop effect of the anomaly is the asymptotic curving of the branes for large v and w. Thus studying the five-brane of Eq. (197) semiclassically is tantamount to having access to its large v, w asymptotics, described by the first two equations in Eq. (197), but not to the shape of the five-brane for small v and w, which is described by the last equation in Eq. (197).

It is therefore interesting that ζ appears in the first two equations only in the combination ζ^{N_c}, while the third equation depends on ζ itself. This means that five-branes of Eq. (197) related by the Z_{N_c} transformation

$$\zeta\to e^{2\pi i/N_c}\zeta \tag{205}$$

look the same asymptotically (or semiclassically) but differ in their detailed shape. Each of the N_c possible values of ζ in Eq. (205) corresponds to a different five-brane and, therefore, to a different vacuum of the quantum theory. The Z_{N_c} symmetry relating them is spontaneously broken. One can think of ζ as the gaugino condensate (139) (Hori, Ooguri, and Oz, 1997; Brandhuber, Itzhaki, *et al.*, 1997a).

In addition to the Z_{N_c} symmetry mentioned above, which acts on v, w, and t as

$$v\to v,$$
$$w\to e^{2\pi i/N_c}w,$$
$$t\to t \tag{206}$$

and which—as explained above—is a symmetry of the first two equations in Eq. (197) but does not leave the third one invariant [or in other words has to be combined with Eq. (205) to become a symmetry], there are two more global symmetries. One is a $U(1)$ R symmetry discussed near Eq. (193),

[32]We also rename $\Lambda_{N=1}\to\Lambda$.-

$$v \to e^{i\delta} v,$$
$$w \to e^{-i\delta} w,$$
$$t \to e^{iN_c\delta} t. \tag{207}$$

As anticipated there, this symmetry, which is preserved semiclassically, is an exact symmetry of the brane configuration. For $N_f > 0$ it corresponds to a symmetry of the low-energy supersymmetric Yang-Mills theory, becoming part of the $N=1$ superconformal algebra in the infrared. In the case considered here, in the absence of matter ($N_f=0$), the SYM fields do not carry charge under this symmetry. It is possible that this $U(1)$ symmetry is still part of the $N=1$ superconformal algebra in the infrared, but pure SYM theory has a mass gap and does not contribute to the extreme infrared conformal field theory. If the brane configuration is to describe SYM physics at low but nonzero energies, any states charged under Eq. (207) must decouple from SYM physics.

There is also a Z_2 symmetry corresponding to exchanging v and w,

$$v \to w,$$
$$w \to v,$$
$$t \to \zeta^{N_c} t^{-1}. \tag{208}$$

This symmetry reverses the orientation of 4–4 strings stretched between different four-branes and therefore acts as charge conjugation. The fact that it is an exact symmetry of the vacuum is in agreement with gauge theory.

Having understood chiral symmetry breaking in the brane language we next turn to confinement (Witten, 1997b). Pure $N=1$ SYM theory is expected to have the property that if one introduces a heavy quark and antiquark into the system, the energy of the pair will grow with their separation as if the two were connected by a string with tension Λ^2. This "QCD string" can thus end on external quarks, but in the absence of quarks it is stable. Since N_c fundamentals of $SU(N_c)$ can combine into a singlet, QCD strings can annihilate in groups of N_c. It is expected that large-N_c QCD can be reformulated in terms of weakly coupled QCD strings. Establishing the existence and studying the properties of QCD strings is one of the major challenges in QCD.

In brane theory it is natural to identify the QCD string with an $M2$-brane ending on the $M5$-brane [Eq. (197)]. We are searching for a membrane that looks like a string to a four-dimensional observer and that is also a string in the space Q labeled by (v,w,t). We can describe the string in Q by an open curve C parametrized by $0 \leq \sigma \leq 1$, such that both of its end points (the points with $\sigma=0,1$) are in Σ. It turns out that the right curve for describing a QCD string is

$$t = t_0 = \text{const},$$
$$v = t_0^{1/N_c} e^{2\pi i \sigma/N_c},$$
$$vw = \zeta. \tag{209}$$

The string in spacetime obtained by wrapping a membrane around the curve C has the following properties:

(1) Groups of N_c (but not fewer) strings can annihilate.
(2) The QCD string can end on an external quark.
(3) For a particular choice of t_0, C has minimal length.

The fact that QCD strings annihilate in groups of N_c can be seen by aligning strings described by curves C_j of the form (209) with $2(j-1)\pi \leq 2\pi\sigma \leq 2j\pi$ $(j=1,\ldots,N_c)$. The N_c strings form a long closed string in Q that can detach from the five-brane and shrink to a point. At the same time, the strings corresponding to different C_j are all equivalent as they can be mapped into each other by continuously varying the phase of t_0, $t_0 \to t_0 \exp(2\pi i\alpha)$ with $0 \leq \alpha \leq 1$.

To minimize the length of C one notes that t is constant along it, while v and w change by amounts of order $t_0^{1/N_c}/N_c$ and $\zeta t_0^{-1/N_c}/N_c$, respectively, (for large N_c). The length is minimized for $t_0 \sim \zeta^{N_c/2}$; it is of order $l_C \sim \zeta^{1/2}/N_c$. The tension of the QCD string is obtained by multiplying l_C by the tension of the $M2$-brane $1/l_p^3$. Restoring dimensions in Eq. (204), $\zeta = N_c l_p^6 \Lambda^3/R_{10}$, we find that the tension of the QCD string is

$$T \sim \left(\frac{\Lambda^3}{R_{10} N_c}\right)^{1/2}. \tag{210}$$

In SYM physics one expects the tension of the QCD string to be of order $T \sim \Lambda^2$. Comparing to Eq. (210) we see that for agreement with SYM theory we must choose

$$R_{10} \sim \frac{1}{N_c \Lambda}. \tag{211}$$

For such values of R_{10} there is no decoupling of the four-dimensional SYM physics from Kaluza-Klein excitations carrying momentum in the x^{10} direction. One might think that, due to Eq. (211), at least for large N_c, the Kaluza-Klein scale would be much higher than the QCD scale Λ. Unfortunately, since the Riemann surface Σ winds N_c times around the x^{10} direction, the Kaluza-Klein modes see an effective radius $N_c R_{10}$ and have energies of order Λ. Thus decoupling fails even in the large-N_c limit.

From the discussion in previous sections it is clear what went wrong. The QCD string is not a BPS-saturated object and therefore its tension is not protected by the usual nonrenormalization theorems. The estimate (210) of its tension is semiclassical in nature and is valid when the supergravity approximation for describing membranes and five-branes is applicable. We are discovering that in this regime the system does not describe decoupled SYM physics. The regime corresponding to SYM theory is described by Eqs. (93)–(95); in that regime it is not clear at present how to study properties of the QCD string, such as the tension, but there is no reason for the formula (210) to be valid. It is known (de Boer *et al.*, 1998a) that other nonholomorphic SYM features, such as the Kähler potential for mesons and baryons, depend sensitively on R_{10}, L_6, and there is no reason to expect that the tension of QCD strings is any different.

In addition to QCD strings, one can construct using branes domain walls separating regions in space corresponding to different vacua (different values of ζ). A domain wall occurs as $x^3 \to -\infty$, when the configuration approaches one value of ζ while as $x^3 \to \infty$ it approaches another. The resulting $M5$-brane interpolates between the two solutions (197). It is known in gauge theory that such domain walls are BPS saturated and their tension is the difference between the values of the superpotential (140) between the different vacua. At large N_c it thus goes like $T_D \simeq N_c \Lambda^3$.

Unlike the QCD string, the tension of the BPS-saturated domain wall (or membrane) can be exactly calculated using branes. Witten has shown that the tension of the domain wall goes at large N_c like $T_D \simeq R_{10}|\zeta|/l_p^6$, which, using the form of ζ and R_{10} discussed above, agrees with the gauge-theory analysis. Witten furthermore pointed out that the domain wall behaves in large-N_c gauge theory like a Dirichlet two-brane in string theory; its tension goes like N_c, which is the inverse QCD string coupling, and the QCD string can end on it, just as the fundamental string can end on a D-brane.

The above discussion can be generalized by adding N_f fundamental chiral multiplets of $SU(N_c)$ with masses m_i, $i=1,\ldots,N_f$. We saw that these can be described by adding N_f semi-infinite four-branes to the left of the NS5-brane at $v=m_i$. The corresponding Riemann surface Σ takes the form (Brandhuber, Itzhaki *et al.*, 1997a; Witten, 1997b; Hori, Ooguri, and Oz, 1998)

$$v^{N_c} = t\prod_{i=1}^{N_f}\left(1-\frac{v}{m_i}\right),$$

$$vw=\zeta, \tag{212}$$

where

$$\zeta^{N_c} = \Lambda^{3N_c-N_f}\prod_{i=1}^{N_f} m_i. \tag{213}$$

For large m_i the configuration (212) is essentially the same as Eq. (197) and one can think of the quarks with masses m_i as static sources.

Quarks are confined in this system, and one expects the energy of a state with a quark and antiquark separated by a large distance $\delta x \gg \Lambda^{-1}$ to grow like $T\delta x$ where T is the tension of the QCD string. Classically, the quark and antiquark are described by fundamental strings connecting a flavor four-brane to the stack of color four-branes. Quantum mechanically these fundamental strings turn into membranes, and the only stable configuration has them connected by a long QCD string; thus its energy is indeed proportional to the separation of the two quarks, as expected from gauge theory.

To study theories with massless quarks we have to take the limit $m_i \to 0$ in Eq. (213). This was discussed by Hori, Ooguri, and Oz (1998). For $0<N_f<N_c$ massless flavors, the curve one finds in the limit is singular—it is infinitely elongated in the x^6 direction, and therefore the corresponding brane configuration does not describe a four-dimensional field theory. This is consistent with the field-theory analysis: the gauge theory has no vacuum due to the nonperturbative superpotential (152).

For $N_f \geq N_c$ the supersymmetric Yang-Mills theories under consideration have quantum moduli spaces of vacua that were described in Sec. V.A. To study them one needs to replace the semi-infinite four-branes by four-branes ending on six-branes, described as in Sec. IV by an $M5$-brane in the background of a resolved A_{N_f-1} multi-Taub-NUT space. It is then possible, by rotating the $N=2$ SYM curves with matter studied in Sec. IV, to describe the roots of different branches of the moduli space. As an example, the root of the baryonic branch, which exists for all $N_f \geq N_c$, is (formally) described by the factorized curve

$$\Sigma_L: \quad t=v^{N_c-N_f}, \quad w=0,$$

$$\Sigma_R: \quad t=\Lambda^{3N_c-N_f}w^{-N_c}, \quad v=0. \tag{214}$$

It can be shown that deformations of the curve (214) leads to a $[2N_cN_f-(N_c^2-1)]$-complex-dimensional space parametrizing the Higgs branch of the theory, in agreement with field-theory results (with the caveat discussed in Sec. IV.C.4 that one complex modulus appears to be a parameter in the brane description).

It should be emphasized that just as in Sec. IV, when one approaches a singular point in moduli (or parameter) space where the infrared behavior changes, such as Eq. (214), the five-brane degenerates and the supergravity approximation breaks down, even if overall the five-brane (i.e., L_6, R_{10}) is large. Thus one cannot use supergravity to study most aspects of the nontrivial superconformal field theory at the origin of moduli space for $N_f \geq N_c$.

What is done in practice is to resolve the singularity by turning on a superpotential for the quarks that lifts all the flat directions, or study the theory in its fully Higgsed branch. As is standard in gauge theory (Seiberg, 1994), by computing the expectation values of chiral fields as a function of the deformation parameters one can recover the superpotential at the origin of moduli space.

Further study of confinement and extended objects in M-theory QCD appear in several recent works (Fayyazuddin and Spalinski, 1997b; Nam, Oh, and Sin, 1997; Volovich, 1997; Ahn, 1998b; Ahn, Oh, and Tatar, 1998c; Hanany, Strassler, and Zaffaroni, 1998). The duality trajectory of Sec. V.B.3 in M theory has been described by Csaki and Skiba (1997); Furukawa (1997); Schmaltz and Sundrum (1998)).

3. Brane interactions

So far we have discussed the vacuum structure of $N=1$ SQCD by using properties of the M theory five-brane. We saw that many features of the quantum vacuum structure can be understood using five-branes. In particular, M-theory techniques provide a very natural description of the Coulomb branch of various $N=1,2$ SUSY gauge theories. They are also very useful for describing isolated vacua with a mass gap, such as those of SQCD with massive quarks, and for studying properties of BPS saturated states in such vacua.

There are also some drawbacks. One is that the description in terms of large and smooth five-branes is inapplicable in the supersymmetric Yang-Mills limit (93)–(95), where the five-brane in fact degenerates, and at the same time most quantities that one might be interested in calculating in SYM theory depend strongly on R_{10} and L_6. Also, the long-distance behavior at the origin of moduli space is described by singular five-branes for which the supergravity description is not valid. If one considers only the vacuum structure, the global structure of moduli space requires a rather involved description in the M-theory language even for SQCD, which makes it difficult to extract physical consequences and study more complicated situations.

One may also want a more uniform description of the physics in different dimensions. We shall discuss later three-dimensional analogs of the theories studied in this section, which correspond to brane configurations in type-IIB string theory, where the M-theory construction is inapplicable. It is one of the remarkable features of brane dynamics that rather different dynamical systems, such as three- and four-dimensional gauge theories, are described by closely related brane configurations. It is difficult to believe that when the dynamics of branes is eventually understood, the story will be drastically different in different dimensions.

To really solve QCD using webs of branes one needs a much better understanding of the theory on four-branes stretched between five-branes in the appropriate scaling limit. Already for a stack of flat parallel NS five-branes, the worldvolume dynamics is not understood (see Aharony, Berkooz *et al.*, 1998; Aharony, Berkooz, and Seiberg, 1998; Ganor and Sethi, 1998 and references therein for recent work on this problem). It is even less clear what happens when one suspends four-branes between the five-branes and studies the system in the limit of Eqs. (93)–(95).

In the absence of understanding of the theory on the five-brane one may proceed as follows (Elitzur, Giveon, *et al.*, 1997). The quantum vacuum structure of different brane configurations can be thought of as a consequence of interactions between different branes. For theories with eight supercharges such interactions modify the metric on moduli space, while for systems with four supercharges they give rise to forces between different branes that sometimes lift some or all of the classical moduli space.

When the interacting branes are nearby, one expects the resulting forces to be rather complicated, and a more detailed understanding of five-brane dynamics is necessary. For widely separated branes, i.e., far from the origin of moduli space, the interactions should simplify. The purpose of this section is to describe the quantum moduli space of vacua of SQCD with $G=SU(N_c)$ and N_f fundamentals by postulating certain long-range interactions between different branes. In the next section we shall show that these interactions also explain the vacuum structure of $N=2$ SUSY gauge theories in three dimensions.

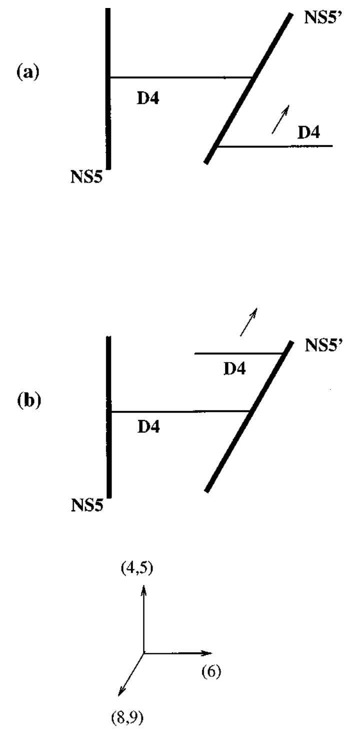

FIG. 36. Widely separated four-branes in configurations with $N=1$ supersymmetry, acting as charged particles: (a) branes that end on opposite sides of a Neveu-Schwarz five-brane attract each other, while (b) those that end on the same side have a repulsive long-range interaction.

Of course, these interactions are not derived from "first principles" but rather guessed by comparison with the gauge-theory results, so on the level of the present discussion they do not necessarily have much predictive power. However, as usual in brane theory, the interactions are local in the sense that they do not depend on the global structure of the configuration in which the branes are embedded. Therefore once the local rules are formulated one can use them in more complicated situations, and even different dimensions, to learn more about gauge dynamics. And, of course, once one is convinced that these rules are valid, they teach us about brane dynamics as well and need to be eventually reproduced by the theory of the five-branes.

The quantum rule of brane dynamics that we shall postulate is (see Fig. 36) as follows: *There is a long-range interaction between a* $D4$*-brane stretched between an* $NS5$*- and an* $NS5'$*-brane, and any other* $D4$*-brane ending on one of the five-branes. It is repulsive if the* $D4$*-branes are on the same side of the five-brane and attractive if they are on different sides.*

Comments:

(1) U duality relates the above rule to many other cases. For example, the classical interaction between $D4$-branes stretched between $D6$- and

$D6'$-branes—discussed in Sec. V.B.4—is related to it by compactifying (say) x^3 and applying the U-duality transformation $U=T_3ST_3$. U relates quantum interactions to classical ones in this case because it involves a strong-weak coupling duality transformation (S). As another example, in the next section we shall discuss the consequences of the above quantum interactions for systems related to the current setup by applying T_3 (i.e., $D3$-branes ending on NS5- and NS5$'$-branes). In the rest of this section we use the quantum interactions to describe the moduli space of vacua of SQCD.

(2) The quantum rules are useful in describing situations where the different branes that interact are widely separated. They provide a qualitative picture of the quantum moduli space and can be used to understand the semiclassical corrections to the superpotential. One can thus see using the quantum rules when runoff to infinity in moduli space will occur; in situations with unlifted quantum moduli spaces, the quantum rules allow one to study the structure of the moduli space far from the origin. The origin of moduli space and, in general, situations where the branes are close to each other need to be studied by different techniques.

We start with electric SQCD described by the brane configuration of Fig. 24. For $N_f=0$ the system contains N_c $D4$-branes stretched between an NS5- and an NS5$'$-brane. The quantum rule formulated above cannot be applied to this case. The $D4$-branes repel each other but are restricted by the geometry to lie on top of each other, and the vacuum structure is determined by short-distance properties of the brane system. In the previous section we saw that the M-theory analysis gave a good description of the vacuum structure for this case.

For $1\leqslant N_f\leqslant N_c-1$ massless flavors, the system develops an instability that can be understood using the quantum rule. Describing the flavors by $D6$-branes intersecting the $D4$-branes [Fig. 24(b)], there is now the possibility for $D4$-branes to break on the $D6$-branes, and the segments of the broken $D4$-branes connecting the NS5$'$-brane to the nearest $D6$-brane are repelled from the remaining color $D4$-branes, which are still stretched between the NS5- and NS5$'$-branes. Since the repulsion is presumed to be long range, these segments run off to $w\to\infty$, and there is no stable vacuum at a finite value of the moduli.

For $N_f\geqslant N_c$ the situation changes. Now there do exist stable configurations of the branes with no repulsive interactions. They correspond to breaking all N_c color four-branes on $D6$-branes, which effectively screens the repulsive interactions and gives rise to a quantum moduli space that looks qualitatively the same as the classical one. Interesting effects occurring near the origin of the quantum moduli space, such as the quantum modification (156) for $N_f=N_c$, again correspond to a regime where the brane interactions are not well understood and have to be studied by different techniques. One consequence of this discussion is that the dimension of the quantum moduli space of electric SQCD is seen in brane theory to be $2N_fN_c-N_c^2$ (presumably $+1$ to account for the difference between $SU(N_c)$ and $U(N_c)$ as discussed above), just like that of the classical theory.

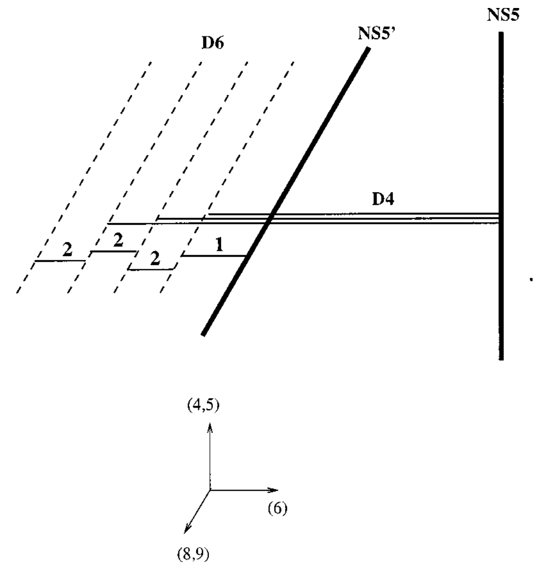

FIG. 37. Quantum brane interactions lifting a part of the magnetic moduli space of Fig. 29, leaving an unlifted $(N_f^2-\bar{N}_c^2)$-dimensional quantum moduli space.

A similar analysis can be performed for the magnetic configuration of Fig. 28 with gauge group $G=SU(\bar{N}_c)$ and N_f flavors. As before, we restrict ourselves to the case $N_f\geqslant\bar{N}_c$. We saw before that the classical moduli space is N_f^2 dimensional, corresponding to giving expectation values to the components of the magnetic meson field M (183), without breaking the gauge group. We also saw that turning on a Fayet-Iliopoulos D term (182) changes the form of the moduli space discontinuously. In particular, for $r\neq0$ the moduli space is $N_f^2-\bar{N}_c^2$ dimensional.

Quantum mechanically the discontinuity in the structure of the moduli space is eliminated. The $\bar{N}_c$ color four-branes are attracted to the N_f flavor four-branes. Hence $\bar{N}_c$ of the N_f flavor four-branes align with the color four-branes and reconnect, giving rise to $\bar{N}_c$ four-branes stretched between the NS5-brane and $\bar{N}_c$ different six-branes (in agreement with the s rule). The remaining $N_f-\bar{N}_c$ flavor four-branes are easily seen to give rise to an $(N_f^2-\bar{N}_c^2)$-dimensional moduli space (see Fig. 37). Furthermore, as is obvious from Fig. 37, the part of the classical moduli space that remains unlifted in the quantum theory is precisely the part that is smoothly connected to the structure at the nonzero Fayet-Iliopoulos D term r [or to the baryonic branch of moduli space, if the gauge group is really $SU(\bar{N}_c)$ and r describes the baryonic branch].

In gauge theory, the lifting of a part of the classical moduli space in the quantum magnetic theory follows from the fact that the classical magnetic superpotential

(183) is corrected quantum mechanically[33] to Eq. (161). The second term in Eq. (161) is due to the fact that when M gets an expectation value, the magnetic quarks become massive due to the classical coupling (183), and a superpotential of the form (154) with $N_c \rightarrow \bar{N}_c$ is generated.

It is not difficult to show that the moduli space corresponding to Eq. (161) is the same as the quantum moduli space of brane configurations (Fig. 37). Thus we have the result that the quantum brane interactions described above can reproduce the consequences of nonperturbative superpotentials of SYM theory.

4. Quantum $N=1$ duality and phase transitions

After understanding the form of the quantum moduli spaces of vacua of the electric and magnetic theories we can complete the demonstration of Seiberg's duality using branes. We saw before that classically the moduli spaces of the electric and magnetic theories agree for nonzero r in Eq. (182), but there is a discrepancy between the structures for $r=0$. We have now seen that quantum mechanically the discrepancy disappears. The electric moduli space is not modified quantum mechanically, while in the magnetic theory quantum effects lift part of the classical moduli space, leaving behind precisely the subspace that connects smoothly to the electric theory via the construction of Sec. V.B.3. This completes the proof of the equivalence of the quantum moduli spaces of the electric and magnetic theories and, therefore, also of the corresponding chiral rings.

In gauge theory one distinguishes between two notions of $N=1$ duality. The weaker version is the statement that members of a dual pair share the same quantum chiral ring and moduli space of vacua, as a function of all possible deformations. In Seiberg's original work this statement was proven for supersymmetric QCD, and we have now rederived it using branes. The stronger version of Seiberg's duality asserts that the full infrared limits of the electric and magnetic theories coincide. In field theory, no proof of this assertion has been given, but it is believed to be correct. One may ask whether the embedding of the problem in brane theory helps to settle the issue.

To show the equivalence of the (in general) nontrivial infrared theories at the origin of the electric and magnetic moduli spaces, one would like to continuously interpolate between them while staying at the origin of moduli space and only varying Λ, or x^6. In the process we pass through a region where the NS5- and NS5′-branes cross. We shall next discuss this region.

In fact, one can ask more generally, what happens to the low-energy physics on webs of branes as some of the branes (which are in general connected by other branes to each other) meet in space and exchange places. We have discussed a few examples of such transitions at various points in the review. Let us summarize the results.

The low-energy physics is smooth when nonparallel Neveu-Schwarz branes connected by four-branes cross (in which case the smoothness of the transition is equivalent to the strong version of Seiberg's duality), and when nonparallel NS and D-branes cross (the Hanany-Witten transition of Fig. 15). When parallel NS five-branes connected by four-branes cross, the transition relates $N=2$ SYM theories with different-rank gauge groups, e.g., $U(N_c)$ and $U(N_f-N_c)$. By construction, these theories have the same fully Higgsed branch but in general different mixed and Coulomb branches, and even different numbers of massless fields. Thus in that case there is a phase transition. Similarly, when parallel D- and NS-branes cross, there is a phase transition. For example, we saw that as a $D6$-brane passes an NS5′-brane, we lose or gain a light-matter multiplet.

In both of the above cases, phase transitions occur in situations where a configuration containing parallel coincident branes is deformed in different directions. An interesting example that superficially shows a different behavior is configurations with rotated six-branes $D6_\theta$, discussed in Sec. V.B.4, where the low-energy physics depends on the order in which different nonparallel six-branes appear along the x^6 axis (different orders corresponding to different superpotentials). A closer look reveals that, in fact, this example follows the same pattern as the others. When *all* branes (the two NS-branes and N_f six-branes) are nonparallel, there is in fact *no* phase transition as different six-branes cross. It is only when some of the six-branes are parallel to other six-branes or to one or more of the NS five-branes, as in the configuration of Fig. 35, that changing the order of the six-branes influences the low-energy dynamics.

For the case in which some of the six-branes become parallel, it is easy to understand the mechanism for the phase transition. Imagine first placing all N_f six-branes at the same value of x^6. In this case, fundamental strings connecting different six-branes give rise to *massless* fields which we shall collectively denote by A. Quarks Q are as usual described by 4–6 strings. The standard three open-string coupling gives rise to cubic superpotentials of the form $W=\tilde{Q}AQ$. As we displace the six-branes relative to each other in x^6 the fields A become massive, and integrating them out gives rise to quartic superpotentials for the quarks, of the general form $W \sim (\tilde{Q}Q)^2$. It is rather easy to see that in the generic case, when no branes are parallel, the superpotential generated this way is the most general one, and the low-energy theory is insensitive to the precise coefficients. When some of the six-branes are parallel, different deformations give superpotentials with inequivalent long-distance behaviors.

The lesson from this example is the following. When branes meet in space, additional degrees of freedom in the theory in general become massless. If these degrees of freedom couple to the gauge theory on the four-

[33]Equation (161) corresponds to $\bar{N}_c=N_f-N_c$, the value relevant for $N=1$ duality.

branes, it is possible that different deformations of the singular point in which branes touch produce different low-energy behaviors. Otherwise the transition is smooth.

What happens in the other cases described above? When two parallel NS-branes approach each other, degrees of freedom corresponding to membranes stretched between them go to zero mass and eventually become tensionless BPS-saturated strings trapped in the five-brane(s). The usual three-membrane vertex in eleven dimensions implies that these tensionless strings interact with the degrees of freedom describing the gauge theory on the four-branes and, therefore, it is not surprising as in the previous case to find that different deformations of the system correspond to different phases.

When two nonparallel five-branes, NS_{θ_1} and NS_{θ_2} with $\theta_1 \neq \theta_2$, approach each other in x^6, membranes stretched between the two five-branes do not lead to BPS-saturated strings inside the five-brane. Hence there is no mechanism for a phase transition to occur as the two five-branes are exchanged.

It is important to emphasize that the above argument does not prove full infrared equivalence of members of a Seiberg dual pair. The fact that membranes stretched between nonparallel Neveu-Schwarz five-branes are not BPS saturated provides another proof of the fact that the *vacuum structure* is smooth. To rule out a change in the full infrared conformal field theory, one needs to understand the interactions of all the light non-BPS modes of a membrane stretched between the NS_{θ_1} and NS_{θ_2} five-branes with the gauge-theory degrees of freedom. This is beyond the reach of available methods.

D. Generalizations

Branes can be used to study the dynamics of a wide variety of $N=1$ supersymmetric gauge theories with different matter contents and superpotentials. In this section we briefly describe a few constructions that have appeared in the recent literature. In situations where a good brane description exists, it leads to new insights both on gauge theory and on brane dynamics. Therefore it is important to enlarge the class of models that can be described this way. This may also provide clues towards the formulation of the five-brane theory.

1. Product groups

In Sec. IV.C.3 we discussed $N=2$ SUSY theories with product gauge groups $G=\Pi_{\alpha=1}^{n} SU(k_\alpha)$, by considering $n+1$ parallel NS5-branes connected by four-branes. $N=1$ configurations of this sort are obtained by performing relative rotations (170) of the five-branes.

As an example, consider the configuration of Fig. 38(a), which was studied by Brodie and Hanany (1997) and Giveon and Pelc (1998). Three NS five-branes denoted by $NS5_L$, NS5, and $NS5_R$ are ordered in the x^6 direction such that the $NS5_L$-brane is the leftmost while the $NS5_R$-brane is the rightmost. We can choose to orient the (middle) NS5-brane as in Eq. (91) and rotate the

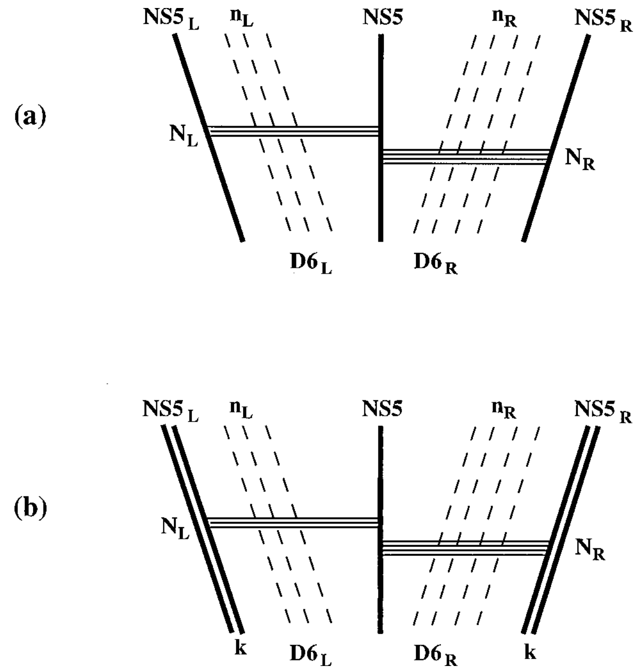

FIG. 38. An $N=1$ SUSY theory with $G=U(N_L)\times U(N_R)$ and matter in the bifundamental and fundamental representations. The bifundamental has (a) a quartic or (b) higher-order polynomial superpotential.

other two relative to it by (θ_L,φ_L) and (θ_R,φ_R) [see Eq. (172)]. N_L or N_R $D4$-branes are stretched in the x^6 direction between the $NS5_L$- and NS5-branes or between the NS5- and $NS5_R$-branes.

The theory on the four-branes is an $SU(N_L)\times SU(N_R)$ gauge theory with two chiral multiplets in the adjoint of the respective gauge groups Φ_L,Φ_R, and bifundamentals $F,\tilde{F}$ in the $(N_L,\bar{N}_R),(\bar{N}_L,N_R)$. The classical superpotential is

$$W=\mu_L \operatorname{Tr}\Phi_L^2+\mu_R \operatorname{Tr}\Phi_R^2+\operatorname{Tr}\tilde{F}\Phi_L F+\operatorname{Tr}F\Phi_R\tilde{F}, \quad (215)$$

where [see Eq. (172)]

$$\mu_L=e^{i\varphi_L}\tan\theta_L, \quad \mu_R=e^{i\varphi_R}\tan\theta_R. \quad (216)$$

Integrating out the massive adjoints we obtain (for generic rotation angles)

$$W\sim \operatorname{Tr}(F\tilde{F})^2. \quad (217)$$

We can add fundamental quarks to the theory by adding to the configuration six-branes and/or semi-infinite four-branes.

A qualitative identification between the parameters and moduli of the field theory on the four-branes and the parameters determining the brane configuration can be made along the lines of this section. The quantum vacuum structure can be studied by starting with the $N=2$ curve (133) and rotating it, following the logic of the discussion of Sec. V.C for a simple group. This was done by Giveon and Pelc (1998).

It is straightforward to find Seiberg dual configurations by interchanging the order of the NS-branes, a procedure that is expected to preserve the long-distance

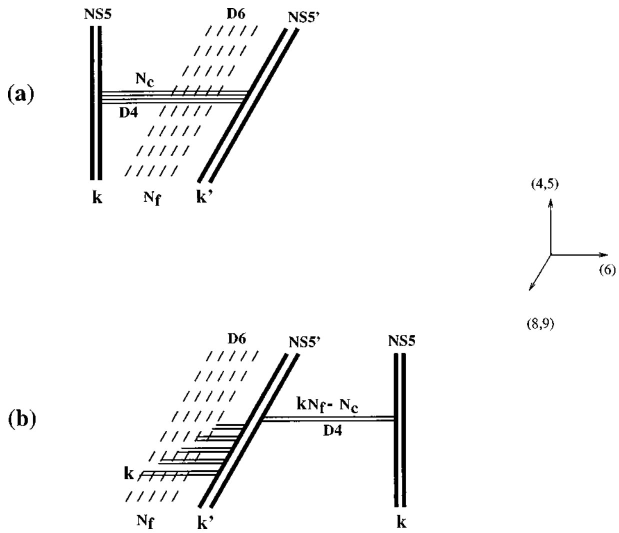

FIG. 39. Two dual models: (a) a theory with $G=U(N_c)$, two adjoint superfields with polynomial superpotentials, and fundamentals; (b) the Seiberg dual model with $\overline{G}=kN_f-N_c$.

physics as long as the five-branes being exchanged are not parallel [which is the case for generic θ_L, θ_R in Eq. (216)]. In particular, if we start in the "electric" configuration of Fig. 38(a) with n_L and n_R flavors of $SU(N_L)$ and $SU(N_R)$, respectively, exchanging the NS5_L- and NS5_R-branes leads (Brodie and Hanany, 1997) to a magnetic theory with $G=SU(n_L+2n_R-N_R)\times SU(n_R+2n_L-N_L)$ and the same number of flavors, in agreement with the field-theory results (Intriligator, Leigh, and Strassler, 1995).

2. Landau-Ginzburg superpotentials

Brane configurations containing $D4$-branes ending on a stack of parallel NS five-branes are interesting since the theory on the five-branes is in this case nontrivial in the IR [it is the (2,0) theory discussed before], and it is interesting to see how this is reflected in the structure of the theory on the four-branes.

Consider (Elitzur, Giveon, and Kutasov, 1997; Elitzur, Giveon, *et al.*, 1997), as an example, a configuration of k coincident NS5-branes connected by N_c $D4$-branes to k' coincident NS5$'$-branes, with N_f $D6$-branes located between the NS5- and NS5$'$-branes [see Fig. 39(a)]. $N=1$ SQCD corresponds to the case $k=k'=1$. The classical low-energy theory on the four-branes is in this case $N=1$ SYM theory with gauge group $U(N_c)$, N_f fundamental flavors Q^i, $\tilde{Q}_i$, and two adjoint superfields Φ, Φ'. The classical superpotential is

$$W=\frac{s_0}{k+1}\mathrm{Tr}\,\Phi^{k+1}+\frac{s_0'}{k'+1}\mathrm{Tr}\,\Phi'^{k'+1}+\mathrm{Tr}[\Phi,\Phi']^2+\tilde{Q}_i\Phi' Q^i. \qquad (218)$$

Here Φ and Φ' can be thought of as describing fluctuations of the four-branes in the w and v directions, respectively. They are massless, but the superpotential (218) implies that there is a polynomial potential for the corresponding fluctuations, allowing only infinitesimal deviations from the vacuum at $\Phi=\Phi'=0$. The couplings s_0, s_0' should be thought of as very large: $s_0,s_0'\to\infty$. This can be deduced, for example, on the basis of the transformation properties of Eq. (218) under the R symmetries $U(1)_{45}$ and $U(1)_{89}$.

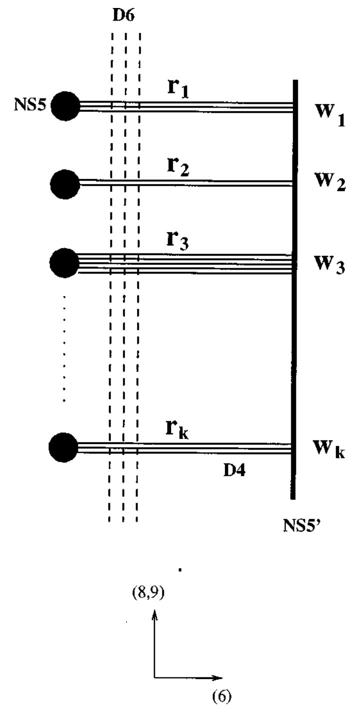

FIG. 40. Displacement of the k NS5-branes in w, giving rise to a rich vacuum structure labeled by the numbers of $D4$-branes attached to the different NS5-branes, r_j.

To see that the configuration of branes constructed above indeed describes a gauge theory with the stated matter content and, in particular, to see the origin of the adjoint fields Φ, Φ' one matches the deformations of the brane configuration with those of the gauge theory (218). Consider first the case $k'=1$ for which Φ' is massive and can be integrated out. For large s_0' this amounts to putting $\Phi'=0$ in Eq. (218).

An interesting deformation of the brane configuration of Fig. 39(a) corresponds to displacing the k NS5-branes in the (x^8,x^9) plane to k different points w_j, $j=1,\ldots,k$. Since the NS5$'$-brane is extended in w, this gives rise to many possible supersymmetric configurations, labeled by sets of non-negative integers $(r_1,\ldots,r_k)$ with $\Sigma_j r_j=N_c$, which specify the number of four-branes stretched between the jth NS5-brane and the NS5$'$-brane (Fig. 40).

When all the $\{w_j\}$ are distinct, the low-energy physics described by the configuration of Fig. 40 corresponds to k decoupled SQCD theories with gauge groups $U(r_i)$ and N_f flavors of quarks. As we approach the origin of parameter space, $w_j=0$, the full $U(N_c)$ gauge group is restored.

To translate the above discussion to the language of gauge theory on the four-brane, one notes that in displacing the NS5-branes in w, the four-branes attached to them are displaced as well. The locations of the four-branes in w correspond to the expectation value of an adjoint of $U(N_c)$, Φ, describing fluctuations of the four-branes in (x^8,x^9). In a vacuum labeled by $(r_1,\ldots,r_k)$ the expectation value of Φ is $\langle\Phi\rangle=\mathrm{diag}(w_1^{r_1},\ldots,w_k^{r_k})$. Furthermore, in the brane construction the $\{w_j\}$ correspond to locations of heavy objects (the five-branes) and thus they are expected to appear as parameters rather than moduli in the gauge-theory description.

The gauge theory that achieves all of the above is the one described by Eq. (218). Generic $\{w_j\}$ correspond to a polynomial superpotential for Φ,

$$W=\sum_{j=0}^{k}\frac{s_j}{k+1-j}\mathrm{Tr}\,\Phi^{k+1-j}. \tag{219}$$

For generic $\{s_j\}$ the superpotential has k distinct minima $\{w_j\}$ related to the parameters in the superpotential via the relation

$$W'(x)=\sum_{j=0}^{k}s_jx^{k-j}\equiv s_0\prod_{j=1}^{k}(x-w_j). \tag{220}$$

The integers $(r_1,\ldots,r_k)$ introduced above are the numbers of eigenvalues of the matrix Φ residing in the different minima of the potential $V=|W'(x)|^2$. Thus the set of $\{r_j\}$ and $\{w_j\}$ determines the expectation value of the adjoint field Φ, in agreement with the brane picture. When all $\{w_j\}$ are distinct the adjoint field is massive, the gauge group is broken,

$$U(N_c)\to U(r_1)\times U(r_2)\times\cdots\times U(r_k), \tag{221}$$

and the theory splits in the infrared into k decoupled copies of SQCD with gauge groups $\{U(r_i)\}$ and N_f flavors of quarks. The brane description makes this structure manifest.

For $k'>1$, the above discussion can be repeated for the parameters corresponding to the locations of the k' NS5$'$-branes in the v plane. These k' complex numbers can be thought of as parametrizing the extrema of a polynomial superpotential in Φ' of order $k'+1$, in complete analogy to Eqs. (219) and (220). The only new element is that when we displace the k' NS5$'$-branes in the v directions, leaving the N_f $D6$-branes fixed, we make the quarks Q, $\tilde{Q}$ massive with masses of order $\langle\Phi'\rangle$. This is the origin of the Yukawa coupling in the superpotential [the last term on the right-hand side of Eq. (218)]. One can also consider situations in which both NS5- and NS5$'$-branes are displaced in the w and v directions, respectively, and study the moduli space of vacua of the theory (218) for general k and k'.

A Seiberg dual of the system (218) can be obtained by interchanging the NS5- and NS5$'$-branes in x^6 [Fig. 39(b)]. For $k\geq1$, $k'=1$ one derives this way (Elitzur, Giveon, and Kutasov, 1997) the dual description (167) obtained in field theory by Kutasov (1995) and Kutasov and Schwimmer (1995). For $k=1$, $k'>1$ one finds a perturbation of this duality that was discussed in field theory by Aharony, Sonnenschein, and Yankielowic (1995). For general k, k' the brane construction predicts a new duality that was not previously known in field theory (Elitzur, Giveon, *et al.*, 1997).

Quantum mechanically, the type-IIA configuration of Fig. 39(a) is again replaced by a smooth $M5$-brane. For $k=1$ and general k' this five-brane was obtained by de Boer and Oz (1997) by rotating an $N=2$ SUSY configuration. It was shown that monopole and meson expectation values computed from M theory match the results obtained in field theory via confining phase superpotentials (Elitzur, Forge, *et al.*, 1996).

More generally, one may consider chains of stacks of coincident five-branes, separated in the x^6 direction as before and rotated with respect to each other. An example that was discussed by Brodie and Hanany (1997) and that is depicted in Fig. 38(b) involves an NS5-brane connected to k NS5$_L$-branes on its left by N_L four-branes, and to k NS5$_R$-branes on its right by N_R four-branes. n_L and n_R six-branes are located to the left and the right of the NS5-brane, respectively.

For generic orientations θ_L, θ_R, this brane configuration corresponds to an $SU(N_L)\times SU(N_R)$ gauge theory with n_L (n_R) fundamental quarks of $SU(N_L)$ ($SU(N_R)$) and bifundamentals $F,\tilde{F}$, with the classical superpotential

$$W\sim(F\tilde{F})^{k+1}. \tag{222}$$

The dual configuration is obtained by interchanging the NS5$_{\theta_L}$- and NS5$_{\theta_R}$-branes. The magnetic gauge group is $SU[(k+1)(n_L+n_R)-n_L-N_R]\times SU[(k+1)(n_L+n_R)-n_R-N_L]$, in agreement with field theory (Intriligator, Leigh, and Strassler, 1995). The case $k=1$ was discussed after Eq. (217).

Finally, note that configurations containing coincident NS five-branes provide an example of a phenomenon mentioned above: different deformations of the configuration describe different low-energy theories. For example, the configuration of k coincident NS5-branes connected by four-branes to an NS5$'$-brane (Fig. 39) can be deformed in two different directions. Separating the five-branes in w we find a theory that is well described by the gauge theory with an adjoint superfield Φ and a polynomial superpotential (218) described in this section. On the other hand, separating the NS5-branes in x^6 leads to a low-energy description in terms of a product group of the general sort described in the previous section. The two configurations are clearly inequivalent and are continuously connected through a transition which involves crossing parallel NS5-branes. We conclude that, as in the other examples mentioned above, a phase transition occurs when the NS5-branes coincide. This transition is apparently related to the nontrivial conformal field theory on $k>1$ five-branes; to understand the nature of the transition a better understanding of the (2,0) theory on k parallel five-branes will probably be required.

3. Orthogonal and symplectic gauge groups from orientifolds

Just as for $N=2$ SUSY configurations, many new theories are obtained by adding an orientifold plane. In

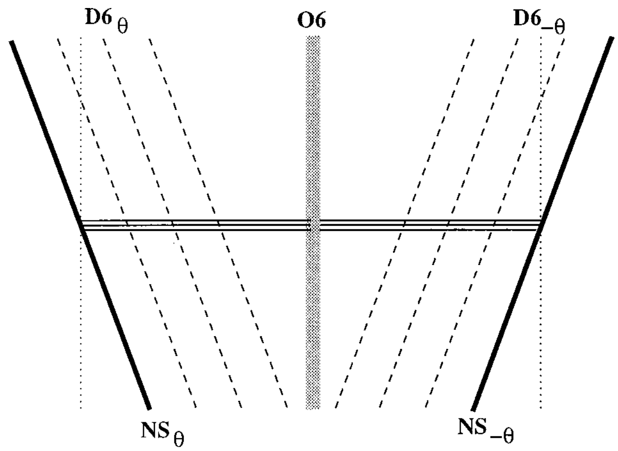

FIG. 41. $N=1$ supersymmetric QCD with orthogonal and symplectic groups, realized using rotated Neveu-Schwarz five-branes near an orientifold six-plane.

this section we list a few examples of such theories and mention some of their properties. We start with an $O6$-plane and then move on to an $O4$-plane.

a. Orientifold six-plane

The simplest configurations to consider are again rotated $N=2$ ones. Starting with the configuration of Fig. 17 and rotating the NS5-brane by a generic angle θ, to an NS_θ-brane[34] gives a mass to the adjoint chiral multiplet. The resulting configuration has $N=1$ supersymmetry and light matter in the fundamental representation of the gauge group, which we recall is $SO(N_c)$ for positive orientifold charge and $Sp(N_c/2)$ for negative charge. If we leave the $D6$-branes parallel to the orientifold, we find a theory with a quartic superpotential for the quarks. To switch off the superpotential we rotate the $D6$-branes as well until they are parallel to the NS_θ-brane (and their mirrors are parallel to the mirror $\mathrm{NS}_{-\theta}$-brane; see Fig. 41).

The moduli space of vacua can be studied by combining the discussion of orientifolds in theories with $N=2$ supersymmetry (Sec. IV.C.2) and the results of this section on the reduction to $N=1$. We leave the details to the reader.

One can also analyze Seiberg's duality for these systems by exchanging five-branes and orientifolds in x^6. In the absence of orientifolds a quick way to find the dual is to exchange the branes while requiring conservation of the linking number (97). The linking number for five-branes near an orientifold six-plane is also given by Eq. (97) with the understanding that an $O6_\pm$ plane contributes to L_{NS} like ± 2 $D6$-branes. Using this, it is not difficult to show that the electric configuration of Fig. 41 is connected by duality to a magnetic one with gauge group $SO(\bar{N}_c)$ with $\bar{N}_c = N_f - N_c + 4$ for $O6_+$, and $Sp(\bar{N}_c/2)$ with $\bar{N}_c = N_f - N_c - 4$ for $O6_-$.

If the rotation angle θ above is tuned to $\theta=\pi/2$, the five-brane and its mirror image turn into NS5′-branes and become parallel to the orientifold. The resulting SYM theory on the four-branes is an $SO(N_c)$ gauge theory with $2N_f$ chiral superfields in the vector representation, a chiral superfield S in the symmetric representation, and $W=0$. Motions of $D4$-branes along the NS5′-brane (in w) correspond to expectation values of S which parametrize an N_c-dimensional moduli space along which $SO(N_c)$ is generically completely broken. Reversing the charge of the orientifold replaces $SO(N_c)\to Sp(N_c/2)$ and $S\to A$, with A a chiral multiplet in the antisymmetric tensor representation of $Sp(N_c/2)$.

The last two models are direct analogs of the $SU(N_c)$ theory with an adjoint, fundamentals, and $W=0$, discussed in Sec. V.B.4, with the symmetric of an orthogonal group (S) or antisymmetric of a symplectic group (A) playing the role of the adjoint field Φ. As in the $SU(N_c)$ theory, one can turn on a polynomial superpotential for the (anti-) symmetric tensor. For example, for the case of an $SO(N_c)$ gauge group this is obtained by studying the following configuration: k coincident $\mathrm{NS5}_\theta$-branes to the left (in x^6) of an $O6_+$ plane, connected to their mirror images (which are k coincident $\mathrm{NS5}_{-\theta}$-branes) by N_c four-branes, with N_f six-branes parallel to the $\mathrm{NS5}_\theta$-branes placed between the five-branes and the orientifold. The SYM theory on the four-branes is an $SO(N_c)$ gauge theory with $2N_f$ vectors, a symmetric flavor S, and

$$W \sim \mathrm{Tr}\, S^{k+1}. \tag{223}$$

The magnetic theory in the brane picture is obtained by interchanging the k five-branes with their mirror images while preserving the linking number (97) (Elitzur, Giveon, *et al.*, 1997). The magnetic theory has $G_m = SO[k(2N_f+4)-N_c]$, $2N_f$ magnetic quarks, magnetic mesons, and an appropriate superpotential, in agreement with field theory (Intriligator, Leigh, and Strassler, 1995).

b. Orientifold four-plane

As for the $N=2$ SUSY case discussed in Sec. IV.C.2, the situation is less well understood than that for $O6$-planes, so we shall be brief.

The basic configuration that describes $N=1$ SQCD with an orthogonal or symplectic gauge group and matter in the fundamental representation includes N_c four-branes stretched between an NS5- and an NS5′-brane, with $D6$-branes between them; all objects are stuck on an $O4$-plane, although the $D6$- and $D4$-branes could leave it in pairs (see Fig. 42). As discussed in Sec. IV.C.2, the charge of the orientifold flips sign each time it passes through an NS five-brane. If the charge between the NS5- and NS5′-branes is positive, the gauge group is $Sp(N_c/2)$; negative charge corresponds to $SO(N_c)$. The moduli space of vacua can be analyzed as in Sec. IV.C.2; we shall not describe the details here.

[34]The mirror image of the NS-brane is necessarily rotated by the angle $-\theta$ and becomes an $\mathrm{NS}_{-\theta}$-brane.

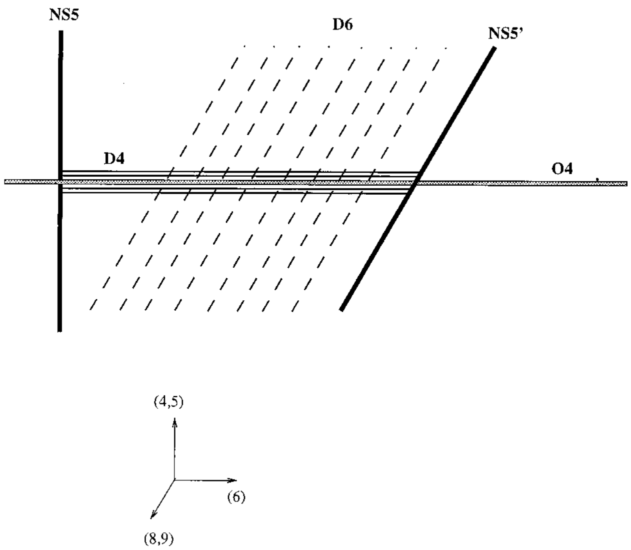

FIG. 42. Branes near an $O4$-plane providing an alternative description of $N=1$ supersymmetric Yang-Mills theories with orthogonal and symplectic gauge groups.

The fully Higgsed branch for both signs of the orientifold plane is illustrated in Fig. 43.

To analyze the smooth transition that corresponds in brane dynamics to Seiberg's duality we need to understand how to compute linking numbers in the presence of the $O4$-plane. Again, Eq. (97) is essentially correct as long as we take into account the contributions of the $O4$-plane. An $O4_{\pm}$ plane contributes like ± 1 $D4$-branes. Using this result, one can verify that Seiberg's duality is reproduced in this system (Elitzur, Giveon, and Kutasov, 1997; Elitzur, Giveon, *et al.*, 1997; Evans, Johnson, and Shapere, 1997).

One can also study generalizations, e.g., replacing the single NS5-brane by $2k+1$ five-branes leads to orthogonal and symplectic gauge theories with a massless adjoint field, with the polynomial superpotential $W \sim \mathrm{Tr}\, \Phi^{2(k+1)}$. Placing a sequence of NS$_\theta$-branes with different $\theta=\theta_i$ along the orientifold leads to theories with product gauge groups of the form $SO(k_1)\times Sp(k_2/2) \times SO(k_3)\times\cdots$ (Tatar, 1998).

4. Unitary gauge groups with two-index tensors

$N=1$ supersymmetric Yang-Mills theories with an $SU(N_c)$ gauge group and chiral superfields in the (anti-) symmetric tensor representation can be constructed by starting with an $N=2$ configuration of branes near an $O6$-plane—mentioned at the end of Sec. IV.C.3—and applying to it all the operations described in other cases. It is again sufficient to describe the theory for one sign of the orientifold charge (we shall choose the case of positive sign). To get the theory corresponding to the other sign, one simply replaces symmetric tensors by antisymmetric ones, or vice versa.

Consider an NS5-brane that is stuck on an $O6_+$ plane. An NS5$'$-brane located to the left of the orientifold (in x^6) is connected to the NS5-brane by N_c four-branes. As usual, we place N_f six-branes between the five-branes.

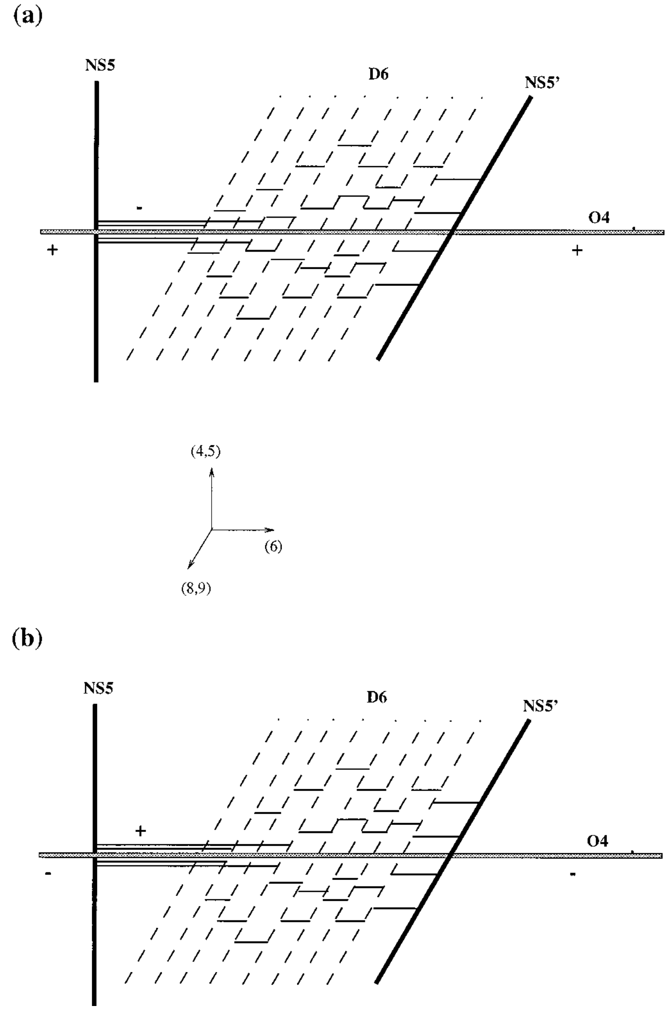

FIG. 43. The fully Higgsed branch of moduli space corresponding to (a) $G=SO(4)$ and $N_f=8$ fundamentals; (b) $G=Sp(2)$ and $N_f=8$ fundamentals.

The theory on the four-branes is classically a $U(N_c)$ gauge theory with N_f fundamental flavors, two symmetric flavors S, $\tilde{S}$, and $W=0$. The analysis of the brane moduli space is easily seen to reproduce that of the proposed gauge theory. In particular, motions of the $D4$-branes in w, away from the NS5-brane, parametrize the N_c-dimensional moduli space of the theory along which S, $\tilde{S}$ get expectation values and the gauge group is typically completely broken. When all the four-branes meet at a point in the w plane that is not the position of the NS5-brane, an $SO(N_c)$ gauge symmetry is restored and one recovers the theory with $G=SO(N_c)$, a symmetric tensor, fundamentals, and $W=0$, described in the previous section. Turning on the Fayet-Iliopoulos D term in the $U(N_c)$ theory [or entering the baryonic branch of the moduli space of the $SU(N_c)$ one] has a similar effect.

Rotating the external NS5$'$-brane to an NS5$_\theta$-brane, and at the same time rotating the $D6$-branes so that they are parallel to the NS5$_\theta$-branes, leads to a theory with the same matter content as before, but now with a classical superpotential for the symmetric tensor,

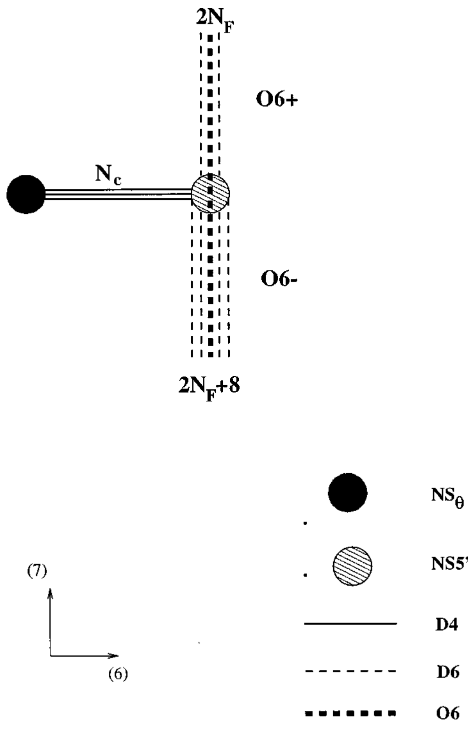

FIG. 44. A chiral brane configuration in which an NS5′-brane is stuck at an $O6$-plane and is connected to an NS_θ-brane outside of the orientifold.

$$W\sim\frac{1}{\mu}\mathrm{Tr}(S\tilde{S})^2. \tag{224}$$

The previous case corresponds to $\mu=\infty$. Rearranging the branes leads to a dual configuration with gauge group $G_m=SU(3N_f+4-N_c)$ and matter that can be easily analyzed as above. The resulting theory agrees with the field-theory analysis (Intriligator, Leigh, and Strassler, 1995).

If there are k coincident $\mathrm{NS5}_\theta$-branes outside the orientifold, one finds a similar theory but with Eq. (224) replaced by

$$W\sim\mathrm{Tr}(S\tilde{S})^{k+1}. \tag{225}$$

Brane rearrangement leads to the Seiberg dual gauge group $SU[(2k+1)N_f+4k-N_c]$, again in agreement with field theory.

5. Chiral models

Generic $N=1$ SYM theories are chiral. Such theories are interesting both because of their relevance to phenomenology and because of their rich dynamics. Their exploration using branes is in its infancy. Here we discuss a few families of brane configurations in the presence of orientifolds and orbifolds leading to chiral models that have appeared in the recent literature.

The first family was studied by Brunner *et al.* (1998), and Elitzur, Giveon, *et al.* (1998b), Landsteiner, Lopez, and Lowe (1998). The brane configuration shown in Fig. 44 involves an NS5′-brane that is embedded in an $O6$-plane, say at $x^7=0$. The NS5′-brane divides the $O6$-plane into two disconnected regions, corresponding to positive and negative x^7. As we saw before, in this situation the Ramond-Ramond charge of the orientifold jumps, from $+4$ to -4, as we cross the NS5′-brane. The part of the orientifold with negative charge (which we shall take to correspond to $x^7<0$) has furthermore eight semi-infinite $D6$-branes embedded in it. The presence of these $D6$-branes is required for charge conservation or, equivalently, vanishing of the six-dimensional anomaly.

In addition to the eight semi-infinite $D6$-branes, we can place on the orientifold any number of parallel infinite $D6$-branes extending all the way from $x^7=-\infty$ to $x^7=\infty$. We shall denote the number of such $D6$-branes by $2N_f$.

Then, an NS_θ five-brane[35] located at a distance L_6 in the x^6 direction from the NS5′-brane, but at the same value of x^7, is connected to the NS5′-brane by N_c $D4$-branes stretched in x^6. N_c must be even for consistency. The mirror image of the NS_θ five-brane, which is an $\mathrm{NS}_{-\theta}$ five-brane, is necessarily also connected to the NS5′-brane.

We can also place any number of $D6$-branes oriented at arbitrary angles θ_i [Eq. (49)] between the NS_θ five-brane and the orientifold (in x^6). We shall mainly discuss the case where such branes are absent, but it is easy to incorporate them.

We shall next describe the gauge theory described by the above brane configuration. Before studying the general case we describe the structure for $\theta=0$ (when the external $\mathrm{NS}_{\pm\theta}$ five-branes are NS5-branes), and $\theta=\pi/2$ (when they are NS5′-branes). We shall only state the result, referring the reader to Elitzur, Giveon, *et al.* (1998b) for further discussion and derivations.

a. The case $\theta=0$

The theory on the $D4$-branes has classical gauge group $U(N_c)$ with a symmetric tensor $\tilde{S}$, an antisymmetric tensor A, $2N_f+8$ quarks Q in the fundamental representation, and $2N_f$ quarks $\tilde{Q}$ in the antifundamental representation. The superpotential is

$$W=Q\tilde{S}Q+\tilde{Q}A\tilde{Q}. \tag{226}$$

Fundamental chiral multiplets of the gauge group come from 4–6 strings connecting the $D4$-branes to $D6$-branes ending on the NS5′-brane from below (in x^7), while antifundamentals arise from $D6$-branes ending on the NS5′-brane from above. The global symmetry of the system is determined by the gauge symmetry on the $D6$-branes, $Sp(N_f)\times SO(2N_f+8)$. The superpotential (226) is the unique one consistent with this symmetry.

The theory is chiral and potentially anomalous as there are eight more fundamental than antifundamental chiral multiplets. The superpotential (226) implies that the symmetric tensor $\tilde{S}$ is in fact a symmetric bar (i.e., a

[35]An NS_θ five-brane is an NS5-brane rotated as in Eq. (49) by the angle θ.

symmetric tensor with two antifundamental indices). Thus the total anomaly $(2N_f+8)-2N_f+(N_c-4)-(N_c+4)$ vanishes, as one would expect for a consistent vacuum of string theory.

As a further check on the identification of the brane configuration and the chiral gauge theory one can analyze the moduli space of vacua as a function of various parameters one can add to the Lagrangian. An example is the Fayet-Iliopoulos D term, which from the brane point of view corresponds to displacements in x^7 of the NS5-brane relative to the NS5$'$-brane. In the gauge theory, adding to the Lagrangian a Fayet-Iliopoulos D term for the $U(1)$ vectormultiplet $\mathrm{Tr}\,V$, $r\int d^4\theta\,\mathrm{Tr}\,V$, modifies the D-flatness vacuum conditions:

$$AA^\dagger-\tilde{S}\tilde{S}^\dagger+QQ^\dagger-\tilde{Q}\tilde{Q}^\dagger=-r. \tag{227}$$

Setting the quarks Q, $\tilde{Q}$ to zero we see that, when r is positive, S gets an expectation value which breaks $U(N_c)\to SO(N_c)$. Due to the superpotential (226), the $2N_f+8$ chiral multiplets Q^i as well as $\tilde{S}$ become massive and one is left with the $N=2$ spectrum and interactions for gauge group $SO(N_c)$, with the antisymmetric tensor A playing the role of the adjoint of $SO(N_c)$. All of this is easily read off from the brane configuration. In particular, the fact that the $2N_f+8$ quarks Q^i are massive is due to the finite length (proportional to r) of the corresponding 4–6 strings.

Similarly, for negative r Eq. (227) implies that A gets an expectation value, breaking $U(N_c)$ to $Sp(N_c/2)$. The quarks $\tilde{Q}$ get a mass and we end up with an $N=2$ gauge theory with $G=Sp(N_c/2)$ and $2N_f+8$ light quarks.

b. The case $\theta=\pi/2$

In this case the external five-brane and its mirror image are NS5$'$-branes. In addition to the matter discussed for the previous case there is now an adjoint field Φ parametrizing fluctuations of the four-branes in the w plane. The classical superpotential is

$$W=\mathrm{Tr}\,\tilde{S}\Phi A+Q\tilde{S}Q+\tilde{Q}A\tilde{Q}. \tag{228}$$

As a check on the gauge theory we can again study the D-term perturbation corresponding to relative displacement in x^7 of the NS5$'$-branes. For positive r we now find an $SO(N_c)$ gauge theory with $2N_f$ fundamental chiral multiplets, a symmetric tensor, and vanishing superpotential. This can be understood by analyzing the D-flatness conditions (227) in the presence of the superpotential (228). As before, the symmetric tensor $\tilde{S}$ gets an expectation value, which for unbroken $SO(N_c)$ must be proportional to the identity matrix. The first term in the superpotential (228) then gives rise to the mass term $W\sim\Phi A$. Since A is antisymmetric, this term gives a mass to the antisymmetric part of Φ (as well as to A). The symmetric part of Φ becomes the symmetric tensor mentioned above. Clearly, it does not couple to the $2N_f$ fundamental chiral multiplets. In the brane description the fact that fluctuations of the four-branes in w are described by a symmetric tensor is a direct consequence of the action of the orientifold projection (Gimon and Polchinski, 1996).

c. The general case

For generic rotation angle θ [Eq. (49)] the adjoint field Φ discussed in the previous point is massive. Its mass $\mu(\theta)$ varies smoothly between zero at $\theta=\pi/2$ and ∞ for $\theta=0$. The superpotential describing this system is

$$W=Q\tilde{S}Q+\tilde{Q}A\tilde{Q}+\Phi A\tilde{S}+\mu(\theta)\Phi^2. \tag{229}$$

For nonzero μ we can integrate Φ out and find the superpotential,

$$W=Q\tilde{S}Q+\tilde{Q}A\tilde{Q}+\frac{1}{\mu(\theta)}(A\tilde{S})^2, \tag{230}$$

for the remaining degrees of freedom. When $\theta\to0$, $\mu\to\infty$, and Eq. (230) approaches Eq. (226). When $\theta=\pi/2$, the mass μ vanishes and it is inconsistent to integrate Φ out.

For generic θ none of the NS five-branes in the configuration are parallel, and one can interchange them to find a dual magnetic theory. The magnetic gauge group one finds is $U(2N_f+4-N_c)$. A careful field-theory analysis leads to the same conclusion (Elitzur, Giveon, *et al.*, 1998b).

A second family of chiral models was studied by Lykken, Poppitz, and Trivedi (1998a). It has a gauge group that is a product of unitary groups with matter in the bifundamental of different pairs. It is obtained from brane configurations in Z_n orbifold backgrounds in the following way. Start with nN_c four-branes stretched between two NS5-branes. The low-energy theory on the four-branes is an $N=2$ SYM theory with gauge group $G=SU(nN_c)$. We now mod out by the Z_n symmetry (see footnote 28) acting on v and w as

$$(v,w)\to(v\exp(2\pi i/n),w\exp(-2\pi i/n)). \tag{231}$$

Orbifolding breaks half of the supercharges and leads to an $N=1$ SUSY gauge theory with gauge group $SU(N_c)_1\times SU(N_c)_2\times SU(N_c)_3\times\cdots\times SU(N_c)_n$ with matter fields F_i, $i=1,\ldots,n$, in the bifundamental $(\mathbf{N}_c,\overline{\mathbf{N}}_c)$ of $SU(N_c)_i\times SU(N_c)_{i+1}$ [where $SU(N_c)_{n+1}\equiv SU(N_c)_1$]. This theory is chiral for $n>2$. The curve describing its moduli space was obtained by Lykken, Poppitz, and Trivedi (1997a).

An interesting variant of this theory is obtained by stretching nN_C $D4$-branes between an NS5-brane and n rotated five-branes located at

$$v=\mu w,\quad v=\mu e^{4\pi i/n}w,\quad v=\mu e^{8\pi i/n}w,\ \cdots$$
$$v=\mu e^{[4(n-1)\pi i]/n}w \tag{232}$$

[of course these five-branes are identified after orbifolding by Eq. (231), and so really describe a single five-brane on R^4/Z_n]. After modding out by the Z_n group [Eq. (231)] one finds a gauge theory that is similar to that described above, but with a tree-level superpotential

$$W=\mu\,\mathrm{Tr}\,F_1\cdots F_n\,. \tag{233}$$

This superpotential lifts the moduli space, in agreement with the brane picture in which for $\mu\neq 0$ the four-branes are stuck at $v=w=0$. Adding nN_f six-branes and interchanging the NS5-brane with the n rotated five-branes leads to a magnetic $SU(N_f-N_c)^n$ dual gauge theory.

A third class of models was also studied by Lykken, Poppitz, and Trivedi (1998b). It corresponds to webs of branes in the presence of orientifold planes and orbifold fixed points. As an example, one can start with the configuration of Fig. 19, that was shown in Sec. IV.C.2 to describe SO or Sp theories with $N=2$ supersymmetry (depending on the sign of the orientifold charge), and then mods out by the Z_3 symmetry (231) with $n=3$. The resulting gauge group is either $SO(N+4)\times SU(N)$ or $Sp(M)\times SU(2M+4)$, with matter in the following representations. For the first case (an $SO\times SU$ gauge group) there is an antisymmetric tensor field A in the $\frac{1}{2}\mathbf{N}(\mathbf{N}-\mathbf{1})$ of $SU(N)$ [it is a singlet under $SO(N+4)$], a field $\bar{Q}$ in the bifundamental $(\mathbf{N+4},\bar{\mathbf{N}})$ of $SO(N+4)\times SU(N)$, and fundamentals of both groups, whose number is partly constrained by anomaly cancellation. The second case [an $Sp(M)\times SU(2M+4)$ gauge group] is related to the first one by replacing the antisymmetric tensor A by a symmetric one S but is otherwise similar.

The theories obtained this way have a vanishing superpotential. Rotating one of the NS5-branes in a way compatible with both the Z_2 orientifold projection and the Z_3 orbifold one, as in Eq. (232) (with $n=3$), leads to the appearance of a superpotential of the form $W\sim(\bar{Q}A\bar{Q})^2$ or $W\sim(\bar{Q}S\bar{Q})^2$ for the two cases. In the presence of a superpotential one can study $N=1$ duality, recovering results first obtained in field theory by Intriligator, Leigh, and Strassler (1995).

VI. THREE-DIMENSIONAL THEORIES

So far in this review we have focused on brane configurations realizing four-dimensional physics. However, it is clear that the framework naturally describes field-theory dynamics in different dimensions. In the remainder of the review we shall study some brane configurations describing field theories in two, three, five, and six dimensions.

We shall see that these theories exhibit many interesting phenomena which can be studied using branes. Apart from the intrinsic interest in strongly coupled dynamics of various field theories in different dimensions and its realization in string theory, the main reason for including this discussion here is that it adds to the "big picture" and, in particular, emphasizes the generality and importance of universality and hidden relations between different theories.

(1) "*Universality.*" One of the interesting features of the four-dimensional analysis was the fact that understanding a few local properties of branes allowed the study of a wide variety of models with various matter contents and numbers of supersymmetries. These were obtained by combining branes in different ways in a sort of flat-space "geometric engineering." We shall in fact see that this universality may allow one to understand[36] in a uniform way theories in different dimensions. This should be contrasted with the situation in field theory where the physics is described in terms of perturbations of weakly coupled fixed points, whose nature depends strongly on the dimensionality.

(2) *Hidden relations between different theories.* In Sec. III we saw how viewing a brane configuration from different points of view provides a relation between gauge theories in different dimensions with different amounts of supersymmetry. In that case, a relation between four-dimensional $N=4$ SYM theory and two-dimensional $N=(4,4)$ SYM theory provided an explanation of Nahm's construction of multimonopole moduli space. In this and the next two sections we shall see that this is an example of a much more general phenomenon.

In this section we discuss three-dimensional field theories, starting with the case of eight supercharges ($N=4$ supersymmetry), followed by four supercharges ($N=2$ supersymmetry). In the next two sections we discuss five-, six-, and two-dimensional theories. The presentation is more condensed than in the four-dimensional case above. We only explain the basic phenomena in the simplest examples, referring the reader to the original papers for more extensive discussion.

A. $N=4$ supersymmetry

The main purpose of this subsection is to describe the explanation using branes of two interesting field-theory phenomena:

(1) The Coulomb branch of a three-dimensional $N=4$ SUSY gauge theory is often identical to the moduli space of monopoles in a *different* gauge theory.

(2) Three-dimensional $N=4$ SUSY gauge theories often have "mirror partners" such that the Higgs branch of one theory is the Coulomb branch of its mirror partner and vice versa.

To study three-dimensional gauge dynamics we consider, following Hanany and Witten (1997), configurations of $D3$-branes suspended between NS5-branes in the presence of $D5$-branes. Using Eqs. (6), (18), and (19) it is not difficult to check that any combination of two or more of the following objects,

$$\begin{aligned}&\mathrm{NS5:}\;(x^0,x^1,x^2,x^3,x^4,x^5),\\&D3:\;(x^0,x^1,x^2,x^6),\\&D5:\;(x^0,x^1,x^2,x^7,x^8,x^9),\end{aligned} \tag{234}$$

in type-IIB string theory preserves 8 of the 32 supercharges and gives rise to an $N=4$ SUSY theory in the

[36]This program is not complete as of this writing; we shall mention some open problems in the discussion section.

$(1+2)$-dimensional spacetime common to all branes (x^0,x^1,x^2). One can think of the branes (234) as obtained from Eq. (91) by performing T duality in x^3.

As a first example, consider a configuration containing k $D3$-branes stretched between two NS5-branes separated by a distance L_6 in x^6. As discussed at length above, the low-energy theory on the three-branes is a three-dimensional $N=4$ SUSY gauge theory with gauge group $G=U(k)$ and no additional light matter. The three-dimensional gauge coupling is

$$\frac{1}{g^2}=\frac{L_6}{g_s}. \tag{235}$$

Motions of the k three-branes along the NS5-branes in (x^3,x^4,x^5) together with the duals of the k photons, corresponding to the Cartan subalgebra of G, parametrize the $4k$-dimensional Coulomb branch of the $N=4$ SUSY gauge theory $\mathcal{M}_k$. Relative displacements of the two NS5-branes in (x^7,x^8,x^9) are interpreted as in Eq. (98) as Fayet-Iliopoulos D terms. Note that the theory under consideration here can be thought of as a dimensional reduction of four-dimensional $N=2$ SYM theory or six-dimensional $N=1$ SYM theory, in both cases without hypermultiplets, and thus much of the discussion of Sec. IV.C.1 applies to it. The R symmetry, which is $SU(2)_R$ in six dimensions and $SU(2)_R\times U(1)$ in four dimensions, is enhanced by the reduction to three dimensions to $SU(2)_R\times SU(2)_{R'}$, where $SU(2)_{R'}$ acts as an $SO(3)$ rotation symmetry on (x^3,x^4,x^5).

From the point of view of the theory on the five-branes, the $4k$-dimensional moduli space of BPS-saturated deformations of the brane configuration $\mathcal{M}_k$ has a rather different interpretation. The situation is very similar to that discussed in Sec. III. The worldvolume theory on the five-branes is a gauge theory with $N=(1,1)$ supersymmetry (16 supercharges) and gauge group $G=U(2)$, broken down to $U(1)\times U(1)$ by an expectation value of one of the worldvolume scalars on the type-IIB five-brane discussed in Sec. II. This expectation value is proportional to the separation of the five-branes L_6.

The massive $SU(2)$ gauge bosons correspond to D strings connecting the two NS5-branes. $D3$-branes stretched between the NS5-branes are magnetic $SU(2)$ monopoles charged with respect to the unbroken $U(1)\subset SU(2)$ [the other $U(1)$ corresponding to joint motion of the five-branes does not play a role and will be ignored below]. In compact space they are U dual to D strings stretched between $D3$-branes, which were shown in Sec. III to describe monopoles in a broken $SU(2)$ gauge theory. The $4k$-dimensional moduli space of brane configurations $\mathcal{M}_k$ is, from the point of view of the five-brane theory, the moduli space of k monopoles.

Thus we learn that the two spaces in question—the Coulomb branch of $N=4$ SUSY $U(k)$ gauge theory in $2+1$ dimensions and the moduli space of k monopoles in $SU(2)$ gauge theory broken to $U(1)$—are closely related; both are equivalent to the moduli space of SUSY brane configurations of Fig. 9. The $U(1)\subset U(k)$ corresponding to the center of mass of the k monopole system gives rise to a trivial $R^3\times S^1$ part of the moduli space. The space of vacua of the remaining $SU(k)$ gauge theory corresponds to the moduli space of centered monopoles.

A closer inspection reveals that the two spaces related above are actually not identical; rather they provide descriptions of the moduli space of brane configurations in two different limits, which we describe next.

As we saw before, to study gauge physics using branes one needs to consider a limit in which gravity and massive string modes decouple. The relevant limit in this case is

$$L_6,l_s,g_s\to 0 \tag{236}$$

with L_6/g_s (235) held fixed.

From the point of view of the theory on the three-brane, the typical energy scale is set by the Higgs expectation values parametrizing the Coulomb branch. These are related using Eq. (7) to the relative displacements of the three-branes along the five-branes δx by $\langle\phi\rangle\sim\delta x/l_s^2$. Thus the typical distances between the three-branes in the gauge-theory limit are

$$\delta x\sim\left(\frac{l_s^2 g_s}{L_6^2}\right)L_6. \tag{237}$$

To have a reliable $(2+1)$-dimensional picture one would like to require $\delta x\ll L_6$, i.e.,

$$Y\equiv\left(\frac{l_s}{L_6}\right)^2 g_s\ll 1. \tag{238}$$

The parameter Y is clearly arbitrary in the limit (236) and when it satisfies Eqs. (236) and (238) the brane configuration is well described by $(2+1)$-dimensional field theory.

The scale (237) is natural from the point of view of the five-brane theory as well. The (massive) charged W bosons correspond to D strings stretched between the two NS5-branes. Their mass is

$$M_W=\frac{L_6}{g_s l_s^2}. \tag{239}$$

The magnetic monopoles are much heavier. The gauge coupling of the $(5+1)$-dimensional five-brane theory is [Eq. (20)] $g_{SYM}^2=l_s^2$; thus the effective coupling in the $(1+3)$-dimensional spacetime (x^0,x^3,x^4,x^5) is

$$\frac{1}{g^2}=\frac{V_{12}}{l_s^2}, \tag{240}$$

where V_{12} is the volume of the (x^1,x^2) plane which is eventually taken to infinity. Magnetic monopoles have mass

$$M_{\rm mon}\simeq\frac{M_W}{g^2}=\frac{L_6 V_{12}}{g_s l_s^4} \tag{241}$$

in agreement with their interpretation as $D3$-branes stretched between the NS5-branes. Recall that the *size* of a magnetic monopole is $\simeq M_W^{-1}$, much larger than its Compton wavelength $M_{\rm mon}^{-1}$ for weak coupling.

Thus we see that the scale δx (237) is nothing but the Compton wavelength of a charged W boson [Eq. (239)] or, equivalently, the size of a magnetic monopole. The five-dimensional description as the moduli space of monopoles is appropriate when the scale of $SU(2)$ breaking M_W [Eq. (239)] is much smaller than the scale of Kaluza-Klein excitations of the strings and three-branes stretched between five-branes $1/L_6$. Requiring $M_W \ll 1/L_6$ leads to the constraint

$$Y \gg 1 \tag{242}$$

on the parameter Y defined in Eq. (238). This is the opposite limit from that in which the $(2+1)$-dimensional picture is valid [Eq. (238)].

We see that, rather than being identical, the three- and five-dimensional descriptions of the brane configuration are appropriate in different limits. As $Y \to 0$ the description of the space of vacua as the Coulomb branch of a three-dimensional $SU(k)$ gauge theory becomes better and better, while as $Y \to \infty$ the five-dimensional description becomes the appropriate one.

The dependence of the metric on Y has not been analyzed. Presumably, as in other cases considered in previous sections, supersymmetry ensures that the metric on $\mathcal{M}_k$ does not depend on Y and, therefore, its form for large Y (where it is interpreted as the metric on the moduli space of k monopoles) and for small Y [where it is thought of as the metric on the Coulomb branch of a $d=2+1$, $N=4$ SUSY $SU(k)$ gauge theory] must coincide. It would be interesting to make this more precise.

The relation between monopoles and vacua of $(2+1)$-dimensional field theories can be generalized in many directions. To study monopoles in higher-rank gauge theories we can consider, as in Sec. IV.C.3, chains of NS5-branes connected by $D3$-branes. For example, the configuration of Fig. 22 with the $D4$-branes [Eq. (91)] replaced by $D3$-branes [Eq. (234)] and no $D5$-branes ($d_\beta=0$) describes monopoles in broken $U(n+1)$ gauge theory. The monopoles carry charges under the n unbroken $U(1)$'s in $SU(n+1)$. In a natural basis the magnetic charge of the configuration is $(k_1, k_2-k_1, k_3-k_2, \ldots, k_n-k_{n-1}, -k_n)$.

From the point of view of the three-branes the configuration describes a $2+1$ gauge theory with gauge group $G=U(k_1)\times U(k_2)\times\cdots\times U(k_n)$ with hypermultiplets transforming in the bifundamental representation of adjacent factors in the gauge group, $(k_\alpha, \overline{k_{\alpha+1}})$ of $U(k_\alpha)\times U(k_{\alpha+1})$ ($\alpha=1,\ldots,n-1$). The moduli space of vacua of this gauge theory is identical to the space of monopoles in broken $SU(n+1)$ gauge theory as discussed above.

The second field-theory phenomenon that we should like to understand using branes is mirror symmetry (Intriligator and Seiberg, 1996b), which has been studied in string theory and M theory (Gomez, 1996; de Boer *et al.*, 1997a, 1997b; Hanany and Witten, 1997; Porrati and Zaffaroni, 1997). As pointed out by Hanany and Witten, this symmetry is a manifestation of the S duality of the underlying $(9+1)$-dimensional type-IIB string theory. We shall next illustrate the general idea in an example.

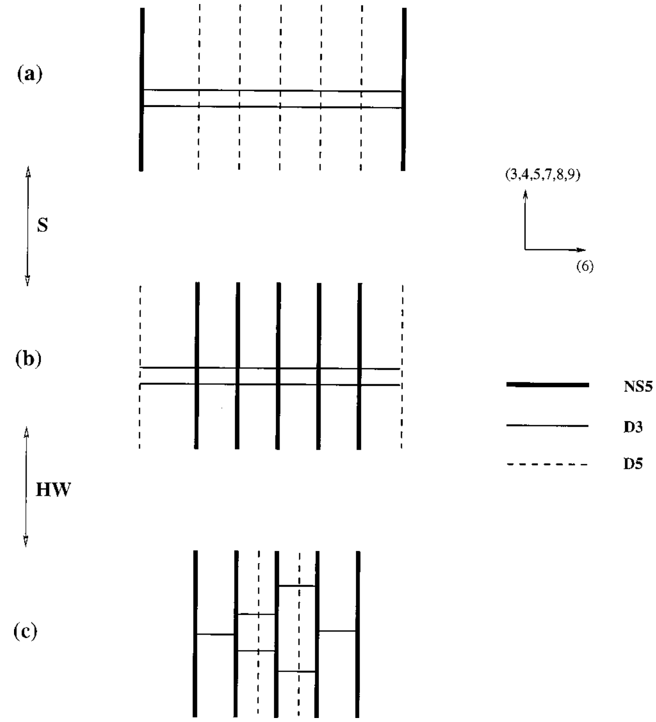

FIG. 45. S duality of type-IIB string theory implying mirror symmetry of the three-dimensional $N=4$ supersymmetric Yang-Mills theory on $D3$-branes.

$N=4$ supersymmetric gauge theory with $G_e=U(N_c)$ and N_f hypermultiplets in the fundamental representation of the gauge group can be studied as in Sec. IV. We consider N_c $D3$-branes stretched between two NS5-branes, in the presence of N_f $D5$-branes placed between the NS5-branes. All branes are oriented as in Eq. (234).

This theory has, like its four-dimensional $N=2$ SUSY analog, a rich phase structure of mixed Higgs-Coulomb phases, which can be studied classically as in Sec. IV.

Under S duality,[37] the NS5-branes are exchanged with the $D5$-branes while the $D3$-branes remain invariant. The original configuration is replaced by one in which N_c $D3$-branes are stretched between two $D5$-branes with N_f NS5-branes located between the two $D5$-branes (see Fig. 45).

This is a configuration that should by now be familar. To exhibit the gauge group we have to reconnect three-branes stretched between the two $D5$-branes into pieces connecting $D5$-branes and NS5-branes, and other pieces connecting different NS5-branes. In doing that one has to take into account the s rule, which implies that the N_c three-branes attached to say the left $D5$-brane have to end on different NS5-branes. Thus if we break the first three-brane on the leftmost NS5-brane we have to break the second on the second leftmost, etc. A similar constraint has to be taken into account on the right $D5$-brane.

The maximal gauge symmetry one can obtain depends

[37] S duality here corresponds to inverting the coupling and exchanging $(x^3,x^4,x^5)\leftrightarrow(x^7,x^8,x^9)$.

on N_f, N_c. The analysis is simplest for $N_f \geqslant 2N_c$ and we shall describe only this case here. The generalization to $N_f < 2N_c$ is simple.

Breaking the N_c three-branes on the NS5-branes in the most general way consistent with the s rule leads in this case to a magnetic gauge theory with $G_m = U(1)\times U(2)\times\cdots U(N_c-1)\times U(N_c)^{N_f-2N_c+1}\times U(N_c-1)\times\cdots U(2)\times U(1)$. To see the hypermultiplets, it is convenient to move the left $D5$-brane past the leftmost N_c NS5-branes (to which it is connected) and similarly for the right $D5$-brane.

The hypermultiplets can now be read off from the brane configuration [Fig. 45(c)]. They transform under G_m as $(1,\overline{2})\oplus(2,\overline{3})\oplus\cdots\oplus(k-1,\overline{k})\oplus k\oplus(k,\overline{k})\oplus\cdots\oplus(k,\overline{k})\oplus k\oplus(k,\overline{k-1})\oplus\cdots\oplus(3,2)\oplus(2,1)$.

The original electric brane configuration at a certain g_s must, by S duality, describe identical physics to the magnetic one at $\tilde{g}_s=1/g_s$ [but the same value of $l_{10}=l_s g_s^{1/4}$; see after Eq. (45)]. In the low-energy limit $E \ll 1/l_{10}$ the electric configuration reduces to the electric gauge theory with gauge group G_e, while the magnetic one reduces to the magnetic gauge theory with gauge group G_m (and the specified matter). Thus the two theories are clearly closely related.

However, as before, to go from one to the other, one has to tune a parameter describing the brane configuration to rather different values. In the electric theory the energy scale we want to hold fixed as we take $l_{10}\to 0$ is set by the three-dimensional gauge coupling (235). To ignore Kaluza-Klein excitations on the three-branes, we must require $g_s/L_6 \ll 1/L_6$, i.e., $g_s \ll 1$. Similarly, in the magnetic theory we must have $\tilde{g}_s \ll 1$ to be able to ignore Kaluza-Klein excitations.

When g_s is small, there exists an energy scale for which all the complications of string theory can be neglected and the running gauge coupling of the electric gauge theory is still very small, so that we are in the vicinity of the ultraviolet fixed point of the gauge theory. The physics of the brane configuration below this energy is well described by gauge theory. Similarly, the magnetic gauge theory provides a good description of the low-energy behavior of the brane configuration for large g_s (or small $\tilde{g}_s$).

To relate the two gauge theories we must go to strong coupling $g_s \simeq 1$. In this regime the brane configuration is still described in the infrared by the same fixed point, but there is no longer an energy range in which it is well approximated by the full RG trajectory of either the electric or the magnetic gauge theories. The Kaluza-Klein excitations of the three-branes modify the RG flow at energies above $1/L_6 \simeq g^2$, and one would expect the correspondence between the two gauge theories to break down.

In effect, the brane construction provides a deformation of the RG trajectories of both the electric and magnetic gauge theories that flow to the same IR fixed point, but with different UV behavior. In particular, the three-dimensional dynamics is embedded in a four-dimensional setting; the fourth (compact) dimension decouples in the extreme infrared but cannot be ignored at finite energies or for large Higgs expectation values. Thus the brane construction shows that the low-energy behavior of the electric and magnetic theories is identical in the strong-coupling limit $g\to\infty$; equivalently, it shows that the infrared limits of the two models coincide for Higgs expectation values $\langle\phi\rangle \ll g^2$.

In gauge theory, mirror symmetry maps the Coulomb branch of the electric theory to the Higgs branch of the magnetic one and vice versa. It also exchanges mass perturbations with Fayet-Iliopoulos D terms. All this is manifest in the brane construction. As should be familiar by now, the Coulomb branch is described by motions of three-branes suspended between NS5-branes, while the Higgs branch corresponds to motions of three-branes stretched between $D5$-branes. Since under S duality NS5-branes are exchanged with $D5$-branes, the Coulomb branch is exchanged with the Higgs branch. In the example discussed in detail above, it is not difficult to check that the (complex) dimensions of the electric Coulomb and Higgs branches are N_c and $2N_c(N_f-N_c)$, respectively, while in the magnetic theory they are reversed.

Similarly, since masses correspond in the brane language to relative displacements of $D5$-branes and Fayet-Iliopoulos D terms are described by relative displacements of NS5-branes, S duality permutes the two.

B. $N=2$ supersymmetry

In this section we shall study three-dimensional $N=2$ SQCD. We start with a summary of field-theoretic results followed by the brane description.

1. Field theory

Consider $N=2$ SQCD with gauge group $G=U(N_c)$ and N_f flavors of chiral multiplets $Q^i,\tilde{Q}_i (i=1,\ldots,N_f)$ in the fundamental representation of G. This theory can be obtained from $N=1$ SQCD in four dimensions by dropping the dependence of all fields on x^3. The vector multiplet (69) gives rise upon reduction to three dimensions to a gauge field, a real scalar field in the adjoint representation of $G, X\equiv A_3$, and fermions. The chiral multiplets $(Q,\tilde{Q})$ reduce in an obvious way. The four-dimensional gauge interaction (74) leads in three dimensions to a potential for the bosonic components of $Q,\tilde{Q}$:

$$V\sim\sum_i |XQ^i|^2+|X\tilde{Q}_i|^2. \tag{243}$$

More generally, one can compactify x^3 on a circle of radius R and interpolate smoothly between four-dimensional $(R\to\infty)$ and three-dimensional $(R\to 0)$ physics. The three- and four-dimensional gauge couplings are related (classically) by $1/g_3^2=R/g_4^2$. Below, we describe the vacuum structure of the theory as a function of R.

The classical theory has an N_c-complex-dimensional Coulomb branch. At generic points in the classical Coulomb branch the light degrees of freedom are the N_c

photons and scalars in the Cartan subalgebra of $U(N_c), A_\mu^{ii}$ and $X^{ii}(i=1,\ldots,N_c)$. Dualizing the photons,

$$\partial_\mu \gamma^{ii} = \epsilon_{\mu\nu}^{\lambda} \partial^\nu A_\lambda^{ii}, \tag{244}$$

gives rise to a second set of scalar fields γ^{ii} that together with X^{ii} form N_c complex chiral superfields whose bosonic components are

$$\Phi^j = X^{jj} + i\gamma^{jj}. \tag{245}$$

The expectation values of Φ^j parametrize the classical Coulomb branch.

In the three-dimensional limit $R=0$ the scalars Φ^j live on a cylinder $R\times S^1$. X^{jj} are noncompact, while γ^{jj} live on a circle of radius g_3^2. For finite R, Φ^j live on a torus since then $\mathrm{Re}\,\Phi^j$ also live on a circle of radius $1/R$. In the four-dimensional limit $R\to\infty$, holding the four-dimensional gauge coupling fixed, the torus shrinks to zero size and the Coulomb branch disappears. The quarks are generically massive on the Coulomb branch [Eq. (243)].

For $N_f \geq N_c$ the theory has a $(2N_cN_f - N_c^2)$-dimensional Higgs branch with completely broken gauge symmetry (whose structure is the same as in four dimensions and, in particular, independent of R). There are also mixed Higgs-Coulomb branches corresponding to partially broken gauge symmetry.

In addition to the complex mass terms, described by a quadratic superpotential $W=m\tilde{Q}Q$, upon compactification to three dimensions one can write a "real-mass" term for the quarks,

$$\int d^4\theta Q^\dagger e^{m_r\theta\bar{\theta}} Q. \tag{246}$$

We have encountered these real-mass terms before, in the previous section, where we saw that the mass parameters in brane configurations describing three-dimensional $N=4$ SUSY gauge theories have three components, and in Eq. (243), which describes a real-mass term for the quarks proportional to $\langle X\rangle$.

Quantum mechanically, the gauge coupling is a relevant (=super-renormalizable) perturbation and thus the theory is strongly coupled in the infrared. Most or all of the Coulomb branch, and in some cases part of the Higgs branch, are typically lifted by strong-coupling quantum effects. We next turn to a brief description of these effects as a function of N_f. More detailed discussions may be found in articles by Affleck, Harvey, and Witten (1982); Aharony, Hanany, *et al.* (1997); and de Boer, Hori, and Oz (1997).

a. $N_f=0$

The dynamics of $U(1)\subset U(N_c)$ is trivial in this case since there are no fields charged under it. It gives a decoupled factor $R\times S^1$ in the quantum moduli space corresponding to $(1/N_c)\Sigma\Phi^j$ (245). The $SU(N_c)$ dynamics is nontrivial. A nonperturbative superpotential is generated by instantons, which in three dimensions are the familiar monopoles of broken $SU(N_c)$ gauge theory. By using the symmetries of the gauge theory and the results of Veneziano and Yankielowicz (1982) and Affleck, Dine, and Seiberg (1984, 1985; for a review see Intriligator and Seiberg, 1996a and references therein) for $N_c=2$ one can compute this superpotential exactly.

Any point in the Coulomb branch can be mapped by a Weyl transformation to the Weyl chamber $X^{11}\geq X^{22}\geq\cdots\geq X^{N_cN_c}$. In this wedge the natural variables are[38]

$$Y_j = \exp\left(\frac{\Phi^j - \Phi^{j+1}}{g_3^2}\right);\ \ j=1,\ldots,N_c-1 \tag{247}$$

and one can show that the exact superpotential is

$$W = \sum_{j=1}^{N_c-1} \frac{1}{Y_j}. \tag{248}$$

This theory has no stable vacuum. The superpotential (248) tends to push the moduli Φ^i away from each other to infinity.

When the radius R of compactification from four to three dimensions is nonzero the analysis is modified. The exact superpotential for finite R is

$$W = \sum_{j=1}^{N_c-1} \frac{1}{Y_j} + \eta \prod_{j=1}^{N_c-1} Y_j \tag{249}$$

where η is related to the four-dimensional QCD scale Λ_4:

$$\eta \sim \exp\left(-\frac{1}{Rg_3^2}\right) \sim \exp\left(-\frac{1}{g_4^2}\right) \sim \Lambda_4^{3N_c-N_f}. \tag{250}$$

As $R\to 0$ at fixed g_3, $\eta\to 0$, while in the four-dimensional limit $(R\to\infty)$ η turns into an appropriate power of the QCD scale [Eq. (250)].

The superpotential (249) is stable. Vacua satisfy $\partial_j W=0$, which leads to

$$Z^{N_c}\eta^{N_c-1} = 1; \quad Z \equiv \prod_{i=1}^{N_c-1} Y_i. \tag{251}$$

Thus for all $R\neq 0$ there are N_c vacua corresponding to different phases of Z. As $R\to 0$, the vacua (251) recede to infinity. Since η remains finite as $R\to\infty$, the N_c solutions persist in the four-dimensional limit.

As we add light fundamentals $Q,\tilde{Q}$, the vacuum structure becomes more intricate due to the appearance of Higgs branches and additional parameters such as real and complex masses and Fayet-Iliopoulos D terms. As in four dimensions, classically there is already a difference between $N_f\geq N_c$ and $N_f<N_c$ massless fundamentals—in the former case the gauge group can be broken completely, while in the latter the maximal breaking is $U(N_c)\to U(N_c-N_f)$. We next turn to the quantum structure in the two cases.

[38]More precisely, the relation below is valid far from the edges of the wedge and for $R=0$; in general there are corrections to the relation between Y_j and Φ^j.

b. $N_f \leqslant N_c$

The theory with $N_f = N_c$ and vanishing real masses at finite R is described at low energies by a sigma model for N_c^2+2 chiral superfields $V_\pm, M_{\tilde{i}}^{i}$, with the superpotential

$$W = V_+ V_- (\det M + \eta). \tag{252}$$

M should be thought of as representing the meson field $M_{\tilde{i}}^{i} = Q^i \tilde{Q}_{\tilde{i}}$, $V_\pm$ parametrize the Coulomb branch, and η is given by Eq. (250). Note that most of the classical N_c-complex-dimensional Coulomb branch is lifted in the quantum theory; its only remnants are $V_\pm$. The description (252) is arrived at by a combination of holomorphicity arguments, analysis of low N_c, and inspired guesswork which we shall not review here (see Aharony, Hanany, *et al.*, 1997).

Varying Eq. (252) with respect to the fields $V_\pm$, M gives rise to the equations of motion

$$V_\pm (\det M + \eta) = 0; \quad V_+ V_- (\det M)(M^{-1})_i^{\tilde{i}} = 0. \tag{253}$$

Consider first the three-dimensional case $(R = \eta = 0)$. There are three branches of moduli space:

(1) $V_+ = V_- = 0$; M arbitrary.
(2) $V_+ V_- = 0$; M has rank at most $N_c - 1$.
(3) V_+, V_- arbitrary; M has rank at most $N_c - 2$.

The first branch can be thought of as a Higgs branch, while the last two are mixed Higgs-Coulomb branches. The three branches meet on a complex hyperplane on which the rank of M is $N_c - 2$ and $V_+ = V_- = 0$.

The understanding of the theory with $N_f = N_c$ allows us to study models with any $N_f \leqslant N_c$ by adding masses to some of the flavors and integrating them out. When we add a complex quark mass term $W = -mM$ to Eq. (252), the following structure emerges. If the rank of m is one, we find in the IR a theory with $N_f - 1 = N_c - 1$ massless flavors. Integrating out the massive flavor we find a moduli space of vacua with

$$V_+ V_- \det M = 1, \tag{254}$$

where M is the $(N_f - 1) \times (N_f - 1)$ matrix of classically massless mesons. Equation (254) implies that the classically separate Coulomb and Higgs branches merge quantum mechanically into one smooth moduli space. If the rank of m is larger than one, we find a superpotential with a runaway behavior. For example, if we add two nonvanishing masses,

$$W = V_+ V_- \det M - m_1^1 M_1^1 - m_2^2 M_2^2, \tag{255}$$

we find, after integrating out the massive mesons $M_1^i, M_j^1, M_2^i, M_j^2$,

$$W = -\frac{m_1^1 m_2^2}{V_+ V_- \det M}, \tag{256}$$

where, again, M represents the $(N_f - 2)^2$ classically massless mesons. Clearly, the superpotential (256) does not have a minimum at finite values of the fields; there is no stable vacuum.

For a mass matrix m of rank N_f,

$$W = V_+ V_- \det M - m_i^{\tilde{i}} M_{\tilde{i}}^{i}, \tag{257}$$

we make contact with the case $N_f = 0$. Integrating out the massive meson fields M gives rise to the superpotential

$$W = -(N_f - 1)\left(\frac{\det m}{V_+ V_-}\right)^{1/(N_f - 1)}. \tag{258}$$

This superpotential can be obtained from Eq. (248) by integrating out the Y_j keeping Z [Eq. (251)], and identifying it with $Z = V_+ V_-$.

When the radius of the circle is not strictly zero ($\eta \neq 0$), the analysis of Eq. (253) changes somewhat. There are now only two branches:

(1) $V_+ = V_- = 0$; M arbitrary.

(2) $V_+ V_- = 0$; $\det M = -\eta$.

In particular, there is no analog of the third branch of the three-dimensional problem. The two branches meet on a complex hyperplane on which $\det M = -\eta$ and $V_+ = V_- = 0$. The structure for all $\eta \neq 0$ agrees with the four-dimensional analysis of Sec. V.

If we add to Eq. (252) a complex mass term $W = -mM$ with a mass matrix m whose rank is smaller than N_f, the vacuum is destabilized (including the case of a mass matrix of rank one where previously there was a stable vacuum). If the rank of m is N_f, so that the low-energy theory is pure $U(N_c)$ SYM theory, there are $N_c (= N_f)$ isolated vacua which run off to infinity as the radius of the circle R goes to zero [there is also a decoupled moduli space for the $U(1)$ piece of the gauge group]. All this can be seen by adding to Eq. (258) the term proportional to Z, $\eta V_+ V_- = \eta Z$, and looking for extrema of the superpotential

$$W = -(N_f - 1) Z^{-1/N_f - 1} + \eta Z. \tag{259}$$

We next turn to the dependence of long-distance physics on the real masses of the quarks. As we saw before, real-mass terms are described by D terms (246); therefore the effective low-energy superpotential (252) is independent of these terms.

The effect of the real masses is to make some of the low-energy degrees of freedom in Eq. (252) massive. To see this, consider weakly gauging the (vector) $SU(N_f)$ flavor symmetry of Eq. (252). The real-mass matrix $m_i^{\tilde{i}}$ corresponds to the expectation values of the scalars in the $SU(N_f)$ vector multiplet. A term analogous to Eq. (243) in the Lagrangian of the $SU(N_f)$ theory will make some of the components of M massive. For a diagonal mass matrix

$$(m_r) = \mathrm{diag}(m_1, m_2, \ldots, m_{N_f}) \tag{260}$$

the off-diagonal components $M_{\tilde{i}}^{i}$ get a mass proportional to $|m_i - m_{\tilde{i}}|$. When all the real masses m_i are different, the low-energy limit is described by a sigma model for the $N_f + 2$ fields $V_+, V_-, M_1^1, M_2^2, \ldots, M_{N_f}^{N_f}$ with the superpotential [compare with Eq. (252)]

$$W = V_+ V_- (M_1^1 M_2^2 \cdots M_{N_f}^{N_f} + \eta). \tag{261}$$

More generally, if

$$(m) = \mathrm{diag}(m_1^{n_1}, m_2^{n_2}, \ldots, m_k^{n_k}), \tag{262}$$

where $\{n_i\}$ are the degeneracies of m_i and $\Sigma_i n_i = N_f$, the low-energy limit includes $V_\pm$ and k matrices M_i whose size is $n_i \times n_i$ $(i=1,\ldots,k)$. The corresponding superpotential is

$$W = V_+ V_- (\det M_1 \det M_2 \cdots \det M_k + \eta). \tag{263}$$

The moduli space corresponding to Eq. (263) is rather complicated in general. We have discussed the case of equal real masses $k=1$ before. We shall next describe the other extreme case, $k=N_f$, leaving the general analysis to the reader.

In the three-dimensional limit $\eta \to 0$, Eq. (261) describes $\binom{N_f+2}{2}$ branches in each of which two of the N_f+2 fields $\{V_\pm, M_i^i\}$ vanish. For nonzero R (or η), there are three branches:

(1) $V_\pm = 0$; M_i^i arbitrary.

(2) $V_+ = 0$, $V_- \neq 0$; $\Pi_{i=1}^{N_f} M_i^i = \eta$.

(3) $V_- = 0$, $V_+ \neq 0$; $\Pi_{i=1}^{N_f} M_i^i = \eta$.

c. $N_f > N_c$

In this case, there is no (known) description of the low-energy physics in terms of a sigma model without gauge fields. For vanishing real masses, instanton corrections again lift all but a two-dimensional subspace of the Coulomb branch, which can be parametrized, as before, by two chiral superfields $V_\pm$. The Higgs branch is similar to that of the four-dimensional theory; it is parametrized by the meson fields $M_{\tilde{i}}^i = Q^i \tilde{Q}_{\tilde{i}}$ subject to classical compositeness constraints (such that only $2N_f N_c - N_c^2$ of the N_f^2 components of M are independent). An attempt to write a superpotential for V_+, V_- and $M_{\tilde{i}}^i$ using holomorphicity and global symmetries leads in this case to

$$W = (V_+ V_- \det M)^{1/(N_f - N_c + 1)}, \tag{264}$$

which is singular at the origin, clearly indicating that additional degrees of freedom that have been ignored become massless there.

For nonvanishing real masses the phase structure becomes quite intricate and has not been analyzed using gauge-theory methods. We shall see later using brane techniques that when all the real masses are different there are $\binom{2N_f - N_c + 2}{N_c}$, N_c-dimensional mixed Higgs-Coulomb branches intersecting on lower-dimensional manifolds.

There are at least two other theories that have the same infrared limit as $N=2$ SQCD. One is the "mirror," which, like $N=4$ SYM theory, is easiest to describe using branes (de Boer, Hori, Oz, and Yin, 1997; Elitzur, Giveon, and Kutasov, 1997); we shall do this later. The other is the "Seiberg dual" (Aharony, 1997; Karch, 1997), which we shall also describe using branes below. This is a gauge theory with $G_m = U(N_f - N_c)$, N_f flavors of magnetic quarks q_i, $\tilde{q}^{\tilde{i}}$, and singlet fields $M_{\tilde{i}}^i$, $V_\pm$ which couple to the magnetic gauge theory via the superpotential

$$W = M_{\tilde{i}}^i q_i \tilde{q}^{\tilde{i}} + V_+ \tilde{V}_- + V_- \tilde{V}_+, \tag{265}$$

where $\tilde{V}_\pm$ are the effective fields parametrizing the unlifted quantum Coulomb branch of the magnetic gauge theory.

It should be emphasized that the three-dimensional "Seiberg duality" is different from its four-dimensional analog in at least two respects. The first is that it is not really a strong-weak coupling duality. In three dimensions both the electric and the magnetic descriptions are strongly coupled [with the exception of the case $N_f = N_c > 1$, where the superpotential (252) is dangerously irrelevant and, therefore, the sigma model is weakly coupled in the IR, at least at the origin of moduli space]. Thus it is less useful as a tool to study strong-coupling dynamics.

The second is that the magnetic theory is not well formulated throughout its RG trajectory. In particular, the fields $\tilde{V}_\pm$ are effective low-energy degrees of freedom that emerge after taking into account nonperturbative gauge dynamics. They are ill defined in the high-energy limit in which the magnetic theory is (asymptotically) free.

Of course, the equivalence of the electric and magnetic theories is expected to hold only in the IR, so this is not necessarily a problem for the duality hypothesis. However, it does seem to suggest that the three-dimensional Seiberg duality is a low-energy manifestation of a relation between different theories that reduce to low-energy supersymmetric Yang-Mills theory in the IR but have quite different high-energy properties. We shall next argue that the relevant theories are theories on branes.

2. Brane theory

To study four-dimensional $N=1$ SQCD compactified on a circle of radius R using branes (Elitzur, Giveon, *et al.*, 1997) we can simply compactify the corresponding type-IIA configuration (i.e., take $x^3 \sim x^3 + 2\pi R$ in Fig. 24). At large R we recover the results of Sec. V. For small R it is convenient to perform a T duality on the x^3 circle; this transforms type IIA to type IIB and turns $D4$-branes wrapped around x^3 into $D3$-branes at points on the dual circle of radius

$$R_3 = \frac{l_s^2}{R}. \tag{266}$$

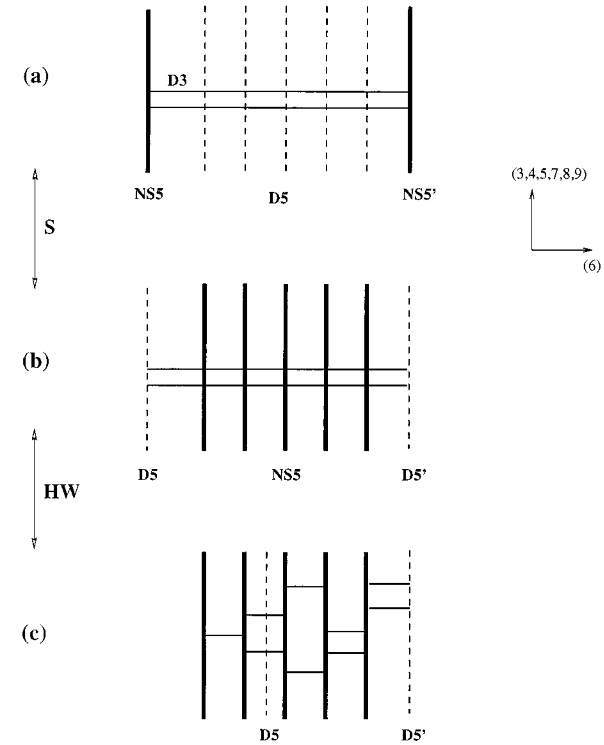

FIG. 46. The brane realization of mirror symmetry in three-dimensional $N=2$ supersymmetric Yang-Mills theory.

The NS5 and NS5$'$-branes transform to themselves, while the $D6$-branes turn into $D5$-branes at points in (x^3,x^4,x^5,x^6). We shall mostly use the type-IIB language to describe the physics.

The type-IIB brane configuration corresponding to three-dimensional $N=2$ SQCD is depicted in Fig. 46. The classical analysis of deformations and moduli mirrors closely the discussion of Sec. V. Compactification to three dimensions gives rise to a new branch of moduli space—the Coulomb branch, and new parameters in the Lagrangian—the real masses. The former correspond in the brane language to locations in x^3 of $D3$-branes stretched between the NS5- and NS5$'$-branes. The latter are given by the positions in x^3 of the N_f $D5$-branes.

Note that due to Eq. (266), as $R_3\to\infty$ we recover the three-dimensional $N=2$ SQCD theory with $\eta=0$, which was discussed in the previous section, while the four-dimensional limit corresponds to $R_3\to 0$. The three-dimensional gauge coupling is given in type-IIB language by Eq. (235), $1/g_3^2=L_6/g_s$. The four-dimensional gauge coupling is related to it by $1/g_3^2=R/g_4^2$ or, using Eq. (266),

$$\frac{1}{g_4^2}=\frac{L_6R_3}{g_s l_s^2}. \tag{267}$$

The gauge-theory limit corresponds to Eq. (236). To get a three-dimensional theory further requires $R_3\to\infty$ with L_6/g_s held fixed; the four-dimensional limit is $R_3\to 0$ with g_4 in Eq. (267) held fixed. The instanton effects which give rise to the term proportional to $\exp(-1/g_4^2)$ in Eq. (249) arise from Euclidean D strings that are stretched between the NS5- and NS5$'$-branes and are wrapped around the x^3 circle.

As discussed in the previous section, the infrared dynamics of the gauge theory in question has at least two alternative descriptions, the mirror and the Seiberg dual. Both are easy to understand using branes. To construct the mirror, we apply an S duality transformation to the electric configuration; the result is described in Fig. 46(b). The NS5- and NS5$'$-branes are exchanged with $D5$- and $D5'$-branes, while $D3$-branes are invariant. The mirror brane configuration is very similar to that found for $N=4$ SQCD in Sec. VI.A. The only difference is that one of the two $D5$-branes has been rotated into a $D5'$-brane. $N=2$ mirror symmetry was suggested by Elitzur, Giveon, and Kutasov (1997) and further investigated by de Boer, Hori, Oz, and Yin (1997).

To find the gauge symmetry of the mirror theory we break the three-branes on the NS5-branes in the most general way [Fig. 46(b)]. This leads to the gauge group $U(1)\times U(2)\times\cdots\times U(N_c-1)\times U(N_c)^{N_f-N_c}$. There is still matter in bifundamental representations of adjacent factors of the gauge group, and since $D3$-branes stretched between NS5-branes actually preserve $N=4$ supersymmetry, there are also chiral multiplets transforming in the adjoint of each factor. The $D3$-branes stretched between the $D5'$-brane and the closest NS5-brane give rise as usual to N_c scalars M_α, which couple via a cubic superpotential to N_c fundamentals of the "last" $U(N_c)$ factor. The analysis of the magnetic theory involves no new elements; details are given by de Boer, Hori, Oz, and Yin (1997).

The Seiberg dual is obtained as usual by exchanging five-branes in x^6. Since the two Neveu-Schwarz five-branes are not parallel, we expect the transition to be smooth and the resulting theory to be equivalent in the infrared to the original one. The magnetic brane configuration one is led to is in fact very similar to that obtained in the four-dimensional case, with $D4$- and $D6$-branes replaced by $D3$- and $D5$-branes. In particular, classically it seems to correspond to a $G_m=U(N_f-N_c)$ gauge theory with magnetic quarks and N_f^2 singlet mesons M with the superpotential $W=Mq\tilde{q}$. Comparing to the gauge-theory result [Eq. (265)], we seem to be missing the two fields $V_\pm$ and their couplings to the gauge degrees of freedom.

What saves the day is the fact that the equivalence between the electric and magnetic theories is expected to be a quantum feature, while our analysis of the magnetic brane configuration so far has been purely classical. Thus our next task is to study the quantum vacuum structure corresponding to the electric and magnetic brane configurations. We shall first describe the structure for the electric theory and, in particular, reproduce the gauge-theory results of the previous section. We shall then turn to the magnetic theory and show that in fact the fields $V_\pm$ are secretly present in the three-dimensional analog of Fig. 28 (but are not geometrical, like the adjoint field with a polynomial superpotential discussed in Sec. V). We shall also see evidence of the superpotential (265).

The tool we shall use to analyze the vacuum structure is the quantum brane interaction rules described in Sec. V.C.3. As explained there, these rules allow one to analyze the moduli space for widely separated branes. The behavior for branes that are close to each other has to be addressed by other means. Unfortunately, the M-theory analysis is inapplicable for type-IIB configurations and there are at present no known alternatives.

Consider the electric configuration of Fig. 46(a) with $N_f=0$. The N_c three-branes stretched between the NS five-branes repel each other; therefore the classical N_c-dimensional Coulomb branch is lifted. The repulsive potential between pairs of adjacent three-branes can be thought of in this case as due to Euclidean D strings stretched between the NS5- and NS5′-branes and between the $D3$-branes (as in Fig. 34). They correspond to instantons in the low-energy three-dimensional gauge theory. Since there are two fermionic zero modes in the presence of these instantons, they lead to a superpotential on the classical Coulomb branch.

In the three-dimensional theory [with $R_3=\infty$, or $\eta=0$ in Eq. (250)], the long-range repulsion between three-branes leads to runaway behavior, since there is no stable vacuum with the three-branes at finite distances; this is in agreement with the gauge-theory analysis of the superpotential (248). For finite R_3 (or η) the three-branes arrange around the x^3 circle at equal spacings, maximizing the distances between them and leading to an isolated vacuum. The fact that there are N_c vacua (251) has to do with the dual of the three-dimensional gauge field [Eqs. (244) and (245)] and is not expected to be seen geometrically in the current setup. As $R_3\to\infty$ the vacua run off to infinity, and we recover the previous results.

In the presence of massless quarks, in the brane description there are $D5$-branes in the system that can "screen" the interactions between the three-branes. This screening can be seen directly by studying Euclidean D strings stretched between $D3$-branes. If the worldsheet of such a D string intersects a $D5$-brane, two additional zero modes appear and the contribution to the superprotential vanishes.

For $1\leq N_f\leq N_c-2$ massless flavors we saw before that the gauge theory is unstable and exhibits a runaway superpotential [given by Eq. (256) for $N_f=N_c-2$]. In the brane picture we have N_c three-branes stretched between NS5- and NS5′-branes, and N_c-2 $D5$-branes located at the same value of x^3 (we consider only the case of vanishing real masses for now) between the NS5- and NS5′-branes.

Due to the repulsion between unscreened three-branes stretched between NS5- and NS5′-branes, N_c-2 of the N_c three-branes must break on different $D5$-branes. The s rule implies that once this has occurred, no additional three-branes attached to the NS5-brane can break on these $D5$-branes. We are left with two unbroken three-branes, one on each side of the $D5$-branes (in x^3). These three-branes repel each other, as well as the pieces of the broken three-branes stretched between the NS5′-brane and the $D5$-brane closest to it. There is no screening in this situation, since all N_c-2 $D5$-branes are connected to the NS5-brane; hence they can be removed by moving them past the NS5-brane in x^6, using the Hanany-Witten transition. The system is unstable, and some or all of the three-branes mentioned above must run away to infinity.

This is in agreement with the gauge-theory analysis of the superpotential (256). One can think of $V_\pm$ as the positions in x^3 of the two three-branes stretched between NS5- and NS5′-branes mentioned above (as usual, together with the dual of the three-dimensional gauge field). The potential obtained from Eq. (256) indeed suggests a repulsion between the different three-branes.

It is clear that the arguments above continue to hold when the radius of the circle on which the three-branes live is finite. While the two three-branes stretched between the NS5- and NS5′-branes can no longer run away to infinity in the x^3 direction, those connecting the NS5′-brane to a $D5$-brane (representing components of M) can, and there is still no stable vacuum. This is in agreement with gauge theory; adding the term $W=\eta V_+V_-$ to Eq. (256) and integrating out $V_\pm$ leads to a superpotential of the form $W\sim(\det M)^{-1/2}$.

The above discussion can be repeated with the same conclusions for all $1\leq N_f\leq N_c-2$.

For $N_f=N_c-1$ the gauge-theory answer is different; there is still no vacuum in the four-dimensional case $\eta\neq 0$, while in three dimensions there is a quantum moduli space of vacua with $V_+V_-\det M=1$. In brane theory there are now N_c-1 $D5$-branes, and the interaction between the D three-branes stretched between NS5- and NS5′-branes can be screened. Indeed, consider a situation in which N_c-2 of the N_c three-branes stretched between NS5- and NS5′-branes break on $D5$-branes. This leaves two three-branes and one $D5$-brane that is not connected to the NS5-brane. If $R_3=\infty$ (i.e., $\eta=0$), the single $D5$-brane can screen the repulsion between the two three-branes. If the three-brane is at $x^3=0$, then using the rules of Sec. V.C.3 we deduce that any configuration where one of the three-branes is at $x^3>0$ while the other is at $x^3<0$ is stable. The locations in x^3 of the two three-branes give the two moduli $V_\pm$. Thus the brane picture correctly predicts the existence of the quantum moduli space and its dimension. The precise shape of the moduli space (the relation $V_+V_-\det M=1$) is a feature of nearby branes and thus is expected to be more difficult to see; nevertheless, it is clear that due to the repulsion there is no vacuum when either V_+ or V_- vanish.

If the radius of the fourth dimension R is not zero, there is a qualitative change in the physics. Since R_3 is now finite, the two three-branes stretched between NS5 and NS5′-branes are no longer screened by the $D5$-brane—they interact through the other side of the circle. Thus one of them has to break on the remaining $D5$-brane, and one remains unbroken because of the s rule. The repulsion between that three-brane and the three-branes stretched between the NS5′-brane and a

$D5$-brane which is no longer screened leads to vacuum destabilization, in agreement with the gauge-theory analysis.

For $N_f=N_c$ (and vanishing real masses) the brane-theory analysis is similar to the previous cases, and the conclusions are again in agreement with gauge theory. For $R_3=\infty$ one finds three phases corresponding to a pure Higgs phase in which there are no three-branes stretched between NS5- and NS5$'$-branes, and two mixed Higgs-Coulomb phases in which there are one or two three-branes stretched between the NS5- and NS5$'$-branes; the locations of the three-branes in x^3 are parametrized by $V_\pm$. When there are two unbroken three-branes, they must be separated in x^3 by the $D5$-branes, which provide the necessary screening.

For finite R_3, the structure is similar, except for the absence of the branch with two unbroken three-branes, which is lifted by the same mechanism as that described in the case $N_f=N_c-1$ above. $N_f>N_c$ works in the same way as in the four-dimensional case described earlier.

So far we have discussed the electric theory with vanishing (more generally equal) real masses for the quarks. Turning on real masses gives rise to a rich phase structure of mixed Higgs-Coulomb branches. We have seen an example in the theory with $N_f=N_c$. The three-dimensional theory with $\eta=0$ and vanishing real masses, which is equivalent in the infrared to the sigma model [Eq. (252)], has three branches described after Eq. (253). When all the real masses are different, the corresponding sigma model [Eq. (261)] has $(N_f+1)(N_f+2)/2$ N_f-dimensional branches intersecting on lower-dimensional manifolds. For $N_f>N_c$ the problem has not been analyzed in gauge theory.

Branes provide a simple way of studying the phase structure. As an example we shall describe it for the case of arbitrary $N_f\geqslant N_c$ with all N_f real masses different. The generalization to cases in which some of the real masses coincide is straightforward.

The brane configuration includes in this case N_f $D5$-branes at different values of x^3. The N_c $D3$-branes can either stretch between the NS5- and NS5$'$-branes or split into two components on $D5$-branes. Different branches of moduli space correspond to different ways of distributing $D3$-branes between the two options, in such a way that there are no unscreened interactions.

Stability implies that any two $D3$-branes stretched between the NS5- and NS5$'$-branes must be separated (in x^3) by a $D5$-brane that screens the repulsive interaction between them. Similarly, a $D3$-brane stretched between the NS5- and NS5$'$-branes and a second one that is broken into two components on a $D5$-brane must be separated by such a $D5$-brane.

To describe the different branches of the quantum moduli space of vacua we have to place the N_c $D3$-branes in such a way that there are no repulsive interactions. Each three-brane can either be placed between two $D5$-branes or on top of one. There are therefore $N_f+(N_f+1)=2N_f+1$ possible locations for $D3$-branes corresponding to the N_f $D5$-branes and the N_f+1 spaces between and around them. The quantum interactions between branes mean that one cannot place two three-branes at adjacent locations. The number of branches is, therefore, the number of ways to distribute N_c identical objects between $2N_f+1$ slots, with at most one object per slot and no two objects sitting in adjacent slots. It is not difficult to show that this number is

$$\binom{n}{k}; \quad n\equiv 2N_f-N_c+2, \quad k\equiv N_c. \tag{268}$$

We next turn to the magnetic brane configuration. Naively one expects it to describe a $U(\bar{N}_c)$ gauge theory with N_f fundamentals $q,\tilde{q}$, a magnetic meson M, and the standard superpotential

$$W=M\tilde{q}q. \tag{269}$$

To see whether this is in fact correct, we shall study the resulting theory for the special case $\bar{N}_c=N_f$ and compare the vacuum structure of the gauge theory to that of the brane configuration, which is described by a three-dimensional analog of Fig. 28. We shall furthermore discuss only the three dimensional limit $\eta=0$. Classically, the two definitely agree. The theory has an $(N_f^2+N_f)$-dimensional moduli space of vacua; in the brane language it corresponds to independent motions of the color branes along the Neveu-Schwarz five-branes and to breaking of flavor three-branes on different $D5$-branes. In gauge theory it is parametrized by expectation values of M and the Φ^j [Eq. (245)]. Quantum mechanically, one finds a discrepancy, which we describe next.

Before turning on the Yukawa superpotential (269), we can describe the low-energy dynamics of the gauge theory in question by the sigma model (252) for the fields $\tilde{M}\equiv\tilde{q}q$ and $\tilde{V}_\pm$ which parametrize the potentially unlifted Coulomb branch of the theory. Coupling the "magnetic quarks" $q,\tilde{q}$ to the singlet meson M leads to a low-energy sigma model with the superpotential

$$W=\tilde{V}_+\tilde{V}_-\det\tilde{M}+M\tilde{M}. \tag{270}$$

Varying with respect to M sets $\tilde{M}=0$. Therefore we conclude that the quantum gauge theory in question has a two-complex-dimensional moduli space of vacua parametrized by arbitrary expectation values of the fields $\tilde{V}_\pm$.

On the other hand, the brane configuration has a unique vacuum at the origin where all $\bar{N}_c=N_f$ three-branes are aligned and can be thought of as stretching between the NS5-brane and the N_f $D5$-branes. This is the only stable configuration, taking into account the repulsive interactions between color three-branes and the attractive interactions between color and flavor three-branes.

We conclude that the gauge theory leading to the low-energy sigma model [Eq. (270)] cannot provide a full description of the physics of the brane configuration of Fig. 28. Elitzur, Giveon, *et al.*, 1997 propose that the magnetic brane configuration is in fact described by the above gauge theory, but there are two more fields $V_\pm$ that are singlets under the $U(\overline{N}_c)$ gauge group and that contribute the term

$$W_V = V_+ \widetilde{V}_- + V_- \widetilde{V}_+ \quad (271)$$

to the low-energy superpotential. Combining Eq. (270) with Eq. (271) clearly gives the right quantum vacuum structure for $\overline{N}_c = N_f$ and also more generally. Note that the term (271) is just what has been seen to be needed in gauge theory to generalize Seiberg's duality to three dimensions [Eq. (265)]! In particular, it can be used to make sense of the dual theory, which as we discussed in the previous section is not really well defined as a local quantum field theory. We see that the high-energy theory that underlies Eq. (265) is best thought of as the theory on the web of branes described by the three-dimensional Fig. 28.

The fields $V_\pm$ and their interactions (271) are not seen geometrically in the brane configuration. This makes it more difficult in general to compare the vacuum structure of three-dimensional Seiberg duals. Nevertheless, in all cases that have been checked, no disagreement has been found, supporting the proposed duality. Some tests of the equivalence of the theories with vanishing real masses have been made by Aharony (1997). We have further checked the other extreme case of N_f *different* real masses in a few examples and find agreement. For $N_f = N_c$ the magnetic theory can be shown to reduce to the sigma model (252), or when all the real masses are different, Eq. (261). As we have seen above, the magnetic moduli space has in this case $(N_f+2)(N_f+1)/2$ branches, in agreement with the electric theory [Eq. (268)].

For $N_f = N_c + 1$, the magnetic theory reduces in the infrared to a $U(1)$ gauge theory with N_f flavors and the superpotential (265). By using the results of Aharony, Hanany, *et al.* (1997) one can check that the phase structure of the magnetic theory is again in agreement with Eq. (268). It would be interesting to check agreement for arbitrary $N_f > N_c$.

VII. TWO-DIMENSIONAL THEORIES

A. Field-theory results

Two-dimensional gauge theories with $N=(4,4)$ supersymmetry can be obtained by the dimensional reduction of six-dimensional $N=(1,0)$ theories (or four-dimensional $N=2$ theories). The $(1+5)$-dimensional Lorentz symmetry is broken to $SO(1,1)\times\text{spin}(4)$ and the latter combines with the R symmetry to a spin(4) $\times SU(2)_R$ global symmetry group.

As in four-dimensional $N=2$ theories, two-dimensional (4,4) gauge theories have two massless representations: a hypermultiplet and a vector multiplet (also called a twisted multiplet). In terms of an $N=(2,2)$ superalgebra the hypermultiplets decompose into two chiral multiplets [see, for example, Witten, 1993 for a review of two-dimensional $N=(2,2)$ theories]. The scalars in these multiplets parametrize a "Higgs branch"[39] which is a Hyper-Kähler manifold. The vector multiplet decomposes into a chiral multiplet and a twisted chiral multiplet. The scalars in these superfields parametrize the "Coulomb branch," which is characterized by a generalized Kähler potential determining the metric and torsion on target space (Gates, Hull, and Roček, 1984).

Next we consider $U(1)$ gauge theories with N_f "electron" hypermultiplets. The Coulomb branch is parametrized by the expectation values $\vec{\phi} \in R^4$ of the four scalars in the twisted multiplet. Classically, the metric on the Coulomb branch is flat. Quantum mechanically, the metric receives a contribution whose form is fixed by hyper-Kähler geometry and whose normalization can be determined by an explicit one-loop computation. In the massless case the metric is (Roček, Schoutens, and Sevrin, 1991)

$$ds^2 = \left(\frac{1}{g_2^2} + \frac{N_f}{\vec{\phi}^2}\right) d\vec{\phi}^2, \quad (272)$$

where g_2 is the $2d$ gauge coupling. The coefficient in front of $d\vec{\phi}^2$ is the effective gauge coupling.

Turning on bare masses $\vec{m}_i$, $i=1,\dots,N_f$, to the hypermultiplets the metric becomes

$$ds^2 = \left(\frac{1}{g_2^2} + \sum_{i=1}^{N_f} \frac{1}{|\vec{\phi}-\vec{m}_i|^2}\right) d\vec{\phi}^2. \quad (273)$$

One notes that this is precisely the form of the metric of a $2d$ conformal field theory describing the propagation of a string near N_f parallel NS five-branes (17). We shall see in the next section that this is not an accident. Moreover, the torsion $H = dB$ on the Coulomb branch is also given by Eq. (17).

As in previous sections, it will also be interesting to consider a compactification, in this case from three to two dimensions on a circle of radius R. The Coulomb branch is four dimensional and is parametrized by the expectation values of the three scalars in the vector multiplet, $\vec{\rho} \in R^3$, and the scalar σ dual to the $3d$ gauge field; σ lives on a circle, $\sigma \sim \sigma + 1/R$. The metric on the Coulomb branch now takes the form (Diaconescu and Seiberg, 1997)

$$ds^2 = 2\pi R \left(\frac{1}{g_3^2} + \sum_{i=1}^{N_f} \frac{1}{|\vec{\rho}-\vec{m}_i|} \times \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} e^{-2\pi R n |\vec{\rho}-\vec{m}_i|} \cos[2\pi R n(\sigma-\sigma_i)] \right\} \right) \times (d\vec{\rho}^2 + d\sigma^2). \quad (274)$$

The coefficient in front of $d\vec{\rho}^2 + d\sigma^2$ is the effective gauge coupling of a $3d$ theory compactified to $2d$ on S_R^1

[39] One should keep in mind that there is no moduli space in two dimensions and we thus work in the Born-Oppenheimer approximation.

and g_3 is the three-dimensional coupling constant, which is related to the two-dimensional coupling constant by standard dimensional reduction:

$$\frac{1}{g_2^2}=\frac{2\pi R}{g_3^2}. \tag{275}$$

When $R\to 0$ the metric Eq. (274) approaches Eq. (273) with $\vec{\phi}\equiv(\vec{\rho},\sigma)$. For large compactification radius, $R\gg 1/|\vec{\rho}|$, the effective gauge coupling becomes

$$\frac{1}{g_2^2}+\pi R\sum_{i=1}^{N_f}\frac{1}{|\vec{\rho}-\vec{m}_i|}. \tag{276}$$

This is similar to the metric (23) on an ALE space with a resolved A_{N_f-1} singularity, which appeared when we discussed the metric felt by a string in the presence of N_f parallel Kaluza-Klein monopoles. Again, as we shall discuss later, this is not an accident.

For $SU(2)\simeq Sp(1)$ gauge group with N_f "quark" hypermultiplets, the metric on the Coulomb branch of the two-dimensional (4,4) theory is (Diaconescu and Seiberg, 1997)

$$ds^2=\left(\frac{1}{g_2^2}+\sum_{i=1}^{N_f}\left\{\frac{1}{|\vec{\phi}-\vec{m}_i|^2}+\frac{1}{|\vec{\phi}+\vec{m}_i|^2}\right\}-\frac{2}{\vec{\phi}^2}\right)d\vec{\phi}^2. \tag{277}$$

This metric is related to an asymptotically locally Euclidean (ALE) space with a resolved D_{N_f} singularity for reasons that we shall point out later.

Next we turn to $N=(2,2)$ supersymmetric gauge theories in two dimensions. (2,2) superconformal field theories in $2d$ were studied, in particular, in the context of standard perturbative string compactifications since they lead to spacetime supersymmetric vacua. Here we shall only touch upon a small class of (2,2) theories.

Two-dimensional $N=(2,2)$ theories can be obtained by dimensional reduction of four-dimensional $N=1$ supersymmetric theories. Since anomaly constraints are milder in $2d$, generic *chiral*, anomalous $4d$ gauge theories typically lead to consistent $2d$ theories. Therefore we may consider gauged linear sigma models like a $U(N_c)$ gauge theory with n quarks Q in the fundamental representation $\mathbf{N}_c$ and $\tilde{n}$ antiquarks $\tilde{Q}$ in the antifundamental $\overline{\mathbf{N}}_c$, where n is *not* necessarily equal to $\tilde{n}$ (for a recent review and further references on such theories see Hanany and Hori, 1998). When $\tilde{n}=0$ and $N_c=1$ this is the CP^{n-1} model. When $\tilde{n}=0$ and $N_c>1$ this theory is called the "Grassmanian model." The space of its classical vacua is the complex Grassmanian manifold $G(N_c,n)$. The dynamics of vacua of the sigma model with target space $G(N_c,n)$ is described by the $U(N_c)/U(N_c)$ gauged Wess-Zumino-Witten model with the level k of $SU(N_c)$ being

$$k=n-N_c. \tag{278}$$

In this case there is a "level-rank duality" (see Nakanishi and Tsuchiya, 1992, and references therein) which exchanges

$$N_c\leftrightarrow k. \tag{279}$$

This duality is the statement that the space of conformal blocks of an $SU(N_c)$ WZW model at level k is identical to the one of $SU(k)$ at level N_c and, therefore, the topological theory $U(N_c)/U(N_c)$ at level k is equivalent to the theory $U(k)/U(k)$ at level N_c.

B. Brane theory I: (4,4) theories

Two-dimensional unitary gauge theories with (4,4) supersymmetry appear on D strings near $D5$-branes. In particular, the low-energy theory on a $D1$-brane stretched in (x^0,x^1) near N_f parallel $D5$-branes stretched in $(x^0,x^1,x^2,x^3,x^4,x^5)$ is a $U(1)$ gauge theory with N_f flavors. The metric on the Coulomb branch of the theory—parametrized by the location of the $D1$-brane in the four directions transverse to the $D5$-branes $l_s^2\vec{\phi}=(x^6,x^7,x^8,x^9)$—should be equal to the background metric of a D string in the presence of N_f parallel $D5$-branes located at $l_s^2\vec{m}_i$, $i=1,\dots,N_f$. This type-IIB system is S dual to a fundamental string in the presence of N_f parallel NS five-branes. This explains the relation between the metric (273) (and torsion) on the Coulomb branch and those of a string propagating in the background of solitonic five-branes [Eq. (17)].

Three-dimensional gauge theories with eight supercharges compactified to two dimensions on S_R^1 can be studied on $D2$-branes stretched in (x^0,x^1,x^6) near N_f $D6$-branes stretched in $(x^0,x^1,x^2,x^3,x^4,x^5,x^6)$, both wrapping a circle of radius R in the x^6 direction. Consider a single $D2$-brane. T_6 duality (42) maps it to a $D1$-brane near N_f $D5$-branes at points on a transverse circle of radius

$$R_6=l_s^2/R. \tag{280}$$

The background metric of a five-brane transverse to a circle in the x^6 direction and located, say, at $(x^6,x^7,x^8,x^9)=0$ can be obtained by considering an infinite array of five-branes separated by a distance $2\pi R_6$. It gives rise to an H-monopole background with metric and torsion given by (Gauntlett, Harvey, and Liu, 1993)

$$\begin{aligned}
&G_{IJ}=e^{2(\Phi-\Phi_0)}\delta_{IJ};\quad I,J,K,M=6,7,8,9,\\
&H_{IJK}=-\epsilon_{IJKM}\partial^M\Phi,\\
&e^{2(\Phi-\Phi_0)}-1=\sum_{n=-\infty}^{\infty}\frac{l_s^2}{\vec{x}^2+(x^6-2\pi R_6 n)^2}\\
&\qquad=\frac{l_s^2}{2R_6x}\frac{\sinh(x/R_6)}{\cosh(x/R_6)-\cos(x^6/R_6)}\\
&\qquad=\frac{l_s^2}{2R_6x}\left\{1+\sum_{n=1}^{\infty}e^{-nx/R_6}\cos(nx^6/R_6)\right\},
\end{aligned} \tag{281}$$

where $x=|\vec{x}|$, $\vec{x}=(x^7,x^8,x^9)$. From Eqs. (280) and (281) we see that the metric on the Coulomb branch (274) is precisely the metric of N_f H monopoles located at

$(x_i^6,\vec{x}_i)=2\pi l_s^2(\sigma_i,\vec{m}_i)$, $i=1,\ldots,N_f$. We thus see again how the geometry is probed by D-branes.

In the limit $R_6\to\infty$ we obtain a system of N_f five-branes in noncompact space which was discussed above. This is compatible with the fact that in this limit (281) behaves like $1/\vec{\phi}^2$, where $\vec{\phi}=(x^6,\vec{x})/2\pi l_s^2$.

On the other hand, in the limit $R\to\infty$ we get on the $D2$-brane a $(1+2)$-dimensional $N=2$ SUSY $U(1)$ gauge theory with N_f flavors. The background metric of the $D6$-branes should be related to the metric on the Coulomb branch of that gauge theory. As discussed in Sec. II, $D6$-branes are Kaluza-Klein monopoles in M theory, and they are described by the same metric as Kaluza-Klein monopoles in type-IIA string theory. A Kaluza-Klein monopole with charge R/l_s (Taub-NUT) is related by T duality (in an appropriate sense) to an H monopole (281) for any value of R (Gregory, Harvey, and Moore, 1997). In particular, when $R\to\infty$ Eq. (281) behaves like $R/|\vec{\rho}|$, where $\vec{\rho}=\vec{x}/2\pi l_s^2$, which is compatible with a Coulomb branch with effective coupling (276). Metrics similar to these showed up already in other (related) situations in this review, such as in Secs. III.D and VI.A.

A (4,4) SUSY gauge theory in two dimensions with gauge group $Sp(1)\simeq SU(2)$ can be obtained on a $D1$-brane (and its mirror image) near an orientifold five-plane parallel to N_f $D5$-branes (and their N_f mirror images). On a transverse circle of radius R_6 it describes a compactification from three to two dimensions on a circle of radius R (280). T duality in this transverse direction gives instead an $O6$-plane parallel to $2N_f$ $D6$-branes, whose background metric is related to an ALE space with resolved D_{N_f} singularity (Seiberg, 1996a; Seiberg and Witten 1996; Sen, 1997d). T dualizing back to the original system (and taking $R_6\to\infty$) gives rise to the metric (277) in agreement with $2d$ field theory.

An alternative way to study (4,4) $2d$ theories on branes is to allow branes to end on branes, as in previous sections. A typical configuration involves N_c $D2$-branes stretched between two NS5-branes, with N_f $D4$-branes located between them (or, equivalently by a Hanany-Witten transition, N_f $D4$-branes to the left of the left NS-brane or to the right of the right NS-brane, each connected to the NS-brane by a single $D2$-brane). This configuration is T dual to configurations preserving eight supercharges which were studied in previous sections (Figs. 11 and 14). The branes involved here have worldvolumes

$$\begin{aligned} &\text{NS5:} \quad (x^0,x^1,x^2,x^3,x^4,x^5),\\ &D2: \quad (x^0,x^1,x^6),\\ &D4: \quad (x^0,x^1,x^7,x^8,x^9). \end{aligned} \tag{282}$$

The low-energy theory on the $D2$-branes is a $U(N_c)$ gauge theory with N_f quark flavors and classical gauge coupling:

$$\frac{1}{g_2^2}=\frac{L_6 l_s}{g_s}, \tag{283}$$

where L_6 is the distance between the two NS-branes in x^6 (as before, we consider the limit $g_s,l_s\to 0$ such that g_2 is held fixed). The locations of the $D2$-branes along the NS-branes $\vec{r}_a=(x_a^2,x_a^3,x_a^4,x_a^5)$, $a=1,\ldots,N_c$, parametrize the Coulomb branch of the theory. The locations of the $D4$-branes $\vec{r}_i=(x_i^2,x_i^3,x_i^4,x_i^5)$, $i=1,\ldots,N_f$, are the bare masses of quark hypermultiplets. Higgsing corresponds to breaking $D2$-branes on $D4$-branes, and the relative motion of the two NS-branes in (x^7,x^8,x^9) corresponds to a Fayet-Iliopoulos D term.

When $\vec{r}_a=\vec{r}_i=0$, the brane configuration is invariant under rotations in (x^2,x^3,x^4,x^5) and (x^7,x^8,x^9). These $\mathrm{spin}(4)_{2345}$ and $SU(2)_{789}$ rotations, respectively, are associated with the global R symmetries of the (4,4) gauge theory.

The interpretation of the torsion on the Coulomb branch in the brane picture is the following (Brodie 1997). A $D2$-brane ending on an NS five-brane looks like a string in the (2,0) six-dimensional theory on the five-brane. Strings in six dimensions couple to the self-dual two-form B, which is identified with the $2d$ B field. Each fundamental hypermultiplet corresponds to a $D2$-brane ending on an NS-brane and contributes to the torsion.

Quantum mechanically, the NS-branes bend due to Coulomb-like interactions in four dimensions [Eq. (51)]. For simplicity, we consider the $U(1)$ theory: a single $D2$-brane located at $\vec{r}$. As in previous sections, the resulting effective gauge coupling is given by the distance between the NS-branes in x^6 as a function of $\vec{r},\vec{r}_i$:

$$\frac{x^6 l_s}{g_s}=\frac{1}{g_{eff}^2}=\frac{1}{g_2^2}+\sum_{i=1}^{N_f}\frac{l_s^4}{|\vec{r}-\vec{r}_i|^2}. \tag{284}$$

This is indeed the exact effective coupling in field theory (273) with $\vec{\phi}\equiv\vec{r}/l_s^2$, $\vec{m}_i\equiv\vec{r}_i/l_s^2$.

As usual, the type-IIA configuration at finite g_s is equivalent to M theory on a compact circle of radius $R_{10}=g_s l_s$. The relative location of the "NS five-branes" in x^{10} corresponds to a "θ angle." This θ parameter together with the Fayet-Iliopoulos D term—the relative position of the five-branes in (x^7,x^8,x^9,x^{10})—combines into a "quaternionic Kähler form."

In M theory, the $SU(2)_R$ symmetry is enhanced to a $\mathrm{spin}(4)_{7,8,9,10}$. Indeed, Witten (1997c) argues that this should happen in field theory. It thus seems from the brane picture that quantum mechanically there is a "mirror symmetry" interchanging masses with the Fayet-Iliopoulos D term and theta parameters, and the Coulomb branch with the Higgs branch. For more details we refer the reader to Brodie (1997).

Brane configurations giving rise to three-dimensional gauge theories compactified to two dimensions on a circle of radius R can be studied using the above configurations by compactifying x^2 on a circle of radius $R_2=l_s^2/R$ (or their T_2-dual versions). In particular, the NS five-branes now bend due to Coulomb-like forces in $R^3\times S^1$. The solution to the Laplace equation in this case gives rise to a distance in x^6 that is compatible with the field-theory effective gauge coupling given in Eq. (274).

Finally, we may add to the configurations above an orientifold two-plane (four-plane) parallel to the $D2$-branes ($D4$-branes) and obtain symplectic or orthogonal gauge groups in two dimensions. For example, considering two $D2$-branes stretched between the NS5-branes in the presence of an $O2$-plane, together with $2N_f$ $D4$-branes, gives rise to either an $SO(2)\simeq U(1)$ or an $Sp(1)\simeq SU(2)$ gauge theory, depending on the sign of the orientifold charge. The sign flip of the orientifold charge and the Coulomb-like interactions associated with $D2$-branes, their mirror images, and the $O2$-plane, give rise to a bending of the $NS5$-branes, which is in agreement with the field-theory results (273) and (277):

$$\frac{x^6 l_s}{g_s}=\frac{1}{g_{eff}^2}=\frac{1}{g_2^2}+\sum_{i=1}^{N_f}\left\{\frac{l_s^4}{|\vec{r}-\vec{r}_i|^2}+\frac{l_s^4}{|\vec{r}+\vec{r}_i|^2}\right\} -\frac{(1+1)(1+1)l_s^4}{|2\vec{r}|^2}\pm\frac{(1/2+1/2)l_s^4}{|\vec{r}|^2}. \tag{285}$$

The second term on the right-hand side of Eq. (285) is due to the N_f flavors and their N_f mirror images, the third term is due to a $D2$-brane at $\vec{r}$ having its mirror image at $-\vec{r}$, and the last term is due to the $O2$-plane—the "$\pm$" corresponding to orthogonal or symplectic projections, respectively. Obviously, this discussion can be generalized to other dimensions and to compactifications from high to lower dimensions performing the analysis with an orientifold charge and Coulomb-like interactions in the appropriate dimension.[40]

Brane configurations corresponding to two-dimensional (4,4) gauge theories were also considered by Alishahiha (1998) and Ito and Maru (1998).

C. Brane theory II: (2,2) theories

We may now rotate branes in the configurations of the previous section and get at low-energy two-dimensional $N=(2,2)$ supersymmetric gauge theories on the $D2$-branes. As an example, we shall examine a configuration of an NS5-brane connected to an NS5$'$-brane by N_c $D2$-branes in the presence of N_f $D4$-branes. The worldvolumes of the various objects are given in Eqs. (173) and (282).

A new ingredient that appears in such a brane configuration is the possibility of putting a $D4$-brane at the same x^6 location as the NS5$'$-brane, then breaking it and separating the two semi-infinite pieces along the NS5$'$-brane in the σ direction:

$$\sigma\equiv x^2+ix^3. \tag{286}$$

One may break all the N_f $D4$-branes on the NS5$'$-brane and take part of the semi-infinite $D4$-branes to infinity. One then obtains a configuration in which, say, n semi-infinite $D4$-branes—stretched in $x^7>0$—are located at σ_i, $i=1,\ldots,n$, along the NS5$'$-brane, and $\tilde{n}$ semi-infinite $D4$-branes—stretched in $x^7<0$—are located at $\sigma_{\tilde{i}}$, $\tilde{i}=1,\ldots,\tilde{n}$ (see Fig. 47).

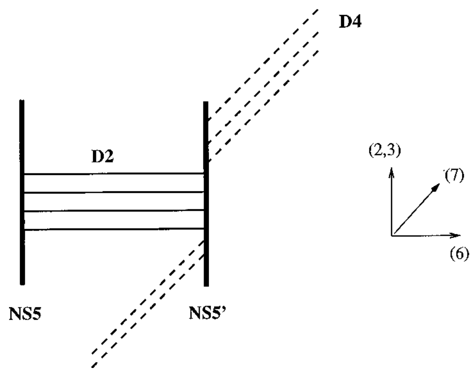

FIG. 47. $D4$-branes oriented as explained in the text, splitting into two disconnected components and separating along a Neveu-Schwarz five-brane in (x^2,x^3).

The low-energy theory on the $D2$-branes is a two-dimensional (2,2), $U(N_c)$ gauge theory with n quarks and $\tilde{n}$ antiquarks. There is a manifest chiral flavor symmetry $U(n)\times U(\tilde{n})$ which is broken for generic values of σ_i, $\sigma_{\tilde{i}}$. One can check that the classical moduli space of vacua and deformations of the brane configuration (almost) agree with a field-theory analysis (Hanany and Hori, 1997).

In M theory, the NS5-brane and $D4$-branes ending on the NS5$'$-brane turn into two disconnected $M5$-branes. The type-IIA $D2$-branes turn into $M2$-branes connecting these $M5$-branes. The dynamics of such open membranes stretched between two five-branes is not completely understood; nevertheless, chiral features of the quantum (2,2) theories can be studied in this way. We refer the reader to Hanany and Hori (1998) for details.

We should remark that the Coulomb-like interactions associated with the $D4$-branes ending on the NS5$'$-brane give rise to terms that are logarithmic in σ and that contribute to the quantum low-energy $2d$ effective superpotential. Logarithmic effective superpotentials are indeed familiar in such two-dimensional (2,2) theories (namely, in gauged linear sigma models). Other relations between the parameter space of (2,2) theories in two dimensions and the moduli space of $N=2$ four-dimensional theories—associated with the $D4$-branes ending on the NS5$'$-brane—are discussed by Hanany and Hori (1998).

Finally, let us consider a duality trajectory interchanging the NS5- and NS5$'$-brane in the x^6 direction. The details of this process can be worked out along the lines of previous sections [up to an ambiguity which is resolved quantum mechanically in M theory (Hanany and Hori, 1998)]. Here we shall only state the result in the case $\tilde{n}=0$, namely, for a $G(N_c,n)$ model (see Sec. VII.A). The duality trajectory takes an electric $U(N_c)$ theory with n quarks to a magnetic $U(n-N_c)$ theory with n quarks, i.e.,

[40]This was done explicitly in an unpublished work (A. Giveon and M. Roček, 1997)

$$G(N_c,n)\leftrightarrow G(n-N_c,n), \tag{287}$$

providing a brane realization of the level-rank duality (278) and (279), discussed in Sec. VII.A.

VIII. FIVE- AND SIX-DIMENSIONAL THEORIES

A. Five-dimensional field-theory results

$N=1$ supersymmetric gauge theories in five dimensions have eight supercharges and an $SU(2)_R$ global symmetry. The two possible multiplets in the theory are the vector multiplet in the adjoint of the gauge group G—containing a vector field, a real scalar ϕ, and fermions—and the hypermultiplet in a representation R_f of G—containing four real scalars and fermions. The Coulomb branch is parametrized by the scalar components of the vector multiplet ϕ^i, $i=1,\ldots,$ rank G, in the Cartan subalgebra of G.

The low-energy theory is determined by the prepotential $\mathcal{F}(\phi)$, which is required to be at most cubic due to $5d$ gauge invariance (Seiberg, 1996b). The exact quantum prepotential is given by (Intriligator, Morrison, and Seiberg, 1997).

$$\mathcal{F}=\frac{1}{2g_0^2}\phi^i\phi_i+\frac{c_{cl}}{6}d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}\left(\sum_{\alpha}|\alpha_i\phi^i|^3-\sum_f\sum_{w\in R_f}|w_i\phi^i+m_f|^3\right). \tag{288}$$

Here g_0 is the bare coupling of the gauge theory and d_{ijk} is the third-rank symmetric tensor: $d_{abc}=\mathrm{Tr}(T_a\{T_b,T_c\})/2$. The first sum in Eq. (288) is over the roots of G and the second sum is over the weights of the representation R_f of G; m_f are the (real) masses of the hypermultiplets in R_f. c_{cl} is a quantized parameter of the theory, related to a $5d$ Chern-Simons term. In terms of $\mathcal{F}$ the effective gauge coupling is

$$\left(\frac{1}{g^2}\right)_{ij}=\frac{\partial^2\mathcal{F}}{\partial\phi^i\partial\phi^j}. \tag{289}$$

From now on we discuss simple groups G with N_f hypermultiplets in the fundamental representations of G, as follows:

- $G=SU(N_c)$ $(N_c>2)$: The Coulomb branch of the moduli space is given by $\phi=\mathrm{diag}(a_1,\ldots,a_{N_c})$ with $\Sigma_{i=1}^{N_c} a_i=0$. The prepotential in this case is

$$\mathcal{F}=\frac{1}{2g_0^2}\sum_{i=1}^{N_c}a_i^2+\frac{1}{12}\left(2\sum_{i<j}^{N_c}|a_i-a_j|^3 +2c_{cl}\sum_{i=1}^{N_c}a_i^3-\sum_{f=1}^{N_f}\sum_{i=1}^{N_c}|a_i+m_f|^3\right). \tag{290}$$

The conditions on N_c, N_f, and c_{cl} in Eq. (288) are

$$c_{cl}+\frac{1}{2}N_f\in Z \tag{291}$$

$$SU(N_c):\quad N_f+2|c_{cl}|\leq 2N_c. \tag{292}$$

- $G=SU(2)$: This case is somewhat special. There are two pure gauge theories labeled by a Z_2-valued theta angle, since $\pi_4[SU(2)]=Z_2$. c_{cl} is irrelevant since $d_{ijk}=0$ in Eq. (288), and the number of flavors allowed is $N_f\leq 7$.
- $G=SO(N_c)$ $(Sp(N_c/2))$: In this case $c_{cl}=0$ and

$$SO(N_c)(Sp(N_c/2)),\quad N_f\leq N_c-4(N_f\leq N_c+4). \tag{293}$$

The inequalities in Eqs. (292) and (293) are necessary conditions to have nontrivial fixed points, which one can use to define the $5d$ gauge theory.

In five dimensions there are no instanton corrections to the metric and therefore the exact results considered above are obtained already at one loop. However, compactifying the theory gives rise to nonperturbative corrections. Supersymmetric $5d$ gauge theories compactified to four dimensions on a circle have been studied by Ganor (1997); Ganor, Morrison, and Seiberg (1998); Nekrasov (1998); and Nekrasov and Lawrence (1998). The perturbative contributions to $\mathcal{F}$ from Kaluza-Klein modes were found, and were shown to obtain the correct behavior in the five- and four-dimensional limits. Nonperturbative corrections to $\mathcal{F}$ are conjectured to be related to spectral curves of relativistic Toda systems.

B. Webs of five-branes and five-dimensional theories

We begin by considering a $D5$-brane ending (classically) on an NS5-brane (Aharony and Hanany, 1997). Recall (Sec. II.D) that type-IIB five-branes sit in a (p,q) multiplet of the $SL(2,Z)$ S-duality group, where the NS5-brane is a (0,1) five-brane while the $D5$-brane is a (1,0) five-brane. We choose the worldvolumes of these five-branes to be

$$\begin{aligned}&\mathrm{NS5}(0,1):\quad (x^0,x^1,x^2,x^3,x^4,x^5),\\ &D5(1,0):\quad (x^0,x^1,x^2,x^3,x^4,x^6).\end{aligned} \tag{294}$$

Classically, we may let the $D5$-brane end on the NS five-brane, say from the left in x^6. Such a configuration is allowed—as discussed in Sec. II.E.3—and it is T dual to situations where $D4$- or $D3$-branes are ending on NS five-branes as in previous sections. Therefore this configuration preserves eight supercharges.

Quantum mechanically the NS five-brane bends. Its bending is due to the fact that the $D5$-brane ending on the NS five-brane looks like an electric charge in one dimension. This causes a linear Coulomb-like interaction [see Eq. (51)], which leads to the bending of the NS5-brane in the (x^5,x^6) plane into the location

$$x^6=\frac{g_s}{2}(|x^5|+x^5). \tag{295}$$

Here and below we set $a=0$ in the complex type-IIB string coupling τ (35), for simplicity. Moreover, without loss of generality, we chose the intersection of the five-

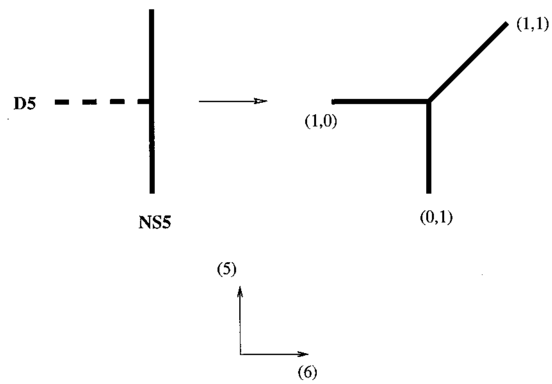

FIG. 48. The classical configuration of a $D5$-brane ending on a Neveu-Schwarz five-brane, replaced for finite g_s by a vertex in which (1,0) and (0,1) five-branes merge into a (1,1) five-brane.

branes to be located at the origin $(x^5,x^6)=0$ and the extension of the NS5-brane to be as in Eq. (294) when $x^5<0$.

While we started classically from a semi-infinite five-brane ending on an infinite five-brane, what we have obtained instead—quantum mechanically—is an intersection of three semi-infinite five-branes (Fig. 48). All five-branes share the same $1+4$ directions (x^0,x^1,x^2,x^3,x^4) and are real straight lines in the (x^5,x^6) plane. In the example above we have three semi-infinite five-branes meeting at the origin. The semi-infinite five-brane located at $x^6=0$ and stretched along $x^5<0$ is the NS5-brane. The semi-infinite five-brane located at $x^5=0$ and stretched along $x^6<0$ is the $D5$-brane. The third semi-infinite five-brane is located (295) at $x^6=g_sx^5$, $x^5,x^6>0$.

Which is the five-brane located at $x^6=g_sx^5$? Clearly, it is a (1,1) five-brane! Charge conservation does not really allow a $D5$-brane to end on an NS5-brane. Instead, at the intersection point the (1,0) and (0,1) five-branes merge together to form a (1,1) five-brane. In order for this "new" (1,1) five-brane not to break supersymmetry any further, it must merge from the intersection point at an angle, as described above in Eq. (295).

In the same way general vertices of (p,q) five-branes are permitted provided that (p,q) charge is preserved. To write down a charge conservation condition we have to pick up an orientation for the five-branes. If we fix the orientation of all n five-branes in the direction towards the vertex the charge conservation reads

$$\sum_{i=1}^{n} p_i=\sum_{i=1}^{n} q_i=0. \tag{296}$$

Moreover, requiring the vertex to preserve eight supercharges implies that the (p,q) five-brane is stretched along the semi-infinite line in the (x^5,x^6) plane located at

$$qx^6=g_spx^5. \tag{297}$$

This condition is equivalent to the zero Coulomb-like force condition required for the stability of the vertex.

We can easily extend the discussion above to a situation in which n_L $D5$-branes end on a five-brane from the left and n_R $D5$-branes end on the five-brane from the right. Let a_i, $i=1,\dots,n_L$ be the x^5 locations of the $D5$-branes from the left and b_j, $j=1,\dots,n_R$ the locations of the $D5$-branes from the right. The bending of the five-brane—generalizing Eq. (295)—is

$$x^6=\frac{g_s}{2}\left(\sum_{i=1}^{n_L}|x^5-a_i|-\sum_{j=1}^{n_R}|x^5-b_j|+(n_L-n_R)x^5\right). \tag{298}$$

This equation has the interpretation of a five-brane that is an NS5-brane at large negative x^5, and it changes its charge and angle in (x^5,x^6) in places where a $D5$-brane ends on it. The change of charge and angles is dictated by the conditions (296) and (297).

The presence of such (p,q) five-branes breaks the $(1+9)$-dimensional Lorentz group of the ten-dimensional type-IIB string to

$$SO(1,9)\rightarrow SO(1,4)\times SO(3)_{789}. \tag{299}$$

The $SO(1,4)$ is the five-dimensional Lorentz symmetry preserved in the (x^0,x^1,x^2,x^3,x^4) directions—common to all five-branes—while $SO(3)_{789}$ is the three-dimensional rotation symmetry preserved in the (x^7,x^8,x^9) directions—transverse to all five-branes. The double cover of this group will be identified with the five-dimensional R symmetry: $SU(2)_{789}\equiv SU(2)_R$.

To study five-dimensional gauge theories on type-IIB five-branes we need to describe webs of (p,q) five-branes in which some of the branes are finite in one direction, say in x^6. A web of five-branes includes vertices (where five-branes intersect), legs (the segments of five-branes), and faces. In each vertex charge conservation is obtained and the zero-force condition (297) is applied to fix the appropriate angles. In what follows we shall not specify the orientation choices for the legs, which should be understood from the charge assignments given in each case.

For example, we study webs describing an $SU(N_c)$ gauge theory. Consider N_c parallel $D5$-branes—with worldvolume as in Eq. (294)—stretched between two other five-branes separated in the x^6 direction, which we choose to be (p_L,q_L) and (p_R,q_R) five-branes for large negative values of x^5. The left and right five-branes are broken into segments between $x^5=-\infty$ and the lower (in x^5) $D5$-brane, between $D5$-branes, and between the upper $D5$-brane and $x^5=\infty$.

In other words, we consider a web with N_c horizontal internal legs (stretched in the x^6 direction), which are connected to each other by N_c-1 internal legs on the left (in x^6) and N_c-1 internal legs connecting them on the right. In addition, there are four external legs, two from above (in x^5) and two from below. The lower left and right legs have charges (p_L,q_L) and (p_R,q_R), respectively.

Charge conservation (296) implies that the left and right internal legs between the ath and $a+1$st $D5$-branes have charges (p_L-a,q_L) and (p_R+a,q_R), respectively. This means that for large positive values of

x^5 the left and right five-branes will have charges $(p_L - N_c, q_L)$ and $(p_R + N_c, q_R)$, respectively. The different left and right five-brane segments are oriented in different directions in the (x^5, x^6) plane, in accordance with the zero-force conditions in each vertex [Eq. (297)], separately. The precise bending of the left and right five-branes can be obtained by using an appropriate version of Eq. (298).

As we saw in the four- and three-dimensional cases, to study the gauge physics using branes we need to consider a limit in which gravity and massive string modes decouple. The relevant limit in this case is $L_{max}, l_s, g_s \to 0$, where L_{max} is the largest length of an internal leg. If the gauge coupling at some scale L satisfying $L \gg l_s \gg L_{max}$ is finite, at larger distances gravity decouples, massive Kaluza-Klein modes can be integrated out, and the dynamics on the brane configuration is governed by gauge theory.

At low energy the theory on the $D5$-branes is a pure $N=1$ supersymmetric $SU(N_c)$ gauge theory in $1+4$ dimensions. Deformations of the web that do not change the asymptotic locations of the external legs correspond to moduli in the field theory. Such locations a_i, $i=1,\ldots,N_c$ of the $D5$-branes along the x^5 direction parametrize the Coulomb branch of the theory.

When $g_s \to 0$ the configuration tends to N_c parallel $D5$-branes stretched between two parallel five-branes, and the classical gauge group is $U(N_c)$ with gauge coupling

$$\frac{1}{g_0^2} = \frac{L_6}{g_s l_s^2}. \tag{300}$$

Here L_6 is the distance between the left and right five-branes. To keep g_0 finite we need to take $L_6 \to 0$ such that the ratio $L_6/g_s l_s^2$ is finite. The N_c values of a_i are independent and parametrize the Coulomb branch of $U(N_c)$.

For finite g_s quantum effects cause the five-branes to bend—as described above—and "freeze" the $U(1)$ factor, as in the four-dimensional theories considered in Secs. IV and V. One of the N_c independent classical motions of the N_c $D5$-branes is indeed frozen—once the left and right five-branes bend $\Sigma_{i=1}^{N_c} a_i = 0$ is required to keep the asymptotic locations of external legs fixed—leaving a total of $N_c - 1$ real motions parametrizing the Coulomb branch of $SU(N_c)$, as in Eq. (290).

The asymptotic positions of the external legs are associated with the gauge coupling. The classical gauge coupling $1/g_0^2$ can be obtained geometrically as follows. We set $a_i = 0$, that is, we deform the $D5$-branes to a position where they are coincident without changing the locations of the external legs. Then the length of the $D5$-branes L_6 is related to g_0 by Eq. (300)—now with nonzero g_s and L_6. For general a_i, L_6 is still the distance between the point where the ("imaginary") continuation of the left external legs meet and the point where the continuation of right external legs meet (see Fig. 49).

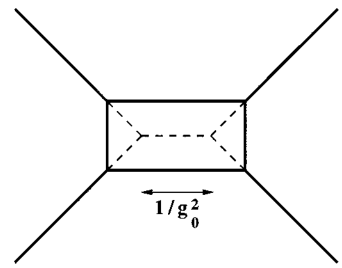

FIG. 49. A five-dimensional $SU(2)$ gauge theory described using five-branes.

The semiclassical $SU(N_c)$ gauge coupling (which in this case is exact) is related to the "size" of the brane configuration in x^6 as in the four- and three-dimensional cases (see Secs. IV, V, and VI). Indeed, it is linear in a_i, as predicted by the bending (297) and (298), and in agreement with the field-theory result obtained from Eqs. (289) and (290), with $N_f = 0$.

Not every charge assignment is allowed to be given to the external legs while still describing an $SU(N_c)$ gauge theory on the $D5$-branes. In this respect, two questions are interesting:

(1) What is the gauge-theory meaning of the charges (p_L, q_L) and (p_R, q_R) on the external legs?

(2) Which values of (p_L, q_L) and (p_R, q_R) are permitted?

The answer to question (2) is clear. The permitted values (p_L, q_L) and (p_R, q_R) are such that the external legs do not cross each other. If the external legs do cross each other, the brane configuration has more crossings of five-branes than those required to describe the Coulomb branch of $N=1$, $SU(N_c)$ gauge theory in five dimensions.

To find the independent "legal" $SU(N_c)$ webs—obeying all the above conditions—and their moduli, it is convenient to describe a web by its dual grid diagram. The grid has points, lines, and polygons, which are dual to the faces, legs, and vertices of the web. One can show (Aharony, Hanany, and Kol, 1998) that for $N_c > 2$ there are $2N_c + 1$ inequivalent webs. Indeed, this is precisely the number of allowed values of c_{cl} as obtained from the conditions (291) and (292) for $N_f = 0$: $c_{cl} = -N_c, -N_c + 1, \ldots, N_c - 1, N_c$. This answers question (1): the $2N_c + 1$ different legal webs are in one-to-one correspondence with the different allowed values of c_{cl}.

Each different allowed (p_L, q_L), (p_R, q_R) corresponds to a different allowed c_{cl}. The web corresponding to $-c_{cl}$ is obtained from the web corresponding to c_{cl} by the use of $SL(2,Z)$ S duality together with a rotation—a Z_2 reflection. Indeed, in field theory there is a Z_2 symmetry of the spectrum under the reflection $c_{cl} \to -c_{cl}$. In particular, the configuration corresponding to $c_{cl} = 0$ is the web invariant under reflection, while configurations corresponding to $|c_{cl}| = N_c$ are the two

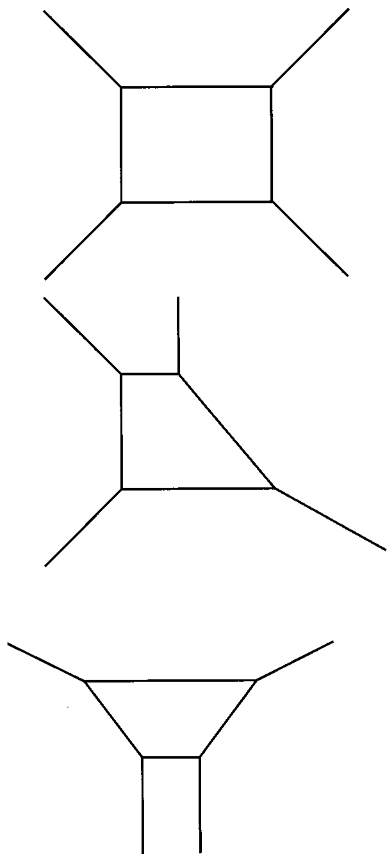

FIG. 50. The three possible configurations corresponding to $SU(2)$ gauge theories. Only the first two appear to give rise to nontrivial fixed points.

webs with parallel external legs. In the latter case an equality holds in the field-theory constraint (292).[41]

The $N_c=2$ case is special. Here one finds three independent, apparently legal, webs (Fig. 50). Each web has (generically) four vertices, four external legs, and four internal legs forming a single face. One of the webs has two parallel external legs. It is claimed, however, that in this case parallel external legs do not correspond to a web describing an $SU(2)$ gauge theory. The remaining two webs—one with a rectangular face and the other with a right-angle trapeze—correspond to the two $SU(2)$ gauge theories found in field theory, as discussed in Sec. VIII.A.

We see that the webs and grids are useful in classifying five-dimensional $N=1$, $SU(N_c)$ gauge theories. We refer the reader to Aharony, Hanany, and Kol (1998) for a detailed description of the classification of $5d$ theories using grids.

In the webs above each vertex had three intersecting legs. However, displacing the $D5$-branes in x^5 and/or changing the locations of external legs may lead to situations where faces shrink to zero size, thus forming vertices with more than three intersecting legs. For instance, the $SU(2)$ diagrams can be deformed to a single vertex with four legs (see Fig. 51). In particular, the gauge coupling in such configurations tends to infinity, and we describe a strong-coupling fixed point of the theory.

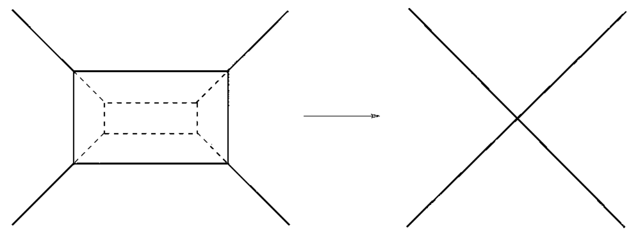

FIG. 51. Nontrivial fixed points described by vertices with more than three external legs.

The webs of five-branes may thus be useful in classifying five-dimensional $N=1$ superconformal fixed points. Such attempts have been initiated (Aharony and Hanany, 1997; Aharony, Hanany, and Kol, 1998). Indeed, many configurations corresponding to known $5d$ superconformal field theories (SCFT) were identified, as well as webs that lead to new SCFT's.

To add N_f fundamental flavors we allow the inclusion of a total of N_f semi-infinite $D5$-branes, N_L of which are connected to the left of the left five-brane and N_R of which are connected to the right of the right five-brane $(N_L+N_R=N_f)$.[42] The locations m_f, $f=1,\ldots,N_f$ of the semi-infinite D five-branes in the x^5 directions parametrize the N_f real masses of quarks. Of course, these additional $D5$-branes affect the bending of the left and right five-branes [see Eq. (298)]. In particular, for large positive values of x^5 the left and right five-branes will have charges $(p_L+N_L-N_c,q_L)$ and $(p_R+N_c-N_R,q_R)$, respectively.

As before, $SU(N_c)$ legal configurations are those in which the left and right five-branes do not cross each other. As a result, the number of inequivalent webs describing $SU(N_c)$ with N_f flavors and obeying all the required conditions is indeed the one indicated by the field-theory conditions (291) and (292). Allowed values of c_{cl} are in one-to-one correspondence with such inequivalent legal webs.

Again, the gauge coupling related to the brane configuration—along the lines of the four- and three-dimensional discussion—is in agreement with the field-

[41]Webs with parallel external legs seem to be inconsistent as $5d$ theories and perhaps should not be considered; when two external legs are parallel a string corresponding to either a gauge boson or an instanton (see below) can ''leak'' out of the web (Aharony, Hanany, and Kol, 1998).

[42]Alternatively, we could add $D7$-branes stretched in $(x^0,x^1,x^2,x^3,x^4,x^7,x^8,x^9)$. But seven-branes affect the asymptotic behavior of spacetime and we shall not consider them here. Without $D7$-branes we shall not be able to see the complete structure of the Higgs branches of the theory geometrically. Some Higgsing can be obtained, however, by deforming a subweb in the (x^7,x^8,x^9) directions corresponding, classically, to a Fayet-Iliopoulos D term. This is possible when a configuration is reducible, that is, when it can be considered to consist of two independent webs. This may happen at the roots of the Higgs branches.

theory result obtained from Eqs. (289) and (290) for general N_c, N_f, m_f, and c_{cl}. This can be seen by using the relation between the bending (298) and the gauge coupling as discussed in Secs. IV, V, and VI.

In addition to classifications and the study of the structure of moduli space, more aspects of five-dimensional gauge theories can also be considered in the brane configurations. In particular, there are BPS-saturated monopole strings, which arise from $D3$-branes stretched along faces in the brane configuration, and instantons—BPS-saturated particles in five dimensions—corresponding to D strings parallel to (or inside) the $D5$-branes and ending on the left and right five-branes. Moreover, one can stretch (p,q) string webs ending on the five-branes.[43] These and other BPS states can be studied in the brane configurations considered above; we refer the reader to the literature (Aharony, Hanany, and Kol, 1998; Kol and Rahmfeld, 1998) for details.

C. Compactifying from five to four dimensions

In this section we compactify the five-dimensional gauge theory with spacetime (x^0,x^1,x^2,x^3,x^4) to four dimensions (x^0,x^1,x^2,x^3) on a circle of radius R. In the type-IIB brane configuration of the previous section this is also obtained by compactifying x^4 on a radius R circle.

The semiclassical results of the previous section are no longer exact for $R<\infty$. To study the exact nonperturbative corrections in the brane configuration we lift the web into an M-theory curve (an $M5$-brane) (Brandhuber, Itzhaki, *et al.*, 1997b; Kol, 1997; Aharony, Hanany, and Kol, 1998). For that purpose, we need to identify the type-IIB parameters g_s, l_s, and R in terms of the M-theory parameters l_p and R_i, $i=1,\dots,10$.

To do that we first perform a T duality in the x^4 direction T_4 which takes the compactified type-IIB string to a type-IIA string compactified on a circle of radius R_4 (see Sec. II.D),

$$T_4:\quad R\to R_4=\frac{l_s^2}{R}, \tag{301}$$

and string coupling

$$g_A=\frac{g_s l_s}{R}. \tag{302}$$

Moreover, as explained in Sec. II.C, the $D5$- and $NS5$-branes with worldvolumes as in Eq. (294) transform under T_4 to the type-IIA $D4$- and NS5-branes, respectively, with worldvolumes

$$\begin{aligned}&\text{NS5:}\quad (x^0,x^1,x^2,x^3,x^4,x^5),\\ &D4:\quad (x^0,x^1,x^2,x^3,x^6).\end{aligned} \tag{303}$$

[43]Webs of (p,q) strings can also be stretched between $D3$-branes; in the context of Sec. III they describe 1/4 BPS states in 4-d, $N=4$ SYM theory (Bergman, 1998).

Therefore a web describing $SU(N_c)$ with N_f flavors, namely, N_c finite and N_f semi-infinite $D5$-branes—as considered in the end of the previous section—transforms into a type-IIA configuration in which N_c four-branes are stretched between two NS five-branes while N_L semi-infinite four-branes are connected to the left of the left five-brane and N_R semi-infinite four-branes are connected to the right of the right five-brane, $N_L+N_R=N_f$. In the following we shall usually take $N_L=0$, $N_R=N_f$.

Equivalently, for finite g_A [Eq. (302)] we have M theory compactified on a rectangular two-torus in the x^4 and x^{10} directions with sizes R_4 and R_{10}, respectively, where

$$R_{10}=l_s g_A \tag{304}$$

and with an 11-dimensional Planck scale

$$l_p^3=R_{10}l_s^2. \tag{305}$$

As explained in Sec. II.C [and as can be rederived from Eqs. (301)–(305)], the type-IIB compactification radius and string coupling are related to the M-theory parameters via

$$R=\frac{l_p^3}{R_4R_{10}} \tag{306}$$

and

$$g_s=\frac{R_{10}}{R_4}. \tag{307}$$

The ten-dimensional type-IIB limit is obtained by taking $R_4R_{10}\to 0$ while keeping g_s (307) fixed. Indeed, Eq. (306) implies that in this limit $R\to\infty$ and we recover the five-dimensional field-theory configurations of the previous section.

As before, in M theory the type-IIA brane configuration is an $M5$-brane with worldvolume $R^{1,3}\times\Sigma$, where $R^{1,3}$ is the $1+3$ spacetime (x^0,x^1,x^2,x^3) and Σ is a two-dimensional surface embedded in the four-dimensional space $Q=S^1\times R^2\times S^1$ in the (x^4,x^5,x^6,x^{10}) directions. Since both x^4 and x^{10} are compact, to find the curve Σ it is convenient to parametrize Q by the single-valued coordinates t and u:

$$\begin{aligned}&t=e^{-s/R_{10}},\quad s=x^6+ix^{10},\\ &u=e^{-iv/R_4},\quad v=x^4+ix^5\end{aligned} \tag{308}$$

and describe the curve by the algebraic equation

$$F(t,u)=0. \tag{309}$$

As in Secs. IV.C.4 and IV.C.5, the form of the curve should be

$$F(t,u)=A(u)t^2+B(u)t+C(u)=0 \tag{310}$$

and we may set

$$A(u)=1 \tag{311}$$

corresponding to all semi-infinite four-branes' being to the right of the right five-brane ($N_L=0$, $N_R=N_f$, see above). Since both $u=\infty$ (that is, $x^5=\infty$) and $u=0$ (x^5

$=-\infty$) correspond to the asymptotic region, the multiplicity of the the zero roots of the polynomials $B(u)$ and $C(u)$ is relevant. Analyzing (Brandhuber, Itzhaki, *et al.*, 1997b) the asymptotic behavior as in Sec. IV.C.4, one finds that curves describing consistent $SU(N_c)$ configurations with N_f fundamental flavors have

$$B(u)=b\prod_{i=1}^{N_c}(u-A_i),$$
$$C(u)=cu^{N_c-N_f/2-c_{cl}}\prod_{f=1}^{N_f}(u-M_f), \quad (312)$$

where a,b,A_i,M_f,c_{cl} are constant parameters and c_{cl} must obey the conditions (291) and (292). Therefore c_{cl} in Eq. (312) corresponds precisely to c_{cl} in field theory. In M theory c_{cl} must obey the condition (291) because otherwise the curve is not holomorphic, and it must obey the condition (292) because otherwise the $M5$-brane describes a type-IIB configuration in which the external five-branes cross each other.

Each monomial $u^n t^m$ in the curve is associated with the point (n,m) in the grid diagram dual to the web, that is, to a face in the web where it is dominant. The curve is just the sum of these monomials (Kol, 1997; Aharony, Hanany, and Kol, 1998) with the coefficients constrained to obey some consistency conditions.

The four-dimensional field-theory limit is obtained at $R\to 0$. To consider this limit in M theory, it is convenient to rewrite the algebraic Eqs. (309)–(312) in terms of v [instead of u in Eq. (308)]. By an appropriate choice of the constants b and c in Eq. (312) one finds the curve

$$t^2+te^{-iN_cv/2R_4}\prod_{i=1}^{N_c}R_4\sin\left(\frac{v-a_i}{2R_4}\right)+e^{-iv(N_c-c_{cl})/R_4}\prod_{f=1}^{N_f}R_4\sin\left(\frac{v-m_f}{2R_4}\right)=0, \quad (313)$$

where the parameters A_i and M_f in Eq. (312) are related to a_i and m_f in Eq. (313), respectively, by

$$A_i=e^{-a_i/R_4}, \quad M_f=e^{-im_f/R_4}. \quad (314)$$

The $R\to 0$ limit means $R_4\to\infty$ [see Eq. (301)] and, therefore, in the four-dimensional limit the curve (313) becomes

$$t^2+t\prod_{i=1}^{N_c}(v-a_i)+\prod_{f=1}^{N_f}(v-m_f)=0. \quad (315)$$

This is precisely the curve of $N=2$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors in four dimensions [Eq. (117)].

D. Some generalizations

To study $N=1$ supersymmetric symplectic (or orthogonal) gauge theories in five dimensions we need to present an orientifold five-plane. For instance, let us introduce an $O5_{+2}$-plane parallel to the $D5$-branes (294) in the type-IIB webs of Sec. VIII.B. This gives rise to an $Sp(N_c/2)$ gauge theory on the $D5$-branes (because the orientifold has a positive charge, thus imposing a symplectic projection on the parallel D five-branes). The brane configuration is necessarily invariant under the orientifold reflection, and therefore it is more constrained than the $SU(N_c)$ configurations. In particular, given N_c and N_f, there is a unique possibility (modulo equivalence transformations) for the orientation of the external legs—the one invariant under the mirror reflection. This single consistent configuration—describing an $Sp(N_c/2)$ gauge theory with N_f fundamental hypermultiplets—corresponds to the unique field theory obeying the condition $c_{cl}=0$ (see Sec. VIII.A). Moreover, as in the unitary case, the field-theory condition (293) translates in the brane construction into the requirement that the external legs not cross each other.

Gauge theories with product gauge groups can also be considered in the brane picture. For example, webs corresponding to a product of unitary gauge groups $SU(N_1)\times SU(N_2)\times\cdots\times SU(N_k)$ have—in the $g_s\to 0$ limit—$k+1$ parallel NS five-branes separated in x^6, and N_i D five-branes connecting the ith NS five-brane (from the left in x^6) to the $i+1$st NS five-brane, $i=1,\ldots,k$. For $g_s\neq 0$, the five-branes bend, according to the rules discussed in this review, describing the exact quantum corrections to the five-dimensional theories. Again, the condition that external legs not cross each other must correspond to appropriate field-theory constraints.

Webs of five-branes can also be used to obtain new $N=2$ 3-d superconformal field theories from $5d$ fixed points. This is done by considering two identical webs separated, say in the x^7 direction, and stretching between them $D3$-branes with worldvolume in (x^0,x^1,x^2,x^7). For more details we refer the reader to Aharony and Hanany (1997).

Finally, we should remark that some deformations of consistent webs may lead to theories with no gauge-theory interpretation.

E. Six-dimensional theories

In this section we discuss brane configurations in the type-IIA string with six-branes ending on five-branes which describe at low-energy six-dimensional (0,1) supersymmetric gauge theories (Brunner and Karch, 1997).

We consider NS five-branes, D six-branes, and orientifold six-planes in the type-IIA string with worldvolume

$$\text{NS5:}\quad (x^0,x^1,x^2,x^3,x^4,x^5),$$
$$D6/O6:\quad (x^0,x^1,x^2,x^3,x^4,x^5,x^6). \quad (316)$$

With these objects we can construct several stable configurations leading to consistent $6d$ supersymmetric gauge theories with eight supercharges, a few examples of which are presented below:

- $SU(N_c)$ *with* $N_f=2N_c$. Let us stretch N_c $D6$-branes between two NS5-branes that are separated in x^6. To the left of the left five-brane we place N_L semi-infinite

$D6$-branes, while to the right of the right five-brane we place N_R semi-infinite $D6$-branes. Since six-branes ending on a five-brane behave like electric charges in zero dimensions, stability implies that the total charge must vanish. This zero-charge condition on the five-branes implies that

$$N_L=N_R=N_c \Rightarrow N_f \equiv N_L+N_R=2N_c. \qquad (317)$$

The low-energy theory on the $D6$-branes is therefore an $SU(N_c)$ gauge theory with $N_f=2N_c$ fundamental hypermultiplets. In field theory, the condition (317) is precisely the one required for anomaly cancellation in (0,1) SUSY six-dimensional theories. Again, we find that the brane configuration is stable if and only if the gauge theory is anomaly free.

We may compactify the theory on a three-torus in the (x^3,x^4,x^5) directions and perform T duality in these directions. The brane configuration considered above is T dual to configurations describing $SU(N_c)$ gauge theories with eight supercharges and $N_f=2N_c$, which were discussed in previous sections. T duality (followed by decompactifications) in the x^5 direction T_5 takes it to a web of five-branes describing a particular $N=1$ five-dimensional gauge theory. T_{45} leads to an $N=2$ four-dimensional configuration, while T_{345} gives an $N=4$ three dimensional case.

- *$SO(N_c)$ with $N_f=N_c-8$.* To get an orthogonal gauge group we add an $O6$-plane and stick the two NS five-branes separated in x^6 on top of the orientifold. As we have learned, there is a sign flip in the Ramond-Ramond charge of the orientifold on the two sides of the five-brane.

  Consider the case in which an $O6_{-4}$-plane (the orientifold six-plane with charge -4) is stretched between the two NS five-branes. To the left of the left five-brane and to the right of the right five-brane we must have semi-infinite $O6_{+4}$-planes. Moreover, between the NS five-branes we stretch N_c D six-branes and to the left and right of them we place N_L and N_R semi-infinite $D6$-branes, respectively. The zero-force condition on the five-branes now implies that

$$N_L=N_R=N_c-8 \Rightarrow N_f \equiv (N_L+N_R)/2=N_c-8. \qquad (318)$$

To obtain the consistency condition (318) we had to take into account the sign flip of the orientifold.
The theory on the six-branes is a (0,1) supersymmetric $SO(N_c)$ gauge theory (because the orientifold segment parallel to the N_c finite six-branes is an $O6_{-4}$, thus imposing an orthogonal projection) with N_f hypermultiplets in the vector representation. The requirement (318) is precisely the anomaly-free condition in such a gauge theory.

- *$Sp(N_c/2)$ with $N_f=N_c+8$.* To get a symplectic gauge group all we need to do is to change the sign of the orientifold in the previous example. The projection of the $O6_{+4}$-plane stretched between the five-branes on the parallel N_c six-branes is the symplectic one, leading to an $Sp(N_c/2)$ supersymmetric gauge theory. Moreover, the zero-force condition implies now that the theory has N_c+8 fundamental hypermultiplets:

$$N_L=N_R=N_c+8 \Rightarrow N_f \equiv (N_L+N_R)/2=N_c+8, \qquad (319)$$

which is precisely the anomaly-free condition in gauge theory.

- *Product groups.* Configurations describing an alternating product of k orthogonal and symplectic gauge groups can be studied by considering $k+1$ NS five-branes separated in x^6 on top of an $O6$-plane. The zero-force condition implies the correct relations of colors and flavors required for anomaly cancellation in field theory.

Very recently, new works discussing branes and six-dimensional theories have appeared (Brunner and Karch, 1998; Hanany and Zaffaroni, 1998b). In these works configurations including eight-branes and orientifolds were considered, leading to classes of $6d$ models with nontrivial fixed points at strong coupling, some of which were studied previously in field theory (Seiberg, 1997a) and using branes at orbifold singularities (Intriligator, 1997).

IX. DISCUSSION

A. Summary

The worldvolume physics of branes in string theory provides a remarkably efficient tool for studying many aspects of the vacuum structure and properties of BPS-saturated states in supersymmetric gauge theories. By embedding it in a much richer dynamical structure, brane dynamics provides a new perspective on gauge theory and in many cases explains phenomena that are known to occur in field theory but are rather mysterious there. We list here some examples of results that can be better understood using branes that were described in this review:

(1) Montonen and Olive's electric-magnetic duality in four-dimensional $N=4$ SUSY gauge theory as well as Intriligator and Seiberg's mirror symmetry in three-dimensional $N=4$ SUSY gauge theory are consequences of the nonperturbative S-duality symmetry of type-IIB string theory (Green and Gutperle, 1996; Tseytlin, 1996; Hanany and Witten, 1997).
(2) Nahm's construction of the moduli space of magnetic monopoles can be derived by using the description of monopoles as D strings stretched between parallel $D3$-branes in type-IIB string theory (Diaconescu, 1997). A similar description leads to a relation between the moduli space of monopoles in one field theory and the quantum Coulomb branch of another (Hanany and Witten, 1997).
(3) The auxiliary Riemann surface whose complex structure was proven by Seiberg and Witten to de-

termine the low-energy coupling matrix of four-dimensional $N=2$ SUSY gauge theory is naturally interpreted as part of the worldvolume of a five-brane (Klemm *et al.*, 1996; Witten, 1997a). Hence it is physical in string theory; moreover, this geometrical interpretation is very useful for studying BPS-saturated states in $N=2$ supersymmetric Yang-Mills theory.

(4) Seiberg's infrared equivalence between different four-dimensional $N=1$ supersymmetric gauge theories is manifest in string theory (Elitzur, Giveon, and Kutasov, 1997; Elitzur, Giveon, *et al.*, 1997). The electric and magnetic theories provide different parametrizations of the same quantum moduli space of vacua. They are related by smoothly exchanging five-branes in an appropriate brane configuration. Many additional features of the vacuum structure of $N=1$ SUSY gauge theories can be reproduced by studying the five-brane configuration (Brandhuber, Itzhaki, *et al.*, 1997a; Hori, Ooguri, and Oz, 1998; Witten, 1997b). In particular, the QCD string of confining $N=1$ supersymmetric Yang-Mills theory appears to be a membrane ending on the five-brane (Witten, 1997b).

(5) The vacuum structure of $N=2$ supersymmetric gauge theories in three dimensions and, in particular, the generalization of Seiberg's duality to such systems, can be understood using branes (Elitzur, Giveon, *et al.*, 1997). An interesting feature of Seiberg's duality in three dimensions is that it relates two theories, one of which is a conventional field theory, while the other does not seem to have a local field-theoretic formulation (but it does have a brane description).

(6) Webs of branes provide a useful description of nontrivial fixed points of the renormalization group in five and six dimensions (Aharony and Hanany, 1997; Aharony, Hanany, and Kol, 1998; Brunner and Karch, 1998; Hanany and Zaffaroni, 1998b).

In fact, one could argue that all the results regarding the vacuum structure of strongly coupled supersymmetric gauge theories obtained in the last four years should be thought of as low-energy manifestations of string theory.

The improved understanding of the vacuum structure obtained by embedding gauge theory in the larger context of string or brane theory is very interesting, but it would be even more important to go beyond the vacuum/BPS sector and obtain new results on nonvacuum low-energy properties, e.g., the masses and interactions of low-lying non-BPS states. In field theory not much is known about this subject, but there are reasons to believe that progress can be made using branes.

The role of branes in describing low-energy gauge theory so far is somewhat analogous to that of Landau-Ginzburg theory in critical phenomena. It provides a remarkably accurate description of the space of vacua of the theory as a function of the parameters in the Lagrangian, including aspects that are quite well hidden in the standard variables, such as strong-weak coupling relations between different theories. As in critical phenomena, to compute critical exponents or, more generally, study the detailed structure of the infrared conformal field theory, one will have to go beyond the analysis of the vacuum. However, if the analogy to statistical mechanics is a good guide, the brane description—which clearly captures correctly the order parameters and symmetries of the theory—should prove to be a very useful starting point for such a study.

B. Open problems

In the course of the discussion we have encountered a few issues that deserve better understanding. Some examples are the following:

1. $SU(N_c)$ versus $U(N_c)$

We have seen in Secs. IV and V that brane configurations describing four-dimensional gauge theory with a unitary gauge group seem to have the peculiar property that while classically the gauge group is $U(N_c)$, quantum mechanically it is $SU(N_c)$, with the gauge coupling of the $U(1)$ factor vanishing logarithmically as we turn on quantum effects. At the same time, an interpretation of the physics in terms of an $SU(N_c)$ gauge theory seems to be in contradiction with certain supersymmetric deformations of the brane configuration, which appear to be parameters in the Lagrangian rather than moduli (Giveon and Pelc, 1998). It would be interesting to resolve this apparent paradox, especially because it is closely related to other issues that one would like to understand better. In particular, as we have seen, some of the features of the infrared physics are not visible geometrically in brane theory. For example, in four-dimensional magnetic $N=1$ SQCD with $N_f=N_c$, the mesons M [Eq. (148)] are clearly visible, while the baryons B, $\tilde{B}$ [Eq. (150)] are more difficult to see. Similarly, in three-dimensional $N=2$ SQCD, the fields $V_\pm$ [Eq. (265)] are a form of dark matter, visible only through their effect on the quantum moduli space of vacua.

2. Nontrivial fixed points, intersecting five-branes, and phase transitions

Brane configurations provide very useful descriptions of the classical and quantum moduli spaces of vacua of different gauge theories, but so far it has proved difficult to use them to study other features of the long-distance behavior. The corresponding M-theory five-brane becomes singular as one approaches a nontrivial IR fixed point, and thus it is not well described by 11-dimensional supergravity. Only aspects of the fixed point that can be studied by perturbing away from it and continuing to unphysical values of L_6, R_{10}, such as the superpotential, dimensions of chiral operators, and global symmetries, can be usefully studied using low-energy M theory.

An important tool for studying the low-energy dynamics of $N=1$ SUSY gauge theories using branes is $N=1$ duality. We have seen that the theory on four-branes stretched between nonparallel five-branes changes smoothly when the five-branes meet in space and exchange places. In the case of parallel five-branes, this process corresponds to a phase transition. It would be very interesting to understand this phenomenon in more detail by studying the theory on parallel versus nonparallel five-branes.

Specifically, we have seen using branes that the quantum moduli spaces of vacua and quantum chiral rings of the electric and magnetic SQCD theories coincide. This leaves open the question whether Seiberg's duality extends to an equivalence of the full infrared theory, since in general the chiral ring does not fully specify the IR conformal field theory. It is believed that in gauge theory the answer is yes, and to prove it in brane theory will require an understanding of the smoothness of the transition when five-branes cross.

It is important to emphasize that the question cannot be addressed using any currently available tools. The M-theory approach fails since the characteristic size of the five-brane becomes small, and the brane interactions relevant for this situation are unknown. As we saw, the fact that when parallel Neveu-Schwarz five-branes cross the theory on four-branes stretched between them undergoes a phase transition, is related to the fact that when the five-branes coincide they describe a nontrivial six-dimensional conformal field theory. It is a hallmark of nontrivial fixed points that the physics seen when one is approaching them from different directions is different. To show smoothness for nonparallel branes one has to understand the theory on intersecting nonparallel NS five-branes. At present such theories are not understood.

3. Orientifolds

Brane configurations involving orientifold four-planes such as those of Figs. 19, 21, and 42, discussed in Secs. IV and V, are still puzzling. It appears that when a D-brane intersects an orientifold and divides it into two disconnected components, the charge of the orientifold flips sign as one crosses the D-brane. Also, upon compactification of such brane configurations on a longitudinal circle and T duality along the circle, one finds brane configurations in type-IIB string theory in which the analysis is severely constrained by S duality. It is not clear how the corresponding analysis is related to the process of compactifying the low-energy gauge theory on the four-branes from four to three dimensions.

4. Future work

Clearly, there is much that remains to be done. The two main avenues for possible progress at the moment seem to be the following:

a. More models

One would like to find the sort of description of the vacuum structure and low-energy physics that we presented for additional models. Specific examples include models with exceptional gauge groups and more general matter representations of the classical groups, such as SO groups with spinors. For example, if one believes that Seiberg's duality is a string-theory phenomenon, it should be possible to find an embedding in string theory of the set of Seiberg dual pairs studied by Pouliot (1995), Berkooz *et al.* (1997), and others.

One way to proceed in the case of four-dimensional $N=2$ supersymmetry is to study configurations in which an $M5$-brane wraps the Seiberg-Witten surface relevant for the particular gauge theory, as has been done by Klemm *et al.* (1996). It would be interesting to understand the relation of these constructions to the sort of configurations studied here. It is worth stressing that one is looking for a brane construction that does not merely share with supersymmetric Yang-Mills theory its vacuum structure. Rather, we want to reproduce the whole RG trajectory corresponding to the particular gauge theory in some limit of string/brane dynamics. This means that there has to be a weakly coupled description of the brane configuration, suitable for studying the vicinity of the UV fixed point of SYM theory.

For $N=1$ supersymmetric theories, it would be interesting to construct large classes of chiral gauge theories that break supersymmetry and study them using branes. This may clarify the general requirements for SUSY breaking and hopefully provide the same kind of conceptual unification of SUSY breaking that was achieved for Seiberg-Witten theory and $N=1$ duality.

In this review we have mostly discussed the worldvolume physics on branes that have finite extent in one noncompact direction. An interesting generalization corresponds to configurations containing branes that are finite or semi-infinite in more than one noncompact direction. The simplest case to examine is that of branes that are finite in *two* directions.

We have seen that such configurations are necessary to describe Euclidean field configurations that give rise to different nonperturbative effects (see, for example, Figs. 8 and 34). Similarly, in Sec. IV configurations of $D2$-branes stretched between two NS5-branes and two $D4$-branes were used to describe magnetic monopoles in four-dimensional $N=2$ SYM theory. Using U duality, such configurations could be mapped to other interesting configurations. For example, one could study Dp-branes [with worldvolume, say, in $(x^0,\ldots,x^{p-2},x^4,x^8)$, $p=2,3,4,5$] stretched between a pair of NS5-branes separated by a distance L_8 in x^8 and a pair of NS5$'$-branes separated by a distance L_4 in x^4 [see Eqs. (91) and (173) for the conventions], or $D3$-branes stretched between two NS5-branes separated in x^6 and two $D5$-branes [Eqs. (294)] separated in x^5.

In the latter case, it is easy to check that the two-dimensional low-energy theory on the three-branes has (4,0) SUSY and is thus chiral. Therefore these configurations and their generalizations offer a useful laboratory for the study of chiral field theories. A large class of generalizations can be obtained by studying ‘‘chess board’’ configurations in which branes finite in two directions stretch like rugs between different segments of

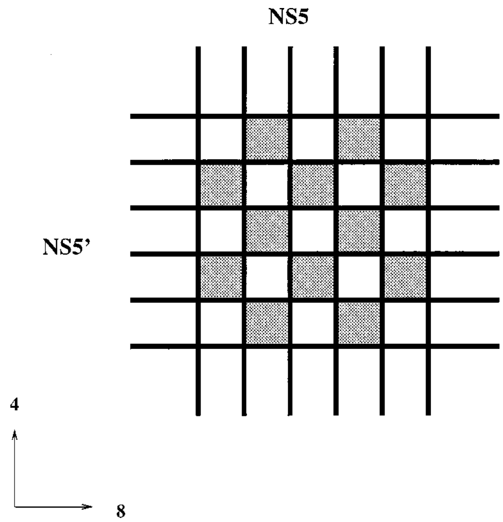

FIG. 52. Does chess play a role in string theory?

a two-dimensional network of intersecting branes (Fig. 52). Particular configurations of this sort[44] are dual to some of the chiral models discussed in Sec. V.D.5, using the duality relating an ALE space with a Z_n orbifold singularity to a vacuum containing $n-1$ solitonic five-branes on R^4/Z_n.

Clearly, it would be interesting to study configurations of this type with more branes and/or orientifolds, as well as consider branes that are finite in more dimensions, which should lead at low energies to many new models and hopefully also to some new understanding.

b. The dynamics of five-branes

It is clearly important to develop tools to study the dynamics of five-branes in string theory. Four-dimensional $N=4$ SYM theory can be thought of as the six-dimensional conformal field theory on N_c five-branes compactified on a two-torus whose modulus τ is related to the four-dimensional supersymmetric Yang-Mills coupling (Witten, 1995b). $N=1,2$ SYM theory can be thought of as compactifications of the (2,0) conformal field theory from six down to four dimensions on the Seiberg-Witten Riemann surface Σ (Witten, 1997b).

Recently, the (2,0) theory on $R^{5,1}$ and the compactified theory on $R^{3,1}\times T^2$ were studied using matrix theory (Aharony, Berkooz, *et al.*, 1998; Aharony, Berkooz, and Seiberg, 1998; Ganor and Sethi, 1998). These attempts are still at an early stage and it is not clear whether they will eventually provide efficient techniques for studying these theories. In any case, matrix descriptions of theories like four-dimensional $N=4$ SYM theory are also useful as a testing ground for matrix theory in general, as the theory that one is trying to describe is in this case well defined and understood (at least in certain corners of parameter space), unlike eleven-dimensional M theory for which matrix theory was originally proposed.

[44]See Hanany and Zaffaroni (1998) for a recent discussion.

Another promising direction is to understand the theory of the QCD string. At large N_c the string coupling of the QCD string is expected to be small ('t Hooft, 1974) and one may hope that the theory can be described by a more or less conventional worldsheet formalism (Polyakov, 1987). What kind of theory does one expect to find? The brane construction suggests a theory that lives in six dimensions, but is Lorentz invariant in only four of these. There is a nontrivial metric in the remaining two directions, Φ, which suppresses fluctuations of the string in these directions. The resulting picture is very reminiscent of noncritical superstrings that were constructed by Kutasov and Seiberg (1990) and of the recent work of Polyakov (1998). It would of course be very interesting to make this more precise.

A long-standing puzzle in the theory of QCD strings is related to the work of Kutasov and Seiberg (1991), who pointed out that in fundamental string theory IR stability of the vacuum (absence of tachyons) and unitarity imply asymptotic supersymmetry of the spectrum. Confining large-N_c gauge theories are traditionally expected to have a string description even in the absence of supersymmetry ('t Hooft, 1974; Polyakov, 1987). The new ideas on QCD string theory and, in particular, the relation of the QCD string to the fundamental string, might help resolve the puzzle. Perhaps a description of QCD in terms of continuous worldsheets requires asymptotic supersymmetry. This may be related to recent speculations that supersymmetry appears to play a deep role in string dynamics (Banks *et al.*, 1997; Douglas *et al.*, 1997). For example, there are indications that locality in string theory is a consequence of asymptotic supersymmetry.

Eventually, one would like to use branes to study the infrared dynamics of nonsupersymmetric theories like QCD. At present, brane constructions shed no light on strongly coupled nonsupersymmetric gauge theory. Thus, if SUSY is dynamically broken for a particular brane configuration, one can generally say very little about the physics of the nonsupersymmetric ground state. It seems quite likely that progress on one of the fronts mentioned above will also allow one to study nonsupersymmetric gauge theories.

ACKNOWLEDGMENTS

We thank O. Aharony, A. Hanany, J. Harvey, B. Kol, E. Martinec, Y. Oz, O. Pelc, R. Plesser, E. Rabinovici, M. Roček, A. Schwimmer, N. Seiberg, and especially S. Elitzur for discussions. We are grateful to M. Dine for his encouragement and comments on the manuscript. This work is supported in part by the Israel Academy of Sciences and Humanities-Centers of Excellence Program. The work of A.G. is supported in part by BSF-American-Israel Bi-National Science Foundation. A.G. thanks the Enrico Fermi Institute at the University of Chicago, where part of this work was done, for its warm hospitality, and the Einstein Center at the Weizmann Institute for partial support. D.K. is supported in part by a DOE OJI grant.

REFERENCES

Affleck, I., M. Dine, and N. Seiberg, 1984, "Dynamical Supersymmetry Breaking in SQCD," Nucl. Phys. B **241**, 493.

Affleck, I., M. Dine, and N. Seiberg, 1985, "Dynamical Supersymmetry Breaking in Four Dimensions and its Phenomenological Implications," Nucl. Phys. B **256**, 557.

Affleck, I., J. A. Harvey, and E. Witten, 1982, "Instantons and (Super)symmetry Breaking in (2+1)-Dimensions," Nucl. Phys. B **206**, 413.

Aharony, O., 1997, "IR Duality in $d=3$ $N=2$ Supersymmetric $USp(2N_c)$ and $U(N_c)$ Gauge Theories," Phys. Lett. B **404**, 71.

Aharony, O., M. Berkooz, S. Kachru, N. Seiberg, and E. Silverstein, 1998, "Matrix Description of Interacting Theories in Six Dimensions," Adv. Theor. Math. Phys. **1**, 148.

Aharony, O., M. Berkooz, and N. Seiberg, 1998, "Light-Cone Description of (2,0) Superconformal Theories in Six Dimensions," Adv. Theor. Math. Phys. **2**, 119.

Aharony, O., and A. Hanany, 1997, "Branes, Superpotentials and Superconformal Fixed Points," Nucl. Phys. B **504**, 239.

Aharony, O., A. Hanany, K. Intriligator, N. Seiberg, and M. J. Strassler, 1997, "Aspects of $N=2$ Supersymmetric Gauge Theories in Three Dimensions," Nucl. Phys. B **499**, 67.

Aharony, O., A. Hanany, and B. Kol, 1998, "Webs of (p,q) 5-branes, Five Dimensional Field Theories and Grid Diagrams," J. High Energy Phys. **01**, 002.

Aharony, O., S. Kachru, and E. Silverstein, 1997, "New $N=1$ Superconformal Field Theories in Four Dimensions from D-brane Probes," Nucl. Phys. B **488**, 159.

Aharony, O., J. Sonnenschein, and S. Yankielowicz, 1995, "Flows and Duality Symmetries in $N=1$ Supersymmetric Gauge Theories," Nucl. Phys. B **449**, 509.

Aharony, O., J. Sonnenschein, S. Yankielowicz, and S. Theisen, 1997, "Field Theory Questions for String Theory Answers," Nucl. Phys. B **493**, 177.

Ahn, C., 1998a, "Geometry, D-Branes and $N=1$ Duality in Four Dimensions II," Phys. Lett. B **419**, 91.

Ahn, C., 1998b, "Confining Phase of $N=1$ $Sp(N_c)$ Gauge Theories via M Theory Fivebrane," Phys. Lett. B **426**, 306.

Ahn, C., and K. Oh, 1997, "Geometry, D-Branes and $N=1$ Duality in Four Dimensions I," Phys. Lett. B **412**, 274.

Ahn, C., K. Oh, and R. Tatar, 1997, "Branes, Geometry and $N=1$ Duality with Product Gauge Groups of SO and Sp," preprint hep-th/9707027.

Ahn, C., K. Oh, and R. Tatar, 1998b, "$Sp(N_c)$ Gauge Theories and M Theory Fivebrane," Phys. Rev. D **58**, 086002.

Ahn, C., K. Oh, and R. Tatar, 1998b, "M Theory Fivebrane Interpretation for Strong Coupling Dynamics of $SO(N_c)$ Gauge Theories," Phys. Rev. B **416**, 75.

Ahn, C., K. Oh, and R. Tatar, 1998c, "M Theory Fivebrane and Confining Phase of $N=1$ $SO(N_c)$ Gauge Theories," J. Geom. Phys. **28**, 163.

Ahn, C., and R. Tatar, 1997, "Geometry, D-branes and $N=1$ Duality in Four Dimensions with Product Gauge Group," Phys. Lett. B **413**, 293.

Alishahiha, M., 1998, "$N=(4,4)$ 2D Supersymmetric Gauge Theory and Brane Configuration," Phys. Lett. B **420**, 51.

Alvarez-Gaume, L., and S. F. Hassan, 1997, "Introduction to S-Duality in $N=2$ SUSY Gauge Theories: A Pedagogical Review of the Work of Seiberg and Witten," Fortschr Phys. **45**, 159.

Amati, D., K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, 1988, "Nonperturbative Aspects in Supersymmetric Gauge Theories," Phys. Rep. **162**, 169.

Argyres, P. C., and M. R. Douglas, 1995, "New Phenomena in SU(3) Supersymmetric Gauge Theory," Nucl. Phys. B **448**, 93.

Argyres, P. C., M. R. Plesser, and N. Seiberg, 1996, "The Moduli Space of Vacua of $N=2$ SUSY QCD and Duality in $N=1$ SUSY QCD," Nucl. Phys. B **471**, 159.

Argyres, P. C., M. R. Plesser, N. Seiberg, and E. Witten, 1996, "New $N=2$ Superconformal Field Theories in Four Dimensions," Nucl. Phys. B **461**, 71.

Argyres, P. C., M. R. Plesser, and A. Shapere, 1995, "The Coulomb Phase of $N=2$ Supersymmetric QCD," Phys. Rev. Lett. **75**, 1699.

Argyres, P. C., M. R. Plesser, and A. Shapere, 1997, "$N=2$ Moduli Spaces and $N=1$ Dualities for SO(N(C)) and USP(2N(C)) Super QCD," Nucl. Phys. B **483**, 172.

Bachas, C. P., M. R. Douglas, and M. B. Green, 1997, "Anomalous Creation of Branes," J. High Energy Phys. **07**, 002.

Bachas, C. P., and M. B. Green, 1998, "A Classical Manifestation of the Pauli Exclusion Principle," J. High Energy Phys. **01**, 015.

Bachas, C. P., M. B. Green, and A. Schwimmer, 1998, "(8,0) Quantum Mechanics and Symmetry Enhancement in Type I' Superstrings," J. High Energy Phys. **01**, 006.

Banks, T., 1998, "Matrix Theory," Nucl. Phys. B, Proc. Suppl. **67**, 180.

Banks, T., M. R. Douglas, and N. Seiberg, 1996, "Probing F-theory With Branes," Phys. Lett. B **387**, 278.

Banks, T., W. Fischler, S. H. Shenker, and L. Susskind, 1997, "M Theory As a Matrix Model: a Conjecture," Phys. Rev. D **55**, 5112.

Banks, T., N. Seiberg, and E. Silverstein, 1997, "Zero and One-dimensional Probes with $N=8$ Supersymmetry," Phys. Lett. B **401**, 30.

Barbon, J. L. F., 1997, "Rotated Branes and $N=1$ Duality," Phys. Lett. B **402**, 59.

Barbon, J. L. F., and A. Pasquinucci, 1998a, "$D0$-Branes, Constrained Instantons and $D=4$ Super Yang-Mills Theories," Nucl. Phys. B **517**, 125.

Barbon, J. L. F., and A. Pasquinucci, 1998b, "Softly Broken MQCD and the Theta Angle," Phys. Lett. B **421**, 131.

Bergman, O., 1998, "Three-Pronged Strings and 1/4 BPS States in $N=4$ Super-Yang-Mills Theory," Nucl. Phys. B **525**, 104.

Bergman, O., M. R. Gaberdiel, and G. Lifschytz, 1998a, "Branes, Orientifolds and the Creation of Elementary Strings," Nucl. Phys. B **509**, 194.

Bergman, O., M. R. Gaberdiel, and G. Lifschytz, 1998b, "String Creation and Heterotic-Type I' Duality," Nucl. Phys. B **524**, 524.

Bergshoeff, E., E. Sezgin, and P. K. Townsend, 1988, "Properties of the Eleven-Dimensional Supermembrane Theory," Ann. Phys. (N.Y.) **185**, 330.

Berkooz, M., P. Cho, P. Kraus, and M. J. Strassler, 1997, "Dual Descriptions of $SO(10)$ SUSY Gauge Theories with Arbitrary Numbers of Spinors and Vectors," Phys. Rev. D **56**, 7166.

Berkooz, M., M. R. Douglas, and R. G. Leigh, 1996, "Branes Intersecting at Angles," Nucl. Phys. B **480**, 265.

Bershadsky, M., A. Johansen, T. Pantev, V. Sadov, and C. Vafa, 1997, ''F-theory, Geometric Engineering and $N=1$ Dualities,'' Nucl. Phys. B **505**, 153.

Bershadsky, M., V. Sadov, and C. Vafa, 1996a, ''D-strings on D-manidolds,'' Nucl. Phys. B **463**, 398.

Bershadsky, M., V. Sadov, and C. Vafa, 1996b, ''D-branes and Topological Field Theories,'' Nucl. Phys. B **463**, 420.

Bigatti, D., and L. Susskind, 1998, ''Review of Matrix Theory,'' Phys. Lett. B **425**, 351.

Bilal, A., 1996, ''Duality in $N=2$ SUSY $SU(2)$ Yang-Mills Theory: A Pedagogical Introduction to the Work of Seiberg and Witten,'' 1996, preprint hep-th/9601007.

Blum, J. D., and K. Intriligator, 1997a, ''Consistency Conditions for Branes at Orbifold Singularities,'' Nucl. Phys. B **506**, 223.

Blum, J. D., and K. Intriligator, 1997b, ''New Phases of String Theory and 6D RG Fixed Points via Branes at Orbifold Singularities,'' Nucl. Phys. B **506**, 199.

Brandhuber, A., N. Itzhaki, V. Kaplunovsky, J. Sonnenschein, and S. Yankielowicz, 1997a, ''Comments on the M Theory Approach to $N=1$ SQCD and Brane Dynamics,'' Phys. Lett. B **410**, 27.

Brandhuber, A., N. Itzhaki, J. Sonnenschein, S. Theisen, and S. Yankielowicz, 1997b, ''On the M-Theory Approach to (Compactified) 5D Field Theories,'' Phys. Lett. B **415**, 127.

Brandhuber, A., J. Sonnenschein, S. Theisen, and S. Yankielowicz, 1997a, ''Brane Configurations and 4D Field Theory Dualities,'' Nucl. Phys. B **502**, 125.

Brandhuber, A., J. Sonnenschein, S. Theisen, and S. Yankielowicz, 1997b, ''M Theory and Seiberg-Witten Curves: Orthogonal and Symplectic Groups,'' Nucl. Phys. B **504**, 175.

Brodie, J. H., 1998, ''Two-Dimensional Mirror Symmetry from M-Theory,'' Nucl. Phys. B **517**, 36..

Brodie, J. H., and A. Hanany, 1997, ''Type IIA Superstrings, Chiral Symmetry, and $N=1$ 4D Gauge Theory Dualities,'' Nucl. Phys. B **506**, 157.

Brunner, I., A. Hanany, A. Karch, and D. Lust, 1998, ''Brane Dynamics and Chiral-Nonchiral Transitions,'' Nucl. Phys. B **528**, 197.

Brunner, I., and A. Karch, 1997, ''Branes and Six Dimensional Fixed Points,'' Phys. Lett. B **409**, 109.

Brunner, I., and A. Karch, 1998, ''Branes at Orbifolds versus Hanany Witten in Six Dimensions,'' J. High Energy Phys. **03**, 003.

Callan, Jr., C. G., J. Harvey, and A. Strominger, 1991a, ''Worldbrane Actions for String Solitons,'' Nucl. Phys. B **367**, 60.

Callan, Jr., C. G., J. Harvey, and A. Strominger, 1991b, ''World Sheet Approach to Heterotic Instantons and Solitons,'' Nucl. Phys. B **359**, 611.

Callan, Jr., C. G., J. Harvey, and A. Strominger, 1991c, ''Supersymmetric String Solitons,'' preprint hep-th/9112030.

Callan, Jr., C. G., and J. M. Maldacena, 1998, ''Brane Dynamics from the Born-Infeld Action,'' Nucl. Phys. B **513**, 198.

Cardy, J. L., 1982, ''Duality and the θ Parameter in Abelian Lattice Models,'' Nucl. Phys. B **205**, 17.

Cardy, J. L., and E. Rabinovici, 1982, ''Phase Structure of Z_p Models in the Presence of a θ Parameter,'' Nucl. Phys. B **205**, 1.

Csaki, C., and W. Skiba, 1997, ''Duality in Sp and SO Gauge Groups from M Theory,'' Phys. Lett. B **415**, 31.

Danielsson, U., G. Ferretti, and I. R. Klebanov, 1997, ''Creation of Fundamental Strings by Crossing D-branes,'' Phys. Rev. Lett. **79**, 1984.

de Alwis, S. P., 1997, ''A Note on Brane Creation,'' Phys. Lett. B **413**, 49.

de Boer, J., K. Hori, H. Ooguri, and Y. Oz, 1997a, ''Mirror Symmetry in Three-Dimensional Gauge Theories, Quivers and D-branes,'' Nucl. Phys. B **493**, 101.

de Boer, J., K. Hori, H. Ooguri, Y. Oz, and Z. Yin, 1997c, ''Mirror Symmetry in Three-Dimensional Gauge Theories, $SL(2,Z)$ and D-Brane Moduli Spaces,'' Nucl. Phys. B **493**, 148.

de Boer, J., K. Hori, H. Ooguri, and Y. Oz, 1998a, ''Kähler Potential and Higher Derivative Terms from M Theory Five-brane,'' Nucl. Phys. B **518**, 173.

de Boer, J., K. Hori, H. Ooguri, and Y. Oz, 1998b, ''Branes and Dynamical Supersymmetry Breaking,'' Nucl. Phys. B **522**, 20.

de Boer, J., K. Hori, and Y. Oz, 1997, ''Dynamics of $N=2$ Supersymmetric Gauge Theories in Three Dimensions,'' Nucl. Phys. B **500**, 163.

de Boer, J., K. Hori, Y. Oz, and Z. Yin, 1997, ''Branes and Mirror Symmetry in $N=2$ Supersymmetric Gauge Theories in Three Dimensions,'' Nucl. Phys. B **502**, 107.

de Boer, J., and Y. Oz, 1998, ''Monopole Condensation and Confining Phase of $N=1$ Gauge Theories Via M Theory Fivebrane,'' Nucl. Phys. B **511**, 155.

de Witt, B., P. G. Lauwers, R. Philippe, S. Q. Su, and A. Van Proyen, 1984, ''Gauge and Matter Fields Coupled to $N=2$ Supergravity,'' Phys. Lett. B **134**, 37.

Diaconescu, D.-E., 1997, ''D-branes, Monopoles, and Nahm Equations,'' Nucl. Phys. B **503**, 220.

Diaconescu, D.-E., and N. Seiberg, 1997, ''The Coulomb Branch of (4,4) Supersymmetric Field Theories in Two Dimensions,'' J. High Energy Phys. **07**, 001.

Dijkgraaf, R., E. Verlinde, and H. Verlinde, 1997, ''BPS Quantization of the Five-Brane,'' Nucl. Phys. B **486**, 89.

Di Vecchia, P., 1997, ''Duality In Supersymmetric Gauge Theories,'' Surveys High Energ. Phys. **10**, 119.

Douglas, M. R., 1995, ''Branes within Branes,'' preprint hep-th/9512077.

Douglas, M. R., 1996, ''Gauge Fields and D-branes,'' preprint hep-th/9604198.

Douglas, M. R., D. Kabat, P. Pouliot, and S. H. Shenker, 1997, ''D-branes and Short Distances in String Theory,'' Nucl. Phys. B **485**, 85.

Douglas, M. R., S. Katz, and C. Vafa, 1997, ''Small Instantons, Del Pezzo Surfaces and Type I′ Theory,'' Nucl. Phys. B **497**, 155.

Douglas, M. R., and M. Li, 1996, ''D-Brane Realization of $N=2$ Super Yang-Mills Theory in Four Dimensions,'' preprint hep-th/9604041.

Douglas, M. R., D. A. Lowe, and J. H. Schwarz, 1997, ''Probing F-theory with Multiple Branes,'' Phys. Lett. B **394**, 297.

Douglas, M. R., and G. Moore, 1996, ''D-branes, Quivers, and ALE Instantons,'' preprint hep-th/9603167.

Duff, M. J., R. R. Khuri, and J. X. Lu, 1995, ''String Solitons,'' Phys. Rep. **259**, 213.

Elitzur, S., A. Forge, A. Giveon, K. Intriligator, and E. Rabinovici, 1996, ''Massless Monopoles Via Confining Phase Superpotentials,'' Phys. Lett. B **379**, 121.

Elitzur, S., A. Forge, A. Giveon, and E. Rabinovici, 1995, "More Results in $N=1$ Supersymmetric Gauge Theories," Phys. Lett. B **353**, 79.

Elitzur, S., A. Forge, A. Giveon, and E. Rabinovici, 1996, "Effective Potentials and Vacuum Structure in $N=1$ Supersymmetric Gauge Theories," Nucl. Phys. B **459**, 160.

Elitzur, S., A. Giveon, and D. Kutasov, 1997, "Branes and $N=1$ Duality in String Theory," Phys. Lett. B **400**, 269.

Elitzur, S., A. Giveon, D. Kutasov, E. Rabinovici, and A. Schwimmer, 1997, "Brane Dynamics and $N=1$ Supersymmetric Gauge Theory," Nucl. Phys. B **505**, 202.

Elitzur, S., A. Giveon, D. Kutasov, and E. Rabinovici, 1998a, "Algebraic Aspects of Matrix Theory on T^d," Nucl. Phys. B **509**, 122.

Elitzur, S., A. Giveon, D. Kutasov, and D. Tsabar, 1998b, "Branes, Orientifolds and Chiral Gauge Theories," Nucl. Phys. B **524**, 251.

Erlich, J., A. Naqvi, and L. Randall, 1998, "The Coulomb Branch of $N=2$ Supersymmetric Product Group Theories from Branes," Phys. Rev. D **52**, 046002.

Evans, N., C. V. Johnson, and A. D. Shapere, 1997, "Orientifolds, Branes, and Duality of 4D Gauge Theories," Nucl. Phys. B **505**, 251.

Evans, N., and M. Schwetz, 1998, "The Field Theory of Non-Supersymmetric Brane Configurations," Nucl. Phys. B **522**, 69.

Fayyazuddin, A., and M. Spalinski, 1997a, "The Seiberg-Witten Differential from M-Theory," Nucl. Phys. B **508**, 219.

Fayyazuddin, A., and M. Spalinski, 1997b, "Extended Objects in MQCD," preprint hep-th/9711083.

Font, A., L. E. Ibanez, D. Lust, and F. Quevedo, 1990, "Strong-Weak Coupling Duality and Non-Perturbative Effects in String Theory," Phys. Lett. B **249**, 245.

Forste, S., D. Ghoshal, and S. Panda, 1997, "An Orientifold of the Solitonic Five-brane," Phys. Lett. B **411**, 46.

Furukawa, J., 1997, "Relation between S-duality in $N=2$ SQCD and Non-Abelian Duality in $N=1$ SQCD," preprint hep-th/9706121.

Ganor, O. J., 1997, "Toroidal Compactification of Heterotic 6D Non-Critical Strings Down to Four Dimensions," Nucl. Phys. B **488**, 223.

Ganor, O. J., D. R. Morrison, and N. Seiberg, 1997, "Branes, Calabi-Yau Spaces, and Toroidal Compactification of the $N=1$ Six-Dimensional E_8 Theory," Nucl. Phys. B **487**, 93.

Ganor, O. J., and S. Sethi, 1998, "New Perspectives on Yang-Mills Theories with Sixteen Supersymmetries," J. High Energy Phys. **01**, 007.

Gates, S. J., 1984, "Superspace Formulation of New Non-Linear Sigma Models," Nucl. Phys. B **238**, 349.

Gates, S. J., M. T. Grisaru, M. Roček, and W. Siegel, 1983, *Superspace or One Thousand and One Lessons in Supersymmetry* (Benjamin/Cummings, New York).

Gates, S. J., C. M. Hull, and M. Roček, 1984, "Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **248**, 157.

Gauntlett, J. P., J. A. Harvey, and J. T. Liu, 1993, "Magnetic Monopoles in String Theory," Nucl. Phys. B **409**, 363.

Gibbons, G. W., 1998, "Born-Infeld Particles and Dirichlet p-branes," Nucl. Phys. B **514**, 603.

Gibbons, G. W., and N. S. Manton, 1995, "The Moduli Space Metric for Well Separated BPS Monoples," Phys. Lett. B **356**, 32.

Gimon, E. G., and J. Polchinski, 1996, "Consistency Conditions for Orientifolds and D-Manifolds," Phys. Rev. D **54**, 1667.

Giveon, A., 1996, "On Non-Perturbative Results in Supersymmetric Gauge Theories—A Lecture," preprint hep-th/9611152.

Giveon, A., and O. Pelc, 1998, "M Theory, Type IIA String and 4D $N=1$ SUSY $SU(N_L)\times SU(N_R)$ Gauge Theory," Nucl. Phys. B **512**, 103.

Giveon, A., M. Porrati, and E. Rabinovici, 1994, "Target Space Duality in String Theory," Phys. Rep. **244**, 77.

Gomez, C., 1996, "D-Brane Probes and Mirror Symmetry," preprint hep-th/9612104.

Gomez, C., 1997, "On the Geometry of Soft Breaking Terms and $N=1$ Superpotentials," preprint hep-th/9706131.

Green, M. B., and M. Gutperle, 1996, "Comments on Three-Branes," Phys. Lett. B **377**, 28.

Green, M. B., and M. Gutperle, 1997, "Effects of D-instantons," Nucl. Phys. B **498**, 227.

Green, M. B., J. H. Schwarz, and E. Witten, 1987, *Superstring Theory* (Cambridge University, Cambridge, England).

Green, M. B., and P. Vanhove, 1997, "D-instantons, Strings and M-theory," Phys. Lett. B **408**, 122.

Greene, B. R., C. Vafa, and N. P. Warner, 1989, "Calabi-Yau Manifolds and Renormalization Group," Nucl. Phys. B **324**, 371.

Gregory, R., J. A. Harvey, and G. Moore, 1997, "Unwinding Strings and T-duality of Kaluza-Klein and H-Monopoles," preprint hep-th/9708086.

Hanany, A., and K. Hori, 1998, "Branes and $N=2$ Theories in Two Dimensions," Nucl. Phys. B **513**, 119.

Hanany, A., and Y. Oz, 1995, "On the Quantum Moduli Space of Vacua of $N=2$ Supersymmetric $SU(N_c)$ Gauge Theories," Nucl. Phys. B **452**, 283.

Hanany, A., M. J. Strassler, and A. Zaffaroni, 1998, "Confinement and Strings in MQCD," Nucl. Phys. B **513**, 87.

Hanany, A., and E. Witten, 1997, "Type IIB Superstrings, BPS Monopoles, and Three-Dimensional Gauge Dynamics," Nucl. Phys. B **492**, 152.

Hanany, A., and A. Zaffaroni, 1998a, "Chiral Symmetry from Type IIA Branes," Nucl. Phys. B **509**, 145.

Hanany, A., and A. Zaffaroni, 1998b, "Branes and Six Dimensional Supersymmetric Theories," Nucl. Phys. B **529**, 180.

Hanany, A., and A. Zaffaroni, 1998, "On the Realization of Chiral Four Dimensional Gauge Theories Using Branes," J. High Energy Phys. **05**, 001.

Harvey, J. A., 1996, "Magnetic Monopoles, Duality, and Supersymmetry," preprint hep-th/9603086.

Hashimoto, A., 1997, "The Shape of Branes Pulled by Strings," preprint hep-th/9711097.

Henningson, M., and P. Yi, 1998, "Four-dimensional BPS-spectra via M-theory," Phys. Rev. D **57**, 1291.

Ho, P.-M., and Y.-S. Wu, 1998, "Brane Creation in M(atrix) Theory," Phys. Lett. B **420**, 43.

Hori, K., H. Ooguri, and Y. Oz, 1998, "Strong Coupling Dynamics of Four-Dimensional $N=1$ Gauge Theories from M Theory Fivebrane," Adv. Theor. Math. Phys. **1**, 1.

Hori, K., and Y. Oz, 1997, "F-Theory, T-Duality on K3 Surfaces and $N=2$ Supersymmetric Gauge Theories in Four Dimensions," Nucl. Phys. B **501**, 97.

Howe, P. S., N. D. Lambert, and P. C. West, 1998, ''Classical M-Fivebrane Dynamics and Quantum $N=2$ Yang-Mills,'' Phys. Lett. B **418**, 85.

Hull, C. M., and P. K. Townsend, 1995, ''Unity of Superstring Dualities,'' Nucl. Phys. B **438**, 109.

Intriligator, K. 1997, ''RG Fixed Points in Six Dimensions via Branes at Orbifold Singularities,'' Nucl. Phys. B **496**, 177.

Intriligator, K., R. G. Leigh, and M. J. Strassler, 1995, ''New Examples of Duality in Chiral and Non-Chiral Supersymmetric Gauge Theories,'' Nucl. Phys. B **456**, 567.

Intriligator, K., D. R. Morrison, and N. Seiberg, 1997, ''Five-Dimensional Supersymmetric Gauge Theories and Degenerations of Calabi-Yau Spaces,'' Nucl. Phys. B **497**, 56.

Intriligator, K., and N. Seiberg, 1996a, ''Lectures on Supersymmetric Gauge Theories and Electric-Magnetic Duality,'' Nucl. Phys. B, Proc. Suppl. **45**, 1.

Intriligator, K., and N. Seiberg, 1996b, ''Mirror Symmetry in Three Dimensional Gauge Theories,'' Phys. Lett. B **387**, 513.

Ito, K., and N. Maru, 1998, ''Matrix String Theory from Brane Configuration,'' Phys. Lett. B **426**, 43.

Johnson, C. V., 1997, ''On the Orientifolding of Type II NS-Fivebranes,'' Phys. Rev. D **56**, 5160.

Kachru, S., A. Klemm, W. Lerche, P. Mayr, and C. Vafa, 1996, ''Nonperturbative Results on the Point Particle Limit of $N=2$ Heterotic String Compactifications,'' Nucl. Phys. B **459**, 537.

Karch, A., 1997, ''Seiberg Duality in Three Dimensions,'' Phys. Lett. B **405**, 79.

Kastor, D., E. Martinec, and S. Shenker, 1989, ''RG Flow in $N=1$ Discrete Series,'' Nucl. Phys. B **316**, 590.

Katz, S., A. Klemm, and C. Vafa, 1997, ''Geometric Engineering of Quantum Field Theories,'' Nucl. Phys. B **497**, 173.

Katz, S., and C. Vafa, 1997, ''Geometric Engineering of $N=1$ Quantum Field Theories,'' Nucl. Phys. B **497**, 196.

Kiritsis, E., 1997, ''Introduction to Superstring Theory,'' preprint hep-th/9709062.

Klemm, A., 1997, ''On the Geometry behind $N=2$ Supersymmetric Effective Actions in Four Dimensions,'' preprint hep-th/9705131.

Klemm, A., W. Lerche, P. Mayr, C. Vafa, and N. Warner, 1996, ''Self-Dual Strings and $N=2$ Supersymmetric Field Theory,'' Nucl. Phys. B **477**, 746.

Kol, B., 1997, ''$5d$ Field Theories and M Theory,'' preprint hep-th/9705031.

Kol, B., and J. Rahmfeld, 1998, ''BPS Spectrum of 5 Dimensional Field Theories, (p,q) Webs and Curve Counting,'' J. High Energy Phys. **08**, 006.

Kutasov, D., 1995, ''A Comment on Duality in $N=1$ Supersymmetric Non-Abelian Gauge Theories,'' Phys. Lett. B **351**, 230.

Kutasov, D., 1996, ''Orbifolds and Solitons,'' Phys. Lett. B **383**, 48.

Kutasov, D., and A. Schwimmer, 1995, ''On Duality in Supersymmetric Yang-Mills Theory,'' Phys. Lett. B **354**, 315.

Kutasov, D., A. Schwimmer, and N. Seiberg, 1996, ''Chiral Rings, Singularity Theory and Electric-Magnetic Duality,'' Nucl. Phys. B **459**, 455.

Kutasov, D., and N. Seiberg, 1990, ''Non-Critical Superstrings,'' Phys. Lett. B **251**, 67.

Kutasov, D., and N. Seiberg, 1991, ''Number of Degrees of Freedom, Density of States and Tachyons in String Theory and CFT,'' Nucl. Phys. B **358**, 600.

Lambert, N. D., and P. C. West, 1998, ''Gauge fields and M-Five-brane Dynamics,'' Nucl. Phys. B **524**, 141.

Landsteiner, K., and E. Lopez, 1998, ''New Curves from Branes,'' Nucl. Phys. B **516** 273.

Landsteiner, K., E. Lopez, and D. A. Lowe, 1997, ''$N=2$ Supersymmetric Gauge Theories, Branes and Orientifold,'' Nucl. Phys. B **507**, 197.

Landsteiner, K., E. Lopez, and D. A. Lowe, 1998, ''Duality of Chiral $N=1$ Supersymmetric Gauge Theories via Branes,'' J. High Energy Phys. **02**, 007.

Lee, S., A. Peet, and L. Thorlacius, 1998, ''Brane-Waves and Strings,'' Nucl. Phys. B **514**, 161.

Lerche, W., 1997, ''Introduction to Seiberg-Witten Theory and its Stringy Origin,'' Nucl. Phys. B, Proc. Suppl. **55**, 83.

Lykken, J., E. Poppitz, and S. P. Trivedi, 1998a, ''Chiral Gauge Theories from D-Branes,'' Phys. Lett. B **416**, 286.

Lykken, J., E. Poppitz, and S. P. Trivedi, 1998b, ''M(ore) on Chiral Gauge Theories from D-Branes,'' Nucl. Phys. B **520**, 51..

Maldacena, J. M., 1996, ''Black Holes in String Theory,'' preprint hep-th/9607235.

Martinec, E., 1989, ''Algebraic Geometry and Effective Lagrangians,'' Phys. Lett. B **217**, 431.

Mikhailov, A., 1998, ''BPS States and Minimal Surfaces,'' Nucl. Phys. B **533**, 243.

Montonen, C., and D. Olive, 1977, ''Magnetic Monopoles as Gauge Particles,'' Phys. Lett. B **72**, 117.

Morrison, D. R., and N. Seiberg, 1997, ''Extremal Transitions and Five-Dimensional Supersymmetric Field Theories,'' Nucl. Phys. B **483**, 229.

Nahm, W., 1980, ''A Simple Formalism for the BPS Monopoles,'' Phys. Lett. B **90**, 413.

Nakanishi, T., and A. Tsuchiya, 1992, ''Level Rank Duality of WZW Models in Conformal Field Theory,'' Commun. Math. Phys. **144**, 351.

Nakatsu, T., K. Ohta, and T. Yokono, 1998, ''On the Baryonic Branch Root of $N=2$ MQCD,'' Phys. Rev. D **58**, 026003.

Nakatsu, T., K. Ohta, T. Yokono, and Y. Yoshida, 1998a, ''Higgs Branch of $N=2$ SQCD and M theory Branes,'' Nucl. Phys. B **519**, 159.

Nakatsu, T., K. Ohta, T. Yokono, and Y. Yoshida, 1998b, ''A proof of Brane Creation via M-theory,'' Mod. Phys. Lett. A **13**, 293.

Nam, S., K. Oh, and S.-J. Sin, 1997, ''Superpotentials of $N=1$ Supersymmetric Gauge Theories from M-theory,'' Phys. Lett. B **416**, 319.

Nekrasov, N., 1998, ''Five-Dimensional Gauge Theories and Relativistic Integrable Systems,'' Nucl. Phys. B **531**, 323.

Nekrasov, N., and A. Lawrence, 1998, ''Instanton Sums and Five-Dimensional Gauge Theories,'' Nucl. Phys. B **513**, 239.

Ohta, N., T. Shimizu, and J.-G. Zhou, 1998, ''Creation of Fundamental String in M(atrix) Theory,'' Phys. Rev. D **57**, 2040.

Olive, D., 1995, ''Exact Electromagnetic Duality,'' preprint hep-th/9508089.

Ooguri, H., and C. Vafa, 1996, ''Two-Dimensional Black Hole and Singularities of CY Manifolds,'' Nucl. Phys. B **463**, 55.

Ooguri, H., and C. Vafa, 1997, ''Geometry of $N=1$ Dualities in Four Dimensions,'' Nucl. Phys. B **500**, 62.

Peet, A., 1998, ''The Bekenstein Formula and String Theory,'' Quant. Grav. **15**, 3291.

Peskin, M., 1997, ''Duality in Supersymmetric Yang-Mills Theory,'' preprint hep-th/9702094.

Polchinski, J., 1994, "Combinatorics of Boundaries in String Theory," Phys. Rev. D **50**, 6041.

Polchinski, J., 1995, "Dirichlet-Branes and Ramond-Ramond Charges," Phys. Rev. Lett. **75**, 4724.

Polchinski, J., 1996, "TASI Lectures on D-Branes," preprint hep-th/9611050.

Polchisnki, J., S. Chaudhuri, and C. V. Johnson, 1996, "Notes on D-Branes," preprint hep-th/9602052.

Polyakov, A., 1987, *Gauge Fields and Strings* (Harwood, Academic, Chur, Switzerland).

Polyakov, A., 1998, "String Theory and Quark Confinement," Nucl. Phys. Proc. Suppl. **68**, 1.

Porrati, M., and A. Rozenberg, 1998, "Bound States at Threshold in Supersymmetric Quantum Mechanics," Nucl. Phys. B **515**, 184.

Porrati, M., and A. Zaffaroni, 1997, "M-Theory Origin of Mirror Symmetry in Three Dimensional Gauge Theories," Nucl. Phys. B **490**, 107.

Pouliot, P., 1995, "Chiral Duals of Non-Chiral SUSY Gauge Theories," Phys. Lett. B **359**, 108.

Roček, M., K. Schoutens, and A. Sevrin, 1991, "Off-Shell WZW Models in Extended Superspace," Phys. Lett. B **265**, 303.

Schmaltz, M., and R. Sundrum, 1998, "$N=1$ Field Theory Duality from M-theory," Phys. Rev. D **57**, 6455.

Schwarz, J. H., 1995, "An $SL(2,Z)$ Multiplet of Type IIB Superstrings," preprint hep-th/9508143.

Schwarz, J. H., 1997a, "Lectures on Superstring and M Theory Dualities," Nucl. Phys. B, Proc. Suppl. **55**, 1.

Schwarz, J. H., 1997b, "The Status of String Theory," preprint hep-th/9711029.

Seiberg, N., 1988, "Supersymmetry and Non-Perturbative Beta Functions," Phys. Lett. B **206**, 75.

Seiberg, N., 1994, "Exact Results on the Space of Vacua of Four-Dimensional SUSY Gauge Theories," Phys. Rev. D **49**, 6857.

Seiberg, N., 1995a, "Electric-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories," Nucl. Phys. B **435**, 129.

Seiberg, N., 1995b, "The Power of Duality—Exact Results in 4D SUSY Field Theory," preprint hep-th/9506077.

Seiberg, N., 1996a, "IR Dynamics on Branes and Space-Time Geometry," Phys. Lett. B **384**, 81.

Seiberg, N., 1996b, "Five Dimensional SUSY Field Theories, Non-trivial Fixed Points and String Dynamics," Phys. Lett. B **388**, 753.

Seiberg, N., 1997a, "Non-trivial Fixed Points of The Renormalization Group in Six Dimensions," Phys. Lett. B **390**, 169.

Seiberg, N., 1998a, "Notes on Theories with 16 Supercharges," Nucl. Phys. Proc. Suppl. **67**, 158.

Seiberg, N., 1998b, "Matrix Description of M-theory on T^5 and T^5/Z_2," Phys. Lett. B **408**, 98.

Seiberg, N., and E. Witten, 1994a, "Monopole Condensation, and Confinement in $N=2$ Supersymmetric Yang-Mills Theory," Nucl. Phys. B **426**, 19.

Seiberg, N., and E. Witten, 1994b, "Monopoles, Duality and Chiral Symmetry Breaking in $N=2$ Supersymmetric QCD," Nucl. Phys. B **431**, 484.

Seiberg, N., and E. Witten, 1996, "Gauge Dynamics and Compactification to Three Dimensions," preprint hep-th/9607163.

Sen, A., 1996, "F-theory and Orientifolds," Nucl. Phys. B **475**, 562.

Sen, A., 1997a, "BPS States on a Three Brane Probe," Phys. Rev. D **55**, 2501.

Sen, A., 1997b, "A Non-perturbative Description of the Gimon-Polchinski Orientifold," Nucl. Phys. B **489**, 139.

Sen, A., 1997c, "F-theory and the Gimon-Polchinski Orientifold," Nucl. Phys. B **498**, 135.

Sen, A., 1997d, "A Note on Enhanced Gauge Symmetries in M- and String Theory," J. High Energy Phys. **09**, 001.

Shifman, M., 1997, "Non-Perturbative Dynamics in Supersymmetric Gauge Theories," Prog. Part. Nucl. Phys. **39**, 1.

Sierra, G., and P. K. Townsend, 1983, in *An Introduction to $N=2$ Rigid Supersymmetry*, in *Supersymmetry and Supergravity 1983*, edited by B. Milewski (World Scientific, Singapore), p. 396.

Strominger, A., 1996, "Open P-Branes," Phys. Lett. B **383**, 44.

Tatar, R., 1998, "Dualities in 4D Theories with Product Gauge Groups from Brane Configurations," Phys. Lett. B **419**, 99.

Taylor, T., G. Veneziano, and S. Yankielowicz, 1983, "Supersymmetric QCD and its Massless Limit: an effective Lagrangian Analysis," Nucl. Phys. B **218**, 439.

Terashima, S., 1998, "Supersymmetric Gauge Theories with Classical Groups via M Theory Five-brane," Nucl. Phys. B **526**, 163.

't Hooft, G., 1974, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72**, 461.

Thorlacius, L., 1998, "Born-Infeld String as a Boundary Conformal Field Theory," Phys. Rev. Lett. **80**, 1588.

Townsend, P. K., 1996, "Four Lectures on M-theory," preprint hep-th/9612121.

Townsend, P. K., 1997, "M-Theory from its Superalgebra," preprint hep-th/9712004.

Tseytlin, A. A., 1996, "Self-duality of Born-Infeld Action and Dirichlet 3-brane of Type IIB Superstring Theory," Nucl. Phys. B **469**, 51.

Vafa, C., 1996, "Evidence for F-Theory," Nucl. Phys. B **469**, 403.

Vafa, C., 1997, "Lectures on Strings and Dualities," preprint hep-th/9702201.

Vafa, C., and B. Zwiebach, 1997, "$N=1$ Dualities of SO and USp Gauge Theories and T-Duality of String Theory," Nucl. Phys. B **506**, 143.

Veneziano, G., and S. Yankielowicz, 1982, "An Effective Lagrangian for the Pure $N=1$ Supersymmetric Yang-Mills Theory," Phys. Lett. B **113**, 231.

Verlinde, E., 1995, "Global Aspects of Electric-Magnetic Duality," Nucl. Phys. B **455**, 211.

Volovich, A., 1997, "Domain Wall in MQCD and Supersymmetric Cycles in Exceptional Holonomy Manifolds," preprint hep-th/9710120.

Wess, J., and J. Bagger, 1992, *Supersymmetry and Supergravity* (Princeton University, Princeton, NJ).

Witten, E., 1993, "Phases of $N=2$ Theories in Two Dimensions," Nucl. Phys. B **403**, 159.

Witten, E., 1995a, "String Theory Dynamics in Various Dimensions," Nucl. Phys. B **443**, 85.

Witten, E., 1995b, "Some Comments on String Dynamics," preprint hep-th/9507121.

Witten, E., 1996a, "Bound States of Strings and p-Branes," Nucl. Phys. B **460**, 335.

Witten, E., 1996b, "Small Instantons in String Theory," Nucl. Phys. B **460**, 541.

Witten, E., 1997a, ''Solutions of Four-Dimensional Field Theories Via *M* Theory,'' Nucl. Phys. B **500**, 3.
Witten, E., 1997b, ''Branes and the Dynamics of QCD,'' Nucl. Phys. B **507**, 658.
Witten, E., 1997c, ''On the Conformal Field Theory of the Higgs Branch,'' J. High Energy Phys. **07**, 003.
Yoshida, Y., 1998, ''Geometrical Analysis of Brane Creation via *M*-Theory,'' Prog. Theor. Phys. **99**, 305.
Youm, D., 1997, ''Black Holes and Solitons in String Theory,'' preprint hep-th/9710046.